Characterizations of Input-to-State Stability for systems with multiple invariant sets

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***Abstract*—We generalize the theory of Input-to-State Stability (ISS) and of its characterizations by means of Lyapunov dissipation inequalities to the study of systems admitting invariant sets, which are not necessarily stable in the sense of Lyapunov but admit a suitable hierarchical decomposition. It is the latter which allows to greatly extend the class of systems to which ISS theory can be applied, allowing in a unified treatement to deal with oscillators in Euclidean coordinates, almost globally asymptotically stable systems on manifolds, systems with multiple equilibria in $\mathbb{R}^n$ just to name a few.**

I. Introduction and motivations

A well established technique for the study of stability and robustness of nonlinear systems, which are described by a set of differential equations globally defined in Euclidean space, is the Input-to-State Stability approach, (see [17] and references therein). The classical definition potentially allows to formulate and characterize stability properties with respect to arbitrary compact invariant sets (and not simply equilibria). The implicit requirement that these sets should be simultaneously Lyapunov stable and globally attractive, however, makes the basic theory not applicable for a global analysis of many dynamical behaviours of interest, such as multistability or periodic oscillations, just to name a few, and only local analysis remains possible [6]. In fact, it is well-known that such systems, when defined in Euclidean space, normally admit invariant sets (such as additional equilibria) that fail to be Lyapunov stable.

As an attempt to overcome such limitations for the case of nonlinear autonomous systems, the almost global stability property was introduced, [13], and short afterwards, almost Input-to-State Stability, [2], for systems admitting exogenous disturbances. In particular, for the case of almost ISS, sufficient criteria based on a combination of dual Lyapunov techniques [13] and classical dissipation inequalities were proposed (see [18] for an application of such tools to stability analysis of rotational motions). The key idea of the dual approach is to replace Lyapunov functions by suitable density functions and to impose a monotonicity condition on the way these are propagated by the flow. While converse dual Lyapunov results have appeared in the literature short afterwards, [14], some difficulties in the explicit construction of density functions for systems involving unstable equilibria have also emerged [1].

More recently, in [3], the need for conditions involving density functions was removed in the case of systems with exponentially unstable equilibria thanks to a careful application of integral manifold theory. While geometric tools involving manifolds and dimensionality arguments provide a very fine structure to the stability properties, it is also clear that they depart quite fundamentally in spirit from the standard ISS paradigm which is essentially an analytical theory.

In this paper we make the point that the most natural way of relaxing Input-to-State Stability for systems with disconnected invariant sets is in fact to relax the Lyapunov stability requirement [7] (rather than the global nature of the attractivity property). This, under relatively mild additional assumptions, in order to avoid classical counter-examples of globally attractive systems not admitting smooth Lyapunov functions, [5], will allow the characterization of the ISS property in terms of classical Lyapunov-like inequalities, thus generalizing the standard ISS theory as well as related literature on time-invariant dynamical systems on compact spaces, [12].

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II. Basic Definitions

Let M be an n-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold without boundary and D be a closed subset of $\mathbb{R}^m$ containing the origin. Consider the map:

$$f(x,d) : M \times D \to T_x M$$

which we assume to be of class $\mathcal{C}^1$ ($T_x M$ denotes as customary the tangent space of M at x). We deal with nonlinear systems of the following form:

$$\dot{x}(t) = f(x(t), d(t)) \tag{1}$$

with state x taking value in M. We denote by $X(t, x; d(\cdot))$ the uniquely defined solution of (1) at time t fulfilling $x(0) = x$ (under the input $d(.)$ which is a locally essentially bounded and measurable signal). Solutions may fail to be defined for all $t \in \mathbb{R}$, however, for the remainder of the Section we assume (without loss of generality) that solutions of the unperturbed system are globally defined backwards and forward in time.

The symbol $\delta(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M. We are now equipped to define a notion of convergent dynamics for systems as in (1). The unperturbed system is defined by the following set of equations:

$$\dot{x}(t) = f(x(t), 0). \tag{2}$$

We say that $S \subset M$ is invariant for the unperturbed system (2) if, for all $x \in S$, $X(t, x; 0) \in S$ for all $t \in \mathbb{R}$. For a set $S \subset M$ define $|\cdot|_S$ as

$$|x|_S = \inf_{a \in S} \delta(x, a).$$

For a point $\mathbf{x}_0 \in M$ selected as "the origin" on M, denote $|x| = |x|_{\{\mathbf{x}_0\}}$. For a measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define its infinity norm

$$\|d\|_\infty = \operatorname*{ess\,sup}_{t \geq 0} |d(t)|.$$

A. Decompositions

Let $\Lambda \subset M$ be a compact invariant set for (2). In order to highlight the structure of the flow of the unperturbed system it is useful to decompose Λ and explicitly denote existence of solutions travelling between different components of its decomposition, as carried out in the following definitions:

Definition 1: An open decomposition for Λ is a finite, disjoint family of open sets, $W_1, W_2, \ldots, W_k \subset M$ such that $\Lambda \subset \bigcup_{i=1}^k W_i$.

An open decomposition can be associated with any invariant set Λ. The roughest decomposition is M itself, while the finest is made up by open neighborhoods of each connected component of Λ.

However, what qualifies a decomposition for its treatment by means of Lyapunov-like analytical tools is the absence of cycles, as detailed in the definitions that follow. For an open set $W \subset M$, define:

$$W^s = \{x_0 \in M : \exists\, t \geq 0 : X(t, x_0; 0) \in W\}.$$

Definition 2: If $W_1, \ldots, W_k$ is an open decomposition of Λ, then

1) An r-cycle, for $r \geq 2$ is an ordered r-tuple of distinct indices $i_1, \ldots, i_r$ such that

$$W_{i_j} \cap W^s_{i_{j+1}} \neq \emptyset \quad (j = 1, \ldots, r-1)$$
$$W_{i_r} \cap W^s_{i_1} \neq \emptyset$$

2) A 1-cycle is an index i such that

$$\exists x \notin W_i, \exists s, t > 0 : X(-s, x; 0) \in W_i \text{ and } X(t, x; 0) \in W_i.$$

Informally an r-cycle is therefore a collection of r disjoint open sets that are reached from one another in a loop by a suitable concatenation of systems solutions. The above definitions are taken from [12] where authors also give the following two definitions and the related proposition.

Definition 3: Let $\Lambda \subset M$ be a compact and invariant set for (2). A decomposition of Λ is a finite, disjoint family of compact invariant sets $\Lambda_1, \ldots, \Lambda_k$ such that

$$\Lambda = \bigcup_{i=1}^{k} \Lambda_i.$$

For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$\mathfrak{A}(\Lambda) = \{x \in M : |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to +\infty\},$$
$$\mathfrak{R}(\Lambda) = \{x \in M : |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to -\infty\}.$$

Define a relation on $\mathcal{W} \subset M$ and $\mathcal{D} \subset M$ by $\mathcal{W} \prec \mathcal{D}$ if $\mathfrak{A}(\mathcal{W}) \cap \mathfrak{R}(\mathcal{D}) \neq \emptyset$ (this relation implies that there is a solution connecting set $\mathcal{D}$ with set $\mathcal{W}$).

Definition 4: Let $\Lambda_1, \ldots, \Lambda_k$ be a decomposition of Λ, then

1) An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \ldots, i_r$ such that $\Lambda_{i_1} \prec \ldots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
2) A 1-cycle is an index i such that $[\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i)] - \Lambda_i \neq \emptyset$.
3) A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

As we can conclude from Definition 4, existence of an r-cycle with $r \geq 2$ is equivalent to existence of a heteroclinic cycle for (2) [9]. And existence of a 1-cycle implies existence of a homoclinic orbit for (2) [9]. In general, existence of cycles has to be checked on separatrix configurations [11].

Proposition 1: Let $\mathcal{W}$ be a compact invariant set containing all α and ω limit sets of (2). Then a necessary and sufficient condition for $\mathcal{W}$ to be the maximal invariant set of an open decomposition with no cycles is that $\mathcal{W}$ have a decomposition with no cycles.

According to this result, for any compact invariant set $\mathcal{W}$ containing all α and ω limit sets, the two notions of decomposition without cycles are equivalent. Notice that in most examples one might be able to choose $\mathcal{W} = \mathcal{A} \cup \mathcal{R} \cup \mathcal{H}$, where the set $\mathcal{A}$ is composed by locally asymptotically stable invariant sets, the set $\mathcal{R}$ contains locally anti-stable invariant sets and $\mathcal{H} = \mathcal{H}^+ \cap \mathcal{H}^-$ is an hyperbolic invariant set ($\mathcal{H}^+$ and $\mathcal{H}^-$ constitute stable and unstable invariant submanifolds for $\mathcal{H}$), some of these sets may be empty. Hyperbolicity is however not a requirement for the subsequent discussion. This makes a large class of systems amenable to be analysed with the techniques described below. Our main assumption on $\mathcal{W}$ which will be used throughout is the following:

Assumption 1: The compact invariant set $\mathcal{W}$ admits a finite decomposition without cycles[1], $\mathcal{W} = \bigcup_{i=1}^k \mathcal{W}_i$ for some non-empty disjoint compact sets $\mathcal{W}_i$, which form a filtration ordering of $\mathcal{W}$, as detailed in definitions 3 and 4.

An open decomposition of $\mathcal{W}$ without cycles will be used in the proof of the main results below, due to Proposition 1 the existence of such an open decomposition follows from Assumption 1. The formulation of Assumption 1 is based on a finite decomposition since verification of this condition is more simple in examples, see Section IV.

B. Robust stability notions for decomposable $\mathcal{W}$

Our object of study is the following robustness notion for the system (1):

Definition 5: The system (1) has the practical asymptotic gain (pAG) property if there exist $\eta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t \to +\infty} |X(t, x; d)|_\mathcal{W} \leq \eta(\|d\|_\infty) + q. \tag{3}$$

If $q = 0$, then we say that the asymptotic gain (AG) property holds.

Definition 6: The system (1) has the limit property (LIM) with respect to $\mathcal{W}$ if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t \geq 0} |X(t, x; d)|_\mathcal{W} \leq \mu(\|d\|_\infty). \tag{4}$$

Definition 7: We say that the system (1) has the practical global stability (pGS) property with respect to $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the following holds for all $t \geq 0$:

$$|X(t, x; d)|_\mathcal{W} \leq q + \beta(\max\{|x|_\mathcal{W}, \|d\|_\infty\}). \tag{5}$$

Note that (5) is equivalent to

$$|X(t, x; d)|_\mathcal{W} \leq \tilde{\beta}(\max\{|x|_\mathcal{W} + c, \|d\|_\infty\})$$

for some $\tilde{\beta} \in \mathcal{K}_\infty$ and $c \geq 0$.

We would like to characterize (3) in terms of Lyapunov functions. The following notion is appropriate:

Definition 8: A $\mathcal{C}^1$ function $V : M \to \mathbb{R}$ is a practical ISS-Lyapunov function for (1) if there exists $\mathcal{K}_\infty$ functions $\alpha_1, [\alpha_2], \alpha$ and γ, and $q \geq 0$ [and $c \geq 0$] such that:

$$\alpha_1(|x|_\mathcal{W}) \leq V(x) \leq [\alpha_2(|x|_\mathcal{W} + c)],$$

and the following dissipation inequality holds for all $(x, d) \in M \times D$:

$$DV(x)f(x, d) \leq -\alpha(|x|_\mathcal{W}) + \gamma(|d|) + q. \tag{6}$$

If (6) holds for $q = 0$, then V is said to be an ISS-Lyapunov function.

If, in addition, the set $\bigcup_{x \in \mathcal{W}_i} \{V(x)\}$ (denoted for short $V(\mathcal{W}_i)$) is a singleton for all $i \in \{1, 2, \ldots, k\}$, then V is said to be an ISS-Lyapunov function constant on invariant sets.

Notice that α_2 and c are in brackets as their existence follows (without any additional assumptions) by standard continuity arguments.

Under Assumption 1, whenever $\mathcal{W}_i$ are recurrent invariant sets of the unperturbed system, any Lyapunov function which is non-increasing along solutions of (2) also takes constant values on any $\mathcal{W}_i$, $1 \leq i \leq k$ from the decomposition of $\mathcal{W}$.

In addition, if $q = 0$, then existence of an ISS Lyapunov function as in Definition 8 (*viz.* strictly decreasing outside $\mathcal{W}$) is possible for a set $\mathcal{W}$ admitting a decomposition without cycles only.

[1]This rules out cycles of any length.

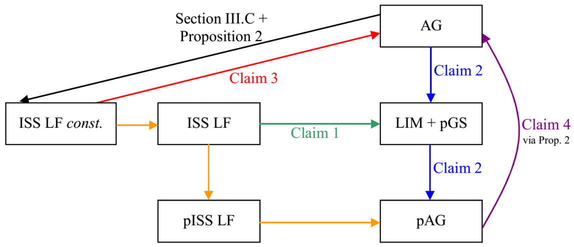

Fig. 1. The road-map of the main result proof

III. Main result and proof's discussion

We are now ready to state our main result:

Theorem 1: Consider a nonlinear system as in (1) and let $\mathcal{W}$ be a compact invariant set containing all α and ω limit sets of (2) as in Assumption 1. Then the following facts are equivalent:

1) The system enjoys the AG property;
2) The system admits an ISS Lyapunov function;
3) The system admits an ISS Lyapunov function constant on invariant sets;
4) The system admits a practical ISS Lyapunov function;
5) The system enjoys the pAG property;
6) The system enjoys the LIM property and the pGS.

The system as in (1) that satisfies these properties will be called ISS with respect to the set $\mathcal{W}$.

Remark 1: The paper [12] (Proposition 6) proves 1. $\Rightarrow$ 2. for the case of systems without disturbances and evolving on compact manifolds. Notice that in this case, extension of the result to systems with inputs is actually trivial and follows by a standard continuity argument:

$$DV(x)f(x,d) \leq DV(x)f(x,0) + |DV(x)[f(x,d)-f(x,0)]| \leq -\alpha(|x|_{\mathcal{W}}) + \gamma(|d|)$$

where

$$\gamma(r) := \max_{|d|\leq r, x\in M} |DV(x)[f(x,d)-f(x,0)]|.$$

Remark 2: Traditionally ISS is formulated for systems with a single equilibrium. When such systems fulfill the existence of a decomposition without cycles (as required in our set-up) and asymptotic gain, then, they also happen to be stable (in fact globally) and therefore our notions are genuinely equivalent to ISS in the classical set-up.

The equivalences of Theorem 1 appear to be non-trivial new results even in the case of systems evolving in $\mathbb{R}^n$. The only straightforward relations given without a proof are:
ISS Lyapunov function $\Rightarrow$ practical ISS Lyapunov function $\Rightarrow$ pAG.

A diagram for the proof of other implications is given in Fig. 1.

A. Direct Lyapunov argument

To illustrate the consequences of existence of an ISS Lyapunov function we need the following intermediate results.

Claim 1: If the system (1) admits an ISS Lyapunov function, then it admits LIM and pGS properties.

Proof. First, the LIM property is satisfied for $\mu(s) = \alpha^{-1}\circ 2\gamma(s)$; if this is not the case then for some solution $X(t,x;d)$ and all $t \geqslant 0$ we have $\dot{V}(X(t,x;d)) \leq -\alpha(|X(t,x;d)|_{\mathcal{W}})/2$. That ensures the asymptotic convergence of $V(X(t,x;d))$ to a limit and in turn convergence of $\alpha(|X(t,x;d)|_{\mathcal{W}})$ to 0, given compactness of sublevel sets of V. The latter implies asymptotic convergence to zero of $|X(t,x;d)|_{\mathcal{W}}$ which is a contradiction. This concludes the proof of the LIM property.

As for the practical GS property notice that the Lyapunov dissipation inequality also implies the following stability notion for all $x\in M$:

$$|X(t,x;d)|_{\mathcal{W}} \leq \kappa(|x|_{\mathcal{W}},t) + \sigma(\|d\|_\infty) + q, \quad \forall t\geq 0, \tag{7}$$

where $\kappa\in\mathcal{KL}$, $\sigma\in\mathcal{K}$ and $q>0$. Indeed, $\alpha(|x|_{\mathcal{W}}/2 + c/2) \leq \alpha(|x|_{\mathcal{W}}) + \alpha(c)$ and

$$DV(x)f(x,d) \leq -\alpha(|x|_{\mathcal{W}}/2 + c/2) + \alpha(c) + \gamma(|d|) \leq -\alpha\circ\frac{1}{2}\alpha_2^{-1}(V(x)) + \alpha(c) + \gamma(|d|)$$

that leads to the time-domain estimate (7) or pGS property.

Claim 2: For the system (1) the following implications hold:
AG $\Rightarrow$ (LIM and pGS) $\Rightarrow$ pAG.

Proof. The LIM property follows by its definition and fulfillment of the pGS property is proven in Lemma 1 (see Appendix). Notice that from equation (5) and LIM property we have

$$\limsup_{t\to+\infty} |X(t,x;d)|_{\mathcal{W}} \leq q + \sigma(\|d\|_\infty), \tag{8}$$

with $\sigma(s) = \beta(\max\{\mu(s),s\})$.

Claim 3: If the system admits an ISS Lyapunov function constant on invariant sets then it enjoys the AG property.

Proof. From claims 1 and 2 the system admits the pAG property (8) in this case. This, for any $\varepsilon>0$, proves AG with respect to input signals $d(\cdot)$ with $\|d\|_\infty \geq \varepsilon$ and gain $\bar{\eta}_\varepsilon(s) = \sigma(s) + qs/\varepsilon$ as follows considering the inequality below:

$$q + \sigma(\|d\|_\infty) \leq q\|d\|_\infty/\varepsilon + \sigma(\|d\|_\infty) = \bar{\eta}_\varepsilon(\|d\|_\infty).$$

Hence, we only need to show that there exists a sufficiently small $\bar{\varepsilon}>0$ such that AG holds for all input signals d with $\|d\|_\infty \leq \bar{\varepsilon}$. Let us focus, without loss of generality, on inputs with $\|d\|_\infty \leq 1$. Asymptotically $X(t,x;d)$ enters a compact set, $\mathcal{X} := \{x : |x|_{\mathcal{W}} \leq q+\sigma(1)+1\}$. Let F be:

$$F := \max_{x\in\mathcal{X}, |d|\leq 1} |f(x,d)|_x < +\infty \tag{9}$$

where $|\cdot|_x$ denotes the norm on T_xM which induces the Riemannian metric δ. Notice that F is finite by continuity of f and compactness of $\mathcal{X}\times\{d : |d|\leq 1\}$. Since $X(t,x;d)$ eventually enters $\mathcal{X}$ whenever $\|d\|_\infty \leq 1$, it holds, $|f(X(t,x;d),d)|_{X(t)} \leq F$ for all sufficiently large $t\geq 0$.

Consider next the minimum distance between the elements of the decomposition:

$$\bar{D} := \min_{\substack{1\leq i\neq j\leq k,\\ x_a\in\mathcal{W}_i,\ x_b\in\mathcal{W}_j}} \delta(x_a,x_b) > 0 \tag{10}$$

Notice that the minimum exists and it is strictly positive by finiteness of the decomposition and compactness of the $\mathcal{W}_i$s. For all $\Delta>0$ such that $\alpha^{-1}(2\gamma(\Delta)) = \mu(\Delta) \leq \bar{D}/4$ it holds that the sets $N_i(\Delta)$ defined below:

$$N_i(\Delta) := \{x\in M : |x|_{\mathcal{W}_i} \leq \alpha^{-1}(2\gamma(\Delta))\}$$

are disjoint and at least at distance $\bar{D}/2$ from each other. By LIM property for $\|d\|_\infty \leq \Delta < 1$, for all $x\in M$ there is a time instant $t'\geq 0$ such that $x(t')\in N_i(\Delta)$ for some $1\leq i\leq k$. Next, for all sufficiently large times, solutions take at least $\bar{D}/2F$ in order to travel between two of the $N_i(\Delta)$ sets. Notice moreover that for all $x\in\mathcal{X}\backslash\bigcup_{i=1}^k N_i(\Delta)$ and all d with $|d|\leq\Delta\leq 1$ it holds:

$$DV(x)f(x,d) \leq -\alpha(|x|_{\mathcal{W}}) + \gamma(|d|) \leq -\gamma(\Delta),$$

where the last inequality follows considering that for all $x \in M$ there exists i_x so that $|x|_{\mathcal{W}} = |x|_{\mathcal{W}_{i_x}}$. Hence the Lyapunov function $V(x)$ along any solution that travels between two distinct sets $N_i(\Delta)$ and $N_j(\Delta)$ decreases at least by $\bar{D}\gamma(\Delta)/2F$ between the time that it last leaves $N_i(\Delta)$ and the one that it first enters $N_j(\Delta)$.

Consider next the function $G : [0, +\infty) \to [0, +\infty)$ defined below:

$$G(r) := \max_{i \in 1,\dots,k} \left\{ \max_{x \in N_i(r)} V(x) - \min_{x \in N_i(r)} V(x) \right\} + r. \tag{11}$$

Notice that G is continuous and increasing, moreover $G(0) = 0$ as $V(\mathcal{W}_i)$ is a singleton for all $i \in \{1, 2, \dots, k\}$. This function is a bound from above to the difference in values of V between points within the same neighborhood of radius r of the $\mathcal{W}_i$s. For any $\Delta > 0$ one may pick $\Delta_1 \leq \gamma^{-1}\left(\frac{\alpha}{2}\left(\frac{\alpha^{-1}(2\gamma(\Delta))}{2}\right)\right) \leq \Delta$ positive and sufficiently small so as to fulfill $G(\Delta_1) \leq \bar{D}\gamma(\Delta)/4F$ for instance by letting:

$$\Delta_1 := \min\left\{ G^{-1}\left(\frac{\bar{D}\gamma(\Delta)}{4F}\right), \gamma^{-1}\left(\frac{\alpha}{2}\left(\frac{\alpha^{-1}(2\gamma(\Delta))}{2}\right)\right)\right\} := \tilde{\gamma}(\Delta).$$

The rationale for this peculiar expression will be clearer after the following arguments are developed.

We claim that with such a choice, all solutions corresponding to input signals with $\|d\|_\infty \leq \Delta_1$ cannot visit twice the same neighborhood $N_i(\Delta_1)$ if in between they have visited another set $N_j(\Delta_1)$. The proof of this fact is sketched below.

Notice that for inputs with infinity norm less than Δ_1, $V(x)$ decreases outside $\bigcup_i N_i(\Delta_1)$, moreover as $N_i(\Delta_1) \subset N_i(\Delta)$ for all is, the Lyapunov function decreases at least by $\bar{D}\gamma(\Delta)/2F$ when traveling between two distinct sets $N_i(\Delta_1)$ and $N_j(\Delta_1)$. On the other hand, while inside such sets, the Lyapunov function can at most grow by $G(\Delta_1) \leq \bar{D}\gamma(\Delta)/4F$. Overall, if a solution could visit in a cycle a number of distinct sets $N_{i_1}(\Delta_1), N_{i_2}(\Delta_1), \dots, N_{i_1}(\Delta_1)$ this would lead to a net decrease of V which is larger than the maximum gap $G(\Delta_1)$ allowed between values of $V(x)$ within the set $N_{i_1}(\Delta_1)$. This is a contradiction and therefore we may conclude that all solutions corresponding to input signals d with $\|d\|_\infty \leq \Delta_1$ eventually keep visiting a single set $N_i(\Delta_1)$.
Notice that:

$$\Delta_1 \leq \gamma^{-1}\left(\frac{\alpha}{2}\left(\frac{\alpha^{-1}(2\gamma(\Delta))}{2}\right)\right) \Rightarrow \frac{\alpha^{-1}(2\gamma(\Delta))}{2} \geq \alpha^{-1}(2\gamma(\Delta_1)).$$

Consider next the function $\rho(r) := \alpha^{-1}(2\gamma(r))/2$. Our choice of Δ_1 implies that:

$$\rho(\Delta) = \alpha^{-1}(2\gamma(\Delta)) - \frac{\alpha^{-1}(2\gamma(\Delta))}{2} \leq \alpha^{-1}(2\gamma(\Delta)) - \alpha^{-1}(2\gamma(\Delta_1)),$$

so that $\rho(\Delta)/F$ is a lower bound to the travel time between any two points at distance $\alpha^{-1}(2\gamma(\Delta_1))$ and $\alpha^{-1}(2\gamma(\Delta))$ respectively from any of the $\mathcal{W}_i$s. Pick next $0 < \Delta_2 \leq \Delta_1$ so as to fulfill:

$$\frac{G(\Delta_2)}{\gamma(\Delta_1)} \leq \rho(\Delta)/2F. \tag{12}$$

The left-hand side is an upper bound to the time that solutions corresponding to inputs of amplitude less than Δ_2 can consecutively spend outside the set $N_i(\Delta_1)$. In particular as this upper-bound is smaller than the minimum time to reach the boundary of $N_i(\Delta)$ (from the boundary of $N_i(\Delta_1)$), then any such solution (leaving a set $N_i(\Delta_2)$) will never leave the set $N_i(\Delta)$. This fact allows us to establish existence of a suitable asymptotic gain function for all inputs of sufficiently small amplitude. In fact we may notice that $\Delta = \tilde{\gamma}^{-1}(\Delta_1)$ and $G(\Delta_2) \leq \rho(\tilde{\gamma}^{-1}(\Delta_1))\gamma(\Delta_1)/2F$. Thus for all sufficiently small Δ_1 we have:

$$\Delta_2 \leq G^{-1}(\rho(\tilde{\gamma}^{-1}(\Delta_1))\gamma(\Delta_1)/2F) := \hat{\gamma}(\Delta_1).$$

A suitable asymptotic gain function for $\|d\|_\infty \leq \Delta_2$ is therefore given by:

$$\underline{\eta}(\|d\|_\infty) = \alpha^{-1}(2\gamma(\tilde{\gamma}^{-1}(\hat{\gamma}^{-1}(\|d\|_\infty)))).$$

Take $\eta(s) = \max\{\underline{\eta}(s), \overline{\eta}_{\Delta_2}(s)\}$, this concludes the proof of Claim 3 and of the implication 3. ⇒ 1. of Theorem 1.

Remark 3: As it has been shown, existence of an ISS Lyapunov function for the system (1) implies the AG property (*i.e.* the inequality (3) is true for $q = 0$) and estimate (7), which is satisfied for all $t \geq 0$. One might wonder if it is possible to combine both results, in order to obtain another estimate

$$|X(t, x; d)|_{\mathcal{W}} \leq \kappa'(|x|_{\mathcal{W}} + q', t) + \sigma'(\|d\|_\infty), \quad \forall t \geq 0,$$

where $\kappa' \in \mathcal{KL}$, $\sigma' \in \mathcal{K}$ and $q' > 0$, which would mimic the conventional ISS theory. However, it is possible to show that in the general framework considered, even for $q' > 0$, existence of an estimate like this is not possible.

B. Converse Lyapunov theorem for (2)

The objective of this subsection is to present an auxiliary useful result on existence of Lyapunov functions for the system (2), and in particular to provide an extension to non-compact manifolds of the following result from [12]:

Proposition 2: Let Assumption 1 be satisfied for a set $\mathcal{W}$ and suppose for all $x_0 \in M$

$$\lim_{t \to +\infty} |X(t, x_0, 0)|_{\mathcal{W}} = 0.$$

Then, there exists a smooth Lyapunov function $L : M \to \mathbb{R}_+$ such that:

- $DL(x) = 0$ for all $x \in \mathcal{W}$, $L(x_i) = L(x_i') \neq L(x_j)$ for any $x_i, x_i' \in \mathcal{W}_i$, $x_j \in \mathcal{W}_j$ and all $1 \leq i \neq j \leq k$;
- $\upsilon(|x|_{\mathcal{W}}) \leq L(x)$, $DL(x)f(x, 0) \leq -\varpi(|x|_{\mathcal{W}})$ for all $x \in M$ and some $\upsilon \in \mathcal{K}$ and a positive definite $\varpi : \mathbb{R}_+ \to \mathbb{R}_+$.

If the set $\mathcal{W}$ contains only purely attracting or repelling subsets $\mathcal{W}_i$, then this proposition has been proven in [7]. Below a general case is studied adapting the arguments proposed in [12].

C. Converse Lyapunov arguments for (1)

The remaining part of the Section will be devoted to the main steps in the proof of the implication 1. ⇒ 3. of our Theorem 1. This will be carried out in several steps:

- First of all we remark that existence of an ISS Lyapunov function as in Definition 8 is equivalent to the following type of dissipation inequality:

$$|x|_{\mathcal{W}} \geq \chi(|d|) \Rightarrow DV(x)f(x, d) \leq -\rho(|x|_{\mathcal{W}})$$

  with χ of class $\mathcal{K}_\infty$ and ρ positive definite (or equivalently $\mathcal{K}_\infty$). This follows by a standard continuity argument and it is shown for instance in [15] for the case of systems defined in Euclidean space and $\mathcal{W}$ being a single equilibrium. The same proof applies here, as the $\mathcal{K}_\infty$ upper-bound of $V(x)$ in terms of $|x|$ is never needed in the proof.
- From Lemma 2 (see appendices) it follows that there exists β of class $\mathcal{K}_\infty$ such that differential inclusion

$$\dot{z} \in \bigcup_{v \in D : |v| \leq \beta^{-1}(|z|_{\mathcal{W}}/2)} f(z, v). \tag{13}$$

  fulfills a global attractivity property:

$$\limsup_{t \to +\infty} |Z(t, z_0)|_{\mathcal{W}} = 0.$$

Notice that, due to the lack of stability and of $\mathcal{KL}$ bounds we cannot assume β to be directly related to the asymptotic gain η, see Lemmas 1 and 2. In other words the gain margin may be much slimmer than the asymptotic gain (in fact transient overshoots could be much larger than the asymptotic gain).

- Consider a monotonically increasing sequence of compact subsets $M_1 \subset M_2 \subset \ldots \subset M_n \ldots \subset M$ with the property that:

$$M = \bigcup_{n=1}^{+\infty} \mathrm{int}(M_n),$$

where $\mathrm{int}(S)$ denotes the interior of the set S.
We denote by $Z(t, S)$ the attainable set of (13) at time t from initial conditions in S. Consider next:

$$\mathcal{D} = \bigcup_{n=1}^{+\infty} \bigcap_{t\geq 0} Z(t, M_n) \tag{14}$$

- The following properties of $\mathcal{D}$ should be crucial (see Lemma 3 for a proof):
  1) $\mathcal{D}$ is positively invariant,
  2) $\mathcal{D}$ is compact,
  3) $\mathcal{D}$ is globally asymptotically stable for (13),
  4) $\mathcal{W} \subset \mathcal{D}$
  5) moreover due to compactness of $\mathcal{D}$ the following is also true: $|x|_\mathcal{D} \leq |x|_\mathcal{W} \leq |x|_\mathcal{D} + c$ for some non-negative real c.
- By a standard converse Lyapunov argument (see for instance [10] where the Euclidean case is treated) we show existence of a smooth $W : M \to \mathbb{R}$ such that for all $|v| \leq 1$:

$$DW(x)f(x, \beta^{-1}(|x|_\mathcal{W}/2)v) \leq -\alpha(|x|_\mathcal{D})$$

for some $\alpha \in \mathcal{K}_\infty$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ which bound from above and below W as follows:

$$\alpha_1(|x|_\mathcal{D}) \leq W(x) \leq \alpha_2(|x|_\mathcal{D}).$$

- From Proposition 2 under the stated assumptions there exists a Lyapunov function $U(x)$ (constant on invariant sets $\mathcal{W}_i$) for the unperturbed system (2) fulfilling:

$$DU(x)f(x, 0) \leq -\varpi(|x|_\mathcal{W}). \tag{15}$$

and $\upsilon'(|x|_\mathcal{W}) \leq U(x)$ with $\upsilon' \in \mathcal{K}$ and a positive definite $\varpi : \mathbb{R}_+ \to \mathbb{R}_+$.
- By continuity of the function $DU(x)f(x, d)$ and the equation (15) we see that the following holds for (1):

$$DU(x)f(x, d) \leq -\varpi(|x|_\mathcal{W}) + \hat{\gamma}_1(1 + |x|_\mathcal{W})\hat{\gamma}_2(|d|).$$

for suitable $\hat{\gamma}_1, \hat{\gamma}_2$ of class $\mathcal{K}_\infty$, for all $x \in M$ and all $d \in D$.
- As $U(x)$ is a semi-proper function (it may be bounded), we need to add U and W in order to obtain a function with a class $\mathcal{K}_\infty$ lower-bound with respect to $|x|_\mathcal{W}$. Define $\tilde{U}(x) = U(x) + W(x)$, then $\tilde{\upsilon}(|x|_\mathcal{W}) \leq \tilde{U}(x)$, where

$$\tilde{\upsilon}(s) = \begin{cases} \upsilon'(s) & \text{if } s \leq c \\ \upsilon'(c) + \alpha_1(s - c) & \text{if } s > c \end{cases}$$

is a function from class $\mathcal{K}_\infty$ (since α_1 is from this class), and

$$D\tilde{U}(x)f(x, 0) \leq -\tilde{\alpha}(|x|_\mathcal{W}), \tag{16}$$

where

$$\tilde{\alpha}(s) = \begin{cases} \frac{s}{1+s} \inf_{s\leq r\leq c} \varpi(r) & \text{if } s \leq c \\ \frac{c}{1+c}\varpi(c) + \alpha(s - c) & \text{if } s > c \end{cases}$$

is also a function from class $\mathcal{K}_\infty$ (since $\alpha \in \mathcal{K}_\infty$). Moreover $\tilde{U}$ is constant on invariant sets $\mathcal{W}_i$ as U is such and W is identically 0 on $\mathcal{D} \supset \mathcal{W}$.
- Next we want to add $\tilde{U}$ and W (suitably rescaled) in order to get a Lyapunov function as desired. To this end consider any Lipschitz continuous $\mathcal{K}_\infty$ function δ fulfilling:

$$\delta(r) \leq \begin{cases} \hat{\gamma}_2^{-1}(\alpha(r - c)) & \text{if } r \geq 2c \\ \hat{\gamma}_2^{-1}\left(\frac{\tilde{\alpha}(r)}{2\hat{\gamma}_1(1+2c)}\right) & \text{if } r \leq 2c \end{cases}.$$

Rescaling W as

$$\tilde{W}(x) = \int_0^{W(x)} q(r)dr,$$

where the function $q \in \mathcal{K}$ will be defined later, yields:

$$\begin{aligned} D\tilde{W}(x)f(x, \beta^{-1}(|x|_\mathcal{W}/2)v) &\leq -q(W(x))\alpha(|x|_\mathcal{D}) \\ &\leq -q(\alpha_1(|x|_\mathcal{D}))\alpha(|x|_\mathcal{D}). \end{aligned}$$

- Define $V(x) = \tilde{W}(x) + \tilde{U}(x)$. Clearly $V(x)$ is lower-bounded by a $\mathcal{K}_\infty$ function of $|x|_\mathcal{W}$. Moreover, it fulfills for all $|v| \leq 1$:

$$DV(x)f(x, \min\{\beta^{-1}(|x|_\mathcal{W}/2), \delta(|x|_\mathcal{W})\}v) \leq -\tilde{\alpha}(|x|_\mathcal{W})/2,$$

which follows considering separately two cases:
  1) Case $|x|_\mathcal{W} \leq 2c$:

$$\begin{aligned} & DV(x)f(x, \min\{\beta^{-1}(|x|_\mathcal{W}/2), \delta(|x|_\mathcal{W})\}v) \\ \leq\ & D\tilde{U}(x)f(x, \min\{\beta^{-1}(|x|_\mathcal{W}/2), \delta(|x|_\mathcal{W})\}v) \\ \leq\ & -\tilde{\alpha}(|x|_\mathcal{W}) + \hat{\gamma}_1(1 + 2c)\hat{\gamma}_2(\delta(|x|_\mathcal{W})) \\ \leq\ & -\tilde{\alpha}(|x|_\mathcal{W}) + \tilde{\alpha}(|x|_\mathcal{W})/2 = -\tilde{\alpha}(|x|_\mathcal{W})/2. \end{aligned}$$

  2) Case $|x|_\mathcal{W} \geq 2c$:

$$\begin{aligned} & D\tilde{W}(x)f(x, \min\{\beta^{-1}(|x|_\mathcal{W}/2), \delta(|x|_\mathcal{W})\}v) \\ \leq\ & -q(\alpha_1(|x|_\mathcal{D}))\alpha(|x|_\mathcal{D}) \\ \leq\ & -q(\alpha_1(|x|_\mathcal{W} - c))\alpha(|x|_\mathcal{W} - c). \end{aligned}$$

  Moreover:

$$\begin{aligned} & D\tilde{U}(x)f(x, \min\{\beta^{-1}(|x|_\mathcal{W}/2), \delta(|x|_\mathcal{W})\}v) \\ &\leq -\tilde{\alpha}(|x|_\mathcal{W}) + \hat{\gamma}_1(1 + |x|_\mathcal{W})\hat{\gamma}_2(\delta(|x|_\mathcal{W})) \\ &\leq -\tilde{\alpha}(|x|_\mathcal{W}) + \hat{\gamma}_1(1 + |x|_\mathcal{W})\alpha(|x|_\mathcal{W} - c). \end{aligned}$$

  Hence, it is enough to take:

$$q(\alpha_1(r - c)) = \hat{\gamma}_1(1 + r)$$

  for all $r \geq 2c$, in order to get:

$$DV(x)f(x, \min\{\beta^{-1}(|x|_\mathcal{W}/2), \delta(|x|_\mathcal{W})\}v) \leq -\tilde{\alpha}(|x|_\mathcal{W})$$

  for all $|v| \leq 1$ and $|x|_\mathcal{W} \geq 2c$.
- The obtained inequality for V implies that for all $x \in M$ and $d \in R^m$

$$|x|_\mathcal{W} \geq \chi(|d|) \ \Rightarrow\ DV(x)f(x, d) \leq -\tilde{\alpha}(|x|_\mathcal{W})/2$$

for $\chi^{-1}(s) = \min\{\beta^{-1}(s/2), \delta(s)\}$, therefore V is an ISS Lyapunov function constant on invariant sets.

In order to complete the proof of Theorem 1 we just need to show that practical AG implies AG. This is discussed below.

Claim 4: If the system (1) enjoys the pAG property, then it also fulfills the AG property.

Proof. Let us consider input signals with infinity norm less or equal to 1. From the definition of pAG property, there exists $\eta \in \mathcal{K}_\infty$ and $q \geq 0$ such that the set $\Omega = \{x \in M : |x|_\mathcal{W} \leq \eta(1) + q + 1\}$

traps in finite time all solutions of (1). Moreover, in the proof above it has been established that for the set $\mathcal{W}$ satisfying restrictions of Assumption 1 there is a smooth Lyapunov function $\tilde{U} : M \to \mathbb{R}^n$ (constant on invariant sets) such that (16) is true for the unperturbed system (2) with $\tilde{v}, \tilde{\alpha} \in \mathcal{K}_\infty$. Thus by standard continuity arguments for the system (1) the estimate

$$D\tilde{U}(x)f(x,d) \leq -\tilde{\alpha}(|x|_{\mathcal{W}}) + \hat{\gamma}_1(1+|x|)\hat{\gamma}_2(|d|)$$

is satisfied for some $\hat{\gamma}_1, \hat{\gamma}_2 \in \mathcal{K}_\infty$. Since $|x| \leq |x|_{\mathcal{W}} + c$ for some $c \geq 0$, then for all $x \in \Omega$ and all $d \in D$ with $|d| \leq 1$, the following inequality is valid:

$$D\tilde{U}(x)f(x,d) \leq -\tilde{\alpha}(|x|_{\mathcal{W}}) + \hat{\gamma}_1(\eta(1)+q+c+2)\hat{\gamma}_2(|d|). \quad (17)$$

Thus, in this case, $\tilde{U}$ is a kind of local ISS Lyapunov function for (1). Hence given any initial condition $x_0 \in M$ and any input d with $\|d\|_\infty \leq 1$, equation (17) eventually holds along the solution and an asymptotic gain estimate η_1 follows along the steps of the proof of Claim 3. For inputs of larger infinity norm one may use the upper-bound provided by the pAG property. Overall a single $\mathcal{K}_\infty$ asymptotic gain exists, for instance $\max\{\eta_1(r), \eta(r)+qr\}$, just by patching the two cases.

IV. Examples and counterexamples

The main results of this note largely improve the range of systems to which Input-to-State stability techniques can be applied. We illustrate this point through simple and effective examples that show the power of this extended framework.

A. Multistable systems

Multistable systems cannot be treated by standard ISS theory as this only applies to globally asymptotically stable attractors. The presence of multiple stable equilibria, in fact, typically entails also existence of unstable equilibria or other attractors which cannot be accomodated by the standard theory. We illustrate this through a simple scalar example, although similar considerations apply to much more general systems:

$$\dot{x} = -x^3 + x + d. \quad (18)$$

The state manifold M is in this case the Euclidean line $\mathbb{R}$. The unperturbed system has 3 equilibria in -1, 0 and 1 respectively. The equilibria in 1 and -1 are locally asymptotically stable and almost-globally attractive. A standard ISS argument would necessarily involve the interval $I = [-1, 1]$ which is the only compact globally asymptotically stable set. In particular, a Lyapunov function such as $|x|_I^2$ could be used to prove ISS with respect to the set I. This however provides a very rough estimate of where solutions belong asymptotically, especially for zero or small disturbance amplitudes. In higher dimensional examples this issue becomes even more critical as explicit knowledge of the smallest globally asymptotically stable attractor is normally not possible. On the contrary, Theorem 1 allows to analyze system's robustness by letting $\mathcal{W} = \{-1, 0, 1\}$. Assumption 1 is trivially satisfied for this set. A suitable candidate ISS-Lyapunov function is $V(x) = (x-1)^2(x+1)^2$. Notice that $|x|_{\mathcal{W}} = \min\{|x-1|, |x|, |x+1|\}$, and therefore: $|x|_{\mathcal{W}}^4 \leq V(x)$. Moreover, taking derivatives along solutions of (18), we have:

$$\frac{\partial V}{\partial x}(x)[-x^3+x+d] = -2(x-1)^2(x+1)^2x^2 + 4dx(x-1)(x+1)$$

$$\leq -(x-1)^2(x+1)^2x^2 + 4d^2 \leq -|x|_{\mathcal{W}}^6 + 4d^2.$$

This proves that V is an ISS-Lyapunov function and by Theorem 1 the system enjoys the AG property. System (18) is also amenable to be analyzed along the lines of [3], in particular, due to the exponential instability of the equilibrium in 0, it is possible to conclude that almost all solutions converge to a neighborhood of $\{-1, 1\}$ of size "proportional" to the infinity norm of d.

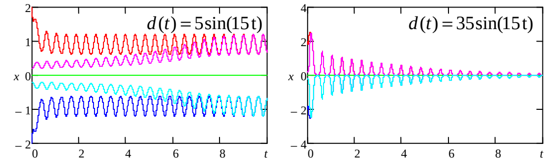

Fig. 2. The results of simulation for (19)

Next we consider a slight modification of the system (18):

$$\dot{x} = -x^5 + x^3 + xd. \quad (19)$$

The unperturbed system still exhibits 3 equilibria, in $-1, 0, 1$, two of which (those in -1 and 1) are asymptotically stable. The equilibrium in 0 is antistable, but non-hyperbolic. This makes the almost global stability theory of [3] unapplicable. In fact a small disturbance is capable of locally stabilizing the 0 equilibrium and creating a basin of attraction of non-zero measure around it.

Despite this, let $\mathcal{W} = \{-1, 0, 1\}$ and define $V(x) = (x-1)^2(x+1)^2$. One may verify that V still serves as an ISS-Lyapunov function for equation (19).

$$\frac{\partial V}{\partial x}(x)[-x^5+x^3+xd] = -4x^4(x-1)^2(x+1)^2 + 4x^2(x-1)(x+1)d$$

$$\leq -2x^4(x-1)^2(x+1)^2 + 2d^2 \leq -2|x|_{\mathcal{W}}^8 + 2d^2.$$

Finally, we would like to show by simulation that even a "large" perturbation d may stabilize the unstable equilibrium at the origin. The results of simulation of (19) for different initial conditions and $d(t) = 5\sin(15t)$ or $d(t) = 35\sin(15t)$ are shown in Fig. 2. As we can see, for a "small" perturbation $d(t) = 5\sin(15t)$ almost all trajectories converge to neighborhoods of equilibriums -1 and 1 (size of the neighborhoods is proportional the amplitude of d), but a "big" perturbation $d(t) = 35\sin(15t)$ makes the origin attractive, which is not an intuitively awaited behavior.

B. A planar example: pendulum with friction

Consider the following set of differential equations, describing the motion of a forced pendulum with friction which was also used as a motivating example in [3]:

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -a\sin(\theta) - b\omega + d. \end{aligned} \quad (20)$$

We regard them as a system with state $x = [\theta, \omega]$ taking values on the cylinder $M := \mathbb{S} \times \mathbb{R}$ affected by some exogenous disturbance $d(t)$, whereas a, b are constant positive parameters. Overall the unperturbed system admits two equilibria $[0, 0]$ and $[\pi, 0]$ the latter being a saddle-point. It is shown in [3] that this system is almost-globally Input-to-State Stable with respect to the downwards pendulum equilibrium $x = [0, 0]'$. The same Lyapunov functions used in [3] can be used in order to prove Input-to-State stability with respect to the set $\mathcal{W} = \{[0, 0], [\pi, 0]\}$. Consider the mechanical energy of the pendulum, that is $V(x) = \omega^2/2 - a\cos(\theta) + a$. For the case $d = 0$ the dissipation equality $\dot{V} = -b\omega^2$ holds, which together with the fact that there is no trajectory on the line $\omega = 0$ connecting the equilibriums imply Assumption 1. Notice that $|x|_{\mathcal{W}} = \sqrt{\omega^2 + \min\{|\theta|, |\theta - \pi|\}^2}$. Therefore $\varepsilon|x|_{\mathcal{W}}^2 \leq V(x)$ for some sufficiently small $\varepsilon > 0$. Moreover, taking derivatives of V along solutions of (20) yields:

$$\begin{array}{rcl} \dot{V}(x) & = & -b\omega^2 + \omega d \leq -\frac{b}{2}\omega^2 + \frac{1}{2b}d^2 \\ & = & -\frac{b}{2}V(x) - \frac{ab}{2}\cos(\theta) + \frac{1}{2b}d^2 \\ & \leq & -\frac{b}{2}V(x) + c + \frac{1}{2b}d^2 \end{array}$$

with constant $c := ab/2$. Therefore system (20) admits a practical ISS Lyapunov function. By Theorem 1 system (20) is Input-to-State Stable with respect to $\mathcal{W}$.

C. Non decomposable invariant set

Consider the system:

$$\begin{array}{rcl} \dot{\theta} & = & 1 - \cos(\theta) + d \\ \dot{z} & = & -z + d \end{array} \tag{21}$$

with state $x = [\theta, z]$ taking values in the cylinder $\mathbb{S}\times\mathbb{R}$. In the absence of disturbances all solutions converge to the unique equilibrium $x = [0, 0]$ (up to multiples of 2π in the first coordinate) as it is easy to check by considering separately the θ and z equation. The equilibrium $[0, 0]$ is not asymptotically stable, however, and in fact, the singleton $\{[0, 0]\}$ does not admit a decomposition without cycles. This means that in order to apply our main result we need to enlarge the set $\mathcal{W}$. We can in fact define $\mathcal{W} = \mathbb{S} \times \{0\}$. This is an invariant and asymptotically stable set for the unperturbed system. Moreover, letting $V(x) = z^2$, yields along solutions:

$$\dot{V}(x) = 2z(-z + d) \leq -z^2 + d^2 = -|x|^2_{\mathcal{W}} + d^2$$

where the last equality follows since $|x|_{\mathcal{W}} = |z|$. Moreover, $|x|^2_{\mathcal{W}} \leq V(x)$; therefore, the system admits an ISS Lyapunov function and thanks to Theorem 1 is Input-to-State Stable with respect to the set $\mathcal{W}$. One may wonder whether tighter characterizations of the robustness properties of system (21) could be possible, for instance if ISS with respect to the set $\{0\}$ be fulfilled regardless of it exhibiting homoclinic cycles. To show that this is not possible take any positive and vanishing disturbance which is not in L_1, such as $d(t) = 1/(1 + t)$, and consider the solution $[\theta(t), z(t)] := X(t, [0, 0]; d)$. If ISS would be true, solutions should converge to 0 by the converging-input converging-state property. While it is easy to see that $z(t) \to 0$ as $t \to +\infty$ it is also clear that

$$\int_0^{+\infty} |\dot{\theta}(t)|dt = \int_0^{+\infty} 1 - \cos(\theta(t)) + d(t)\, dt \geq \int_0^{+\infty} d(t)dt = +\infty,$$

thus proving that solutions never stop describing full rotations around the circle. Hence, the choice of $\mathcal{W} = \mathbb{S} \times \{0\}$ is in fact the tightest possible.

D. Van der Pol oscillator

Consider the following set of equations, describing the state evolution of a system evolving in $M = \mathbb{R}^2$:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 - \varphi(x_2) + d \end{array} \tag{22}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies $\varphi(0) = 0$, $\varphi'(0) < 0$ and

$$\varphi(y) \to +\infty \text{ as } y \to +\infty, \qquad \varphi(y) \to -\infty \text{ as } y \to -\infty.$$

This set of equations encompasses, for a peculiar choice of φ, the so called Van der Pol oscillator. It was shown in [4] that a practical asymptotic gain property holds for (22) (as well as for a broader class of systems). For $\varphi(x_2) = x_2 - \frac{x_2^3}{3}$ a Lyapunov function to establish the practical asymptotic gain property with respect to the origin is given in [8]. Moreover, it is well known that Van der Pol oscillator in the absence of external forcing admits 2 invariant sets, namely the origin and the limit cycle $\mathcal{L}$. Hence, we may define $\mathcal{W} = \{0\}\cup\mathcal{L}$. As any solution converges to $\mathcal{L}$, except for the one initiated at 0, which is antistable, we can conclude that $\mathcal{W}$ admits a decomposition without cycles. As a consequence we can claim existence of an ISS-Lyapunov function and of a class $\mathcal{K}_\infty$ function γ such that:

$$\limsup_{t\to+\infty} |X(t, \xi; d)|_{\mathcal{W}} \leq \gamma(\|d\|_\infty).$$

E. FitzHugh-Nagumo model

This model is a two-dimensional simplification of the Hodgkin-Huxley model of spike generation:

$$\begin{array}{rcl} \dot{x}_1 & = & x_1 - x_1^3 - x_2 + d \\ \dot{x}_2 & = & \tau^{-1}(x_1 - a - bx_2) \end{array} \tag{23}$$

where $x_1 \in \mathbb{R}$ is the membrane potential, $x_2 \in \mathbb{R}$ is a recovery variable, and d is the magnitude of stimulus current. The model (22) is a particular case of (23) for $a = b = 0$. It is well known fact that for any constant d this model has an equilibrium x_d and almost globally attracting set Γ containing oscillating trajectories (depending on values of parameters). Thus Assumption 1 is satisfied for $\mathcal{W} = \{x_d\}\cup\Gamma$, but as for $\mathcal{L}$ in (22), it is hard to find an analytical expression for characterization of Γ as a function of x. Therefore, similarly to (22) we can establish the practical asymptotic gain property with respect to the origin using the function

$$V(x) = 0.5(x_1^2 + \tau x_2^2),\ \dot{V} \leq -0.5(x_1^2 + bx_2^2) + 1 + 0.5a^2/b + 0.5d^2,$$

then the system (23) admits the practical asymptotic gain property with respect to $\mathcal{W}$ since $|x|_{\mathcal{W}} \leq |x|$. Next, according to result of Theorem 1 the FitzHugh-Nagumo model is ISS with respect to $\mathcal{W}$ considering stimulus current as input.

A similar consideration can be repeated for the Hindmarsh-Rose model of neuronal activity.

F. A model with a continuum of equilibria

Consider the system defined on the cylinder $M = \mathbb{R} \times \mathbb{S}$ by the following set of differential equations

$$\begin{array}{rcl} \dot{z} & = & -z + 2z\cos^2(\theta) + d \\ \dot{\theta} & = & z^2\cos(\theta). \end{array} \tag{24}$$

with state $x = [z, \theta]$ expressed in coordinates, with the usual convention that points x whose second coordinate differs by a multiple of 2π are identified. It is easy to prove that the set $\mathcal{W} = \{0\} \times \mathbb{S}$ is the equilibrium set for the unperturbed system. This is an invariant connected set that does not admit finer decompositions. Take the Lyapunov function candidate V given below:

$$V(z, \theta) = z^2 + 4 - 4\sin(\theta).$$

Clearly, $V(x, \theta) \geq z^2 = |x|^2_{\mathcal{W}}$. Moreover, differentiating V along solutions of (24) yields:

$$\begin{array}{rcl} \dot{V}(z, \theta) & = & 2z(-z + 2z\cos^2(\theta) + d) - 4z^2\cos(\theta)^2 \\ & = & -2z^2 + 2zd \leq -z^2 + d^2 = -|x|^2_{\mathcal{W}} + |d|^2. \end{array}$$

Hence V is an ISS Lyapunov function and the system is ISS with respect to $\mathcal{W}$. Notice that $V(\mathcal{W}) = [0, 8]$ and is therefore not a singleton. Finding a Lyapunov function constant on invariant sets appears to be more challenging. In particular, $V(x) = z^2$ is not appropriate as its derivative is not negative definite already for the unperturbed dynamics.

V. Conclusion

The paper proposes definitions and characterizations of input-to-state stability for systems with multiple attractors, in the cases when the invariant sets of the system are not connected by homoclinic and heteroclinic trajectories or, alternatively, these are included in the invariant set themselves. The invariant sets under consideration may contain disjointed subsets, some of which may be unstable in the Lyapunov sense. It is shown that under such assumptions the practical stability notions are equivalent to "conventional" ones. Necessary and sufficient characterizations of input-to-state stability in terms of Lyapunov function existence are given. Applicability of the proposed framework is demonstrated on several examples of popular systems.

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References

[1] D. Angeli, "Some remarks on density functions for dual Lyapunov methods", *Proc. of the IEEE Conf. on Decision and Control*, vol. 5, pp. 5080-5082, 2003.

[2] D. Angeli, "An almost global notion of input-to-state stability", *IEEE Trans. on Automatic Control*, Vol. 49, N. 6, pp. 866-874, 2004.

[3] D. Angeli and L. Praly, "Stability robustness in the presence of exponentially unstable isolated equilibria", *IEEE Transactions on Automatic Control*, Vol. 56, N. 7, pp. 1582–1592, 2012.

[4] M. Arcak and A. Teel, "Input-to-State Stability for a class of Lurie systems", *Automatica*, Vol. 38, pp. 1945–1949, 2002.

[5] N.P. Bhatia and G.P. Szegö, "Stability Theory of Dynamical Systems", Springer-Verlag, Berlin, pp. 59, 1970.

[6] M. Chaves, T. Eissing and F. Allgower, "Bistable biological systems: A characterization through local compact input-to-state stability", IEEE Trans. Automat. Control, 45, pp. 87–100, 2008.

[7] D. Efimov, "Global Lyapunov analysis of multi-stable nonlinear systems", *SIAM Journal on Control and Optimization*, Vol. 50, N. 5, pp. 3132-3154, 2012.

[8] D. Efimov, A.L. Fradkov, "Oscillatority of Nonlinear Systems with Static Feedback", *SIAM Journal on Optimization and Control*, Vol. 48, N. 2, pp. 618-640, 2009.

[9] J. Guckenheimer and P. Holmes, "Structurally stable heteroclinic cycles", *Math. Proc. Camb. Phil. Soc.*, Vol. 103, pp:189–192, 1988.

[10] Y. Lin, E.D. Sontag, and Y. Wang, "A smooth converse Lyapunov theorem for robust stability", *SIAM Journal on Control and Optimization*, Vol. 34, N.1, pp:124–160, 1996.

[11] L. Markus, "Global structure of ordinary differential equations in the plane", *Trans. Am. Math. Soc.*, Vol. 76, pp:127–148, 1954.

[12] Z. Nitecki and M. Shub, "Filtrations, Decompositions, and Explosions", *American Journal of Mathematics*, Vol. 97, N. 4, pp. 1029–1047, 1975.

[13] A. Rantzer, "A dual to Lyapunov's stability theorem", *Systems and Control Letters*, Vol. 42, pp. 161-168, 2001.

[14] A. Rantzer, "An converse theorem for density functions", *Proc. of the IEEE Conf. on Decision and Control*, vol. 2, pp. 1890-1891, 2002.

[15] E.D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property", *Systems and Control Letters*, Vol. 24, N. 5, pp:351–359, 1995.

[16] E.D. Sontag and Y. Wang, "New characterizations of input-to-state stability", *IEEE Trans. Automat. Control*, Vol. 41, N. 9, pp:1283–1294, 1996.

[17] E.D. Sontag, "Input to state stability: Basic concepts and results", in P. Nistri and G. Stefani, editors, *Nonlinear and Optimal Control Theory*, pp. 163-220, Springer-Verlag, Berlin, 2007.

[18] J. F. Vasconcelos, A. Rantzer, C. Silvestre and P. J. Oliveira, "Combination of Lyapunov and Density Functions for Stability of Rotational Motion", *IEEE Trans. on Automatic Control*, Vol. 56, N. 11, pp. 2599-2607, 2011.

[19] F.W. Wilson, "Smoothing derivatives of functions and applications", *Trans. of the American Mathematical Society*, Vol. 139, pp. 413–428, 1969.

Appendix

A. Some technical Lemmas

Lemma 1: Consider a nonlinear system as in (1), and assume that it enjoys the AG property (3). Then, there exists β of class $\mathcal{K}_\infty$ and $q \geq 0$ such that for all $x \in M$ and all measurable inputs $d(\cdot)$ the pGS property holds.

Proof. Define, for all $r \geq 0$ the following open set:

$$\Omega(r) := \{x \in M : |x|_{\mathcal{W}} < \eta(r) + 1\} \tag{25}$$

where η denotes the asymptotic gain. For all solutions $X(\cdot, x; d)$ we define the hitting time as follows:

$$\tau_{x,d}(r) = \inf\{t \geq 0 : X(t, x; d) \in \Omega(r)\}. \tag{26}$$

Notice that, for all d with infinity norm less or equal to r, $\tau_{x,d}(r) < +\infty$ by virtue of the asymptotic gain property. By virtue of Corollary III.3 in [16], given any compact set $C \subset M$ of initial conditions,

$$\mathcal{T}_{C,r} := \sup_{x \in C,\, d(\cdot):\|d\|_\infty \leq r} \tau_{x,d}(r) < +\infty.$$

(notice that the corollary is stated for systems evolving in Euclidean space, but the same proof applies to systems on manifolds). Define next the reachable set:

$$\mathcal{R}^T(C, r) := \{z \in M : \exists\, t \in [0, T], \exists\, x \in C, \exists\, d(\cdot) : \\ \|d\|_\infty \leq r : z = X(t, x; d)\}.$$

Clearly, if $C \supset \Omega(r)$ we have:

$$\mathcal{R}(C) := \bigcup_{T \geq 0} \mathcal{R}^T(C, r) = \mathcal{R}^{\mathcal{T}_{C,r}}(C, r). \tag{27}$$

By virtue of Proposition 5.1 in [10], $\mathcal{R}^{\mathcal{T}_{C,r}}(C, r)$ is bounded.

Define, the set C_r as follows,

$$C_r = \{z \in M : |z|_{\mathcal{W}} \leq \max\{r, \eta(r) + 1\}\}.$$

Notice that by construction $C_r \supset \Omega(r)$. Let, for any $r \geq 0$, the function Γ be defined as follows:

$$\Gamma(r) := \sup_{z \in C_r, d(\cdot):\|d\|_\infty \leq r,\, t \geq 0} |X(t, z; d)|_{\mathcal{W}} \tag{28}$$

Hence, by boundedness of $\mathcal{R}^{\mathcal{T}_{C_r,r}}(C_r, r)$, Γ is a well defined non-decreasing function. Moreover, for any $x \in M$ and any bounded input d we may let $r = \max\{|x|_{\mathcal{W}}, \|d\|_\infty\}$ and by equation (28) we see that:

$$|X(t, x; d)|_{\mathcal{W}} \leq \Gamma(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}) \qquad \forall\, t \geq 0.$$

Hence, for some $\mathcal{K}_\infty$ function β and $q = \Gamma(0)$:

$$|X(t, x; d)|_{\mathcal{W}} \leq q + \beta(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}) \qquad \forall\, t \geq 0.$$

Lemma 2: Consider a system as in (1) and fulfilling all assumptions of Lemma 1. Then, provided $\beta(\cdot) \geq \eta(\cdot)$ (which can be assumed without loss of generality) the differential inclusion:

$$\dot{z} \in \bigcup_{v \in D: |v| \leq \beta^{-1}(|z|_{\mathcal{W}}/2)} f(z, v). \tag{29}$$

has uniformly bounded solutions. Moreover, all solutions converge asymptotically to $\mathcal{W}$.

Proof. Consider $z \in M$ arbitrary and let $Z(t, z)$ denote any solution of (29) (maximally defined over some open interval I including 0). By selection, there exists μ, measurable and $\|\mu\| \leq 1$ such that, $Z(t, z)$ is a solution of:

$$\dot{z} = f(z, \beta^{-1}(|z|_{\mathcal{W}}/2)\mu).$$

By Lemma 1, Z fulfills for all $t \geq 0$ in I:

$$|||Z(\cdot,z)|_{\mathcal{W}}\|_{[0,t]} \leq q + \beta(\max\{|z|_{\mathcal{W}}, \beta^{-1}(|||Z(\cdot,z)|_{\mathcal{W}}\|_{[0,t]}/2)\}).$$

Then we have for all $t \in I$:

$$|Z(t,z)|_{\mathcal{W}} \leq 2q + 2\beta(|z|_{\mathcal{W}}). \tag{30}$$

Hence, solutions of (29) are uniformly bounded and defined for all $t \geq 0$. It follows from the asymptotic gain property that:

$$\limsup_{t\to+\infty} |X(t,x;d)|_{\mathcal{W}} \leq \eta\left(\limsup_{t\to+\infty} |d(t)|\right).$$

Applying this inequality to the solution Z previously defined we have:

$$\limsup_{t\to+\infty} |Z(t,z)|_{\mathcal{W}} \leq \eta\left(\limsup_{t\to+\infty} \beta^{-1}(|Z(t,z)|_{\mathcal{W}}/2)\right) \leq \limsup_{t\to+\infty} |Z(t,z)|_{\mathcal{W}}/2.$$

Hence, $|Z(t,z)|_{\mathcal{W}} \to 0$ as $t \to +\infty$.

Lemma 3: The set $\mathcal{D}$ defined in (14) is bounded, closed, forward invariant and globally asymptotically stable. Moreover it contains $\mathcal{W}$.

Proof. Consider the differential inclusion (29) and let $\mathcal{R}^T(S)$ denote its reachable set up to time T for initial conditions in S, and $\mathcal{R}(S) := \bigcup_{T\geq 0} \mathcal{R}^T(S)$; by Lemma 2 each solution converges to $\mathcal{W}$ and therefore enters in finite time the set $\mathcal{Z} := \{z \in M : |z|_{\mathcal{W}} \leq 1\}$ as well as a suitable compact subset of int$(\mathcal{Z})$.

There exists index n such that $\mathcal{Z} \subset M_n$. Indeed, $\{\text{int}(M_n)\}_{n=1}^{+\infty}$ is an open cover of M, and therefore, due to paracompactness of M, it admits a locally finite refinement, *i.e.* any point x in M has a neighboorhood U_x that intersects only finitely many sets in the refinement $\{V_k\}_{k=1}^{+\infty}$. Denote by n_k the integer (as a function of k) such that $V_k \subset \text{int}(M_{n_k})$. Any compact subset K of M is covered by the $\{U_x : x \in K\}$, and in particular (by compactness) by a finite number of them:

$$K \subset U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_N}.$$

Each one of the U_{x_i}s is in its turn contained in a finite number of sets from the refinement $\{V_k\}_{k=1}^{+\infty}$ so that, overall, every compact set K is covered by a finite number of V_ns. Let $\bar{n}$ be the maximum of the indexes ks involved in such a cover. Then $K \subset \bigcup_{k=1}^{\bar{n}} V_k \subset \bigcup_{k=1}^{\bar{n}} \text{int}(M_{n_k}) = M_{\bar{N}}$, where

$$\bar{N} = \max_{k\in\{1\ldots\bar{n}\}} n_k$$

and the last equality follows by monotonicity of the M_n sequence. This shows that every compact set is contained in some M_n. Without loss of generality assume $M_1 \supset \mathcal{Z}$.

By virtue of Corollary III.3 in [16] for each $n \in \mathbb{N}$ there exists $T_n < +\infty$, such that for all $z \in M_n$ and any solution $Z(\cdot,z)$ of (29) there exists some time $t_{Z(\cdot),n} \leq T_n$, such that $Z(t_{Z(\cdot),n}, z) \in \mathcal{Z}$. Hence, for all $t \geq T_n$ the following holds:

$$Z(t, M_n) \subset \bigcup_{T\geq 0} \mathcal{R}^T(\mathcal{Z}) = \mathcal{R}^{T_1}(\mathcal{Z})$$

where the last equality follows since $M_1 \supset \mathcal{Z}$. Notice that $\mathcal{R}^{T_1}(\mathcal{Z})$ is a compact set, by forward completeness of (29) and compactness of $\mathcal{Z}$. This entails that $\mathcal{R}(\mathcal{Z})$ is also compact.

Hence, $\bigcap_{t\geq 0} Z(t, M_n) \subset \mathcal{R}^{T_1}(\mathcal{Z})$ and

$$\mathcal{D} = \bigcup_{n=1}^{+\infty} \bigcap_{t\geq 0} Z(t, M_n) \subset \bigcup_{n=1}^{+\infty} \mathcal{R}^{T_1}(\mathcal{Z}) = \mathcal{R}^{T_1}(\mathcal{Z}),$$

thus showing boundedness of $\mathcal{D}$.

By boundedness of $\mathcal{D}$, there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$ it holds $M_n \supset \mathcal{D}$ and $M_n \supset \mathcal{W}$. Moreover, assuming without loss of generality Lipschitzness of the differential inclusion (29) (this can be done as any $\mathcal{K}_\infty$ function admits a Lipschitz and $\mathcal{K}_\infty$ lower bound), it holds $Z(t, \mathcal{W}) = \mathcal{W}$, for all $t \geq 0$. Hence:

$$\mathcal{D} = \bigcup_{n\geq\bar{n}} \bigcap_{t\geq 0} Z(t, M_n) \supset \bigcup_{n\geq\bar{n}} \bigcap_{t\geq 0} Z(t, \mathcal{W}) = \mathcal{W}. \tag{31}$$

Notice that, for all n large enough, (say larger than $\hat{n}$) it holds:

$$M_n \supset \mathcal{R}(\mathcal{Z}).$$

Moreover, we see that:

$$\bigcap_{t\geq 0} Z(t, \mathcal{R}(\mathcal{Z})) \subset \bigcap_{t\geq 0} Z(t, M_n) \subset \bigcap_{t\geq T_n} Z(t, M_n) \subset \bigcap_{t\geq 0} Z(t, \mathcal{R}(\mathcal{Z})).$$

where the last inclusion holds since any solution originating in M_n ends up in $\mathcal{R}(\mathcal{Z})$ at time T_n (having touched $\mathcal{Z}$ in the meanwhile). Hence, the following holds for all n large enough:

$$\bigcap_{t\geq 0} Z(t, \mathcal{R}(\mathcal{Z})) = \bigcap_{t\geq 0} Z(t, M_n).$$

We exploit this fact by remarking that:

$$\mathcal{D} = \bigcup_{n\geq\hat{n}} \bigcap_{t\geq 0} Z(t, M_n) = \bigcup_{n\geq\hat{n}} \bigcap_{t\geq 0} Z(t, \mathcal{R}(\mathcal{Z})) = \bigcap_{t\geq 0} Z(t, \mathcal{R}(\mathcal{Z})).$$

Notice the following monotonicity property, $Z(t, \mathcal{R}(\mathcal{Z})) \subset \mathcal{R}(\mathcal{Z})$ (for $t \geq 0$), as any solution initialted in the reachable set $\mathcal{R}(\mathcal{Z})$ is after t units of time still inside $\mathcal{R}(\mathcal{Z})$. More generally:

$$t_1 \geq t_2 \geq 0 \;\Rightarrow Z(t_1, \mathcal{R}(\mathcal{Z})) \subset Z(t_2, \mathcal{R}(\mathcal{Z}))$$

as it follows considering that:

$$Z(t_1, \mathcal{R}(\mathcal{Z})) = Z(t_2, Z(t_1 - t_2, \mathcal{R}(Z))) \subset Z(t_2, \mathcal{R}(\mathcal{Z})).$$

This can be used in order to see that for all $\tau \geq 0$:

$$\begin{aligned} Z(\tau, \mathcal{D}) &= Z(\tau, \bigcap_{t\geq 0} Z(t, \mathcal{R}(\mathcal{Z}))) \\ &= \bigcap_{t\geq\tau} Z(t, \mathcal{R}(\mathcal{Z})) = \bigcap_{t\geq 0} Z(t, \mathcal{R}(\mathcal{Z})) = \mathcal{D}. \end{aligned}$$

Finally, $\mathcal{D}$ is closed as it can be written as an intersection of closed sets, (viz. $Z(t, \mathcal{R}(\mathcal{Z}))$ for $t \geq 0$, where each of the set is closed by forward completeness of the flow and compactness of $\mathcal{R}(\mathcal{Z})$. In order to show Global Asymptotic Stability it is enough to remark that this is equivalent to uniform Attraction (as in [10]). Notice also that the latter follows Corollary 3.3 in [16].

B. Converse Lyapunov theorems for dichotomy systems on manifolds

In the proof of Proposition 2 (see Appendix C) we will need the following auxiliary lemmas dealing with two forward invariant sets $A_i, B_i \subset M$ (the case of a compact set A_i has been treated in [7]). Define $\rho^i_{\max} = \inf_{x\in A_i, y\in B_i} \delta(x,y)$. As in Proposition 2, it is assumed that Assumption 1 is satisfied, then the set $A_i \cup B_i$ admits a finite decomposition without cycles.

Lemma 4: Let A_i, B_i be forward invariant sets, which are asymptotically stable for forward and backward flows of (2) in $M \setminus B_i$ and $M \setminus A_i$ respectively. Then for any $0 < \rho < \rho^i_{\max}$ there exists a locally Lipschitz continuous function $V : M \backslash D_\rho \to \mathbb{R}_+$, $D_\rho = \{x \in M : |x|_{B_i} < \rho\}$ (continuous on the set A_i) such that $\alpha_1(|x|_{A_i}) \leqslant V(x)$ for all $x \in M \backslash D_\rho$ for $\alpha_1 \in \mathcal{K}_\infty$, and $DV(x)f(x,0) \leqslant -\beta(V(x), |x|)$, $\beta \in \mathcal{KL}$ for a.e. $x \in M \backslash D_\rho$.

Proof. For any $x_0 \in M \backslash D_\rho$, define

$$v(x_0) = \sup_{t\geqslant 0} |X(t, x_0, 0)|_{A_i},$$

by construction $|x_0|_{A_i} \leqslant v(x_0)$ and $v(x) = 0$ for $x \in A_i$ due to forward invariance of A_i. From attractivity of A_i and continuity of $X(t, \cdot, 0)$, for any $x_0 \in M \backslash D_\rho$ there exists $T_{x_0} \in \mathbb{R}_+$ such that $v(x_0) = \sup_{0 \leqslant t \leqslant T_{x_0}} |X(t, x_0, 0)|_{A_i}$. To analyze continuity of the function v, consider

$$\begin{aligned} |v(x_1) - v(x_2)| &= |\sup_{t \geqslant 0} |X(t, x_1, 0)|_{A_i} - \sup_{t \geqslant 0} |X(t, x_2, 0)|_{A_i}| \\ &= |\sup_{0 \leqslant t \leqslant T_{x_1}} |X(t, x_1, 0)|_{A_i} - \sup_{0 \leqslant t \leqslant T_{x_2}} |X(t, x_2, 0)|_{A_i}| \\ &\leqslant \sup_{0 \leqslant t \leqslant T} ||X(t, x_1, 0)|_{A_i} - |X(t, x_2, 0)|_{A_i}|, \end{aligned}$$

where $T = \max\{T_{x_1}, T_{x_2}\}$ and $x_1, x_2 \in M \setminus D_\rho$. Due to Lipschitz continuity of the system (2), for any compact set of initial conditions $\mathcal{E} \subset M \setminus D_\rho$ and any time $0 \leqslant T < +\infty$, there exist $K \in \mathbb{R}_+$, $L \in \mathbb{R}_+$ such that

$$\delta(X(t, x_1, 0), X(t, x_2, 0)) \leqslant K\, \delta(x_1, x_2)$$

and

$$||X(t, x_1, 0)|_{A_i} - |X(t, x_2, 0)|_{A_i}| \leqslant L\, \delta(x_1, x_2),$$

for all $0 \leqslant t \leqslant T$ and any $x_1, x_2 \in \mathcal{E}$. For all $0 < \rho < \rho^i_{\max}$ and for any compact $\mathcal{E} \subset M \setminus D_\rho$ there exists $T_\rho = \sup_{x_0 \in \mathcal{E}} T_{x_0}$ with the property $T_\rho < +\infty$ (due to local repelling property of the set B_i, for any $0 < \rho < \rho^i_{\max}$ there exists $0 < \rho' \leqslant \rho$ such that trajectories initiated into the set $\mathcal{E}$ never reach the set $D_{\rho'}$). Keeping this in mind we see that

$$\begin{aligned} |v(x_1) - v(x_2)| &\leqslant \sup_{0 \leqslant t \leqslant T_\rho} ||X(t, x_1, 0)|_{A_i} - |X(t, x_2, 0)|_{A_i}| \\ &\leqslant L\, \delta(x_1, x_2) \end{aligned}$$

for all $x_1, x_2 \in \mathcal{E}$, and the function v is locally Lipschitz continuous on the set $\mathcal{E}$ for any fixed $0 < \rho < \rho^i_{\max}$ as claimed.

Moreover, the function v is not increasing on any trajectory of the system (2), indeed for any $x_0 \in M \setminus D_\rho$:

$$\begin{aligned} v(X(t, x_0, 0)) &= \sup_{\tau \geqslant 0} |X(\tau, X(t, x_0, 0), 0)|_{A_i} = \sup_{\tau \geqslant t} |X(\tau, x_0, 0)|_{A_i} \\ &\leqslant \sup_{\tau \geqslant 0} |X(\tau, x_0, 0)|_{A_i} = v(x_0). \end{aligned}$$

Now, define a new function for all $x_0 \in M \setminus D_\rho$:

$$V(x_0) = \sup_{t \geqslant 0} \{v(X(t, x_0, 0))k(t)\},$$

where $k : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable function satisfying $0 < \kappa_1 \leqslant k(t) \leqslant \kappa_2 < +\infty$ and $\dot{k}(t) \geqslant \kappa_3(t) > 0$ for all $t \geqslant 0$, where κ_3 is a monotonically decreasing function. An example of such a function is

$$k(t) = (\kappa_1 + \kappa_2 t)(1 + t)^{-1},\ \dot{k}(t) = (\kappa_2 - \kappa_1)(1 + t)^{-2}.$$

The function V has a lower bound $\kappa_1 |x_0|_{\mathcal{W}} \leqslant V(x_0)$ and $V(x) = 0$ for all $x \in A_i$. Again, for any $x_0 \in M \setminus D_\rho$ there exists $T_{x_0} \in \mathbb{R}_+$ such that $V(x_0) = \sup_{0 \leqslant t \leqslant T_{x_0}} \{v(X(t, x_0, 0))k(t)\}$. This claim follows from the non-strict decreasing of the function $v(X(t, x_0, 0))$ to zero with $t \to +\infty$. Next, for all $x_1, x_2 \in M \setminus D_\rho$

$$\begin{aligned} |V(x_1) - V(x_2)| &= |\sup_{t \geqslant 0} \{v(X(t, x_1, 0))k(t)\} - \sup_{t \geqslant 0} \{v(X(t, x_2, 0))k(t)\}| \\ &= |\sup_{0 \leqslant t \leqslant T} \{v(X(t, x_1, 0))k(t)\} - \sup_{0 \leqslant t \leqslant T} \{v(X(t, x_2, 0))k(t)\}| \\ &\leqslant \sup_{0 \leqslant t \leqslant T} |k(t)[v(X(t, x_1, 0)) - v(X(t, x_2, 0))]| \\ &\leqslant \kappa_2 \sup_{0 \leqslant t \leqslant T} |v(X(t, x_1, 0)) - v(X(t, x_2, 0))|, \end{aligned}$$

where $T = \max\{T_{x_1}, T_{x_2}\}$. For all $0 < \rho < \rho^i_{\max}$ and for any compact $\mathcal{E} \subset M \setminus D_\rho$ there exists $T_\rho = \sup_{x_0 \in \mathcal{E}} T_{x_0}$ such that $T_\rho < +\infty$ and

$$\begin{aligned} |V(x_1) - V(x_2)| &\leqslant \kappa_2 \sup_{0 \leqslant t \leqslant T_\rho} |v(X(t, x_1, 0)) - v(X(t, x_2, 0))| \\ &\leqslant \kappa_2 L\, \delta(X(t, x_1, 0), X(t, x_2, 0)) \\ &\leqslant \kappa_2 L K\, \delta(x_1, x_2) \end{aligned}$$

for all $x_1, x_2 \in \mathcal{E}$. The function V is locally Lipschitz continuous on the set $M \backslash D_\rho$ for any $0 < \rho \leqslant \rho^i_{\max}$ and strictly decreasing for any $x_0 \in M \backslash \{A_i \cup D_\rho\}$:

$$\begin{aligned} V(X(t, x_0, 0)) &= \sup_{\tau \geqslant 0} \{v(X[\tau, X(t, x_0, 0), 0])k(\tau)\} \\ &= \sup_{\tau \geqslant t} \{v(X[\tau, x_0, 0])k(\tau - t)\} \\ &< \sup_{\tau \geqslant 0} \{v(X[\tau, x_0, 0])k(\tau)\} = V(x_0), \end{aligned}$$

In addition, V equals zero on the set A_i. Denote by

$$L_{f(x_0,0)} V(x_0) := \lim\sup_{h \to 0} h^{-1}[V(X(h, x_0, 0)) - V(x_0)],$$

then

$$L_{f(x_0,0)} V(x_0) < 0$$

for a.e. $x_0 \in M \backslash \{A_i \cup D_\rho\}$. Define for some $r \in \mathbb{R}_+$ the set $G_r = \{x \in M : |x| \leq r\} \setminus D_\rho$ and the time $T_r = \sup_{x_0 \in G_r} T_{x_0}$. The time T_r is well defined and finite since the set G_r is compact, in addition

$$T_r \leq \varphi_0 + \varphi_1(r)$$

for all $r \in \mathbb{R}_+$ and some $\varphi_0 \in \mathbb{R}_+$, $\varphi_1 \in \mathcal{K}$. By definition

$$\begin{aligned} V(X(h, x_0, 0)) &= \sup_{t \geqslant 0} \{v(X[t, X(h, x_0, 0), 0])k(t)\} \\ &= \sup_{h \leqslant t \leqslant T_{|x_0|}} \{v(X[t, x_0, 0])k(t - h)\} \\ &= \sup_{h \leqslant t \leqslant T_{|x_0|}} \{v(X[t, x_0, 0])k(t)k(t)^{-1}k(t - h)\} \\ &\leq \sup_{h \leqslant t \leqslant T_{|x_0|}} \{v(X[t, x_0, 0])k(t)\} \sup_{h \leqslant t \leqslant T_{|x_0|}} \{k(t)^{-1}k(t - h)\} \\ &\leqslant V(x_0) \sup_{h \leqslant t \leqslant T_{|x_0|}} \{k(t)^{-1}k(t - h)\} \end{aligned}$$

for a.e. $x_0 \in M \backslash \{A_i \cup D_\rho\}$. Further

$$\begin{aligned} &\lim_{h \to 0} h^{-1}[V(X(h, x_0, 0)) - V(x_0)] \\ &= \lim_{h \to 0} h^{-1}[V(x_0) \sup_{h \leqslant t \leqslant T_{|x_0|}} \{k(t)^{-1}k(t - h)\} - V(x_0)] \\ &= V(x_0) \lim_{h \to 0} h^{-1}[\sup_{h \leqslant t \leqslant T_{|x_0|}} \{k(t)^{-1}k(t - h)\} - 1] \\ &= V(x_0) \lim_{h \to 0} h^{-1} \sup_{h \leqslant t \leqslant T_{|x_0|}} k(t)^{-1}\{k(t - h) - k(t)\} \\ &\leqslant V(x_0) \sup_{0 \leqslant t \leqslant T_{|x_0|}} k(t)^{-1} \lim_{h \to 0} h^{-1}\{k(t - h) - k(t)\} \\ &= V(x_0) \sup_{t \geqslant 0} \{-k(t)^{-1}\dot{k}(t)\} \leqslant -\kappa_2^{-1}\kappa_3(T_{|x_0|})V(x_0). \end{aligned}$$

Due to properties of the function κ_3 (it is a strictly decreasing function from a constant $(\kappa_2 - \kappa_1)$ to zero), the inequality $L_{f(x,0)} V(x) \leqslant -\kappa_2^{-1}\kappa_3(\varphi_0 + \varphi_1[|x|])V(x)$ has been approved for a.e. $x \in M \backslash \{A_i \cup D_\rho\}$. This inequality is additionally valid on the set A_i, then we obtain

$$L_{f(x,0)} V(x) \leqslant -\beta(V(x), |x|)$$

for all $x \in M \backslash D_\rho$ for a function $\beta \in \mathcal{KL}$.

Lemma 5: Let A_i, B_i be forward invariant sets, which are asymptotically stable for forward and backward flows of (2) in $M \setminus B_i$ and $M \setminus A_i$ respectively. Then for any $0 < \rho < \rho^i_{\max}$ there exists a locally Lipschitz continuous function $V : D_\rho \to \mathbb{R}_+$, $D_\rho = \{x \in M : |x|_{B_i} < \rho\}$ (continuous on the set B_i) such that $\alpha_1(|x|_{B_i}) \leqslant V(x) \leqslant \alpha_2(|x|_{B_i}) < 1$, $L_{f(x,0)} V(x) \geqslant \beta(V(x), |x|)$, for $\alpha_1, \alpha_2 \in \mathcal{K}$, $\beta \in \mathcal{KL}$ and a.e. $x \in D_\rho$.

Proof. For any $0 < \rho < \rho^i_{\max}$ and all $x_0 \in D_\rho \backslash B_i$ there exists $T^\rho_{x_0} \in \mathbb{R}_+$ such that $X(t, x_0, 0) \notin D_\rho$ for all $t \geqslant T^\rho_{x_0}$ (the set A_i is asymptotically stable). Then there exists $0 \leq T_{x_0} \leq T^\rho_{x_0}$ such that $\inf_{t \geq 0} |X(t, x_0, 0)|_{B_i} = \inf_{0 \leqslant t \leqslant T_{x_0}} |X(t, x_0, 0)|_{B_i}$, define

$$v(x_0) = \inf_{0 \leqslant t \leqslant T_{x_0}} |X(t, x_0, 0)|_{B_i},$$

by construction $v(x_0) \leqslant |x_0|_{B_i} < \rho$ and $v(x) = 0$ iff $x \in B_i$ To analyze continuity of the function v, consider

$$\begin{aligned}|v(x_1) - v(x_2)| &= |\inf_{0\leqslant t\leqslant T_{x_1}} |X(t, x_1, 0)|_{B_i} - \inf_{0\leqslant t\leqslant T_{x_2}} |X(t, x_2, 0)|_{B_i}| \\ &\leqslant \sup_{0\leqslant t\leqslant T} ||X(t, x_1, 0)|_{B_i} - |X(t, x_2, 0)|_{B_i}|,\end{aligned}$$

where $T = \max\{T_{x_1}, T_{x_2}\}$. Due to Lipschitz continuity of solutions of the system (2), for any compact set of initial conditions $\mathcal{E} \subset M$ and time $0 \leqslant T < +\infty$, there exist $K \in \mathbb{R}_+$, $L \in \mathbb{R}_+$ such that

$$\begin{gathered}\delta(X(t, x_1, 0), X(t, x_2, 0)) \leqslant K\,\delta(x_1, x_2), \\ ||X(t, x_1, 0)|_{B_i} - |X(t, x_2, 0)|_{B_i}| \leqslant L\,\delta(x_1, x_2),\end{gathered}$$

for all $0 \leqslant t \leqslant T$ and any $x_1, x_2 \in \mathcal{E}$. For any compact set $\mathcal{E} \subset M$ define $T_\rho = \sup_{x_0\in\mathcal{E}\cap D_\rho} T_{x_0}$, then

$$|v(x_1)-v(x_2)| \leqslant \sup_{0\leqslant t\leqslant T_\rho} ||X(t, x_1, 0)|_{B_i}-|X(t, x_2, 0)|_{B_i}| \leqslant L\,\delta(x_1, x_2)$$

for all $x_1, x_2 \in \mathcal{E}\cap D_\rho$, and the function v is locally Lipschitz continuous on the set $D_\rho\backslash B_i$ and continuous on D_ρ. The function v is not decreasing on any trajectory of the system (2), indeed for any $x_0 \in D_\rho$:

$$\begin{aligned}v(X(t, x_0, 0)) &= \inf_{0\leqslant\tau\leqslant T_{X(t,x_0,0)}} |X(\tau, X(t, x_0, 0), 0)|_{B_i} \\ &= \inf_{t\leqslant\tau\leqslant T_{x_0}} |X(\tau, x_0, 0)|_{B_i} \\ &\geqslant \inf_{0\leqslant\tau\leqslant T_{x_0}} |X(\tau, x_0, 0)|_{B_i} = v(x_0).\end{aligned}$$

Therefore, $\delta'(|x_0|_{B_i}) \leqslant v(x_0)$ for $\delta'(s) = s(1 + s)^{-1} \inf_{|x|_{B_i}=s} v(x)$, $\delta' \in \mathcal{K}$ and all $x_0 \in D_\rho$.

Now, define a new function for all $x_0 \in D_\rho$:

$$V(x_0) = \inf_{0\leqslant t\leqslant T_{x_0}} \{v(X(t, x_0, 0))k(t)\},$$

where $k : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable function with properties $0 < \kappa_1 \leqslant k(t) \leqslant \kappa_2 < +\infty$ and $\dot{k}(t) \leqslant -\kappa_3(t) < 0$ for all $t \geqslant 0$, where κ_3 is a monotonically decreasing function. An example of such a function is

$$k(t) = \kappa_1 + (\kappa_2 - \kappa_1)e^{-t},\ \dot{k}(t) = (\kappa_1 - \kappa_2)e^{-t}.$$

The function V has bounds $\kappa_1\delta'(|x_0|_{B_i}) \leqslant V(x_0) \leqslant \kappa_2|x_0|_{B_i}$ and $V(x) = 0$ for all $x \in B_i$. Next, for all $x_1, x_2 \in D_\rho$,

$$\begin{aligned}& |V(x_1) - V(x_2)| \\ = & |\inf_{0\leqslant t\leqslant T_{x_1}} \{v(X(t, x_1, 0))k(t)\} - \inf_{0\leqslant t\leqslant T_{x_2}} \{v(X(t, x_2, 0))k(t)\}| \\ = & |\inf_{0\leqslant t\leqslant T} \{v(X(t, x_1, 0))k(t)\} - \inf_{0\leqslant t\leqslant T} \{v(X(t, x_2, 0))k(t)\}| \\ \leqslant & \sup_{0\leqslant t\leqslant T} |k(t)[v(X(t, x_1, 0)) - v(X(t, x_2, 0))]| \\ \leqslant & \kappa_2 \sup_{0\leqslant t\leqslant T} |v(X(t, x_1, 0)) - v(X(t, x_2, 0))|,\end{aligned}$$

where $T = \max\{T_{x_1}, T_{x_2}\}$. For any compact set $\mathcal{E} \subset M$ define $T_\rho = \sup_{x_0\in\mathcal{E}\cap D_\rho} T_{x_0}$ as before and

$$\begin{aligned}|V(x_1) - V(x_2)| &\leqslant \kappa_2 \sup_{0\leqslant t\leqslant T_\rho} |v(X(t, x_1, 0)) - v(X(t, x_2, 0))| \\ &\leqslant \kappa_2 L\,\delta(X(t, x_1, 0), X(t, x_2, 0)) \leqslant \kappa_2 LK\,\delta(x_1, x_2)\end{aligned}$$

for all $x_1, x_2 \in \mathcal{E}\cap D_\rho$. Then the function V is locally Lipschitz continuous on the set $D_\rho\backslash B_i$ and continuous on D_ρ. This function is strictly increasing for any $x_0 \in D_\rho\backslash B_i$:

$$\begin{aligned}V(X(t, x_0, 0)) &= \inf_{0\leqslant\tau\leqslant T_{X(t,x_0,0)}} \{v(X[\tau, X(t, x_0, 0), 0])k(\tau)\} \\ &\geqslant \inf_{t\leqslant\tau\leqslant T_{x_0}} \{v(X[\tau, x_0, 0])k(\tau - t)\} \\ &> \inf_{0\leqslant\tau\leqslant T_{x_0}} \{v(X[\tau, x_0, 0])k(\tau)\} = V(x_0),\end{aligned}$$

$V(t)$ equals zero on any trajectories into the set B_i, then

$$L_{f(x_0,0)}V(x_0) = \lim_{h\to 0} h^{-1}[V(X(h, x_0, 0)) - V(x_0)] > 0$$

for a.e. $x_0 \in D_\rho\backslash B_i$. Define for some $r \in \mathbb{R}_+$ the set $G_r = \{x \in M : |x| \leq r\} \cap D_\rho$ and the time $T_r = \sup_{x_0\in G_r} T_{x_0}$. The time T_r is well defined and finite since the set G_r is compact, in addition

$$T_r \leq \varphi_0 + \varphi_1(r)$$

for all $r \in \mathbb{R}_+$ and some $\varphi_0 \in \mathbb{R}_+$, $\varphi_1 \in \mathcal{K}$. By definition

$$\begin{aligned}& V(X(h, x_0, 0)) \\ = & \inf_{0\leqslant t\leqslant T_{X(h,x_0,0)}} \{v(X[t, X(h, x_0, 0), 0])k(t)\} \\ = & \inf_{h\leqslant t\leqslant T_{|x_0|}} \{v(X[t, x_0, 0])k(t - h)\} \\ = & \inf_{h\leqslant t\leqslant T_{|x_0|}} \{v(X[t, x_0, 0])k(t)k(t)^{-1}k(t - h)\} \\ \geqslant & \inf_{h\leqslant t\leqslant T_{|x_0|}} \{v(X[t, x_0, 0])k(t)\} \inf_{h\leqslant t\leqslant T_{|x_0|}} \{k(t)^{-1}k(t - h)\} \\ \geqslant & V(x_0) \inf_{h\leqslant t\leqslant T_{|x_0|}} \{k(t)^{-1}k(t - h)\}.\end{aligned}$$

Finally,

$$\begin{aligned}& \lim_{h\to 0} h^{-1}[V(X(h, x_0, 0)) - V(x_0)] \\ \geqslant & \lim_{h\to 0} h^{-1}[V(x_0) \inf_{h\leqslant t\leqslant T_{|x_0|}} \{k(t)^{-1}k(t - h)\} - V(x_0)] \\ = & V(x_0) \lim_{h\to 0} h^{-1}[\inf_{h\leqslant t\leqslant T_{|x_0|}} \{k(t)^{-1}k(t - h)\} - 1] \\ = & V(x_0) \lim_{h\to 0} h^{-1} \inf_{h\leqslant t\leqslant T_{|x_0|}} k(t)^{-1}\{k(t - h) - k(t)\} \\ \geqslant & V(x_0) \inf_{0\leqslant t\leqslant T_{|x_0|}} k(t)^{-1} \lim_{h\to 0} h^{-1}\{k(t - h) - k(t)\} \\ = & V(x_0) \inf_{0\leqslant t\leqslant T_{|x_0|}} \{-k(t)^{-1}\dot{k}(t)\} \geqslant \kappa_2^{-1}\kappa_3(T_{|x_0|})V(x_0).\end{aligned}$$

for a.e. $x \in D_\rho$ (the inequality is additionally valid on the set B_i). Since κ_3 is a monotonically decreasing function, the following inequality has been substantiated:

$$L_{f(x,0)}V(x) \geqslant \kappa_2^{-1}\kappa_3(\varphi_0 + \varphi_1(|x|))V(x) \geq \beta(V(x), |x|)$$

for a.e. $x \in D_\rho$ for some $\beta \in \mathcal{KL}$.

Remark 4: Note that if the set A_i (or B_i) is compact, then there exists constant $m_{A_i} > 0$ ($m_{B_i} > 0$) such that for any $x \in M$ it holds $|x| \leq |x|_{A_i} + m_{A_i}$ ($|x| \leq |x|_{B_i} + m_{B_i}$) and therefore

$$\begin{gathered}DV(x)f(x, 0) \leqslant -\beta(V(x), |x|) \leq -\beta(V(x), |x|_{A_i} + m_{A_i}) \\ (DV(x)f(x, 0) \geqslant \beta(V(x), |x|) \geq \beta(V(x), |x|_{B_i} + m_{B_i}))\,.\end{gathered}$$

Therefore, we can introduce a function $\tilde{V}(x) = \int_0^{V(x)} \tilde{\sigma}(r)dr$ for a suitably defined $\tilde{\sigma} \in \mathcal{K}_\infty$ such that

$$\begin{gathered}\tilde{\alpha}_1(|x|_{A_i}) \leqslant \tilde{V}(x),\ D\tilde{V}(x)f(x, 0) \leqslant -\tilde{\beta}(\tilde{V}(x)) \\ \left(\tilde{\alpha}_1(|x|_{B_i}) \leqslant \tilde{V}(x) \leqslant \tilde{\alpha}_2(|x|_{B_i}),\ D\tilde{V}(x)f(x, 0) \geqslant \tilde{\beta}(\tilde{V}(x))\right)\end{gathered}$$

for a.e. $x \in M\backslash D_\rho$ ($x \in D_\rho$) and some $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta} \in \mathcal{K}_\infty$, see also Lemma 3 in [7].

C. Proof of Proposition 2

In the following assume, without loss of generality, that the unperturbed system $\dot{x} = f(x, 0)$ is backwards and forward complete, *viz.* its solutions are globally defined in $\mathbb{R}$. If this is not the case just replace it by the system

$$\dot{x} = \tilde{f}(x, 0) = f(x, 0)/(1 + |f(x, 0)|_x).$$

The dissipation inequality proved for $\tilde{f}$ will *a fortiori* hold for f.

Consider the following sets for any $1 \leq i \leq k$:

$$A_i = \bigcup_{j\leq i} \mathfrak{R}(\mathcal{W}_j),\ B_i = \bigcup_{i<l} \mathfrak{A}(\mathcal{W}_l).$$

We have the following properties for these sets:

A Since the sets $\mathcal{W}_i$, $1 \leq i \leq k$ create a filtration ordering of $\mathcal{W}$, then the set A_i represents the initial conditions attracted in the backward time by $\mathcal{W}_i$ and all its sinks $\mathcal{W}_j$ ($\mathcal{W}_j \prec \mathcal{W}_i$ for $j < i$). The set A_i is a union of "repulsion" sets $\mathfrak{R}(\mathcal{W}_j)$ of $\mathcal{W}_j$ with $j \leq i$; if $\mathcal{W}_i$ is a purely attracting set, that is always the case for $i = 1$ under conditions of the proposition, then $A_i = \mathcal{W}_i$. The set B_i corresponds to the initial conditions attracted by $\mathcal{W}_{i+1}$ and all its sources $\mathcal{W}_l$ ($\mathcal{W}_{i+1} \prec \mathcal{W}_l$ for $i+1 < l$) in the forward time.

B The sets A_i and B_i are forward invariant for (2) and $A_i \subset C_i$, where

$$C_i = \bigcup_{j \leq i} \mathfrak{A}(\mathcal{W}_j)$$

is a forward invariant set. Indeed, if a set $\mathcal{W}_i$ is purely attracting then $\mathfrak{R}(\mathcal{W}_i) = \mathcal{W}_i \subset \mathfrak{A}(\mathcal{W}_i) \subset C_i$. If a set $\mathcal{W}_i$ is not purely attracting, then by the definition of filtration ordering (see Definition 4) all its sinks $\mathcal{W}_j$ have $j < i$, therefore $\mathfrak{R}(\mathcal{W}_i) \subset \bigcup_{j<i} \mathfrak{A}(\mathcal{W}_j) \subset C_i$.

C Take an $x \in \text{clos}[\mathfrak{R}(\mathcal{W}_i)]$, then $X(t, x, 0)$ has its α-limit set inside some $\mathcal{W}_j$ with $j \leq i$ by the filtration ordering definition, therefore $\text{clos}[\mathfrak{R}(\mathcal{W}_i)] \subset A_i$ for any $i \geq 1$. Now let us apply these arguments to the reverse flow $X(-t, x, 0)$. We see that:

$$\begin{aligned} \text{clos}[\textstyle\bigcup_{j>i} \mathfrak{A}(\mathcal{W}_j)] &= \textstyle\bigcup_{j>i} \text{clos}[\mathfrak{A}(\mathcal{W}_j)] \\ &\subseteq \textstyle\bigcup_{j>i} \bigcup_{l \geq j} \mathfrak{A}(\mathcal{W}_l) = \bigcup_{j>i} \mathfrak{A}(\mathcal{W}_j). \end{aligned}$$

Then the set $\bigcup_{j>i} \mathfrak{A}(\mathcal{W}_j)$ is closed and the set $C_i = M \setminus \bigcup_{j>i} \mathfrak{A}(\mathcal{W}_j)$ is open.

D The set A_i is attractive for all initial conditions in an open and forward invariant set C_i. Indeed, the set $C_i \setminus A_i \subset \bigcup_{j \leq i} \mathfrak{A}(\mathcal{W}_j) \setminus \mathcal{W}_j$ and it does not contain limit invariant solutions of the system (2), therefore all trajectories initiated at $C_i \setminus A_i$ go to A_i, which contains all $\mathcal{W}_j$ for $j \leq i$. Thus for any $\varepsilon > 0$ and all $x_0 \in C_i$ there is a $0 \leq T_{x_0,\varepsilon} < +\infty$ such that $|X(t, x_0, 0)|_{A_i} \leq \varepsilon$ for all $t \geq T_{x_0,\varepsilon}$ (note that $X(t, x_0, 0) \in C_i$ for all $t \geq 0$ since C_i is invariant).

E The set A_i is locally Lagrange stable, *i.e.* for any $\delta > 0$ there is an $\epsilon > 0$ such that $|X(t, x_0, 0)|_{A_i} \leq \epsilon$ for all $t \geq 0$ and for all $x_0 \in C_i$ with $|x_0|_{A_i} \leq \delta$. Indeed, take a set $D_\rho = \{x \in C_i : |x|_{A_i} \leq \rho\}$ for some $\rho > 0$ and consider $\nu(\rho) = \sup_{x_0 \in D_\rho} \sup_{t \geq 0} |X(t, x_0, 0)|_{A_i}$. Remark that $\nu(\rho)$ is a monotone function. Assume that $\nu(\rho) < +\infty$ for any such $\rho > 0$, then the set A_i is stable with $\delta = \rho$ and $\epsilon = \nu(\rho)$. On the contrary, assume that there is a $\rho' > 0$ such that $\nu(\rho') = +\infty$, then it means that there exists a sequence of points $x'_q \in D_{\rho'}$, $q \in \mathbb{N}_+$ such that $\sup_{q \geq 0} \sup_{t \geq 0} |X(t, x'_q, 0)|_{A_i} = +\infty$. Take an $\varepsilon < \rho'$, then by attractiveness of A_i there are $0 \leq T_{x'_q,\varepsilon} < +\infty$ such that $|X(t, x'_q, 0)|_{A_i} \leq \varepsilon$ for all $t \geq T_{x'_q,\varepsilon}$. Define $T' = \sup_{q \geq 0} T_{x'_q,\varepsilon}$, let $T' < +\infty$, then the trajectories $X(t, x'_q, 0)$ leave to infinity and approach a vicinity of the set A_i in a finite time T', that is a contradiction due to the system continuity and its forward completeness. Finally, assume that $T' = +\infty$, then it means that there is a point $x_\infty \in C_i$ such that $x'_q \to x_\infty$ when $q \to +\infty$ such that $T_{x_\infty,\varepsilon/2} = +\infty$, which contradicts the set A_i attractiveness in C_i. Therefore, $\nu(\rho) < +\infty$ for any such $\rho > 0$ and the set A_i is locally Lagrange stable in C_i.

F The set A_i is locally Lyapunov stable, *i.e.* for any $\epsilon > 0$ there is a $\delta > 0$ such that $|X(t, x_0, 0)|_{A_i} \leq \epsilon$ for all $t \geq 0$ and for all $x_0 \in C_i$ with $|x_0|_{A_i} \leq \delta$. Indeed, by the definition above, $\nu(\rho_1) \leq \nu(\rho_2)$ for $\rho_1 \leq \rho_2$. In addition $\lim_{\rho \to 0} \nu(\rho) = 0$, assume it is not, then it means that there exists an $\upsilon > 0$ such that for any $\iota > 0$ there is $x_0 \in C_i$ with $|x_0|_{A_i} \leq \iota$ such that $|X(t', x_0, 0)|_{A_i} \geq \upsilon$ for some $t' \geq 0$. Since $\iota > 0$ can be chosen arbitrary, it implies that there exist some trajectories, which exit from A_i into C_i and return back to A_i (the set A_i is attractive in C_i). Therefore, the set A_i has 1-cycle, that is a contradiction and $\lim_{\rho \to 0} \nu(\rho) = 0$. Thus there is a function $\tilde{\nu} \in \mathcal{K}_\infty$ such that $\nu(\rho) \leq \tilde{\nu}(\rho)$, then the set A_i is locally Lyapunov stable in C_i: for any $\epsilon > 0$ if $x_0 \in D_{\tilde{\nu}^{-1}(\epsilon)}$ then $|X(t, x_0, 0)|_{A_i} \leq \epsilon$. Since A_i is also attractive by the consideration above, the set A_i is asymptotically stable with the domain of attraction C_i [10].

G Note that $M = \bigcup_{i=1}^k \mathfrak{A}(\mathcal{W}_i)$ and $C_i = M \setminus B_i$, therefore the set A_i is uniformly asymptotically stable with the domain of attraction $M \setminus B_i$. Applying the above arguments for the flow of (2) in the backward time (*i.e.* $X(-t, \cdot, 0)$) we can prove that B_i is asymptotically stable for the backward flow with the domain of attraction $M \setminus A_i$.

Now we need to recall Lemma 4 and Lemma 5, applying these lemmas to the sets A_i and B_i, and using the smoothing arguments from [10] (Theorem B1) or [19] for $0 < \rho_1 < \rho_2 < \rho^i_{\max}$ we may obtain two smooth functions $V_1 : M \backslash B_{\rho_1} \to \mathbb{R}_+$, $V_2 : B_{\rho_2} \to \mathbb{R}_+$ such that:

- $\alpha_1(|x|_{A_i}) \leqslant V_1(x)$ for all $x \in M \backslash B_{\rho_1}$ for $\alpha_1 \in \mathcal{K}_\infty$;
- $\alpha_2(|x|_{B_i}) \leqslant V_2(x) \leqslant \alpha_3(|x|_{B_i}) < 1$ for all $x \in B_{\rho_2}$ for $\alpha_2, \alpha_3 \in \mathcal{K}_\infty$;
- $DV_1(x) f(x, 0) \leqslant -\beta_1(V_1(x), |x|)$, $\beta_1 \in \mathcal{KL}$ for all $|x|_{B_i} \geqslant \rho_1$;
- $DV_2(x) f(x, 0) \geqslant \beta_2(V_2(x), |x|)$, $\beta_2 \in \mathcal{KL}$ for all $|x|_{B_i} \leq \rho_2$.

Note that

$$\begin{aligned} DV_1(x) f(x, 0) \leqslant -\beta_1(\alpha_1(|x|_{A_i}), |x|),\ & D[1 - V_2(x)] f(x, 0) \\ & \leqslant -\beta_2(\alpha_2(|x|_{B_i}), |x|). \end{aligned}$$

Next, it is necessary to unite the functions V_1 and V_2 using the covering property of these functions into the set $\Upsilon = \{x \in M : \rho_1 < |x|_{B_i} < \rho_2\}$. The obstacle there is that B_i can be non-compact in a general case, and the function V_1 may take unbounded values on Υ (recall that $V_1(x) \leq \alpha_4(|x| + m)$ for some $m > 0$). To avoid this issue let us introduce a semi-proper function $V_3(x) = \theta(V_1(x))$, where $\theta(s) = \int_0^s (1 + \chi(r))^{-1} dr$ is a bounded function for a suitable selected $\chi \in \mathcal{K}_\infty$, then for all $|x|_{B_i} \geqslant \rho_1$

$$\alpha_5(|x|_{A_i}) \leqslant V_3(x),\ DV_3(x) f(x, 0) \leqslant -\beta_3(|x|_{A_i}, |x|)$$

for $\alpha_5(s) = \theta \circ \alpha_1(s) \in \mathcal{K}$ and $\beta_3(s, r) = \beta_1(\alpha_1(s), r)/(1 + \chi(\alpha_4(r + m)))\in \mathcal{KL}$. There exist $v_1, v_2 \in \mathbb{R}_+$(under an appropriate rescaling of V_2) such that $\Upsilon' = \{x \in M : v_1 < V_2(x) < v_2 < 0.5\} \subset \Upsilon$, define $v_3 = \sup_{x \in \Upsilon} V_3(x)$ (by construction $v_3 < +\infty$) and take a smooth function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(s) = 0$ for $s \leq v_1$, $\phi(s) = 1$ for $s \geq v_2$ and $d\phi(s)/ds > 0$ for $s \in (v_1, v_2)$ then $L_i(x) = v_4 \phi(V_2(x)) V_3(x) + (1 - \phi(V_2(x)))(1 - V_2(x))$ with $v_4 = (0.5 - v_2) v_3^{-1} > 0$ is a smooth Lyapunov function for the system. Indeed, $L_i(x) = v_4 V_3(x)$ for $V_2(x) \geq v_2$ and $L_i(x) = 1 - V_2(x)$ for $V_2(x) \leq v_1$, while on the set Υ' we have

$$\begin{aligned} & DL_i(x) f(x, 0) \\ = \;& v_4 \left.\frac{d\phi(s)}{ds}\right|_{s=V_2(x)} V_3(x) DV_2(x) f(x, 0) + v_4 \phi(V_2(x)) DV_3(x) f(x, 0) \\ & - \left.\frac{d\phi(s)}{ds}\right|_{s=V_2(x)} (1 - V_2(x)) DV_2(x) f(x, 0) \\ & -(1 - \phi(V_2(x))) DV_2(x) f(x, 0) \\ = \;& \left.\frac{d\phi(s)}{ds}\right|_{s=V_2(x)} [v_4 V_3(x) - 1 + V_2(x)]\, DV_2(x) f(x, 0) \\ & + v_4 \phi(V_2(x)) DV_3(x) f(x, 0) - (1 - \phi(V_2(x))) DV_2(x) f(x, 0) \end{aligned}$$

and all terms are negative in the last expression since $v_4 V_3(x) - 1 + V_2(x) \leq v_4 v_3 - 1 + v_2 = -0.5$. Thus we have proven the following auxiliary result.

Lemma 6: Let A_i, B_i be forward invariant sets asymptotically stable for forward and backward flows of (2) in $M \setminus B_i$ and $M \setminus A_i$ respectively. Then there exists a smooth function $L_i : M \to \mathbb{R}_+$ such that

- $\alpha'(|x|_{A_i}) \leq L_i(x)$ for all $x \in M$ and some $\alpha' \in \mathcal{K}$;
- $L_i^{-1}[0] = A_i$, $B_i \subset L_i^{-1}[1]$, $DL_i(x)f(x,0) = 0$ for all $x \in G_i = A_i \cup B_i$;
- $DL_i(x)f(x,0) \leq -\beta'(|x|_{G_i}, |x|)$ for all $x \in M$ and some $\beta' \in \mathcal{KL}$.

Finally to prove Proposition 2 select $L(x) = \Sigma_{i=1}^{k-1} L_i(x)$, which fulfils all our requirements. Indeed, $\bigcap_{i=1}^{k} A_i \subset \mathcal{W}$ and $\mathcal{W} = \bigcap_{i=1}^{k} A_i \cup \bigcap_{i=1}^{k} B_i$, then there are functions $\upsilon \in \mathcal{K}$ and $\varpi \in \mathcal{KL}$ such that

$$\upsilon(|x|_{\mathcal{W}}) \leq L(x),\ DL(x)f(x,0) \leq -\varpi(|x|_{\mathcal{W}}, |x|)$$

for all $x \in M$, while the sets $\mathcal{W}_i$ contain critical points of L and belong to different constant levels of L. Since the set $\mathcal{W}$ is compact there is $y \in \mathbb{R}_+$ such that $|x| \leq |x|_{\mathcal{W}} + y$, then $DL(x)f(x,0) \leq -\varpi'(|x|_{\mathcal{W}})$ for all $x \in M$ and some positive definite function ϖ'.

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