

ALGEBRAIC COMPUTING

IN GENERAL RELATIVITY AND SUPERGRAVITY:

Space-Time Embeddings and Higher Dimensional Theories

by

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To Mohammad and Thamina
My Parents..

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ABSTRACT

The true dimension of space time, according to most contemporary theories, is more than four! This is suggested in order to explain the intricate physics of elementary particles and gravity. A study of three ways for enlarging space-time dimension, with the help of algebraic computing, constitutes the major part of my PhD thesis. These are :

1- The embedding theory : Which evolved naturally from the fact that Riemannian spaces, The mathematical "models" of space-time, can be considered as surfaces in Euclidean spaces of sufficiently large dimension. The theory of normal deformations of space-times is discussed, and a new mathematical method for computing these deformations is presented. The formalism obtained is applied to specific cosmologies, with emphasis on the physical significance.

2- Kaluza-Klein : Here, "real" space is obtained from the observed (i.e. 4-dimensional) space-time by adding to it a compact space, which is appropriately chosen so that it will account for particle symmetries (gauge-fields). A specific ansatz for the full metric is presented, and relevant parts of the Lagrangian of the theory are computed using a computer program that can handle compactifications.

3- Supergravity : Assuming Fermions-Bosons symmetry under the supersymmetric group action, (in a superspace) the Lagrangian of the theory is constructed for the massive $N=2$ supergravity model, using the symmetric and antisymmetric representations of the $OSP(4/2)$ group. Then using spontaneous symmetry breaking the physical spectrum of the theory is revealed: a massless spin 2 graviton, two massive spin 3/2 gravitinos, one massive vector-boson and two massive scalar fields.

Algebraic computing was applied for performing tedious computations, and a presentation of the subject is given in a separate chapter.

Thesis Outline

This thesis comprises of four main chapters and a pseudo-chapter (chapter zero!). The latter presents broad physical arguments for justifying the research presented in the rest of the thesis.

The first chapter includes a review of the application of algebraic computing to research in theoretical physics, and in particular, to general relativity and other gravitational theories. Section (1-1) discusses the motivation for using the computer. The requirements of constructing an algebraic computing package are discussed in section (1-2). The different types of computing languages are presented in section (1-3), and attention is concentrated on the language LISP. The available algebraic computing packages are reviewed in (1-4) with closer examination of the packages SHEEP and STENSOR which are of exceptional relevance to general relativistic applications. To demonstrate the power of these packages, two non-trivial applications are discussed in (1-5), the Rainich conditions of general relativity, and the Clifford's algebra of the Dirac gamma-matrices.

Chapter tow tackles the classical problem of local isometric embedding of space-times. A general introduction to embedding is given in (2-1), then, in (2-2), mathematical structure of the theory of submanifolds is explored. Section (2-3) reviews the different types of embeddings, and their connection to general relativity. In (2-4) attention is

turned to the problem of normal deformations of space-times, a new strikingly simple formalism is developed, and applications to four well known exact solutions of Einstein's field equations are discussed.

Kaluza-Klein theories are presented in chapter three. The introductory section (3-1) clarifies the relationship between Kaluza-Klein theories, and the theory of embedding, it also discusses the relevance of these theories to contemporary theoretical physics. The original (5-dimensional) Kaluza-Klein theory is presented in (3-2), while (3-3) discusses the generalization of this theory to the $n+4$ -dimensional case, using what is known as the Kaluza-Klein ansatz. Section (3-4) contains a new ansatz for the metric of the total space, which generalizes completely all the previous work in this field. A computer program is constructed for calculating some relevant parts of the Lagrangian, this program, which is applied to this new ansatz, is of a very general nature, and can be used for any other ansatz. Some of the difficulties facing the Kaluza-Klein theories are presented in (3-5), together with a discussion of the physical interpretation of the additional dimensions and the link to supergravity.

Chapter four is devoted to the study of massive $N=2$ supergravity from a geometrical point of view, this is done by considering the super-fibre-bundle theory with an $OSP(4/2)$ right action group. A brief introduction to supergravity, and its relation with higher dimensional theories is given

in section (4-1). Section (4-2) constructs the mathematical apparatus for the study of geometrical supergravity. The Orthosymplectic groups are introduced in (4-3) and their symmetric and antisymmetric representations are given, with particular attention to the transformation rules of these representations under the action of the super-group. The Lagrangian of the massive N=2 supergravity theory is constructed and analyzed in section (4-4), this Lagrangian is manifestly invariant under the action of $OSP(4/2)$. When the $OSP(4/2)$ symmetry is spontaneously broken, the physical spectrum of the theory becomes clear, this turns out to consist of a massless spin 2 graviton, two massive spin 3/2 gravitinos, a massive gauge vector-boson and two massive scalar fields.

Appendices A, C and D contain listings of some computer programs and results. While appendix B is devoted for the notational conventions.

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Chapter Zero

INTRODUCTION**0-1 GRAVITY THEORY AND UNIFICATION ATTEMPTS**

The gravitational force is the oldest force known to man, yet, when compared to the other known forces of nature (electromagnetism, weak and strong nuclear forces), it is the least understood!. This curious fact is mainly attributed to the extreme smallness of the gravitational constant ($\sim 6.6 \times 10^{-8}$ c.g.s. units, dyne-cm/gm) which makes gravitational experiments very difficult to perform with the presently available lab-equipments and energies. However, for the theoreticians, the source of the same problem is much harder to identify, and is still a disputed issue.

Einstein's geometrical theory of general relativity is, so far, the most successful classical theory for describing low-energy gravity. Experimental data known hitherto, confirm this theory and thus it is likely that any future quantum theory of gravitation should have general relativity as its low energy limit, rather than replacing it completely. Nevertheless, a full understanding of general relativity and its implications has not been achieved yet, which is obviously a serious hurdle in the way of quantizing and/or generalizing it.

Being essentially a geometrical theory, general relativity differs radically from the usual field theories of physics on fundamental issues, like the nature of the "force" itself: whether it is a description of space-time curvature, or rather, an exchange of a particle?

The methods of conventional "field theories" have been, apparently, quite successful in describing nongravitational forces. For electromagnetic and weak interactions, there is the unified quantum field theory of $SU(2) \times U(1)$ of Glashow, Salam, Ward, Weinberg and others. At the same time, "Quantum Chromo-Dynamics" QCD theory of interacting gluons (Yang-Mills bosons) with quarks, presents a good model for studying strong interactions. There even exist grand unified schemes (GUTs) in which these two theories are unified in a world of color and flavor, but without gravity!! Note that the predictive power of these quantum field theories is a consequence of their renormalizability.

These successes tempt theoreticians to treat gravity on the same footing, i.e. considering it a field theory with spin 2 massless particle as the "force carrier", as a first step towards the ultimate unified theory of nature (the Utopia of physicists!), however, this scheme faces the grave problem of non-renormalizability of gravity, due to the dimensionality of its coupling constant.

The alternative procedure, of "geometrizing" field theories is also equally problematic, which casts doubts on the unification idea as a whole. However, the difference in basic

mathematical tools used for studying different forces of nature did not compel most physicists to give up the unification dream, especially after the emergence of new theories that claim to possess the remedy. The most important of these theories is a combination of two types of theories:

i) Supergravity theories: which are basically field theories, that hope to provide an alternative to renormalizability through automatic (magic) cancellation of infinities.

ii) Kaluza-Klein theories: which are of geometrical nature, they explain the non-gravitational fields as geometrical symmetries in extra (hidden) dimensions of space-time.

It is interesting to note that one important formulation of supergravity is done in superspace, whose dimensionality is more than (the usual) 4. This observation helped to investigate the geometrical nature of some supergravity theories (see chapters 3 and 4), which should provide the recipe for forging a link with Kaluza-Klein theories, that would hopefully lead to the total unification of forces.

From the above discussion, The importance of higher-dimensional theories (i.e. theories which regard the dimension of the universe to be greater than 4) is clear, since they are becoming the "standard approach" for modern theoretical physics, and these will provide the major theme of this thesis, which will present, from a geometrical point of view, specific problems facing the above mentioned

schemes, and attempt to solve them. However, before going into these "super-theories", a presentation of the most natural higher-dimensional theory is needed, that is the embedding theory, which has not been fully understood yet, particularly the question of normal deformations of embedded space-times of the classical general relativity theory.

Another common thing between these theories is the extreme complexity of the mathematical calculation involved!!, which lead us to adopting algebraic computing as another theme of the thesis. This however does not mean that all the thesis will be filled with computer terminology, since these are only restricted to the first chapter, and some appendices, while only results are given in other chapters.

0-2 EMBEDDING AND HIGHER DIMENSIONAL THEORIES

More and more physicists are beginning to question the true dimensionality of space-time (as discussed above), to which the number 4 is usually associated as "a priori". In this section some interesting facts concerning space-time dimensionality are presented, together with a preliminary introduction to the embedding theory.

It is quite remarkable to note that the quantum field theories are much more successful in d -dimensional space-times with $d < 2$ or $d > 4$ rather than $d = 4$, for which they are notorious for being ill-defined! one quoted example is the dimensional "regularization" technique [1], which avoids

the divergencies in 4-dimensions by working in d -dimensions and then taking the limit $d \rightarrow 4$ in the final answer!.

Another intriguing fact is that the basic postulates of Einstein's theory of general relativity are independent of the dimensionality of space-time and therefore one can study d -dimensional general relativity using the same formalism. However, general relativity theories in $d < 4$ dimensions have completely unintuitive implications, which are qualitatively different from well established classical physics [2], therefore they can be dismissed.

Embedding is the most natural higher-dimensional theory, since additional dimensions are implied by the curvature of the space-time, rather than being inserted by hand. It should provide a clear and intuitive geometrical picture for space-times, making their properties transparent. For example, if we consider the 2-sphere S^2 , it is clearly much easier and useful to study it as a surface in 3-dimensional Euclidean space, rather than a mere abstract manifold. However, this scheme faces severe computational problems, over and above the fundamental physical problem of the lack of the manifold which correctly describes the universe. On the other hand, other higher-dimensional theories (including supergravity and Kaluza-Klein) "implant" the extra dimensions artificially, so as to (hopefully) extract the correct physical theory.

Despite the huge amount of literature present, it is fair to claim that the study of higher-dimensional theories is

still in its infancy, especially regarding the development of the mathematical tools needed in such type of work. A simple example that could be cited is the work of Teli [3], who used Hexon-algebra for formulating the classical Maxwell's equations in 6-dimensions, however his algebra did not close [4]! and an alternative construction using Octons was suggested in [4] to retain the results of [3].

0-3 THE ROLE OF ALGEBRAIC COMPUTING

Another thing common to all higher-dimensional theories (as well as most theories of physics) is the extreme complexity of the mathematical calculation involved!!, this lead me to adopt algebraic computing as another major theme for this thesis. This however does not mean that this thesis will be filled with computing terminology (which is naturally not familiar to theoretical physicists), in fact an attempt is made to restricted all computing material to the first chapter, and the appendices A, C and D, while only results are given in other chapters.

It is no exaggeration to describe the 20th century as the "computer age" for mankind, because of the undeniable impact of this machine on all human activities. The main effect is felt in the research areas.

Computers are traditionally known as the "stupid¹ efficient hard-worker"! therefore, they are well suited for solving

¹"Artificial intelligence" is being slowly build into them!!

heavy calculations far beyond human capacity. In particular, using computers for routine-type work can pay back with significantly improved efficiency.

However, the use of computers, especially for algebraic calculations, is not as widely spread as one would hope to see, and hence the need to introduce this subject in a separate chapter (the first chapter), because of its fundamental role in obtaining the results presented in this thesis.

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Chapter One

ALGEBRAIC COMPUTING

1-1 Motivation

It was reported that Einstein, the founder of the present theory General Relativity [1], was rather surprised to learn that Schwarzschild [2] presented an exact solution to his field equations, only a few weeks after these equations were actually published. His surprise stems from his knowledge of the cumbersome nature of the required calculations.

Hundreds of exact solutions have since been found [3], still it is generally accepted that working on General Relativity involves extensive and difficult calculations.

Fortunately though, the great bulk of General Relativity calculations are of an algebraic nature, which makes it possible to use electronic computers for carrying out the heavy computations. At this stage it is important to note that what has been said so far is equally true for all branches of theoretical physics, where a lot of time is employed in carrying out algebraic calculations of different levels of complexity, of routine nature.

To use the computer, one is compelled to formulate his problem in a language understood by that machine, and a natural choice for a theoretical physicist, is to use a high level programming language, since machine code and assembly languages are subjects of specialization in their own right, and they are more likely to vary from one machine to

another. Nevertheless, one has to narrow down the possible programming language choice even further.

For historical (and may be psychological) reasons, most of the well known high-level languages are of a "procedural" nature designed mainly for number-crunching computations, and thus are not entirely adequate for algebraic calculations (symbol-crunching!), and despite earlier attempts to construct algebraic computing packages based on a procedural language (two important examples are FORMAC [4] and ALTRAN [5], which are FORTRAN-based systems) the present trend is heavily dependent on "functional" languages, which are themselves receiving much renewed interest and attention in computer science for their fundamental role in developing "artificial intelligence", which is very much needed for the building of the "Fifth Generation" computer [6].

1-2 REQUIREMENTS

Before deciding on the most suitable "language" for algebraic computing, it is essential to examine the problems involved in algebraic manipulation itself:

1-2-1 Data Structure

Historically, computers are thought to be number-crunching machines, thus, the data input, processed and then outputted, was supposed to consist of numbers only, and consequently, procedural languages were designed mainly to handle numerical data. Clearly such data are not adequate for algebraic calculations, where symbols are needed for most of

formatting. Functional languages, on the other hand, provide much greater liberty in defining a multitude of data-types; This includes both numerical and symbolic data. Hence, it is obviously beneficial to start off with a functional language as a base for an efficient algebraic computing package.

1-2-2 Garbage collection!

The complexity of an iterative numerical calculation increases at worst power-wise, whereas in an iterative algebraic calculation, involving symbolic data, the complexity grows exponentially. This would easily lead to a memory "blow-up", i.e. the computer storage would be choked up with expressions, however, a close examination reveals that most of the data responsible for creating this problem is obsolete. This can be succinctly illustrated by the following example:

Suppose we would like to expand the expression $(x+7)^{10}$ iteratively, we then have:

$$\begin{aligned}
 (x+7)^{10} &= (x+7)\dots(x+7) = \dots & (1.1) \\
 &= x^{10} + 70x^9 + 2205x^8 + 41160x^7 + 504210x^6 + 4235364x^5 + 24706290x^4 \\
 &\quad + 98825160x^3 + 259416045x^2 + 403536070x + 282475240
 \end{aligned}$$

Clearly, if the computer continues to hold on to the data of the intermediate steps, then a small number of similar operations would exhaust the memory of even large (main-frame) computers. Hence the need to have our chosen language equipped with a special device to round up

discarded information and reclaim the store for further use, such a device is technically called a "garbage collector".

1-2-3 Simplification

According to Fitch [7], there are three major reasons for simplification:

i) Compactness, which is concerned with making the expression small to save on the store, and to speed up the subsequent calculations.

ii) Intelligibility of algebraic expressions: This is not a well defined concept, because it has a subjective element for deciding the most appropriate form in which an algebraic expression should be presented to the user.

iii) Checking identities, or in other words, checking whether a certain expression is identical to zero, something which was proven to be untenable within a finite number of steps, indeed, Richardson [8] proved rigorously that, in general, no algorithm exists for deciding if an algebraic expression, containing terms from the class formed from the action of addition, multiplication division, subtraction, forming trigonometric functions, exponentiation and logarithm modules on x (a variable), π , $\log 2$ and other elementary constants, vanishes identically. Thus, even if people would agree on the simplest form of every expression, it is clear that constructing a general algorithm for simplification is impossible.

However, there are well defined classes of algorithmic expressions that has canonical forms (the class of polynomials over rational numbers is one example) i.e. they can be written in a unique fashion by assigning an arbitrary rule for canonizing the terms in the given expression. The identity problem for such expression is decidable [9]. The above reasons for simplification are obviously not entirely independent, since, for example, if one has the expression:

$$x^2 - y^2 = (x-y)(x+y) \quad (1.2)$$

The decision on whether the left hand side or the right hand side form of the same expression should be used, can vary depending on its position relative to the rest of the calculation, in particular, if the next step would involve dividing by $x+y$, the right hand side is more desirable.

The best way out, for solving simplification problems examined above is to provide an interactive environment for doing the calculations, then the user will be able to interfere with the work of the computer at suitable times, and tell it where and when to expand a given expression.

Therefore, it is very desirable to chose a computing language that can run interactive programs, as well as Batch¹ jobs.

¹ "Batch job" is a technical term, used for non-interactive jobs that are submitted as a single bundle to the computer.

1-2-4 Substitutions

These are needed in algebraic computing for dividing the calculation into stages, this would help to increase the speed and storage efficiency of the manipulation.

Substitutions are of three different types:

i) Expansion, where one term is replaced by its full value, which in turn might contain terms that are expandable. This used to break down the given algebraic expression and is mostly needed at the start of the manipulation.

ii) Algorithmic substitutions, are those substitutions which are carried out in all basic computer systems, like:

a- gathering substitutions, which are used to gather a few terms of an expression in one term. These are generally needed at the end of simplification, although they can also be useful in intermediate stages of work to reduce the number of terms to be dealt with. b-

Algebraic ring substitutions, e.g. if A is a ring over F:

$$x+0=x$$

$$x.1=x$$

$$x.0=0$$

$$x^0=1 \quad (\text{for } x \neq 0)$$

$$-(-x)=x$$

$$x-x=0$$

$$x^{-y}x^y=1, \quad y \text{ is finite in } A.$$

For all $x \in A$. $0, 1 \in F$ have the usual meaning.

Therefore, studying the "substitutions" problem, reveals another aspect of the usefulness, or rather necessity, of interactive computing.

iii) Sum substitutions: These are probably the most difficult type of substitutions, since the relation given will be of the form:

$$a+b+c=0 \quad (1.3)$$

Where a , b and c could themselves be complicated expressions. A very important example of such substitutions is the trigonometric relation:

$$\cos^2 x + \sin^2 x - 1 = 0 \quad (1.4)$$

The difficulty in such substitution arises from the fact that there is no standard way of deciding which term should be removed and replaced by the other two terms! A judicious use of (1.4) would extensively help in simplifying trigonometric polynomials, but on the other hand, the simple-minded procedure of using:

$$\cos^2 x \rightarrow 1 - \sin^2 x \quad \cos^3 x \rightarrow \cos x - \cos x \sin^2 x \quad \text{etc}$$

can sometimes be a recipe for disaster!.

Although there exist some fairly successful trigonometric simplifiers, that exploit the sum relation (1.4) optimally (see for e.g. [10]), still the best current expedient for handling an arbitrary sum substitution is the user-machine interaction. One example where "foresight" (something which cannot be done by an unaided computer) is needed, is

the observation that: $bgf+cdh-ahf-bdi-cge$ may be written as : $\Delta-aei$, where $\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

1-2-5 Interactibility

From what has been mentioned so far, interactibility is clearly a very desirable quality in the candidate language for constructing an efficient algebraic computing package. While in numerical computing there is nothing but the straight-forward process of number crunching, it is obvious from what has been said so far that algebraic computing involves, in addition to straight forward processes, a lot of intelligent decision-making and guess-work which is not always algorithmic, and hence not programmable. Therefore, the most effective way round this problem is to let the user share the work with the machine, by intervening when necessary to give directing commands about the method in which the work should be carried out. It is important here to concede that the user may err in his suggestions but the advantage of using the computer to do the hard labour is that the price that has to be paid for a wrong guess is the loss of only a few minutes (or, at most hours) rather than days or months!.

The need for interactive computing has risen in more than one field of study, and at the time of writing this thesis, most popular programming languages in use have been updated to allow for this facility, and to make the most of it.

One should note that the interactiveness problem is linked

to the problem of intelligibility, that is the user should be able to read off the stage of calculation reached by the computer as easily as possible, to reduce the chances of making a wrong assessment of the current situation, which would inevitably lead to issuing misleading "hints" to the computer.

It is true that interactive computing generally takes longer time than batch computing, because the computer has to loose valuable "seconds" just waiting for the user to enter his command. In fact, even if the user did not have to think about the commands he is typing in at all, still the time needed to type the characters on the terminal could be enough for the computer to execute thousands of elementary jobs! but still, for work that requires intermediate guessing, interactive computing saves a lot of time for the user, who would not have to wait each time for the computer to finish the whole calculation before finding out whether his guess was lucky!

1-2-6 Calculus and Polynomial Division

Calculus is a very important mathematical tool, which is essential for serious calculations in all branches of science and engineering. It is therefore natural to expect that all algebraic computing packages would provide calculus facilities, at varying levels, depending on the special purpose of the package concerned.

Differentiation, which is a straight forward algorithmic

process, is available in almost all algebraic computing packages. It has attracted the attention of the earliest system builders because of its algorithmic nature. However the inverse of derivation, namely, indefinite integration, is not algorithmic. Indeed not all functions, satisfying the usual smoothness conditions, have integrals that can be drawn from the set of elementary functions. Earlier integration programs had severe restrictions [11], despite using artificial intelligence techniques, but more recently improved algorithms have been used, and overall, the situation looks more hopeful, especially since the development of a new algorithm [12] that decides whether an expression, from a fairly wide class of expressions, is integrable in terms of that class, and if so, the integral can be worked out, this procedure has already been implemented in MACSYMA and REDUCE (see section (1-4) on packages). However the integration program is so large at present, that it is considered a special purpose program, that is, unless a calculation involves extensive amount of integration, it is not recommended to load this program onto the user's working space in the computer. Immediately related to integration, is the problem of polynomial factorization. This problem is basically algorithmic, since it is related to finding the common factor, (or the GCD) of polynomials [15], but it involves much testing, and thus consumes a lot of computer time. This would also classify factorization programs as "special purpose".

1-3 CHOSING THE COMPUTING LANGUAGE

1-3-1 More Constraints!

Examining the requirements for algebraic computing is not enough, on its own, to pinpoint a specific programming language as the ultimate choice for building the desired system. This unfortunate fact is due to practical difficulties, which, despite being traditionally disregarded by theoreticians, play an important role in the process of engineering (i.e. building a real apparatus from a theoretical model). These physical constraints can be summerized in one word: "availability", or rather, the non-availability of certain facilities. For example, no interactiveness can be achieved if the appropriate hardware (in this case VDU) is not available. It is arguable that divergence in the computing languages used in constructing algebraic computing packages (as will be seen in (1-4-1)) is mainly due to the problem of availability! This means that there is little hope in unifying those systems, mainly because computer manufacturers and scientists are too far apart to agree.

1-3-2 Procedural Versus Applicative Languages

The high level programming languages can be divided into two main groups "procedural" and "functional", whose major properties are presented below:

i) Procedural languages: are the more traditional type of languages, such as FORTRAN, BASIC, PASCAL etc, they were designed mainly for numerical analysis, with generally two

data-types available: integers and floating-point numbers! even in their modern versions, defining symbolic data requires a lot of intricate formating.

Variables in procedural languages behave in a peculiar way, completely different from the traditional mathematical understanding of the nature of variables! this is mainly because of the "procedural" nature of these languages. More precisely, a variable in a procedural language is a name given to an allocation in the memory of the computer, hence, statements of the type:

$$x = x + 1 \quad (1.5)$$

are quite sensible!

The command (1.5) means: add one to the content of store 'x', and replace the result in that same store.

Obviously, this makes it more difficult to adapt these languages to symbolic and algebraic computing, where $x = x + 1$ is meaningless!

These languages however are quite powerful in numerical computing, with a speed and precision far beyond human capacities. They have also been used to build algebraic computing systems (as shall be seen in (1-4-1)) mainly because they are widely available, in contrast to "applicative" or "functional" languages, which were thought to be of less importance.

ii) Functional languages are programming languages in which variables behave in the usual mathematical way. These

languages have varying design philosophies, for example APL is organized around the concept of arrays as the building blocks, for accomplishing the mathematical notation for algorithms, as described by K. Iverson [38], while LISP is based on lists, in the widest sense of the word [39].

All functional languages accept symbolic data, but the flexibility of the data structure may vary from one language to another. And since algebraic computing generally demands the use of highly complex data-types (like tensors and operators), it is important to be selective, even when considering functional languages.

Recently, a lot of research has been taking place in the field of functional languages and a new breed of language is emerging, such as the language HOPE [40], which provides an extremely flexible data structure, where the introduction of new data-types is at the fingertips of the user, in addition to a lot of other interesting features. The main problem with these languages is that they are, in general, still developing, and thus should be regarded as computing-languages of the future!

Table (T1) contains a general comparison of the major features of the rather well known functional languages. This table shows that LISP is apparently more favorable to fulfill the requirements needed for constructing an algebraic computing system, this was also the conclusion of most system-builders, as shall be seen in (1.4). Hence, the rest of this section will be devoted for explaining LISP and its

relevant advantages and disadvantages.

1-3-3 The Language LISP

LISP, an acronym for LIST Processing, was designed by John McCarthy and co-workers for symbol manipulation algorithms. Atoms are the building blocks of LISP; an atom can be an integer, a floating point number, or a symbol composed of one or more characters (but not starting with a numeric character²) also the use of some special symbols in atoms are permitted. Atoms can be grouped together to form lists. A list is either empty, or contains a collection of atoms and/or lists. For example:

Language	<u>APL</u>	<u>LISP</u>	<u>PROLOG</u>	<u>SASL</u>
Building Blocks	Arrays	Lists	Relations	Lists
Notation	Infix	Prefix	Infix	Infix
Garbage Collection	No	Yes	Yes	Yes
Datatypes Available	Integers + Symbols	Open	Integers + Strings	Reals + Strings
Important Features	Large and Confusing Character Set	Input and Output are of the Type (Lists)	Very small Library	Difficult to Learn

Table T1: Well Known Applicative Languages

² Since the LISP-reader uses the first character to discriminate between variables and numbers.

1 , 0.206 , A1F ... are atoms

() , (3) , (A (3 I) MATH) ... are lists.

A running LISP constitutes of an infinite READ-EVAL loop, i.e. it reads strings from the standard input and evaluate them, then waits for more input.

The basic LISP functions are only 8 in number, and understanding these is enough for writing LISP programs! these functions are:

(1) QUOTE (or just ') returns the value of its argument! in other words, it shields its argument from the LISP evaluator. For example, if you type: (QUOTE NAME1) to LISP, it responds: NAME1

(2) ATOM is a predicate, which returns NIL if its argument is not an atom, otherwise it returns T (for true). e.g.: The input (ATOM 'ALPHA) has the value T (for true), while (ATOM '(ALPHA)) has the value NIL.

(3) EQ This function should be applied to two arguments, it is also a predicate, which checks whether its arguments are Equal.

(4) CAR returns the first member of the argument list. e.g. :

(CAR '((HEAD OF) THE (LIST))) = (HEAD OF) It has no meaning if applied to an atom, and will give an error.

(5) CDR returns its argument list, after chopping off the first member (i.e. the CAR), for example:

(CDR '((HEAD OF) THE (LIST))) = (THE (LIST))

(6) CONS CONStructs a new list, whose CAR is the first

argument, and CDR the second, e.g.:

`(CONS 10 '(20 30)) = (10 20 30)` clearly the second argument must be a list.

(7) COND this function takes any number of lists as arguments. It first checks if the CAR of the first list is empty, if not, it will evaluate its CDR and stop, otherwise, it will do the same work on the next argument list, and so on.

(8) DEFUN this function helps to introduce new functions to LISP, it takes three arguments: the first is the name of the function being defined, the second is the list of variables and the third is the algorithm of the function. Using the same syntax, different function-types can be defined in LISP, the most important are:

i) Expr: These functions take only a specific number of arguments and evaluate them (via the LISP EVAL reader), before doing its job. Thus it is important to QUOTE the argument for which pre-evaluation is not desired. As an example, consider the built-in addition function EQ, If one types: `(EQ ALPHA ALPHA)`, the computer will complain, since ALPHA can not be evaluated! the answer for this operation is neither T nor NIL!! However, if we quote both ALPHA's by ' we have: `(EQ 'ALPHA 'ALPHA)` which gives T. Alternatively, we may set ALPHA to a value, say A1, using the SETQ command (more about SETQ in the next section), then the above command will be equivalent to checking whether A1 is the same as A1, which is

obviously true, giving the output T. All the functions presented so far, except QUOTE of course, are Exprs.

ii) Fexpr: Takes, in theory, one argument (which can be a list of arbitrary number of entries!), and do not pre-evaluate it. This is extremely useful for introducing functions with a flexible argument-list. Consider for example the built-in function PLUS which takes two arguments and add them up. To allow the addition of more than two numbers in one go, we can define the Fexpr PLUSG (G for general), as follows:

```
(DF PLUSG (X)
  (COND ((ATOM X) X)
        ((NULL X) 0)
        (T (PLUS (CAR X) (PLUSG (CDR X))))))
```

Note that for defining Fexprs one should use DF instead of DE.

Needless to say that in addition to these functions, LISP knows all the traditional functions of one variable (log, exp, sin, ...) the arithmetic operations (+, x, ...) and the Boolean operations (AND, NOT, ...). There are also other very useful built-in functions, like SETQ, PUT, APPEND etc (for more details on these and other functions, see for eg [35]). Here we shall only describe the command SETQ:

This is an Expr which takes two arguments (and quotes the first argument automatically). Its job is to set the second argument as the LISP-value of the first. Therefore, if we type: (SETQ ALPHA 'A1) Any subsequent use of un-quoted

ALPHA, will be understood by LISP to be A1, in particular, if we enter only: ALPHA, the computer will respond by printing A1.

Clearly, the syntax of LISP is extremely simple, and thus, LISP itself is a very easy language to learn. No previous knowledge of other programming languages is needed, on the contrary, having experience in another language can be harmful, since different languages do things differently and "functional-to-function" translation may lead to unnecessary complications.

Although the traditional ideas of "programs" and "loops" are known to LISP, "purists" try to avoid using them, they recommend instead the use of the very powerful tool of recursion, which in most cases, supercedes the traditional iterative techniques. For example, one can compare the following two ways of defining the factorial of an integer:

```
i)      (DE FACTORIAL (N)
          (COND ((EQ N 0) 1)
                (T (TIMES (FACTORIAL (DIFFERENCE N 1)) N))))

ii)     (DE FACTORIAL (N)
          (PROG (A) (SETQ A 1)
                LOOP (COND ((EQ N 0) (RETURN A))
                            (T (SETQ A (TIMES N A))))
                (SETQ N (DIFFERENCE N 1))
                GO LOOP))
```

Particularly, in symbolic manipulation, recursion is more suitable for most applications, while iteration may be of

some use when working with arrays.

Another characteristic of LISP, is the simplicity with which program-generating programs can be written. This is substantially helped by the fact that both, programs and data are constructed from the same structure; namely lists. This property is extremely useful in algebraic computing, where it might be necessary to have different levels of data abstractness, such that the passing from one level to another is algorithmic. An example of this is TCOMP program in STENSOR, which will be described in section (1-4-3).

The major alleged disadvantages of LISP are only two:

i) According to M. Ward [41], it has been alleged that LISP stands for: "Lots of Irritating Single Parentheses", which appears as a fair accusation to an outsider. Experience, however, show that this is rather an artificial disadvantage of LISP, similar in nature to the fear from water for beginners in learning swimming!, and as soon as this initial fear is overcome, its hidden beauty becomes clear.

ii) The second disadvantage of LISP is a real one, but is by no means confined to LISP only. This is the problem of "dialects"!. It is due to the continuous succession of contributions and "improvements" to local versions of LISP at different research sites, as well as the constraints imposed by the hardware structure of different computers!. This impedes the standardization of LISP, so necessary for the development of easy software interaction.

The original versions of McCarthy (LISP1.5 , LISP1.55 , LISP1.6 etc) are obsolete, and at present, the most important versions in use are: i) MACLISP [42]: developed at MIT, which has in turn split into different subdialects, the most important of which is the FRANZ LISP [43]. ii) Interlisp [44]. iii) UCI LISP [45]. iv) SLISP [46]; which was intended to be Standard LISP!

1-4 IMPORTANT PACKAGES

1-4-1 Historical Development

Originally, algebraic computing packages had two distinct sources: researchers (particularly in areas of theoretical physics) and computer scientists. The packages designed by computer scientists were aimed at quite general algebraic procedures, with no specific application in mind; On the other hand, the packages designed by research workers were much less sophisticated and more to the point, because there research workers had specific problems that required solution. Generally, the latter packages were more successful, judging from the amount of usage generated. However, at present, systems are being developed by a new generation of scientists, who should be called "computational-physicists, possessing extensive knowledge of various branches of physics, or engineering and expertise in computing. Thus the impact of new algebraic systems is expected to change radically the way research is carried out in a lot of theoretical fields of science. Especially since many of the new

generation packages are designed to work on mini and personal desk-top computers³, which are relatively cheap, and within the budget of small research firms and small university departments

R d'Inverno [9 and 16] reviewed extensively the historical development in algebraic computing, giving attention to almost all algebraic packages, including those now out of date. Here however, I attempt a more pragmatic analysis, considering critically only the "surviving" packages and systems, in order to select the most appropriate one for our work. Nevertheless, for the sake of completeness, table T2 below exhibits all the systems and packages known to the me.

The word "package" usually refers to a collection of programs, with development time measured in days of human work. While a "system" is a collection of packages, often built from a scratch, having their development time measured in years!. Systems, therefore have wider applications, but the price to pay is that the storage-size needed is much bigger than that of a package, restricting their use to relatively large computers.

Although it has been argued above that the language LISP is the best candidate for construct an efficient algebraic computing system, there are nowadays a few other systems and packages in use, which are not LISP-based, important exam-

³For example REDUCE is already available on Darkstar-Sirius.

ples are (see table T2): the system ALTRAN and the packages GOEDEL [17] and GRATOS [18] which are FORTRAN-based. There is also CAMAL, which is written in a low-level language, where the data structure that originally accommodated integers and floating-point numbers, was modified to accept symbols instead of the latter data-type [19]. Despite many drawbacks, CAMAL has been very successful as a system [17], mainly due to the active support group it enjoys. However, it is interesting to note that the main author of CAMAL, J. Fitch, declared in a lecture at Imperial College (on 2-2-1982) his preference to LISP over other computing languages, in the field of symbolic manipulation!, and he is at the time of writing this these involved in constructing a new version of LISP: the Cambridge-LISP. The view of the author is that, if only because of the need for standardization, CAMAL may not survive much longer.

At the end of this, rather quick, process of elimination, one is left with mainly three LISP-based systems: MACSYMA [20], REDUCE [14] and SHEEP [21]. It is actually very hard to choose between these systems, since they all have very great virtues, especially in their latest versions. Nevertheless, after careful consideration of the types of applications intended, which are in the area of gravitational theories, and keeping in mind the limitations on the computer storage available for such a project. SHEEP emerges as the most suitable system for use for research in general relativity, since its main rivals MACSYMA and REDUCE

being both large systems, intended for a variety of algebraic computing requirements, (including integration and factorization packages), which as discussed before, would make their storing and running expensive, in terms of computer time. Though, it is only fair to remark that both of the rival systems have been used in gravitational and general relativistic calculations, and have produced very important results (see for e.g. [22], [23] and [24]) and they are still being used, and developed in many research institutes.

Package/ System	Specialized For General Relativity	Base Language	Interactive	Obsolete
ALAM	Yes	L	No	Yes
ALPAK	No	F	No	Yes
ALTRAN	No	F	No	No
CAMAL	No	O	No	No
CLAM	Yes	L	No	Yes
FORMAC	No	F	No	Yes
GOEDEL	No	F	No	No
GRATOS	Yes	F	No	No
LAM	Yes	L	No	Yes
MACSYMA	No	L	Yes	No
METASAC	No	F	No	No
ORTOCARTAN	Yes	L	No	No
REDUCE	No	L	Yes	No
SAC-1	No	F	No	Yes
SCHOONSCHIP	No	O	No	No
SCRATCHPAD	No	L	Yes	No
SHEEP	Yes	L	Yes	No
SML	No	O	No	No
SYMBAL	No	O	No	Yes

Key: L = LISP, F = FORTRAN and O = Other.

Table T2: Algebraic Computing Packages

1-4-2 The Package SHEEP

SHEEP was written by I. Frick [21] to become the latest member of the LAM family of packages (Lisp Algebraic Manipulators), developed by R d'Inverno [5] and [26], who described the new comer as: "a very different "animal" indeed" ! [9]. It supercedes its "predecessors" with many improvements in basic algorithms and important additional functions, like tensor manipulation, tetrad calculations, truncated power series and complex arithmetic. However, the most important new feature, is that SHEEP can be used interactively. With respect to efficiency, SHEEP competes favorably with respect to other existing packages and systems, both in computing storage and in speed [9].

Although my decision to use the system SHEEP is now made, it still is useful to present a brief account of its advantages and disadvantages!. This will contribute to a better understanding of the system, its capacity and limitations, which would help in applying it more efficiently.

The major advantages of SHEEP are:

- i) The "grown up LAM(B)" has a comparatively small size: typically 30k words (of 32 bit word), which leaves a lot of free memory for the user's work. The small size should, in principal, be helpful in making SHEEP available for a greater number of machines, including those with limited storage capacity, but unfortunately, there are still other problems impeding portability (more about portability in the section on disadvantages).

ii) SHEEP is a very fast "animal" when it was put to the "standard" test of calculating the Einstein tensor for the Bondi radiating metric [27], SHEEP took about 30 seconds to compute and print out the components of the Christoffel symbols, the Riemann tensor and scalar, and the Einstein tensor. Also, it gave a print-out of the covariant and contravariant components of the metric (for confirmation). Throughout this job, only 35k of memory was used. SHEEP is thus 10 times faster than REDUCE.

iii) Interaction is a very important property of SHEEP, not only for facilitating the work, but also to make some calculations at all possible, as has been discussed earlier.

iv) However, for efficient interaction, neat and clear formatting is needed for inputting and outputting data. SHEEP has probably the most readable output, compared to all other systems [9], and as far as inputting is concerned, the notation of SHEEP is quite straight forward. Also, data can be entered interactively, so that the user can verify instantly whether any typing mistakes were made.

v) The internal layout of SHEEP is organized as a series of self-contained components, called modules. This means that only the modules which are needed for the task in hand will be "loaded" into the working space of the computer so contributing to the remarkable efficiency of SHEEP. Also, this helps debugging⁴ the system (when

needed), and makes it much easier to add more modules in the future to provide extra facilities.

vi) SHEEP is very easy to learn at the user level, since the knowledge of only a few basic functions can start the user off. So that in a very short time he becomes able to tackle interesting research problems [28]. This has been a major factor in increasing the number of its users quite rapidly, and significantly helps the developing of the system, in accordance with the feedback from the users.

The major disadvantages of SHEEP:

i) The LISP base of SHEEP has obviously played an important role in providing the above mentioned advantages, yet the same LISP is responsible for one of the major disadvantages of SHEEP, namely: its availability.

SHEEP was written in SLISP dialect, the Standard LISP, or rather what was intended to be the standard LISP! Therefore, at present, SHEEP can only run on PDP10 and VAX machines of DEC computers; for which translators of the implemented LISP (1.6 or Franz) were written by I. Frick, the author of SHEEP. However, more adaptable (portable) versions of SHEEP are being prepared.

ii) Common to all algebra systems, is the problem of running into store difficulties, and SHEEP is no exception to this, although it generally holds a bit longer than

⁴ Debugging: is a technical term in computer science which is used for the process of removing the sources of error (Bugs!) from a program.

others!

iii) SHEEP has no symbolic integration program, which is a price that had to be paid for compactness, since all known integration programs are extensively large. However, the lack of integration facility does not seriously handicap SHEEP, since it is not essential for most general relativistic applications, which constitute the intended domain for SHEEP.

iv) Another missing facility is polynomial division and factorization, which is usually more needed than integration. To compensate for this SHEEP has got a built-in program, which will detect if a denominator appearing in a standard computation consists of an irreducible sum of terms, and if so it will replace that denominator by a single new term before proceeding with calculations. This procedure has to be done manually in case of a non-standard type of calculation.

The last two disadvantages (iii) and (iv) can be overcome in the case when REDUCE and SHEEP live together in the same machine, since the formatting of SHEEP is compatible with that of REDUCE, so that one can easily direct the output of the calculation of one system to be used as an input to the other. But, the efficiency of such a procedure is questionable.

SHEEP has already been used for solving many general relativistic problems, even exceeding the scope predicted by its author [29]. At present, an estimated 30 research

institutes are using it, for yet more applications and developments.

Two very important modules have been added to SHEEP, namely: CLASSI [30] written by J. Aman, and STENSOR [10] written by L. Hornfeldt. The current research using SHEEP, includes the study of the "equivalence problem" [31 and 32] and building a data-base for all exact solutions of Einstein's field equations, the relativity research group at Queen Mary College (London) under M.H.A. MacCallum's.

1-4-3 Tensor Manipulator: STENSOR

As indicated above, SHEEP is run by loading onto LISP some basic packages for general definitions of algebraic computing, then if needed, more modules can be added subsequently, depending on the specific application intended. This contributes substantially to the speed and efficiency of SHEEP, and it gives it a powerful ingredient for expandability. Figure (F1) below illustrates this property of SHEEP, and gives a comparison with the sizes of MACSYMA and REDUCE.

STENSOR module shall be considered in more detail, because of its importance for the rest of this thesis.

Basically, STENSOR is a special-purpose algebraic manipulator, designed to handle any type of indicial quantities, i.e. quantities that have symbolic indices; these include: spinors, tensors and operators. STENSOR can perform all known tensor-calculus operations, such as addition, multiplication, partial differentiation, covariant differentia-

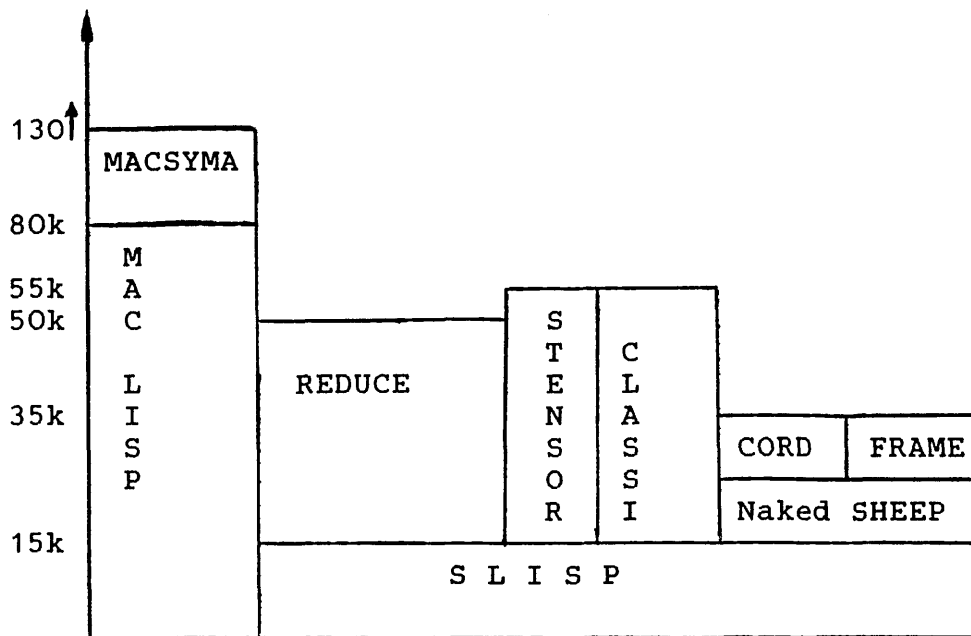


Figure 1: The approximate storage needed for the package SHEEP and associated packages, compared to other systems, in Kilo-bytes.

tion etc. It also uses efficiently the symmetries of different indices during computation. For example, if we define the tensors A_{ij} and S^{ij} to be respectively antisymmetric and symmetric in the pair of indices i and j , then the contracted quantity $A_{ij}S^{ij}$ will be given the value zero automatically.

STENSOR can handle anticommuting, as well as non-commuting objects, and it possesses a very sophisticated substitution program, which uses the interactivity of SHEEP quite efficiently.

Another important sub-program of STENSOR is the trigonometric simplifier, which exploits optimally the trigonometric sum-relation (1.4). By adjusting few switches, the user can direct the process of simplification which combined with interactiveness, makes this program the best trigonometric simplifier presently known in the field of

algebraic computing.

Probably, the most important feature of STENSOR is the automatic generation of tensor algorithms: there is a compiler that from any given tensor formula, generates a program for computing individual tensor components (via SHEEP) by actually performing contractions, derivations, matrix inversion etc. This means that SHEEP's work is not any more restricted to calculating the traditional terms of general relativity, using the metric, but also new tensors and scalars can be defined, and computed, which might not be related to the metric at all! this opens up the way for all sorts of new applications to this system.

The input notation of STENSOR is extremely easy to learn, since it is very similar to ordinary textbook conventions. For example, if we want to define the tensor TE_{ijk} in terms of tensors A_i , B_j and C_{ij} , where:

$$TE_{ijk} = A_i B_{j,k} + C_{ij;k} \quad (1.6)$$

using the command PDEF (Put DEFINition), we have to input only the following line:

$$(PDEFTE) \langle AI \rangle \langle BJ, K \rangle + \langle CIJ; K \rangle \$ \quad (1.7)$$

Obviously, the resemblance between (1.6) and (1.7) is unmistakable! and it hardly takes any time at all for the potential user to get acquainted with STENSOR, and start using it in interesting projects. Note that the '\$' sign at the end of (1.6) is an indication to the computer that the input has ended. Another way to end input is by a ';'.

(semicolon), which (as in SHEEP) will echo on the computer terminal (printer/VDU) the inputted formula, ended by a question mark '?', and then it will wait for the user to confirm the formula, by typing another ';', or to enter the formula again, if an error was detected.

STENSOR has been successfully applied in investigating many problems in the fields of classical relativity (see for example [33]), quantum gravity [23] and supergravity [34]. In the next section, more recent applications of STENSOR are reported in the course of illustrating the properties of STENSOR (and SHEEP as well) discussed above.

1-5 APPLICATIONS

1-5-1 Introduction

Applications are the crucial test for the usefulness and efficiency of computing systems. Some of them also serve to add to the power of these systems, and their facilities, such applications are usually referred to as the "user library".

In what follows, two of the applications that were carried out at Imperial College are discussed. These applications were selected because they demonstrate the most important features of STENSOR and SHEEP, while serving as a simple tutorial for the system.

1-5-2 The Rainich Conditions

G.Y. Rainich [35] analyzed the relationship between the Riemann (curvature) and the Maxwell (electromagnetic)

tensors. He arrived at a set of constraints on the first tensor, that should be satisfied to allow the presence of a non-null electromagnetic tensor field in the corresponding Einstein field equations. These constraints are equivalent to demanding the vanishing of the following two tensors:

$$\text{RAIN1}_a^c = R_a^b R_b^c - (1/4) \delta_c^a R_d^b R_b^d = 0 \quad (1.8a)$$

$$\text{RAIN2}_{ab} = \alpha_{[a,b]} = 0 \quad (1.8b)$$

Where R_{ab} = mixed Ricci tensor, and $\alpha_a = \frac{1}{R_d^b R_b^d} \epsilon_{abcd} R_e^{b,ed;c}$

These tensors provide a straight forward method for testing whether a given solution of Einstein's equations includes electromagnetism, or not. It is clear however, that the calculation involved can be horrendous! especially for complicated metrics, for which the computer's help is very necessary.

A very simple program is written in STENSOR, introducing the RAIN1 and RAIN2 tensors to SHEEP. This program is needed only once since, after loading it to STENSOR, the command SAVBU (SAVE BULf⁵) can be applied to the tensors concerned and this command, in turn, will use the TCOMP (Tensor COMPiler) command to generate a program for computing the individual components of the desired tensors, via SHEEP, for any given metric. The generated program is saved on a separate file, which can be loaded onto a "naked"

⁵Bulf is a technical term which refers to "automatic program-generation" program.

version of SHEEP (i.e. SHEEP on its own, where it is not necessary to have STENSOR running as well), and after entering the component of the metric $g_{\alpha\beta}$, the command WMAKE can be applied to RAIN1 and RAIN2 tensors, to obtain their exact values. The file containing the definitions of RAIN1 and RAIN2 tensors is exhibited in figure (F2), the simple and straight forward nature of the commands used is apparent. Note that the "%" means that the rest of the line is a comment, i.e. it is there to clarify the program to the reader, and will be overlooked by the computer.

To use this program, the following should be done:

1- Log-on to the computer and start STENSOR by typing 'stensor'.

2- Load the file containing the necessary definitions, using the SLISP command (LOAD "RAIN.DEF"), the response of the computer during the loading process is self-explanatory, (cf. the listing on page 174 of appendix C).

3- Use the command SAVBU as suggested: (SAVBU RAINICH RAIN1 RAIN2), STENSOR will then generate programs for computing RAIN1 and RAIN2, then save them in a file called "rainich.blf".

To apply the Rainich test to a given metric, one needs to enter the components of this metric, via the command RPL (see SHEEP manual), then using the command (WMAKE RAIN1 RAIN2), and leaving the rest to the computer!

One interesting application was the verification that, contrary to his claim, Wilson's solution [36] does not satisfy

```

% *** rain.def ***
(PRIN2
% The next three lines will be printed when this file is loaded.
"The Rainich Conditions: RAIN1=RAIN2=0
=====
(Exact Solutions P. 72 (5.20,21))
RAIN1 and RAIN2 are computed in frame, from RICMF, HUU and GAMF")
% define a new tensor RIC2 to simplify calculations:

% PDEF command is invoked, with the name of the tensor
% to be defined as an argument.
(PDEF RIC2)
<RICMF A B><RICMF B A> $
% RICMF is the built-in Mixed RICci tensor in the
% Frame version of SHEEP.

% The first, Algebraic Rainich Condition (IBID (5.20)):

(PDEF RAIN1 1)
<RICMF A B><RICMF B C>-1/4<DEL C A><RIC2 > $
% DEL is the usual Kroneker delta.

(PDEF RICU S12 1 2)
<RICMF A B><HUU B C> $ % HUU is the frame metric.
% Take the co-variant derivative:
(PDEF RICUC S12 1 2 R) <RICU A B ;C> $
% Below EDEF this RICUC-TSR!!
(PDEF GAMU 1 S23) <GAMF K B C><HUU A K> $
% Only GAMF known in FRAME
(PDEF RICUU S12 1 2 3 R)<RICUC A B C><HUU C D> $
(PDEF ALF) 1/<RIC2 ><EPS4 A B C E><RICMF B D><RICUU D E C>$

% The second, Analytic Rainich Cond (IBID (5.21)):

(PDEF RAIN2 A12) <ALF A ,B>-<ALF B ,A> $

% Write and Evaluate DEFinition:
(WEDEF RICUC)
% Must EDEF this RICUC-TSR!! do EDEF by hand if not reloaded
% as said above. EDEF expands the COVAR-DIFF(;) into GAMU`s.
% Write and Simplify DEFinition:
(WSDEF ALF)
%If TCOMP complains, do SDEF to canonicalize formulea.

% Now do (TCOMP GAMU) explicitly,
% if yo do this in cord (to replace old BLF).
% Then, do (SAVBU rainich RAIN1 RAIN2),
% and answer Y(es) on TCOMPilation.

```

Figure F2: A listing of the Rainich program

the Rainich conditions. In fact the second Rainich tensor turned out to be:

$$\text{Rain2}_{12} = -(3/x)^{1/2} \quad , \quad \text{and} \quad \text{Rain1} = 0 \quad (1.9)$$

1-5-3 The Gamma-Algebra

Another more involved application of STENSOR, is the creation of an algebraic system for the γ -matrices of field theory. A description of this system will be given here, while the listings of the basic programs, and results are left for a separate appendix (appendix A).

The γ -matrices (known as Dirac matrices in 4-dimensions) are of great importance in theoretical physics. They always feature in scattering cross-section computations, and all sorts of calculations in supergravity and field theory.

The noncommutativity of these matrices, which form a Clifford's algebra, makes their handling extremely cumbersome. It is no exaggeration to say that the computation of some expressions involving γ -matrices, by hand, is practically impossible!. Hence the need for a computerized system arises.

To start off, the only relation needed is:

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab} \quad (1.10)$$

which is the basic relation in Clifford's algebra. Noting that the matrix indices are suppressed here, in fact $\gamma_a \equiv \gamma_a^{ij}$, and (1.10) should read:

$$\gamma_{(a}^i \gamma_{b)}^j = \eta_{ab} \delta^{ij} \quad (1.11)$$

where, $a, b, \dots = 1, \dots, n$ and $i, j, \dots = 1, \dots, N$.

Using (1.10), one can easily find (still suppressing the matrix indices):

$$\eta^{ab} \gamma_a \gamma_b = \gamma_a \gamma^a = n \quad (1.12)$$

By definition, $\text{tr } \gamma_a = 0$, and using (1.10) again, implies: straight forward calculation yields the general formulae for tracing:

$$\text{tr}(\gamma_{a_1} \gamma_{a_2} \dots \gamma_{a_p}) = \sum_{i=2}^p (-1)^i \eta_{a_1 a_i} \text{tr}(\gamma_{a_2} \dots \hat{\gamma}_{a_i} \dots \gamma_{a_p}) \quad (1.13)$$

If 'p' is even.

Where the hat '^' symbol over γ_{a_i} indicates that this matrix is missing from the product. This recursive relation reflects the mounting complexity of computations for large values of p, since the number of terms in each tracing is given by: $\frac{(p-1)!}{2^{p/2-1} (p/2-1)!}$, and each term contains p/2 different η 's which, for p = 8, becomes 105, and for p = 10, is 945 terms!, each containing 5 gammas's!!.

The γ -algebra system was built in stage-wise process, during which, the results obtained by the computer at each stage of the calculation, were added to the input information, thus, facilitating the calculations of the next stage, and so on.

In the first instance, the computer was instructed not to commute any two terms automatically, except the Kronecker delta (since otherwise, ordinary tensor algebra would apply

where STENSOR arranges factors according to lexicographical order, as a part of simplification). This was done by the commands⁶: (ON FOINS AUTONEVERCOMMUTE) and (ALWAYS COMMUTE DEL).

Other switches are also turned on, to help in simplification. The basic relation (1.10) is not given to the computer in its sum-substitution form, but rather, it was used as an ordinary substitution with an additional requirement imposed on the order of the indices:

```
(SETSUB SW1 (ORDERASK A B))
```

```
<T B<<T A<    $
```

```
-<T A<<T B< + 2 <DEL A B> $
```

Where SW1 is the name of the substitution list which contains the given substitution, the second and third lines of input are respectively the mask and the substitute. This substitution, whenever used in calculations, will first search for all occurrences of the "mask", which is any product of two γ 's, then will check whether the indices are in the wrong lexicographical order, if so, it will do the substitution, otherwise it will not!. Checking the order of indices can be done interactively by turning on the ORDERASK switch, then the user's decision on ordering supercedes the lexicographical implications.

Another basic substitution is:

⁶ For a complete explanation of the STENSOR commands used here, refer to the STENSOR manual [10]

$$\gamma^a \gamma_a = n \quad (1.14)$$

For the first stage of work, no more input is needed, except defining the quantities that we wish to calculate.

Suppose that we want to evaluate the contraction:

$$\gamma^c \gamma_{abc} \quad (1.15)$$

Where a γ with more than one index represents the totally antisymmetrized product of the appropriate number of single γ -matrices.

The intuitive approach would be to expand γ_{abc} fully, and then on applying SW1, we will have terms of the form: $\gamma_a \gamma_b \gamma^c \gamma_c$ and $\gamma_a \gamma_b \dots$ then using (1.14) we get the answer:

$$\gamma^c \gamma_{abc} = (d-2) \gamma_{ab} \quad (1.16)$$

The relation (1.16) can be defined as a new substitution, and then used in future calculations (for example in computing $\gamma^{ac} \gamma_{bdc}$ etc).

It is profitable to place different types of substitutions in different substitution lists, and then to use the suitable sub-list at the appropriate stage of the calculation. This would eliminate ambiguity, and put less strain on the simplification algorithm, by limiting the number of "masks" it has to match.

After a few similar exercises, the user will acquire enough experience to be able to set beforehand the sequence of substitutions needed for a particular exercise, which would cut the time needed for interaction.

Iterating the process described above, a large library was built, containing almost all basic possible combinations of γ 's that may be required in dimensions < 7 calculations. However, keeping the dimensions $= n$ throughout the work, without specifying its value, makes the interpolation to $n > 7$ rather academic!. The value for n can actually be fixed at the last step of the calculation.

The collection of results obtained are saved on separate files, which can be loaded to the system whenever required, rather having to recompute them from a scratch. Some of these results are probably at the limit of human capacity: for example, it is expected that not many people would try to verify by hand the result: $\gamma^{abc}\gamma_{ijk}\gamma_{abc} = (228+128n+21n^2+n^3)\gamma_{ijk}$. obtained by the computer, interactively, in about 5 minutes.

A special file of this "library" was devoted for the tracing relations.

Although a special recursive function could be constructed for to compute the trace of any combination of γ 's, it is more efficient from computing-time point of view to do these traces up to a certain level (say a product of 10 γ 's) only once, and save the results on a file. Since, on the one hand, using the results obtained so far, practically all physical computations can be reduced to traces of a finite product of γ 's. While on the other hand, it is not practical to repeat the same computation again and again, as is necessary for the recursive definition: because, for

evaluating the trace of the product of n - γ 's, the computer has to evaluate $\frac{(p-1)!}{2^{p/2-1}(p/2-1)!}$ traces of $(n-1)$ - γ 's each, and so on, until the number of γ 's in each trace reduces to 0, i.e. we are left with tracing the delta function, which has the known result: $\text{tr}\delta_{ij} = N$.

A typical scattering x-section calculation, would involve terms of the form:

$$\text{tr}(\gamma_a \not{A} \gamma_b \not{B}) \quad \text{and} \quad \text{tr}(\gamma_a \not{A} \gamma_b \not{B} \gamma^a \not{C} \gamma^b \not{D}) \quad (1.17)$$

where, $\not{A} = A_a \gamma^a + M$ etc.

Both of these terms were calculated in less than a minute! (a listing of the input and output files is given in appendix A).

Another important addition to the γ -algebra, are the Fierz substitutions [37]. For example, one of the Fierz rearrangements in 7-dimensions is:

$$\begin{aligned} \eta_e \eta_b \gamma_\gamma \eta_a &= (1/48) \gamma^{ijk} \eta_a \eta_b \gamma_\gamma \eta_{ijk} \eta_e - (1/16) \gamma^{ij} \eta_a \eta_b \gamma_\gamma \eta_{ij} \\ &\quad - (1/8) \gamma^i \eta_a \eta_b \gamma_\gamma \eta_i \eta_e + (1/8) \eta_a \eta_b \gamma_\gamma \eta_e \end{aligned} \quad (1.18)$$

where η is a spinor, and γ_γ stands for a γ of any order. This can be defined in STENSOR, using the WILDCARD command (see STENSOR manual).

Obviously, Fierz rearrangements are not particularly easy to handle! even with the help of the computer, they may still be very tricky, and only careful interactive manipulation can be effective. A portion of the substitution program for Fierz rearrangements is listed in table T3.

What has been presented in this section is meant to give a flavor of the way the computer algebra system SHEEP (and its extensions) operate. This would hopefully help deciding whether it would be suitable for a given problem or not, but in no way can replace the "manual", or even claim to exhibit all the features.

```

% *** fiertz.sbs
% Turn the following switches on:
(ON POTSIM FOINS AUTONEVERCOMMUTE)
% POTSIM the "power" simplification.
% FOINS the formula input reader.
% and AUTONEVERCOMMUTE (self explanatory).
% The exception for never-commute is of course delta:
(ALWAYS COMMUTE DEL)
(USESETUL FZ)      % declare a new substitution list FZ
(REMSP XI YD)     % remove previous substitution properties
                  % on XI and YD.

% ?T stands for any gamma matrix, with unspecified number
% of indices.
% ==== Headed by XI: =====
(PDEF TENO) <XI <      <N E< <Y B<<! ?T <      <N A < $
(PDEF FIENO) 1/8      <N A< <Y B<<! ?T <      <N E <
    - 1/8 <T I <      <N A< <Y B<<! ?T < <T I < <N E <
    - 1/16 <T I J <  <N A< <Y B<<! ?T < <T I J <<N E <
    + 1/48 <T I J K <<N A< <Y B<<! ?T < <T I J K <<N E < $
(SETSUB FZ (ORDERASK A E))
<:TEN0 A B E> $ <XI <<:FIENO A B E> $

(PDEF TEN2) <XI <<! ?T < <N E< <Y B<<! ?T <      <N A < $
% Here we have TWO different ?T's in the "mask" definition,
% they can be distinguished in the "substitute" by assing
% the number 2 for calling the 2nd of them:
(PDEF FIEN2) 1/8      <N A< <Y B<<! ?T 2>      <! ?T <<N E<
    - 1/8 <T I <      <N A< <Y B<<! ?T 2> <T I <<! ?T <<N E<
    - 1/16 <T I J <  <N A< <Y B<<! ?T 2> <T I J <<! ?T <<N E<
+ 1/48 <T I J K <<N A< <Y B<<! ?T 2> <T I J K <<! ?T <<N E< $

(SETSUB FZ (ORDERASK A E))
<:TEN2 A B E> $ <XI <<:FIEN2 A B E> $

% Note that for making the character ? acceptable to LISP
% it had to be quoted by a '!'.

```

Table T3: The Fietz substitutions program

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Chapter Two

EMBEDDINGS**2-1 INTRODUCTION**

Einstein's theory of general relativity establishes a very intimate connection between gravitational fields, and the geometry of a class of Riemannian manifolds, particularly, 4-dimensional manifolds with Lorentzian signature.

An n -dimensional abstract manifold is defined as a paracompact Hausdorff topological space², which admits an atlas, i.e. it can be covered by a collection of local-coordinate patches with coordinates: $(x^\alpha, \alpha=1,2\dots d)$. For Riemannian manifolds extra structure is imposed, namely, covariant differentiation and the metric 'g', for making them adequate as physical models.

Historically, the idea of Riemannian manifolds evolved from the concept of surfaces in Euclidean spaces (of arbitrary dimension). On such surfaces, the number of independent coordinates (i.e. the effective dimension of the surface) is less than the dimension of the total space, such a surface can be parametrized (locally) by the coordinates $(x^\alpha, \alpha=1,2\dots d)$. If $(y^A, A=1,2\dots d')$ are the coordinates of the total Euclidean space, then one can relate these to the

¹Some authors exclude either paracompactness or Hausdorffness, or both, when defining manifolds. However, for our work both of these are needed for the rigorous proofs of theorems that will be assumed

x^α 's on the surface, the set of these relations is nothing but the "equation" of the surface:

$$y^A = y^A(x^\alpha) \quad (2.1)$$

The metric on such surfaces is non-trivial (i.e. not of Euclidean type), in fact the differential distance between two neighbouring points on the surface is given by:

$$ds^2 = \eta_{AB} y_{,\alpha}^A(x) y_{,\beta}^B(x) dx^\alpha dx^\beta \quad (2.2)$$

From (2.2) one can immediately read off the metric:

$$g_{\alpha\beta}(x) = \eta_{AB} y_{,\alpha}^A(x) y_{,\beta}^B(x) \quad (2.3)$$

Hence the obvious generalization is to consider spaces which admit, locally, metric-tensors of the form: $g_{\alpha\beta}(x)$ (we do not know yet whether functions $y^A(x)$ can be found for which the relation (2.3) holds), the only restrictions on such metrics are that they should be symmetric w.r.t. the indices α and β , and they should be non-degenerate.

From what has been said so far, it is natural to enquire whether the reverse problem can be solved? i.e. whether abstract manifolds can be, in general, embedded in Euclidean spaces of sufficiently high dimensions?. This leads to the study of embedding, which is a sub-case of the theory of submanifolds.

There are other "geometrical", as well as physical motivations for studying the embedding problem: clearly, embedding offers a better insight into the structure of a given manifold and its global properties, it provides a tool to inves-

tigating singularities, by eliminating those caused by a "bad choice" of parametrization [1], and to find the maximal extension of the manifold considered [2]. In principle, global embedding should pave the way for assigning a suitable boundary for the manifolds and thus "completing" it. A possible link between elementary particle symmetries and extrinsic invariants of embedded space-times was suggested in [4], while Fronsdal [5] used the embedding techniques to study elementary particle properties on a background manifold of constant curvature.

However, despite the apparently very interesting nature of the embedding problem, research in this field does not attract many relativists, this is mainly due to the profound complexity of the mathematics involved. And unfortunately, there is so far no clear physical interpretation of the mathematical quantities and invariants introduced by this theory.

2-2 SUBMANIFOLDS

2-2-1 Existence Theorems

Rigorously speaking, an immersion of a manifold M^d into another manifold $M'^{d'}$ is a map $f: M \rightarrow M'$, such that: $\text{rank}(f_*) = d$, at all points $x \in M^d$. Furthermore, the immersion is called C^k $k=1,2,\dots$, if f is of differentiability class C^k .

Embedding is defined as an injective immersion.

For example, if we consider the printed characters as maps from a real segment $I \subset \mathbb{R}$ into the two-dimensional plane \mathbb{R}^2 , then C, O and S are embeddings, α , ρ and ι are immersions (since they contain double-points) while v , w and Y are neither because they have cusps which make $\text{rank}(f_*) > 1$.

If the metric forms¹ of M and M' are related via:

$$g(X,Y) = g'(f_*X, f_*Y) \quad X, Y \in T_x(M) \quad (2.4)$$

then f is an isometric embedding at the point $x \in M$. If (2.4) holds for all $x \in U \subset M$, then the isometric embedding is local, it is called global if $U=M$.

In local embedding, (2.4) can be expressed as:

$$g_{AB} = g'_{\alpha\beta} y^A_{,\alpha} y^B_{,\beta} \quad (2.5)$$

where (x^α) are the coordinates of $x \in M$ and (y^A) are the coordinates of $f(x) \in M'$. And the components of the metric form are defined as usual by: $g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$. (note that (2.5)

¹See appendix B for the notational conventions adopted in this thesis.

is identical to (2.3))

In most cases, the question of existence of embedding was settled positively. The following results, concerning existence, are quoted:

i) For abstract manifolds:

Whitney's embedding theorem [10]: Every d -dimensional manifold can be embedded in E^{2d} .

ii) For Riemannian manifolds:

(a) Local embedding:

Generalized Cartan-Janet theorem [11]: Any analytic Riemannian manifold $M^{(p,q)}(g)$, locally can be analytically and isometrically embedded in any $M^{(p',q')}(g')$, provided that: $d' > d(d+1)/2$, [for C^∞ manifolds, this should read [12]: $d' > d(d+3)/2$] preserving $p' > p$ and $q' > q$, simultaneously.

(b) Global embedding:

(Improved) Nash theorem [13]: Any C^∞ -Riemannian manifold $M^{(p,q)}$ with C^k -Riemannian metric of rank r and signature s , has a C^∞ -global and isometric embedding into an Euclidean space $E^{(p',q')}$, with:

$$p' > d - (1/2)(r+s) + 1$$

and $q' > d(3d+11)/2$ for compact M

$$q' > (d/2)(2d+37) + (5/2)d + 1 \quad \text{for non-compact } M.$$

if the metric is C^∞ , then [13]:

$$p' = q' > d(d+5)/2 \quad \text{for compact } M$$

and $d' = q' > 2d(2d+1)(d+3)$ for non-compact M

It is interesting to note that the existence theorems

displayed above, impose no restriction on the choice of the metrics g and g' (except for conditions on dimension and signature) nor they restrict the global topological structure of the manifolds.

The embedding formalism can be easily constructed within the frame work of fibre bundles. Here the manifold M is considered as already embedded in M' , i.e. a submanifold of M' , therefore, no explicit reference to the embedding function f is needed, except for translating results to coordinate-dependent forms. Hence, no distinction will be made between $x \in M$ and $f(x) \in M'$.

Define $N_x(M)$ the normal space to M at a point $x \in M$, w.r.t. its embedding in M' , to be the complement to $T_x(M)$ in $T_x(M')$, (more rigorously $T_{f(x)}(M')$). i.e. $T_x(M') = T_x(M) \oplus N_x(M)$.

And let $N(M) = \bigcup_{x \in M} N_x(M)$ be the normal bundle. It is easy to see that:

$$T_M(M') = T(M) \oplus N(M)$$

$$\text{where } T_M(M') := (\pi')^{-1}(M) = \bigcup_{x \in M} T_x(M')$$

The projection maps τ and ν are defined to separate the tangential and transverse parts respectively:

$$\tau : T_M(M') \rightarrow T(M) \quad \text{and} \quad \nu : T_M(M') \rightarrow N(M)$$

Let ξ_i $i = 1, 2, \dots, d'-d$, be $d'-d$ x -sections of $N(M)$ that form a basis in each fibre, with corresponding dual 1-forms ω^i , defined by: $\omega^i(\xi_j) = \delta_j^i$ and $\omega^i(X) = 0$ for all $X \in T(M)$

Therefore we can write $\nu(X) = \omega^i(X)\xi_i$, for $X \in T_M(M')$

obviously: $\tau(X) = X - \nu(X)$.

2-2-2 The 2nd Fundamental Forms

The torsion-free linear connection ∇' on M' induces, in a natural way [14], a torsionless connection ∇ on M via the map τ . Let X and Y be sections of $T(M)$, then $\tau(\nabla'_X Y) = \nabla_X Y$.

The 2nd fundamental form α is defined to be the transverse component of $\nabla'_X Y$ i.e. $\alpha(X, Y) := \nu(\nabla'_X Y)$

Which implies that:

$$\nabla'_X Y = \nabla_X Y + \alpha(X, Y) \quad (2.6a)$$

We may locally write: $\alpha(X, Y) = \alpha^i(X, Y)\xi_i$

The forms $\alpha^i(X, Y)$ are known as the 2nd FFs in classical literature. Since ∇' is torsion-free, it follows that the map $\alpha : T(M) \times T(M) \rightarrow N(M)$ is symmetric consequently the α^i 's should be also symmetric.

Let ξ be a section of $N(M)$. The covariant derivative of ξ w.r.t. X can be decomposed into tangential and normal components in a similar fashion:

$$\nabla'_X \xi = D_X \xi + A(X, \xi) \quad (2.6b)$$

here the normal component $D_X \xi$ is the covariant derivative w.r.t. the induced connection on the normal bundle [14], also known as the "shape operator" in classical literature. And for a fixed ξ , A is the symmetric linear transformation of $T(M)$ with respect to the metric, which corresponds to the function α on $T(M) \times T(M)$, i.e.:

$$g'(\alpha(X, Y), \xi) = -g'(A(X, \xi), Y) \quad (2.7)$$

It is trivial to show that α and A are bilinear.

Analogous relations can be obtained for 1-forms on M :

Locally, on M , chose an "adapted" frame (X_α, ξ_i) , $\alpha=1, \dots, d$ and $i=1, \dots, d'-d$, with X_α 's tangent to M . Then, define the matrices of one-forms:

$$[\alpha]_\alpha^i = \alpha^i(X_\alpha, \cdot) \quad (2.8a)$$

$$[D]_j^i = \omega^i(D_\cdot \xi_i) \quad (2.8b)$$

$$[A]_i^\alpha = \omega^\alpha A(\cdot, \xi_i) \quad (2.8c)$$

From (2.4) we get:

$$g_{ij} \alpha_\alpha^j = -g_{\alpha\beta} A_i^\beta \quad (2.9)$$

These matrices help in expressing the results (2.6) in a more compact form, which allows great simplifications in future calculations.

Let Y' be a x -section of $T_M(M')$ then, we can write:

$$Y' = \tau(Y') + \nu(Y') = Y + \xi$$

using a locally adapted frame: $Y' = Y^\alpha X_\alpha + Y^i \xi_i$ Therefore:

$$\begin{aligned} \nabla_Z Y' &= \nabla_Z Y + Y^i A_i^\alpha(Z) X_\alpha \\ &\quad + \alpha^i(Z, Y) \xi_i + Y^i D_Z \xi_i + (\nabla_Z Y^i) \xi_i \\ &= \nabla_Z Y - Y^i g_{ij} g^{\alpha\beta} \alpha_\beta^i(Z) X_\alpha \\ &\quad + Y^\alpha \alpha_\alpha^i(Z) \xi_i + Y^j D_j^i(Z) \xi_i + (\nabla_Z Y^i) \xi_i \end{aligned}$$

If d' stands for covariant exterior differentiation of tensor valued p -forms of M' (w.r.t. the connection ∇'), we have:

$$d'Y' = dY' + \begin{pmatrix} 0 & A \\ \alpha & D \end{pmatrix} \wedge Y' \quad (2.10)$$

treating Y' as a column vector: $\begin{pmatrix} Y^\alpha \\ Y^i \end{pmatrix}$ (2.10) can be written as:

$$d' = d + \Gamma \wedge \quad (2.11)$$

The product law \wedge , is a generalization of the usual \wedge , allowing (implicitly) for matrix product, therefore anticommutation should be used with care!.

Here the connection one-form Γ is the block-matrix $\begin{pmatrix} 0 & A \\ \alpha & D \end{pmatrix}$.

2-2-3 The GCR Equations

The integrability conditions for (2.11) can be found by applying a second exterior differentiation, this yields:

$$d'^2 Y' = d^2 Y' + \Gamma \wedge \Gamma \wedge Y' + d\Gamma \wedge Y' \quad (2.12)$$

Let Ω' be the Riemannian two form on M' :

$$\Omega'_{\quad D}^{\quad C} = (1/2) R'_{\alpha\beta D}^{\quad C} dx^\alpha dx^\beta \quad (2.13)$$

and let $\bar{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix}$. Using the relation $\Omega \times Y = d^2 Y$ we can rewrite (2.12) as:

$$\Omega' = \bar{\Omega} + \Gamma \wedge \Gamma + d\Gamma \quad (2.14)$$

substituting for the value of Γ , we get after some algebra:

$$\Omega' = \begin{pmatrix} \Omega'_{11} & \Omega'_{21} \\ \Omega'_{12} & \Omega'_{22} \end{pmatrix} \quad (2.15)$$

where

$$\Omega'_{11} = \Omega + A \wedge \alpha$$

$$\Omega'_{12} = dA + A \wedge D$$

$$\Omega'_{21} = d\alpha + D \wedge \alpha$$

$$\Omega'_{22} = dD + D \wedge D + \alpha \wedge A$$

To calculate the explicit components of the Riemann tensor, R'_{ABCD} relative to a given frame of local coordinates (x^α) , using (2.13) and (2.14), the following 4 equations are obtained:

$$R'^{\alpha}_{\beta\gamma\delta} = R^{\alpha}_{\beta\gamma\delta} + 2\alpha^i_{\beta}[\gamma^A_{\alpha\delta}]_i \quad (2.16a)$$

$$(1/2)R'^i_{\beta\gamma\delta} = \alpha^i_{\beta}[\gamma;\delta] - D^i_j[\gamma^{\alpha j}_{\delta}]_{\beta} \quad (2.16b)$$

$$(1/2)R'^{\alpha}_{i\gamma\delta} = A_{\alpha i}[\gamma;\delta] + D^j_i[\gamma^A_{\alpha\delta}]_j \quad (2.16c)$$

$$(1/2)R'^i_{j\gamma\delta} = D^i_j[\gamma;\delta] - D^i_k[\gamma^D^k_{\delta}]_j - \alpha^i_{\alpha}[\gamma^A_{\alpha\delta}]_j \quad (2.16d)$$

Where $\alpha^i_{\beta\gamma} dx^\gamma := \alpha^i_{\beta}$ etc.

Upon replacing A by its value relative to α (cf (2.9)) it is clear that (2.16b) and (2.16c) are identical, and the only independent equations are:

$$R'^{\alpha}_{\beta\gamma\delta} = R^{\alpha}_{\beta\gamma\delta} - 2g_{ij}\alpha^i_{\beta}[\gamma^{\alpha j}_{\delta}]_{\alpha} \quad (2.17a)$$

$$R'^i_{\beta\gamma\delta} = 2\alpha^i_{\beta}[\gamma;\delta] - 2D^i_j[\gamma^{\alpha j}_{\delta}]_{\beta} \quad (2.17b)$$

$$R'^i_{j\gamma\delta} = 2D^i_j[\gamma;\delta] - 2D^i_k[\gamma^D^k_{\delta}]_j + 2\alpha^i_{\alpha}[\gamma^{\alpha k}_{\delta}]_{\beta} g^{\alpha\beta} g_{jk} \quad (2.17c)$$

Equations (2.17a, b and c) are better known as the Gauss Codazzi and Ricci equations (GCR for short). For the particular case when the embedding space is Euclidean, i.e. $R' = 0$, the GCR equations are necessary and sufficient

conditions for the embedding to exist [14].

Nevertheless, the GCR equations are not completely independent! this was discussed as early as the year 1936 by Thomas [15], who was looking for a set of algebraic conditions, necessary and sufficient for embedding a d -dimensional Riemannian manifold in a $d+1$ -dimensional Euclidean space, when he found that if the second fundamental form was of the full rank, the Gauss equations imply the Codazzi equations, for $d > 3$. This result was generalized to embeddings into Euclidean spaces of $n+2$ -dimensions in [17], where it was found that, provided that one of the 2nd fundamental forms is of full rank, the Gauss equations (2.17a), together with one set of the Codazzi equations (taking for example $i=1$ in (2.17b)), imply the other set of Codazzi (i.e. for $i=2$ in (2.17b)) and the Ricci equations.

On a more general level, the Bianchi identities:

$$R'_{AB}[CD;E] = 0 \quad (2.18)$$

provide the interdependence constraints to the GCR equations. These constraints were explicitly presented by Blum [18] and later generalized to the indefinite case by Goenner [19], who used the modern coordinate-independent notation. Using our notation however, provide the simplest way for deriving the interdependence relations, since the Bianchi identities (2.18) are equivalent to:

$$d'\Omega' = 0 \quad (2.19)$$

Therefore only a single exterior derivation should be

applied to the Riemann 2-form (2.14).

First, define the two-form Π to be:

$$\Pi = \Omega' - \bar{\Omega} - \Gamma \wedge \Gamma - d\Gamma \quad (2.20)$$

with the block-diagonal form:

$$\Pi = \begin{pmatrix} G & C \\ C^t & H \end{pmatrix} \quad (2.21)$$

where the matrices of 2-forms: G , C and H will vanish if the Gauss, Codazzi and Ricci equations are respectively satisfied.

Apply exterior differentiation on (2.20), this gives:

$$d\Pi = d\Omega' - d\bar{\Omega} - d\Gamma \wedge \Gamma + \Gamma \wedge d\Gamma - \bar{\Omega} \wedge \Gamma + \Gamma \wedge \bar{\Omega} \quad (2.22)$$

The Bianchi identities for Ω and Ω' are:

$$d\Omega = 0 \quad (=> \quad d\bar{\Omega} = 0) \quad (2.23a)$$

$$\text{and } d'\Omega' = d\Omega' + \Gamma \times \Omega' - \Omega' \times \Gamma = 0 \quad (2.23b)$$

replacing (2.23) into (2.22), we get:

$$\begin{aligned} d\Pi &= -\Gamma \wedge \Omega' + \Omega' \wedge \Gamma - d\Gamma \wedge \Gamma + \Gamma \wedge d\Gamma - \bar{\Omega} \wedge \Gamma + \Gamma \wedge \bar{\Omega} \\ &= -\Gamma \times \Pi + \Pi \times \Gamma \end{aligned} \quad (2.24)$$

This can be expanded to three equations:

$$dG = C \wedge \alpha - A \wedge C^t \quad (2.25a)$$

$$dC^t = H \wedge \alpha - \alpha \wedge G - D \wedge C^t \quad (2.25b)$$

$$dH = H \wedge D - D \wedge H + C^t \wedge A - \alpha \wedge C \quad (2.25c)$$

The local-coordinate form of (2.25) is:

$$G_{\alpha\beta[\gamma\delta;\epsilon]} = C_{\alpha i[\gamma\delta\alpha^i|\beta|\epsilon]} - \alpha^i_{\alpha}[\gamma^C|\beta i|\delta\epsilon] \quad (2.26a)$$

$$C_{i\alpha[\beta\gamma;\delta]} = H^i_j[\beta\gamma\alpha^i|\alpha|\delta] - \alpha^i_e[\beta^G|\alpha|\beta\delta] - D^i_j[\beta^C|\alpha j|\gamma\delta] \quad (2.26b)$$

$$H_{ij[\alpha\beta;\delta]} = H_{ik[\alpha\beta^D^k|j|\delta]} - D_{ik}[\alpha^H^k|j|\beta\delta] + C_{\gamma i[\alpha\beta\alpha^\gamma|j|\delta]} - \alpha_{i\gamma}[\alpha^C|j|\beta\delta] \quad (2.26c)$$

The explicit values of G , C and H can be looked up from the relations (2.17). Blum noted that (2.26) do not depend explicitly on the embedding therefore this result for flat embeddings could be extended to any embedding type.

These results can be used to reduce the number of equations that should be solved in the GCR system, by eliminating redundancies, for example: suppose that a set 2nd fundamental forms is found that satisfies the Gauss equations, i.e. $G^\alpha_{\beta\gamma\delta}$ and $G^\alpha_{\beta\gamma\delta;\epsilon} = 0$, this implies that (2.26a) reduces to a system of linear and homogeneous equations in the C 's which restricts the number of independent components of the Codazzi tensor. In particular, for $M^d_{cM^{d'+1}}$ (respectively $M^d_{cM^{d'+2}}$) the result of Thomas [16] (respectively Gupta and Goel [17]) is verified.

It is interesting to note that Gupta and Goel [17] did not know of Blum's result, although it was published two and a half decades before their work!.

2-3 SPACE-TIMES EMBEDDINGS

2-3-1 The Class

In general relativity the embedding procedure can be used for obtaining arithmetic invariants characterizing the intrinsic geometry of a given space-time. The existence theorems presented in (2-2) suggest the investigation of a particular arithmetic invariant, namely the "embedding class"; which is defined as follows:

The embedding class of a d -dimensional manifold M^d , is the minimum number ' p ' such that M could be embedded in an Euclidean space $E^{d'}$ of dimension $d' = d + p$.

Clearly, other types of embedding class numbers can be defined, by varying the choice of the embedding space, in general, these class numbers are different for the same M , for example, the well known extended Schwarzschild space-time has a class number $p = 1$ if the embedding manifold is conformally decomposable [3], while its class is $p = 2$ when embedded in a flat space [6, p200]. Therefore for the sake of conformity, and for reducing ambiguity, the structure of the embedding space should be fixed in advance, and in this work, only flat embedding will be considered. This choice is based on intuitive and subjective inclinations rather than on physical arguments! simply because a clear physical understanding of the "class" is still lacking. In fact, previous work on embedding include the study of embeddings into spaces of constant curvature [6], conformally flat [7]

or Ricci-flat [9], etc.

The argument for adopting minimal embedding is also very loosely supported by physical interpretations!. Indeed, non-minimal flat embedding have been used (more or less successfully) in studying geometrical problems like the geodesics in Godel-Synge spaces [20], physical problems, like the scalar and Neutrino fields in the Godel universe [21], and even in a new approach to the embedding itself, defined as algebraic embedding (see later). However, it transpires that relaxing the minimality requirement from embedding without imposing alternative conditions, introduces further new degrees of freedom that could be unexplainable! for even Minkowski space-time can be embedded in any $E^{d'}$ for $d' > 4$ (for example, consider the embedding: $y^1 = \cos(t)$, $y^2 = \sin(t)$, $y^3 = x$, $y^4 = y$, $y^5 = z$. Then clearly: $e_i (dy^i)^2 = dt^2 - dx^2 - dy^2 - dz^2$, with $e_1 = e_2 = -e_3 = -e_4 = -e_5 = 1$. Therefore, minimal embedding could be adopted, at least, for being rather less problematic than other embedding that one can use.

Arithmetic extrinsic invariants can also be defined, e.g. the type-number [22] which is identical to the rank of the 2nd fundamental form for class one embedding.

2-3-2 Algebraic and Global Embedding

Besides minimal l.i.e., other types of embedding were studied, two of these types appear to give help for clarifying the underlying geometrical structure of space-time, these

are the algebraic local isometric embedding and the global isometric embedding.

In algebraic embedding, space-time is considered as an algebraic 4-dimensional variety of E^d that would naturally exclude multiple points. This type of embedding was investigated by Ferraris and Francaviglia [32] for a few simple metrics, where they established its usefulness in explaining some geometric properties of the considered space-times. However, no systematic approach for obtaining the algebraic local isometric embedding of a given manifold is known, which greatly reduces the credibility of this scheme.

The interpretation of singularities as "boundary points" of space-time has contributed immensely to the interest in global embedding, since clearly, the best way for visualizing the boundary of a space-time is through regarding it as a subset of an Euclidean space. Unfortunately however, as in the case of algebraic embedding, no general method for solving global embedding is known, therefore, physicists resort to techniques for extending the local isometric embedding of the given space, this approach suffers many disadvantages, and works only in limited number of cases, it also cannot provide, in general, a way for completing the manifold, i.e. there would still be geodesics running into singularities! consequently, the important question of fixing global topology remains unresolved.

2-3-3 Embedding Techniques

From a physicists point of view, studying the local isometric embedding of a given space-time M^d , is usually linked to determining the embedding function itself, i.e. finding the values of $y^A(x^\alpha)$ that describe the surface M^d in E^d . This is called "explicit embedding". However, no short-cut technique is available for solving for these functions in general. One therefore have to resort to "guess work"! which when combined with the suggestions offered by Rosen [23] for embedding certain patterns of line-elements, will lead to an embedding in most cases, but will not guarantee the minimality. When the explicit embedding of a space-time is known it is possible to calculate the 2nd fundamental forms and the Ricci vectors, using the equations (2.44a,b).

A more natural approach to the embedding problem is to find first the implicit embedding, i.e. to solve for the 2nd fundamental forms and Ricci vectors, using the GCR equations and then using these, one try to look for an explicit embedding. However, this method is generally not adopted in literature, except for the odd cases when the GCR are relatively intelligible, like for example, the cases of Schwarzschild [24], Narliker-Karmakar [25] spaces and plane-fronted gravitational waves [28].

Looking back at the local coordinate form of the GCR equations (2.4), it is clear that these equations are highly non-linear, the number of unknown functions is significantly large: For a class 'p' embedding, there are $pd(p+d)/2$ unknown functions determining the 2nd fundamental forms and the

Ricci torsion vectors. Therefore, it is not surprising that even for a seemingly simple metric, like Godel's space, no solution has yet been found.

Attempts were made to find necessary and sufficient conditions, of algebraic nature, for the existence of a solution for the GCR equations. These have been successful only in the simplest case of class 1 embedding [16], where necessary and sufficient conditions in the form of tensor equations were determined [26]. For class 2 embedding, only necessary conditions were found [27] while no significant results are known for higher classes!.

Since a general solution of the GCR equations is unobtainable, attention was given to specialized cases, in the hope that these cases would be sufficient for testing the usefulness of the embedding approach. One way of testing a special case is by restricting the choice of the 2nd fundamental form, which would simplify (and possibly linearize) the GCR equations.

Barnes [29] wrote down the Jordan canonical forms for the rank 2 symmetric tensor, corresponding to the different Segre types in a 4-dimensional space with Lorentzian signature as follows:

$$A_{\alpha\beta} = -\lambda_0 u_{\alpha} u_{\beta} + \lambda_1 s_{\alpha} s_{\beta} + \lambda_2 e_{\alpha} e_{\beta} + \lambda_3 f_{\alpha} f_{\beta} \quad \text{for type [1111]}$$

$$A_{\alpha\beta} = \alpha(-u_{\alpha} u_{\beta} + s_{\alpha} s_{\beta}) + 2\beta u_{(\alpha} s_{\beta)} + \lambda_2 e_{\alpha} e_{\beta} + \lambda_3 f_{\alpha} f_{\beta} \quad \text{for type [zz\bar{1}1]}$$

$$A_{\alpha\beta} = +2\lambda_1 l_{(\alpha} n_{\beta)} + l_{\alpha} l_{\beta} + \lambda_2 e_{\alpha} e_{\beta} + \lambda_3 f_{\alpha} f_{\beta} \quad \text{for type [211]}$$

$$A_{\alpha\beta} = +\lambda_1 \left[2l_{(\alpha} n_{\beta)} + e_{\alpha} e_{\beta} \right] + 2l_{(\alpha} e_{\beta)} + \lambda_3 f_{\alpha} f_{\beta} \quad \text{for type [31]}$$

where the coefficients λ_i , $i = 0, \dots, 3$ are the eigenvalues of A , α and β are the real and imaginary parts of the complex eigenvalue, and the frame vectors satisfy: $-u_{\alpha} u^{\alpha} = s_{\alpha} s^{\alpha} = e_{\alpha} e^{\alpha} = l_{\alpha} l^{\alpha} = 1$ and all other inner products vanish.

He used these forms to link the Segre classification of the 2nd fundamental forms of class 1 embedding, to the algebraic classifications (Petrov types) of the Ricci and Weyl tensors, and consequently to study the case of perfect fluids of class one.

The relationship between these canonical forms and the Petrov classification for empty space-times (which are of embedding class 2) have been used by Hodgkinson [30] to determine the possible algebraic structure of the two 2nd fundamental tensors. It was hoped that this would provide a mechanism for obtaining all possible empty space-times [31], and a program was written for STENSOR to obtain the differential equations that would determine the frame vectors for each possible case. The resulting equations turned out to be harder than the GCR equations themselves! and thus not particularly useful.

2-4 SPACE DEFORMATIONS

2-4-1 More Embedding Equations!

So far, differentiation in the normal space have not been used, simply because the classical interpretation of embedding excludes variations in the directions transverse to the manifold itself. However, interesting results can be obtained from studying space deformations in the normal directions, which may have physical relevance, as shall be discussed in this section. Unfortunately though, there has been virtually no research into this area! and to the my knowledge, only the work of Kerner [34] exists in this field. The following will include a full generalization of Kerner's work.

Parallel to what has been done in the previous section, concerning computations in the tangent bundle, we shall assume the following structure for the covariant differentiation w.r.t. the "normal" vectors:

$$\nabla'_{\xi}\eta = \tilde{\partial}_{\xi}\eta + \beta(\xi,\eta) \quad (2.27a)$$

$$\nabla'_{\xi}X = \hat{\partial}_{\xi}X + B(\xi,X) \quad (2.27b)$$

Locally, we may chose the normal vector fields to be geodesics in the normal bundle, without loss of generality, this would make $\nabla'_{\xi}\eta = 0$ always.

Using the torsionlessness property of the Riemann spaces, we can relate the normal and tangent components of $\nabla'_{\xi}X$, namely $\hat{\partial}_{\xi}X$ and $B(\xi,X)$ to the 2nd fundamental form[~]and the

Ricci torsion-vectors. Noting that X and ξ commute always (which is a consequence of splitting M' in the said way), we have:

$$\hat{\partial}_{\xi} X \equiv D_X \xi \quad (2.28a)$$

$$B(\xi, X) \equiv A(X, \xi) \quad (2.28b)$$

On M , the metric form of a tangent vector and a normal one should vanish, however this is not necessarily true elsewhere. Indeed, expanding $\nabla'_{\xi} g'(X, \xi) = 0$, we get:

$$\xi . g'(X, \xi) = g'(D_X \xi, \eta) \quad (2.29)$$

in local coordinates, (2.29) becomes:

$$g'_{i\alpha, j} = g'_{ik} D^k_{j\alpha} \quad (2.30)$$

Consider now a pair of commuting X -sections of $N(M)$ η and ζ , from the definition of connection we have:

$$2g'(\nabla'_{\eta} \zeta, X) = \zeta . g'(\eta, X) + \eta . g'(\zeta, X) - X . g'(\zeta, \eta) = 0 \quad (2.31)$$

since $\nabla'_{\eta} \zeta = 0$. Deriving (2.31) w.r.t. ξ yields:

$$\xi \zeta g'(\eta, X) + \xi \eta g'(\zeta, X) = 0 \quad (2.32)$$

similarly, we have:

$$\zeta \xi g'(\eta, X) + \zeta \eta g'(\xi, X) = 0 \quad (2.33)$$

subtracting (2.33) from (2.32) gives:

$$\eta \left[\xi g'(\zeta, X) - \zeta g'(\xi, X) \right] = 0 \quad (2.34)$$

but we also have:

$$2g'(\nabla'_{X} \xi, \zeta) = X . g'(\zeta, \xi) + \xi . g'(\zeta, X) - \zeta . g'(X, \xi)$$

Therefore (2.34) can be written as:

$$\eta g'(\nabla'_x \xi, \zeta) = 0 \quad (2.35a)$$

this implies that $g'(\nabla'_x \xi, \zeta)$ is constant w.r.t. transverse deformations.

Using (2.6b) and (2.30) we obtain the following local coordinate form for (2.35a):

$$D_{j i \alpha, l} = -\alpha_{j \alpha \beta} D_{i l}^{\beta} \quad (2.35b)$$

As far as the 2nd fundamental form is concerned, straight forward computation reveal that:

$$g'(\alpha(X, Y), \xi) = -(1/2) \xi \cdot g'(X, Y) \quad (2.36a)$$

or locally:

$$g_{i j} \alpha_{\alpha \beta}^i = -(1/2) g_{\alpha \beta, j} \quad (2.36b)$$

from equation (2.36) we can readily interpret the 2nd fundamental form as the rate of metric deformation along normal directions, with a factor of $-1/2$. Each of the classical 2nd fundamental forms $\alpha_{\alpha \beta}^i$ is associated with the deformation of the metric $g_{\alpha \beta}$ along a particular normal direction ξ_i .

Computing the curvature tensor, using the same method as in the previous section, we arrive at two more (new) equations, which can be written in the local-coordinate system chosen above as:

$$R'_{i \alpha \beta j} = \alpha_{i \alpha \beta, j}^i + \alpha_{j \beta}^{\gamma} \alpha_{i \gamma \alpha} - D_{i k \alpha} D_{j \beta}^k - D_{i j \alpha; \beta} \quad (2.37)$$

$$R'_{\gamma \alpha \beta i} = -\alpha_{\gamma \beta i; \alpha} + \alpha_{j \beta}^{\gamma} \alpha_{i \gamma \alpha} - D_{i j \beta} \alpha_{\gamma \alpha}^j + \Gamma_{\alpha \beta \gamma, i} \quad (2.38)$$

In the case of flat embedding, $R' = 0$, antisymmetrizing w.r.t. α and β , (2.37) becomes the Ricci equation (2.17c), while (2.38) becomes the Codazzi equation (2.27b). Therefore, it is clear that (2.37) and (2.38) are consistent with the standard GCR set of equations (as they should).

Obviously, adding (2.35), (2.37) and (2.38) to the GCR system (2.17) will only make the system more complicated, even the number of variables will increase to d' instead of d . But then, the total system will be describing the normal deformation of the embedded space in addition to the embedding itself.

2-4-2 The Deformation Formalism

To avoid confusion, we shall henceforth use a tilde sign "~" to denote the general (deformation-dependent) tensor, while the un-tilded tensor will be its restriction to M .

Now, consider an infinitesimal transverse deformation of magnitude ϵ and direction V^i , using Taylor's expansion to the 2nd order in ϵ , we can write:

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon g_{\alpha\beta, j} V^j + \epsilon^2/2 g_{\alpha\beta, jk} V^j V^k + O(\epsilon^3) \quad (2.39)$$

and similarly for $\alpha, D \dots$

Considering flat embedding, then $R' = 0$ and in a suitable choice of normal coordinates: $g_{ij} = \eta_{ij}$. Using (2.36b) and (2.37) we can write (2.39) as:

(2.40)

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - 2\epsilon \eta_{ij} V^i \alpha^i_{\alpha\beta} + \epsilon^2 \eta_{ij} (\alpha^i_{\alpha\gamma} \alpha^\gamma_{k\beta} - D_{j\alpha}^i D_{k\beta}^j - D_{k\alpha; \beta}^i) V^i V^k + O(\epsilon^3)$$

The transverse deformation formalism can be used to gen-

erate solutions to Einstein's field equations "near" a given solution, these new solutions would satisfy Einstein's equations up to a fixed order in ϵ , the deformation magnitude (incidentally this would also suggest a procedure for quantizing space-times, by quantizing the deformation field [33]).

The perturbed form of the Riemann tensor can be most easily obtained from the Gauss equation (2.17a) and the perturbation of the 2nd fundamental forms, up to the first order in ϵ we have:

$$\begin{aligned} \tilde{R}_{\alpha\beta\gamma\delta} = & R_{\alpha\beta\gamma\delta} - 2\epsilon\eta_{ik}V^j\{D_{j\beta}^k;[\gamma^{\alpha i}\delta]_{\alpha} \\ & + D_{l\beta}^k D_j^l[\gamma^{\alpha i}\delta]_{\alpha} - D_{j\alpha}^i;[\gamma^{\alpha k}\delta]_{\beta} - D_{l\alpha}^i D_j^l[\gamma^{\alpha k}\delta]_{\beta} \\ & + \eta_{jl}g^{ef}(\alpha^k_{\beta e}\alpha^i_{\alpha}[\gamma^{\alpha l}\delta]_f - \alpha^k_{\alpha e}\alpha^i_{\beta}[\gamma^{\alpha l}\delta]_f)\} + O(\epsilon^2) \end{aligned} \quad (2.41)$$

where the new equation (2.37) have been used for calculating the value of the derivative of the 2nd fundamental form in the normal direction.

Now, assuming a similar perturbation development for the cosmological constant $\tilde{\lambda}$ and the energy-momentum tensor $\tilde{T}_{\alpha\beta}$, we can expand Einstein's field equations w.r.t. ϵ . First define the (new) Einstein-equations tensor by:

$$E_{\alpha\beta} = R_{\alpha\beta} - ((1/2)R + \lambda)g_{\alpha\beta} - \chi_0 T_{\alpha\beta} \quad (2.42)$$

Clearly $E_{\alpha\beta}$ vanishes if the Einstein field equations are satisfied.

Let $g_{\alpha\beta}$ be a given metric satisfying $E_{\alpha\beta} = 0$ for a given λ and $T_{\alpha\beta}$, and assume that the embedding of $g_{\alpha\beta}$ known. Then \tilde{g} is also known (at least formally). In general, \tilde{g} does not satisfy $\tilde{E} = 0$, however, this can be used as a constraint for

restricting the deformation options. Working to the first order, we have:

$$\begin{aligned} \tilde{E}_{\alpha\beta} = & \epsilon\eta_{ij}V^k\{g^{\gamma\delta}{}_{\alpha}{}^i{}_{\beta}[\delta^{\alpha}{}^j]_{\gamma,k} + g^{\gamma\delta}{}_{\alpha}{}^i{}_{\delta}[\alpha^{\alpha}{}^j]_{\gamma}\beta,k + \delta_k^i R^{\alpha}{}^j{}_{\alpha\beta} \\ & + g^{\gamma\delta} g^{ef} g_{\alpha\beta}{}^i{}_{f}[e^{\alpha}{}^j]_{\gamma,k} + 2\delta_k^i g^{\gamma\delta} g^{ef} R_{\gamma\alpha\beta e}{}^{\alpha}{}^j{}_{\delta f} + 2\delta_k^i{}_{\alpha}{}^j{}_{\alpha\beta} \\ & + 2\delta_k^i g^{cd} g^{ef} g_{\alpha\beta} R_{ce}{}^{\alpha}{}^j{}_{df}\} - g_{\alpha\beta}\lambda_1 - \chi_0 T_{1\alpha\beta} \end{aligned} \quad (2.43)$$

where the derivative of $\tilde{g}^{\alpha\beta}$ was obtained via the relation:

$$\tilde{g}^{\alpha\beta,\gamma} = -g^{\alpha\delta} g^{\beta\epsilon} g_{\delta\beta,\epsilon}$$

(2.43) provides a powerful tool for studying transverse deformations of space-times, especially when compared to the formalism in [37], which involves solving second order differential equations!. Therefore, hopefully more interesting space-times can be studied, and as examples we study Einstein static, deSitter, Robertson-Walker and Schwarzschild spaces.

2-4-3 Topology of Space-Times

One of the most interesting, and yet most difficult problems of general relativity is the study of the topology of the set of solutions of Einstein's field equations, this would provide the most natural way of classifying space-times and would settle the question of equivalence [38] of any two given solutions. The fundamental question of defining "distance" for the proposed topology introduced in the space of symmetric tensor fields of type (0,2) on M, can be stated as follows: Given an exact solution g of (2.42), do there exist another exact solution $\tilde{g}_{\alpha\beta}$ of these same equa-

tions, which can be regarded as being near g ? and are there any "isolated points" [40] in the set of all solutions of these equations?.

It is easy to demonstrate that embedding formalism is indispensable for a complete analysis of this problem through the following example:

Consider the Minkowskian metric $\eta_{ij} = \text{diag}(1, -1, -1, -1)$, this metric can be realized on a flat space E^4 , or on a cylinder C^4 embedded in E^5 , with:

$$z_1 = t,$$

$$z_2 = x,$$

$$z_3 = y,$$

$$z_4 = R \cos(z/R)$$

$$z_5 = R \sin(z/R).$$

Both manifolds (E^4 and C^4) are exact solutions of Einstein's field equations, however, they clearly cannot be considered as "near" each other, because of the fundamental difference in their topologies. Hence, the 4-dimensional picture on its own is not adequate for a complete understanding of this problem, and in general, there is no guarantee that the two metrics, arbitrarily near each other in a suitable topology, can be realized on the 4-dimensional manifold of the same kind.

Kerner [37], proposed the method of deformation of embeddings for investigating this problem, however, his formalism involves very hard and complicated calculations and works

only for Einstein spaces [40] which satisfy the relation $R_{\alpha\beta} = (1/4) g_{\alpha\beta}$, which imposes a severe restriction on the class of metrics that could be studied, and the only examples he gave were the Minkowski, Einstein and deSitter spaces, which are, at most, of embedding class one!.

Using the equations developed in the previous section, we are able to propose an alternative scheme to that of Kerner, for doing the same calculations. This scheme has, in principle, no restrictions on the types of metrics that it can handle, and although the algebra involved is still very hard, this formalism is completely algebraic, and does not require solving differential equations of any order. This very fact meant that we could give the new formalism to the computer and have no further worries about the computational aspect of the problem!. A full listing of the input file is given in a separate appendix (see appendix C).

Our formalism also involves the 2nd fundamental forms and the Ricci vectors, i.e. the implicit embedding must be known beforehand. This is still a less severe demand than finding the explicit embedding (which is required for Kerner's approach). In particular if the explicit embedding function is known, straight forward algebra can be used to find the 2nd fundamental forms and the Ricci vectors (which calls again for the use of computer algebra).

From the definition of the 2nd fundamental forms and Ricci vectors given above, we have:

$$\begin{aligned}\alpha^i_{\alpha\beta} &= g'(\xi_i, \alpha(\partial_\alpha, \partial_\beta)) \\ &= g'_{AB} n_i^A (y^B_{;\alpha\beta} + \Gamma'^B_{CD} y^C_{,\alpha} y^D_{,\beta})\end{aligned}\quad (2.44a)$$

$$\begin{aligned}D_{ij\alpha} &= g'(\xi_i, D_\alpha \xi_j) \\ &= g'_{AB} n_i^A (n^B_{j,\alpha} + \Gamma'^B_{CD} n^C_{j,\alpha} y^D_{,\alpha})\end{aligned}\quad (2.44b)$$

In particular, for flat embedding, $g' \rightarrow \eta$ and $\Gamma' \rightarrow 0$. Obviously, the first step for calculating and t is finding the normal vectors n_i 's. This is done in the following manner: 1st the p.d (where $p = d' - d$) relations:

$$z_1 := \eta_{AB} y^A_{,\alpha} n_i^A = 0 \quad (2.45)$$

which state that the normal vectors are actually perpendicular to the space-time, should be fully exploited, then the $p/2$ ($p + 1$) orthonormality relations of the set of normal vectors: $\eta_{AB} n_i^A n_j^B = 0$ are employed to determine the most of the unknown components of the normals, the rest of the unknown components (if any) can be found via careful analysis of the Gauss equation, which can be constructed from the information available hitherto (this step is not needed for class one embeddings, $p = 1$). Interactive computing is invaluable for carrying out such a calculation as is shown in the following example for computing the normal vector field for the deSitter universe [42]:

2-4-4 Applications

(i) Deformation of deSitter space-time

deSitter's universe is a static spherically symmetric and homogeneous Einstein universe [41, from page 335], the line

element is given by:

$$ds^2 = (1-A) dt^2 - (1-A)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \quad (2.46)$$

where: $A = (r/C)^2$ and $C = \text{constant}$, the radius of the universe.

This can be readily embedded in a 5-dimensional Euclidean space, with the embedding function given by:

$$\begin{aligned} z_1 &= (C^2 - r^2)^{1/2} \text{sh}(t/C) \\ z_2 &= (C^2 - r^2)^{1/2} \text{ch}(t/C) \\ z_3 &= r \cos(\theta) \\ z_4 &= r \cos(\varphi) \sin(\theta) \\ z_5 &= r \sin(\varphi) \sin(\theta) \end{aligned} \quad (2.47)$$

the metric for this space is: $\eta_{AB} = \text{diag}(1, -1, -1, -1, -1)$.

Computing the value of Z_1 , we get:

$$Z_{10} = (C^2 - r^2)^{1/2} [\text{ch}(t/C)n^1 - \text{sh}(t/C)n^2]/C \quad (2.48a)$$

$$\begin{aligned} Z_{11} &= -r(C^2 - r^2)^{-1/2} [\text{sh}(t/C)n^1 - \text{ch}(t/C)n^2] - [\cos\theta]n^3 \\ &\quad - [\sin\theta \cos\varphi]n^4 - [\sin\theta \sin\varphi]n^5 \end{aligned} \quad (2.48b)$$

$$Z_{12} = r[\sin\theta]n^3 - r[\cos\theta \cos\varphi]n^4 - r[\cos\theta \sin\varphi]n^5 \quad (2.48c)$$

$$Z_{13} = r \sin\theta ([\sin\theta]n^4 - [\cos\theta]n^5) \quad (2.48d)$$

The system $Z_1 = 0$ can be easily solved interactively, by eliminating the components of n w.r.t. each other gradually, the first of such eliminations is clearly: replacing this value for $n^5 = [\tan\theta]n^4$ will cause Z_{13} to vanish immediately, and Z_{12} will depend on n^3 and n^4 only. Simplifying Z_1 and

repeating the above step we enable to calculate n^A up to a multiplicative factor, which in turn can be fixed by the fact that the "length" of n^A is unity, i.e. $\eta_{AB} n^A n^B = -1$. Eventually we get the value of n^A , which turn out to be proportional to the embedding functions y^A ; $n^A = (1/C) y^A$.

The computation of the 2nd fundamental form becomes straight forward and we find that it is proportional to the metric; $\alpha_{\alpha\beta} = -(1/C) g_{\alpha\beta}$

Using this result in the deformation equation, we find the magnitude V of the deformation to be constant, proportional to the ratio of the deformation of the cosmological constant to that constant itself:

$$V = (C/2\lambda) \lambda_1 \quad (2.49)$$

which is identical to Kerner's result, up to a scaling factor.

(ii) Repeating the same work for Einstein's space-time [43], whose line element is given by:

$$ds^2 = dt^2 - (1-A)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \quad (2.50)$$

we get the 2nd fundamental form: $\alpha_{\hat{\alpha}\hat{\beta}} = -(1/C) g_{\hat{\alpha}\hat{\beta}}$ with $\hat{\alpha} = 1, 2, 3$ and $\alpha_{0\alpha} = 0$ and the deformation is identical to the that for deSitter.

(iii) Taking now a more complicated metric which generalizes the previous metrics, namely the non-static homogeneous model (also known as Robertson-Walker) with line element [44]:

$$ds^2 = dt^2 - f^{-1}(t) [dr^2 - \sin^2 r (d\theta^2 + \sin^2 \theta d\varphi^2)] \quad (2.51)$$

The 2nd fundamental form depends on f :

$$(\alpha_{\alpha\beta}) = (1+f^2)^{1/2} \begin{pmatrix} -\dot{f} (1+f^2)^{-1} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & f \sin^2 r & 0 \\ 0 & 0 & 0 & f \sin^2 r \sin^2 \theta \end{pmatrix} \quad (2.52)$$

The deformation factor becomes:

$$V = \lambda_1 \frac{f^2 (1+f^2)^2}{4\dot{f}^2 - (1+f^2)(4\dot{f} - 2f\lambda)} \quad (2.53)$$

This clearly gives back the previous results for $f = 1$.

(iv) Finally, we apply our scheme to a more interesting space-time, the Schwarzschild universe, which is clearly beyond Kerner's approach because it is of embedding class 2 and it does not satisfy the requirements of Einstein spaces.

The Schwarzschild metric, which describes a static, spherically symmetric universe, has the following line element:

$$ds^2 = (1-A) dt^2 - (1-A)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (2.54)$$

where: $A = (2M/r)$, $M = \text{constant}$,

Being the oldest known exact solution for Einstein's field equations, it received a lot of attention, and although it is not very realistic from the cosmological point of view, it has been very helpful in understanding important physical problems, at least qualitatively (for example the general relativistic perihelion shift in the planetary orbits [34]).

A local isometric embedding of Schwarzschild space was obtained by Kasner [35], soon after Schwarzschild published

his result. Much later, Fronsdal [36] embedded Schwarzschild into a flat space with Lorentzian signature. Other authors have investigated the embedding of Schwarzschild and its implications, and probably the most complete analysis is given in [24], since there, the implicit embedding is solved first, yielding an explicit embedding which includes the embeddings of Kasner and Fronsdal (and others) as special cases!.

Let us turn now to the deformation problem of Schwarzschild space¹ Here we have two 2nd fundamental forms, $\alpha^i_{\alpha\beta} = 1,2$ which can be calculated as prescribed above. We find

$$(\alpha^1_{\alpha\beta}) = (MB/2r^3)^{1/2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.55)$$

$$(\alpha^2_{\alpha\beta}) = (Br)^{-1}(M/2r)^{1/2} \begin{pmatrix} B^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2Br^2 & 0 \\ 0 & 0 & 0 & 2Br^2 \sin^2 \theta \end{pmatrix}$$

where: $B = 1-2M/r$

For this particular choice of 2nd fundamental forms, the Ricci vector does not vanish and it is given by:

$$D_{12\alpha} = (0, -3\sqrt{B/2r}, 0, 0) \quad (2.56)$$

Although it is well known, from the works in [24] and [27]

¹Here we use a new implicit embedding of Schwarzschild, different from that in [24]

(and others) that the Ricci vector can be made equal to zero by an appropriate choice of the normal vectors, in this work, the vanishing of D_α will not be demanded because, on the one hand it is not actually needed, and on the other hand its presence helps to demonstrate the potential of our formalism, which can work equally well for the most general cases.

Here we have two deformation vectors, corresponding to deforming the space-time in the directions of the 2 normals. We obtain the following:

$$\begin{aligned}
 \Delta E_{00} &= (3B/2r^3)(2MB/r)^{1/2}(1-8M/r)V_1 - (3BM/r^4)(2M/r)^{1/2}(1-8M/r)V_2 \\
 &\quad - B\lambda_1 \\
 \Delta E_{11} &= (1/2r^3)(2M/Br)^{1/2}(3-8M/r)V_1 + (3/2Br^3)(2M/r)^{1/2}(1-4M/r)V_2 \\
 &\quad + B^{-1}\lambda_1 \\
 \Delta E_{22} &= (1/8r)(2M/Br)^{1/2}(4M/r + 6BM/r - 27B^2)V_1 + \\
 &\quad (9/8r)\sqrt{\frac{2M}{r}}(1 - 6M/r)V_2 + r^2\lambda_1 \\
 \Delta E_{33} &= (\Delta E_{22}) \sin^2\theta \tag{2.57}
 \end{aligned}$$

2-4-5 Discussion

A full answer to the question : what does the normal deformation of space-times mean physically? is not yet known; Kerner (the only person to deal with the question of normal deformation, prior to my work) did not attempt to discuss the physical implications of this technique at all. In the following, only superficial explanation of the above results is offered, hoping that future research will clarify the deeper implications of this scheme.

(i) DeSitter: for this space-time, we can easily see that

the 1st order deformed metric is proportional to the metric (of the unperturbed space), in fact:

$$g_{\alpha\beta} = -2Vg_{\alpha\beta}$$

Which implies that the deformed metric (up to first order) is:

$$\tilde{g}_{\alpha\beta} = (1-2V\epsilon)g_{\alpha\beta} + O(\epsilon^2)$$

Using this, it is easy to show that the effect of such deformation can be obtained by rescaling the coordinates t and r by a factor $\sqrt{1-2V\epsilon}$ each! Therefore, a normal deformation of de'Sitter is equivalent to a rescaling of the time and radial coordinates, and the cosmological constant. This implies that all physical and cosmological properties of de'Sitter are preserved under normal deformations. Also, the knowledge of the scales of the coordinates would determine the cosmological constant λ precisely and vice versa.

The same result can be easily obtained for the Einstein's universe, except that the time (t) coordinate will be left unscaled.

(ii) Robertson-Walker: Here the situation is more complicated because of the use of an arbitrary function $f(t)$. It is clear from the deformation equation above (2.53) that in general, V can depend on time, which means that one cannot absorb it in the deformed part of the cosmological constant (as was done in the previous cases) therefore, a non-trivial energy-momentum tensor must be added to balance this result, i.e. a matter field should be created! This is a very

interesting consequence of normal deformation that should receive a lot of attention.

(iv) Schwarzschild: We had originally no cosmological constant, and clearly the cosmological constant (introduced via perturbation) can not explain the rest of the terms in (2.57), hence we are forced to introduce a matter-field, in the form of the first order expansion of the, otherwise empty!, energy-momentum tensor. Clearly, this result is consistent with the well known fact that the Schwarzschild solution of Einstein's equations, which is the only empty space-time which is static and spherically symmetric [8].

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Chapter Three

KALUZA-KLEIN THEORIES**3-1 INTRODUCTION**

3-1-1 Embedding and Higher Dimensional Theories

The profound complexity of the embedding formalism, exhibited in the previous chapter, was a "convincing argument" for most physicists to forsake it! however, the basic theme of working in higher dimensional spaces (i.e. spaces of dimension > 4) is still held by many as a possible framework for obtaining the ultimate grand unified theory, which should explain gravity, electromagnetism and elementary particle physics, in one go!. This time, however, the additional dimensions are chosen manually, with the particular properties that would provide the desired additional fields to the Lagrangian of the theory. It is important at this stage to clarify the relationship between such theories and embedding:

- i) Theories described above are still embedding theories, in the broad sense of the term. i.e. these theories involve the study of manifolds that are embedded in larger manifolds, with the restriction that the normal bundle is chosen to (at least) include a tangent bundle of an appropriately chosen manifold (in Kaluza-Klein theories, the inclusion becomes identification).

ii) Clearly therefore, these theories as they stand, have no relation to flat embeddings, except in the instance when the flat embedding of one of the manifolds involved was to be studied separately (as suggested in [2]).

iii) Additional dimensions are completely physical, and should be placed on equal footing with the "intuitive" 4-dimensions of space-time. This is an important observation for studying the consistency of such theories [1]. To explain the fact that these additional dimensions are not obvious (to say the least!) in the daily experience of mankind, it is alleged that they are of periodic nature with a very small amplitude (of the order of Planck's length).

3-1-2 Kaluza-Klein and Unified Theories

The use of Kaluza-Klein theories in the quest for unifying the fundamental forces of nature is not new; in fact the very first paper by Kaluza [3] was directed the particular problem of unifying Maxwell's electromagnetic field with Einstein's gravitational field (of general relativity). The electromagnetic gauge transformations were interpreted as coordinates transformations in the extra dimension, in such a way that the invariance of the 5-dimensional line element is preserved. (This model will be explained in more details in the next section).

Kaluza-Klein theory spent a long time in obscurity, mostly because of Einstein's declared opinion; that this theory "do

not bring us nearer to the true solution of the fundamental problem", which is, according to him, the logically distinct structures of the gravitational field, and the matter (including electromagnetic) fields [30].

The first interesting generalization of the Kaluza-Klein theory was suggested by deWitt [4], who considered theories in higher-dimensions (higher than 5) with non-Abelian gauge fields coming into the stage as part of the generalized metric. The need for non-Abelian gauge fields became apparent after the introduction of the Salam [5]-Weinberg [6] unified model for weak and electromagnetic interactions with gauge group $SU(2) \times U(1)$.

Further interest in higher-dimensional field theories arose with the advent of supergravity theories and the corresponding unification schemes. Aspects of these theories will be discussed in chapter 4, while in the present chapter, we will only study the use of supergravity in constructing important Kaluza-Klein models, while concentrating mainly on the techniques of dimensional reduction and spontaneous compactification.

3-1-3 Kaluza-Klein and Supergravity

Using the term Kaluza-Klein "theories", in the plural tense, should be clear by now; since more than one theory have been so far mentioned, which adopts the basic Kaluza-Klein idea. Even when restricting ourselves to Kaluza-Klein formulation of supergravity, we are left with a number of

options, corresponding to the freedom in choosing the topology and dimension of the additional manifold, and the metric for the total space.

This chapter however, will be restricted to the special case of $N = 8$ supergravity theory, obtained from $N = 1$ supergravity over an 11-dimensional total space, taking the "squashed seven-sphere" S^7_S , as our compact subspace. Unfortunately, there are no compelling theoretical arguments for favoring this particular theory!, however, there are some indications supporting our choice of a 7-dimensional compact space. The most important of these observations is given by Witten [7]:

The number 11 is the maximum number of dimensions allowed for space-time admitting a supergravity theory, since for $d > 11$, spin > 2 fields will have to be used (which cannot be accounted for physically).

This same number " $11 = 4 + 7$ ", is also the minimum number of dimensions required for accommodating an isometry group $G = SU(3) \times SU(2) \times U(1)$, (which is needed for the unification of strong and electro-weak interactions), in the additional compact space.

In section (3-2) of this chapter, the original Kaluza-Klein theory in 5-dimensions is presented, as a simple introduction to Kaluza-Klein basic techniques. Then in (3-3), a generalization of the work in (3-2) is given, and the basic question of finding an adequate ansatz for the metric is addressed. (3-4) and (3-5) contain the calculation done

using a new ansatz, together with analysis and conclusions.

3-2 ORIGINAL KALUZA-KLEIN IN 5-DIMENSIONS

3-2-1 Formalism

The original theory proposed by Kaluza [3], and Klein [8-9], was simply to consider a 5-dimensional space-time, consisting of the usual 4-dimensional space-time and an additional space-like dimension with the topology of a circle (i.e. periodic). Thus the 5-dimensional manifold is, at least locally, of the form $M \times S^1$.

The Hilbert action is generalized in a natural way to 5-dimensions,

$$I_{EH} = (\kappa^{-2}) \int \sqrt{-g'} R' d^5 X' \quad (3.2.1)$$

and the idea is simply to integrate away the 5th dimension to reduce the action to a 4-dimensional one, describing Einstein-Maxwell theory. To be able to achieve this goal, the ansatz for the 5-dimensional metric was carefully constructed to be:

$$g'_{AB} = (\varphi)^{-1/3} \begin{bmatrix} g_{\alpha\beta} - \kappa^2 \varphi A_\alpha A_\beta & -\kappa \varphi A_\beta \\ -\kappa \varphi A_\alpha & -\varphi \end{bmatrix} \quad (3.2.2)$$

where $\varphi = g'_{55}(x^\alpha, x^5)$ is a scalar field, A_μ is a vector field, which plays the role of the 4-dimensional electromagnetic field. i.e. it has units of mass, therefore $\kappa = 4 \sqrt{\pi G}$ ($G =$ the gravitational constant) is needed to render κA_μ dimensionless.

To see that A_μ identifies with the electromagnetic gauge field of Maxwell's theory, we consider the isometry

transformation of the line element:

$$ds'^2 = (\varphi)^{-1/3} ds^2 - (\varphi)^{2/3} (dx^5 + \kappa A_\alpha dx^\alpha)^2 \quad (3.2.3)$$

obtained from the metric (3.2.2), where $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ is the usual 4-dimensional line element.

Assuming that A_α would transform as a covariant vector field w.r.t. general coordinate transformations on M , we shall only study its transformation under a change of the 5th coordinate

$$x^5 \rightarrow \tilde{x}^5(x^5, x^\alpha)$$

We shall assume φ and A_α to depend only on x^α for the time being, concentrating on the study of the massless sector. Therefore, the possible transformation of the 5th coordinate is:

$$\tilde{x}^5 = x^5 + f(x^\alpha) \quad (3.2.4)$$

where f is an arbitrary function of x^α .

Replacing this in the last term of (3.2.3), we get:

$$dx^5 + \kappa A_\alpha dx^\alpha \rightarrow dx^5 + [\kappa \tilde{A}_\alpha(x) + \partial_\alpha f(x)] dx^\alpha \quad (3.2.5)$$

The invariance requirement fixes the transformation law for A :

$$A_\alpha \rightarrow \tilde{A}_\alpha = A_\alpha(x) - \kappa^{-1} \partial_\alpha f(x) \quad (3.2.6)$$

Which is nothing but a gauge transformation for the vector field A_α !.

3-2-2 The Lagrangian

An explicit calculation of the 5-dimensional Ricci scalar

(R') yields:

$$R' = (\varphi)^{1/3} \left[R - (\kappa/4)^2 \varphi F_{\alpha\beta} F^{\alpha\beta} - (1/6\varphi^2) \partial_{\alpha}\varphi \partial^{\alpha}\varphi \right] \quad (3.2.7)$$

where $F_{\alpha\beta} = 2\partial_{[\alpha}A_{\beta]}$ is the usual field strength in the electromagnetic theory. R is the 4-dimensional Ricci scalar.

Replacing (2.3.7) into the Einstein-Hilbert (EH) action (3.2.1), and performing the "trivial" integration w.r.t. the 5th coordinate, we get:

$$I_{\text{EHYM}} = (\kappa^{-2}) \int d^4x \sqrt{-g} \left[R - (\kappa/4)^2 \varphi F_{\alpha\beta} F^{\alpha\beta} - (1/6\varphi^2) \partial_{\alpha}\varphi \partial^{\alpha}\varphi \right] \quad (3.2.8)$$

This action describes a 4-dimensional theory of massless spin 2 graviton ($g_{\alpha\beta}$), a massless spin 1 photon (A_{α}) and a massless spin 0 (scalar) field (φ).

Assuming x^5 dependence for $g_{\alpha\beta}$, A_{α} , and φ we can expand these fields in a Fourier series of the form:

$$F(x^{\alpha}, x^5) = \sum_{n=-\infty}^{+\infty} F_n(x^{\alpha}) e^{inx^5}$$

The ground state Lagrangian (for $n = 0$ in the Fourier expansion) provides the same expression as the massless sector (3.2.8), and the other terms, will add an infinite tower of massive states [10].

It is interesting to remark that, choosing the "amplitude" of the fifth dimension to be very small (of the order $\sim 10^{-33}$ cm) would serve at least three purposes in Kaluza-Klein theory:

i) Explains the hiding of the 5th dimension from the human eye!.

ii) Gives correct value for the 4-dimensional gauge

coupling constant.

iii) The massive states obtained in the Lagrangian have masses in excess of 10^{19} GEV, which explains why they are not observed in present-day labs [11].

3-2-3 Discussion

The periodicity of the 5th dimension, which results from the compactness of S^1 , plays a crucial role in the Kaluza-Klein theory as is clear from above. First, it leads to the quantization of the electric charge in a natural and geometrical way. Secondly, it allows the expansion of the different fields into a sum of harmonics (in this case: Fourier harmonics), with coefficients depending only on the "observed" (x^α) coordinates, and thus allowing the interpretation of the transformations in the fifth dimension as gauge transformations in the 4-dimensional physical world.

Generalizing this idea to non-Abelian gauge transformations of Yang-Mills (YM) theory appears to be straight forward; all what is needed is a higher dimensional compact space (dimension $d > 1$) to replace the circle S^1 in the above work. The topology of this space will be prescribed in such a way that it includes the necessary symmetry groups that provide the desired gauge fields, which are determined by the killing vectors of the compact space.

For our Kaluza-Klein theory to include strong and electroweak interactions the internal compact space must have $SU(3) \times SU(2) \times U(1)$ as an isometry group. This does not neces-

sarily imply that the above group should be a subset of the compact space, in fact such a choice would force us to excess in the dimensions unacceptable by virtue of the Witten observation discussed earlier. The most natural choice would therefore be a coset space (of the form G/H with H a subgroup of G), since this will have G as an isometry group, and $\dim(G/H) = \dim(G) - \dim(H)$, which is the most economic choice for the extra needed dimensions, especially if H was chosen to be the maximal proper subgroup of G . In our work, we shall take $G = SO(8)$, $H = SO(7)$, and hence $G/H = S^7$ (seven-sphere). Note that although $SO(8)$ does not include $SU(3) \times SU(2) \times U(1)$ as a subgroup, we still get an $SU(3) \times SU(2) \times U(1)$ theory where $SU(3)$ comes from a Lorentz group, and the rest come from breaking the $SO(8)$ gauge symmetry.

It is important to note at this stage that our scheme does not lead to a "Grand Unified Theory" (GUT) since $SU(3) \times SU(2) \times U(1)$ group is not contained in a simple group [12]. For that matter, it seems that Kaluza-Klein theories are not well suited for such schemes, since the choice of $G = SU(5)$ grand unified group, with $H = U(4)$ will give $\dim(G/H) = 8$, i.e. unacceptable!.

3-3 GENERALIZED KALUZA-KLEIN

3-3-1 Geometry of Coset Spaces

Since coset spaces are of special interest, a brief summary of the necessary mathematics is given in this subsection.

A coset space G/H is the set of all equivalence classes of the form: $[g] = \{ g' \in G \mid \exists h \in H \text{ and } g \in G; g' = gh \}$, H being a closed subgroup of G . Clearly, G acts transitively on G/H .

Consider the Killing vector fields K_i on G/H generated by the left¹ action $L: G \times Y \rightarrow Y$ of G on the homogeneous space Y diffeomorphic to G/H . Then the commutation relations of these fields will be:

$$[K_i, K_j] = - C_{ij}^k K_k \quad (3.3.1)$$

where the C_{ij}^k are the structure constants of G . Obviously the K_i 's will depend on the coordinates in $Y: (y^\alpha)$.

Following Salam and Strathdee [2], we label the points in coset space by $L_y \in [L_y]$. From transitivity of the action one can show that [2]:

$$K_i(y) = L D_i^{\hat{\alpha}}(L_y) e_{\alpha}^{\beta} \frac{\partial}{\partial y^{\beta}} \quad (3.3.2)$$

where L = dimensional constant, e_{α}^{β} = vierbein of G/H and $D_i^{\hat{\alpha}}$ = matrices of adjoint representation of G . the index $\hat{\alpha} = 1, \dots, d = \dim(G/H)$ is a numbering index.

Since the matrix D is of full rank, it is easy to see that

¹We are using the notation of [13], where the - sign in (3.3.1) is essential

one can chose the first d killing vectors to be linearly independent, and hence to form a basis for the tangent bundle of G/H .

For the product space, we can thus define the following vector fields in the tangent bundle $T(M) + T(Y)$:

$$e_i(x, y) = K_i(y) \quad (3.3.3a)$$

$$e_\mu(x, y) = \left(\frac{\partial}{\partial x^\mu}\right)_x - A_\mu^i(x)e_i(x, y) \quad (3.3.3b)$$

(3.3.3) serves as a basis for the tangent bundle, when the index i is restricted to α in (3.3.3a).

(3.3.3b) is nothing but the "gauge-derivative" vector field, with $A_\mu^i(x)$ the usual Y-M potential.

It is easy to show that the commutation relations for this basis are:

$$[e_\alpha, e_\beta] = -C_{\alpha\beta}^i K_i^\gamma(y) e_\gamma(x, y) \quad (3.3.4a)$$

$$[e_\mu, e_\alpha] = A_\mu^i C_{i\alpha}^k K_k(y) \quad (3.3.4b)$$

$$[e_\mu, e_\nu] = F_{\mu\nu}^i(x) K_i(y) \quad (3.3.4c)$$

Where $K_i^\gamma(y)$ are the coefficients of expanding K_i w.r.t. K_γ basis, i.e. $K_i = K_i^\gamma(y) K_\gamma$, and $F_{\mu\nu}^i = -2\partial_{[\mu} A_{\nu]}^i - C_{jk}^i A_\mu^j A_\nu^k$ are the field strengths of Y-M potential. We have fixed $[\partial_\mu, e_\beta] = 0$. This basis is very convenient for doing calculations in Kaluza-Klein theory.

3-3-2 The Kaluza-Klein Ansatz

The immediate generalization of the 5-dimensional Kaluza-

Klein metric (3.2.2) to higher dimensions is given by:

$$g_{AB} = \begin{bmatrix} g_{\mu\nu}(x) - A_{\mu}^i(x)A_{\nu}^j(x)K_i^{\alpha}(y)K_j^{\beta}(y)\dot{g}_{\alpha\beta}(y) & -A_{\nu}^jK_j^{\beta}\dot{g}_{\alpha\beta} \\ -A_{\nu}^jK_j^{\beta}\dot{g}_{\alpha\beta} & -\dot{g}_{\alpha\beta} \end{bmatrix} \quad (3.3.5)$$

where $\dot{g}_{\alpha\beta}$ is the killing metric on G/H , i.e. $\dot{g}_{\alpha\beta} = K_i^{\alpha}K_{i\beta}$

For the present time, we chose to work in the horizontal lift basis (as defined in [14]) given by the 1-forms:

$$\begin{aligned} \omega^{\mu} &= dx^{\mu} \\ \omega^{\alpha} &= dy^{\alpha} + K_i^{\alpha}(y)A_{\mu}^i(x)dx^{\mu} \end{aligned} \quad (3.3.6)$$

In this frame, the metric takes the simple block-diagonal form:

$$g_{AB} = \begin{bmatrix} g_{\mu\nu}(x) & 0 \\ 0 & -\dot{g}_{\alpha\beta} \end{bmatrix} \quad (3.3.7)$$

The frame itself is given by the dual to (3.3.6):

$$\begin{aligned} e_{\mu} &= \partial_{\mu} - A_{\mu}^i(x)K_i^{\alpha}e_{\alpha} \\ e_{\alpha} &= \partial_{\alpha} \end{aligned} \quad (3.3.8)$$

The above ansatz is known in the literature as the Kaluza-Klein ansatz. The commutation relations of the frame (3.3.8) are given by:

$$[e_{\alpha}, e_{\beta}] = 0 \quad (3.3.9a)$$

$$[e_{\mu}, e_{\alpha}] = A_{\mu}^i K_{i,\alpha}^{\gamma} e_{\gamma} \quad (3.3.9b)$$

$$[e_{\mu}, e_{\nu}] = F_{\mu\nu}^i K_i^{\gamma} e_{\gamma} \quad (3.3.9c)$$

where $F_{\mu\nu}^i = -2\partial_{[\mu}A_{\nu]}^i - C_{jk}^i A_{\mu}^j A_{\nu}^k$ is the Y-M field strength, as above.

Using the same techniques as in (3-2-1), we show that the line element defined via (3.3.5) is invariant under x^μ -dependent left action of the form:

$$\begin{aligned} x^\mu &\rightarrow x^\mu \\ y^\alpha &\rightarrow y^\alpha + f^i(x)K_i^\alpha(y) \end{aligned} \quad (3.3.10)$$

only if the transformation law of $A_\mu^i(x)$ was:

$$A_\nu^k(x) \rightarrow A_\nu^k(x) - C_{ij}^k f^i(x)A_\nu^j(x) - f_{,\nu}^k \quad (3.3.11)$$

which may be recognized as the Y-M gauge transformation of the non-Abelian gauge field. Hence, the first hope that the gauge fields can be obtained from higher-dimensional Kaluza-Klein theories is fulfilled, which supply us with more courage to examine other aspects of this theory.

3-3-3 Mathematical Derivations

Here we attempt to calculate the action of the theory, and we start by computing the Christofel symbols given by²:

$$\begin{aligned} \Gamma_{BC}^A &= (1/2) \{g^{AD}(e_C g_{BD} + e_B g_{CD} - e_D g_{BC}) - \omega^A[e_C, e_D] \\ &\quad - g^{AD}g_{BE}\omega^E[e_C, e_D] - g^{AD}g_{EC}\omega^E[e_B, e_D]\} \end{aligned} \quad (3.3.12)$$

This computation can be very easily performed with the help of STENSOR's splitting facility, which allows introducing a multitude of index-types, each with a different range, with the possibility of considering a particular index-type as the direct sum of two (or more) other index-types, then com-

²Following the notation of Kobayachi and Nomizo (see volume I in reference [14] of Chapter 2)

pactification can be done in a natural way, via replacing by its (symbolic) constituent indices (for example, a sum over such an index will be replaced by two (or more) sums over the ranges of the sub-indices). We get the following result:

$$\begin{aligned}
\Gamma_{\nu\varrho}^{\mu} &= \Gamma_{\nu\varrho}^{\mu} \\
\Gamma_{\nu\alpha}^{\mu} &= (-1/2) F^{i\mu}{}_{\nu} K_i^{\beta} \overset{\circ}{g}_{\alpha\beta} = \Gamma_{\alpha\nu}^{\mu} \\
\Gamma_{\mu\nu}^{\alpha} &= (1/2) F_{\mu\nu}^i K_i^{\alpha} \\
\Gamma_{\beta\mu}^{\alpha} &= A_{\mu}^i K_{i,\beta}^{\alpha} \\
\Gamma_{\beta\gamma}^{\alpha} &= \Gamma_{\beta\gamma}^{\alpha}
\end{aligned} \tag{3.3.13}$$

Using (3.3.13) we proceed to compute the Ricci scalar, given by:

$$R = R_M + R_{G/H} + g^{\mu\nu} R_{\mu\alpha\nu}^{\alpha} - \overset{\circ}{g}^{\alpha\beta} R_{\alpha\mu\beta}^{\mu} \tag{3.3.14}$$

where R_M and $R_{G/H}$ are the Ricci scalars for the 4-dimensional space-time M and the coset space G/H respectively.

Note that in our notation the Riemann tensor is:

$$\begin{aligned}
R^A{}_{BCD} &= e_C \Gamma_{DB}^A - e_D \Gamma_{CB}^A + \Gamma_{DB}^E \Gamma_{CE}^A - \Gamma_{CB}^E \Gamma_{DE}^A \\
&\quad - \Gamma_{EB}^A \omega^E [e_C, e_D]
\end{aligned} \tag{3.3.15}$$

The result for the Ricci scalar is:

$$R = R_M + R_{G/H} - (1/4) \overset{\circ}{g}_{\alpha\beta} K_i^{\alpha} K_j^{\beta} F^{i\mu\nu} F_{\mu\nu}^j \tag{3.3.16}$$

Note that $R_{G/H}$ will be a positive number for closed compact spaces with positive definite metric $\overset{\circ}{g}_{\alpha\beta}$.

As in the 5-dimensional theory, we write a higher-dimensional analogue of the Einstein-Hilbert action:

$$I_{EH} = (\kappa^2 V_{G/H})^{-1} \int_M \sqrt{-g} d^4 x \int_{G/H} (\dot{g})^{1/2} d^m y R \quad (3.3.17a)$$

where $V_{G/H} = \int_{G/H} (\dot{g})^{1/2} d^m y =$ "volume" of G/H . This implies

that, for $R_{G/H} =$ constant:

$$(3.3.17b)$$

$$I_{EH} = (\kappa^{-2}) \int_M \sqrt{-g} d^4 x (R_M + R_{G/H}) + (B_{ij}) (4\kappa^2 V_{G/H})^{-1} \int_M \sqrt{-g} d^4 x F^{i\mu\nu} F_{\mu\nu}^i$$

where $B_{ij} = \int_{G/H} (\dot{g})^{1/2} d^m y (\dot{g}_{\alpha\beta} K_i^\alpha K_j^\beta)$

B_{ij} is the average measure over G/H . To first order in the expansion of $K_i^\alpha K_j^\beta$, B_{ij} is directly proportional to δ_{ij} , hence contracting with δ^{ij} gives:

$$\delta^{ij} B_{ij} = L^2 \int_{G/H} (\dot{g})^{1/2} d^m y \dot{g}_{\alpha\beta} \dot{g}^{\alpha\beta} = mVL^2$$

$$\Rightarrow B_{ij} = L^2 (d_{G/H}/d_G) v \delta_{ij}$$

taking $L^2 = (\kappa d_{G/H})/d_G$, we finally get the Einstein-Hilbert-Yang-Mills action:

$$(3.3.17c)$$

$$I_{EHYM} = \int_M \sqrt{-g} d^4 x \{ (\kappa^{-2}) (R_M + R_{G/H}) - (1/4) F^{i\mu\nu} F_{\mu\nu}^i \}$$

The term $R_{G/H}$ plays the role of a cosmological constant.

So far, the "size" of the compact space is not fixed, however, it is possible to relate it to the gauge coupling constant, except when $U(1)$ is a factor of the gauge group [15].

3-3-4 Harmonic Expansion

In this section we show that a scalar field on higher dimensional Kaluza-Klein background $M \times G/H$ is equivalent to an infinite tower of scalar fields on the effective 4-dimensional world, with masses quantized in unites of the "radius" of the compact space G/H . This is done by expanding

the higher-dimensional field in terms of the appropriate harmonics on G/H (which generalizes Fourier expansion).

The action for a charged scalar field φ on $M \times G/H$ is:

$$I = \int \sqrt{-g} d^d x \{ g^{AD} e_A F^* e_D \varphi - m^2 F^* \varphi - \epsilon R F^* \varphi \} \quad (3.3.18)$$

where m = mass and ϵ = curvature coupling coefficient.

Expanding the field $\varphi(x, y)$ in terms of scalar harmonics on G/H we have [2]:

$$\varphi(x, y) = V^{-1/2} \sum_n \sqrt{d_n} D^{(n)}_{\alpha_n i_n} (L_Y^{-1}) \varphi^{(n)}_{i_n \alpha_n}(x) \quad (3.3.19)$$

where the index "n" refers to a particular representation of the group G , with a dimension d_n and matrices $D^{(n)}_{\alpha_n i_n}(g)$ for $g \in G$. Obviously, the indices α and i in (3.3.19) are summed, and n runs over an infinite number of representations.

Let $(T^{(n)})_{i_n j_n}$ be the matrices of the d_n -dimensional representation of the Lie algebra of G , relative to the representation "n". Then one can show [14 and 30] that:

$$K_i^\alpha \partial_\alpha D^{(n)}_{\alpha_n j_n} (L_Y^{-1}) = - D^{(n)}_{\alpha_n k_n} (L_Y^{-1}) (T^{(n)})_{i_n k_n} j_n \quad (3.3.20)$$

It is also known [2], that the group representations obey the orthonormality conditions:

$$\int_{G/H} dV_Y D^{(n)}_{\alpha_n i_n} (L_Y^{-1}) D^{(m)}_{\alpha_m i_m} (L_Y^{-1}) = (V/d_n) \delta_{i_n i_m} \delta_{\alpha_n \alpha_m} \delta_{nm} \quad (3.3.21)$$

substituting (3.3.19) into (3.3.18), and using the identities (3.3.20) and (3.3.21), and expanding R as in (3.3.15), we can integrate away the y -dependence in all but one term

of the Lagrangian, namely the term containing $\dot{g}_{\alpha\beta} K_i^\alpha K_j^\beta F_{\mu\nu}^j$, which can be integrated after employing the same trick as in (3.3.17). After some manipulation we find:

$$\begin{aligned} I = & \int \int \sqrt{-g} d^4x \left\{ (\nabla_\mu \varphi_{i_n \alpha_n}^{(n)})^* (\nabla^\mu \varphi_{i_n \alpha_n}^{(n)}) \right. \\ & - [L^{-2} C_G(n) + m^2 + \epsilon R_{G/H}] \varphi_{i_n \alpha_n}^{(n)*} \varphi_{i_n \alpha_n}^{(n)} \\ & \left. - \epsilon R \varphi_{i_n \alpha_n}^{(n)} \varphi_{i_n \alpha_n}^{(n)} + [(4d_{G/H})/(d_G)] \epsilon \kappa^2 F_{\mu\nu}^i F^{i\mu\nu} \varphi_{i_n \alpha_n}^{(n)*} \varphi_{i_n \alpha_n}^{(n)} \right\} \end{aligned} \quad (3.3.22)$$

where:

$$\nabla_\mu \varphi_{i_n \alpha_n}^{(n)} = \partial_\mu \varphi_{i_n \alpha_n}^{(n)} + \kappa L^{-1} A_\mu^i (T^{(n)}_i)_{j_n k_n} \varphi_{j_n \alpha_n}^{(n)}$$

is a generalized covariant derivative and $C_G(n)$ is the quadratic Casimir invariant for the representation "n" of the group, given by:

$$(T^{(n)}_i)_{j_n k_n} (T^{(n)}_i)_{j_n k_n} = - C_G(n) \delta_{i_n k_n} \quad (3.3.23)$$

which is proportional to the eigenvalue of the Laplace-Beltrami operator for scalars on G/H [16].

The effective 4-dimensional action (3.3.22) clearly describes an infinite number of scalar fields $\varphi_{i_n \alpha_n}^{(n)}$, the mass of the mode $\varphi_{j_n \alpha_n}^{(n)}$ is given by:

$$M_n^2 = m^2 + \epsilon R_{G/H} + L^{-1} C_G(n) \quad (3.3.24)$$

Thus, the quantum masses for our modes will be very large, if the "radius" of the coset space was chosen to be very small, as discussed above. Another observation is that the coupling between the gauge fields and the scalar fields is not minimal (for $\epsilon = 0$) due to the last term in (3.3.22).

3-3-5 Problems With Kaluza-Klein Ansatz

So far the "rosy" side of Kaluza-Klein ansatz has been exhibited, nevertheless, one must concede that the above approach suffers from a lot of difficulties, especially when used for constructing models of "nature"!. The most important of these problems can be summarized as follows:

i) Absence of fermions!: This problem is not related to the choice of the ansatz, but unfortunately is of more fundamental nature. The previous work allowed us to obtain Einstein-Maxwell-Yang-Mills theory (3.3.17c), where all propagating (physical) fields were of integer spin (0, 1 and 2). One would naturally hope to include Fermions (i.e. half-integer spin fields) in Kaluza-Klein theories, however, the Lichnerowicz theorem [17] states that there are no zero eigenmodes of the Dirac operator on spaces which admit positive scalar curvature, which apply to all coset spaces with their standard metrics, save the uninteresting case of the n -torus T^n . Hence, no spin 1/2 Dirac particles can be included in a natural way.

ii) The cosmological constant causes a two-fold problem; first, the choice of the sign: To illustrate this problem, let us work with the maximally symmetric (Einstein) spaces, with:

$$R_{AB} = \left(2\frac{\Lambda}{d-2}\right)g_{AB} \quad (3.3.25)$$

then, in the ground state, we have $R_{\mu\nu} = C_1 g_{\mu\nu}$ and $R_{\alpha\beta} = C_2 g_{\alpha\beta}$ with $C_1 C_2 > 0$ always (i.e. C_1 and C_2 have the

same sign).

The Myers theorem of differential geometry states that Einstein spaces with positive definite metrics and $C_2 > 0$ are always compact, while on the other hand, according to Yano [18], such spaces with $C_2 < 0$ have no symmetries!. Therefor we need $C_2 > 0$, which forces $C_1 > 0$, i.e. we should use the deSitter space-time for our manifold M , which is not very satisfactory, since deSitter does not have a positive energy theorem and cannot be used for a supersymmetric theory!.

The second independent problem with the cosmological constant is related to its magnitude; the requirement of assigning Plank size to the extra dimensions forces a cosmological constant for space-time 120 orders of magnitude greater than the observed upper limit [19] !, this is known as the cosmological constant puzzle, and is not confined to Kaluza-Klein theories.

iii) Consistency of the Ansatz: It is quite a common mistake in the literature to start off a Kaluza-Klein computation with an inconsistent ansatz for the ground state total metric. The underlying reason is that the extra dimensions are not treated as physical, but rather regarded as a mere mathematical device.

As a simple example, consider the original 5-dimensional theory: many authors take $g_{55} = 1$ (see for example the review article [14]), this is inconsistent with the $R_{55} = 0$ component of the Einstein equation, since this would

force $F_{\mu\nu}F^{\mu\nu}$ to vanish!. However, with our choice of $g_{\alpha\beta}$ (see (3.2.3) in the previous section), the 5,5 component of Einstein's equations implies:

$$\square(\log\varphi) \approx \varphi F_{\mu\nu}F^{\mu\nu} \quad (3.3.26)$$

Therefor, consistency can be restored by retaining the scalar field φ .

In higher-dimensional theories, the process of restoring consistency forces the inclusion of all the massive modes in the harmonic expansion in most cases [1], this means that no consistent truncation of the high energy part of the theory can be done, which would clearly limit the predictive power of this theory for the low energy scale. Fortunately, a few exceptions exist, amongst these are certain S^7 compactifications of 11-dimensional $N = 1$ supergravity (which shall be considered later).

We will not attempt to solve these problems explicitly, however, it is useful to introduce the model which is held by many to be a possible break-through, namely the $N = 1$, $d = 11$ supergravity on an (anti-deSitter) $\times S^7$ background, spontaneously broken to $N = 8$, $d = 4$ supergravity [20]. In this model Fermions (of spins 1/2 and 3/2) are introduced via the spin 3/2 11-dimensional gravitino field ψ_A [20], while the introduction of the generalized "electromagnetic" field A_{ABC} will resolve the sign-problem of the gravitational constant by introducing Bosonic matter with non-trivial VEV (Vacuum Expectation Value) in the ground state, the Einstein equa-

tions will read [21]:

$$R_{\mu\nu} = -12m^2 g_{\mu\nu}, \quad \text{i.e. } C_1 < 0 \quad (3.3.27a)$$

$$R_{\alpha\beta} = -12m^2 g_{\alpha\beta}, \quad \text{i.e. } C_2 > 0 \quad (3.3.27b)$$

here $m = \text{constant}$. Where the Lagrangian of this system is given by [23]:

$$\begin{aligned} L_0 = & (1/4) (\det e_A^M) \{ e_N^B e_M^A R_{AB}^{MN}(\omega) - 2i\bar{\psi}_A \Gamma^{ABC} D_B [(\omega + \hat{\omega})/2] \psi_C \\ & - (1/12) F_{ABCD} F^{ABCD} + 8(12)^{-4} \epsilon^{A_1 A_2 \dots A_{11}} F_{A_1 \dots A_4} F_{A_5 \dots A_8} F_{A_9 \dots A_{11}} \\ & + 3(12)^{-14} (\bar{\psi} \Gamma^{ABCDEFG} \psi_B + 12 \bar{\psi} \Gamma^{CDE} \psi^F) (F_{CDEF} + \hat{F}_{CDEF}) \end{aligned}$$

with e_N^B being the "elfbein" on the 11-dimensional manifold, $F_{ABCD} = 4\partial[A^A BCD]$ is the invariant field strengths associated with the field A_{BCD} . and

$$\begin{aligned} \omega_{BMN} = & (1/2) (-\Omega_{MNB} + \Omega_{NMB} - \Omega_{BMN}) \\ & + (i/4) [-\bar{\psi}_A \Gamma^{AC} \psi_C + 2(\bar{\psi}_B \Gamma_N \psi_M - \\ & \bar{\psi}_B \Gamma_M \psi_N + \bar{\psi}_N \Gamma_B \psi_M)] \end{aligned}$$

with

$$\Omega_{AB}^M = 2\partial[B^M A]$$

$$\bar{\psi} = \psi^+ \Gamma_0$$

$$D_A(\omega) = \partial_A + (1/4)\omega_A^{MN} \Gamma_{MN}$$

$$\hat{\omega}_{AMN} = \omega_{AMN} + (i/4)\bar{\psi}_B \Gamma_{AMN}^{BC} \psi_C$$

$$\hat{F}_{ABCD} = F_{ABCD} - 3\bar{\psi}[A \Gamma_{BC} \psi_D]$$

The Γ symbol is the generalized Dirac Γ -matrix in 11-

dimensions, and Γ with more than one index is the fully antisymmetrized product of the appropriate number of Γ 's.

On the other hand it has been suggested [22] that the Fermionic condensates and the Ricci-flattening torsion, which exists whenever the extra manifold admits a Killing-spinor, may resolve the "fine-tuning" puzzle of the cosmological constant.

Finally, the consistency of this theory is retained in the zero mode restriction [1 and 10].

We conclude by writing down the 11-dimensional field equations for the bosonic fields [23]:

$$\begin{aligned}
 R_{AB} - (1/2) g_{AB} R &= (1/3) (F_{ACDE} F_B{}^{CDE} - (1/8) g_{AB} F^2) \\
 \nabla_A F^{ABCD} &= -(1/576) \epsilon^{A_1 A_2 \dots A_8 BCD} F_{A_1 \dots A_4} F_{A_5 \dots A_8}
 \end{aligned}
 \tag{3.3.29}$$

3-4 THE NEW ANSATZ

3-4-1 The Computer Program

In the previous section the credibility of the Kaluza-Klein theories was defended. However, the number of degrees of freedom in constructing such theories is still unfortunately very large, and we are still in need of the "trial and error" techniques for refining the input ansatz. The immediate problem that confront us is the tremendous amount of calculation required for testing individual "guesses" which can deter the keenest of physicists from contemplating such a technique for improving the ansatz of the Kaluza-Klein theory, such type of work.

The rest of this chapter will be devoted to the study of a particular new ansatz, which can be used for constructing a Kaluza-Klein theory from any coset space G/H , with no restrictions on the total number of dimensions, or the geometry of the 4-dimensional manifold used. For this purpose, a computer program was developed to handle the computational aspects of the work. However, this program was designed to work equally well in the general cases and can be used for computing the necessary physical quantities for any ansatz, therefore it can be of great help in the search for the "ultimate" ansatz.

The program computes the connection coefficients, the Riemann and Ricci tensors and the curvature scalar of a given d -dimensional metric, via the usual spontaneous com-

pactification techniques. It utilizes the "index-splitting" facility of STENSOR; this allows defining different index-types, which would run over different ranges of integers (always starting with zero). One index-type can be subsequently split into two (or more) other index-types, whose combined ranges are equal to the range of the initial index-type. This facility can be extremely useful in other problems of theoretical physics, such as the 3+1 formalism of the Cauchy problem in general relativity.

A full listing of our program is provided in appendix D.

The first test of this program was to re-calculate the Ricci scalar for the important ansatz of M. Awada [24] (which helped clearing a lot of confusion in previous literature), the computer did more than verifying the results in [24], it actually combined two complicated terms in the expression of $R_{G/H}$, which were not combined in [24], which proves the superiority of the machine "noticing" power!. More important of course was the time-saving involved, it transpires that such a computation takes an enormous duration of time to be performed by hand: Dr. Awada revealed to the author that he spent more than a month to complete his calculations, while the computer managed to finish the same work in less than 10 minutes¹, and these include the time used for interaction, which is naturally responsible for any time-wasting!.

¹The time needed for writing the software itself is not counted, of course!

3-4-2 The Ansatz

Further generalization of the above work was suggested [25]; the main change is to multiply the total contravariant metric by a "weight" factor which depends on both (x^α, y^α) coordinates (cf the 5-dimensional case). It shall be shown later on (see section (3-3-4)) that this weight is nothing but the determinant of the old $g_{\alpha\beta}$ metric raised to the power $-1/9$ i.e. $W(x,y) = (\det g^{\alpha\beta})^{-1/9}$, however, for the time being we may leave it as an unknown.

The new ansatz metric would therefore be:

$$g_{AB} = W(x,y)^{-1} \begin{bmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{\alpha\beta}(x,y) \end{bmatrix} \quad (3.4.1)$$

where $g_{\alpha\beta}$ is the inverse of $g^{\alpha\beta}(x,y)$ defined by"

$$g^{\alpha\beta}(x,y) = K_i^\alpha(y) K_i^\beta(y) \varphi^{ij}(x) \quad (3.4.2)$$

The reasons behind the choice (3.4.2) lies in the fact that it provides the most natural generalization of the ansatz in ref [24], since it is known that on the one hand $g_{\mu\nu}(x)$ should be the lowest order mode in the expression of the g_{AB} harmonic expansion, after restricting the indices $A=\mu$ and $B=\nu$, in order that the ground state of our theory should describe the usual general relativity metric, and on the other hand, (3.4.2) is by construction ([24 and 25]), the 1st order term of the harmonic expansion of g_{AB} , $A=\alpha$, $B=\beta$. Therefore, to get maximal generalization, we should multiply the metric by (so far) an arbitrary factor $W(x,y)$ which

depends on both sets of variables, with the only constraint that its zeroth-order mode should be identical to one, allowing us to retrieve the initial ansatz in the zeroth order.

3-4-3 Computations

Using the program described above, the calculation proceeds as follows:

i) Computing the Lie derivative of the metric tensor, with respect to the basis vectors: This is done by splitting $e_A g_{BC}$ into its constituents: $e_A g_{\alpha\beta}$ and $e_A g_{\mu\nu}$ (note that $g_{\mu\alpha}=0$), then evaluating these individually and storing the results for future use in computing the connection coefficients.

One useful relation is employed for computing $e_\mu g_{\alpha\beta}$, namely:

$$e_\mu g_{\alpha\beta} = -g_{\alpha\gamma} g_{\beta\delta} e_\mu g^{\delta\gamma} \quad (3.4.3)$$

since the form of $g^{\alpha\beta}$ is known.

ii) The next step is evaluating the connection coefficients, using (3.3.9 and 12), the results obtained by STENSOR are displayed in figure F3 below.

Two important remarks concerning F3: The first is that the Group indices are typed as small Roman indices, from the beginning of the alphabet, in order to distinguish them from the space-time capital Roman indices. This is necessary since SHEEP can not handel Greek letters. The second remark is that the numerical indices 0 and 1 denote the

different indextypes and have no other significance.

iii) Finally, we evaluate the Ricci scalar, which is needed for the Kaluza-Klein Lagrangian, from (3.3.9 and 15) and the connection. The result is displayed in figure F4.

After appropriately rearranging the terms we have:

$$\begin{aligned}
 R = & WR_M + v_{G/H} + W \overset{\dagger}{\nabla}_\mu g^{\mu\nu} (g_{\gamma\delta} K_l^\gamma K_k^\delta \nabla_\nu \varphi^{kl}) \\
 & - (W/4) g^{\mu\nu} K_k^\epsilon K_l^\gamma K_i^\alpha K_j^\beta (g_{\epsilon\gamma} g_{\alpha\beta} + g_{\epsilon\alpha} g_{\epsilon\beta}) (\nabla_\mu \varphi^{kl}) (\nabla_\nu \varphi^{ij}) \\
 & - (W/4) g^{\mu\nu} g^{\rho\sigma} g_{\alpha\beta} K_i^\alpha K_j^\beta F_{\mu\rho}^i F_{\nu\sigma}^j \quad (3.4.4)
 \end{aligned}$$

where

$$\nabla_\mu \varphi^{ij} = \partial_\mu \varphi^{ij} + A_{\mu k}^i S_{jk} \quad (3.4.5)$$

and the gauge covariant derivative

$$\overset{\dagger}{\nabla}_\mu = \partial_\mu + \Gamma_\mu \quad (3.4.6)$$

and

$$v_{G/H} = W v_{old} + f(W, W_\alpha, W_\mu) \quad (3.4.7)$$

where f is a non-trivial function of W and its derivatives, which vanishes for $W=1$, and:

$$v_{old} = [v_{G/H}]_{W=1} \quad (3.4.8)$$

The term v_{old} is used to denote the result of [24].

The potential for the scalars is given by:

$$U[\delta, \varphi^{ij}] = (v_{G/H})^{-1} \int d^m y \sqrt{g(x, y)} v_{G/H} \quad (3.4.9)$$

iv) Integration of the y -dependence: The expression of $v_{G/H}$ is too complicated to be integrated for an arbitrary

$$\begin{aligned}
T_{00}^0 &= -1/2d \begin{matrix} q \\ p \end{matrix} MW \begin{matrix} R \\ p \end{matrix} -1/2d \begin{matrix} R \\ p \end{matrix} MW \begin{matrix} q \\ q \end{matrix} +1/2W \begin{matrix} g \\ S \end{matrix} \begin{matrix} g \\ qR \end{matrix} \begin{matrix} pS \\ +T \\ qR \end{matrix} \\
T_{01}^0 &= -1/2g \begin{matrix} pR \\ ca \end{matrix} \begin{matrix} i \\ qR \end{matrix} \begin{matrix} a \\ i \end{matrix} K \begin{matrix} q \\ p \end{matrix} -1/2Md \begin{matrix} W \\ c \end{matrix} \\
T_{10}^0 &= -1/2g \begin{matrix} pq \\ ba \end{matrix} \begin{matrix} i \\ Rq \end{matrix} \begin{matrix} a \\ i \end{matrix} K \begin{matrix} R \\ p \end{matrix} -1/2Md \begin{matrix} W \\ b \end{matrix} \\
T_{11}^0 &= 1/2g \begin{matrix} pq \\ q \end{matrix} DFI \begin{matrix} ij \\ ba \end{matrix} \begin{matrix} a \\ cd \end{matrix} \begin{matrix} d \\ i \end{matrix} K \begin{matrix} K \\ j \end{matrix} +1/2Mg \begin{matrix} W \\ q \end{matrix} \begin{matrix} g \\ bc \end{matrix} \\
T_{00}^1 &= 1/2F \begin{matrix} i \\ qR \end{matrix} \begin{matrix} a \\ i \end{matrix} K \begin{matrix} ab \\ b \end{matrix} +1/2Mg \begin{matrix} W \\ qR \end{matrix} \begin{matrix} g \\ qR \end{matrix} \\
T_{01}^1 &= -A \begin{matrix} i \\ q \end{matrix} \begin{matrix} a \\ j \end{matrix} K \begin{matrix} C \\ ci \end{matrix} \begin{matrix} c \\ a \end{matrix} -1/2Md \begin{matrix} W \\ q \end{matrix} \begin{matrix} ij \\ cb \end{matrix} \begin{matrix} a \\ i \end{matrix} K \begin{matrix} K \\ j \end{matrix} \\
T_{10}^1 &= T_{10}^a \\
T_{11}^1 &= -1/2g \begin{matrix} ad \\ ce \end{matrix} \begin{matrix} e \\ i \end{matrix} K \begin{matrix} C \\ bd \end{matrix} -1/2g \begin{matrix} ad \\ be \end{matrix} \begin{matrix} e \\ i \end{matrix} K \begin{matrix} C \\ cd \end{matrix} \\
&\quad -1/2g \begin{matrix} ad \\ d \end{matrix} \begin{matrix} ij \\ be \end{matrix} \begin{matrix} e \\ cf \end{matrix} \begin{matrix} K \\ i \end{matrix} \begin{matrix} K \\ j \end{matrix} -1/2K \begin{matrix} C \\ i \end{matrix} \begin{matrix} a \\ bc \end{matrix} +1/2S \begin{matrix} g \\ c \end{matrix} \begin{matrix} K \\ bd \end{matrix} \begin{matrix} K \\ i \end{matrix} \begin{matrix} a \\ j \end{matrix} \begin{matrix} d \\ j \end{matrix} \\
&\quad +1/2S \begin{matrix} ij \\ b \end{matrix} \begin{matrix} a \\ cd \end{matrix} \begin{matrix} d \\ i \end{matrix} \begin{matrix} K \\ j \end{matrix} +1/2Mg \begin{matrix} W \\ d \end{matrix} \begin{matrix} g \\ bc \end{matrix} -1/2Md \begin{matrix} W \\ a \end{matrix} \begin{matrix} c \\ b \end{matrix} - \\
&\quad -1/2Md \begin{matrix} a \\ b \end{matrix} \begin{matrix} W \\ c \end{matrix}
\end{aligned}$$

Figure F3: The computer output for the connection coefficients. Note that the term M is a shorthand for (1/W) and a W with an index is the derivative of W w.r.t. that index.

$$\begin{aligned}
R = & (-1+NC)g \begin{matrix} ab & & ij & kl & a & b \\ e & W & +W & H & S & S & K & K & +W & R \\ a & b & & ij & a & b & k & l & & M \end{matrix} \\
& +(1-NC)g \begin{matrix} ab & c & i & & ab & cd & i & j \\ W & K & C & -1/4Wg & g & H & C & C & - \\ a & i & bc & & & ij & ac & bd \end{matrix} \\
& Wg \begin{matrix} ab & c & i & j & ab & c & d & j & i & ab & c & d & i & j \\ K & C & C & +1/2Wg & K & K & C & C & -Wg & K & K & C & C \\ i & aj & bc & & i & j & ac & bd & & i & j & ac & bd \end{matrix} \\
& +(1/2-1/4NC-1/4NC)Mg \begin{matrix} 2 & ab & & ik & jl & a & b \\ W & W & +1/2W & H & S & S & K & K & + \\ a & b & & ij & a & b & l & k \end{matrix} \\
& -5/4Wg \begin{matrix} ab & & ik & jl & ab & & ij & kl \\ H & H & S & S & -1/4Wg & H & H & S & S & + \\ ij & kl & a & b & & ij & kl & a & b \end{matrix} \\
& + (-1/2+1/2NC)g \begin{matrix} ab & & ij & & ab & & ik & j \\ W & H & S & +2Wg & H & S & C \\ a & ij & b & & ij & a & bk \end{matrix} \\
& +Wg \begin{matrix} ab & ij & c & k & ij & a & b & k & ij & a & b & k \\ H & S & K & C & +WS & K & K & C & +WS & K & K & C \\ ij & a & k & bc & a & k & i & jb & a & i & k & jb \end{matrix} \\
& +(1-NC)W \begin{matrix} ij & a & b & ab & ik & c & j & ab \\ S & K & K & +Wg & H & S & K & C & + & 4 & g & e & W \\ a & b & i & j & ij & a & k & bc & & a & b \end{matrix} \\
& - (3/4NC)g \begin{matrix} pq & & & pq & ij \\ W & W & +(3-1/2NC)g & DFI & W & H \\ p & q & & p & q & ij \end{matrix} \\
& -4g \begin{matrix} ab & c & i & & ab & & ij & & pq & RS & i & j \\ W & K & C & + & 2 & g & W & H & S & -1/4Wg & g & H & F & F \\ a & i & bc & & a & ij & b & & & ij & pR & qS \end{matrix} \\
& +(3/4)Wg \begin{matrix} pq & ij & & pq & ij & kl \\ DFI & W & H & -1/4Wg & DFI & DFI & H & H \\ p & q & ij & & p & q & ik & jl \end{matrix} \\
& -1/4Wg \begin{matrix} pq & ij & kl & & 2 & pq \\ DFI & DFI & H & H & +(1/2NC-NC-1/4NC)Mg & W & W \\ p & q & ij & kl & & p & q \end{matrix} \\
& +(-2NC-5)Mg \begin{matrix} 2 & ab & & ij & a & b \\ W_a & W_b & - & 4 & W_a & S & b & K & i & K & j \end{matrix}
\end{aligned}$$

$$+W \left(-2g^{pq} e^a T^p_{aq} + 2g^{pq} R^a_{pq} \right) - 2WS^{ij} K^a_{ij} K^b_{K^c} K^c_{bc}$$

Figure F4: The Ricci Scalar

coset space G/H. However, restricting ourselves to zeroth order in the expansion of the K_i^α 's [24], we can integrate the kinetic term in (3.4.4).

First we rescale $\varphi^{ij}(x)$ to make its determinant equal to one by: $\varphi^{ij} \rightarrow \delta^{-1}(x)\varphi_1^{ij}$ where $\det\varphi_1^{ij} = 1$.

Therefore, we should have $g_{\alpha\beta} \rightarrow \delta(x)g_{1\alpha\beta}$ using:
 $V_{G/H} =: \int_{G/H} d^m y \sqrt{g_1(x,y)}$ we can evaluate: EQ I (3.4.10)

$$\lambda_{ij} = \int_{G/H} d^m y \sqrt{g_1(x,y)} g_{1\alpha\beta} K^i_{\alpha}(y) K^j_{\beta}(y) \tag{3.4.9}$$

as follows:

Contracting with φ^{ij} yields: $\varphi^{ij}\lambda_{ij} = mV_{G/H}$ then writing $\lambda_{ij} = (\varphi_{ij} + \sigma_{ij}) V_{G/H}$ and contracting again, implies:

$$\varphi^{ij}\sigma_{ij} = m - n \tag{3.4.11}$$

expressing φ^{ij} as a perturbation of the group metric:

$$\varphi^{ij} = g^{ij} + \epsilon\psi^{ij} \tag{3.4.12}$$

Here the ψ^{ij} are the deformations around G associated with the scalar fields of our theory [24]. Similarly we can write:

$$\sigma_{ij} = T_{1ij} + \epsilon T_{2ij} \tag{3.4.11} \text{ implies: } g^{ij}T_{1ij} + \epsilon(g^{ij}T_{2ij} + \psi^{ij}T_{1ij}) = m - n \text{ this identity in}$$

turn implies:

$$g^{ij}T_{1ij} = m - n \quad (3.4.13a)$$

$$g^{ij}T_{2ij} + \psi^{ij}T_{1ij} = 0 \quad (3.4.13a)$$

the first of these relations imply that we may have:

$$T_{1ij} = -(1 - m/n) g_{ij} \quad (3.4.14)$$

(3.4.13b and 14) give: $T_{2ij} = -(1 - m/n) \psi_{ij}$

$$(3.4.14)$$

and finally:

$$\lambda_{ij} = (m/n)\psi_{ij}V_{G/H} \quad (3.4.15)$$

to the first order in ϵ .

Therefore we have:

$$\begin{aligned} I = \int_M d^4x \sqrt{-g} \{ \delta^{m/2} R_M + U[\delta, \psi^{ij}] - \\ (m/4n) g^{\mu\nu} g^{\rho\sigma} \delta^{(1+m/n)} \psi_{ij} F^i_{\mu\rho} F^j_{\nu\sigma} + (m/2n)^2 \delta^{m/2} g^{\mu\nu} (\nabla_\mu \psi_{ij}) (\nabla_\nu \psi^{ij}) \\ - (m^2(n+1)/4n) \delta^{-(2-m/2)} g^{\mu\nu} (\partial_\mu \delta) (\partial_\nu \delta) \} \quad (3.4.16) \end{aligned}$$

3-4-4 Discussion

The Kaluza-Klein model presented above provide a new and consistent theory that generalizes completely all previous work in this field. This makes it the best candidate for testing important features of the Kaluza-Klein theories in general. The "weight" factor $W(x,y)$ which multiply the ansatz metric is a very important ingredient, since it opens the door for formulating the general (non-linear)

supergravity (see below). The fixing of W is done as follows:

From (3.4.4) we have W as a factor multiplying the "traditional" 4-dimensional Ricci scalar, Therefore demanding that our theory should include the usual Lagrangian term for Einstein's general relativity theory in all modes (and not only zeroth mode, as used in integration of (3.4.3)), we must impose:

$$\begin{aligned} \sqrt{-g}W &= (-g_M)^{1/2} \\ \Rightarrow (g_{G/H})^{1/2} W^{-11/2} \cdot W &= 1 \\ \Rightarrow W &= (g_{G/H})^{1/9} \end{aligned} \quad (3.4.17)$$

The scalar fields in our work are described by φ^{ij} or rather ψ^{ij} , the deformation around the group metric (3.4.12), these fields are not singlets in general, and it is easy to show that, when we restrict them to singlets only (by choosing ψ^{ij} proportional to g^{ij}) we obtain the results of ref [26] exactly.

It is very difficult to check whether our potential for the scalars (3.4.9) is bounded from below, except for simple special cases; for example, if $g^{\alpha\beta} = \varphi \delta^{\alpha\beta}$, then $U[\delta, \varphi]$ is proportional to $\delta^{m/2} \varphi$, with a positive proportionality constant, which means that it is bounded below in this case.

3-5 CONCLUSIONS

3-5-1 Difficulties

A clear analysis of the difficulties facing a physical theory provide the best means for gauging its importance and future!. The most important problems with Kaluza-Klein theories, presented above, are the following:

i) The understanding of the full content of the potential for the scalar fields (3.4.7 and 8) is almost impossible in the general case, which limits the physical interpretations to special cases only!. Fortunately, however, in the interesting case of S^7 , the killing tensors (which are products of killing vectors [27]) describing the scalars have unique properties that simplify integration over S^7 .

ii) In general, it is necessary for the ansatz to include non-geometrical fields (i.e. matter) coupled to the metric field, as in the case of the work above, for "triggering" spontaneous compactification. The presence of these ad-hoc fields detracts from the simplicity of the purely geometrical theory, and may indicate that the geometrical component is not necessarily of fundamental nature. Supergravity might provide a justification for these additional fields, since it allows for such fields through the requirement of supersymmetry [21].

iii) The restrictions imposed on the choice of the ansatz are still not enough to limit the class of possible metrics to a well defined set. Indeed, there is even no

theoretical justification for restricting the choice of the additional compact space to coset spaces (although this choice is favored because of the hand-wavy arguments in the conclusion of section (3-2) above). Therefore work is concentrated on what are considered "hopeful" models, linked to supergravity theories, while physicists are keeping their fingers crossed!.

3-5-2 Interpretation of additional dimensions

The ultimate aims of Kaluza-Klein theories is to translate all physics into geometry! and in the same way that energy and momentum are associated with translational symmetry in 4-dimensional Minkowski space-time, the internal quantum numbers, such as electric charge, would be associated with symmetry in the extra dimensions. This would mean that gravity is the "only" force of nature, which has in reality $4+m$ -dimensions, the differences in the topological properties of the 4 and m -dimensions is responsible for the different manifestations of the gravitational force!. Thus, each point of the 4-dimensional space that we observe is a compact m -dimensional manifold by itself! which is so small in size (of the order of Planck's length) that it escapes all attempts of detection!. The only "appearance" of the extra dimensions takes place in the form of multiplets of the associated symmetry groups, which is better known as the physics of elementary particles.

In addition to the benefits of associating Planck's length

to the radius of the compact space, there is an important drawback; namely that this would render Kaluza-Klein theories untestable! since the excitations above ground state are too massive to be accessible at energies that would be available in the accelerators of the foreseeable future. One would hope to discover exceptional cases, where for reasons of symmetry, the mass contribution is oppressed (as in the case with the vector A_μ which is prevented from acquiring any mass by the gauge symmetry), this phenomenon is known as "space invaders" [22 and 28].

3-5-3 N=8 Supergravity

One instance of partial success for the Kaluza-Klein theories is the "Duff-Pope theory" of spontaneous compactification of $d = 11$ $N = 1$ Kaluza-Klein supergravity on the S^7 compact manifold, which reproduced a non-trivial $N = 8$ supergravity theory with local $SO(8)$ invariance on a 4-dimensional anti-deSitter background. Furthermore, "squashing" S^7 leads to spontaneously breaking $N = 8$ to $N = 0$ or 1 supergravity (depending on whether S^7 is "right" or "left" squashed [28 and 10]).

The ansatz for massless fields, at the linearized level, was fully determined [20] and used to prove that, at the linearized level, the truncated Duff-Pope theory is equivalent to the gauged $N = 8$ supergravity theory of DeWitt and Nicolai [29].

It is remarkable to note here that our ansatz can be a good

candidate for the complete non-linear ansatz, since, to the first order, we have:

$$|\det g^{\alpha\beta}(x,y)|^{1/9} g^{\alpha\beta}(x,y) \cong g^{\alpha\beta}(x,y) + 1/9 \delta^{\alpha\beta} g^{\gamma\gamma}$$

which is the exact expression obtained for 1st order linearization. One can show that this is also true for other fields as well. This reveals the importance of the (1/9) exponent, which was calculated (see (3-4-4)), rather than put by hand in our work. However, the final verdict on our ansatz can only be reached after it is supplemented by another ansatz for the "generalized" electromagnetic term A_{ABC} , and further complicated calculation is done.

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Chapter Four

MASSIVE N=2 SUPERGRAVITY**4-1 INTRODUCTION**

Since its invention in mid 1970's¹, supergravity theory has been gaining more and more attention from theoretical physicists, because it is the only known gauge theory in which space-time curvature is obtained as a natural consequence of the demand for local supersymmetry. Although supergravity theories were not originally designed as higher-dimensional theories, their most interesting formulation is done in "superspace", which assumes extra spinorial coordinates in addition to the usual x-y-z-t coordinates. There are various approaches to superspace, based on different geometrical ideas and using different bundles for constructing the theory, however, they all use the notion of anticommuting Grassmann coordinates.

Supergravity manages to avoid the stricture of the "no-go theorem" of Coleman and Mandula² by generalizing groups of

¹ The first 4-dimensional theory with linearly realized supersymmetry was that of Wess and Zumino [16], while supergravity itself was introduced later by Deser and Zumino [17]

² This theorem states roughly that the only allowed symmetry groups for a realistic, relativistic S-matrix are locally isomorphic to the direct product of Poincare group and an internal symmetry group, and hence changes in spin and mass are prohibited [1]

symmetry to "graded groups" (supergroups), which are characterized by graded Lie algebras, i.e. algebras whose composition rules contain anticommutators, as well as commutators [5]. These groups assume a new kind of symmetry between bosonic and fermionic fields, which is known as supersymmetry. In fact supergravity is defined as the gauge theory of local supersymmetry.

It is well known that the traditional renormalization techniques used in quantum field theories, cannot be used in gravitational theories because of the dimensionality of the coupling constant in general relativity, which is a consequence of the equivalence principle. Therefore, the only hope for a quantum-gravity theory with predictive power is through a mechanism in which infinities in the S-matrix cancel without the need for renormalization! Remarkably, supergravity does just that for 1st and 2nd order quantum corrections and one would like to conjecture that these cancellations persist in all higher order corrections, due to the symmetry between bosons and fermions.

Supergravity theories are generally classified according to the number of fermionic companions to the graviton (called gravitinos) in each theory, thus we have $N = 0$ supergravity which is nothing but general relativity, $N = 1$ "simple" supergravity with one gravitino, and $N = i$ "extended" supergravity where $i = 2, 3, \dots, 8$. For $i > 8$, spin $5/2$ particles enter the theory, for which no consistent coupling to particles with other spins is possible. The N -extended

supergravity can be viewed as an extension to pure general relativity (i.e. "general relativistic vacuum" without matter) in which there is one super-particle, whose "polarizations" are: the single graviton, N -gravitinos, $N(N - 1)/2$ vectors etc. This would lead to unification of all forces, since forces arise from the exchange of particles.

The most important case is the $N = 8$ supergravity which "predicts" amongst other things, massive spin $1/2$ quarks with $SU(3)$ symmetry and correct fractional charges [2]! However, one should concede that the very high predictive power of this scheme might be its downfall!!

In what follows, we shall study the geometrical structure of supergravity, using fibre bundle approach to study specific models for constructing massive supergravity theories.

4-2 FIBRE BUNDLES AND GENERAL RELATIVITY

The analysis of the intrinsic geometry behind the structure of supergravity theories rests heavily on fibre bundle techniques (or rather super fibre bundle); such a purely geometrical approach provides an easy technique for constructing supergravity in a natural way. It was also shown [3] that fibre bundle formulation of supergravity helps overcoming certain difficulties that arise from using the superspace approach.

4-2-1 Principle Super-Fibre-Bundle

A principle fibre bundle P is a triplet (E, π, M) , and an action Lie group G (which is also a manifold), where E is a bundle space, M is the base manifold and $\pi: E \rightarrow M$ is the natural projection map, satisfying the following conditions [4]:

- i) The right G -action on E , defined by $(X, g) \in E \times G \rightarrow Xg \in E$ is free, i.e. if for some $X \in E$, $Xg = X$ this implies that $g = e$ (the identity of G).
- ii) $M \cong E/G$ and π is the canonical projection (differentiable).
- iii) E is locally trivial, i.e. for every $x \in M$ \exists a neighbourhood $U \subset M$ such that, $\pi^{-1}(U) \cong U \times G$.
- iv) Every fibre is diffeomorphic to G .

A connection Γ in P is a map:

$$X \in E \rightarrow Q_X \subset T_X(E) \quad , \text{ such that:}$$

$$i) \quad T_X(E) = G_X \oplus Q_X$$

- ii) $Q_{Xg} = (R_X)_* Q_X$ for every $X \in E$, $g \in G$
- iii) Q depend differentiably on X where R_a is the transformation of E induced by the right action of G , and G_X is the subspace of $T_X(E)$ consisting of vectors tangent to the fibre through X .

Given a connection Γ in P , one can define the connection 1-form ω to be the algebra-valued one-form on E , such that:

- i) For $u_X^* \in G_X$, $\omega(u_X^*) = u \in \text{Lie}(G)$. where u is the unique element of the Lie algebra of G that induces the vector field u_X^* at the point $X \in E$.
- ii) $\omega(Q_X) = 0$.

One can show that the connection 1-form satisfies the relation:

$$(R_g)^* \omega = \text{ad}(g^{-1})\omega$$

If $h: T_X(E) \rightarrow Q_X$ is the horizontal projection of $T_X(E)$, we define the exterior covariant differentiation D by: $D\varphi = (d\varphi).h$ For a r -form φ , $D\varphi$ is $(r+1)$ -form and d is the usual exterior derivative operator.

The curvature two-form of a given connection Γ is defined by: $\Omega = D\omega$

The curvature two-form can be calculated via the structure equation:

$$\Omega(X, Y) = d\omega(X, Y) + (1/2) [\omega(X), \omega(Y)] \quad (4.2.1)$$

or symbolically: $\Omega = d\omega + (1/2) [\omega, \omega]$.

It also obeys the Bianchi identity: $D\Omega = 0$.

The standard general relativity theory evolves from the definitions given above. The bundle used is the bundle of vierbeins, or linear frames $L(M)$, which is invari-

ant under the action of the Poincare group (for spaces with Lorentzian signature of course). Thus, a connection Γ in this bundle may be expressed as:

$$\Gamma = (1/2) \omega^{ab} J_{ab} + \theta^a P_a \quad (4.2.2)$$

where P and J are the usual translation and rotation in the Lie algebra (boson + $SL(2, C)$ subgroups) which obey the following commutation laws:

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} \quad (4.2.3a)$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \quad (4.2.3b)$$

From (4.2.1) and (4.2.3) we have:

$$\Omega = (1/2) \Omega^{ab} J_{ab} + T^a P_a \quad (4.2.4)$$

Let $\sigma : M \rightarrow L(M)$ be a local x-section, then the pull-back of Ω is given by:

$$\sigma^* \Omega^{ab} = (1/2) R_{\mu\nu}^{ab} dx^\mu \wedge dx^\nu \quad (4.2.5a)$$

$$\sigma^* T^{ab} = (1/2) T_{\mu\nu}^a dx^\mu \wedge dx^\nu \quad (4.2.5b)$$

where:

$$R_{\mu\nu}^{ab} = 2\omega_{[\nu, \mu]}^{ab} + 2\omega_{[\mu}^{ac} \omega_{c| \nu]}^b$$

and

$$T_{\mu\nu}^a = 2e_{[\nu, \mu]}^a + 2\omega_{[\mu}^a | c| \nu]}^c$$

are the usual curvature and torsion tensors, relative to the vierbein e_μ^a , with the Levi-Civita connection ω_μ^{ab} .

The Lagrangian is constructed by demanding that its variation w.r.t. ω_μ^{ab} , should lead to an equation stating that the

torsion is zero. And since it is a 4-form, we have [3, 7]:

$$\delta L_{\theta=Cst} = 2\delta\omega^{ab} \times T^a \times \theta^b \epsilon_{abcd} \quad (4.2.6)$$

Remembering that $T^a = D\theta^a$ and $D\Omega^{ab} = 0$, we can integrate (4.2.6) to get:

$$L = \theta^a \times \theta^b \times \Omega^{cd} \epsilon_{abcd} \quad (4.2.7)$$

whose pull-back is

$$L = e R \quad (4.2.8)$$

where $e = \det(e_{\mu}^a)$ and $R = \text{Ricci scalar}$.

4-2-2 Supergravity and Super Fibre Bundles

Attempts to generalize the work in the previous section to supergravity should start at the fundamental level of generalizing the fibre bundle to super fibre bundles, with a super-group action. The new versions of the definitions in (4-2-1) will have "super-group" replacing all occurrences of the term "group", and "graded Lie algebra" replacing "Lie algebra". In particular, when defining the super-curvature two-form via the structure equation, the operator $[,]$ in (4.2.1) should be interpreted in the sense of Nijenhuis [6], i.e. it can be either a commutator or an anticommutator, depending on the grading of the algebra.

As a concrete example, consider [7] the $N = 2$ super Poincare group, with a single central charge Z as the structure group, the super-algebra is given by the relations (4.2.3) and (4.2.9) below:

$$\{Q_{\alpha i}, Q_{\beta j}\} = (C\gamma^a)_{\alpha\beta} \delta_{ij} P_a + C_{\alpha\beta} \epsilon_{ij} Z \quad (4.2.9a)$$

$$[J_{ab}, Q_{\alpha i}] = (\sigma_{ab})_{\alpha i}^{\beta j} Q_{\beta j}, \quad \text{where } \sigma_{ab} = (1/4)[\gamma_a, \gamma_b] \quad (4.2.9b)$$

where $Q_{\alpha i}$ are the generators of fermionic translations, Z is the central charge generator of $U(1)$, and $C_{\alpha\beta}$ is the antisymmetric charge conjugate matrix (other commutators vanish).

The connection in this bundle is Γ_S :

$$\Gamma_S = \Gamma + \psi^{\alpha i} Q_{\alpha i} + AZ \quad (4.2.10)$$

where Γ is the connection given in (4.2.2), $\psi^{\alpha i}$ is a one-form on the super-bundle E , taking values in the odd part of a Grassmann algebra ($i = 1, 2$) and A is a one-form on E . The pull-backs of $\psi^{\alpha i}$ and A are (the usual spin 3/2 fields) $\psi_{\mu}^{\alpha i}$ and A_{μ} (the electromagnetic field) respectively.

Repeating the calculation as for the classical case, we arrive at:

$$\Omega = (1/2) \Omega_E^{ab} J_{ab} + T^a P_a + e^{\alpha i} Q_{\alpha i} + HZ \quad (4.2.11)$$

where

$$\Omega_E^{ab} = \Omega^{ab} - (1/3) f^{mn} f_{mn} \theta^a \wedge \theta^b \quad (4.2.12a)$$

$$T^a = T_E^a + (1/2) \psi^i \wedge C\gamma^a \psi_i \quad (4.2.12b)$$

$$e^{\alpha i} = d\psi^{\alpha i} + (1/2) \omega^{bc} \wedge (\sigma_{bc})^{\alpha i} \psi \quad (4.2.12c)$$

$$H = dA + (1/2) \psi^i \wedge \psi^j \epsilon_{ij} \quad (4.2.12d)$$

where $f_{mn} = (1/2) F_{mn}$ is a zero-form matter field, antisymmetric tensor representation of $SO(1,3)$. The pull-backs are

given by:

$$T_{\mu\nu}^a = T_{\mu\nu}^a + (1/2) \psi_{\mu}^i C\gamma^a \psi_{\nu i} \quad (4.2.13a)$$

$$e_{\mu\nu}^{\alpha i} = -2\psi_{[\mu, \nu]}^{\alpha i} + (1/2) \omega_{[\mu}^{bc} (\sigma_{bc} \psi_{\nu]}^{\alpha i}) \quad (4.2.13b)$$

$$\begin{aligned} F_{\mu\nu} &= e_{\mu}^a e_{\nu}^b F_{ab} \\ &= -2A_{[\mu, \nu]} + (1/2) \psi_{[\mu}^i C\psi_{\nu]}^i \epsilon_{ij} \end{aligned} \quad (4.2.13c)$$

The Bianchi identities imply:

$$d\Omega^{ab} + \omega_c^a \wedge \Omega^{cb} - \Omega^{ac} \wedge \omega_c^b = 0 \quad (4.2.14a)$$

$$dT^a + \omega^{ab} \wedge T_b - \Omega^{ab} \wedge \theta_b + \psi^i \wedge C\gamma^a e_i \quad (4.2.14b)$$

$$de^{\alpha i} + d\psi^{\alpha i} + (1/2) \omega^{bc} \wedge (\sigma_{bc} e)^{\alpha i} - (1/2) \Omega^{bc} \wedge (\sigma_{bc} \psi)^{\alpha i}$$

$$dH + \psi^i \wedge C e^i \epsilon_{ij} \quad (4.2.14d)$$

The formulation of N = 2 supergravity proceeds from these equations, to obtain the Lagrangian of the theory [7]:

$$\begin{aligned} L_{N=2} &= L_0 - (1/3) f^{mn} f_{mn} \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d \epsilon_{abcd} - 4\psi^i \wedge C\gamma_5 \gamma_b e_i \wedge \theta^b \\ &\quad - (\alpha/4) \left[4dA + \psi^i \wedge C\psi^j \epsilon_{ij} \right] \wedge \psi^k \wedge C\gamma_5 \psi^l \epsilon_{kl} + \alpha Df^{cd} \wedge \theta^b \wedge \theta^a \wedge A \epsilon_{abcd} \\ &\quad + \alpha f^{cd} \wedge \psi^i \wedge C\gamma^a \psi_i \wedge \theta^b \wedge A \epsilon_{abcd} + \epsilon_{ij} (\alpha/2) f^{cd} \psi^i \wedge C\psi^j \wedge \theta^a \wedge \theta^b \epsilon_{abcd} \end{aligned} \quad (4.2.15)$$

Further generalization is possible by extending the group of the bundle to the Orthosymplectic group OSP(4/2) (more about these groups in the coming section), whose generators now satisfy:

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} \quad (4.2.16a)$$

$$[P_a, P_b] = J_{ab} \quad (4.2.16b)$$

$$[P_a, Q_{\alpha i}] = (1/2) (\gamma_a)^{\beta j}_{\alpha i} Q_{\beta j} \quad (4.2.16c)$$

$$[J_{ab}, Q_{\alpha i}] = (1/2) (\sigma_{ab})^{\beta j}_{\alpha i} Q_{\beta j} \quad (4.2.16d)$$

$$\{Q_{\alpha i}, Q_{\beta j}\} = (\sigma^{ab})_{\alpha\beta} \delta_{ij} J_{ab} + (C\gamma^a)_{\alpha\beta} \delta_{ij} P_a + C_{\alpha\beta} \epsilon_{ij} \quad (4.2.16e)$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \quad (4.2.16f)$$

$$[Q_{\alpha i}, Z] = \epsilon_{ij} Q_{\alpha i} \quad (4.2.16g)$$

The $N = 2$ super Poincare algebra can be obtained from (4.2.16) via a Wigner-Inonu contraction, where the generators J_{ab} and $Q_{\alpha i}$ are rescaled to $m \cdot J_{ab}$ and $m \cdot Q_{\alpha i}$, then taking the limit as $m \rightarrow 0$.

4-3 ORTHOSYMPLECTIC GROUPS

The Orthosymplectic groups play a very important role in the geometric formulation of supergravity theories, it is therefore useful to review their major properties:

4-3-1 Definitions:

Following ref [8], consider a graded vector space $\{Z^A\}$ with m commuting (bosonic) coordinates, and $2n$ anticommuting (fermionic) coordinates. The metric of this space is of the form: $\eta_{AB} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$ where C is an antisymmetric root of the $2n \times 2n$ unit matrix and I is the $m \times m$ unit matrix. It follows that this metric have the same symmetry property as $Z_A Z_B$, i.e.

$$\eta_{AB} = (-1)^{i_A i_B} \eta_{BA} \quad (4.3.1)$$

where $i_A = 0$ for $A = 1, 2, \dots, 2n$

$$= 1 \text{ for } A = 2n+1, \dots, 2n+m.$$

Note that for η to be invertible, the number of anticommuting coordinates must be even ($2n$). The dot-product is defined as $(Z, Z') = Z^t \eta Z'$.

The $OSP(2n/m)$ group is defined as the group of linear transformations on the space $\{Z^A\}$ leaving the dot-product w.r.t. η invariant.

The dimension of $OSP(2n/m)$ is the total of the sum of $n(2n + 1)$ due to $SP(2n)$ part, $m(m - 1)/2$ due to $O(m)$ and the grading dimension $2nm$. Since $OSP(2n/m)$ has $SP(2n) \times O(m)$ as its

ordinary subgroup (the Bose sector [8]).

To study the generators of the Orthosymplectic group, consider the infinitesimal variation to be of the form:

$$\delta Z^A = -Z^{Bu} \omega_B^A \quad (4.3.2)$$

where ω is the gauge parameter of the group. Then the invariance of (Z, Z') requires that:

$$\delta(Z, Z') = 0 = \omega_C^A \eta_{AB} + (-1)^{i_B(i_A+i_B)} \eta_{CA} \omega_B^A \quad (4.3.3)$$

where ω_B^A is assumed to have the commutativity properties as $Z^A Z_B$, i.e.

$$Z^{Bu} \omega_B^A = (-1)^{i_B(i_A+i_B)} \omega_B^A Z^{Bu} \quad (4.3.4)$$

(4.3.3) implies the following symmetry properties for ω :

$$\omega_a^b = -\eta \omega_d^c \eta^{db} = -\omega_c^d \quad (4.3.5a)$$

$$\omega_\alpha^\beta = -C_{\alpha\gamma} \omega_\delta^\gamma (C^{-1})^{\delta\beta} \quad (4.3.5b)$$

$$\omega_a^\beta = \eta_{ac} \omega_\delta^c (C^{-1})^{\delta\beta} \quad (4.3.5c)$$

$$\omega_\gamma^d = -C_{\gamma\alpha} \omega_b^\alpha \eta^{bd} = \eta^{bd} + \omega_b^\alpha C_{\alpha\gamma} \quad (4.3.5d)$$

The submatrices $(\omega\eta)_{ab}$ and $(\omega C)_{\alpha\beta}$ are antisymmetric and symmetric respectively, they belong to the subalgebra of $SP(2n) \times O(m)$, while the $(2n)m$ graded parameters provide the graded extension for the algebra.

4-3-2 Irreducible Representations

Consider the matrix representation of the Orthosymplectic group $OSP(2n/m)$: ϕ_A^B which transforms like $Z_A Z^{Bu}$, i.e.

$$\delta\phi_A^B = \omega_A^C \phi_C^B - \phi_A^C \omega_C^B \quad (4.3.6)$$

Notice

that

$$(-1)^{i_A} \delta\phi_A^A = (-1)^{i_A} \omega_A^C \phi_C^A - (-1)^{i_A} \phi_A^C \omega_C^A = 0$$

which implies that the trace: $(-1)^{i_A} \phi_A^A = -\phi_\alpha^\alpha + \phi_a^a$ is invariant.

The representation ϕ_A^B can be split into two irreducible tensors:

$$\begin{aligned} \phi_{AB} &= \phi_{AB}^S + \phi_{AB}^a \\ &= (1/2)(\phi_{AB} + (-1)^{i_{AB}} \phi_{BA}) + (1/2)(\phi_{AB} - (-1)^{i_{AB}} \phi_{BA}) \end{aligned} \quad (4.3.7)$$

The respective dimensionalities are clearly:

$$\phi_{AB}^S \rightarrow \begin{array}{l} \phi(ab) + \phi[\alpha\beta] + \phi_{a\beta} \\ m(m+1)/2 + n(2n-1) + 2nm \end{array}$$

$$\phi_{AB}^a \rightarrow \begin{array}{l} \phi[ab] + \phi(\alpha\beta) + \phi_{a\beta} \\ m(m-1)/2 + n(2n+1) + 2nm \end{array}$$

These provide the symmetric and antisymmetric (adjoint) representations of $OSP(2n/m)$ (clearly ϕ_{AB}^S is not faithful in general).

For the special cases of $OSP(4/1)$ and $OSP(4/2)$, we have [9]:

$$\phi_{AB}^{SB} = \begin{pmatrix} ((1/4)\varphi + \varphi_5 \gamma_5 + \varphi_{\mu 5} i \gamma_\mu \gamma_5)^\beta & \psi_\alpha \\ -\bar{\psi}^\alpha & \varphi \end{pmatrix} \quad (4.3.8a)$$

$$\phi_{\text{A}}^{\text{aB}} = \begin{bmatrix} (i/2)(\varphi_{\mu}\gamma_{\mu} + (1/2)\varphi_{\mu\nu}\sigma_{\mu\nu})^{\beta} & \psi_{\alpha} \\ \bar{\psi}^{\alpha} & 0 \end{bmatrix} \quad (4.3.8b)$$

$$\phi_{\text{A}}^{\text{sB}} = \begin{bmatrix} ((1/4)\varphi + \varphi_5\gamma_5 + \varphi_{\mu 5}i\gamma_{\mu}\gamma_5)^{\beta} & \psi^i_{\alpha} \\ -\bar{\psi}^{\alpha i} & \varphi - \varphi_1 & \varphi_2 \\ & \varphi_2 & \varphi_1 \end{bmatrix} \quad (4.3.9a)$$

$$\phi_{\text{A}}^{\text{aB}} = \begin{bmatrix} (i/2)(\varphi_{\mu}\gamma_{\mu} + (1/2)\varphi_{\mu\nu}\sigma_{\mu\nu})^{\beta} & \psi^i_{\alpha} \\ \bar{\psi}^{\alpha i} & 0 & -A \\ & A & 0 \end{bmatrix} \quad (4.3.9b)$$

where φ (respectively ψ) are bosonic (respectively fermionic) fields, and stands for the Dirac matrices. In both cases, the gauge matrix is of the form: $\Omega = \exp\left[\phi_{\text{A}}^{\text{aB}}\right]$, (after replacing ψ^i by ϵ^i the usual supersymmetric parameter and φ by ω).

4-3-3 The Transformation Rules

Before constructing a gauge invariant Lagrangian for the theory, the transformation laws under the action of the Orthosymplectic symmetry group must be studied for the basic ingredients of the theory. These are the gauge potentials for local $OSP(2n/m) \simeq \left[\phi_{\text{A}}^{\text{aB}}\right]$ and the matrix $C \simeq \left[\phi_{\text{A}}^{\text{sB}}\right]$. Here we restrict ourselves to the $OSP(4/2)$ case, remembering that the $OSP(4/1)$ case can be immediately obtained by a suitable truncation of the $OSP(4/2)$ theory.

The gauge potentials and the matrix C are given locally by:

$$\phi_\mu = \begin{bmatrix} (i/2)(\kappa^{-1}e_{\mu a} + (1/2)B_{\mu ab}\sigma_{ab})^\beta & \kappa^{1/2}\psi^{i\mu\alpha} \\ \kappa^{1/2}\bar{\psi}^{\alpha i} & 0 \quad -A_m \\ A_\mu & 0 \end{bmatrix} \quad (4.3.10)$$

$$C = \begin{bmatrix} ((1/4)\pi + \varphi\gamma_5 + V_a i\gamma_a\gamma_5)^\beta & \lambda_\alpha^i \\ -\lambda^{b\alpha i} & \begin{matrix} \pi-\sigma & m \\ m & \sigma \end{matrix} \end{bmatrix} \quad (4.3.11)$$

where $i = 1, 2$ is an $O(2)$ index, A_μ is a gauge field describing a spin 1 particle, κ is the gravitational constant, $\mu, \nu = 1, \dots, 4$ are the usual space-time indices. $\psi^{i\mu\alpha}$ are gravitino fields, $e_{\mu a}$ is the vierbein field, $\sigma_{ab} = i/2 [\gamma_a, \gamma_b]$, π and σ are two pseudo-scalar fields.

Let Ψ be a matter multiplet, which transforms under the full group as

$$\Psi(x) \rightarrow \Omega(x)\Psi(x) \quad (4.3.12)$$

Then the covariant derivation of Ψ , defined by:

$$\nabla_\mu \Psi(x) = \partial_\mu \Psi(x) + \phi_\mu \Psi(x) \quad (4.3.13)$$

transforms as:

$$\nabla_\mu \Psi(x) \rightarrow \Omega(x)\nabla_\mu \Psi(x) \quad (4.3.14)$$

with the connection obeying:

$$\phi_\mu \rightarrow \Omega\phi_\mu\Omega^{-1} + \Omega\partial_\mu\Omega^{-1} \quad (4.3.15)$$

For the matrix C , we have:

$$C \rightarrow \Omega C \Omega^{-1} \quad (4.3.16)$$

We may define the following covariant derivative for C :

$$\nabla_\mu C = \partial_\mu C + [\phi_\mu, C] \quad (4.3.17)$$

which obviously transforms as:

$$\nabla_{\mu} C \rightarrow \Omega \nabla_{\mu} C \Omega^{-1} \quad (4.3.18)$$

The field strengths of the gauge potentials ϕ_{μ} , is defined by:

$$\begin{aligned} \phi_{\mu\nu} &= \partial_{\mu} \phi_{\nu} - \partial_{\nu} \phi_{\mu} + [\phi_{\mu}, \phi_{\nu}] \\ &= \begin{bmatrix} (i/2) X_{[\mu\nu]}^{\beta} & \kappa^{1/2} \psi^{i[\mu\nu]\alpha} \\ \kappa^{1/2} \bar{\psi}_{[\mu\nu]}^{\alpha i} & \tilde{F}_{\mu\nu} \epsilon_j^i \end{bmatrix} \end{aligned} \quad (4.3.19a)$$

where:

$$\begin{aligned} X_{[\mu\nu]}^{\beta} &= (\partial_{\mu} X_{\nu} - \partial_{\nu} X_{\mu} + (i/2) [X_{\mu}, X_{\nu}])_{\alpha}^{\beta} \\ &\quad + (2/i) (\psi^{\alpha i}_{\mu} \bar{\psi}_{i\nu}^{\beta} - \psi^{\alpha i}_{\nu} \bar{\psi}_{i\mu}^{\beta}) \end{aligned} \quad (4.3.19b)$$

$$\psi^{i[\mu\nu]\alpha} = \partial_{\mu} \psi^{\alpha i\nu} - \partial_{\nu} \psi^{\alpha i\mu} + (i/2) (X_{\mu} \psi^i_{\nu} - X_{\nu} \psi^i_{\mu})_{\alpha} \quad (4.3.19c)$$

$$\tilde{F}_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + \kappa (\bar{\psi}_{\mu}^i \psi^j_{\nu} - \bar{\psi}_{\nu}^i \psi^j_{\mu}) \epsilon_{ij} \quad (4.3.19d)$$

$$\epsilon_j^i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (4.3.19e)$$

$$X_{\mu} = \kappa^{-1} e_{\mu a} \gamma_a + (1/2) B_{\mu ab} \sigma_{ab} \quad (4.3.19f)$$

From (4.3.15), it is easy to show that the transformation rule for $\phi_{\mu\nu}$ is:

$$\phi_{\mu\nu} \rightarrow \Omega \phi_{\mu\nu} \Omega^{-1} \quad (4.3.20)$$

The transformation laws for individual fields in ϕ_{μ} and C can be evaluated by using (4.3.6) and the value of Ω for expanding (4.3.15) and (4.3.16) in component form, this yields:

$$\delta e_{\mu a} = -\partial_{\mu} w_a + w_b B_{\mu ab} - w_{ab} e_{\mu b} + (i/2) \kappa^2 \bar{\epsilon}^i \gamma_a \psi^i_{\mu} \quad (4.3.21a)$$

$$\delta B_{\mu ab} = -\partial_{\mu} w_{ab} + \kappa^{-2} (w_a e_{\mu b} - w_b e_{\mu a}) - (w_{ac} B_{\mu cb} - w_{bc} B_{\mu ca}) - (i/2) \kappa \bar{\epsilon}^i \sigma_{ab} \psi^i_{\mu} \quad (4.3.21b)$$

$$\delta \psi^{\alpha i}_{\mu} = -\partial_{\mu} \epsilon^i_{\alpha} + (i/2) [\kappa^{-1} w_a \gamma_a + (1/2) w_{ab} \sigma_{ab}] \psi^i_{\mu} + \epsilon^{ij} \epsilon^j_{\alpha} A_{\mu} - \Lambda \epsilon^{ij} \psi^j_{\mu \alpha} - (i/2) [\kappa^{-1} e_{\mu a} \gamma_a + (1/2) B_{\mu ab} \sigma_{ab}] \epsilon^i_{\alpha} \quad (4.3.21c)$$

$$\delta A_{\mu} = -\partial_{\mu} \Lambda + \epsilon^{ij} \bar{\epsilon}^{\alpha i} \psi^j_{\alpha} \quad (4.3.21d)$$

$$\delta \varphi = -\kappa^{-1} w_a V_a - (1/2) \kappa^{1/2} \bar{\epsilon}^i \gamma_5 \lambda^i \quad (4.3.22a)$$

$$\delta V_a = -\kappa^{-1} w_a \varphi - w_{ab} V_b - (1/2) \kappa^{1/2} \bar{\epsilon}^i \gamma_a \gamma_5 \lambda^i \quad (4.3.22b)$$

$$\delta \lambda^i_{\alpha} = (i/2) [\kappa^{-1} w_a \gamma_a + (1/2) w_{ab} \sigma_{ab}] \lambda^i_{\alpha} - \kappa^{1/2} ((1/4) \pi + \varphi \gamma_5 + V_a \gamma_a \gamma_5) \epsilon^i_{\alpha} - \kappa^{1/2} \epsilon^{ij} \Lambda \lambda^i_{\alpha} + \kappa^{1/2} \delta^i_1 (\epsilon^1_{\alpha} (\pi - \sigma) + \epsilon^2_{\alpha} m) + \kappa^{1/2} \delta^i_2 (\epsilon^1_{\alpha} m + \epsilon^2_{\alpha} \sigma) \quad (4.3.22c)$$

$$\delta m = \kappa^{1/2} (\bar{\epsilon}^{\alpha 1} \lambda^2_{\alpha} + \bar{\epsilon}^{\alpha 2} \lambda^1_{\alpha}) \quad (4.3.22d)$$

$$\delta \sigma = 2\kappa^{1/2} \bar{\epsilon}^{\alpha 2} \lambda^2_{\alpha} - 2m\Lambda \quad (4.3.22e)$$

$$\delta \pi = 2\kappa^{1/2} \bar{\epsilon}^i \lambda_i \quad (4.3.22f)$$

4-4 APPLICATIONS

4-4-1 Construction of The Lagrangian

Equations (4.2.16, 18 and 20) provide us with the ingredients for constructing a gauge invariant Lagrangian density from ϕ_μ and C , which is also invariant under the general coordinate transformations. The use of the matrix C constitutes the main departure from the traditional Yang-Mills theory. In this construction, the metric tensor, $g_{\mu\nu} = e_{\mu a} e^a_\nu$ shall not be explicitly used, since, although it can be defined, it is not invariant under the action of the full group. This would limit us to an almost unique Lagrangian density, which is manifestly Orthosymplectic invariant [9]:

$$L = \epsilon^{\mu\nu\rho\sigma} \{ g_1 \text{Tr}(C \phi_{\mu\nu} \phi_{\rho\sigma}) + g_2 \text{Tr}(C^2) \text{Tr}(C \nabla_\mu C \nabla_\nu C \nabla_\rho C \nabla_\sigma C) + g_3 \text{Tr}(C \nabla_\mu C \nabla_\nu C \phi_{\rho\sigma}) \} \quad (4.4.1)$$

The factors g_1 , g_2 , and g_3 are (so far) arbitrary dimensional coupling constants. The $\text{Tr}(C^2)$ factor has been included in the 2nd term of the Lagrangian in order to give a π , φ and m dependence to the potential of the scalar fields (which will be calculated later), in such a way that the VEVs (VEV is the short-hand notation for: "Vacuum Expectation Value") for these scalars would vanish. Other powers of the matrix C can also be added, however, this does not seem to be particularly useful.

It has been shown [9] that, for the $OSP(4/1)$ case, the first term of (4.4.1) gives the usual $N = 1$ supergravity

Lagrangian, plus a cosmological term and a "masslike" term for the ψ_μ field (after fixing the VEV of C to be: $\langle C \rangle = \langle \varphi \rangle \begin{bmatrix} \gamma_5 & 0 \\ 0 & 0 \end{bmatrix}$), this demonstrates the versatility of the above choice of Lagrangian.

4-4-2 The Gauge Fixing

To expand the Lagrangian (4.4.1) fully in terms of the component fields, would obviously involve an enormous number of traces to be calculated. This could be unrewarding, since we still have the gauge freedom at hand. Therefore, it is more natural to choose our gauge-fixing first.

From the transformation laws of the fields, (see previous section) it is clear that the Lie parameters ω_{ab} can be used to make $e_{\mu a}$ symmetric, while ε_α^i can be used to gauge away the spinor fields λ_α^i [10]. This can be seen through the following argument: Assume that the VEVs of π , m , σ , V_a and λ_α^i are vanishing (this will be proved later, when minimizing the potential), then from (4.3.22b,c) we see that:

$$\langle \delta V_a \rangle = -\kappa^{-1} \omega_a \langle \varphi \rangle \text{ and } \langle \delta \lambda_\alpha^i \rangle = -\gamma_5 \langle \varphi \rangle \varepsilon_\alpha^i \quad (4.4.2)$$

Indeed, V_a and λ_α^i are Goldstone fields, corresponding to the spontaneous breaking of the Orthosymplectic symmetry to the Lorentzian symmetry [9]. The gauge defined above is called the Unitary gauge. In this gauge, the matrix C takes the simple form:

$$C = \begin{bmatrix} ((1/4)\pi + \varphi\gamma_5 + V_a i\gamma_a \gamma_5)^\beta_\alpha & 0 \\ 0 & \begin{matrix} \pi - \sigma & m \\ m & \sigma \end{matrix} \end{bmatrix} \quad (4.4.3)$$

and $\nabla_\mu C$ becomes:

$$\nabla_\mu C = \begin{bmatrix} \Sigma_{\mu\alpha}^\beta & -\kappa^{1/2} R_{\alpha j}^{\gamma i} \psi^j_{\mu\gamma} \\ -\kappa^{1/2} \bar{\psi}_{\mu j} \gamma_{\gamma i}^{\alpha j} & A_\mu S_j^i + \bar{B}_{\mu j}^i \end{bmatrix} \quad (4.4.4)$$

where:

$$\begin{aligned} \Sigma_{\mu\alpha}^\beta &= (1/4) \pi + (\partial_\mu + i\kappa^{-1} e_\mu) \varphi \gamma_5 \\ &+ (i\gamma_a \partial_\mu - \kappa^{-1} e_{\mu a} + iB_{\mu ab} \eta_a [b^\gamma c]) V_a \gamma_5 \end{aligned} \quad (4.4.5a)$$

$$S_j^i = \begin{bmatrix} 2m & 2\sigma - \pi \\ 2\sigma - \pi & -2m \end{bmatrix} \quad (4.4.5b)$$

$$\bar{B}_j^i = \begin{bmatrix} \pi - \sigma & m \\ 2m & \sigma \end{bmatrix} \quad (4.4.5c)$$

$$R_{\gamma j}^{\alpha i} = \bar{B}_j^i \delta_\gamma^\alpha - \delta_j^i T_\gamma^\alpha \quad (4.4.5d)$$

$$T_\gamma^\alpha = (1/4) \pi + \varphi \gamma_5 \quad (4.4.5e)$$

$$\bar{B}_{\mu j}^i = \partial_\mu \bar{B}_j^i \quad (4.4.5f)$$

Substituting (4.3.19), (4.4.3) and (4.4.4), we get the full Lagrangian in component form:

$$L = L_{g_1} + L_{g_2} + L_{g_3} \quad (4.4.6)$$

with:

$$L_{g_1} = g_1 \varepsilon^{\mu\nu\rho\sigma} \{ -(1/4) \text{Tr}(\psi \gamma_5 X_{\mu\nu} X_{\rho\sigma}) - (1/16) \pi \text{Tr}(X_{\mu\nu} X_{\rho\sigma}) \\ - \kappa M_{\mu\nu}^{i\bar{j}} \psi^i \psi^{\bar{j}} \varepsilon^{\rho\sigma} - \kappa \psi_{\mu\nu}^{\bar{j}} \gamma_5 \psi^i \varepsilon^{\rho\sigma} + \pi \tilde{F}_{\mu\nu} \tilde{F}_{\rho\sigma} \} \quad (4.4.7a)$$

where:

$$M_{\mu\nu}^{i\bar{j}} \psi^i \psi^{\bar{j}} \varepsilon^{\rho\sigma} = \text{Tr}(\bar{B}_{\mu\nu}^{i\bar{j}} \psi^i \psi^{\bar{j}} \varepsilon^{\rho\sigma})$$

$$L_{g_2} = g_2 \varepsilon^{\mu\nu\rho\sigma} \{ (3/4) \pi^2 + 4\varphi^2 + 2(\sigma^2 + m^2 - \sigma\pi) \{ \\ \text{Tr}((T\Sigma_{\mu\nu} \Sigma_{\rho\sigma})_{\alpha}^{\alpha} + \kappa \psi_{\nu i}^{\bar{\delta}} (T_{\alpha}^{\beta} R_{\beta}^{\eta} R_{\delta}^{\gamma})_{\bar{j}}^i \psi^{\mu\eta} \bar{\psi}_{\rho 1}^{\varepsilon} (R_{\gamma}^{\xi} R_{\varepsilon}^{\alpha})_{ubm} \psi_{\sigma\xi}^m \\ - 2\kappa (T\Sigma_{\mu\nu})_{\alpha}^{\gamma} \bar{\psi}_{\rho j}^{\delta} (R_{\gamma}^{\beta} R_{\delta}^{\alpha})_{\bar{j}}^i \psi^j \sigma^{\beta} - \kappa (T\Sigma_{\mu} R_{\nu}^i)_{\alpha}^{\beta} \psi^j \psi_{\rho 1}^{\beta\gamma} (R_{i\sigma}^1)_{\gamma}^{\alpha} \\ - \kappa T_{\alpha}^{\beta} (R_{\beta}^{\gamma} (A_{\nu} S + \bar{B}_{\nu}))_{\bar{j}}^i \psi^j \psi_{\rho k}^{\mu\gamma\delta} R_{i\delta}^k \Sigma_{\alpha\varepsilon}^{\alpha} \\ + \kappa (T\Sigma_{\mu} R_{\nu}^i)_{\alpha}^{\delta} \psi^j \psi_{\rho k}^{\nu\delta\gamma} ([A_{\sigma} S + \bar{B}_{\sigma}] R_{i\sigma}^1)_{\gamma}^{\alpha} \\ + \kappa T_{\alpha}^{\beta} (R_{\beta}^{\gamma} (A_{\nu} S + \bar{B}_{\nu}))_{\bar{j}}^i \psi^j \psi_{\rho k}^{\mu\gamma\delta} ([A_{\sigma} S + \bar{B}_{\sigma}] R_{\delta}^{\alpha})_i^k \\ + \kappa \psi_{\mu j}^{\bar{\gamma}} \bar{B}_{\bar{k}}^i (R_{\bar{j}}^i \Sigma_{\mu} \Sigma_{\rho} R_{\bar{l}}^m)_{\gamma}^{\delta} \psi_{\sigma\delta}^l \\ + \kappa \psi_{\nu 1}^{\bar{\delta}} (\bar{B} (A_{\mu} S + \bar{B}_{\mu}) R_{\delta}^{\beta})_k^l (R_{\beta}^{\gamma} (A_{\sigma} S + \bar{B}_{\sigma}))_i^m \psi^i \varepsilon^{\rho\gamma} \\ + \kappa \psi_{\mu i}^{\bar{\gamma}} \bar{B}_{\bar{k}}^j (R_{\bar{j}}^i + \Sigma_{\mu} R_{\bar{m}}^1)_{\gamma}^{\delta} (A_{\sigma} S + \bar{B}_{\sigma})_m^l \psi_{\rho\delta}^n \\ + \kappa \psi_{\nu 1}^{\bar{\gamma}} (\bar{B} (A_{\mu} S + \bar{B}_{\mu}) R_{\gamma}^{\beta})_k^l (\Sigma_{\rho} R_{\bar{n}}^m)_{\beta}^{\delta} \psi_{\sigma\delta}^n \\ - \kappa \psi_{\mu j}^{\bar{\gamma}} (\bar{B} R_{\gamma}^{\alpha})_{\bar{k}}^j R_{\bar{m}\alpha}^l \psi_{\nu\beta}^m \bar{\psi}_{\rho p}^{\delta} (R_{\bar{l}}^p R_{\bar{n}}^i)_{\delta}^{\varepsilon} \psi_{\sigma\varepsilon}^n \\ + \kappa \psi_{\mu j}^{\bar{\gamma}} \bar{B}_{\bar{k}}^n (R_{\bar{n}}^j R_{\bar{m}}^1)_{\gamma}^{\beta} \psi_{\nu\beta}^m (B_{\rho}^1 \bar{B}_{\sigma})_l^i \\ + \kappa (\bar{B} \bar{B}_{\mu} \bar{B}_{\nu})_k^l \bar{\psi}_{\rho m}^{\delta} (R_{\bar{l}}^m R_{\bar{n}}^i)_{\delta}^{\varepsilon} \psi_{\sigma\varepsilon}^n + \kappa (\bar{B} \bar{B}_{\mu} \bar{B}_{\nu} \bar{B}_{\rho} \bar{B}_{\sigma})_k^l \} \} \quad (4.4.7b)$$

$$\begin{aligned}
L_{g_3} = & g_3 \varepsilon^{\mu\nu\rho\sigma} \text{Tr} \{ (i/2) (T_{\mu}^{\nu} X_{\rho\sigma})_{\alpha}^{\beta} - \\
& (i/2) \kappa T_{\alpha}^{\delta} \bar{\psi}_{\nu\delta}^{\varepsilon} (R_{\delta}^{\gamma} R_{\varepsilon}^{\eta})_{\beta}^{\eta} \psi^{\mu\gamma} X_{\rho\sigma\eta}^{\beta} + \\
& \kappa (T_{\mu}^{\nu})_{\alpha}^{\gamma} R_{\beta}^{\delta} \bar{\psi}_{\rho\sigma}^{\delta} \psi_{\nu\delta}^{\beta} - \kappa (T_{\alpha}^{\delta} R_{\delta}^{\gamma} (A_{\nu} S + \bar{B}_{\nu}))_{\beta}^{\gamma} \bar{\psi}_{\rho\sigma}^{\delta} \psi_{\nu\delta}^{\beta} \\
& + \kappa \bar{\psi}_{\mu j}^{\gamma} (\bar{B} R_{\gamma}^{\alpha})_{\beta}^{\alpha} \psi_{\nu\alpha}^{\beta} \psi_{\rho\sigma}^{\beta} + \kappa (\bar{B} (A_{\mu} S + \bar{B}_{\mu}))_{\beta}^{\gamma} \bar{\psi}_{\nu\delta}^{\delta} R_{\rho\sigma}^{\beta} \psi_{\nu\delta}^{\beta} \\
& + \kappa \bar{\psi}_{\mu j}^{\gamma} (\bar{B} R)_{\beta}^{\alpha} (R_{\varepsilon}^{\eta})_{\gamma}^{\delta} \psi_{\nu\delta}^{\beta} \tilde{F}_{\rho\sigma}^{\varepsilon\eta} \} + \pi (\pi_{\mu} - 2\sigma_{\mu}) m_{\mu} \tilde{F}_{\rho\sigma}
\end{aligned} \tag{4.4.7c}$$

where: $\pi_{\mu} = \partial_{\mu} \pi$; $m_{\mu} = \partial_{\mu} m$ and so on.

4-4-3 Minimizing the potential

To explore the physical content of our Lagrangian, we should study its bilinear form, for an infinitesimal shift from the ground state values of the fields; these values are nothing but the values that minimize the potential of the theory for the scalar fields.

The potential of the theory is found by putting all fields in the Lagrangian (4.4.7a,b,c) to zero except the vierbein field e_{μ} , we find:

$$\begin{aligned}
V(\pi, m, \sigma, \varphi, e_{\mu}) = & (96/\kappa^4) (\det e_{\mu a}) \{ (1/4) g_1 \varphi - g_2 \left((3/4) \pi^2 + 4\varphi^2 + 2\sigma^2 \right. \\
& \left. + 2m^2 - 2\sigma\pi \right) \varphi^5 + (1/2) g_3 \varphi^3 \}
\end{aligned} \tag{4.4.8}$$

To minimize the potential we solve the set:

$$\left\langle \frac{\partial V}{\partial \pi} = \frac{\partial V}{\partial m} = \frac{\partial V}{\partial \sigma} = \frac{\partial V}{\partial \varphi} \right\rangle = 0 \tag{4.4.9}$$

Assuming $\det(e_{\mu a}) \neq 0$, and excluding the trivial solution $\langle \varphi \rangle = 0$, since it corresponds to $g_1 = 0$, we get:

$$\langle \pi \rangle = \langle m \rangle = \langle \sigma \rangle = 0 \quad (4.4.10)$$

$$(1/4) g_1 - 28g_2\phi^6 + (3/2) g_3\phi^2 = 0 \quad (4.4.11)$$

Moreover, requiring the sum of the cosmological terms to be zero, we should set the vacuum expectation value of the potential to zero.

$$\langle V(\pi, m, \sigma, \phi, e_\mu) \rangle = 0 \quad (4.4.12)$$

Equation (4.4.12) assures the fact that our physical space is Minkowskian in the flat limit.

From (4.4.10)-(4.4.12) we get the following solutions [6]:

$$\langle \phi \rangle = \left[\frac{-3g_1}{4g_3} \right]^{1/2} \quad (4.4.13)$$

$$g_3^3 = (27/2) g_1^2 g_2$$

$$\langle e_{\mu a} \rangle = \eta_{\mu a}$$

Clearly, the potential (4.4.8) is not bounded from below. To get the physical spectrum of our Lagrangian (4.4.7) we consider its bilinear form: First we expand the fields around their expectation values:

$$\begin{aligned} \phi &= \langle \phi \rangle + \phi' \\ e_{\mu a} &= \eta_{\mu a} + \kappa h_{\mu a}' \\ B_\mu &= B_\mu \\ V_a &= V_a \end{aligned} \quad (4.4.14)$$

$$\pi = \pi$$

$$m = m$$

$$\sigma = \sigma$$

Then writing the Lagrangian in terms of the shifted fields (4.4.14) up to bilinear order, we have:

$$\begin{aligned}
L_{\text{bilinear}} = & \varepsilon^{\mu\nu\kappa\lambda} g_1 \{-\kappa^{-1} \langle \varphi \rangle h_{\nu a} \partial_{\kappa} B_{\lambda c d} \varepsilon_{\mu a c d} + \quad (4.4.15) \\
& (1/2\kappa^2) \langle \varphi \rangle B_{\kappa \nu a} B_{\lambda c d} \varepsilon_{\mu a c d} - (\pi/\kappa^2) \partial_{\kappa} B_{\lambda \mu \nu} \\
& - (2/\kappa^2) \text{phi}' \partial_{\mu} B_{\mu a b} \varepsilon_{a b \kappa \lambda} + (3/\kappa) \langle \varphi \rangle \bar{\psi}_{\nu i} \gamma_5 \gamma_{\mu} \gamma_{\kappa} \psi^{i \lambda} \\
& - 4i \langle \varphi \rangle \bar{\psi}_{\nu i} \gamma_5 \gamma_{\mu} \partial_{\kappa} \psi^{i \lambda} \} - \varepsilon^{\mu\nu\kappa\lambda} g_2 \{ (12/\kappa) \langle \varphi \rangle \bar{\psi}_{\nu i} \gamma_5 \gamma_{\kappa} \gamma_{\lambda} \psi^{i \mu} \\
& - (96/\kappa^2) \varphi^5 \partial_{\mu} V_a \partial_{\nu} V_b \varepsilon_{a b \kappa \lambda} - (64/\kappa^3) \varphi^6 \partial_{\mu} V_a \varepsilon_{a \nu \kappa \lambda} - (32/\kappa^4) \varphi^5 V_{\mu} V_a \varepsilon_{a \nu \kappa \lambda} \\
& + \varepsilon^{\mu\nu\kappa\lambda} g_3 \{ (7/2) \kappa^{-1} \langle \varphi \rangle \bar{\psi}_{\lambda i} \gamma_5 \gamma_{\mu} \gamma_{\nu} \psi^{i \kappa} - (1/\kappa^2) \langle \varphi \rangle^2 \pi \partial_{\mu} B_{\mu \kappa \lambda} \\
& - \varepsilon^{\mu\nu\kappa\lambda} g_3 \{ 2i \langle \varphi \rangle \bar{\psi}_{\lambda i} \gamma_5 \gamma_{\mu} \psi^{i \nu} - (6/\kappa^2) \langle \varphi \rangle^2 \partial_{\mu} B_{\mu a b} \varepsilon_{a b \kappa \lambda} - \\
& - (2/\kappa^2) \varphi \partial_{\mu} V_a \partial_{\nu} V_b \varepsilon_{a b \kappa \lambda} - (4/\kappa^3) \varphi^2 \partial_{\mu} V_a \varepsilon_{a \nu \kappa \lambda} - (2/\kappa^4) \varphi V_{\mu} V_a \varepsilon_{a \nu \kappa \lambda} \} \\
& - (96/\kappa^4) \{ 2g_3 \langle \varphi \rangle \varphi'^2 + (3/4) g_2 \langle \varphi \rangle^5 \pi^2 + 2g_2 \langle \varphi \rangle^5 (\sigma^2 + m^2 + \sigma \pi) \}
\end{aligned}$$

The Bosonic part of (4.4.15) given by:

$$\begin{aligned}
& \varepsilon^{\mu\nu\kappa\lambda} (g_1/2) \left[\kappa^{-1} \langle \varphi \rangle (-2\varepsilon_{\mu a c d} \partial_{\kappa} B_{\lambda c d} h_{\nu a} + \kappa^{-1} \varepsilon_{\mu a c d} B_{\kappa \nu a} B_{\lambda c d} \right. \\
& + (1/2\kappa^2) (\partial_{\mu} \pi) B_{\mu \kappa \lambda} - (5/2\kappa^2) (\partial_{\mu} \varphi') B_{\mu a b} \varepsilon_{a b \kappa \lambda} \quad \left. \right] \\
& - (96/\kappa^4) \langle \varphi \rangle \{ 2g_3 \varphi'^2 + g_2 \langle \varphi \rangle^4 \left((3/4) \pi^2 + 2(\sigma^2 + 2m^2 - \sigma \pi) \right) \} \\
& \quad (4.4.16)
\end{aligned}$$

$$+ (4/\kappa^2)\langle\varphi\rangle g_3 \{((\partial_\mu v_\nu)(\partial_\nu v_\mu) - (\partial_\mu v_\mu)^2) + (1/\kappa^2)v_\mu v_\mu + 2\langle\varphi\rangle(\partial_\mu v_\mu h_{\nu\nu})\}$$

To see whether the scalar fields π , σ and m propagate or not we solve the Euler-Lagrange equations for the Field $B_{\mu ab}$ which gives:

$$B_{e,\sigma\kappa} = \kappa(\partial_\kappa h_{\sigma e} - \partial_\sigma h_{\kappa e}) + (4\langle\varphi\rangle)^{-1} \epsilon_{\mu\kappa e\sigma} \partial_\mu \pi + 5(2\langle\varphi\rangle\kappa^2)^{-1} (\partial_\sigma \varphi' \eta_{inrk} - \partial_\kappa \varphi' \eta_{e\sigma}) \quad (4.4.17)$$

Substituting this back into (4.4.15) we finally have:

$$\begin{aligned} L_{\text{bosonic,bilinear}} = & g_1 \langle\varphi\rangle \{2(\partial_\kappa h_{\nu\nu})(\partial_\lambda h_{\lambda\kappa}) + (\partial_\kappa h_{\lambda\nu})(\partial_\kappa h_{\lambda\nu}) \\ & - 2(\partial_\kappa h_{\kappa\nu})(\partial_\lambda h_{\lambda\nu}) - (\partial_\kappa h_{\nu\nu})(\partial_\kappa h_{\nu\nu})\} \\ & - 75g_1 (2\kappa^2\langle\varphi\rangle)^{-1} [(\partial_\mu \varphi')(\partial_\mu \varphi') - (96/25\kappa^2)\varphi'^2] \\ & - 3g_1 (16\kappa^2\langle\varphi\rangle)^{-1} [(\partial_\mu \pi)(\partial_\mu \pi) - 12\kappa^2\pi^2] \\ & + (4/\kappa^2)\langle\varphi\rangle g_3 \{[(\partial_\mu v_\nu)(\partial_\nu v_\mu) - (\partial_\mu v_\mu)^2] + (1/\kappa^2)v_\mu v_\mu \\ & + 2\langle\varphi\rangle(\partial_\mu v_\mu h_{\nu\nu} - \partial_\mu v_\nu h_{\mu\nu}) - 12\varphi' \partial_\mu v_\mu / \kappa\} \\ & + 6(\langle\varphi\rangle\kappa^4)^{-1} g_1 (\sigma^2 + m^2 - \sigma\pi) \quad (4.4.18) \end{aligned}$$

Let g_1 , g_2 and g_3 , the dimensional coupling constants be expanded in terms of κ , i.e.

$$\begin{aligned} g_1 &= \kappa b_1 \\ g_2 &= \kappa b_2 \\ g_3 &= \kappa b_3 \end{aligned} \quad (4.4.19)$$

Where b_1 , b_2 and b_3 are three dimensionless parameters, then from (4.4.10)-(4.4.12) we have for

$$g_1 \langle\varphi\rangle = -1/2 \quad (4.4.20)$$

The following values:

$$-2b_1^7 b_3 = -3b_1^3 \langle\varphi\rangle = -(2\kappa b_1)^{-1} = b_2 \quad (4.4.21)$$

With b_1 being now a free parameter chosen to be real.

The full rescaled Lagrangian (4.4.15) takes the final form:

$$\begin{aligned}
L_{\text{bilinear}} = & -(1/2) \{ 2(\partial_\kappa h_{\nu\nu})(\partial_\lambda h_{\lambda\kappa}) + (\partial_\kappa h_{\lambda\nu})(\partial_\kappa h_{\lambda\nu}) \\
& - 2(\partial_\kappa h_{\kappa\nu})(\partial_\lambda h_{\lambda\nu}) - (\partial_\kappa h_{\nu\nu})(\partial_\kappa h_{\nu\nu}) \} \\
& - 75b_1^2 [(\partial_\mu \phi')(\partial_\mu \phi') - (96/25\kappa^2)\phi'^2] \\
& - (3/8)b_1^2 [(\partial_\mu \pi)(\partial_\mu \pi) - (12/\kappa^2)\pi^2] \quad (4.4.22) \\
& + (6/\kappa^2)b_1^2 [(\partial_\mu v_\nu)(\partial_\nu v_\mu) - (\partial_\mu v_\mu)^2] + (1/\kappa^2)v_\mu v_\mu \\
& - (\kappa b_1)^{-1} (\partial_\mu v_\mu h_{\nu\nu} - \partial_\mu v_\nu h_{\mu\nu}) - 12\phi' \partial_\mu v_\mu / \kappa \\
& - 12\kappa^{-2} (\sigma^2 + m^2 - \sigma\pi) \\
& + (5i/4) \epsilon^{\mu\nu\kappa\lambda} \{ \bar{\psi}_\nu \gamma_5 \gamma_\mu \partial_\kappa \psi_\lambda + (3i/10) \bar{\psi}_\nu \gamma_5 \gamma_\mu \gamma_\kappa \psi_\lambda \}
\end{aligned}$$

From (4.4.22) we see that the physical spectrum of our Lagrangian (4.4.7a,b,c) is: Two massive spin 3/2 fields, a massless graviton, a massive gauge vector boson, two massive physical scalars ϕ , π , and two auxiliary fields m and σ , which can be eliminated via their equations of motion .

4-4-4 Orthosymplectic breaking

The most important feature of our work, crucial to the emergence of particle spectrum, is the spontaneous breaking of the Orthosymplectic symmetry in the ground state; this breaking provide us with the following lessons:

Our Lagrangian is manifestly invariant under OSP symmetry in the general form; the symmetry was broken only in the ground state.

Two Goldstone fields (spin 1/2 λ 's) arising from this SSB (Spontaneous Symmetry Breaking) were gauged away from the

ground state.

The resulting ground state Lagrangian (which retained Lorentzian symmetry) had no Orthosymplectic symmetry, and it describes massive $N = 2$ supergravity in the presence of auxiliary fields.

This is a manifest example of the supper-Higgs effect.

4-5 GENERALIZATION

4-5-1 Difficulties in generalization

To generalize the above work for $N \geq 3$ supergravity theories, one immediately thinks of the theory with the geometrical group, $OSP(4/3)$. Such theory should have the gauge fields $\theta^a, \psi^{\alpha i}$ and A^i ; $i = 1, 2, 3$, with ω^{ab} being the connection; these are the of the same type as the $OSP(4/2)$ gauge fields (the only difference is the range of the index i) of the $N = 2$ theory. However, such naive generalization runs into immediate trouble, since the form (4.2.15) is not invariant under $OSP(4/3)$! this is due in part to the deficiency in the number of fields to carry an $OSP(4/3)$ transformation. Thus the moral is that introducing extra gauge fields is necessary. In fact the apparent source of trouble is the content of the $N = 3$ theory, since the representation theory of $SO(3)$ should contain a spin $1/2$ field which is not a gauge field (i.e. it does not belong to the super algebra of $OSP(4/3)$). Therefore it is clear that the geometrical group should be greater than $OSP(4/3)$.

4-5-2 Geometric spin $1/2$

The most natural and simple generalization of $OSP(4/3)$ is a semi-direct sum of it with an Abelian group G , i.e. [11] $\Gamma = OSP(4/3) \ltimes G$. The Abelian group G is chosen to have a generator S_α that will carry an extra field θ^α , the latter will represent the "missing" spin $1/2$ field upon imposing a new rheonomic [12] symmetry on the geometrical Lagrangian

[11]. The mechanism for obtaining the spin 1/2 field is by spontaneously breaking the $SO(3,1)$ subgroup of Γ [13], which will give the spin 1/2 field from a gage connection of spin 3/2. This new connection is time-like under the action of the non-compact $SO(3,1)$, while the three "physical" spin 3/2 fields are space-like. The resulting theory is the usual $N = 3$ supergravity (the $SO(3)$ invariance is preserved). The beauty with this construction is that Γ can be embedded in a new super group $OSP(4/3,1)$ [14], it is this group which have been employed in the construction of a geometrical (i.e. all fields are gauge fields) action of $N = 3$ supergravity.

This opens the door for further generalization to our work, for constructing massive $N = 3$ supergravity theories. Since all what is basically needed is to find the antisymmetric and symmetric representation of $OSP(4/3,1)$ and repeating the work done here to generate the Lagrangian of the theory.

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APPENDIX A

The Gamma Algebra

This appendix comprises of a listing of some of the programs related to the algebra of the GAMMA-matrices.

Comments for clarifying the significance of commands are included in this listing, a comment (in LISP) must start by a percent '%' sign. My comments, however, are not meant to replace the full description given in the STENSOR manual.

```
% **** gamalg.sbs
% This file contains the basic substitutions
% characterizing the algebra of the
% the famous Gamma matrices (of field theory).

(ON FOINS AUTONEVERCOMMUTE POTSIM)
(ALWAYS COMMUTE DEL)
(SETQ CODIM 'N)% set to N for general database.
% Then one can afterwards specialize by
% (SETSUB ESUL)N $ n $ and (ON ESUBS)

(LOAD "GAMPRG.STS")
% Make sure its in. (Some help-programs)

% Reverse prod. of T and T. (Commutator relation):
(PDEF TT) <T A<<T B< $

% *** From rank to products:
% These are actually the Definitions of multi-indexed
% gammas:
(PDEF TDD A12 T) <T A<<T B< -<DEL A B>$
% The A12 in PDEF means that the defined tensor TDD is
% Antisymmetric in the indices 1 and 2.
% The term T is used as a print-out name for the gamma,
% since SHEEP cannot use Greek symbols.
(PDEF TDDD A123 T) 1/3 <T (A<<T (B (C< $
(PPDEF TDDD !( ) % Perform Permutations.
(PDEF TDDDD A1234 T) 1/4 <T [A<<T [B [C [D<$
(PPDEF TDDDD ![ )
(PDEF TDDDDD A12345 T) 1/5 <T (A<<T (B (C (D (E<$
(PPDEF TDDDDD !( )
```

```

(PDEF TDDDDDD A123456 T)1/6 <T [A<<T [B [C [D [E [F<$
(PPDEF TDDDDDD ![ )

% Introduce a new substitution list CO:
(USSETSUL CO)
% Remove previous substitutions on TD TDD...
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDDD)
(USSETSUL SW1)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDDD)
(USSETSUL SW2)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDDD)
(USSETSUL SW3)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDDD)
(USSETSUL TSUL)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDDD)
(USSETSUL SSUL)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDDD)

(RELOAD "gamalg.Kef")
% Contains the N-dim. Saved dEFinitions for subs.
(USSETSUL TSUL) % TSUL is the built-in substitution list.
% ***** Always do contractions: *****
(SETSUB TSUL) <T A< <T A< $ <:AO > $
(SETSUB TSUL) <T B< <T A B< $ <:AB1 A> $
(SETSUB TSUL) <T A B< <T B< $ -<:AB1 A> $
(SETSUB TSUL) <T A B< <T C B< $ <:BB2 A C> $
(SETSUB TSUL) <T A B< <T A C< $ <:BB2 B C> $
(SETSUB TSUL) <T A B< <T B C< $ -<:BB2 A C> $
(SETSUB TSUL) <T A B K< <T C K< $ <:CB3 A B C> $
(SETSUB TSUL) <T A B C K<<T D K< $ <:DB3 A B C D> $

% **** conditional SWapping :

(SETSUB SW1 (ORDERASK A B))
<T B<<T A< $ -<T A<<T B< + 2<DEL A B> $

(USSETSUL SW2)
(SETSUB SW2 (AND (ORDERASK A C) (ORDERASK A B)))
<T B C<<T A<$ <:POS1 A B C> $

(SETSUB SW2 (AND (ORDERASK A C) (ORDERASK B C)))
<T C<<T A B<$ <:POS2 A B C> $

(USSETSUL SW3)
(SETSUB SW3 (AND (ORDERASK A B) (ORDERASK B C)))
<T B<<T A C<$ <:POS3 A B C> $

(SETSUB TSUL (AND (ORDERASK A B) (ORDERASK B C)))
<T A C<<T B<$ <:POS4 A B C> $
% The values of the tensors POS1, POS2 and POS3 are
% saved in the file gamalg.kef loaded above. However,
% if STENSOR did not find these defintions, it will
% re-compute them!

```

```

% gamaco.sbs
% This file contains some contraction exercises

% This flag causes a SETSUB to also do a PDEF:
(SETQ PSETSUB T)

% In the substitutions below the contractions
% in the mask will be computed once for all,
% and then assigned to the A0, A1... etc.
% From there on, these results are used through
% the substitution, without STENSOR having to
% recompute them again.
% Furthermore, the values A0,... are saved in
% a file, via SAVDEF command, and can be loaded
% prior to this file next time.

(SETQ PDEFINED NIL)

% Introducing new substitution lists:
(USETSUL CO1)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDD)
(USETSUL CO2)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDD)
(USETSUL CO) %Sets LASUB too
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDD)
(RPLACD CO NIL) (RPLACD CO2 NIL) (RPLACD CO1 NIL)

% Interaction is not necessary if the steps of the
% evaluation are known, in such cases, one can
% program the Evaluation:
(SETQ EVSUB'(CO(O CO)(O TSUL) CO(O TSUL)(O SW2)
              (CO TSUL) SW1 (O GA)(O GA)))
% Here, during evaluation the substitution
% lists CO and CO and TSUL etc. are used.
% In (O GA) only simplify, "No eval".

% Substitute for contractions:
(SETSUB CO) <T I< <T I< $<:AO > $
(SETSUB CO) <T I<<T A< <T I< $<:A1 A> $
(SETSUB CO) <T I<<T A B< <T I< $<:A2 A B> $
(SETSUB CO) <T I<<T A B C< <T I< $<:A3 A B C> $
(SETSUB CO) <T I<<T A B C D<<T I< $<:A4 A B C D>$

(SETSUB CO)<T A B< <T A B<$ <:BO> $
(SETSUB CO)<T A B<<T I<<T A B<$ <:B1 I> $
(SETSUB CO)<T A B<<T I J<<T A B<$ <:B2 I J> $
(SETSUB CO)<T A B<<T I J K<<T A B<$ <:B3 I J K> $
(SETSUB CO)<T A B<<T I J K L<<T A B<$ <:B4 I J K L>$
% B4 needed 1311366 sec.
(SETQ EVSUB '( CO (O CO) (O TSUL) CO(O TSUL)(O SW2)

```

```

(CO TSUL) (O GA1)(O CO TSUL)
SW1 (O GA1) (O CO TSUL)
SW1 (O GA) (O GA)      )

(SETSUB CO) <T I J K<    <T I J K< $   <:CO >      $
(SETSUB CO) <T A B C<<T I<<T A B C<$   <:C1 I>      $
(SETSUB CO) <T A B C<<T I J<<T A B C<$   <:C2 I J>    $
%-was 26min

```

```

% **** gamtra.sbs
% Substitutions for the tracing the Gamma matrices.

(ON FOINS AUTONEVERCOMMUTE POTSIM)
(ALWAYS COMMUTE DEL S N M A B C D)
(SETQ CODIM 'N)

% Remove previous substitutions on TD TDD...

(USETSUL TRACE)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDD TDDDDDDDD)
(USETSUL CO)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDD TDDDDDDDD)
(USETSUL SW1)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDD TDDDDDDDD)
(USETSUL TSUL)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDD TDDDDDDDD)
(USETSUL SSUL)
(REMSP TD TDD TDDD TDDDD TDDDDD TDDDDDD TDDDDDDD TDDDDDDDD)

(USETSUL CO)
(SETSUB CO) <T A<<T A< $ N $

(USETSUL TRACE)
(PDEF TRO) N $
(PDEF TR2) N*<DEL A B> $
(PDEF TR4)
N<DEL A B><DEL C D>-N<DEL A C><DEL B D>+N<DEL A D><DEL B C>$

(SETSUB TRACE)

<TRACE ><LPAREN ><T A<<T B<<T C<
<T D<<T E<<T F<<T G<<T H<<RPAREN >$
% The special terms LPAREN and RPAREN introduce (as expected)
% a left an right parenthesis respectively.

<:TR8 A B C D E F G H> $

(SETSUB TRACE)

<TRACE ><LPAREN ><T A<<T B<<T C<

```



```

<T D<<T E<<T F<<T G<<RPAREN >$
O   $           % Trace of odd number of gammas is zero.

(SETSUB TRACE)
<TRACE ><LPAREN ><T A<<T B<<T C<
<T D<<T E<<T F<<RPAREN >$
<:TR6 A B C D E F> $
(SETSUB TRACE)
<TRACE ><LPAREN ><T A<<T B<<T C<<T D<<T E<<RPAREN >$
O   $

(SETSUB TRACE)
<TRACE ><LPAREN ><T A<<T B<<T C<<T D<<RPAREN >$
<:TR4 A B C D>   $
(SETSUB TRACE)
<TRACE ><LPAREN ><T A<<T B<<T C<<RPAREN >$
O   $
(SETSUB TRACE) <TRACE ><LPAREN ><T A<<T B<<RPAREN >$
<:TR2 A B> $
(SETSUB TRACE) <TRACE ><LPAREN ><T A<<RPAREN > $
O   $
(SETSUB TRACE) <TRACE ><LPAREN ><RPAREN > $
<:TRO > $

% exercises:
(PDEF ASL) <A I><T I< + M $
(PDEF DSL) <D I><T I< + M $
(PDEF CSL) <C I><T I< + M $
(PDEF BSL) <B I><T I< + M $
(PDEF EX1)
<TRACE ><LPAREN ><T I<<ASL ><T J<<BSL ><RPAREN > $
(PDEF EX2)
<TRACE ><LPAREN ><T I<<ASL ><T J<<BSL >
<T I<<CSL ><T J<<DSL ><RPAREN > $

```

```

% *** gamtra.out
% The result for the exercises:
% The execution time is given below each result, note
% that the second tensor was simplified from 198 terms
% to 10 terms in less than one minute!

```

$$EX1 = M \sum_{i,j}^2 N_d \quad -N_A B \sum_{k,i}^j d \quad +N_A B \sum_{j,i} +N_A B \sum_{i,j}$$

```
(Ex.-GC time= (4 567) GC= 0)
```

$$\begin{aligned}
EX2 = & (4M^2N - 4M^2N + M^2N^2)C^2D^2 + (-4M^2N + 6M^2N - M^2N^2)B^2D^2 \\
& + (4M^2N - 4M^2N + M^2N^2)B^2C^2 + (4M^2N - 4M^2N + M^2N^2)A^2D^2 \\
& + (-4M^2N + 6M^2N - M^2N^2)A^2C^2 + (-8N^2 + 6N^2 - N^2)A^2B^2C^2D^2 \\
& + (16N^2 - 10N^2 + N^2)A^2B^2C^2D^2 + (-8N^2 + 6N^2 - N^2)A^2B^2C^2D^2 \\
& + (4M^2N - 4M^2N + M^2N^2)A^2B^2 + (2M^4N^2 - M^4N^2)
\end{aligned}$$

198/ 10TERMS

(Ex.-GC time= (54 400) GC= 0)

APPENDIX B

Notation

The notational conventions described here are adopted throughout the thesis, unless explicitly stated otherwise.

The term Euclidean space is used to refer to all kinds of flat spaces, including hyperbolic spaces (pseudo-Euclidean). The same applies to Riemannian spaces.

A d -dimensional manifold is denoted by $M^{p,q}(g)$, $M^d(g)$, or simply M , depending on the particular situation, where g is the metric two form, and p (respectively q) is the number of positive (respectively negative) eigenvalues of the metric form g . Clearly, $p + q = d$, and $p - q = s := \text{signature}$.

If x is a point in M , then $T_x(M)$ and $N_x(M)$ denote the tangent and normal spaces (relative to a particular embedding) to M at x respectively. $\bigcup_{x \in M} T_x(M) = T(M)$ is the tangent bundle over M , with projection π , being the natural projection that takes each vector to the point on M at which this vector is defined. (More about Bundles in Chapter 4)

If $f : M \rightarrow M'$ is a map, then then its induced differential map is defined as:

$$\begin{array}{ccc} f_* : T_x(M) & \rightarrow & T_{f(x)}(M') \\ X & \mapsto & X' \end{array}$$

such that, if $h : M' \rightarrow R$, then $X'_{f(x)}(h) = X_x(h \circ f)$.

A Riemannian connection over $T(M)$ is denoted by ∇ , with ∇_X denoting the covariant differentiation associated with such

a connection with respect to a cross-section X of $T(M)$. \mathfrak{X} is the Lie-algebra of vector fields on M and $\mathfrak{F}(M)$ the algebra of differentiable functions on M .

Small letters x, y, \dots are used to denote points in M , while capitals X, Y, \dots denote vectors and X -sections in $T(M)$, and ξ, η, \dots are reserved for vectors in the normal bundle.

In local coordinates, a point $x \in M$ is represented by an ordered set of real numbers (x^α) , $\alpha=1, \dots, d$, and tangent vectors can be written as linear combinations of the basis $\partial/\partial x^\alpha =: \partial_\alpha$.

Greek indices α, β, \dots will have the range $1, \dots, d$. Capital Roman indices A, B, \dots run from 1 , to d' , while small Roman indices i, j, \dots run over the range: $1, \dots, d'-d$.

The components of a form can be expressed in local coordinates, in the usual way, for example, the components of the metric form can be written as: $g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$

APPENDIX C

The Deformation Programs

This appendix contains a listing of the file "dform.def" which gives the definition of the program used for computing the normal deformations of space-times, followed by a listing of the file "dform.doc" which is a documentation of the computer's response to the loading of "dform.def", and finally, a sample of the output is given in the file "dform.out".

```
***** dform.def *****
% DeFORMATION formalism using embedding.

% For nicer output, having e (for epsilon),
% n (for eta) and V at first. The ORDER
% command is used to imposes a new ordering
% weight for tensor:
(OORDER -100 EP ETA VU)

% DECLT = Declare Tensors:
(DECLT (ALF A 1 S23 "2nd fundamental forms")
(ETA E S12 "The flat metric")
(EP e "The infinitesimal")
(VU V 1 "Deformation vector")
(D 1 "Ricci vector")
(LAM L! "Cosmological cnst")
(DTS Ds 1 "Derivative of Ricci vector")
(LAM1 L1! "Variation of cosmo cnst"))

% Transverse space, NORMAL to space-time.
(INDEXTYPE NORMAL N I J K L M N O P Q R S)
% The INDEXTYPE command introduces a new index-type
% with the head term of the argumnet-list is a name
% for this type the second term is the dimension,
% i.e. the range of the indices, which are given in
% the rest of the argument-list.
% Tangent space, usual space-time.
(INDEXTYPE USUAL U A B C D E F G H X Y Z)
```

```

(PDEF GAUST Gt "Deformed Riemann Tensor")
  <ETA I J><ALFT I A [C><ALFT J B [D>$
(PPDEF GAUST ![ )
(PDEF GAUS G "Rie tensor")
  <ETA I J><ALF I A [C><ALF J B [D>$
(PPDEF GAUS ![ )
(PDEF RIE1 dG "1st order deformation")
  <GAUST A B C D>-<GAUS A B C D>$
(PDEF (ALFT I A B) At 1 "Deformed ALF")
  <ALF I A B>+<EP ><ALFS I J A B><VU J> $
(PDEF (ALFS I J A B) As 1 "Derivative of ALF")
  <DTS I J A B> + <D I K A><D K J B>-
  <ALF I C A><ALF K B D><GUU C D><ETA K J> $
(PDEF (DTS I J A B) "Derivative of D") <D I J A ;B> $
  % Need only 1st order terms:
(SETSUB TSUL) <EP ><EP > $          O$
(PDEF EINS S12 E "EINStein's field equations")
  <RIC1 A B>-1/2 <RSCL1 ><GDD A B>-
  <LAM1 ><GDD A B>-1/2 <RSCLC ><G1DD A B>
  -<LAM ><G1DD A B> $
(PDEF RIC1 R1 S12 "RICci tensor first order")
  <G1UU A B><RIEC A C D B>+<GUU A B><RIE1 A C D B>$
(PDEF G1UU S12 1 2 g1 "contra-Metric, 1st order")
  2<EP ><VU I><ETA I J><ALF J C D><GUU A C><GUU B D> $
(PDEF G1DD S12 g1 "cov-Metric, 1st order")
  -2 <EP ><VU I><ETA I J><ALF J A B> $
(PDEF RSCL1 R1! "Ricci Scalar, 1st order")
  <GUU A B><RIC1 A B>+<G1UU A B><RICC A B>$
(USSETSUL ARR) (REMSP EP)
(SETSUB ARR)
  <EP ><ETA I J>$ <F I J< $
(SETSUB ARR)
  <F I J<<VU K> $ <F I J ^K< $

```

***** Documentation of Objects in file DFORM.DEF and .BLF *****
Wed Aug 1 22:33:29 1984
Loading dform.def

```

ALF      2nd fundamental forms
ETA      The flat metric
EP       The infintesimal
VU       Deformation vector
D        Ricci vector
LAM      Cosmological cnst
DTS      Derivative of Ricci vector
LAM1     Variation of cosmo cnst
GAUST    Deformed Riemann Tensor

```

$$G_{abcd} = E \begin{matrix} i & j \\ A_{ij} & A_{cd} \end{matrix} a[c \quad bd]$$

GAUS Rie tensor

$$G_{abcd} = E \begin{matrix} i & j \\ A_{ij} & A_{cd} \end{matrix} a[c \quad bd]$$

RIE1 1st order deformation

$$dG_{abcd} = G_{abcd} - G_{abcd}$$

ALFT Deformed ALF

$$A_{ab} = A_{ab} + e A_{ab} V_{jab}$$

ALFS Derivative of ALF

$$A_{jab} = D_s \begin{matrix} i & k \\ A_{ij} & A_{ka} \end{matrix} + d \begin{matrix} i & k \\ d_{ij} & d_{ka} \end{matrix} - A \begin{matrix} i & k \\ A_{ca} & A_{bd} \end{matrix} g \begin{matrix} cd \\ E_{kj} \end{matrix}$$

DTS Derivative of D

$$D_s \begin{matrix} i \\ jab \end{matrix} = D \begin{matrix} i \\ ja;b \end{matrix}$$

EINS EINStein's field equations

$$E_{ab} = R1_{ab}^{-1} - (2) R1_{ab} g_{ab}^{-1} - L1_{ab} g_{ab}^{-1} - (2) RSCLCg1_{ab}^{-1} - L g_{ab}^{-1}$$

RIC1 RICci tensor first order

$$R1_{cd} = g1_{cd} \begin{matrix} ab \\ RIEC_{acdb} \end{matrix} + g_{cd} \begin{matrix} ab \\ dG_{acdb} \end{matrix}$$

G1UU contra-Metric, 1st order

$$g1_{ab} = 2eV \begin{matrix} i & j \\ E_{ij} & A_{cd} \end{matrix} g_{ac} g_{bd}$$

G1DD cov-Metric, 1st order

$$g1_{ab} = -2eV E \begin{matrix} i & j \\ & ij \\ & ab \end{matrix}$$

RSCL1 Ricci Scalar, 1st order

$$R1 = g_{ab} R1_{ab} + g1_{ab} RICC_{ab}$$

dform.def loaded

The Gause-Riemann tensor. (The 't' for tilde!)

$$Gt_{bcd} = -At \begin{matrix} i & j \\ & ad \\ & bc \end{matrix} E_{ij} + At \begin{matrix} i & j \\ & ac \\ & bd \end{matrix} E_{ij}$$

The deformation in Rie. for the first order:

$$\begin{aligned} dG_{abcd} = eV E & \left(-Ds \begin{matrix} i & k \\ & ki \end{matrix} A_{jbc} da + Ds \begin{matrix} i & k \\ & jbd \end{matrix} A_{ac} - d \begin{matrix} k & l \\ & lb \end{matrix} d \begin{matrix} i & j \\ & jc \end{matrix} A_{da} + \right. \\ & + d \begin{matrix} k & l \\ & lb \end{matrix} d \begin{matrix} i & j \\ & jd \end{matrix} A_{ac} + E g \begin{matrix} ef k \\ & jl \end{matrix} A_{i} A_{bc} A_{de} A_{af} - E g \begin{matrix} ef k \\ & jl \end{matrix} A_{i} A_{bd} A_{ae} A_{cf} - \\ & - E g \begin{matrix} ef k \\ & jl \end{matrix} A_{i} A_{be} A_{df} A_{ac} + E g \begin{matrix} ef k \\ & jl \end{matrix} A_{i} A_{be} A_{da} A_{cf} + Ds \begin{matrix} i & k \\ & jac \end{matrix} A_{bd} - \\ & \left. - Ds \begin{matrix} i & k \\ & jda \end{matrix} A_{bc} + d \begin{matrix} i & l \\ & la \end{matrix} d \begin{matrix} k \\ & jc \end{matrix} A_{bd} - d \begin{matrix} i & l \\ & ld \end{matrix} d \begin{matrix} k \\ & ja \end{matrix} A_{bc} \right) \end{aligned}$$

APPENDIX D

The Kaluza-Klein Program-Listing

Only the basic files are listed in this appendix, these are the "job" file, which monitors the interaction with the computer, the "initialization" file, needed for any Kaluza-Klein computation, and finally the two files for defining the connection and the Ricci tensor.

Samples of the output are found in chapter three.

```
% *** job.def ***
% This file is automatically loaded by STENSOR,
% as soon as it is provoked.
% Here it is used to load KK files needed for a
% specific work session. Edit as appropriate.

(LOAD "KKPRG.STS")           % Additional LISP functions.
(OFF EDEFASK)% temporarily stop interactive evaluation.
(SETQ PSETSUB T)
(LOAD "KKINI.DEF")
% the following command asks the user interactively about
% the ansatz file to be loaded:
(COND (
  (YASK "Load Mustafa's ansatz?" "answer Y or N: ")
  (LOAD "MADER.SEF"))
  (T
  (EVALIST 'LOAD
    (CAR (ERASK "Enter alternative-ansatz file"
      "(remember to quote))))))

(ON NOASK NOZERO ALLMSG SYMZERO)

(LOAD "KKCOM.SBS")
(LOAD "KKDER.SBS")
(LOAD "KKTSB.SBS")
(USETSUL TSUL)
(LOAD "KKCON.DEF")
(LOAD "GRCONU.SEF")
(LOAD "KKCUR.DEF")
(USETSUL TSUL)
(OFF NOASK) (ON EDEFASK SPLIT)
```

```

(DOSUM RS)
(EDEF RS)
(PUT 'T110 'DEF (COPY (CONU 1 1 O)))

% defining the following two tensors proved to be quite
% useful in subsequent simplifications, since they are
% standard for all ansatz.

(LPDEF CONU 1 1 O)% here we define only one "component"
                    % of the split connection.
    <T110 A B R> $
(PDEF CDER)
    -2 <GSUU P Q><ESD P><T110 A A Q>
    +2 <GSUU P Q><TUDD R P Q><T110 A A R> $

```

```

%*** kkini.def ***
%Kaluza-Klein INItialization file DEFining basic relations.
(OFF GDD GUU)          % to prevent g & ( ) from commuting.
                        % which is built-in.

(ORDER -200 GUU GSUU GCUU)      % To keep GUU to the left!!!
(SETQ INDEXTYPES '(
(DSPACE ID JD KD LD MD ND PD QD RD SD TD UD VD XD YD ZD)))
(INDEXTYPE GROUP NG
    I J K L M N I1 J1 K1 L1 M1 N1 I2 J2 K2 L2 M2 N2)
    % I,J,.. =1,...,NG. for the Group G.
(INDEXTYPE COSET NC
    A B C D E F G H A1 B1 C1 D1 E1 F1 G1 H1
    A2 B2 C2 D2 E2 F2 G2 H2)
    % A,B,.. =1,...,NC. for the Coset G/H.

% RPL (RePLace ) is a SHEEP command for assigning a value
% for a variable.
(RPL CG) NG - NC $ % To have a specific value for CG, use:
%(SETQ CG (EVAL CG)) % after specifying MC & NC via SETQ.

(INDEXTYPE SUBG CG IX IY IZ)
    % IX,IY,.. =1,...,NC-MC. Sub-Group H.

(INDEXTYPE SPACETIME ST
    P Q R S T X Y Z P1 Q1 R1 S1 T1 X1 Y1 Z1
    P2 Q2 R2 S2)
    % X,Y,..=1,...,ST. Space-Time indices.

(RPL TOT) ST + NC $
(SETQ CODIM TOT)          %(EVAL TOT))
(SETSPLIT DSPACE SPACETIME COSET) % Initialize splitting.
(DECLT (ED e)(ECD e)(ESD e) (DID d)
    (FIUU FI S12 1 2) (COMUDD C 1)
    (CUDD C 1 A23) (KUD K 1) (AUD A 1)
    (FUDD F 1 A23) (GCDD g S12) (GSDD g S12)
    (GSUU g S12 1 2)(SUUD S S12 1 2))
(DIFFOPERATOR (ED ID) (ESD X) (ECD A) (HU ID) (DID X))

```

```

(PDEF HDD h S12 "The metric of the group")
    <GCDD A B><KUD A I><KUD B J> $
(PDEF GUU g S12 1 2)
    <GUU ID JD>$
(SPLIT GUU)
(PDEF GDD g S12)
    <GDD ID JD>$
(SPLIT GDD)
(PDEF ED)
    <ED ID>$
(SPLIT ED)
(LPDEF ED)      % the "split" EDEF command! note that it
                % takes two tensor expressions, since the
                % index I is split to X and A.
                <ESD X> $
                <ECD A> $
(ON POTSIM)    % turn on simplification for interaction.

```

```

% **** kkcon.def ****
%DEFining the CONnection coefficients for the KK theory.

(PDEF CON T)    % define the connection in terms of two
                % new tensors to simplify computations.
                <DG JD KD LD> +<DCM JD KD LD> $

(PDEF DG)
    <LPAREN ><ED [JD]><GDD [ID [KD]><RPAREN > $
(PDEF DG ![ )
(PDEF DCM)
    <GDD LD MD><COMUDD MD JD KD>
    -<GDD JD MD><COMUDD MD KD LD>
    -<GDD KD MD><COMUDD MD JD LD> $
(PDEF CONU T 1)
    1/2 <GUU ID LD><CON JD KD LD> $

%(SPLIT DCM)    % the splitting was needed only once,
%(SPLIT DG)     % the result was saved in a separate
%(SPLIT CONU)   % file. This can be repeated for new
                % ansatz.
(DECLT (TUDD 1 T) (T110 1 T))

```

```

% **** kkcur.def ****
% DEFining the CURvature tensor for KK theory.

% The evaluated CONnection coefficients:
%(LOAD "MACONU.SEF") % load if needed.
(PDEF CUR R "The Ricci tensor")
    <ED LD><CONU LD KD JD> -<ED KD><CONU LD LD JD>
    -<CONU LD KD ID><CONU ID LD JD>
    +<CONU LD LD ID><CONU ID KD JD>
    -<CONU LD ID JD><COMUDD ID LD KD> $

```

```
(PDEF RS R "The Rieman Scalar")
  <GUU ID LD><CUR ID LD> $
```

```
(SPLIT CUR)
(RPL CUR 0 1) 0 $ % put the off-diagonal elements
(RPL CUR 1 0) 0 $ % of the Ricci tensor to zero,
                  % this works in our case because
                  % after all we are only interested
                  % in the trace of this tensor.
```
