A remark on non-local theories of piezoelectric materials

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1 Introduction

A number of years ago Eringen and coworkers (1978 ...) wrote a number of papers on cracks in a non-local elastic continuum theory. This theory was looked at by Atkinson who showed that the authors had obtained numerical results for integral equations for which he could show that no solution satisfying appropriate boundary conditions existed.A simple example of this type was an equation like :

$$I = \int_{-1}^{1} \frac{(x-t)f(t)}{(x-t)^2 + a^2} \mathrm{d}t = 1 \tag{1}$$

for $-1 < x < 1$ on $y = 0$ together with a condition

$$\int_{-1}^{1} f(t)\mathrm{d}t = 0 \tag{2}$$

which implies a crack which closes at both ends ;the crack displacement being

$$\Delta u = \int_{-1}^{x} f(t)\mathrm{d}t \tag{3}$$

and a stress

$$\sigma_y = \int_{-1}^{1} \frac{(x-t)f(t)}{(x-t)^2 + (y+a)^2} \mathrm{d}t \tag{4}$$

say. IF $a = 0$ the integral in equation (1) is a Cauchy principal value and the solution of the integral equation satisfying condition (2) has the form

$$f(t) = \frac{At}{\sqrt{(1-t^2)}} \tag{5}$$

Making a substitution $t = Cos(\theta)$ the integral equation (1) can be solved when $a = 0$ for $-1 < x < 1$ to give $A = 1/\pi$. For $a > 0$ we consider

$$F(z) = \int_{-1}^{1} \frac{(z-t)f(t)}{(z-t)^2 + (a)^2} \mathrm{d}t \tag{6}$$

and treat z as a complex number .As $|z|$ tends to ∞ in the strip $|Imz| < a$ it is easy to show that

$$|F(z)| < \frac{\int_{-1}^{1} f(t)\mathrm{d}t}{|z|} \tag{7}$$

and this equals zero because of condition (2). Also for $a > 0$ the $n'th$ derivative of F with respect to z exists so $F(z)$ is an analytic function which tends to zero as $|z|$ tends to ∞ in $|Imz| < a$ and so cannot equal 1 on $Im(z) = 0$ when $-1 < x < 1$. Hence no solution exists. It is not enough to say that equation (1) is ill posed ; the situation is worse than that . If for example one attempts an approximate solution by substituting from equation (5) into equation (1) and plots the resulting stress from equation (4) on $y = 0$ after making the change of variable $t = Cos(\theta)$ we get

$$\sigma_y = \int_0^\pi \frac{(x - Cos(\theta)}{\pi((x - Cos(\theta)^2 + a^2)} \mathrm{d}\theta \tag{8}$$

If we evaluate this expression for different values of a small we get results as shown

Figure 1: Model reduction framework