Abstract—Transmission of a Gaussian source over a time-varying multiple-input multiple-output (MIMO) channel is studied under strict delay constraints. Availability of a correlated side information at the receiver is assumed, whose quality, i.e., its correlation with the source signal, also varies over time. A block-fading model is considered for the states of the time-varying channel and side information; and perfect state information at the receiver is assumed, while the transmitter knows only the statistics. The high SNR performance, characterized by the distortion exponent, is studied for this joint source-channel coding problem. An upper bound is derived and compared with several lower bounds based on list decoding, hybrid digital-analog transmission, as well as multi-layer schemes, which transmit successive refinements of the source, relying on progressive or superposition transmission with list decoding. The optimal distortion exponent is characterized for the single-input multiple-output (SIMO) and multiple-input single-output (MISO) scenarios by showing that the distortion exponent achieved by multi-layer superposition encoding with joint decoding meets the proposed upper bound. In the MIMO scenario, the optimal distortion exponent is characterized in the low bandwidth ratio regime, and it is shown that the multi-layer superposition encoding performs very close to the upper bound in the high bandwidth ratio regime.

Index Terms—Distortion exponent, joint source-channel coding, time-varying channel and side information, list decoding, broadcast codes, successive refinement.

I. INTRODUCTION

Many applications in wireless networks require transmission of a source signal over a fading channel, e.g., multimedia signals over cellular networks, accumulation of sensor measurements at a fusion center, to be reconstructed with the minimum possible distortion at the destination. In many practical scenarios, the destination receives additional correlated side information about the underlying source signal, either from other transmissions, or through its own sensing devices. For example, measurements from other sensors at a fusion center, signals from repeaters in digital TV broadcasting, or relay signals in mobile networks.

Theoretical benefits of having correlated side information at the receiver for source compression are well known [4]. However, similarly to estimating the channel state information available side information to the transmitter, or may even be impossible in uncoordinated scenarios. Without the knowledge of the instantaneous channel and side information states, a transmitter needs to transmit in an adaptive manner. While the availability of perfect or imperfect channel state information (CSI) has been widely studied in the literature, the impact of time-varying correlated source side information in the joint source-channel coding context has not been considered before. In this work, we introduce a novel system model that brings together time-varying channel and correlated side-information at the receiver, and propose upper and lower bounds on the system performance.

We consider the transmission of a Gaussian source over a multiple-input multiple-output (MIMO) block-fading channel when the receiver has access to time-varying correlated source side information. Both the time-varying channel and the side-information are assumed to follow block-fading models, whose states are unknown at the transmitter. Moreover, strict delay constraints apply requiring the transmission of a block of source samples, for which the side-information state is constant, over a block of the channel, during which the channel state is also constant. The source and channel blocks do not necessarily have the same length, and their ratio is defined as the bandwidth ratio between the channel and the source bandwidths. We are interested in minimizing the average end-to-end distortion of the reconstructed source samples, averaged over many blocks. This may correspond to the average distortion over video frames in a video streaming application, where each frame has to be transmitted under a strict delay constraint.

When the knowledge of the channel and side information states is available at both the transmitter and the receiver (CSI-TR), Shannon’s separation theorem applies [5], assuming that the channel and source blocks are sufficiently long. However, the optimality of separation does not extend to non-ergodic scenarios such as the model studied in this paper, since each source block is required to be transmitted over a single channel block whose state is not known at the transmitter. We note that the suboptimality of separation also depends on the performance criterion. For example, it was shown in [6] that, if, instead of the average distortion, the outage distortion is considered, separate source and channel coding is still optimal.

This problem has been studied extensively in the literature in the absence of correlated side information at the receiver [7]–[9]. Despite the ongoing efforts, the minimum achievable average distortion remains an open problem; however, more conclusive results on the performance can be obtained by studying the distortion exponent, which characterizes the exponential decay of the expected distortion in the high SNR regime [10]. The distortion exponent has been studied for
parallel fading channels in [11], for the relay channel in [12], for point-to-point MIMO channels in [13], for channels with feedback in [14], for the two-way relay channel in [15], for the interference channel in [16], and in the presence of side information that might be absent in [17]. In the absence of source side information at the receiver, the optimal distortion exponent in MIMO channels is known in some regimes of operation, such as the large bandwidth regime [13] and the low bandwidth regime [13]. However, the general problem remains open. In [13] successive refinement source coding followed by superposition transmission is shown to achieve the optimal distortion exponent for high bandwidth ratios in MIMO systems. The optimal distortion exponent in the low bandwidth ratio regime is achieved through hybrid digital-analog transmission [13], [18]. In [19], superposition multi-layer schemes are shown to achieve the optimal distortion exponent for some other bandwidth ratios as well.

The source coding version of our problem, in which the encoder and decoder are connected by an error-free finite-capacity link, is studied in [20]. The single-input single-output (SISO) model in the presence of a time-varying channel and side information is considered for matched bandwidth ratios in [21], where uncoded transmission is shown to achieve the minimum expected distortion for certain side information fading gain distributions, while separate source and channel coding is shown to be suboptimal in general. A scheme based on list decoding at the receiver is proposed in [22] in the context of lossless broadcasting of a common source to multiple receivers with different side information qualities, and it is shown to achieve the optimal performance. This scheme is used in [21] for time-varying channel and side information, where different channel and side information states are considered as virtual users, and therefore, the problem is treated as a broadcasting problem to users with different channel and side information qualities.

Our goal in this work is to find tight bounds on the distortion exponent when transmitting a Gaussian source over a time-varying MIMO channel in the presence of time-varying correlated side information at the receiver. Main contributions of this work can be summarized as follows:

- We derive an upper bound on the distortion exponent by providing the channel state realization to the transmitter, while the source side information state remains unknown.
- We characterize the distortion exponent achieved by the list decoding (LD) scheme. While this scheme achieves a lower expected distortion than SSCC, we show that it does not improve the distortion exponent.
- Based on LD, we consider a hybrid digital-analog list decoding scheme (HDA-LD), as well as a multi-layer extension of LD, in which multiple layers, each carrying successive refinement information for the source sequence, are transmitted over each block. We consider both the progressive (LS-LD) and superposition (BS-LD) transmission of these layers, and derive the respective distortion exponents.

We show that the distortion exponent achieved by BS-LD meets the proposed upper bound for SISO/SIMO/MISO systems, characterizing the optimal distortion exponent in these scenarios. We show that HDA-LD also achieves the optimal distortion exponent in SISO channels.

- In the general MIMO setup, we characterize the optimal distortion exponent in the low bandwidth ratio regime, and show that it is achievable by both HDA-LD and BS-LD. We also show that, in certain regimes of operation, LS-LD outperforms all the other proposed schemes.

We will use the following notation in the rest of the paper. We denote random variables with upper-case letters, e.g., $X$, their realizations with lower-case letters, e.g., $x$, and the sets with calligraphic letters, e.g., $\mathcal{A}$. We denote $E_X[\cdot]$ as the expectation with respect to $X$, and $E_{\mathcal{A}}[\cdot]$ as the expectation over the set $\mathcal{A}$. We denote random vectors as $\mathbf{X}$ with realizations $x$. We denote by $\mathbb{R}^+$ the set of positive real numbers, and by $\mathbb{R}^{++}$ the set of strictly positive real numbers in $\mathbb{R}$, respectively. We define $(x)^\dagger = \max\{0, x\}$. Given two functions $f(x)$ and $g(x)$, we use $f(x) \triangleq g(x)$ to denote the exponential equality $\lim_{x \to \infty} \frac{\log f(x)}{\log g(x)} = 1$, while $\geq$ and $\leq$ are defined similarly.

The rest of the paper is organized as follows. The problem statement is given in Section II. Two upper bounds on the distortion exponent are derived in Section III. Various achievable schemes are studied in Section IV. The characterization of the optimal distortion exponent for certain regimes is relegated to Section V. Finally, the conclusions are presented in Section VI.

**II. PROBLEM STATEMENT**

We wish to transmit a zero mean, unit variance complex Gaussian source sequence $S^m \in \mathbb{C}^m$ of independent and identically distributed (i.i.d.) random variables, i.e., $S_i \sim \mathcal{CN}(0, 1)$, over a complex MIMO block Rayleigh-fading channel with $M_t$ transmit and $M_r$ receiver antennas, as shown in Figure 1. In addition to the channel output, time-varying correlated source side information is also available at the receiver. Time-variations in the source side information are assumed to follow a block fading model as well. The channel and the side information states are assumed to be constant for the duration of one block, and independent of each other, and among different blocks. We assume that each source block is composed of $m$ source samples, which, due to the delay limitations of the underlying application, must be transmitted over one block of the channel, which consists of $n$ channel samples.
uses. We define the bandwidth ratio of the system as
\[ b \triangleq \frac{n}{m} \text{ channel dimension per source sample.} \] (1)

The encoder maps each source sequence \( S^m \) to a channel input sequence \( X^n = [X_1, \ldots, X_n] \in C^{M_t \times n} \) using an encoding function \( f^{(m,n)} : C^m \rightarrow C^{M_t \times n} \) such that the average power constraint is satisfied: \( \sum_{i=1}^n \text{Tr} \{ E[X_i^H X_i] \} \leq n \cdot M_t \).

If codeword \( x^n \) is transmitted, the signal received at the destination is modeled by memoryless slow fading channel
\[ Y_i = \sqrt{\rho} M_t H x_i + N_i, \quad i = 1, \ldots, n, \] (2)

where \( H \in C^{M_t \times M_r} \) is the channel matrix with i.i.d. zero mean complex Gaussian entries, i.e., \( H_{ij} \sim CN(0,1) \), whose realizations are denoted by \( H, \rho \in \mathbb{R}^+ \) is the average signal to noise ratio (SNR) in the channel, and \( N_i \) models the additive noise with \( N_i \sim CN(0,1) \). We define \( M^* = \max\{M_t, M_r\} \) and \( M_t = \min\{M_t, M_r\} \), and consider \( \lambda M_t \geq \cdots \geq \lambda_1 > 0 \) to be the eigenvalues of \( HH^H \).

In addition to the channel output \( Y^n = [Y_1, \ldots, Y_n] \in C^{M_t \times n} \), the decoder observes \( T^m \in C^m \), a randomly degraded version of the source sequence:
\[ T^m = \sqrt{\rho} \Gamma c S^m + Z^m, \] (3)

where \( \Gamma \sim CN(0,1) \) models time-varying Rayleigh fading in the quality of the side information, \( \rho_s \in \mathbb{R}^+ \) models the average quality of the side information, and \( Z_j \sim N(0,1), j = 1, \ldots, m \), models the noise. We define the side information gain as \( \Gamma \triangleq [\Gamma_{c,\Gamma}], \) and its realization as \( \gamma \). Then, \( \Gamma \) follows an exponential distribution with probability density function (pdf):
\[ p_{\Gamma} (\gamma) = e^{-\gamma}, \quad \gamma \geq 0. \] (4)

In this work, we assume that the receiver knows the side information and channel realizations, \( \gamma \) and \( H, \) while the transmitter is only aware of their distributions. The receiver reconstructs the source sequence \( S^m = g^{(m,n)}(Y^n, T^m, H, \gamma) \) with a mapping \( g^{(m,n)} : C^n \times M_t \times C^m \times C^{M_t \times M_r} \times \mathbb{R} \rightarrow C^m \). The distortion between the source sequence and the reconstruction is measured by the quadratic average distortion \( D \triangleq \frac{1}{m} \sum_{i=1}^m |S_i - \hat{S}_i|^2. \)

We are interested in characterizing the minimum expected distortion, \( E[D] \), where the expectation is taken with respect to the source, the side information and channel state realizations, as well as the noise terms, and expressed as
\[ E[D] \triangleq \lim_{n,m \to \infty} \min_{f^{(m,n)}} \text{E}_{g^{(m,n)}} \text{E}_{n} E[D]. \] (5)

In particular, we are interested in characterizing the optimal performance in the high SNR regime, i.e., when \( \rho,\rho_s \to \infty \).

We define \( \nu \) as a measure of the average side information quality in the high SNR regime. We assume that the average side information scales as follows:
\[ \nu = \lim_{\rho_s \to \infty} \frac{\log \rho_s}{\log \rho}. \] (6)

Parameter \( \nu \) captures the increase in the quality of the side information with respect to the average SNR in the channel. For example, if the side information is made available to the receiver through other transmissions, if the average SNR in the channel increases, so does the side information quality.

The performance measure we consider is the distortion exponent, defined as
\[ \Delta(b, \nu) \triangleq -\lim_{\rho,\rho_s \to \infty} \frac{\log E[D]}{\log \rho}. \] (7)

### III. DISTORTION EXPONENT UPPER BOUND

In this section we derive an upper bound on the distortion exponent by extending the bound on the expected distortion \( ED^* \) obtained in [21] to the MIMO setup with bandwidth mismatch, and analyzing the high SNR behavior. The upper bound is constructed by providing the receiver with only the channel state realization, \( H, \) while the side information state, \( \gamma, \) remains unknown. We call this the partially informed encoder upper bound. The optimality of separate source and channel coding is shown in [21] when the side information fading gain distribution is discrete, or continuous and quasiconcave for \( b = 1 \).

As shown in [20], [21], if \( p_{\Gamma}(\gamma) \) is monotonically decreasing, the optimal source encoder ignores the side information completely, and the side-information is used only at the decoder for source reconstruction.

Concatenating this side-information-ignorant source code with a channel code at the instantaneous capacity, the minimum expected distortion at each channel state \( H \) is given by
\[ D_{op}(\rho, \rho_s, b, H) = \frac{1}{\rho_s} e^{\frac{\rho_b}{\rho_s}} E_1 \left( \frac{\rho_b}{\rho_s} \right), \] (8)

where \( E_1(x) \) is the exponential integral given by \( E_1(x) = \int_{x}^{\infty} t^{-1} e^t dt. \) Averaging over the channel state realizations, the expected distortion is lower bounded as
\[ ED^*_p(\rho, \rho_s, b, H) = H[D_{op}(\rho, \rho_s, b, H)]. \] (9)

Then, an upper bound on the distortion exponent is found by analyzing the high SNR behavior of (9). This upper bound will be expressed in terms of the diversity-multiplexing tradeoff (DMT), which measures the tradeoff between the rate and reliability in the transmission of a message over a MIMO fading channel in the asymptotic high SNR regime [23]. For a family of channel codes with rate \( R = r \log \rho \), where \( r \) is the multiplexing gain, the DMT is the piecewise-linear function \( d^*(r) \) connecting the points \((k, d^*(k))\), \( k = 0, \ldots, M_s \), where \( d^*(k) = (M_s - k)(M_s - k) \). More specifically, for \( r \geq M_s \), we

\[ d^*(r) = \left( \frac{r}{M_s} \right)^{M_s}, \quad \text{for } r \geq M_s. \]

Although in our setup the side information state \( \Gamma \) is complex, the receiver can always correct the phase to have an equivalent real side information state with Rayleigh amplitude. Our setup then reduces to the two parallel problems of reconstructing the real and imaginary parts of \( S^m \) with the same side information gain, and all the techniques of [21] can be applied.

We note that when the distribution of the side information is not Rayleigh, the optimal encoder follows a different strategy. For example, for quasiconcave continuous distributions the optimal source code compresses the source aiming at a single target side information state. See [21] for details.
have $d^*(r) = 0$, and for $0 \leq r \leq M_*$ satisfying $k < r \leq k + 1$ for some $k = 0, 1, \ldots, M_* - 1$, the DMT curve is given by

$$d^*(r) \triangleq \Phi_k - \Upsilon_k(r - k),$$

where we have defined

$$\Phi_k \triangleq (M^* - k)(M_* - k) \quad \text{and} \quad \Upsilon_k \triangleq (M^* + M_* - 2k - 1).$$

**Theorem 1.** Let $l = 1$ if $\nu/M_* < M^* - M_* + 1$, and let $l \in \{2, \ldots, M_*\}$ be the integer satisfying $2l - 3 + M^* - M_* \leq \nu/M_* < 2l - 1 + M^* - M_*$ if $M^* - M_* + 1 \leq \nu/M_* < M^* + M_* - 1$. The distortion exponent is upper bounded by

$$\Delta_{up}(b, \nu) = \begin{cases} \nu & \text{if } 0 \leq b < \frac{\nu}{M_*}, \\ bM_* & \text{if } \frac{\nu}{M_*} \leq b < M^* - M_* + 1, \\ \nu + d^*(\frac{\nu}{b}) & \text{if } 1 + M^* - M_* \leq b < \frac{\nu}{M_* - k} \\ \nu + d^*(\frac{\nu}{b}) & \text{if } M^* + M_* - 1 \leq b, \end{cases}$$

where $k \in \{1, \ldots, M_* - 1\}$ is the integer satisfying $2k - 1 + M^* - M_* \leq b < 2k + 1 + M^* - M_*$, and

$$\Delta_{MIMO}(b) \triangleq \sum_{i=1}^{M_*} \min\{b, 2i - 1 + M^* - M_*\}. \quad (14)$$

If $\nu/M_* \geq M^* + M_* - 1$, then

$$\Delta_{up}(b, \nu) = \nu + d^*(\nu/b), \quad (15)$$

where $d^*(r)$ is the DMT characterized in [10,11].

**Proof:** The proof is given in Appendix A. \hfill \blacksquare

A looser bound, denoted as the **fully informed encoder upper bound**, is obtained if both the channel state $H$ and the side information state $\gamma$ are provided to the transmitter. At each realization, the problem reduces to the static setup in [5], and source-channel separation theorem applies; that is, the concatenation of a Wyner-Ziv source code with a source-channel separation theorem applies; that is, the expected distortion is minimized by avoiding outage in source decoding, that is, by not using binning. Then, the optimal SSCC scheme compresses the source sequence at a fixed rate ignoring the source-side information, and transmits the compressed bits over the channel using a channel code at a fixed rate. At the receiver, first the transmitted channel codeword is recovered. If the channel decoding is successful, the compression codeword is recovered, and the source sequence is reconstructed together with the side information. Otherwise, only the side information is used for reconstruction.

Instead of using explicit binning at the source encoder, and decoding a single channel codeword, in LD the channel decoder outputs a list of channel codeword candidates, which are then used by the source decoder together with the source-side information to find the transmitted source codeword. The success of decoding for this scheme depends on the joint quality of the channel and side information states. List decoding is proposed in [22] in the context of lossless broadcast of a common source to multiple receivers with different side information qualities, and shown to achieve the optimal necessary and sufficient conditions. Here, we interpret each channel and side information state pair as a different virtual user, similarly to the approach used by Shamai for fading channels in [24]. Then the problem becomes a joint source-channel broadcasting problem to an infinite number of users each having a different channel and side information quality.

At the encoder, we generate a codebook of $2^{mR_{ld}}$ length-$n$ quantization codewords $W^m(i)$ through a ‘test channel’ given by $W = S + Q$, where $Q \sim \mathcal{CN}(0, \sigma_Q^2)$ is independent of $S$; and an independent Gaussian codebook of size $2^{bR_{ld}}$ with length-$n$ codewords $X(i) \in \mathbb{C}^{M_1 \times n}$, where $X \sim \mathcal{CN}(0, I)$, such that $bR_{ld} = I(S; W) + \epsilon$, for an arbitrarily small $\epsilon > 0$, i.e., with $\sigma_Q^2 = (2^{bR_{ld}} - 1)^{-1}$. Given a source outcome $S^m$, the transmitter finds the quantization codeword $W^m(i)$ jointly typical with the source outcome, and transmits the corresponding channel codeword $X(i)$. The channel decoder looks for the list of indices $I$ of jointly typical codewords $(X^m(l), Y^n)$. Then, the source decoder finds the unique $W^m(i)$ jointly typical with $T^m$ among the codewords $W^m(l)$, $l \in I$.

Joint decoding produces a binning-like decoding: only some $Y^n$ are jointly typical with $X(i)$, generating a virtual bin, or list, of $W^m$ codewords from which only one is jointly typical with $T^m$ with high probability. The size of the list depends on the realizations of $H$ and $\Gamma$ unlike in a Wyner-Ziv scheme, in which the bin sizes are chosen in advance. Therefore, if we disregard the outage event dependencies on the channel and the side information states $(H, \gamma)$, an outage is declared whenever, due to the channel and side information randomness, a unique
codeword cannot be recovered, and is given by
\[
O_{ld} = \{ (\mathbf{H}, \gamma) : I(S; W|T) \geq b I(X; Y) \},
\] (17)
where \( I(X; Y) = \log \det (I + \frac{\rho_s}{\delta} \mathbf{H} \mathbf{H}^H) \) and \( I(S; W|T) = \log (1 + (\rho_s R_{ld} - \epsilon - 1)/\gamma). \)

If \( W^m \) is successfully decoded, the source sequence is estimated with an MMSE estimator using the quantization codeword and the side information sequence, i.e., \( \hat{S}_i = E[S_i|W_i, T_i] \), and reconstructed with a distortion \( D_d(b R_{ld}; \gamma) \), where
\[
D_d(R, \gamma) = (\rho_s \gamma + 2^R - 1)^{-1}. \quad (18)
\]

If there is an outage, only the side information is used in source reconstruction, and the corresponding distortion is given by \( D_d(0, \gamma) \). Then, the expected distortion for LD is expressed as
\[
ED_{ld}(R_{ld}) = E_{O_{ld}} [D_d(b R_{ld}, \Gamma)] + E_{O_{ld}}[D_d(0, \Gamma)]. \quad (19)
\]

**Theorem 2.** The achievable distortion exponent for LD, \( \Delta_{ld}(b, \nu) \), is given by
\[
\Delta_{ld}(b, \nu) = \max \left\{ \nu, b \frac{\Phi_k + k \Upsilon_k + \nu}{k+\nu} \right\}, \quad (20)
\]
for \( b \in \left[ \frac{\Phi_k + \nu}{k+1}, \frac{\Phi_k + \nu}{k} \right], k = 0, 1, ..., M_s - 1, \)
where \( \Phi_k \) and \( \Upsilon_k \) are as defined in (17).

**Proof:** See Appendix B.

LD reduces the probability of outage, and thus, the expected distortion is also reduced, compared to SSCC. Figure 2 shows the expected distortion achievable by SSCC and LD schemes, as well as the partially informed encoder lower bound on the expected distortion in a SISO and a 3 × 3 MIMO system for \( b = 2 \). It is observed that LD outperforms SSCC in both SISO and MIMO scenarios, although both schemes fall short of the expected distortion lower bound, \( ED_{pl} \). We also observe that both schemes keep a constant performance gap as the SNR increases. In fact, the next proposition, given without proof, reveals that both schemes achieve the same distortion exponent.

**Proposition 1.** The distortion exponent of LD, \( \Delta_{ld}(b, \nu) \), is the same as that SSCC, i.e., \( \Delta_{ld}(b, \nu) = \Delta_s(b, \nu) \).

We note that although LD and SSCC achieve the same distortion exponent in the current setting, LD is shown to achieve larger distortion exponents than SSCC in general [21].

Next, we extend the idea of list decoding as a building block for more advanced transmission strategies.

**B. Hybrid digital-analog list decoding scheme (HDA-LD)**

We introduce a hybrid digital-analog (HDA) scheme that quantizes the source sequence, uses a scaled version of the quantization error as the channel input, and exploits list decoding at the decoder. This scheme is introduced in [25], and shown to be optimal in static SISO channels in the presence of side information for \( b = 1 \). HDA-LD is considered in [21] in the SISO fading setup with \( b = 1 \), and is shown to achieve the optimal distortion exponent for a wide family of side information distributions. In this paper, we propose a generalization of HDA-LD in [21] to the MIMO channel and to bandwidth ratios satisfying \( \nu \geq 1/M_s \).

For \( \nu \leq 1/M_s \), we ignore the available side information and use the hybrid digital-analog scheme proposed in [18]. In this scheme, which we denote by superposed HDA (HDA-S), the source sequence is transmitted in two layers. The first layer transmits a part of the source sequence in an uncoded fashion, while the second layer digitally transmits the remaining samples. The two layers are superposed and the power is allocated between the two. At the receiver, the digital layer is decoded treating the uncoded layer as noise. Then, the source sequence is reconstructed using both layers. The distortion exponent achievable by HDA-S is given by \( \Delta_h(b, \nu) = b M_s \) for \( b \leq 1/M_s \) [18].

HDA-S can be modified to include list decoding and to use the available side information at the reconstruction to reduce the expected distortion. However, as we will show in Section V-A, if \( 0 \leq b \leq \nu/M_s \), simple MMSE estimation of the source sequence is sufficient to achieve the optimal distortion exponent, given by \( \Delta^*(b, \nu) = \nu \), while, if \( \nu/M_s \leq b \leq 1/M_s \), HDA-S is sufficient to achieve the optimal distortion exponent. Therefore, HDA-S with list decoding does not improve the distortion exponent in this regime.

**Lemma 1.** The distortion exponent achievable by HDA-S is given by \( \Delta_h(b, \nu) = b M_s \), if \( b \leq 1/M_s \).

Next, we consider the HDA-LD scheme for \( b M_s > 1 \). At the encoder, a quantization codebook of \( 2^m R_b \) length-\( m \) codewords \( W^m(s) \), \( s = 1, ..., 2^m R_b \), is generated with a test channel \( W = S + Q \), where \( Q \sim \mathcal{CN}(0, \sigma_Q^2) \) is independent of \( S \), and the quantization noise variance is chosen such that \( R_b = I(W; S) + \epsilon \), for an arbitrarily small \( \epsilon > 0 \), i.e., \( \sigma_Q^2 \triangleq (2 R_b - \epsilon - 1)^{-1} \). Then, each \( W^m \) is reordered into length-\( M_s \) codewords \( W(s) = [W_1(s), ..., W_{M_s}(s)] \in \mathcal{C}^{M_s} \), where \( W_i(s), i = 1, ..., m/M_s \), is given by \( W_i(s) = [W_{(i-1)M_s+1}(s); ..., W_{iM_s}(s)]^T \). Similarly, we can reorder \( S^m \).
and $Q^m$, and define $S_i$ and $Q_i$ for $i = 1, \ldots, m/M_s$.

We then generate $2^{mR_b}$ independent auxiliary random vectors $U \in \mathcal{C}^{(n - \frac{n}{M_s}) \times M_s}$, distributed as $U_i \sim \mathcal{CN}(0, I)$, for $i = 1, \ldots, n - \frac{n}{M_s}$ and assign one to each $W(s)$ to construct the codebook of size $2^{mR_b}$ consisting of the pairs of codewords $(W(s), U(s))$, $s = 1, \ldots, 2^{mR_b}$. For a given source sequence $S^m$, the encoder looks for the $s$-th codeword $(W(s), U(s))$ such that $(W(s^*), S^m)$ are jointly typical. A unique $s^*$ is found if $M_sR_b > I(W; S)$. Then, the pair $(W(s^*), U(s^*))$ is used to generate the channel input, which is scaled to satisfy the power constraint:

$$X_i = \begin{cases} \sqrt{\frac{b}{M_s}}[S_i - W_i(s^*)], & \text{for } i = 1, \ldots, \frac{m}{M_s}, \\ U_i - \frac{n}{M_s}(s^*), & \text{for } i = \frac{m}{M_s} + 1, \ldots, n. \end{cases}$$  

(21)

Basically, in the first block of $\frac{n}{M_s}$ channel accesses we transmit a scaled version of the error of the quantization codeword $Q_i$, in an uncoded fashion, while in the second block of $n - \frac{n}{M_s}$ accesses we transmit a digital codeword.

At the receiver, list-decoding is successful with high probability if [21]

$$I(W; S) < M_sR_b < I(WU; YT)$$  

(22)

The outage event can be shown to be given, after some algebra, as follows.

$$\mathcal{O}_h = \left\{(H, \gamma) : I(W, S) \geq I(W; Y_W T) \right\}$$

(23)

$$+ (bM_s - 1)I(U; Y_V),$$

where $I(U; Y_V) = \log \det(I + \frac{b}{M_s}HH^H)$ and,

$$I(W; Y_W T) = \log \left(\frac{\left(\xi(1 + \sigma_Q^2)^{M_s} \det(I + \frac{b}{M_s}HH^H)\right)}{\det(I + \sigma_Q^2\left(\frac{b}{M_s}HH^H + \xi I\right))}\right),$$  

(24)

where $\xi \triangleq 1 + \rho_s \gamma$.

If $W^m$ is successfully decoded, each $X^n$ is reconstructed with an MMSE estimator using $Y^n$ and $T^n$ within a distortion given by

$$D_h(\sigma_Q^2, H, \gamma) = \frac{1}{M_s} \sum_{i=1}^{M_s} \left(1 + \rho_s \gamma + \frac{1}{\sigma_Q^2} \left(1 + \frac{b}{M_s} \lambda_i\right)\right)^{-1}$$  

(25)

If an outage occurs, $W^m$ is not decoded, and only $T^n$ is used in the reconstruction, since $X^n$ is uncorrelated with the source sequence by construction, and so is $Y^n$. Using an MMSE estimator, the achievable distortion is given by $D_d(0, \gamma)$.

The expected distortion for HDA-LD is given by

$$ED_h(R_b) = EC_h[D_h(\sigma_Q^2, H, \Gamma)] + EC_h[D_d(0, \Gamma)].$$  

(26)

The distortion exponent of HDA-LD, $\Delta_h(b, \nu)$, is characterized in the next theorem.

**Theorem 3.** Let $bM_s > 1$. The distortion exponent achieved by HDA-LD, $\Delta_h(b, \nu)$, is given by $\nu$ if $\frac{b}{M_s} \leq b < \frac{\nu}{\lambda_s}$ and

$$\Delta_h(b, \nu) = 1 + \frac{(bM_s - 1)(\Phi_k + k\gamma_k - 1 + \nu)}{bM_s - 1 + M_s\gamma_k},$$  

(27)

for

$$b \in \left[\frac{\Phi_k + \gamma}{k + 1} + \frac{\Phi_k - 1 + \nu}{M_s}, \frac{1}{M_s}\right],$$  

(28)

and $k = 0, \ldots, M_s - 1$.

**Proof:** See Appendix C.

C. Progressive multi-layer LD transmission (LS-LD)

In this section, we consider a multi-layer transmission scheme to improve the achievable distortion exponent, in particular in the high bandwidth ratio regime. Multi-layer transmission is proposed in [13] to combat channel fading by transmitting multiple layers that carry successive refinements of the source sequence. At the receiver, as many layers as possible are decoded depending on the channel state. The better the channel state, the more layers can be decoded and the smaller is the distortion at the receiver. We propose to use successive refinement codewords that exploit the side information at the receiver [26]. Then, the refinement codewords are transmitted one after the other over the channel using the LD scheme introduced in Section IV-A. Similarly to [13], we assume that each layer is allocated the same time resources (or number of channel accesses). In the limit of infinite layers, this assumption does not incur a loss in performance.

At the encoder, we generate $L$ Gaussian quantization codebooks, each with $2^{nR_l}$ codewords $W^m_l$ and $bR_l/L = I(S; W_l | W_{l-1}) + \epsilon$, for $l = 1, \ldots, L$, with an arbitrarily small $\epsilon > 0$, such that each Gaussian codebook is a refinement for the previous layers [26]. The quantization codewords $W^m_l$ are generated with a test channel given by $W_l = S + \sum_{l=1}^{L} Q_l$, for $l = 1, \ldots, L$, where $Q_l \sim \mathcal{N}(0, \sigma_l^2)$ is independent of $S$ and each other. Note that $T - S - W_l - W_{l-1} - \cdots - W_1$ form a Markov chain. For a given rate tuple $R \triangleq [R_1, \ldots, R_L]$, with $R_1 \geq \cdots \geq R_L \geq 0$, the quantization noise variances satisfy

$$\sum_{l=1}^{L} \sigma_l^2 = (2\sum_{l=1}^{L} (\frac{b}{L} R_l - \epsilon) - 1)^{-1}, \quad l = 1, \ldots, L.$$  

(30)

We generate $L$ independent channel codebooks, each with $2^{nR_l}$ length-$\frac{b}{L}$ codewords $X^n_l \in \mathcal{C}^{M_s \times nL}$ with $X^n_l \sim \mathcal{CN}(0, I)$. Each successive refinement codeword is transmitted using LD as in Section IV-A. At the destination, the decoder successively decodes each refinement codeword using joint decoding from the first layer up to the $L$-th layer. Then, $l$ layers will be successfully decoded if

$$I(S; W_l | T, W_{l-1}) < \frac{b}{L} I(X; Y) \leq I(S; W_{l+1} | T, W_l),$$  

(31)

that is, $l$ layers are successfully decoded while there is an outage in decoding the $(l+1)$-th layer. Let us define the outage event, for $l = 1, \ldots, L$, as follows

$$\mathcal{O}_l^{\frac{b}{L}} \triangleq \left\{(H, \gamma) : I(S; W_l | T, W_{l-1}) \geq \frac{b}{L} I(X; Y)\right\},$$  

(32)

where $I(X; Y) = \log \det(I + \frac{b}{L} HH^H)$, and, with $R_0 \triangleq 0$,

$$I(S; W_l | W_{l-1}, T) = \log \left(\frac{2^{\sum_{i=1}^{L} \frac{b}{L} R_i + \gamma \rho_s}}{2^{\sum_{i=1}^{L} \frac{b}{L} R_i + \gamma \rho_s}}\right).$$  

(33)
The expected distortion can be expressed as follows.

\[ E D_{l_0}(R) = \sum_{l=0}^{L} E_{(O_{l+1})} \cap O_{l+1} \left[ D_{d} \left( \sum_{i=1}^{l} b R_{i} / L, \gamma \right) \right] \]  

(34)

The distortion exponent achieved by LS-LD is given next.

**Theorem 4.** Let us define

\[ \phi_{k} \triangleq M^{*} - M_{s} + 2k - 1, \quad M_{k} \triangleq M_{s} - k + 1 \]  

(35)

and the sequence \{c_{i}\} as

\[ c_{0} = 0, \quad c_{i} = c_{i-1} + \phi_{k} \ln \left( \frac{M_{i}}{M_{i-1}} \right), \]  

(36)

for \( i = 1, \ldots, M_{s} - 1 \), and \( c_{M_{s}} = \infty \).

The distortion exponent achieved by LS-LD with infinite number of layers is given by \( \Delta_{\lambda}^{*}(b, \nu) = \nu \) if \( b \leq \nu / M_{s} \), and if

\[ c_{k-1} + \frac{\nu}{M_{k}} \leq b < c_{k} + \frac{\nu}{M_{k-1}} \]  

(37)

for some \( k \in \{1, \ldots, M_{s}\} \), the achievable distortion exponent is given by

\[ \Delta_{\lambda}^{*}(b, \nu) = \nu + \sum_{i=1}^{k-1} \phi_{i} + M_{k} \phi_{k} \]  

\[ \times \left( 1 - e^{-b(1-\kappa^{*})-c_{k-1}} \right) \]  

(38)

\[ \times \left( 1 - e^{-b(1-\kappa^{*})-c_{k-1}} \right) \]  

(39)

where

\[ \kappa^{*} = \frac{\phi_{k} b}{b} W \left( \frac{e^{-b(1-\kappa^{*})-c_{k-1}}}{M_{k} \phi_{k}} \nu \right) \]  

(40)

and \( W(z) \) is the Lambert \( W \) function, which gives the principal solution for \( w = z \).

**Proof:** See Appendix [I].

The proof of Theorem 4 indicates that the distortion exponent for LS-LD is achieved by allocating an equal rate among the first \( \kappa^{*} L \) layers to guarantee that the distortion exponent is at least \( \nu \). Then, the rest of the refinement layers are used to further increase the distortion exponent with the corresponding rate allocation. Note that for \( \nu = 0 \), we have \( \kappa^{*} = 0 \), and Theorem 4 boils down to Theorem 4.2 in [I].

**D. Broadcast strategy with LD (BS-LD)**

In this section, we consider the broadcast strategy in which the successive refinement layers are transmitted by superposition, and are decoded one by one with list decoding. The receiver decodes as many layers as possible using successive joint decoding, and reconstructs the source sequence using the successfully decoded layers and the side information sequence.

At the encoder, we generate \( L \) Gaussian quantization codebooks, at rates \( b R_{l} = I(S; W_{l}^{r-1}) + \epsilon, \ l = 1, \ldots, L, \) \( \epsilon > 0 \), as in Section 4.4C and L channel codebooks \( X_{n}^{l}, \ l = 1, \ldots, L, \) i.i.d. with \( X_{l,1} \sim C_{N}(0, I) \). Let \( \rho = [\rho_{1}, \ldots, \rho_{L}, \rho_{L+1}]^{T} \) be the power allocation among channel codebooks such that \( \rho_{l} = \rho^{l+1} - \rho^{l} \) with \( 1 = \xi_{0} \geq \xi_{1} \geq \ldots \geq \xi_{L} \geq 0 \), and define \( \xi \triangleq [\xi_{1}, \ldots, \xi_{L}] \). In the last layer, the layer \( L + 1 \), Gaussian i.i.d. noise sequence with distribution \( N_{l} \sim C_{N}(0, I) \) is transmitted using the remaining power \( \rho_{L+1} = \rho^{L+1} \) for mathematical convenience. The channel input \( X^{n} \) is generated as the superposition of the \( L \) codewords, \( X_{l}^{n} \) with the corresponding power allocation \( \sqrt{\rho^{l}} \) as

\[ X^{n} = \frac{1}{\sqrt{\rho}} \sum_{j=1}^{L} \sqrt{\rho_{j}} X_{l}^{n} + \sqrt{\rho^{L+1}} N^{n}. \]  

(41)

At the receiver, successive joint decoding is used from layer 1 up to layer \( L \), considering the posterior layers as noise. Layer \( L + 1 \), containing the noise, is ignored. The outage event at layer \( l \), provided \( l - 1 \) layers have been decoded successfully, is given by

\[ O_{l}^{0} = \{ (H, \gamma) : I(X_{l}; Y | X_{l-1}^{l-1}) \leq I(S; W_{l}^{T}, W_{l-1}^{T-1}) \} \]  

(42)

If \( l \) layers are decoded, the source is reconstructed at a distortion \( D_{d}(\sum_{i=1}^{l} b R_{i}, \gamma) \) with an MMSE estimator, and the expected distortion is found as

\[ E D_{b,s}(R, \xi) = \sum_{l=1}^{L} E_{C_{l+1}} \left[ D_{d} \left( \sum_{i=0}^{l} b R_{i}, \Gamma \right) \right] \]  

(43)

where \( R \triangleq [R_{1}, \ldots, R_{L}] \) and \( O_{l+1}^{b,s} \) is the set of states in which all the \( L \) layers are successfully decoded.

The problem of optimizing the distortion exponent for BS-LD for \( L \) layers, which we denote by \( \Delta_{b,s}^{(L)}(b, \nu) \), can be formulated as a linear program over the multiplexing gains \( r \triangleq [r_{1}, \ldots, r_{L}] \), where \( R_{l} = r_{l} \log \rho \) for \( l = 1, \ldots, L \), and the power allocation \( \xi \), as shown in (263) in Appendix [B] and can be efficiently solved numerically. In general, the performance of BS-LD is improved by increasing the number of layers \( L \), and an upper bound on the performance, denoted by \( \Delta_{b,s}^{(L)}(b, \nu) \), is given in the limit of infinite layers, i.e., \( L \rightarrow \infty \), which can be approximated by numerically solving \( \Delta_{b,s}^{(L)}(b, \nu) \) with a large number of layers. However, obtaining a complete analytical characterization of \( \Delta_{b,s}^{(L)}(b, \nu) \) and \( \Delta_{b,s}^{(L)}(b, \nu) \) general is complicated. In the following, we fix the multiplexing gains, and optimize the distortion exponent over the power allocation. While fixing the multiplexing gains is potentially suboptimal, we obtain a closed form expression for an achievable distortion exponent, and analytically evaluate its limiting behavior. We shall see that, as the number of layers increases, this analytical solution matches the numerically optimized distortion exponent.

First, we fix the multiplexing gains as \( \hat{r} = [\hat{r}_{1}, \ldots, \hat{r}_{L}] \) where \( \hat{r}_{l} = [(k+1)(\xi_{l-1} - \xi_{l}) - c_{l}] \) for \( l = 1, \ldots, L \), for some \( c_{1} \rightarrow 0 \), and optimize the distortion exponent over \( \xi \). The achievable distortion exponent is given in the next theorem.
The distortion exponent achievable by BS-LD with $L$ layers and multiplexing gain $\hat{r}$, is given by $\Delta_{bs}^{(L)}(b, \nu) = \nu$ for $bM_s \leq \nu$, and by

$$\hat{\Delta}_{bs}^{(L)}(b, \nu) = \nu + \Phi_k \left( \frac{Y_k(b + (k + 1))}{(Y_k + b + (k + 1))\frac{(1 + b + (k + 1))\Gamma_k}{b(k + 1)\Phi_k\Gamma_k} - b(k + 1)\Phi_k\Gamma_k, \right)$$

for

$$b \in \left[ \frac{\Phi_{k+1} + \nu}{k + 1}, \frac{\Phi_k + \nu}{k + 1} \right], \quad k = 0, \ldots, M_s - 1.$$  \hfill (45)

Proof: See Appendix E. \hfill ■

An upper bound on the performance of BS-LD with multiplexing gains $\hat{r}_{i}$ is obtained for a continuum of infinite layers, i.e., $L \to \infty$.

Corollary 1. The distortion exponent of BS-LD with multiplexing gains $\hat{r}$ in the limit of infinite layers, $\hat{\Delta}_{bs}^{\infty}(b, \nu)$, is given, for $k = 0, \ldots, M_s - 1$, by

$$\hat{\Delta}_{bs}^{\infty}(b, \nu) = \max\{\nu, b(k + 1)\}$$

for $b \in \left[ \frac{\Phi_{k+1} + \nu}{k + 1}, \frac{\Phi_k + \nu}{k + 1} \right], \quad k = 0, \ldots, M_s - 1$. \hfill (47)

and

$$\hat{\Delta}_{bs}^{\infty}(b, \nu) = \Phi_k + \nu \left( \frac{b(1 + k) - \Phi_k}{b(1 + k) - \Phi_{k+1}} \right)$$

for $b \in \left[ \frac{\Phi_k + \nu}{k + 1}, \frac{\Phi_k + \nu}{k + 1} \right]$. \hfill (48)

Proof: See Appendix E. \hfill ■

The solution in Theorem 5 is obtained by fixing the multiplexing gains to $\hat{r}$. This is potentially suboptimal since it excludes, for example, the performance of single-layer LD from the set of feasible solutions. By fixing $r$ such that $r_2 = \cdots = r_L = 0$, BS-LD reduces to single layer LD and achieves a distortion exponent given in Theorem 2, i.e., $\Delta_{ld}(b, \nu)$. Interestingly, for $b$ satisfying

$$b \in \left[ \frac{\Phi_k}{k}, \frac{\Phi_k + \nu}{k} \right], \quad k = 1, \ldots, M_s - 1,$$

single-layer LD achieves a larger distortion exponent than $\hat{\Delta}_{bs}^{\infty}(b, \nu)$ in Corollary 1 as shown in Figure 3. Note that this region is empty for $\nu = 0$, and thus, this phenomena does not appear in the absence of side information. The achievable distortion exponent for BS-LD can be stated as follows.

Lemma 2. BS-LD achieves the distortion exponent

$$\hat{\Delta}_{bs}(b, \nu) = \max\{\Delta_{bs}^{\infty}(b, \nu), \Delta_{ld}(b, \nu)\}.$$ \hfill (52)

Next, we consider the numerical optimization $\Delta_{bs}^{(L)}(b, \nu)$, and compare it with the distortion exponent achieved by fixing the multiplexing gain. In Figure 3, we show one instance of the numerical optimization of $\Delta_{bs}^{(L)}(b, s)$ for $3 \times 2$ MIMO and $\nu = 0.5$, for $L = 2$ and $L = 500$ layers. We also include the distortion exponent achievable by single-layer LD, i.e., $\Delta_{ld}(b, \nu)$.
bound. First, we use the upper bound derived in Section [III] to characterize the optimal distortion exponent for bandwidth ratios $0 \leq b \leq \max\{M^*-M_*, \nu\}/M_*$. We show that, when $0 \leq b \leq \nu/M_*$, the optimal distortion exponent is achieved by ignoring the channel, and reconstructing the source sequence using only the side information. If $\nu/M_* \leq b \leq (M^*-M_*+1)/M_*$, the optimal distortion exponent is achieved by ignoring the side information, and employing the optimal transmission scheme in the absence of side information.

Then, we characterize the optimal distortion exponent for MISO/SIMO/SISO scenarios. In MISO/SIMO, i.e., $M_* = 1$, we show that BS-LD meets the upper bound, thus characterizing the optimal distortion exponent. This extends the result of [13] to the case with time-varying source side information. For SISO, i.e., $M^* = M_* = 1$, HDA-LD also achieves the optimal distortion exponent. For the general MIMO setup, none of the proposed schemes meet the upper bound for $b > 1/M_*$. Nevertheless, multi-layer transmission schemes perform close to the upper bound, especially in the high bandwidth ratio regime.

A. Optimal distortion exponent for low bandwidth ratios

First, we consider the MMSE reconstruction of $S^m$ only from the side information sequence $T^m$ available at the receiver, i.e., $\hat{S}_i = E[S|T_i]$. The source sequence is reconstructed with distortion $D_{\alpha}(\gamma) \triangleq (1+\rho_\alpha \gamma)^{-1}$, and averaging over the side information realizations the distortion exponent is found as $\Delta_{\alpha}(b,\nu) = \nu$, which meets the upper bound $\Delta_{up}(b,\nu)$ for $0 \leq b \leq \nu/M_*$, characterizing the optimal distortion exponent.

Lemma 3. For $0 \leq b \leq \nu/M_*$, the optimal distortion exponent $\Delta^*(b,\nu) = \nu$ is achieved by simple MMSE reconstruction of $S^m$ from the side information sequence $T^m$.

Additionally, Theorem 1 reveals that in certain regimes, the distortion exponent is upper bounded by $\Delta_{MIMO}(b)$, the distortion exponent upper bound in the absence of side information at the destination [13] Theorem 3.1]. In fact, for $\nu/M_* \leq b \leq M^*-M_*+1$, we have $\Delta_{up}(b,\nu) = bM_*$. This distortion exponent is achievable for $b$ satisfying $\nu/M_* \leq b \leq (M^*-M_*+1)/M_*$ by ignoring the side information and employing the optimal scheme in the absence of side information, which is given by the multi-layer broadcast transmission scheme considered in [19]. The same distortion exponent is achievable by considering BS-LD ignoring the side information, i.e., $\Delta^*(b,\nu) = \Delta_{MIMO}(b,0)$. If $\nu/M_* \leq b \leq 1/M_*$, the optimal distortion exponent is also achievable by HDA-S and $\Delta^*(b,\nu) = \Delta_{h}(b,\nu)$.

Lemma 4. For $\nu/M_* \leq b \leq (M^*-M_*+1)/M_*$, the optimal distortion exponent is given by $\Delta^*(b,\nu) = bM_*$, and is achievable by BS-LD ignoring the side information sequence $T^m$. If $\nu/M_* \leq b \leq 1/M_*$, the distortion exponent is also achievable by HDA-S.

B. Optimal distortion exponent for MISO/SIMO/SISO

The following distortion exponent is achievable by HDA-S for $\nu \leq b \leq 1$, and by HDA-LD for $b > 1$, in the MISO/SIMO setup.

$$\Delta_h(b,\nu) = \begin{cases} \max\{\nu, b\} & \text{for } b \leq \max\{M^*, \nu\}, \\ \nu + M^* \left(1 - e^{-\frac{\nu b}{M^*}}\right) & \text{for } b > \max\{M^*, \nu\} \end{cases}$$

As seen in Section V-A, HDA-S meets the partially informed upper bound for $b \leq 1$. HDA-LD is in general suboptimal.

For the multi-layer transmission schemes, the distortion exponent achievable by LS-LD is given by

$$\Delta_{\alpha}(b,\nu) = \nu + M^* \left(1 - e^{-\frac{\nu b}{M^*}}\right)$$

$$\kappa^* = \frac{M^*}{b} W\left(\frac{e^{\frac{\nu b}{M^*}}}{M^*}\right).$$

As for BS-LD, considering the achievable rate in Corollary 1, this scheme meets the partially informed encoder lower bound in the limit of infinite layers, i.e., $\Delta_{\alpha}(b,\nu) = \Delta_{up}(b,\nu)$. This fully characterizes the optimal distortion exponent in the MISO/SIMO setup, as stated in the next theorem.

Theorem 6. The optimal distortion exponent $\Delta^*(b,\nu)$ for MISO/SIMO systems is given by

$$\Delta^*(b,\nu) = \begin{cases} \max\{\nu, b\} & \text{for } b \leq \max\{M^*, \nu\}, \\ M^* + \nu \left(1 - \frac{M^*}{b}\right) & \text{for } b > \max\{M^*, \nu\} \end{cases}$$

and is achieved by BS-LD in the limit of infinite layers.

We note that in SISO setups, HDA-LD achieves the optimal distortion exponent for $b \geq 1$, in addition to BS-LD.

Lemma 5. The optimal distortion exponent for SISO channels is achieved by BS-LD, HDA-LD and HDA-S.

In Figure 4 we plot the distortion exponent for a MISO/SIMO channel with $M^* = 4$ and $\nu = 0.5$, with respect to the bandwidth ratio $b$. We observe that, as given in Theorem 6 BS-LD achieves the optimal distortion exponent. We observe that HDA-LD outperforms LD in all regimes, and, although it outperforms the multi-layer LS-LD for low $b$ values, LS-LD achieves higher distortion exponents than HDA-LD for $b \geq 3$. In general we observe that single-layer
schemes perform poorly as the bandwidth ratio increases, as they are not capable of fully exploiting the available degrees-of-freedom in the system.

C. General MIMO

Here, we consider the general MIMO channel with \( M_x > 1 \). Figure 5 shows the upper and lower bounds on the distortion exponent derived in the previous sections for a 2 × 2 MIMO channel with \( \nu = 0.5 \). First, it is observed that the optimal distortion exponent is achieved by HDA-S and BS-LD with infinite layers for \( b \leq 0.5 \), as expected from Section V-A, while the other schemes are suboptimal in general.

For \( 0.5 < b \lesssim 2.4 \), HDA-LD is the scheme achieving the highest distortion exponent, and outperforms BS-LD, in particular, when the performance of BS-LD reduces to that of LD, since HDA-LD outperforms LD in general. For larger \( b \) values, the highest distortion exponent is achieved by BS-LD. Note that for \( b \geq 4 \), \( \Delta^*_b(b, 0.5) \) is very close to the upper bound. We also observe that for \( b \gtrsim 2.4 \) LS-LD outperforms HDA-LD, but it is worse than BS-LD. This is not always the case, as will be seen next.

In Figure 6 we plot the upper and lower bounds for a 4 × 4 MIMO channel with \( \nu = 3 \). We note that, for \( b \leq \max \{1, \nu\} / M_x \), \( \Delta^*(b, 3) = 3 \), which is achievable by using only the side information sequence at the decoder. For this setup, LS-LD achieves the best distortion exponent for intermediate \( b \) values, outperforming both HDA-LD and BS-LD. Again, in the large bandwidth ratio regime, BS-LD achieves the best distortion exponent values, and performs close to the upper bound. We note that for high side information quality, the difference in performance between LD and HDA-LD decreases. Comparing Figure 5 and Figure 6, we observe that, when the side information quality is high, digital schemes better exploit the degrees-of-freedom of the system than analog schemes.

In Figure 7 we plot the upper and lower bounds for a 7 × 7 MIMO channel with \( \nu = 3 \). In comparison with Figure 6, as the number of antennas increases the difference in performance between LD and HDA-LD decreases. This seems to be the case also between BS-LD and LS-LD in the high bandwidth ratio regime. However, LS-LD significantly outperforms BS-LD for intermediate \( b \) values.

VI. CONCLUSIONS

We have introduced the concept of time-varying correlated side-information in the context of joint source-channel coding over a time-varying fading channel. We have derived an upper bound on the high SNR distortion exponent, as well as several lower bounds based on separate and joint source and channel coding, in particular, using list decoding, hybrid digital-analog transmission and multi-layer transmission. We have considered the effects of the bandwidth ratio and the side information quality on the distortion exponent, and shown that the multi-layer superposition transmission meets the upper bound in MISO/SIMO/SISO channels, solving the joint source channel coding problem in the high SNR regime. For general MIMO channels, we have characterized the optimal distortion exponent in the low bandwidth ratio regime, and shown that the multi-layer superposition scheme performs very close to the upper bound in the high bandwidth ratio regime. We have shown that, while separate source and channel coding fails when either the channel or side information quality is low, in list decoding one can compensate for the other, improving the
end-to-end average distortion performance. Further analysis is required for more general models for the distribution of the channel and side information states, as well as models that go beyond the additive Gaussian assumptions for the channel and side information sequences. While joint-source-channel coding based on list decoding is expected to out-perform separation in general, particularly with multi-layer transmission, non-Gaussian codes as well as non-trivial non-linear transformations for analog and hybrid transmission will need to be considered.

**APPENDIX A**

**PROOF OF THEOREM 1**

The exponential integral can be bounded as follows [27, p.229, 5.1.20]:
\[ \frac{1}{2} \ln \left( 1 + \frac{2}{t} \right) < e^{t} E_1(t) < \ln \left( 1 + \frac{1}{t} \right), \quad t > 0. \] (58)

Next, using the lower bound \( \ln(1+t) \geq \frac{t}{1+t} \), for \( t > -1 \), we have
\[ \frac{1}{2} \ln \left( 1 + \frac{2}{t} \right) > \frac{1}{2} \frac{2/t}{1 + 2/t} = \frac{1}{t + 2}. \] (59)

Then, \( ED^*_p \) in (9) is lower bounded by
\[ ED^*_p(\rho, \rho_s, b) \geq \int_{H} \frac{1}{2c(H) + 2\rho_s} p_H(H) dH. \] (60)

Following [23], the capacity of the MIMO channel is upper bounded as
\[ C(H) = \sup_{C_u: \text{Tr}(C_u) \leq M_i} \log \det \left( I + \frac{\rho}{M_t} H C_u H^H \right) \] (61)
\[ \leq \log \det \left( I + \rho H H^H \right), \] (62)

where the inequality follows from the fact that \( M_t I - C_u \succeq 0 \) subject to the power constraint \( \text{Tr}(C_u) \leq M_t \), and the function \( \log \det(\cdot) \) is nondecreasing on the cone of positive semidefinite Hermitian matrices.

Let \( \lambda_{M_t} \geq \cdots \geq \lambda_1 > 0 \) be the eigenvalues of matrix \( HH^H \), and consider the change of variables \( \lambda_i = \rho^{-\alpha_i} \), with \( \alpha_1 \geq \cdots \geq \alpha_{M_t} \geq 0 \). The joint probability density function (pdf) of \( \alpha \triangleq [\alpha_1, \ldots, \alpha_{M_t}] \) is given by [23]:
\[ p_\alpha(\alpha) = K_{M_t,M_r}^{-1} \rho^{M_r} \prod_{i=1}^{M_t} \rho^{-(M_r - M_t + 1) \alpha_i}, \] (63)
\[ \cdot \left[ \prod_{i<j} (\rho^{\alpha_i} - \rho^{\alpha_j})^2 \right] \exp \left( -\sum_{i=1}^{M_t} \rho^{\alpha_i} \right), \] (64)

where \( K_{M_t,M_r}^{-1} \) is a normalizing constant.

We define the high SNR exponent of \( p_\alpha(\alpha) \) as \( S_\alpha(\alpha) \), that is, we have \( p_\alpha(\alpha) \approx \rho^{-S_\alpha(\alpha)} \), where
\[ S_\alpha(\alpha) \triangleq \begin{cases} \sum_{i=1}^{M_t} (2i - 1 + M^* - M_s) \alpha_i & \text{if } \alpha_{M_t} \geq 0, \quad (65) \\ \infty & \text{otherwise.} \end{cases} \]

Then, from (60) and (61) we have
\[ ED^*_p(\rho, \rho_s, b) \geq \int_{H} \frac{1}{\prod_{i=1}^{M_t} (1 + \rho \lambda_i)^{b} + 2\rho_s} p_H(H) dH. \] (67)
\[ = \int_{\alpha} \frac{1}{\prod_{i=1}^{M_t} (1 + (\rho^{1-\alpha_i})^{b} + 2\rho_s} p_A(\alpha) d\alpha, \] (68)
\[ \geq \int_{\alpha^+} G_{\rho}(\alpha) p_A(\alpha) d\alpha, \] (69)

where we define
\[ G_{\rho}(\alpha) \triangleq \left( \prod_{i=1}^{M_t} (1 + (\rho^{1-\alpha_i})^{b} + 2\rho_s) \right)^{-1}, \] (70)

and the set \( \alpha^+ \triangleq \{ \alpha \in \mathbb{R}^{M_t} : 1 \geq \alpha_1 \geq \cdots \geq \alpha_{M_t} \geq 0 \} \) in (69).

Then, the distortion exponent of the partially informed encoder is upper bounded by
\[ \Delta^*_p(b, \nu) \triangleq -\lim_{\rho \to \infty} \frac{\log ED^*_p(\rho, \rho_s, b)}{\log \rho} \] (71)
\[ \leq \lim_{\rho \to \infty} \frac{1}{\log \rho} \log \int_{\alpha^+} \exp \left( \log G_{\rho}(\alpha) - \log \rho \right) p_A(\alpha) d\alpha, \] (73)
\[ = \lim_{\rho \to \infty} \frac{1}{\log \rho} \log \int_{\alpha^+} \exp (G(\alpha) \log \rho) p_A(\alpha) d\alpha, \] (74)

where (74) follows from the application of the Dominated Convergence Theorem [28], which holds since \( G_{\rho}(\alpha) \leq 1 \) for all \( \alpha \), the continuity of the logarithmic and exponential functions, and since we have the following limit
\[ G(\alpha) \triangleq \lim_{\rho \to \infty} \frac{\log G_{\rho}(\alpha)}{\log \rho} = \frac{\log \rho^{b \sum_{i=1}^{M_t} (1-\alpha_i)^{+}} + 2\rho^{b}}{\log \rho}, \] (75)
\[ = -\max \{ \nu, b \sum_{i=1}^{M_t} (1-\alpha_i)^{+} \} \] (76)

where we have used the exponential equalities \( 1 + \rho^{1-\alpha_i} \approx \rho^{(1-\alpha_i)^+} \), and \( \rho_s \approx \rho^{b} \).

From Varadhan’s lemma [29], it follows that the distortion exponent of \( ED^*_p \) is upper bounded by the solution to the following optimization problem,
\[ \Delta_{up}(b, \nu) \triangleq \inf_{\alpha^+} \left[ -G(\alpha) + S(\alpha) \right]. \] (78)

In order to solve (78) we divide the optimization into two subproblems: the case when \( \nu < b \sum_{i=1}^{M_t} (1-\alpha_i) \), and the case when \( \nu \geq b \sum_{i=1}^{M_t} (1-\alpha_i) \). The solution is then given by the minimum of the solutions of these subproblems.

If \( \nu \geq b \sum_{i=1}^{M_t} (1-\alpha_i) \), the problem in (78) reduces to
\[ \Delta_{up}^{(1)}(b, \nu) = \nu + \inf_{\alpha^+} \sum_{i=1}^{M_t} (2i - 1 + M^* - M_s) \alpha_i \] (79)
\[ \text{s.t. } \sum_{i=1}^{M_t} (1-\alpha_i) \leq \frac{\nu}{b}. \] (80)
The optimization in (79) can be identified with the DMT problem in [23 Eq. (14)] for a multiplexing gain of $r = \frac{\nu}{b}$.

Next, we give an explicit solution for completeness.

First, if $bM_s \leq \nu$, the infimum is given by $\Delta_{up}^{(1)}(b, \nu) = \nu$ for $\alpha^* = 0$. Then, for $k \leq \frac{\nu}{b} \leq k + 1$, for $k = 0, ..., M_s - 1$, i.e., $\frac{k}{k+1} \leq b \leq \frac{k+1}{k}$, the infimum is achieved by

$$
\alpha_i^* = \begin{cases} 
1 & \text{for } i = 1, ..., M_s - k - 1, \\
\frac{k+1 - \nu}{b} & \text{for } i = M_s - k, \\
0 & \text{for } i = M_s - k + 1, ..., M_s.
\end{cases}
$$

(81)

Substituting, we have, for $k = 0, ..., M_s - 1$,

$$
\Delta_{up}^{(1)}(b, \nu) = \nu + \Phi_k - \Upsilon_k \left(\frac{\nu}{b} - k\right) = \nu + d^* \left(\frac{\nu}{b}\right),
$$

(82)

where $\Phi_k$ and $\Upsilon_k$ are defined as in (11).

Now we solve the second subproblem with $\nu < b \sum_{i=1}^{M_s} (1 - \alpha_i)$. Since $1 \geq \alpha_1 \geq ... \geq \alpha_{M_s} \geq 0$ we can rewrite (78) as

$$
\Delta_{up}^{(2)}(b, \nu) = \inf \{bM_s - \sum_{i=1}^{M_s} \alpha_i \phi(i) \mid \sum_{i=1}^{M_s} \alpha_i \leq M_s - \frac{\nu}{b}\}.
$$

(83)

where we have defined $\phi(i) \triangleq [b - (2i - 1 + M^* - M_s)]$. Note that $\alpha(1) > ... > \alpha(M_s)$.

First, we note that for $bM_s < \nu$ there is no feasible solution due to the constraint in (83).

Now, we consider the case $\nu \leq M_s (1 + M^* - M_s)$. If $\frac{\nu}{M_s} \leq b < 1 + M^* - M_s$, all the terms $\phi(i)$ multiplying $\alpha_i$’s are negative, and, thus, the infimum is achieved by $\alpha^* = 0$, and is given by $\Delta_{up}^{(2)}(b, \nu) = bM_s$. If $1 + M^* - M_s \leq b < 3 + M^* - M_s$, then $\phi(1)$ multiplying $\alpha_1$ is positive, while the other $\phi(i)$ terms are negative. Then $\alpha_1^* = 0$ for $i = 2, ..., M_s$. From (83) we have $\alpha_1 \leq M_s - \frac{\nu}{b}$. If $b \geq \frac{\nu}{M_s - k}$, the right hand side (r.h.s.) of (83) is greater than one, and smaller otherwise. Then, we have

$$
\alpha_1^* = \begin{cases} 
1 & \text{if } b \geq \frac{\nu}{M_s - k}, \\
M_s - \frac{\nu}{b} & \text{if } b < \frac{\nu}{M_s - k}.
\end{cases}
$$

(85)

Note that $\alpha_1^* \geq 0$ since $b > \frac{\nu}{M_s}$.

When $2k - 1 + M^* - M_s \leq b < 2k + 1 + M^* - M_s$, for $k = 2, ..., M_s - 1$, the coefficients $\phi(i), i = 1, ..., k$, associated with the first $k$ $\alpha_i$ terms are positive, while the others remain negative. Then,

$$
\alpha_i^* = 0, \text{ for } i = k + 1, ..., M_s.
$$

(86)

Since $\phi(i), i = 1, ..., k$, are positive and $\phi(1) > ... > \phi(k)$, we have $\alpha_i^* = 1$ for $i = 1, ..., k - 1$, and the constraint becomes $\alpha_k < M_s - (k - 1) - \frac{\nu}{b}$. If $b \geq \frac{\nu}{M_s - k}$, then the r.h.s. is greater than one, and smaller otherwise. In order for the solution to be feasible, we need $\alpha_k \geq 0$, that is, $M_s - (k - 1) - \frac{\nu}{b} \geq 0$.

Then we have

$$
\alpha_k^* = \begin{cases} 
1 & \text{if } b \geq \frac{\nu}{M_s - k}, \\
M_s - (k - 1) - \frac{\nu}{b} & \text{if } \frac{\nu}{M_s - k} \leq b < \frac{\nu}{M_s - (k - 1)}.
\end{cases}
$$

(87)

If $b < \frac{\nu}{M_s - (k - 1)}$, the solution in (87) is not feasible. Instead, we have $\alpha_k^* = 0$, since $\phi(k) < \phi(k - 1)$, $\alpha_i^* = 0$ for $i = k + 1, ..., M_s$, and $\alpha_i^* = 1$, for $i = 1, ..., k - 2$. Then, the constraint in (83) is given by $\alpha_{k-1} = M_s - (k - 2) - \frac{\nu}{b}$. Since $b < \frac{\nu}{M_s - (k - 1)}$, the r.h.s. is always smaller than one. For the existence of a feasible solution, the r.h.s. is required to be greater than zero. Therefore, we have

$$
\alpha_{k-1}^* = M_s - (k - 2) - \frac{\nu}{b},
$$

(88)

if $\frac{\nu}{M_s - (k - 2)} \leq b < \frac{\nu}{M_s - (k - 1)}$.

(89)

In general, iterating this procedure, for

$$
\frac{\nu}{M_s - (j - 1)} \leq b < \frac{\nu}{M_s - j}, \quad j = 1, ..., k,
$$

(90)

we have

$$
\alpha_j^* = \begin{cases} 
1 & \text{for } i = 1, ..., j - 1, \\
M_s - (j - 1) - \frac{\nu}{b} & \text{for } i = j, \\
0 & \text{for } i = j + 1, ..., M_s.
\end{cases}
$$

(91)

Note that for the case $j = 1$, we have $\alpha_1 = M_s - \frac{\nu}{b}$, which is always feasible.

We now evaluate (83) with the optimal $\alpha^*$ if $2k - 1 + M^* - M_s \leq b < 2k + 1 + M^* - M_s$ for some $k \in \{2, ..., M_s - 1\}$. For $b \geq \frac{\nu}{M_s - k}$, we have $\alpha_1 = ... = \alpha_k = 1$ and $\alpha_{k+1} = ... = \alpha_{M_s} = 0$, and then

$$
\Delta_{up}^{(2)}(b, \nu) = \sum_{i=1}^{M_s} \min\{b, 2i - 1 + M^* - M_s\} = \Delta_{MMIMO}(b).
$$

(92)

$$
= \Delta_{MMIMO}(b).
$$

(93)

For $\frac{\nu}{M_s} \leq b \leq \frac{\nu}{M_s - k}$, substituting (91) into (83) we have

$$
\Delta_{up}^{(2)}(b, \nu) = \nu + (M^* - M_s - 1 + j)(j - 1)
$$

$$
+ \left(\frac{\nu}{b} - j\right) (M_s - (j - 1) - \frac{\nu}{b} \right),
$$

(94)

where

$$
\frac{\nu}{M_s - (j - 1)} \leq b < \frac{\nu}{M_s - j}, \text{ for some } j \in \{1, ..., k\}.
$$

(95)

Note that with the change of index $j = M_s - j'$, we have, after some manipulation,

$$
\Delta_{up}^{(2)}(b, \nu) = \nu + (M^* + j')(M_s - j') - \left(\frac{\nu}{b} - j\right) (M_s + M_s - 2j' - 1),
$$

(96)

in the regime

$$
\frac{\nu}{j' + 1} \leq b < \frac{\nu}{j'}, \quad j' = M_s - k, ..., M_s - 1.
$$

(98)

This is equivalent to the value of the DMT curve in [10] at multiplexing gain $r = \frac{\nu}{b}$. Then, for $\frac{\nu}{M_s} \leq b < \frac{\nu}{M_s - k}$ we have

$$
\Delta_{up}^{(2)}(b, \nu) = \nu + d^* \left(\frac{\nu}{b}\right).
$$

(99)

If $b \geq M^* + M_s - 1$, the infimum is achieved by $\alpha_1^* = 1$, for $i = 1, ..., M_s - 1$, and $\alpha_{M_s} = 1 - \frac{\nu}{b}$ if $b \geq \nu$. If $b < \nu$, this solution is not feasible, and the solution is given by (91). Therefore, in this regime we also have

$$
\Delta_{up}^{(2)}(b, \nu) = \nu + d^* \left(\frac{\nu}{b}\right).
$$

(100)
Putting all these results together, for \( \nu \leq M_*(M^* - M_* + 1) \) we have

\[
\Delta_{up}^{(2)}(b, \nu) = \begin{cases} 
  bM_* & \text{for } \nu \leq b < M^* - M_* + 1, \\
  \nu + d^*(\frac{\nu}{M_*}) & \text{for } M^* - M_* + 1 \leq b < \frac{\nu}{M_* - k}, \\
  \Delta_{MIMO}(b) & \text{for } \frac{\nu}{M_* - k} \leq b < M^* + M_* - 1, \\
  \nu + d^*(\frac{\nu}{b}) & \text{for } b \geq M^* + M_* - 1,
\end{cases}
\] (101)

where \( k \in \{1, \ldots, M_* - 1\} \) is the integer satisfying \( 2k - 1 + M^* - M_* \leq b < 2k + 1 + M^* - M_* \).

Now, we solve (98) for \( M_*(M^* - M_* + 1) \leq \nu \leq M_*(M^* + M_* - 1) \). Let \( l \in \{2, \ldots, M_*\} \) be the integer satisfying \( M_*(2l - 1) - 1 + M^* - M_* \leq \nu \leq M_*(2l - 1 + M^* - M_*) \) for some \( l \in \{2, \ldots, M_*\} \). Then, the optimal \( \Delta_{up}(b, \nu) \) in (91) is linear and increasing in \( \alpha \), and hence, the solution is such that the constraint is satisfied with equality, i.e.,

\[
\nu \leq \sum_{i=1}^{M_*} b(1 - \alpha_i).
\]

That is, \( \Delta_{up}^{(2)}(b, \nu) \leq \Delta_{up}^{(1)}(b, \nu) \) whenever both solutions exist in the same \( \alpha \) region. Then, the minimizing \( \alpha \) will be one such that either \( \Delta_{up}^{(1)}(b, \nu) < \Delta_{up}^{(2)}(b, \nu) \), or the one arbitrarily close to the boundary \( \nu = \sum_{i=1}^{M_*} b(1 - \alpha_i) \), where \( \Delta_{up}^{(1)}(b, \nu) = \Delta_{up}^{(2)}(b, \nu) \). Consequently, \( \min \{\Delta_{up}^{(1)}(b, \nu), \Delta_{up}^{(2)}(b, \nu)\} = \Delta_{up}^{(1)}(b, \nu) \), whenever they are defined in the same region. Putting all the results together we complete the proof.

**APPENDIX B**

**PROOF OF THEOREM 2**

Applying the change of variables \( \lambda_i = \rho^{-\alpha_i} \) and \( \gamma = \rho^{-\beta} \), and considering a rate \( R_{ld} = r_{ld} \log \rho, r_{ld} > 0 \), the outage event in (77) can be written as

\[
\mathcal{O}_{ld} = \{(H, \gamma) : 1 + \frac{2^{-r_{ld}} - 1}{\gamma^2} \geq \prod_{i=1}^{M_*} (1 + \rho \lambda_i)^b \}
\] (106)

\[
= \{(\alpha, \beta) : 1 + \frac{2^{-r_{ld}} - 1}{\rho^{(1 - \alpha_i)}} \geq \prod_{i=1}^{M_*} (1 + \rho^{1 - \alpha_i})^b \}
\] (107)

For large \( \rho \), we have

\[
1 + \frac{2^{-r_{ld}} - 1}{\rho^{(1 - \alpha_i)}} = \frac{1 + \rho^{br_{ld}} - (\nu - \beta)^+}{\rho^{\sum_{i=1}^{M_*} (1 - \alpha_i)^+}} \leq \rho^{(br_{ld} - (\nu - \beta)^+ + b \sum_{i=1}^{M_*} (1 - \alpha_i)^+)}
\] (108)

Therefore, at high SNR, the achievable expected end-to-end distortion for LD is found as,

\[
E D_{ld}(br_{ld} \log \rho) = \int_{\mathcal{O}_{ld}} D_d(0, \rho^{-\beta}) p_A(\alpha) p_B(\beta) d\alpha d\beta
\] (110)

\[
= \int_{\mathcal{A}_{ld}} D_d(0, \rho^{-\beta}) p_A(\alpha) p_B(\beta) d\alpha d\beta
\] (111)

\[
\leq \int_{\mathcal{A}_{ld}} \rho^{-\min((\nu - \beta)^+, br_{ld})} (\rho^{-S(\alpha) + \beta}) d\alpha d\beta
\] (112)

\[
\leq \rho^{-\Delta_{up}^{(2)}(r_{ld}, (\nu - \beta)^+)} \rho^{-S(\alpha) + \beta} d\alpha d\beta
\] (113)

\[
\leq \rho^{-\Delta_{up}^{(1)}(r_{ld}) + \Delta_{up}^{(2)}(r_{ld})}
\] (114)

\[
\leq \rho^{-\Delta_{up}^{(1)}(r_{ld}) + \Delta_{up}^{(2)}(r_{ld})}
\] (115)

\[
\leq \rho^{-\Delta_{up}^{(2)}(r_{ld})}
\] (116)

\[
\leq \Delta_{ld}(r_{ld})
\] (117)

where \( D_d(R, \gamma) \) is as defined in (18), and we have used \( D_d(r \log \rho, \beta) \leq \rho^{-\min((\nu - \beta)^+, 2r)} \). We have also defined the high SNR equivalent of the outage event as

\[
\mathcal{A}_{ld} = \{ (\alpha, \beta) : (br_{ld} - (\nu - \beta)^+) \geq b \sum_{i=1}^{M_*} (1 - \alpha_i)^+ \}
\] (118)
We have applied Varadhan’s lemma to each integral to obtain
\[
\Delta_{id}^{(1)}(r_{id}) \triangleq \inf_{A_{id}} \max\{\nu - \beta, br_{id}\} + \beta + S_A(\alpha),
\]
and
\[
\Delta_{id}^{(2)}(r_{id}) \triangleq \inf_{A_{id}} (\nu - \beta)^+ + \beta + S_A(\alpha).
\]
Then, the distortion exponent of LD is found as
\[
\Delta_{id}(r_{id}) = \min\{\Delta_{id}^{(1)}(r_{id}), \Delta_{id}^{(2)}(r_{id})\}.
\]
We first solve (119). We can constrain the optimization to \(\alpha \geq 0\) and \(\beta \geq 0\) without loss of optimality, since for \(\alpha, \beta < 0\) we have \(S_A(\alpha) = S_B(\beta) = +\infty\). Then, \(\Delta_{id}^{(1)}(r_{id})\) is minimized by \(\alpha^* = 0\) since this minimizes \(S_A(\alpha)\) and enlarges \(A_{id}\). We can rewrite (119) as
\[
\Delta_{id}^{(2)}(r_{id}) = \inf_{\beta \geq 0} \max\{\nu - \beta, br_{id}\} + \beta
\]
\[
s.t. (br_{id} - (\nu - \beta)^+) < bM_*.
\]
If \(br_{id} < (\nu - \beta)^+\), the minimum is achieved by any \(0 \leq \beta < \nu - r_{id}b\), and thus \(\Delta_{id}^{(2)}(r_{id}) = \nu\) for \(\nu > br_{id}\). If \(br_{id} \geq (\nu - \beta)^+\), then
\[
\Delta_{id}^{(2)}(r_{id}) = \inf_{\beta \geq 0} \max\{\nu - \beta, br_{id}\} + \beta
\]
\[
s.t. (br_{id} - (\nu - \beta)^+) < bM_*.
\]
If \(\beta > \nu\), the problem is minimized by \(\beta^* = \nu + \epsilon, \epsilon > 0\), and \(\Delta_{id}(r_{id}) = br_{id} + \nu + \epsilon\), for \(r_{id} \leq M_*\). For \(0 \leq \beta \leq \nu\), we have \(\beta^* = (\nu - r_{id}b)^+\), and \(\Delta_{id}^{(2)}(r_{id}) = \max\{br_{id}, \nu\}\) if \(br_{id} \leq bM_*\). Putting all these together, we obtain
\[
\Delta_{id}^{(1)}(r_{id}) = \max\{br_{id}, \nu\} \quad \text{if} \quad br_{id} \leq \nu + bM_*.
\]
If \(br_{id} > \nu + bM_*\), \(A_{id}\) is empty, and there is always outage. Next we solve the second optimization problem in (120).

With \(\beta = \nu\), \(\Delta_{id}^{(2)}(r_{id})\) is minimized and the range of \(\alpha\) is enlarged. Then, the problem to solve reduces to
\[
\Delta_{id}^{(2)}(r_{id}) = \inf \nu + S(\alpha)
\]
\[
s.t. r_{id} \geq \sum_{i=1}^{M_*} (1 - \alpha_i)^+,\n\]
which is the DMT problem in (79). Hence, \(\Delta_{id}^{(2)}(r_{id}, b) = \nu + d^*(r_{id})\). Bringing all together,
\[
\Delta_{id}(b, \nu) = \max\{\min\{\nu, br_{id}, \nu + d^*(r_{id})\}\}\]
Since \(d^*(r_{id}) = 0\) for \(r_{id} > M_*\), the constraint in (129) can be reduced to \(0 \leq r_{id} \leq M_*\), without loss of optimality since \(\Delta_{id}(b, \nu) = \nu\) for any \(r_{id} \geq M_*\). Then, the maximum achieved when the two terms inside \(\min\{\}\) are equal, i.e., \(\max\{br_{id}, \nu\} = \nu + d^*(r_{id})\). We chose a rate \(r_{id}\) such that \(br_{id} > \nu\) and \(r_{id} < M_*\), as otherwise, the solution is readily given by \(\Delta_{id}(b, \nu) = \nu\). Note that for \(bM_* \leq \nu\) this is never feasible, and thus, \(\Delta_{id}(b, \nu) = \nu\), and if \(\nu \geq b \cdot d^*(M_*)\), the intersection is always at \(br_{id} = \nu\). Assuming \(k \leq r_{id} \leq k + 1\), \(k = 0, ..., M_* - 1\), the optimal \(r_{id}\) satisfies at \(br_{id} = d^*(r_{id})+\nu\), or, equivalently, \(br_{id} = \nu + \Phi_k - (r_e - k)\Upsilon_k\), and we have
\[
r_{id}^* = \frac{\Phi_k + k\Upsilon_k + \nu}{\Upsilon_k + b},
\]
\[
\Delta_{id}(b, \nu) = \frac{\Phi_k + k\Upsilon_k + \nu}{\Upsilon_k + b}.
\]
Since solution \(r_{id}^*\) is feasible whenever \(k < r_{id} \leq k + 1\), this solution is defined in
\[
b \in \left[\frac{\Phi_{k+1} + \nu}{k + 1}, \frac{\Phi_k + \nu}{k}\right], \quad \text{for} \quad k = 0, ..., M_* - 1.(132)
\]
where we have used \(\Phi_{k+1} = \Phi_k - \Upsilon_k\). Notice that, whenever \(\Delta_{id}(b, \nu) \leq \nu\) in (131), we have \(br_{id}^* \leq \nu\), which is not feasible, and therefore \(\Delta_{id}(b, \nu) = \nu\). Remember that for \(bM_* \leq \nu\) we also have \(\Delta_{id}(b, \nu) = \nu\). Putting all these cases together completes the proof of Theorem 2.

APPENDIX C
PROOF Theorem 2

In this Appendix we derive the distortion exponent achieved by HDA-LD. The outage region in (23) is given by
\[
O_h = \left\{ \left( H, \gamma \right) : \left( 1 + \frac{1}{\alpha Q} \right)^{M_*} \geq \left( 1 + \frac{1}{\alpha Q} \right)^{M_*} \prod_{i=1}^{M_*} \frac{1}{1 - \frac{\rho}{\alpha} \lambda_i (1 + \frac{\rho}{\alpha} \sigma^2_0)} \right\}.
\]
Similarly to the analysis of the previous schemes, we consider the change of variables \(\lambda_i = \rho^{-\alpha_i}\), and \(\gamma = \rho^{-\beta}\), and a rate \(R_h = r_h \log \rho\), for \(r_h \geq 0\). Then, we start by finding the equivalent outage set in the high SNR regime. We have,
\[
\prod_{i=1}^{M_*} \left( 1 + \frac{\rho}{M_*} \lambda_i \right)^{bM_*} \geq \frac{\rho^{bM_*} \sum_{i=1}^{M_*} (1 - \alpha_i)^+}{\prod_{i=1}^{M_*} \frac{1}{1 - \frac{\rho}{\alpha} \lambda_i (1 + \frac{\rho}{\alpha} \sigma^2_0)}}.
\]
and
\[
\prod_{i=1}^{M_*} \left( 1 + \frac{\rho}{M_*} \lambda_i + (1 + \rho \sigma^2_0) \right)^{\gamma Q}
\]
\[
\geq \prod_{i=1}^{M_*} \frac{1}{1 - \frac{\rho}{\alpha} \lambda_i (1 + \rho \sigma^2_0)}
\]
where we use \(\sigma^2_0 = (2R_h - \epsilon - 1)^{-1} = (2R_h - \epsilon)^{-1} \leq \rho-rh\). For the outage condition in (133), we have
\[
\left( 1 + \frac{1}{\alpha Q} \right)^{M_*} \prod_{i=1}^{M_*} \frac{1}{1 - \frac{\rho}{\alpha} \lambda_i + (1 + \rho \sigma^2_0) \gamma Q}
\]
\[
= \left((1 + \rho \gamma) (1 + \sigma^2_0) \right)^{M_*} \prod_{i=1}^{M_*} \frac{1}{1 - \frac{\rho}{\alpha} \lambda_i (1 + \frac{\rho}{\alpha} \sigma^2_0)}
\]
\[
= \frac{\rho^{bM_* \gamma Q} \sum_{i=1}^{M_*} (1 - \alpha_i)^+}{\prod_{i=1}^{M_*} \frac{1}{1 - \frac{\rho}{\alpha} \lambda_i (1 + \rho \sigma^2_0)}}
\]
\[
= \prod_{i=1}^{M_*} \left( (1 - \alpha_i)^+ + (1 - \alpha_i)^+ - r_h \right)
\]
\[
= \rho^{bM_* \gamma Q} \sum_{i=1}^{M_*} (1 - \alpha_i)^+,(1 - \alpha_i)^+ - r_h
\]
\[
= \rho^{bM_* \gamma Q} \sum_{i=1}^{M_*} (1 - \alpha_i)^+,(1 - \alpha_i)^+ - r_h
\]
Therefore, in the high SNR regime, the set $O_h$ is equivalent to the set given by
\begin{equation}
A_h = \left\{ (\alpha, \beta)^+ : \sum_{i=1}^{M_h} (r_{ih} - (\nu - \beta)^+ + (1 - \alpha_i)) = 1 \right\}.
\end{equation}

On the other hand, in the high SNR regime, the distortion achieved by HDA-LD is equivalent to
\begin{equation}
D_h(\sigma^2_h, H, \gamma) = \frac{1}{M_h} \sum_{i=1}^{M_h} \left( 1 + \rho \gamma + \frac{1}{\sigma^2_h} \left( 1 + \frac{\rho}{M_h} \lambda_i \right)^{-1} \right)^{-1},
\end{equation}
\begin{equation}
\frac{\sum_{i=1}^{M_h} \left( 1 + \rho \gamma + \frac{1}{\sigma^2_h} \left( 1 + \frac{\rho}{M_h} \lambda_i \right)^{-1} \right)^{-1}}{M_h} + \sum_{i=1}^{M_h} \left( 1 + \rho \gamma + \frac{1}{\sigma^2_h} \left( 1 + \frac{\rho}{M_h} \lambda_i \right)^{-1} \right)^{-1},
\end{equation}
\begin{equation}
\text{where the last equality follows since } \alpha_1 \geq \ldots \geq \alpha_{M_h} \geq 0.
\end{equation}

Then, in the high SNR regime, the expected distortion for HDA-LD is given as
\begin{equation}
ED_h(r_h \log \rho) = \int_{O_h} D_h(\sigma^2_h, H, \gamma) p_h(H) p_T(\gamma) dH d\gamma + \int_{A_h} D_d(\gamma) p_h(H) p_T(\gamma) dH d\gamma,
\end{equation}
\begin{equation}
\text{Similarly to the proof of Theorem 2, applying Varadhan's lemma, the exponent of each integral is found as}
\end{equation}
\begin{equation}
\Delta_h^{(1)}(r_h) = \inf_{A_h} \max\{ (\nu - \beta)^+, r_h + 1 - \alpha \} + S_A(\alpha) + \beta,
\end{equation}
\begin{equation}
\text{and}
\end{equation}
\begin{equation}
\Delta_h^{(2)}(r_h) = \inf_{A_h} \nu + S_A(\alpha) + \beta,
\end{equation}
\begin{equation}
\text{First we solve } \Delta_h^{(1)}(r_h). \text{ The infimum for this problem is achieved by } \alpha^* = 0 \text{ and } \beta^* = 0, \text{ and is given by}
\end{equation}
\begin{equation}
\Delta_h^{(1)}(r_h) = \max\{ \nu, r_h + 1 \} \text{ for } r_h \leq M_h b - 1 + \nu(152).
\end{equation}

Now we solve $\Delta_h^{(2)}(r_h)$ in (151). By letting $\beta^* = \nu$, the range of $\alpha$ is enlarged while the objective function is minimized. Thus, the problem reduces to
\begin{equation}
\Delta_h^{(2)}(r_h) = \inf \nu + S(\alpha)
\end{equation}
\begin{equation}
s.t. \frac{b M_h - 1}{M_h} \sum_{i=1}^{M_h} (1 - \alpha_i).\end{equation}

Again, this problem is a scaled version of the DMT curve in (79). Therefore, we have
\begin{equation}
\Delta_h^{(2)}(r_h) = \nu + d^* \left( \frac{b M_h - 1}{M_h} \right)^{-1} \frac{1}{1 - \gamma},
\end{equation}
\begin{equation}
The distortion exponent is given by optimizing over $r_h$ as
\end{equation}
\begin{equation}
\Delta_h(b, \nu) = \max \{ \Delta_h^{(1)}(r_h), \Delta_h^{(2)}(r_h) \},
\end{equation}
\begin{equation}
The maximum distortion exponent is obtained by letting $\Delta_h^{(1)}(r_h) = \Delta_h^{(2)}(r_h)$. We assume $r_h > 1 - \nu$ since otherwise $\Delta_h(b, \nu) = \nu$, and then, we have $r_h + 1 > \nu + d^*(b - \frac{1}{M_h})^{-1} r_h$. Let $r_h' = r_h(b - \frac{1}{M_h})^{-1}$. Using (10), for $k \leq r_h' \leq k + 1, k = 0, \ldots, M_h - 1$, the problem is equivalent to $r_h' b - \frac{1}{M_h} + 1 = \nu + \Phi_k - (r_h' - k) \Gamma_k$, where $\Phi_k$ and $\Gamma_k$ are given as in (11). The $r_h'$ satisfying the equality is given by
\begin{equation}
r_h' = \frac{\Phi_k + k \Phi_k - 1 + \nu}{b - \frac{1}{M_h} + \Phi_k},
\end{equation}
\begin{equation}
and the corresponding distortion exponent is found as
\end{equation}
\begin{equation}
\Delta_h(b, \nu) = 1 + \frac{\Phi_k + k \Phi_k - 1 + \nu}{b M_h - 1 + M_h \Gamma_k},
\end{equation}
\begin{equation}
\text{for } b \in \left[ \frac{\Phi_k+1 - 1 + \nu}{k+1} \frac{\Phi_k + k - 1 + \nu}{k+1} \right],
\end{equation}
\begin{equation}
Note that we have $r_h' > 0$ whenever $\Delta_h(b, \nu) > \nu$. Otherwise, $r_h'$ is not feasible and $\Delta_h(b, \nu) = \nu$. Note also that if $\nu > b M_h$, the distortion exponent is given by $\Delta_h(b, \nu) = \nu$.

APPENDIX D

PROOF OF THEOREM 2

In this section we obtain the distortion exponent for LS-LD. Let us define $R_{l}^{1} = \sum_{i=1}^{L} R_i$. First, we consider the outage event. For the successive refinement codebook the l.h.s. of (32) is given by
\begin{equation}
I(S; W_l | W_{l-1}^{-1}, Y)
\end{equation}
\begin{equation}
\text{where }\Phi_l \approx \sum_{i=1}^{L} \Phi_i, \text{ and (a) is due to the Markov chain } T - S - W_l - \ldots - W_1, \text{ and (b) is due to the independence of } \Phi_l \text{ from } S \text{ and } T, \text{ and finally (c) follows since } H(W_l | T) = \frac{1}{2} \log \left( \sum_{i=1}^{L} \sigma_i^2 + \frac{1}{1 + \gamma \rho_s} \right) \text{ for } l = 1, \ldots, L. \text{ We also have}
\end{equation}
\begin{equation}
I(S; W_l | T) = \log \left( 1 + \frac{1}{1 + \gamma \rho_s} \sum_{i=1}^{L} \sigma_i^2 \right),
\end{equation}
\begin{equation}
Substituting (30) into (164), we have
\end{equation}
\begin{equation}
I(S; W_l | W_{l-1}^{-1}, T) = \log \left( \frac{\sum_{i=1}^{L} \frac{1}{2} R_i - \epsilon + \gamma \rho_s}{2 \sum_{i=1}^{L} \frac{1}{2} R_i - \epsilon + \gamma \rho_s} \right).
\end{equation}
Then, the outage condition in (32) is given by
\[
\log \left( \frac{2 \sum_{i=1}^L \frac{1}{L_i} R_i - \epsilon + \gamma \rho_s}{2 \sum_{i=1}^L \frac{1}{L_i} R_i - \epsilon + \gamma \rho_s} \right) \geq \frac{b}{L} \log \prod_{i=1}^{M_s} \left( 1 + \frac{\rho}{M_s} \lambda_i \right) \tag{167}
\]
Therefore, in the high SNR regime, we have, for \( l = 1, \ldots, L \)
\[
2 \sum_{i=1}^L \frac{1}{L_i} R_i - \epsilon + \gamma \rho_s = \frac{\rho \sum_{i=1}^L \frac{1}{L_i} r_i + \rho^{\nu - \beta}}{\rho \sum_{i=1}^L \frac{1}{L_i} r_i + \rho^{\nu - \beta}} = \frac{\rho \sum_{i=1}^L \frac{1}{L_i} r_i - (\nu - \beta) + 1}{\rho \sum_{i=1}^L \frac{1}{L_i} r_i - (\nu - \beta) + 1} \tag{168}
\]
\[
\frac{\rho \sum_{i=1}^L \frac{1}{L_i} r_i - (\nu - \beta) + 1}{\rho \sum_{i=1}^L \frac{1}{L_i} r_i - (\nu - \beta) + 1} \leq \frac{\rho \sum_{i=1}^L \frac{1}{L_i} r_i - (\nu - \beta)}{\rho \sum_{i=1}^L \frac{1}{L_i} r_i - (\nu - \beta)} \leq \frac{\rho \sum_{i=1}^L \frac{1}{L_i} r_i - (\nu - \beta)}{\rho \sum_{i=1}^L \frac{1}{L_i} r_i - (\nu - \beta)} + 1 \tag{169}
\]
(170)
and
\[
\frac{b}{L} \log \prod_{i=1}^{M_s} \left( 1 + \frac{\rho}{M_s} \lambda_i \right) \leq \frac{b}{L} \sum_{i=1}^{M_s} (1 - \alpha_i)^+ \tag{171}
\]
The outage set (32) in the high SNR regime is equivalent to
\[
A_{ls}^1 \triangleq \left\{ (\alpha, \beta) : \frac{b}{L} \sum_{i=1}^{M_s} (1 - \alpha_i)^+ < \left( \frac{b}{L} \sum_{i=1}^{M_s} r_i - (\nu - \beta) \right)^+ \right\} \tag{172}
\]
Now, we study the high SNR behavior of the expected distortion. It is not hard to see that (32) is given by
\[
ED_{ls}(R) = \sum_{l=0}^{L} E_{O_{ls}^1} \left[ D_r \left( \frac{b}{L} R_l, \gamma \right) \right] \tag{173}
\]
(174)
where \( O_{ls}^1 \triangleq \emptyset \) and \( O_{L+1}^1 \triangleq R^{M_s+1} \). For each term in (173), we have
\[
E_{O_{ls}^1} \left[ D_r \left( \frac{b}{L} R_l, \gamma \right) \right] = \int_{A_{ls}^1} \rho^{-\max\{\frac{b}{L} \sum_{i=1}^{L_i} r_i, (\nu - \beta)^+\}} \rho^{-S_A(\alpha)} \rho^{-\beta} d\alpha d\beta, \tag{175}
\]
(176)
where the outage set in the high SNR regime is given by (172).
Applying Varadhan’s lemma to (175), the exponential behavior of (175) for \( l = 0, \ldots, L - 1 \), is found as the solution to
\[
\Delta_l^+ \triangleq \inf_{A_{ls}^1} \max\{b/L R_l, (\nu - \beta)^+\} + S_A(\alpha) + \beta, \tag{177}
\]
where we define \( R_l \triangleq \sum_{i=1}^{L_i} r_i \). Similarly, applying Varadhan’s lemma to (176), the exponential behavior of (176) for \( l = 0, \ldots, L - 1 \) is given by
\[
\Delta_l^+ \triangleq \inf_{A_{ls}^1} \max\{b/L R_l, (\nu - \beta)^+\} + S_A(\alpha) + \beta. \tag{178}
\]
Since \( r_1 \leq r_2 \leq \cdots \leq r_L \) we have \( A_{ls}^1 \subseteq A_{l+1}^1 \), and therefore \( \Delta_l^+ \geq \Delta_{l+1}^+ \). Then, from (173), we have
\[
ED_{ls}(R) = \sum_{l=0}^{L} \rho^{-\Delta_l^+} \rho^{-\Delta_l^+} \leq \sum_{l=0}^{L} \rho^{-\Delta_l^+}. \tag{179}
\]
We define \( A_{ls}^1(r) \triangleq A_{ls}^1 \), where \( r \triangleq [r_1, \ldots, r_L] \). Then, the distortion exponent of LS-LD is given as follows:
\[
\Delta_{ls}^1(b, \nu) = \max_{\nu} \min_{\nu} \Delta_{ls}^1(r). \tag{180}
\]
For \( l = 0 \), i.e., no codeword is successfully decoded, we have
\[
\Delta_{ls}^1(r) = \inf_{\nu} (\nu - \beta)^+ + \beta + S_A(\alpha) \tag{181}
\]
\[
\text{s.t.} \quad \frac{b}{L} \sum_{i=1}^{M_s} (1 - \alpha_i)^+ < \left( \frac{b}{L} \sum_{i=1}^{M_s} r_i - (\nu - \beta) \right)^+. \tag{182}
\]
The infimum is achieved by \( \beta = \nu \) and using the DMT in (79), we have
\[
\Delta_{ls}^1(r) = \nu + d^*(r_1). \tag{183}
\]
The distortion exponent when \( l \) layers are successfully decoded is found as
\[
\Delta_{ls}^1(r) = \inf_{\nu} \max_{\nu} \left\{ \frac{b}{L} R_1, (\nu - \beta)^+ \right\} + \beta + S_A(\alpha) \tag{184}
\]
\[
\text{s.t.} \quad \frac{b}{L} \sum_{i=1}^{M_s} (1 - \alpha_i)^+ \leq \left( \frac{b}{L} \sum_{i=1}^{M_s} r_i - (\nu - \beta) \right)^+. \tag{185}
\]
If \( \frac{b}{L} R_1 \geq \nu \), the infimum of (184) is obtained for \( \beta^* = 0 \) and
\[
\Delta_{ls}^1(r) = \inf_{\nu} \frac{b}{L} R_1 + S_A(\alpha) \tag{186}
\]
\[
\text{s.t.} \quad \sum_{i=1}^{M_s} (\xi_i - \alpha_i)^+ < r_{k+1}. \tag{187}
\]
Using the DMT in (79), (186) is minimized as
\[
\Delta_{ls}^1(r) = \frac{b}{L} R_1 + d^*(r_{l+1}). \tag{188}
\]
If \( \frac{b}{L} R_1 \leq \nu \), we have that the minimum of (184) is achieved by \( \beta^* = (\nu - \frac{b}{L} R_1)^+ \) if \( \frac{b}{L} R_1 > (\nu - \beta) \) and is given by
\[
\Delta_{ls}^1(r) = \nu + d^*(r_{l+1}). \tag{189}
\]
If \( \frac{b}{L} R_1 \leq (\nu - \beta) < \frac{b}{L} R_1^+, \) the optimization problem in (184) is equivalent to
\[
\Delta_{ls}^1(r) = \inf_{\nu} (\nu - \beta)^+ + \beta + S_A(\alpha) \tag{190}
\]
\[
\text{s.t.} \quad \frac{b}{L} \sum_{i=1}^{M_s} (1 - \alpha_i)^+ < \left( \frac{b}{L} R_1^+ - (\nu - \beta) \right)^+. \tag{191}
\]
\[
\frac{b}{L} R_1^+ \leq (\nu - \beta) < \frac{b}{L} R_1^+. \tag{192}
\]
The infimum of (190) is achieved by the largest \( \beta \), since increasing \( \beta \) enlarges the range of \( \alpha \). Then, \( \beta^* = (\nu - \frac{b}{L} R_1)^+ \), and we have,
\[
\Delta_{ls}^1(r) = \nu + d^*(r_{l+1}). \tag{193}
\]
Finally, if \( \frac{b}{L} \frac{d+1}{2} \leq (\nu - \beta) \), there are no feasible solutions for \((184)\). Therefore, putting all together we have

\[
\Delta_i^{s}(r) = \inf \max \left\{ \frac{b}{L} \frac{d+1}{2}, \nu \right\} + d^*(r_{i+1}). \tag{194}
\]

Similarly, at layer \( L \), the infimum is achieved by \( \alpha^* = 0 \) and \( \beta^* = 0 \) and is given by

\[
\Delta_L^{s}(r) = \max \left\{ \frac{b}{L} \frac{d}{2}, \nu \right\}, \quad \text{for} \ r_L \leq M_*. \tag{195}
\]

Note that the condition on \( r_L \) always holds.

A. Solution of the distortion exponent

Assume that for a given layer \( \ell \) we have \( \frac{d}{2} \frac{L}{2} \leq \nu \leq \frac{d}{2} \frac{L}{2} \). Then, \( \Delta_i^{s}(r) = \nu + d^*(r_{i+1}) \) for \( l = 0, \ldots, L-1 \). Using the KKT conditions, the maximum distortion exponent is obtained when all the distortion exponents are equal.

From \( \Delta_0^{s}(r) = \cdots = \Delta_{L-1}^{s}(r) \) we have \( r_1 = \cdots = r_L \), and thus, \( r_l^* = \hat{r}_l^* \). Then, the exponents are given by

\[
\Delta_0^{s}(r) = \nu + d^*(r_1) \tag{196}
\]

\[
\Delta_i^{s}(r) = b \frac{\hat{r}_l^*}{L} r_1 + d^*(r_{i+1}) \tag{197}
\]

\[
\Delta_{L-1}^{s}(r) = b \frac{\hat{r}_L^*}{L} r_1 + b \frac{\hat{r}_{L-1}^*}{L} + d^*(r_L) \tag{198}
\]

\[
\Delta_L^{s}(r) = b \frac{\hat{r}_L^*}{L} r_1 + b \frac{\hat{r}_{L-1}^*}{L} + d^*(r_L) \tag{199}
\]

Equating all these exponents, we have

\[
\frac{b}{L} \frac{\hat{r}_L^*}{L} = d^*(r_L) \tag{200}
\]

\[
\frac{b}{L} \frac{\hat{r}_{L-1}^*}{L} + d^*(r_L) = d^*(r_{L-1}) \tag{201}
\]

\[
\frac{b}{L} \frac{\hat{r}_{L-2}^*}{L} + d^*(r_{L-1}) = d^*(r_{L-2}) \tag{202}
\]

\[
\frac{b}{L} \frac{\hat{r}_{L-3}^*}{L} + d^*(r_{L-2}) = d^*(r_{L-3}) \tag{203}
\]

\[
\frac{b}{L} \hat{r}_L^* + d^*(r_{L+1}) = d^*(r_L) + \nu \tag{204}
\]

\[
\frac{b}{L} \hat{r}_{L+1}^* + d^*(r_{L+2}) = d^*(r_{L+1}) + \nu \tag{205}
\]

A geometric interpretation of the rate allocation for LS-LD satisfying the above equalities is the following: we have \( L - \hat{L} \) straight lines of slope \( b/L \) and each line intersects in the \( y \) axis at a point with the same ordinate as the intersection of the previous line with the DMT curve. The more layers we have the higher the distortion exponent of LS-LD can climb. The remaining \( l \) layers allow a final climb of slope \( lb/L \). Note that the higher \( \hat{L} \), the higher the slope, but the lower the starting point \( d^*(r_{\hat{L}+1}) \).

Next, we adapt Lemma 3 from [13] to our setup. Let \( q \) be a line with equation \( y = -\alpha(t - M) \) for some \( \alpha > 0 \) and \( M > 0 \) and let \( g_l = 1, \ldots, L \) be the set of lines defined recursively from \( L \) to 1 as \( y = (b/L)t + d_{L+1} \), where \( b > 0 \), \( d_{L+1} \equiv 0 \), and \( d_i \) is the \( y \) component of the intersection of \( g_i \) with \( q \). Then, sequentially solving the intersection points for \( i = \hat{L} + 1, \ldots, L \) we have:

\[
d_i - d_{i+1} = M \frac{b}{L} \left( \frac{\alpha}{\alpha + b/L} \right)^{L-i+1}. \tag{206}
\]

Summing all the terms for \( i = \hat{L} + 1, \ldots, L \) we obtain

\[
d_i = M\alpha \left[ 1 - \left( \frac{\alpha}{\alpha + b/L} \right)^{L-i+1} \right]. \tag{207}
\]

In the following we consider a continuum of layers, i.e., we let \( L \to \infty \). Let \( \hat{L} = \kappa L \) be the numbers of layers needed so that \( b\kappa L/Lr_1 = b\kappa r_1 = \nu \), that is, from \( 1 = 1 \) to \( \kappa L \).

When \( M_1 = 1 \), the DMT curve is composed of a single line with \( \alpha = M^* \) and \( M = 1 \). In that case, with layers from \( \kappa L + 1 \) to \( L \) the distortion exponent increases up to

\[
d^*(r_{L\kappa+1}) = M\alpha \left[ 1 - \left( \frac{\alpha}{\alpha + b/L} \right)^{(L-\kappa)} \right]. \tag{208}
\]

In the limit of infinite layers, we obtain

\[
\lim_{L \to \infty} d^*(r_{L\kappa+1}) = M\alpha \left( 1 - e^{-\frac{M_\kappa}{\alpha}} \right). \tag{209}
\]

We still need to determine the distortion achieved due to the climb with layers from \( l = 1 \) to \( l = \kappa L \) by determining \( r_1 \), which is found as the solution to \( \Delta_0^{s}(r) = \Delta_{L\kappa}^{s}(r) \), i.e.,

\[
b \kappa r_1 + d^*(r_{L\kappa+1}) = \nu - \alpha(r_1 - M), \tag{210}
\]

Since \( \nu = b\kappa r_1 \), \( r_1 = \nu/b\kappa \), and from \(210 \) we get to

\[
d^*(r_{L\kappa+1}) = -\alpha \left( \frac{\nu}{b\kappa} - M \right), \tag{211}
\]

which, in the limit of infinite layers, solves for

\[
\kappa^* = \frac{M^*}{b} W\left( e^{\frac{\beta \nu}{M^*}} \right), \tag{212}
\]

where \( W(z) \) is the Lambert \( W \) function, which gives the principal solution for \( w \) in \( z = we^w \). The distortion exponent in the MISO/SIMO case is then found as

\[
\Delta_{L\kappa}^{s}(b, \nu) = \nu + M^* \left( 1 - e^{-\frac{\beta(b^{\kappa} - M^*2)}{M^*}} \right). \tag{213}
\]

For MIMO channels, the DMT curve is formed by \( M_\kappa \) linear pieces, each between \( M_\kappa - k \) and \( M_\kappa - k + 1 \) for \( k = 1, \ldots, M_\kappa \). From the value of the DMT at \( M_\kappa - k \) to the value at \( M_\kappa - k + 1 \), there is a gap of \( M^* - M_\kappa + 2k - 1 \) in the \( y \) axis. Each piece of the curve can be characterized by \( y = -\alpha(t - M) \), where for the \( k \)-th interval we have \( \alpha = \phi_k \) and \( M = M_k \) as in \(35 \).

We will again consider a continuum of layers, i.e., we let \( L \to \infty \), and we let \( l = L\kappa \) be the number of lines required to have \( b\kappa r_1 = \nu \). Then, for the remaining lines from \( l + 1 \) to \( L \), let \( L(1 - \kappa)\kappa_k \) be the number of lines with slope \( b\kappa \) required to climb up the whole interval \( k \). Since the gap in the \( y \) axis from the value at \( M_\kappa - k \) to the value at \( M_\kappa - k + 1 \), is \( M^* - M_\kappa + 2k - 1 \), climbing the whole \( k \)-th interval with \( L(1 - \kappa)\kappa_k \) lines requires

\[
d_{L-L[1-(1-\kappa)]\kappa_k} = M^* - M_\kappa + 2k - 1, \tag{214}
\]

where

\[
d_{L-L[1-(1-\kappa)]\kappa_k} = M\alpha \left[ 1 - \left( \frac{\alpha}{\alpha + b/L} \right)^{(L-\kappa)\kappa_k+1} \right] \tag{215}
\]
In the limit we have
\[
\lim_{L \to \infty} d_{L-L(1-\kappa)} = M \alpha \left[ 1 - e^{-\frac{b(1-\kappa)}{M(1-\kappa)}} \right].
\] (216)

Then, each required portion, \(\kappa_k\), is found as
\[
\kappa_k = \frac{M^* - M_* + 2k - 1}{b(1-\kappa)} \ln \left( \frac{M_* - k + 1}{M_* - k} \right).\] (217)

This gives the portion of lines required to climb up the \(k\)-th segment of the DMT curve. In the MIMO case, to be able to go up exactly to the \(k\)-th segment with lines from \(l + 1\) to \(L\) we need to have \(\sum_{j=1}^{k-1} \kappa_j < 1 \leq \sum_{j=1}^{k} \kappa_j\). This is equivalent to the requirement \(c_{k-1} < b(1-\kappa) \leq c_k\) using \(c_i\) as defined in Theorem 2. To climb up each line segment we need \(\kappa_k(1-\kappa)\) lines (layers) for \(k = 1, \ldots, M_* - 1\), and for the last segment climbed we have \((1 - \sum_{j=1}^{k-1} \kappa_j) L\) lines remaining, which gives an extra ascent of
\[
M\alpha \left( 1 - e^{-\frac{b(1-\kappa)(1-\sum_{j=1}^{k-1} \kappa_j)}{M(1-\kappa)}} \right).\] (218)

Then, we have climbed up to the value
\[
d_{L_0+1} = \sum_{i=1}^{k-1} (M^* - M_* + 2i - 1) + (M_* - k + 1)(M^* - M_* + 2k - 1) \left( 1 - e^{-\frac{b(1-\kappa)(1-\sum_{j=1}^{k-1} \kappa_j)}{M(1-\kappa)}} \right).\] (219)

(220)

With the remaining lines, i.e., from \(l = 1\) to \(l = \kappa L\), the extra climb is given by solving \(\Delta_l^{\alpha}(r) = \Delta_{L_l}(r)\), i.e.,
\[
\nu + d^*(r_1) = b\nu r_1 + d_{L_0+1}.\] (222)

The diversity gain \(d^*(r_1)\) at segment \(k\) is given by
\[
d^*(r_1) = -\alpha(r_1 - M) + \sum_{i=1}^{k-1} (M^* - M_* + 2i - 1).\] (223)

Since we have \(b\nu r_1 = \nu\), this equation simplifies to
\[
d^*(\frac{\nu}{bk}) = d_{L_0+1}.\] (224)

Therefore, using \(c_{k-1} = b(1-\kappa) \sum_{j=1}^{k-1} \kappa_j\), we solve \(\kappa\) from
\[
-\alpha \left( \frac{\nu}{bk} - M \right) = M \alpha \left( 1 - e^{-\frac{b(1-\kappa)}{M(1-\kappa)}} \right),\] (225)

and find
\[
\kappa^* = \frac{\alpha}{b} \mathcal{W} \left( \frac{e^{-\frac{b}{M(1-\kappa)}}}{M \alpha} \right).\] (226)

The range of validity for each \(k\) is given by \(c_{k-1} < b(1-\kappa) \leq c_k\). Since for a given \(c\), the solution to \(c = b(1-\kappa^*)\) is found as
\[
b = \frac{\nu c_{k-1} - c}{M} + c,\] (227)

when \(c = c_{k-1}\), we have
\[
b > \frac{\nu}{M} + c_{k-1} = c_{k-1} + \frac{\nu}{M_* - k + 1}.\] (228)

When \(c = c_k\), since \(c_{k-1} - c_k = \alpha \ln(M/(M_* - k))\), we have
\[
b \leq \frac{\nu c_{k-1} - c_k}{M} + c_k = c_k + \frac{\nu}{M_* - k}.\] (229)

Putting all together, we obtain the condition of the theorem and the corresponding distortion exponent.

**Appendix E: Proof of Theorem 5**

We consider the usual change of variables, \(\lambda_i = \rho^{-\alpha_i}\), and \(\gamma = \rho^{-\beta}\). Let \(r_1\) be the multiplexing gain of the \(l\)-th layer and \(r = [r_1, \ldots, r_L]\), such that \(R_i = r_i \log \rho\), and define \(r_1'\) as in (168).

First, we derive the outage set \(O_l^{\rho}\) for each layer in the high SNR regime, which we denote by \(L_l^*\). For the power allocation \(r_i' = \rho^\alpha_i(r_i - \beta_i)\), the l.h.s. of the inequality in the definition of \(O_l^{\rho}\) in (42) is given by
\[
I(X_l; Y|X_1^{\rho}) = I(X_l^{\rho}; Y|X_1^{\rho}) - I(X_l^{\rho+1}; Y|X_1^{\rho}) = \log \frac{\det (I + \rho^{\alpha_l-\beta_l} HH^H)}{\det (I + \rho^{\alpha_l-\beta_l} HH^H)}\] (230)

(231)

\[
= \log \left( \prod_{i=1}^{M} \frac{1 + \rho^{\alpha_l-\beta_l} \lambda_i}{1 + \rho^{\alpha_l-\beta_l} \lambda_i} \right) \leq \sum_{i=1}^{M} \rho^{\alpha_i-\beta_i} \Delta_{L_1}(r, \xi).
\] (232)

The r.h.s. of the inequality in the definition of \(O_l^{\rho}\) in (42) can be calculated as in (168). Then, from (230) and (168), \(L_l^*\) follows as:
\[
L_l^* \triangleq \left\{ (\alpha, \beta) : b \sum_{i=1}^{M} [\Delta_{L_1}(r, \xi)] \right\} (233)
\]

and find
\[
\kappa^* = \frac{\alpha}{b} \mathcal{W} \left( e^{-\frac{b}{M(1-\kappa)}} \right).\] (226)

The range of validity for each \(k\) is given by \(c_{k-1} < b(1-\kappa) \leq c_k\). Since for a given \(c\), the solution to \(c = b(1-\kappa^*)\) is found as
\[
b = \frac{\nu c_{k-1} - c}{M} + c,\] (227)

when \(c = c_{k-1}\), we have
\[
b > \frac{\nu}{M} + c_{k-1} = c_{k-1} + \frac{\nu}{M_* - k + 1}.\] (228)

where, from Varadhan’s lemma, the exponent for each integral term is given by
\[
\Delta(L_1; \xi) = \inf_{L_1} \{ \Delta(L_1; \xi) + \beta + S_A(\alpha, \beta) \}.
\] (240)
Then, the distortion exponent is found as
\[ \Delta_{bs}(r, \nu) = \max_{r, \xi} \min_{l=0, \ldots, L} \left\{ \Delta^{bs}_l(r, \xi) \right\}. \] (242)

Similarly to the DMT, we consider the successive decoding diversity gain, defined in [13], as the solution to the probability of outage with successive decoding of each layer, given by
\[ d_{ds}(r_l, \xi_{l-1}, \xi_l) \triangleq \inf_{\alpha^+} S_A(\alpha) \] (243)
\[ \text{s.t. } r_l > \sum_{i=1}^{M_l} \left[ (\xi_{l-1} - \alpha_i)^+ - (\xi_l - \alpha_i)^+ \right] \] (244)

Without loss of generality, consider the multiplexing gain \( r_l \) given by \( r_l = k(\xi_{l-1} - \xi_l) + \delta_l \), where \( k \in \{0, 1, \ldots, M_s - 1\} \) and \( 0 \leq \delta_l < \xi_{l-1} - \xi_l \). Then, the infimum for (243) is found as
\[ d_{ds}(r_l, \xi_{l-1}, \xi_l) = \Phi_k \xi_{l-1} - \Upsilon_k \delta_l, \] (245)
with
\[ \alpha^*_l = \begin{cases} \xi_{l-1}, & 1 \leq i < M_s - k, \\ \xi_{l-1} - \delta_l, & i = M_s - k, \\ 0, & M_s - k < i \leq M_s. \end{cases} \] (246)

Now, we solve (241), using (245) for each layer, as a function of the power allocation \( \xi_{l-1} \) and \( \xi_l \), and the rate \( r_l \).

When no layer is successfully decoded, i.e., \( l = 0 \), we have
\[ \Delta^{bs}_0(r, \xi) = \inf(\nu - \beta)^+ + \beta + S_A(\alpha) \] (247)
\[ \text{s.t. } b \sum_{i=1}^{M_0} \left[ (\xi_0 - \alpha_i)^+ - (\xi_1 - \alpha_i)^+ \right] < (br_1 - (\nu - \beta)\xi) \] (248)

The infimum is achieved by \( \beta^* = \nu \) and using (243), we have
\[ \Delta^{bs}_0(r, \xi) = \nu + d_{ds}(r_1, \xi_0, \xi_1). \] (249)

At layer \( l \), the distortion exponent is given by the solution of the following optim
\[ \Delta^{bs}_l(r, \xi) = \inf \max \left\{ b r_1^l, (\nu - \beta)^+ \right\} + \beta + S_A(\alpha) \] (250)
\[ \text{s.t. } b \sum_{i=1}^{M_l} \left[ (\xi_l - \alpha_i)^+ - (\xi_{l+1} - \alpha_i)^+ \right] < (br_{l+1} - (\nu - \beta)\xi) \] (251)

If \( br_1^l \geq \nu \), the infimum is obtained for \( \beta^* = 0 \) and solving
\[ \Delta^{bs}_l(r, \xi) = \inf \max \left\{ b r_1^l, \nu \right\} + S_A(\alpha) \] (252)
\[ \text{s.t. } b \sum_{i=1}^{M_l} \left[ (\xi_l - \alpha_i)^+ - (\xi_{l+1} - \alpha_i)^+ \right] < r_{k+l} \] (253)

Using (243), we obtain the solution as
\[ \Delta^{bs}_l(r, \xi) = \max\{x, br_1^l\} + d_{ds}(r_{l+1}, \xi_l, \xi_{l+1}). \] (254)

If \( br_1^l \leq \nu \), the infimum is given by \( \beta^* = (\nu - br_1^l)^+ \), and again, we have a version of (243) with the distortion exponent
\[ \Delta^{bs}_l(r, \xi) = \nu + d_{ds}(r_{l+1}, \xi_l, \xi_{l+1}). \] (255)

At layer \( L \), the distortion exponent is the solution to the optimization problem
\[ \Delta^{bs}_L(r, \xi) = \inf \max \left\{ b r_1^L, (\nu - \beta)^+ \right\} + \beta + S_A(\alpha) \] (256)
\[ \text{s.t. } b \sum_{i=1}^{M_L} \left[ (\xi_{L-1} - \alpha_i)^+ - (\xi_L - \alpha_i)^+ \right] \] (257)
\[ \geq (br_{L-1}^L - (\nu - \beta)^+) + (br_{L-1}^L - (\nu - \beta))^+ \]

The infimum is achieved by \( \alpha^* = 0 \) and \( \beta^* = 0 \), and is given by
\[ \Delta^{bs}_L(r, \xi) = \max \left\{ b r_{L+1}^L, \nu \right\}, \] (258)
for \( r_{L+1} \leq M_s \xi_{L-1} - \xi \)

Note that the condition on \( r_{L+1} \) always holds.

Gathering all the results, the distortion exponent problem in (242) is solved as the minimum of the exponent of each layer, \( \Delta^{bs}_l(r, \xi) \), which can be formulated as
\[ \Delta^{bs}(r, \nu) = \max \left\{ \Delta^{bs}_l(r_l, \xi_l) \right\}, \] (259)
\[ \text{s.t. } t \leq \nu + d_{ad} (r_1, \xi_0, \xi_1), \] (260)
\[ t \leq \max\{b r_1^l, \nu\} + d_{ad} (r_{l+1}, \xi_l, \xi_{l+1}), \] (261)
for \( l = 1, \ldots, L - 1 \)

If \( \nu \geq br_1^{L+1} \), then \( \max\{b r_1^l, \nu\} = \nu \) for all \( l \), and the minimum distortion exponent is given by \( \Delta^{bs}_l(r, \xi) = \nu \), which implies \( \Delta^{bs}(r, \nu) = \nu \). If \( \nu \leq br_1^1 \), then \( \max\{b r_1^l, \nu\} = br_1^1 \) for all \( l \). In general, if \( br_1^l < \nu \leq br_1^{q+1} \), \( q = 0, \ldots, L \), and \( r_{L+1} \triangleq 0, r_{L+1}^{L+1} \triangleq \infty \), then (259) can be formulated, using \( r_l = k(\xi_{l-1} - \xi_l) + \delta_l, \delta \triangleq [\delta_1, \ldots, \delta_L] \) and \( \xi \), as the following linear optimization program:
\[ \Delta^{bs}(r, \nu) = \min \min \left\{ \Delta^{bs}_l(r_l, \xi_l) \right\} \] (263)
\[ \text{s.t. } t \leq \nu + b \sum_{i=1}^{M_l} \left[ k(\xi_{l-1} - \xi_l) + \delta_i \right] + \Phi_k \xi_l - \Upsilon_k \delta_{l+1}, \] (264)
\[ t \leq \nu + b \sum_{i=1}^{M_l} \left[ k(\xi_{l-1} - \xi_l) + \delta_i \right] + \Phi_k \xi_l - \Upsilon_k \delta_{l+1}, \] (265)
\[ t \leq b \sum_{i=1}^{M_l} \left[ k(\xi_{l-1} - \xi_l) + \delta_i \right], \] (266)
\[ 0 \leq \delta_l \leq \xi_{l-1} - \xi_l, \] (267)
\[ \text{for } l = 1, \ldots, L, \] (268)
\[ 0 \leq \xi_L \leq \xi \leq 0 = 1, \] (269)
\[ \sum_{i=1}^{M_l} [bk(\xi_{l-1} - \xi_l) + \delta_i] < \nu. \] (270)

The linear program (263) can be efficiently solved using numerical methods. In Figure 3, the numerical solution is shown. However, in the following we provide a suboptimal yet more compact analytical solution by fixing the multiplexing gains \( r \).

We fix the multiplexing gains as \( \beta_l = [(k+1)(\xi_{l-1} - \xi_l) - \epsilon_1], \)
\( \epsilon_1 > 0 \) for \( k = 0, \ldots, M_k - 1 \), and \( \delta_l \triangleq (\xi_{l-1} - \xi_l) - \epsilon_1 \), when the bandwidth ratio satisfies

\[
 b \in \left[ \frac{\Phi_{k+1} + \nu}{k}, \frac{\Phi_k + \nu}{k+1} \right], \quad (271)
\]

Assume \( br_1 \geq \nu \). Then, each distortion exponent is found as

\[
 \hat{\Delta}_{bs}^L (r, \xi) = \nu + \Phi_k \xi_l - \Omega_k \delta_{l+1}, \quad (272)
\]

\[
 \hat{\Delta}_l^{bs} (r, \xi) = b r_l + \Phi_k \xi_l - \Omega_k \delta_{l+1}, \quad \text{for } l = 1, \ldots, L - 1, \quad (273)
\]

\[
 \hat{\Delta}_{L}^{bs} (r, \xi) = b r_{L}. \quad (274)
\]

Similarly to the other schemes, for which the distortion exponent is maximized by equating the exponents, we look for the power allocation \( \xi_l \), such that all distortion exponent terms \( \Delta_{bs}^{L} (r, \xi) \) in (242) are equal.

Equating all distortion exponents \( \hat{\Delta}_l^{bs} (r, \xi) \) for \( l = 2, \ldots, L - 1 \), i.e., \( \Delta_{bs}^{L} (r, \xi) = \Delta_{bs}^{L} (r, \xi) \), we have

\[
 d_{sd} (\hat{r}_l, \xi_l - \xi_l - \epsilon_1). \quad (275)
\]

Substituting in (275), we find that the power allocations for \( l \geq 2 \) need to satisfy

\[
 (\xi_l - \xi_l - \epsilon_1) = \eta_l (\xi_l - \xi_l) + O(\epsilon_l), \quad (277)
\]

where \( \eta_l \) is defined in (44) and \( O(\epsilon_l) \) denotes a term that tends to 0 as \( \epsilon_l \to 0 \). Then, for \( l = 2, \ldots, L - 1 \) we obtain

\[
 \xi_l - \xi_l - \epsilon_1 = \eta_l^{-1} (\xi_l - \xi_l) + O(\epsilon_l), \quad (278)
\]

and \( \xi_l \) can be found as

\[
 1 - \xi_l = (1 - \xi_l) + \sum_{i=1}^{l-1} (\xi_l - \xi_l - \epsilon_1) + O(\epsilon_l) \quad (279)
\]

\[
 = (1 - \xi_l) + \sum_{i=1}^{l-1} \eta_l^{-1} (\xi_l - \xi_l - \epsilon_1) + O(\epsilon_l) \quad (280)
\]

\[
 = (1 - \xi_l) + (\xi_l - \xi_l - \epsilon_1) \frac{1 - \eta_l^{-1}}{1 - \eta_l} + O(\epsilon_l). \quad (281)
\]

Then, for \( l = 2, \ldots, L \), we have

\[
 \xi_l = \xi_l - (\xi_l - \xi_l - \epsilon_1) \frac{1 - \eta_l^{-1}}{1 - \eta_l} + O(\epsilon_l). \quad (282)
\]

From \( \hat{\Delta}_L^{bs} (r, \xi) = b r_{L} \), we have

\[
 \hat{\Delta}_{L}^{bs} (r, \xi) = b \sum_{i=1}^{L} (\xi_0 - \xi_l - \epsilon_1) + O(\epsilon_l). \quad (283)
\]

\[
 \hat{\Delta}_{L}^{bs} (r, \xi) = b (k + 1) (\xi_0 - \xi_l - \epsilon_1) + O(\epsilon_l). \quad (284)
\]

Then solving \( \hat{\Delta}_{bs}^{L} (b, \nu) = \hat{\Delta}_{bs}^{L} (r, \xi) \), and letting \( \epsilon_1 \to 0 \), we obtain (45), and

\[
 \xi_1 = \frac{(\Omega_k + \Phi_k \xi_l - \Omega_k \delta_{l+1})}{(\Omega_k + \Phi_k \xi_l - \Omega_k \delta_{l+1}) - \Omega_k \delta_{l+1}}. \quad (289)
\]

\[
 \xi_1 - \xi_2 = \frac{\Phi_k (\Omega_k + \Phi_k \xi_l - \Omega_k \delta_{l+1}) - \Omega_k \delta_{l+1}}{(\Omega_k + \Phi_k \xi_l - \Omega_k \delta_{l+1}) - \Omega_k \delta_{l+1}} \quad (290)
\]

For this solution to be feasible, the power allocation sequence has to satisfy \( 1 \geq \xi_1 \geq \ldots \xi_L \geq 0 \), i.e., \( \xi_1 - \xi_0 \geq 0 \). From (275) we need \( \eta_l \geq 0 \) and \( \xi_l - \xi_l - \epsilon_1 \geq 0 \). We have \( \eta_l \geq 0 \) if \( b \geq \frac{\nu}{k+1} \), which holds in the regime characterized by (271).

Then, \( \xi_1 - \xi_2 \geq 0 \) holds if \( (\Omega_k + \Phi_k \xi_l - \Omega_k \delta_{l+1}) - \Omega_k \delta_{l+1} \geq 0 \).

It can be shown that \( (\Omega_k + \Phi_k \xi_l - \Omega_k \delta_{l+1}) - \Omega_k \delta_{l+1} \) is monotonically increasing in \( b \geq 0 \), and positive for \( k = 0, \ldots, M_k - 1 \). Therefore, we need to check if \( \Omega_k + \Phi_k \xi_l - \Omega_k \delta_{l+1} \geq 0 \). This holds since this condition is equivalent to

\[
 b \geq \frac{\Phi_k + \nu}{k + 1}. \quad (291)
\]

Note that, in this regime, we have \( \xi_1 \geq 0 \). In addition, \( \xi_1 = \xi_1(1 + \xi_1 - \epsilon_2) \Gamma_k \geq 0 \). Therefore, for each \( k \) the power allocation is feasible in the regime characterized by (271). It can also be checked that \( br_1 = \nu \) is satisfied. This completes the proof.

REFERENCES


