An Investigation of the Mechanical Fatigue Behavior of Low Thermal Expansion Lattice Structures

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ABSTRACT

A study of the mechanical fatigue behavior of a Ti-6Al-4V lattice structure designed to exhibit controlled thermal expansion has been performed. Comparison of S-N curves generated under both zero-tension and fully reversed cyclic loading has determined that the fatigue resistance of the lattice is substantially poorer than that of the constituent Ti-6Al-4V material for the same remote applied (macroscopic) stress. In addition, the effect of beta annealing the as-received mill-annealed alloy was also to reduce fatigue life in both the lattices and parent material. This effect is due to significant microstructural changes that occurred during heat treatment. Increasing the stress ratio ($\sigma_{\text{min}}/\sigma_{\text{max}}$) of the cyclic waveform from -1 to 0 had a similar effect. An analytical model has been developed to predict the fatigue life of the lattice structures from the S-N curves of the parent material, by determining the relationship between the macroscopic stresses acting on the lattice structure and the local stresses. The local stresses were then used in a multiaxial fatigue model to determine the fatigue life. The analytical model is able to predict the fatigue life with reasonable accuracy and minimal cost. The Findley multiaxial fatigue parameter for the parent material and lattice structures can be fitted with a power law equation and appears to fall onto a single curve, suggesting the local behavior within the lattice material is similar to the parent material. The analytical tools developed in this study can be hugely beneficial to the design of these lattice structures in the aerospace and communications industries.
1. INTRODUCTION

In future hypersonic aircraft and spacecraft, large temperature gradients can develop across the structural skin of the vehicle because of thermal expansion caused by external frictional heating of the outer surface. The thermal strains associated with this expansion can impose substantial thermal stresses on underlying components and structures. Such stresses can severely compromise the life of the component, and so minimization of these stresses via thermal expansion control is an important consideration. In addition to future hypersonic vehicles, other applications where control of thermal expansion is desirable include communications satellites, space mirrors, thermal protection panels and structural heat pipes.

Bi-material lattice structures that exhibit low thermal expansion behavior are a promising method of achieving this goal. An example of such a lattice, designed by the University of California, Santa Barbara (UCSB) [1, 2], is shown in Fig. 1. The lattice is comprised of a framework manufactured from a material with a relatively low coefficient of thermal expansion (CTE), and triangular inserts made from a material with a higher CTE. The mechanism by which low CTE is achieved is explained in detail elsewhere [1, 2]. However, summarized briefly, overall thermal expansion of the lattice is counteracted by the triangular inserts pushing against the framework, causing expansion into the open spaces of the lattice, via rotation of the lattice members at invariant nodes. Additionally, precisely controlled thermal expansion characteristics of the lattice can be achieved through minor adjustments to the lattice geometry.

Considerable research on the subject has already been performed [1-3]. The theoretical concept has been validated experimentally for bi-metallic lattices comprised of titanium and aluminum [1-3]. The study has also been extended to fiber reinforced polymer (FRP) lattices, and again, the low expansion concept was successfully demonstrated (unpublished research, J. Berger, C. Mercer, R. M. McMeeking and F. W. Zok, UCSB). In addition, the lattice geometries were found to possess satisfactory mechanical robustness under in-plane compression loading, exhibiting plastic yielding rather than elastic buckling at critical stress levels [3].

However, little research on the fatigue behavior of these lattice structures, and lattice structures in general, has been carried out thus far, especially in terms of modeling [4, 5]. Fatigue is an important consideration, since these lattices would be exposed to both mechanical and thermal...
cyclic stresses during service. Work performed by Steeves et al. [1] has demonstrated that shakedown occurs after a modest amount of thermal cycles (~30) and so the lattice reaches stability very quickly. Mechanical cyclic loading, on the other hand, has not been studied and is likely to have a major influence on the lives of these structures during service. Furthermore, it is likely that the fatigue behavior of low expansion lattices would vary greatly depending on various materials/geometry combinations. Therefore, a model with low computational costs that can predict the fatigue behavior of lattice structures from the S-N curve of the parent material will be a very useful tool in the design and analysis of these lattice structures.

Some notable studies on modelling the fatigue behavior of lattice or cellular structures include Huang and Liu, [6, 7] and Masoumi Khalil Abad et al. [5]. Huang and Liu [6, 7] have developed fatigue models for open-cell foams and honeycomb lattices for crack propagation, as well as high cycle and low cycle fatigue. Their analyses were based on dimensional analysis and the fatigue life was found to be a function of relative density and the stress components. Due to the dimensional analysis approach of their model, it cannot be easily extended to allow for the prediction of lattices with different geometries, as the model parameters used to determine the fatigue life do not explicitly involve geometry. On the other hand, Masoumi et al. [5] used homogenization techniques to relate the microscopic stresses within a unit cell of the lattice structure with the macroscopic stresses. They then calculated the fatigue stresses by multiplying the macroscopic yield stress by the ratio of the fatigue stress and yield stress of the parent material. Although this approach is robust, the homogenization requires the use of finite element analysis, which may have a higher computational cost. Hence, there is an imperative to develop a computationally efficient model to predict the fatigue lives of such lattices.

The purpose of this investigation, therefore, is to determine the typical fatigue behavior that could be expected of these types of structures for a representative material/lattice geometry combination; and to develop an analytical model that has the capability of predicting, to a fair degree of accuracy, the fatigue life of such a lattice structure for a particular defined geometry.
2. MATERIAL AND EXPERIMENTAL PROTOCOL

The titanium alloy Ti-6Al-4V was selected for this investigation. There were two primary reasons for this choice of material. To begin with, the alloy is a well-known structural material, widely used in the aerospace industry. Secondly, the material has been employed in previous work on bi-metallic lattices due to its relatively low CTE (~9 ppm/°C) [1-3]. The material was procured in the form of mill-annealed, 280 mm x 88 mm plates with thicknesses of 2 mm and 4 mm. Half of the plates were subjected to the following beta anneal heat treatment schedule:

1. 1035°C for 30 minutes, air cool + 730°C for 2 hours, then air cool.

Beta annealing is often used in industry to improve the toughness of Ti-6Al-4V. In this case, it was also performed to assess the effect of microstructural changes on the fatigue behavior of the lattice structures.

Microstructural characterization of the as-received (mill-annealed) and beta-annealed material was performed using scanning electron microscopy with backscattered (compositional contrast) imaging following standard metallographic preparation procedures.

The Ti-6Al-4V plates were initially surface ground to remove (i) scratches present on the surface of the as-received material, and (ii) any scale or brittle alpha-case formed during beta annealing. Lattice specimens for fatigue experiments were then fabricated from the plates using water jet machining. This technique was chosen in preference to laser cutting, since the latter may have introduced a brittle heat-affected zone to the machined lattice surfaces. The same lattice geometry used in previous investigations [1-3] was chosen, since low expansion behavior for this geometry has been successfully demonstrated. For the purposes of this study, it was considered sufficient to fabricate lattice specimens from a single material, since only the fatigue behavior of the lattice geometry is being explored. This will allow a detailed baseline understanding of the fatigue behavior of this particular geometry to be established without the additional complication of the effects of stress concentrations at joints between dissimilar materials. These stress concentrations are, nevertheless, important and could form the subject of future investigations.

The finished lattice fatigue specimen is shown in Fig. 2. It incorporates a complete unit cell in the center of the specimen to help eliminate boundary effects, and solid regions either side of the lattice area for clamping during testing.
Following specimen fabrication, fatigue experiments were performed under zero-tension (R=σ_min/σ_max=0) and fully reversed (R=σ_min/σ_max=-1) loading conditions. The 4 mm thick plates were used for the fully reversed tests to prevent out-of-plane buckling during the compression part of the fatigue cycle. The cyclic waveform was sinusoidal and the frequency was 10 Hz. The fatigue life of the lattice structure for a range of remote applied stress values was established. The data was then plotted to generate S-N curves (plots of applied stress amplitude versus the number of cycles to failure).

In order to allow a meaningful comparison between the fatigue behavior of the lattice and the constituent material (Ti-6Al-4V), similar fatigue experiments were also performed on dog-bone specimens fabricated from the same Ti-6Al-4V plates. The fatigue behavior of a material is strongly influenced by parameters such as chemical composition, microstructure, processing method and surface condition. Therefore, additional fatigue tests were conducted on smooth (polished), circular-cross section dog-bone specimens machined from 20 mm diameter Ti-6Al4V round bars, in the same mill-annealed and beta-annealed conditions as the plate material, and procured from the same supplier. The purpose of performing experiments on smooth, cylindrical specimens was to fully remove any geometry effects and so ensure a thorough understanding of the role of microstructure on the fatigue characteristics of this lattice geometry/material combination.

3. EXPERIMENTAL RESULTS

The microstructures of the mill-annealed and beta-annealed Ti-6Al-4V alloy are presented in Figs. 3 (a) and (b), respectively. The mill-annealed plate material exhibited a “necklace” structure comprised of alpha grains (dark phase) with a network of very fine-grained beta (light phase) located on the alpha grain boundaries. The alpha grains were found to be somewhat elongated in the rolling direction of the plate. The microstructure of the beta-annealed material, on the other hand, consisted of a Widmanstatten or “basket-weave” structure comprised of relatively large packets of “inter-woven” alpha and beta lamellae.

The results of the fatigue experiments for both the mill-annealed and beta-annealed conditions are shown by the S-N curves on the log-log plot presented in Fig. 4. The markers in Fig. 4
represent the experimental data points, while the lines are curve fits of the data points with Eq. 1 using least square regression. Eq. 1 is the sum of Basquin’s law used to describe high cycle fatigue, which is the interest of this study, with a constant c:

$$\sigma = aN^b + c$$  \hspace{1cm} \text{Eq. 1}

where $a$, $b$, and $c$ are constants.

The fits in Fig 4 are excellent with $R^2$ values no smaller than 0.875. An important point to note is that the failure mechanism of materials at low cycles is generally different from the failure mechanism for high cycle fatigue, especially when plastic strains become significant. Extrapolating the fits to low cycles where there are no experimental data points may lead to inaccurate results; for example the fit that suggest the mill-annealed plate ($R = -1$) will have a fatigue strength of 4 GPa at 1000 cycles. Hence, it is important to be aware that the fits are only valid within the regions where the data points are available and interpreting extrapolated results must be done with care.

It is immediately apparent from the comparison of the lattice structures with the bar and plate materials in Fig. 4 that, under both zero-tension and fully reversed loading, the fatigue resistance of the lattice structure is markedly inferior to that of the constituent material. There was a slight reduction in fatigue life for the flat plate specimen compared to the smooth, cylindrical samples, suggesting that the presence of four 90° corners had a small but not overly significant effect. However, the effect of machining the lattice geometry into the plates was to severely impair the fatigue resistance for the same remote applied (macroscopic) stress. In fact, the lattice specimens were found to exhibit comparable fatigue life to the constituent Ti-6Al-4V alloy at macroscopic stress levels of up to two orders of magnitude lower. However, the result is not particularly surprising, since work performed at UCSB on the mechanical behavior of 304 stainless steel lattices indicated that the compressive strength levels of such structures are only about 10% that of the constituent stainless steel itself [3].

The effect of beta-annealing was to reduce the fatigue performance of both the Ti-6Al-4V material, and consequently, the lattice. This is a known consequence of beta-annealing. The gradient of the S-N curves produced for the beta-annealed condition was also different; a fatigue endurance limit was not observed, as was generally the case for the mill-annealed condition. This
is particularly evident in the curves for the lattices. Interstitial oxygen level and Widmanstatten packet size can have a marked effect on fatigue crack growth in beta-annealed Ti-6Al-4V [8].

The effect of stress ratio, $R (\sigma_{\text{min}}/\sigma_{\text{max}})$, was to reduce the fatigue life for the same applied stress amplitude. This is expected due to the higher mean stress in the experiments conducted at a stress ratio of zero (zero-tension). However, the effect was not noticeably worse in the case of the lattices compared with that of the Ti-6Al-4V material.

All lattice specimens were observed to fail in the same location (Fig. 5 (a)). This was confirmed by finite element analysis (Fig. 5 (b)). The lattice members where failure occurred will be under the greatest localized tensile stress during cyclic loading (see Analytical Modeling section below). It is also apparent from the finite element simulation (Fig. 5 (b)) that the tensile stress in the lattice framework is considerably higher than that in the triangular members.

4. ANALYTICAL MODELLING

4.1. MODELLING APPROACH

In this study, an analytical model to predict the fatigue life of lattice structures using the S-N curve of the parent material was developed. The only additional information that the analytical model requires are the bulk material properties and the design parameters of the lattice structure, which are the length, $L_1$, and skew angle, $\theta$, as seen in Fig. 6; in this case to predict the fatigue life. The model has minimal computational cost compared to the method described in Masoumi Khalil Abad et al. [5].

The first step in the analysis is to construct a model to relate the local stresses within the lattice structure to the macroscopic stresses. The local stresses will then be used in conjunction with a multiaxial fatigue model to determine the fatigue behavior of the struts within the lattice structure at a given macroscopic loading condition. Lastly, the fatigue strength calculated using the multiaxial fatigue model and local stresses will be converted to the S-N curve using the relationship between the local stresses and macroscopic stresses.

4.2. MODELLING THE LATTICE STRUCTURE
The lattice structure and the specimen, as seen in Fig. 7 (a), were modelled using the unit cell of the lattice structure, as shown in Fig. 6. The elastic properties of both the mill- and beta-annealed Ti-6Al-4V material used in the model were assumed to be identical and are described in Table 1.

Table 1. Mechanical properties of Ti-6Al-4V

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus, E</td>
<td>113.80</td>
<td>GPa</td>
</tr>
<tr>
<td>Poisson’s Ratio, ν</td>
<td>0.342</td>
<td></td>
</tr>
</tbody>
</table>

The dimensions of a single unit cell used in this study are summarized in Table 2. The values in Table 2 were determined based on the geometry of the lattice structure of the actual fatigue specimens shown in Fig. 8, and the widths of the strut member was taken to be the average between the two ends (2.80-2.47 mm in Fig 8). Additionally, as seen in Fig. 8, there is a significant amount of additional material at the nodes of the lattice structure, which will result in a stiffer and stronger lattice structure. Since the width of the additional material is significantly larger than the width of the beams in the model, the effects of the additional material were approximated by using an effective length, described by Eq. 2. This is equivalent to treating the portion with the additional material as completely rigid.

\[ L_{1,\text{eff}} = L_1 - l_1 - l_2 \]

where \( L_{1,\text{eff}} \) is the effective lengths of the beams, while \( l_1 \) and \( l_2 \) are the representative lengths of the additional material as seen in Fig. 8.

Table 2. Dimensions of a single unit cell

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of beam, ( L_1 )</td>
<td>23.21</td>
<td>mm</td>
</tr>
<tr>
<td>Skew Angle, ( \theta )</td>
<td>20.3</td>
<td>(^\circ)</td>
</tr>
<tr>
<td>Thickness of specimen, ( b )</td>
<td>2 (for R=0), 4 (for R=-1)</td>
<td>mm</td>
</tr>
<tr>
<td>Width of strut member, ( h )</td>
<td>2.635</td>
<td>mm</td>
</tr>
</tbody>
</table>
In this work, the lattice structure was modelled in a similar manner to Berger et al. [3] but with Timoshenko beams instead of Euler-Bernoulli beams. This is to increase the accuracy of the predictions for cases in which the beams are not slender and the shear effects are significant; for example, in this study where $L_1/h \approx 9.4$. Additionally, the triangle insert of the lattice structure was modelled using a stiffness matrix instead of an equivalent strut as done in Berger et al. [3]. Modelling the triangle insert this way will result in more accurate predictions for the behavior of the triangle insert and allows the deformation behavior of the triangle to be characterized using geometric and material properties. The improvements introduced in this paper is not limited to determining the fatigue life but can also be used to predict the overall deformation of the lattice structure. The modelling of the struts and the triangle inserts of the lattice structure will be described in Sections 4.2.1 and 4.2.2 respectively.

### 4.2.1. MODELLING THE STRUTS

The struts were modelled using Timoshenko beams loaded at both points with the sign convention shown in Fig. 9 (a). The constitutive equations for the Timoshenko beams are described by Eq. 3 and Eq. 4.

$$M = E I \frac{\partial \phi}{\partial x} \quad \text{Eq. 3}$$

$$V = \kappa A G \left( \frac{\partial w}{\partial x} - \phi \right) \quad \text{Eq. 4}$$

where $M$ is the bending moment, $V$ is the shear force, $I$ is the second moment of area, $A$ is the cross sectional area, $G$ is the shear modulus, $w$ is the transverse displacement, $\phi$ is the angle...
of shear rotation, and $\kappa$ is the Timoshenko shear coefficient, which was set to the value in Table 2 for the lattice structure in this study.

Since only the two ends are loaded, the displacements, $w$, and rotations, $\phi$, can be expressed using Eq. 5 and Eq. 6.

$$\phi(x) = \frac{M_0}{EI}x - \frac{V_0}{2EI}x^2 + \phi_0 \quad \text{Eq. 5}$$

$$w(x) = -\frac{V_0}{6EI}x^3 + \frac{M_0}{2EI}x^2 + \left(\phi_0 + \frac{V_0}{\kappa AG}\right)x + w_0 \quad \text{Eq. 6}$$

where subscript 0 is used to denote values at $x = 0$.

Similarly, the axial forces and displacements for the beams are described by Eq. 7 and Eq. 8.

$$T = EA \frac{\partial u}{\partial x} \quad \text{Eq. 7}$$

$$u(x) = \frac{T}{EA}x + u_0 \quad \text{Eq. 8}$$

where $T$ is the axial force and $u$ is the horizontal displacement.

Since the beams in the lattice are at an angle, the coordinate transformation using Eq. 9 and Eq. 10 was applied for the rotation shown in Fig. 9 (b). The signs were corrected for the forces and moments at $x = 0$ by multiplying the rotation matrix by -1.

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} \quad \text{Eq. 9}$$

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} T \\ V \end{bmatrix} \quad \text{Eq. 10}$$

The strains of the beams can be found from the displacements using Eq. 11 and Eq. 12. Since the largest bending stresses will occur at the top and bottom of the beam, only the strains at these two regions were considered.

$$\varepsilon_{xx} = -\frac{\partial \phi}{\partial x} + \frac{\partial u}{\partial x} \quad \text{Eq. 11}$$

$$\varepsilon_{xz} = \kappa \left(\frac{\partial w}{\partial x} - \phi\right) \quad \text{Eq. 12}$$
The local stresses at the beams were then calculated using the strains by applying Hooke’s law, assuming plane stress as stated in Eq. 13, where \( \sigma_y \) is set to 0 as there are no vertical stresses in the beam.

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1 - \nu/2
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

Eq. 13

4.2.2. MODELLING THE TRIANGLE

There is no known analytical solution for the triangle insert between Points B, C and D, illustrated by the green part in Fig. 6. Since the triangle insert was assumed to be loaded only at the three points, it can be modelled using a stiffness matrix, \( K \), as described by Eq. 14.

\[
\begin{bmatrix}
F_{Bx} \\
F_{By} \\
F_{Cx} \\
F_{Cy} \\
F_{Dx} \\
F_{Dy}
\end{bmatrix} = K \begin{bmatrix}
\delta_{Bx} \\
\delta_{By} \\
\delta_{Cx} \\
\delta_{Cy} \\
\delta_{Dx} \\
\delta_{Dy}
\end{bmatrix}
\]

Eq. 14

where \( K \) is the stiffness matrix, \( F \) is the applied force and \( \delta \) is the displacement.

The stiffness matrix in Eq. 14 was found by treating the entire triangle insert as being made from a number of equilateral constant strain triangular finite elements. An analytical relationship between the components of the stiffness matrix, \( K \), and the material and geometrical parameters was found from the finite element triangle, allowing \( K \) to be determined analytically. The procedure used to determine the components for the triangular portion is explained in the electronic supplementary material.

4.2.3. BOUNDARY CONDITIONS AND COMBINING THE STRUTS AND TRIANGLE

With the Timoshenko beam models for the struts and the stiffness matrix developed for the triangle inserts in the previous sections, the full model for a single unit cell was created by combining the equations relating the displacements of each strut (Eq. 3 to Eq. 8) at every point in the lattice, and enforcing equilibrium using the equations for the forces and moments in the struts and the triangle insert. As the ends of the struts are considered to be significantly thicker than the midspan, the shear rotation at each node was assumed to be zero because the stiffness at the
nodes in terms of rotation is expected to be very large, and the shear rotation is small, making this a good approximation. This results in a linear system of equations that describes the forces and displacements for all the struts in the lattices structure, when appropriate boundary conditions of forces and/or displacements are set for Points A, J and F, as shown in Fig. 6. The equations were combined using commercial algebraic software Maple and solved using Matlab.

In this study, the experiment was conducted on the lattice structure with a finite number of unit cells, as seen in Fig. 7 (a); hence the stress-strain behavior will differ from that of a system with an infinite number of unit cells. For example, as shown in Fig 7 (a), the unit cells on the sides of the central cell are incomplete. Hence, they do not carry significant load and as a result, the forces for the central unit cell studied will be approximately twice the forces in a unit cell of an infinite structure. It is important to point out that even for the infinite case, the unit cell will be constrained by adjacent unit cells and suitable homogenization techniques are required to obtain accurate stress-strain behavior instead of using a single unconstrained unit cell.

Due to the difference in the stress-strain state of the finite and infinite case, more than a single unit cell was used in the analysis. In this study, several individual unit cells were combined to produce the finite structure and the boundary conditions were as shown in Fig. 7 (b) in order to resemble the loading conditions in the experiment. The relevant struts in the unit cells on the sides have been removed to match the experiments, as shown in Fig. 7 (b), the bottom points were fixed, while the top points, P₁ and P₂, were set to have vertical displacements equal to unity. The vertical reaction forces at the top points, \( F_{1y} \) and \( F_{2y} \), are then related to the uniaxial stress applied to the lattice using Eq. 15, and since the material was assumed to be elastic, the displacements, \( d_i \), at the individual points can be scaled using Eq. 16. Eq. 15 is the total force acting on the top portion of the plate divided by the total number of struts (two struts for each unit cell) at the top portion of the specimen in Fig. 7 (b). For an infinite structure, Eq. 15 should be replaced with a similar equation that averages the total force by the total number of struts. However, this approach will not be accurate for finite specimens with more than 2 unit cells at the top row of the specimen as the forces of the outermost cells will be lower than the innermost cells. However, the effects of the outer cells will be less significant when there are a large number of unit cells in the specimen because the portion of the outer cells will be low.


$$F_{1y} = F_{2y} = \frac{\sigma_A \text{Specimen}}{4} \quad \text{Eq. 15}$$

$$d_i = \frac{d_{\text{unity}}}{\sigma_{\text{unity}}} \cdot \sigma \quad \text{Eq. 16}$$

where the subscript $i$ denotes the point number, and the subscript unity denotes the values when top vertical displacements are unity.

The local stresses for each strut were then calculated from the displacements of each point and were input to the multiaxial fatigue model. Although the analyses in this study were conducted using multiple unit cells to ensure that the conditions are similar to that of the experiment, homogenization techniques could be implemented on the single unit cell model derived here to study the fatigue life for structures with a large number of unit cells. This is a topic for future study.

4.3. MULTIAXIAL FATIGUE MODEL

The lattice structure was assumed to fail when any one of the struts in the specimen fails and the struts were assumed to have the same fatigue behavior as the parent material. Therefore, a suitable multiaxial fatigue model for Ti-6Al-V4 is required to predict the overall fatigue life of the lattice as the struts will experience multiaxial loads.

A study by Kallmeyer et al. [9] evaluated a total of 14 multiaxial fatigue models for Ti-6Al-4V alloys and concluded that the Findley and Fatemi-Socie-Kurath (FSK) fatigue models provided good predictions of their biaxial experimental data. Furthermore, the Findley model was found to be more successful at correlating both the uniaxial and biaxial datasets compared to the FSK model [9]. Hence, the Findley multiaxial fatigue model was selected for this study.

The Findley model is described by Eq. 17 [10, 11].

$$DP = \Delta \tau + k \sigma_{\text{Normal}}^{\text{Max}} = f(N) \quad \text{Eq. 17}$$

where $DP$ is the damage parameter, $\Delta \tau$ is the maximum shear stress amplitude, $k$ is the parameter determined from experimental measurements, and $\sigma_{\text{Normal}}^{\text{Max}}$ is the maximum normal stress.
Both the Findley and Fatemi-Socie-Kurath (FSK) multiaxial fatigue models are critical plane models \cite{9,11,12}. Critical plane models postulate that cracks nucleate and grow on critical planes based on phenomenological observations of fatigue crack development \cite{9,11,12}. Critical plane models hypothesize that the normal stresses will open fatigue cracks and reduce the friction between the crack surfaces, while the shear stresses cause dislocations to move along slip lines resulting in crack growth and nucleation \cite{9}. In general, the critical planes are the planes that have the maximum shear stress or strain amplitude \cite{11,12}. For the Findley model (Eq. 17) used in this study, the main mechanism for fatigue damage is the maximum stress amplitude, $\Delta \tau$, while the secondary mechanism, which is the maximum normal stress on the critical plane, $\sigma_{\text{Normal Max}}$, is accounted for in the model by multiplying it with an adjustable material parameter, $k$, \cite{9} that characterizes the contribution of the maximum normal stress.

In order to determine $\Delta \tau$ and $\sigma_{\text{Normal Max}}$ required to find the Findley damage parameter in Eq. 17, the stresses in the struts, bar material, and plate material throughout the entire cycle were first calculated. Stress transformation was then performed by rotating the plane across an entire range of angles to find the critical plane, which is the plane that will give the largest $\Delta \tau$ in a cycle. According to \cite{13}, there are a number of definitions for $\sigma_{\text{Normal Max}}$, for example, $\sigma_{\text{Normal Max}}$ can be defined as the stress normal to the critical plane when the shear stress experiences a reversal \cite{13}. In this study, the $\sigma_{\text{Normal Max}}$ is taken to be the maximum magnitude of the stress normal to the critical plane within a cycle, which is also a common definition for $\sigma_{\text{Normal Max}}$ \cite{13}. The definition of $\sigma_{\text{Normal Max}}$ will lead to the largest value of $\sigma_{\text{Normal Max}}$ being used and is considered most damaging, leading to conservative predictions of the fatigue life.

Several functions for the fatigue life calculations, $f(N)$ have been proposed in the literature, for example in Fatemi and Socie \cite{10}, Kallmeyer et al. \cite{9} and Sosie \cite{14}, with Eq. 18 being a common choice. However, in this study, Eq. 19 was used instead because non-linear curve fitting is required for Eq. 18, which was sensitive to the initial guess of the parameters; good curve fits can be obtained using Eq. 19.

$$f(N) = a_1 N^{b_1} + a_2 N^{b_2}$$ \hspace{1cm} \text{Eq. 18}

$$f(N) = a_1 N^{b_1} + c$$ \hspace{1cm} \text{Eq. 19}
where \( a_1, a_2, b_1, b_2 \) and \( c \) are parameters to be determined from experimental results, and \( N \) is the number of cycles until failure.

The parameter, \( k \), is a key parameter in the Findley fatigue model in Eq. 17 that needs to be calibrated. As seen in Eq. 17, the damage parameter is a function of \( \Delta \tau \) and \( \sigma_{\text{Normal}}^{\text{Max}} \), which suggests that the calibration requires experimental data from fatigue tests with at least two different stress states. The calibration of \( k \) in the Findley model usually requires fully reversed torsional and uniaxial fatigue tests [15] as these tests will give independent values of \( \Delta \tau \) and \( \sigma_{\text{Normal}}^{\text{Max}} \) respectively. Unfortunately, torsional fatigue tests were not conducted in this study because suitable torsional test specimens could not be easily manufactured from the Ti-6Al-4V plate material that was used in the experiments. The Findley model takes into account the mean stresses in the fatigue life predictions similar to the FSK model [9, 11, 15] because the maximum normal stresses, \( \sigma_{\text{Normal}}^{\text{Max}} \), will be different when mean stresses are present. Therefore, the value of \( k \) was calibrated in this study by using the uniaxial fatigue data with different stress ratios, \( R \).

This is done by first selecting a nominal value for \( k \) and then using the values of \( \Delta \tau \) and \( \sigma_{\text{Normal}}^{\text{Max}} \) calculated from the stresses in the uniaxial test (for the plate and bar material) or the local stresses using the lattice models (for the lattice materials). Since the Findley fatigue model takes into account the mean stresses, the damage parameter vs life curve data should fall onto a single curve regardless of the stress ratio, \( R \). Therefore, the experimental values for the plate, bar, and lattice materials were curve fitted with Eq. 19 and the relative errors, defined in Eq. 20 for the selected values of \( k \), were calculated.

\[
(Relative \ error)^2 = \sum_i \left( \frac{DP_{\text{Fit},i} - DP_{\text{Exp},i}}{DP_{\text{Exp},i}} \right)^2 \tag{Eq. 20}
\]

where \( DP \) refers to the Findley damage parameter defined in Eq. 17 while the subscripts Fit and Exp refers to the fitted using Eq. 19 and the experimental values respectively.

The process was then repeated for different values of \( k \) ranging from 0.1 to 3.0. The value of \( k \) was set to the value that gives the lowest square of the relative error. This process is analogous to performing a least square regression analysis.

**4.4 COMBINING THE LATTICE AND MULTIAXIAL FATIGUE MODELS**
As described previously, the lattice model will give the displacements and the strains for each beam at a given value of macroscopic stress, \( \sigma \). The local stresses for each beam were then calculated using the approach described in Section 4.2 and the Findley parameter was then calculated using the steps described in Section 4.3. The fatigue life of the lattice structure was set to be equal to that of the beam with the largest damage parameter, \( DP \), at a given unit of macroscopic stress, \( \sigma \). For the lattice structure considered in this study, the upper struts of the lattice structure at the center, indicated in Fig. 7 (b), were found to experience the largest stresses and the largest damage parameter. Therefore, these struts will be the first to fail. This result is consistent with the experimental and finite element results presented in Section 3. Therefore, the damage parameter presented for the lattice structure herein will be that of the central strut.

The ratio of the stress amplitude, \( \sigma_a \), to the strain amplitude, \( \Delta \gamma \), and maximum normal stress, \( \sigma_{\text{Max}}^{\text{Normal}} \), were determined and these ratios, denoted as \( r_1 \) and \( r_2 \) respectively, were used to convert the fatigue life curve for a given loading condition in terms of the Findley damage parameter to the stress amplitude \( \sigma_a \) by solving Eq. 21 for \( \sigma_a \).

\[
DP = \left( \frac{r_1}{2} + k r_2 \right) \sigma_a = r \sigma_a \quad \text{Eq. 21}
\]

\[
\sigma_a = \sigma_{\text{Max}} - \sigma_{\text{Min}} \quad \text{Eq. 22}
\]

\[
r = \left( \frac{r_1}{2} + k r_2 \right) \quad \text{Eq. 23}
\]

5. RESULTS AND DISCUSSION OF ANALYTICAL MODELLING

5.1. CALIBRATING PARAMETER \( k \) FOR MULTIAXIAL FATIGUE MODEL

Fig. 10 shows the values of the relative errors, defined by Eq. 20 that was used to calibrate the parameter \( k \) for the multiaxial model described in Section 4.3. Based on Fig. 10, the value of \( k \) for the mill- and beta-annealed round bar was set to be 0.6 and 0.8 respectively, while the values of \( k \) for the mill- and beta-annealed plate material were set to be 0.4 and 1.6 respectively. These values of \( k \) gives the lowest error for the curve fits. Since the geometry of the plate material...
resembles the lattice material more closely compared to the bar material, the fatigue life behavior of the lattice material was calculated based on the fatigue behavior of the plate material.

5.2. FATIGUE LIFE PREDICTIONS

Figs. 11 (a) and (b) show the Findley damage parameter versus N curve for all the materials that have been tested in this study. The Findley fatigue parameter for the lattice structures in Figs. 11 (a) and (b) were calculated using the lattice model and the stress amplitude from the experimental data to give an indication of the accuracy of the model. As seen in the figures, the experimental data with different stress ratios, R, in Figs. 11 (a) and (b) all fall onto a single curve for both the mill- and beta-annealed material, suggesting that the calibrated values of k in the Findley multiaxial fatigue model are accurate.

As seen in Figs. 11 (a) and (b), there are some discrepancies between the round bar and plate material data points suggesting that there are some differences between the fatigue behaviors of these materials. Furthermore, the lattice material data points in Figs. 11 (a) and (b) are closer to the plate material data points indicating that lattice material behaves more closely to the plate material. This result is expected and indicates that the use of the plate material fatigue data for the lattice calculations was correct.

Additionally, since the Findley fatigue parameter for the lattice structure was calculated based on the stress state at the individual struts and the material behavior of the lattice structure was assumed to be identical to the plate material, the Findley fatigue parameter for the lattice structures should also fall onto the same curve as that of their parent material. However, this was not observed in the results for Figs. 11 (a) and (b), in which the fatigue parameters for the lattice structures are in general slightly higher than both the round bar and plate for the beta-annealed material, and higher than the plate material for the mill-annealed material. This means that the fatigue stresses of the lattice structures will be slightly under-predicted if the fatigue behavior of the plate and the lattice model described in this paper is used to determine the fatigue behavior of the lattice structures.

Suspected causes of errors, such as the presence of stress concentrations due to the sharp corners in the lattice structures, are unlikely the cause of the under-prediction of the fatigue stress (and fatigue life) as these will decrease the fatigue life at a given value of stress. Hence, the source of
the error is likely to be the assumption that the struts are uniform thickness beams connected at the nodes. Although there some errors, the predictions made using the model are fairly accurate as will be further elaborated when inspecting the S-N curve. Lastly, the results also suggest that the reduced fatigue life of the lattice structure is a result of higher local stresses at the struts and the multiaxial stress state.

5.3. S-N CURVES

The predictions in Figs. 11 (a) and (b) were converted into S-N curves for the lattice structures using the procedure described in Section 4.4. The experimental and predicted S-N curves for the mill- and beta-annealed lattices are plotted in Figs. 12 (a) and (b), respectively. The predictions made using the curve fit of the Findley fatigue model on the lattice structure itself, described by the red and blue lines in Figs. 12 (a) and (b), are accurate. This result is expected as the predictions were made based on the lattice itself.

Since the objective of developing the fatigue model for the lattice was to predict the fatigue life of the lattice from the parent material, it is essential to evaluate the predictions made using the data for the plate material. As seen in Figs. 12 (a) and (b), the predictions made from the analytical model developed here are fairly accurate with the experimental data being close to the predictions, especially when there are a number of sources of error, such as the presence of stress concentrations in the lattice, simplification of the geometry in the analytical model, possible differences in the plate and lattice material, possible scatter in the experimental data, and calibration errors of $k$. These sources of error may affect both the stress state of the struts and also the S-N behavior of the base material of the strut, which may also explain a slightly different trend in the S-N behavior of the lattice material compared to the plate material. However, within the tested region, the accuracy of the analytical model is adequate and since the analytical model does not require large computing resources, this makes it a useful tool for the analysis of fatigue in such lattice structures. Furthermore, the framework used to develop this model can be applied to lattice structures with different designs.

6. GENERAL DISCUSSION
The consequence of machining a lattice geometry in a solid plate of Ti-6Al-4V was to seriously impair the fatigue life of the plate for the same remote applied (macroscopic) stress. Although this is an unsurprising result based on a previous investigation of the strength of such structures [3], it is nevertheless something that must be taken into consideration when contemplating the application of such lattices as structural components.

The study has demonstrated that analytical tools can be developed to predict the fatigue lives of these types of lattice structures with a reasonable degree of accuracy. Therefore, it is possible to predict the fatigue behavior for any material/lattice geometry combination, and hence this is a powerful analytical tool to assist design engineers when using these lattice structures in actual aerospace applications.

As stated in the introduction of this paper, lattices intended for applications where low thermal expansion is desirable would be fabricated from two materials, with differing CTEs. If modeling the lattice as a single material is deemed insufficiently accurate, the analytical model developed here can be easily modified to account for two materials. This is done simply by changing the Young’s modulus and Poisson’s ratio used in the triangular portion (for the higher CTE inserts) to that of the relevant material; since the struts will fail prior to the triangular inserts, fatigue data for the higher CTE inserts are not required. Although, in this study, the analytical model was set up to correspond to the loading conditions with a finite number of unit cells, homogenization methods can be implemented to the single unit cell and then the same method of relating the fatigue life to the local stresses described in this study could be employed to study the fatigue behavior of infinite lattice structures.

7. CONCLUDING REMARKS

The mechanical fatigue behavior of a low thermal expansion lattice structure was investigated in this study and several important findings have been made, which are summarized as follows.

- Fatigue experiments have been conducted for Ti-6Al-4V round bars, plates and lattice structures. Based on the experimental results, the lattice structures have been found to
have fatigue strengths that are approximately two orders of magnitude lower than the parent material.

- An analytical model to predict the fatigue behavior of the lattice structures using experimental fatigue lives of the parent material was developed and the predictions were compared with experimental data. Firstly, a model, which is an improvement to the model established by Berger et al. [3], was used to predict the local stresses in the lattice structures. The local stresses were then implemented in a multiaxial fatigue model to determine the fatigue life of the lattice. The predictions made by the analytical model are fairly accurate considering the number of simplifications made to the model.

- The fatigue behavior of the lattice structure can be described using a Findley fatigue parameter versus life curve, which follows a power law relationship. The results suggest that the reduced fatigue strength of the lattice structures are a result of the localized multiaxial stress state, which can be accurately modelled using the Findley multiaxial model.

- The successful development of analytical modelling tools to predict lattice fatigue stresses and S-N curve behavior accurately will hopefully be of great benefit when designing these structures for practical applications in the aerospace or communications industries.

ACKNOWLEDGMENTS

Appreciation is extended to Dr. Satoshi Emura (NIMS) for assistance with the metallography and microscopy work presented in the paper. The experimental part of the investigation was financially supported by the National Institute for Materials Science, Tsukuba, Japan (Project Number CG002).

REFERENCES


FIGURE CAPTIONS

Figure 1. Schematic illustration of a bi-metallic, low thermal expansion lattice structure (designed by the University of California Santa Barbara).

Figure 2. Lattice fatigue specimen fabricated by water jet machining from Ti-6Al-4V plate.

Figure 3. Backscattered electron images showing the microstructures of (a) mill annealed and (b) beta annealed Ti-6Al-4V plate.

Figure 4. S-N curves for (a) mill-annealed and beta annealed Ti-6Al-4V material and (b) mill-annealed and beta annealed lattice structures.

Figure 5. (a) Fractured lattice specimen and (b) finite element simulation, showing failure location.

Figure 6. A unit cell of the lattice structure.

Figure 7. Loading conditions of test specimen.

Figure 8. Dimensions from specimen that are used in model.
Figure 9. Timoshenko beam (a) Sign convention (b) rotation.

Figure 10. Relative error vs $k$ curve.

Figure 11. Findley parameter vs N curves for (a) Mill annealed materials (b) Beta annealed materials.

Figure 12. S-N curves for (a) Mill annealed lattice material (b) Beta annealed lattice material

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