A geometric characterisation of the persistence of excitation condition for sequences generated by discrete-time autonomous systems

Alberto Padoan, Giordano Scarciotti and Alessandro Astolfi

Abstract— **The persistence of excitation condition for sequences generated by time-invariant, discrete-time, autonomous linear and nonlinear systems is studied. A rank condition is shown to be equivalent to the persistence of excitation of sequences generated by the class of systems considered, consistently with the results established by the authors for the continuous-time case. The condition is geometric in nature and can be checked *a priori* for a Poisson stable system, that is, without knowing explicitly the state trajectories of the system. The significance of the ideas and tools presented is illustrated by means of simple examples.**

I. Introduction

The notion of persistence of excitation plays a pivotal role in experiment design [1 – 13]. Historically, the notion of persistence of excitation of a sequence has been introduced in [14] to address the problem of estimating the parameters of a time-invariant, discrete-time, single-input, single-output system described by a linear ordinary difference equation. During the past fifty years an ever-increasing variety of applications of the notion of persistence of excitation have flourished in different fields. Persistently exciting sequences are especially important in the solution of least squares minimization problems arising in the context of system identification and are strongly related to the possibility of devising informative experiments which involve the use of input-output data [1 – 4]. The persistence of excitation condition is also central in the study of the stability properties of specific classes of nonlinear systems which, in turn, allow to establish convergence properties of adaptive control algorithms [5 – 13]. For an in-depth discussions on the numerous applications of the notion of persistence of excitation the reader may consult, for example, [1 – 13] and references therein.

In system identification and adaptive control, the notion of persistence of excitation of a sequence is defined as the uniform positive-definiteness of the matrix given by the area of the sequence times its transpose over every interval of a given length [1 – 13]. For a deterministic stationary sequence the persistence of excitation condition corresponds to the positive-definiteness of its covariance function at the origin [2]. Equivalently, a deterministic stationary sequence is persistently exciting if its spectrum contains a sufficiently large number of harmonics [15].

The main difficulty with testing the persistence of excitation condition is the requirement of the explicit knowledge of the given sequence. To circumvent this shortcoming in many situations the persistence of excitation condition is simply assumed to be satisfied *a priori*.

The present work builds upon and extends to the class of time-invariant, discrete-time, autonomous systems the results established by the authors in [16], where the so-called "excitability rank condition" has been shown to be equivalent to the persistence of excitation of signals generated by the class of time-invariant, continuous-time, autonomous systems. A discrete-time counterpart of the excitability rank condition introduced in [16] is defined herein and it is shown to be equivalent to the persistence of excitation condition. As a consequence of our results, when the sequence is generated by a system described by *linear* ordinary difference equations, checking the persistence of excitation condition boils down to knowing the initial condition and the position of the eigenvalues of the matrix which describes the evolution of the system. However, for systems described by *nonlinear* ordinary difference equations nothing can be said beforehand unless special properties of the system are known. To remedy to this issue we show that for a Poisson stable system the excitability rank condition needs to be checked only at the initial condition, without knowing explicitly the state trajectories of the system. As already stressed above, ideas and tools presented herein and in the companion paper [16] may be of interest for input design or in other data-driven applications, such as the problem of output regulation [17 , 18].

The rest of the paper is organized as follows. Section II provides elementary definitions. Section III contains our main results, which give the discrete-time counterpart of the results established in [16]. Section IV presents simple examples which illustrate the theoretical results developed. Finally, conclusions and future research directions are outlined in Section V.

Notation: $\mathbb{Z}_{\geq 0}$ (resp. $\mathbb{Z}_{>0}$) denotes the set of non-negative (resp. positive) integer numbers. $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{p\times m}$ denote the set of real numbers, of n-dimensional vectors with real entries and of $p \times m$-dimensional matrices with real entries, respectively. $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{R}_{>0}$) denotes the set of non-negative (resp. positive) real numbers. $\mathbb{C}$ denotes the set of complex numbers. i denotes the imaginary unit. I denotes the identity matrix. $\sigma(S)$ denotes the spectrum of the matrix $S \in \mathbb{R}^{\nu\times\nu}$. M' denotes the transpose of the matrix $M \in \mathbb{R}^{p\times m}$. $\|\omega\|$ denotes the standard Euclidean norm of

A. Padoan and G. Scarciotti are with the Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, UK. (Emails: alberto.padoan13@imperial.ac.uk, giordano.scarciotti10@imperial.ac.uk). A. Astolfi is with the Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, UK and with the Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata", Via del Politecnico 1, 00133, Rome, Italy. (Email: a.astolfi@imperial.ac.uk).

the vector $\omega \in \mathbb{R}^\nu$. $\operatorname{span} V$ denotes the set of all finite $\mathbb{R}$-linear combinations of elements of the set V. $s_1 \circ s_2$ denotes the composition of the maps s_1 and s_2, provided it is well-defined. The iterates of the map s are defined recursively as $s^{k+1} = s \circ s^k$ for all $k \in \mathbb{Z}_{\geq 0}$, with s^0 the identity map. No confusion should arise when superscripts are used for powers of matrices, since the meaning of the notation is clear from the context. The positive orbit of the map $s : \mathbb{R}^\nu \to \mathbb{R}^\nu$ passing through $\omega \in \mathbb{R}^\nu$ at time $k = 0$ is denoted by $\gamma_+(\omega)$, *i.e.* $\gamma_+(\omega) = \{s^k(\omega) : k \in \mathbb{Z}_{\geq 0}\}$.

II. Preliminaries

Consider a discrete-time, autonomous, nonlinear system described by equations of the form

$$\omega^+ = s(\omega), \tag{1}$$

in which $\omega(k) \in W$ denotes the state of the system at time $k \in \mathbb{Z}_{\geq 0}$, W is an open subset of $\mathbb{R}^\nu$ containing the origin and ω^+ denotes the successive state of ω. Without loss of generality suppose that the map $s : W \to W$ is continuous, that the set W is invariant under s, *i.e.* $s(W) \subset W$, and that the origin is an equilibrium point for the system, *i.e.* $s(0) = 0$.

Define recursively the maps

$$\theta_{k+1} : W \to W : \omega \mapsto \theta_k \circ s(\omega), \quad k \in \mathbb{Z}_{\geq 0},$$

with θ_0 the identity map. If $\omega(0) = \omega_0$ is the initial condition of the ordinary difference equation (1) then

$$\omega(k) = \theta_k(\omega_0), \quad k \in \mathbb{Z}_{\geq 0}. \tag{2}$$

The sequence defined by (2) is referred to as the state sequence of system (1).

With these premises, the definitions of excitation space, excitation distribution and excitation rank condition for a system governed by equations of the form (1) are similar to the ones given in [16].

Definition 1: Consider the system (1). The *excitation space* of system (1) is defined as $\mathcal{E} = \operatorname{span}\{\theta_k,\ k \in \mathbb{Z}_{\geq 0}\}$.

Definition 2: Consider the system (1). The *excitation distribution* of system (1) is defined as $E(\omega) = \operatorname{span}\{\theta_k(\omega),\ k \in \mathbb{Z}_{\geq 0}\}$ for all $\omega \in W$.

Definition 3: Consider the system (1) and the corresponding excitation distribution E. The system (1) is said to satisfy the *excitation rank condition* at $\omega \in W$ if $\dim E(\omega) = \nu$. The pair (s, ω) is said to be *exciting* if system (1) satisfies the excitation rank condition at $\omega \in W$.

We conclude this preliminary section giving the following notion of persistence of excitation (see, *e.g.*, [1] and references therein).

Definition 4: A sequence $\omega : \mathbb{Z}_{\geq 0} \to \mathbb{R}^\nu$ is *persistently exciting* if there exists a constant $N \in \mathbb{Z}_{>0}$ such that the matrix

$$\mathcal{W}_{[k,k+N-1]} = \sum_{l=k}^{k+N-1} \omega(l)\omega(l)' \tag{3}$$

is positive definite for all $k \in \mathbb{Z}_{\geq 0}$.

For convenience, we use $\mathcal{W}_{[k,k+N-1]}(s, \omega_0)$ to denote the matrix defined in (3) when the sequence $\{\omega(k)\}_{k \in \mathbb{Z}_{\geq 0}}$ is generated by the system (1) with $\omega(0) = \omega_0$, to stress the dependence of such a matrix on the map which describes the system and on the initial condition.

III. Main results

To establish our main results we first consider the case of linear systems and subsequently discuss the extension to nonlinear systems.

A. Linear systems

Consider a discrete-time, autonomous, linear system described by equations of the form

$$\omega^+ = S\omega, \tag{4}$$

in which $\omega(k) \in \mathbb{R}^\nu$ and $S \in \mathbb{R}^{\nu \times \nu}$ is a constant matrix.

The following statement formalises a first geometric characterisation of the persistence of excitation condition.

Theorem 1: Consider the system (4) with initial condition $\omega(0) = \omega_0$. The following statements are equivalent.

- (L1-D) The state sequence of the system is persistently exciting.
- (L2-D) There exists a constant $N \in \mathbb{Z}_{>0}$ such that the matrix $\mathcal{W}_{[0,N-1]}(S, \omega)$ is positive definite for every $\omega \in \gamma_+(\omega_0)$.
- (L3-D) The system (4) satisfies the excitation rank condition at every $\omega \in \gamma_+(\omega_0)$.

Proof: (L1-D) $\Leftrightarrow$ (L2-D). By definition, the state sequence of system (4) is persistently exciting if there exists a constant $N \in \mathbb{Z}_{>0}$ such that the matrix $\mathcal{W}_{[k,k+N-1]}(S, \omega_0)$ is positive definite for all $k \in \mathbb{Z}_{\geq 0}$. Replacing the summation variable by $h = l - k$ in (3) and taking into account that the solution of the ordinary difference equation (4) is $\omega : k \to S^k \omega_0$ yields

$$\mathcal{W}_{[0,N-1]}(S, \omega(k)) = \sum_{h=0}^{N-1} S^h \omega(k)(S^h \omega(k))',$$

for all $k \in \mathbb{Z}_{\geq 0}$. Since the sequence $\{\omega(k)\}_{k \in \mathbb{Z}_{\geq 0}}$ uniquely specifies the positive orbit $\gamma_+(\omega_0)$, the state sequence of system (4) is persistently exciting if and only if there exists a constant $N \in \mathbb{Z}_{>0}$ such that the matrix $\mathcal{W}_{[0,N-1]}(S, \omega)$ is positive definite for every $\omega \in \gamma_+(\omega_0)$.

(L2-D) $\Leftrightarrow$ (L3-D). To prove this equivalence note that the matrix $\mathcal{W}_{[0,N-1]}(S, \omega)$ is the finite reachability Gramian associated with the pair (S, ω) at time $N \in \mathbb{Z}_{>0}$ (see, *e.g.*, [19, Definition 4.9]). Thus, the claim is a direct consequence of a standard result of linear systems theory [19, Corollary 4.11]. ■

The significance of Theorem 1 is that the state of the system (4) is persistently exciting if and only if the pair (S, ω) is exciting for every point ω which belongs to the positive orbit $\gamma_+(\omega_0)$. As a consequence, the state sequence of system (4) is persistently exciting if the state trajectories are known to lie entirely in a region where the excitability rank condition is satisfied. Note that an objection can be raised: it is not always possible to check the validity of (L2) or (L3) for every $\omega \in \gamma_+(\omega_0)$ to conclude that the state sequence of system (4) is persistently exciting, thus making the verification practically infeasible. To resolve this issue a more applicable result can be established by strengthening the hypotheses of Theorem 1 by using the following assumption.

Assumption 1: The eigenvalues of S have unitary modulus and algebraic multiplicity one.

Theorem 2: Consider the system (4) with initial condition $\omega(0) = \omega_0$. Suppose that Assumption 1 holds. The following statements are equivalent.

(L1-D)* The state sequence of the system is persistently exciting.

(L2-D)* There exists a constant $N \in \mathbb{Z}_{>0}$ such that the matrix $\mathcal{W}_{[0,N-1]}(S, \omega_0)$ is positive definite.

(L3-D)* The system (4) satisfies the excitation rank condition at ω_0.

Proof: (L1-D)* $\Rightarrow$ (L2-D)*. This implication is trivial.

(L2-D)* $\Rightarrow$ (L1-D)*. By definition, the state sequence of system (4) is persistently exciting if there exists a constant $N \in \mathbb{Z}_{>0}$ such that the matrix $\mathcal{W}_{[k,k+N-1]}(S, \omega_0)$ is positive definite for all $k \in \mathbb{Z}_{\geq 0}$. Replacing the summation variable by $h = l - k$ in (3) and taking into account that the solution of the ordinary difference equation (4) is $\omega : k \to S^k\omega_0$ yields

$$\begin{aligned}\mathcal{W}_{[k,k+N-1]}(S, \omega_0) &= S^k \left(\sum_{l=0}^{N-1} S^l \omega_0 (S^l \omega_0)' \right) (S^k)' \\ &= S^k \mathcal{W}_{[0,N-1]}(S, \omega_0)(S^k)'.\end{aligned}$$

Selecting $N \in \mathbb{Z}_{>0}$ such that the matrix $\mathcal{W}_{[0,N-1]}(S, \omega_0)$ is positive definite gives

$$\mathcal{W}_{[k,k+N-1]}(S, \omega_0) \geq \alpha S^k (S^k)',$$

with $\alpha \in \mathbb{R}_{>0}$ the smallest singular value of the matrix $\mathcal{W}_{[0,N-1]}(S, \omega_0)$. Taking into account that by Assumption 1 there exists $\beta \in \mathbb{R}_{>0}$ such that

$$S^k (S^k)' \geq \beta I,$$

for every $k \in \mathbb{Z}_{\geq 0}$, the inequality

$$\mathcal{W}_{[k,k+N-1]}(S, \omega_0) \geq \alpha\beta I,$$

holds for every $k \in \mathbb{Z}_{\geq 0}$ and hence the claim.

(L2-D)* $\Leftrightarrow$ (L3-D)*. The proof of this equivalence is a direct consequence of Theorem 1 and hence it is omitted. ■

B. Nonlinear systems

The arguments used in the linear case can be extended to nonlinear systems. To begin with, we establish the following result.

Theorem 3: Consider the system (1) with initial condition $\omega(0) = \omega_0$. The following statements are equivalent.

(NL1-D) The state sequence of the system is persistently exciting.

(NL2-D) There exists a constant $N \in \mathbb{Z}_{>0}$ such that the matrix $\mathcal{W}_{[0,N-1]}(s, \omega)$ is positive definite for every $\omega \in \gamma_+(\omega_0)$.

(NL3-D) The system (1) satisfies the excitation rank condition at every $\omega \in \gamma_+(\omega_0)$.

Proof: (NL1-D) $\Leftrightarrow$ (NL2-D).

The proof follows from the same steps of the proof of the equivalence (L1-D) $\Leftrightarrow$ (L2-D), with the map s playing the role of the matrix S.

(NL2-D) $\Leftrightarrow$ (NL3-D). To prove this equivalence select $\omega \in \gamma_+(\omega_0)$ and define for every $k \in \mathbb{Z}_{>0}$ the matrix

$$\Theta_k(s, \omega) = \left[\begin{array}{cccc} \theta_0(\omega) & \theta_1(\omega) & \cdots & \theta_{k-1}(\omega) \end{array} \right].$$

By definition, the system (1) satisfies the excitation rank condition at ω if there exists an integer constant $N > 0$ such that $\operatorname{rank} \Theta_N(s, \omega) = \nu$. This implies that system (1) satisfies the excitation rank condition at ω if and only if the matrix

$$\Theta_N(s, \omega)\Theta_N(s, \omega)' = \mathcal{W}_{[0,N-1]}(s, \omega)$$

is positive definite. Hence, since $\omega \in \gamma_+(\omega_0)$ is arbitrary, the claim is proved. ■

Remark 1: For a linear system described by equations of the form (4), by the Cayley-Hamilton theorem [20], to check the excitation rank condition one only needs to verify the linear independence of a number of vectors equal to the dimension of the system. On the contrary, for a nonlinear system described by equations of the form (1) one may be forced to check the linear independence of the infinitely many vectors which span the excitation distribution. Thus, while for linear systems it is possible to prove that the state sequence is *not* persistently exciting, for nonlinear systems this is a more difficult issue.

Before proceeding to the next result, a preliminary definition is borrowed from [21, Definition 4.1].

Definition 5: Consider the system (1). A point $\omega \in W$ is said to be *positively Poisson stable* if for every arbitrarily small neighbourhood $\mathcal{N}$ of ω and for every constant $N \in \mathbb{Z}_{\geq 0}$ there exists an integer constant $k > N$ such that $s^k(\omega) \in \mathcal{N}$.

Remark 2: For a discrete-time, autonomous, linear system governed by equations of the form (4), the Assumption 1 given in Theorem 2 implies that all state trajectories of the system are periodic and therefore that every initial condition is positively Poisson stable.

Remark 3: Consider the system (4) with initial condition $\omega(0) = \omega_0$. As a direct consequence of Definition 5, if $\omega_0 \in W$ is a positively Poisson stable point for system (1), then there exists an increasing sequence of non-negative integers $\{j_k\}_{k\in\mathbb{Z}_{\geq 0}}$ such that

(i) $\lim_{k\to\infty} j_k = \infty$,

(ii) $\lim_{k\to\infty} s^{j_k}(\omega_0) = \omega_0$.

To state a nonlinear counterpart of Theorem 2, a technical assumption is made on the sequence $\{j_k\}_{k\in\mathbb{Z}_{\geq 0}}$ described in Remark 3.

Assumption 2: For every sequence $\{j_k\}_{k\in\mathbb{Z}_{\geq 0}}$ that possesses the properties (i) and (ii) there exists a constant $N_1 \in \mathbb{Z}_{>0}$ such that the condition

(iii) $j_{k+1} - j_k \leq N_1$ holds for all $k \in \mathbb{Z}_{>0}$.

Remark 4: The restriction described by Assumption 2 lies in the requirement that $\{j_k\}_{k\in\mathbb{Z}_{\geq 0}}$ can be always chosen so

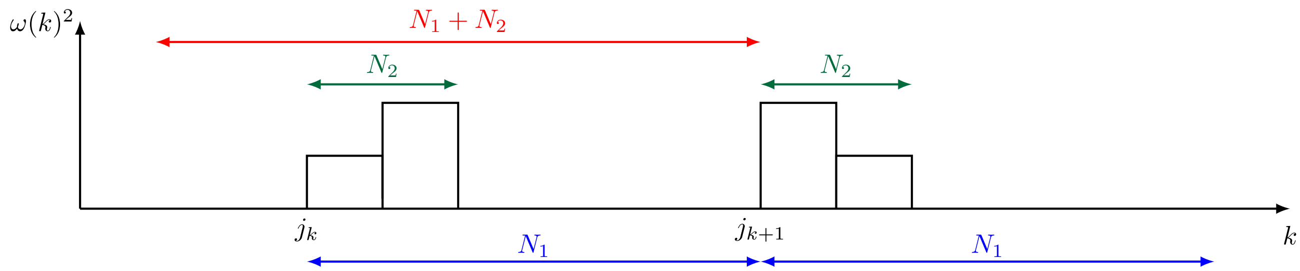

Fig. 1. Diagrammatic illustration of the proof of Theorem 4 for $\nu = 1$.

that condition (iii) holds. This latter condition expresses the fact that the distance between two consecutive time instants $\{j_k\}_{k\in\mathbb{Z}_{\geq 0}}$ of the subsequence $\{s^{j_k}(\omega_0)\}_{k\in\mathbb{Z}_{\geq 0}}$ is bounded by a fixed constant. Condition (iii) is satisfied in cases of practical interest, such as linear systems, periodic systems, almost periodic systems and systems with attractive limit cycles, and is violated if the time required to return to a given neighbourhood of the point ω_0 grows unbounded with time. The authors conjecture that the time-invariance property of the class of systems considered prevents the latter behaviour, but are not aware of results of this nature. The described behaviour, which can occur when dealing with time-varying systems, is further explored in Section IV.

We are now in a position to establish the following result.

Theorem 4: Consider the system (4) with initial condition $\omega(0) = \omega_0$. Suppose that the point ω_0 is positively Poisson stable and that Assumption 2 holds. The following statements are equivalent.

(NL1-D)* The state sequence of the system is persistently exciting.

(NL2-D)* There exists a constant $N \in \mathbb{Z}_{>0}$ such that the matrix $\mathcal{W}_{[0,N-1]}(s,\omega_0)$ is positive definite.

(NL3-D)* The system (1) satisfies the excitation rank condition at ω_0.

Proof: (NL1-D)* ⇔ (NL2-D)*. We only show the implication (NL2-D)* ⇒ (NL1-D)*, since the converse is trivial.

By hypothesis, the point ω_0 is positively Poisson stable and thus, in view of Assumption 2, there exist a constant $N_1 \in \mathbb{Z}_{>0}$ and an increasing sequence of non-negative integers $\{j_k\}_{k\in\mathbb{Z}_{\geq 0}}$ satisfying

$$\lim_{k\to\infty} j_k \to \infty,$$

$$\lim_{k\to\infty} s^{j_k}(\omega_0) = \omega_0,$$

and

$$j_{k+1} - j_k \leq N_1 \quad \text{for all} \quad k \in \mathbb{Z}_{\geq 0}.$$

Let $N_2 \in \mathbb{Z}_{>0}$ be such that the matrix $\mathcal{W}_{[0,N_2-1]}(s,\omega_0)$ is positive definite. By continuity of $\omega \mapsto \mathcal{W}_{[0,N_2-1]}(s,\omega)$ it follows that

$$\lim_{k\to\infty} \mathcal{W}_{[0,N_2-1]}(s, s^{j_k}(\omega_0)) = \mathcal{W}_{[0,N_2-1]}(s,\omega_0). \tag{5}$$

Since the right-hand side of (5) is positive definite by assumption, there exists a constant $N_3 \in \mathbb{Z}_{\geq 0}$ such that for every $j_k \geq N_3$ the inequality

$$\mathcal{W}_{[0,N_2-1]}(s, s^{j_k}(\omega_0)) = \sum_{l=0}^{N_2-1} s^{l+j_k}(\omega_0)s^{l+j_k}(\omega_0)' > 0 \tag{6}$$

holds. Observe that adding positive semi-definite terms to a positive definite matrix preserves the positive definiteness property. As a consequence, as illustrated in Figure 1, since by assumption the distance between two consecutive time instants of the sequence $\{j_k\}_{k\in\mathbb{Z}_{\geq 0}}$ is at most N_2, (6) implies that for every $k \geq N_3$ the sum of all terms of the form $s^k(\omega_0)s^k(\omega_0)'$ over a moving window of length $N_1 + N_2$ is positive definite, *i.e.*

$$\sum_{l=k}^{k+N_1+N_2-1} s^l(\omega_0)s^l(\omega_0)' > 0, \quad k \geq N_3. \tag{7}$$

This implies that the inequality

$$\sum_{l=k}^{k+N_1+N_2+N_3-1} s^l(\omega_0)s^l(\omega_0)' > 0$$

is satisfied for every $k \in \mathbb{Z}_{\geq 0}$. Selecting $N = N_1 + N_2 + N_3$ yields that the matrix $\mathcal{W}_{[k,k+N-1]}(s,\omega_0)$ is positive definite for every $k \in \mathbb{Z}_{\geq 0}$, which proves the claim.

(NL2-D)* ⇔ (NL3-D)*. The proof of this equivalence is a direct consequence of Theorem 3 and hence it is omitted. ■

Remark 5: The authors believe that Assumption 2 is not necessary to prove Theorem 4. This conjecture is supported by experimental evidence and is examined further by means of an academic example in Section IV.

IV. Examples

This section offers simple examples which illustrate the notions and results presented in Sections II and III.

A. Linear systems

Consider a discrete-time, autonomous, linear system described by equations of the form (4) with $\omega(k) \in \mathbb{R}^3$ and $S \in \mathbb{R}^{3\times 3}$ a constant matrix defined as

$$S = \begin{bmatrix} \frac{1}{4}(1+3\kappa_1) & \frac{\sqrt{3}}{4}(1-\kappa_1) & -\frac{\sqrt{3}}{2}\kappa_2 \\ \frac{\sqrt{3}}{4}(1-\kappa_1) & \frac{1}{4}(3+\kappa_1) & \frac{1}{2}\kappa_2 \\ \frac{\sqrt{3}}{2}\kappa_2 & -\frac{1}{2}\kappa_2 & \kappa_1 \end{bmatrix},$$

in which $\kappa_1 = \cos\psi$, $\kappa_2 = \sin\psi$ and $\psi = \frac{1}{\pi}$. Assume $\omega(0) = \omega_0 \neq 0$.

A direct computation shows that $\sigma(S)=\{1,\pm i\psi\}$ and thus Assumption 1 holds. This implies that the matrix S is orthogonal and hence the norm of the state trajectory of the system is constant. As a result the positive orbit $\gamma_+(\omega_0)$ lies entirely on a circle centred at the origin, the radius of which depends upon the initial condition. It is interesting to note that although the state trajectory of the system is not periodic, since $\omega_0 \neq 0$ and $\psi = \frac{1}{\pi}$ is irrational, the positive orbit $\gamma_+(\omega_0)$ is a dense subset of the circle mentioned above and hence the initial condition ω_0 is a positively Poisson stable point.

Suppose $\omega_0 = \frac{1}{2\sqrt{2}}[\,1\ \sqrt{3}\ 2\,]'$. A direct computation shows that the matrix

$$\Theta_3(S,\omega_0) = \left[\ \omega_0 \quad S\omega_0 \quad S^2\omega_0\ \right] \tag{8}$$

is non-singular. This implies that the dimension of the excitation distribution of the system is maximal and thus that the pair (S,ω_0) is exciting. By Theorem 2, the state sequence of the system is therefore persistently exciting. Conversely, assume $\omega_0 = [\,0\ 0\ 1\,]'$. Since in this case the matrix $\Theta_3(S,\omega_0)$, defined as in (8), can be shown to be singular, by the same argument used above and by Remark 1 the pair (S,ω_0) is not exciting. Therefore, in virtue of Theorem 2, the state sequence of the system is not persistently exciting.

The persistence of excitation condition (3) has been numerically verified for the state trajectory of the considered system and the simulations led to consistent results.

B. An academic example

The next academic example has exploratory intentions, as the system considered falls out of the class of systems studied in this paper. The goal of the example is twofold. First, to support the claim that, for a system with positively Poisson stable initial condition, Assumption 2 may be violated only when the system is described by a time-varying difference equation. Second, consistently with what observed in Remark 5, to show that Assumption 2 is presumably not needed to prove that a system with positively Poisson stable initial condition produces a persistently exciting state trajectory. The phenomenon studied in the example is motivated and partly inspired by [22, Section VII].

Consider a discrete-time, autonomous, time-varying, linear system described by an equation of the form[1]

$$\zeta^+ = e^{\frac{i\pi}{k}}\zeta, \tag{9}$$

in which $\zeta(k) \in \mathbb{C}$ denotes the state of the system at time $k \in \mathbb{Z}_{>0}$. Assume $\zeta(0) = 1$.

System (9) models the evolution of a discrete-time system on the unit circle with an oscillatory behaviour and a period that gradually tends to infinity (see Remark 4). In fact, representing equation (9) in polar form, the equations which govern the evolution of the state of the system can be rewritten as

$$\rho^+ = \rho, \quad \varphi^+ = \varphi + \frac{\pi}{k}, \tag{10}$$

in which $\rho(k) \in \mathbb{R}_{>0}$ and $\varphi(k) \in (-\pi,\pi]$ denote modulus and principal value of the argument of the state of the system (9) for all $k \in \mathbb{Z}_{\geq 0}$, respectively, with the usual convention that two angles are identified if they differ by an integer multiple of 2π. The first equation in (10) implies that the state trajectory lies entirely on the unit circle, as the initial condition has unitary modulus and the modulus is constant. The second equation in (10) yields

$$\varphi(k) = \pi H_k, \tag{11}$$

for all $k \in \mathbb{Z}_{>0}$, in which

$$H_k = \sum_{l=1}^{k} \frac{1}{l},$$

with $k \in \mathbb{Z}_{>0}$, is the k-th harmonic number [24, p.75]. Since

$$\lim_{k\to\infty} (H_{k-1} - \ln k) = \gamma,$$

in which $\gamma \approx 0.57721\ldots$ is the Euler-Mascheroni constant [24, p.119], the identity (11) implies that the argument has an asymptotic logarithmic growth. As a result, every point of the unit circle is positively Poisson stable for the system and the oscillatory behaviour gradually "slows down" as $k \to \infty$. Figure 2 displays the principal value of the argument φ of the state of system (9) when the initial condition is $\zeta(0) = 1$ or, what is the same, $\varphi(0) = 0$. To emphasise that the state of the system visits every fixed open subset of the unit circle with a period that increases logarithmically with time, the abscissae are plotted in linear scale (top) and in logarithmic scale (bottom). Nevertheless, since at each time instant the state sequence of system (9) lies on the unit circle, the identity

$$\sum_{l=k}^{k+N-1} |\zeta(l)|^2 = N \tag{12}$$

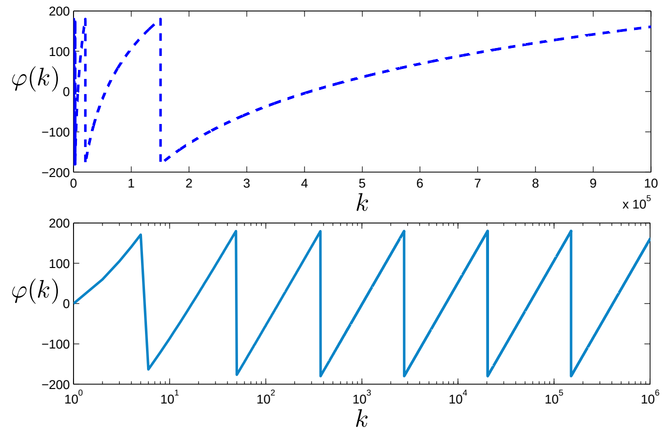

Fig. 2. Time history of the principal value of the argument $\varphi(k)$ of the state of system (9) with $\varphi(0) = 0$. The abscissae are in linear scale (top) and in logarithmic scale (bottom).

[1] $z^{1/k}$, with $k \in \mathbb{Z}_{>0}$, denotes the principal value of the k-th square root of $z \in \mathbb{C}$ [23]. The choice of representing system (9) on the complex plane is without loss of generality, since every point of $\mathbb{C}$ can be uniquely identified with a point of $\mathbb{R}^2$.

holds for every $N \in \mathbb{R}_{>0}$ and thus, in a somewhat broad sense[2], is a persistently exciting sequence.

As already mentioned, since the difference equation (9) is time-varying, the system *does not* belong to the class of systems considered in this work. However, system (9) provides an example which serves to corroborate the conjecture that even if condition (iii) of Assumption 2 is not satisfied a system with positively Poisson stable initial condition should be able to produce a persistently exciting state sequence.

V. Conclusion

The persistence of excitation of sequences generated by time-invariant, discrete-time, autonomous linear and nonlinear systems has been studied. Notions and results presented in [16] for continuous-time systems have been extended to the case of discrete-time systems. The significance of the proposed geometric characterisation has been illustrated by means of simple examples. Future work should explore the applicability of the excitation rank condition in control-related problems, including that of data-driven output regulation [17, 18]. Another important future research endeavour should determine whether or not the conjectures made on Assumption 2 hold.

References

[1] T. D. Söderström and P. G. Stoica, *System identification*. Upper Saddle River, NJ: Prentice Hall, 1989.

[2] L. Ljung, *System identification - Theory for the user (2nd edition)*. Upper Saddle River, NJ: Prentice Hall, 1999.

[3] P. Van Overschee and B. De Moor, *Subspace identification for linear systems: theory - implementation - applications*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1996.

[4] T. Katayama, *Subspace methods for system identification*. London, U.K.: Springer, 2006.

[5] A. P. Morgan and K. S. Narendra, "On the stability of nonautonomous differential equations $\dot{x} = (A + B(t))x$ with skew symmetric matrix $B(t)$," *SIAM J. Control Optim.*, vol. 15, no. 1, pp. 163–176, 1977.

[6] S. Sastry and M. Bodson, *Adaptive control: stability, convergence and robustness*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[7] K. S. Narendra and A. M. Annaswamy, *Stable adaptive systems*. Upper Saddle River, NJ: Prentice-Hall, 1989.

[8] K. J. Åström and B. Wittenmark, *Adaptive control (2nd edition)*. Reading, MA: Addison-Wesley, 1995.

[9] P. A. Ioannou and J. Sun, *Robust adaptive control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[10] H. K. Khalil, *Nonlinear systems (3rd edition)*. Upper Saddle River, N.J.: Prentice Hall, 1996.

[11] G. Tao, *Adaptive control design and analysis*. New York: Wiley-IEEE Press, 2003.

[12] E. Panteley, A. Loría, and A. R. Teel, "Relaxed persistency of excitation for uniform asymptotic stability," *IEEE Trans. Autom. Control*, vol. 46, no. 12, pp. 1874–1886, 2001.

[13] A. Loría, E. Panteley, D. Popović, and A. R. Teel, "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 183–198, 2005.

[14] K. J. Åström and T. Bohlin, "Numerical identification of linear dynamic systems from normal operating records," in *Proc. 2nd IFAC Symp. Self-adaptive Systems*, Teddington, U.K., 1965.

[15] L. Ljung, "Characterization of the concept of 'persistently exciting' in the frequency domain," Div. Autom. Control, Lund Inst. Tech., Lund, Sweden, Tech. Rep. 7119, 1971.

[16] A. Padoan, G. Scarciotti, and A. Astolfi, "A geometric characterisation of persistently exciting signals generated by autonomous systems," *10th IFAC Symp. Nonlinear Control Syst.*, Monterey, CA, USA, 2016, submitted.

[17] Y. Jiang and Z. P. Jiang, "Computational adaptive optimal control for continuous-time linear systems with completely unknown dynamics," *Automatica*, vol. 48, no. 10, pp. 2699–2704, 2012.

[18] T. Bian, Y. Jiang, and Z. P. Jiang, "Adaptive dynamic programming and optimal control of nonlinear nonaffine systems," *Automatica*, vol. 50, no. 10, pp. 2624–2632, 2014.

[19] A. C. Antoulas, *Approximation of large-scale dynamical systems*. Philadelphia, PA: SIAM, 2005.

[20] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge, U.K.: Cambridge University Press, 1985.

[21] F. Albertini and E. D. Sontag, "Discrete-time transitivity and accessibility: analytic systems," *SIAM J. Control Optim.*, vol. 31, no. 6, pp. 1599–1622, 1993.

[22] G. Scarciotti, L. Praly, and A. Astolfi, "Invariance-like theorems and "lim inf" convergence properties," *IEEE Trans. Autom. Control*, vol. 61, no. 3, pp. 648–661, 2016.

[23] J. B. Conway, *Functions of one complex variable*. New York: Springer-Verlag, 1973.

[24] D. E. Knuth, *The Art of Computer Programming-Volume 1: Fundamental Algorithms*. Harlow, England: Addison-Wesley, 1997.

[2] For the notion of persistence of excitation to make sense for a complex sequence the transpose operator in Definition 4 needs to be replaced by the Hermitian (conjugate) transpose operator.