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ALGORITHMS FOR STATE CONSTRAINED CONTROL PROBLEMS WITH DELAY

by

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To my mother and father

ABSTRACT

In this thesis we present algorithms for solving several optimal control problems for systems expressed in terms of a general nonlinear delay-differential equation with control term. The first control problem considered, Problem P1, involves only control constraints and its solution is determined using an algorithm permitting strong variations in control. By enlarging the set of controls on which P1 is defined to relaxed controls we assure that minimising solutions exist. This together with the "linear" nature of relaxed controls, simplify many of the difficulties associated with proving convergence. Indeed, this is shown by designing an algorithm which solves P1 over the set of relaxed controls, denoted $\underset{\sim}{G}$, and obtaining its convergence properties. These are much simpler and readily obtained than the strong variational algorithms which uses ordinary controls. Next, the computationally expensive problem of simulating measure valued relaxed controls is tackled. It is shown that relaxed controls can be approximated to any degree of accuracy using only ordinary controls. An algorithm where this is done is investigated and results show that, if the degree of approximation is kept constant at some chosen value, all limit points generated satisfy optimality conditions to within "delta". On the other hand, if the approximation accuracy is increased indefinitely as the algorithm proceeds, it is shown that limit points then satisfy optimality conditions "exactly". Terminal equality constraints are then added to P1 to give a more general problem, denoted Problem P2. An algorithm which solves P2 is obtained. This uses an exact penalty function to eliminate the

terminal equality constraints and handles the control constraints by minimising an intermediate subproblem over the feasible control set $\underset{\sim}{G}$. A test function which determines the value of the penalty parameter (guaranteed to be finite) needed to ensure feasibility is also included. The last problem considered in this thesis is that obtained by replacing the terminal equality constraints in P2 by a general nonlinear state constraint and is denoted as Problem P3. Two algorithms, one conceptual and the other implementable, which solve P3 are presented. Both these use a method similar to the one used in solving P2. However, a different method of updating the penalty parameter K, which does not require a test function is presented. This method relies on keeping K larger than the multipliers at the solutions to the intermediate problems at each stage of the algorithms. The existence of a finite K which gives feasibility is guaranteed by considering calm problems.

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CONTENTS

CHAPTER 1

FUNDAMENTAL PRELIMINARIES

Introduction

In this chapter we supply the necessary mathematical background and the basic results needed for the remainder of the thesis. It is intended primarily for reference and readers with a grounding in elementary Functional Analysis and Optimal Control Theory will loose nothing by proceeding directly to Chapter 2 after perhaps a glance at the Maximum Principles for delay problems and the section on Relaxed Controls.

For the most part, we shall confine ourselves to stating definitions and theorems, and because of this, the material in this chapter is sketchy at best and in many cases the results are not presented in their full generality. We shall, however, cite references which contain the necessary proofs and details throughout the chapter (and thesis) so that the interested reader may consult the literature available. It is hoped that even in this scanty form, this chapter will provide a valuable insight to readers who are new to the area of optimal control and to make the thesis somewhat self-contained.

The chapter is divided into two parts, Section A and Section B. Section A contains the basic definitions for sets, topological spaces, duals, etc. Extensive discussion of most of this material can be found in Luenberger [LU1], Kingman and Taylor [K1], Warga [W3] and Sutherland [SU1]. Section B contains an introduction to optimal control theory and the existence of minimising solutions. Necessary conditions of optimality for the problems considered later in the thesis are also presented. These are essentially adaptations of the results of Huang [HU1] to a form which is applicable here. The existence of minimising solutions is guaranteed by considering solutions which are relaxed controls - see Warga [W3] and Young [Y1].

Section A

A1. Sets and Functions

We denote sets (also called collections, families and classes) by capital letters (Latin, script or Greek) and the empty set by $\emptyset$. $x \in A$ signifies that x is a member (synonomously element, point) of the set A, i.e. A contains x. If the set A is defined to be all members of the set Δ having a certain property P(x), then we write

$$A = \{x \in \Delta : P(x)\}$$

We also write $\{x_1, x_2, \ldots\}$ for the set with elements $x_1, x_2, \ldots$, or as $\{x_i\}$ or $\{x_i : i \in \mathbb{N}\}$ where $\mathbb{N} = \{1,2,3,\ldots\}$ is the set of all positive integers (or some indexing set). At little risk of ambiguity sequences of points $x_1, x_2, \ldots$ in a set B are also denoted $\{x_i\}$ or $\{x_i\}_{i=o}^{\infty}$ or as "an infinite sequence $\{x_i\} \in B$" with each $x_i \in B$.

Given two sets A and B, then

(i) the union of A and B, written $A \cup B$, comprises of all elements in A or B

(ii) the intersection of A and B, written $A \cap B$, comprises of all elements in A and B

(iii) the difference of A and B, written $A \setminus B$, comprises of all elements in A not in B, similarly the difference of B and A, written $B \setminus A$, comprises of all elements in B not in A

The concept of union and intersection generalizes to apply to families of sets $\{A_i\}$. Here $\bigcup_{i \in \mathbb{N}} A_i$ (or $\bigcup_i A_i$ for short) is the set of all points lying in at least one member of the family $\{A_i\}$ and $\bigcap_{i \in \mathbb{N}} A_i$ (or $\bigcap_i A_i$) is the set of points lying in every A_i. A family of sets $\{A_i\}$ has the finite intersection property if given any finite subset M of the index set $\mathbb{N}$, then $\bigcap_{i \in M} A_i$ is non-empty. Sets with empty intersection are called disjoint.

A is a subset of Λ if each member of the set A is also a member of the set Λ, symbolically we write this as $A \subset \Lambda$. If A and Λ are not the same, we say A is a strict subset of Λ. A set is called countable if it can be put into a one-to-one correspondence with the set of positive integers.

Let A, B be two sets. Then a function (alternatively called map, mapping, operator, transformation) $f: A \longrightarrow B$ (or simply f when A and B are understood) is a rule which assigns to any element $x \in A$ an element $f(x) \in B$. If the element $b \in B$ is assigned to the element $a \in A$, we say that b is the image of a under f (or f takes the value b at a), and write $b = f(a)$. If $D \subset A$, then f(D), defined by

$$f(D) \triangleq \{ b \in B : b = f(d) \text{ for some } d \in D \}$$

is called the image of the set D (under f). If $E \subset B$, then $f^{-1}(E)$, defined by

$$f^{-1}(E) \triangleq \{ a \in A : f(a) \in E \}$$

is called the pre-image of the set E (under f). A is called the domain of f and B is called the co-domain of f. The set $\{b \in B : b = f(a) \text{ for some } a \in A\}$ is called the range of the map f (written R(f)). In the case when R(f) coincides with the co-domain of f, we say that f maps A onto B (or briefly that f is onto). The function f is one-to-one, if for each point $b \in R(f)$ the set $f^{-1}(b)$ contains only one point. In this case, the operation $f^{-1}(.)$ defines a function from R(f) onto A. This function is called the inverse of f.

Suppose again that A, B are sets and $D \subset A$. Let f be a function from A onto B and let g be a function from D onto B. In the case when $d \in D$ implies $f(d) = g(d)$, we say that g is the restriction of f to D. Conversely, we say f is an extension of g to A.

A2. Real Numbers

If a and b are real numbers with b greater than a (written $b > a$, or a less than b which we write as $a < b$), then $[a,b]$ denotes the set of real numbers x satisfying $a \le x \le b$. A round bracket denotes strict inequality in the definition. Hence $(a,b]$ denotes all x with $a < x \le b$.

Let S be a set of real numbers bounded from above, then there is an $x_0 \in S$ which satisfies $x_0 \ge x$ for all $x \in S$. The number

x_0 is called the supremum of S and is denoted sup $\{x : x \in S\}$ or $\sup_{x \in S} \{x\}$. If S is not bounded from above we write $\sup_{x \in S} \{x\} = \infty$. Similarly if S is bounded from below we say $x^* = \inf_{x \in S} \{x\}$ or $x^* = \inf \{x : x \in S\}$.

Let $\{x_i\}$ be an infinite sequence of real numbers. Then we say that $\hat{x}$ is the limit superior of $\{x_i\}_{i=0}^{\infty}$ and write $\hat{x} = \limsup \{x_i\}$ if, given any $\epsilon > 0$,

(i) there exists some positive integer i_0 such that for all $i > i_0$, $x_i < \hat{x} + \epsilon$;

(ii) for any positive integer j_0, $x_i > \hat{x} - \epsilon$ for some $i > j_0$

$\hat{x}$ is the limit inferior of $\{x_i\}_{i=0}^{\infty}$, written $\hat{x} = \liminf \{x_i\}$ if $\hat{x} = -\limsup \{-x_i\}$. If $\liminf \{x_i\} = \limsup \{x_i\} = x^*$, we say x^* is the limit of $\{x_i\}_{i=0}^{\infty}$ and write $x^* = \lim \{x_i\}$.

We denote by $\mathbb{R}$ the real line $\triangleq (-\infty, \infty)$ and by $\mathbb{R}^+$ the extended real line $\triangleq [-\infty, \infty]$.

A3. Vector Spaces

A vector space X is a set of elements called vectors (one of which is the null element $\emptyset$) and two operations, namely addition and multiplication by a real scalar. The set X and these two operations are assumed to satisfy a number of axions which ensure that vector spaces possess many of the features of elementary vector algebra (see for example Luenberger [LU1]).

[Note: we shall only be concerned with vector spaces defined on the field of real numbers.]

A subspace S of X is a subset of X closed under addition and scalar multiplication, i.e. if $x, y \in S$ then $\alpha x + \beta y \in S$ for any real numbers α, β.

If M,N are subspaces of X, we define the sum of M and N to be

$$M+N = \{ x \in X : x = m+n, \text{ some } m \in M, \text{ some } n \in N \}$$

Similarly we define the difference of M and N by

$$M-N = \{ x \in X : x = m-n, \text{ some } m \in M, \text{ some } n \in N \}$$

A linear variety V in X is defined to be a subset of X which can be expressed as $V = x_0 + M$, where x_0 is some fixed element in X and M is a subspace of X. If $V \neq X$, then we say V is a strict linear variety.

A set K in X is said to be convex if for all x_1, $x_2 \in K$, we have that all points of the form $x = \alpha x_1 + (1-\alpha)x_2$ for all $\alpha \in [0,1]$ are in K.

Let S be an arbitrary set in X. The convex hull of S, denoted co(S), is the smallest convex subset in X containing S.

Let X,Y be two vector spaces. We say X and Y are isomorphic if there exists a linear operator T with domain X, range all of Y, and for which the inverse T^{-1} exists (a map $f: X \to Y$ is linear if for any real α_1, α_2 and $x_1, x_2 \in X$ we have

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2))$$

A4. Topological Spaces

A collection $\mathcal{T}$ of subsets of a set X is a topology in X if

(i) $\emptyset \subset \mathcal{T}$, $X \subset \mathcal{T}$

(ii) the union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$

(iii) the intersection of a finite number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$

The couple $(X, \mathcal{T})$ is called a topological space. A topology $\mathcal{T}_1$ is weaker than a topology $\mathcal{T}_2$ (or $\mathcal{T}_2$ is stronger than $\mathcal{T}_1$) if $\mathcal{T}_1 \subset \mathcal{T}_2$.

A common method of defining a topology on a set X is by means of a "base". If $\mathcal{U}$ is a collection of subsets of X, then there exists a unique topology $\mathcal{T}(\mathcal{U})$ in X containing $\mathcal{U}$ and weaker than any other topology containing $\mathcal{U}$; $\mathcal{T}(\mathcal{U})$ is constructed by first forming $\mathcal{V}(\mathcal{U})$ defined as follows:

$$\mathcal{V}(\mathcal{U}) \triangleq \{ \bigcap_{j=1}^{K} A_j \ : \ K \in \mathbb{N} \ , \ A_j \subset \mathcal{U} \}$$

and then defining $\mathcal{T}(\mathcal{U})$ as the set of the unions of all sub-collections of $\mathcal{V}(\mathcal{U})$. In this case $\mathcal{U}$ is called a subbase of $\mathcal{T}(\mathcal{U})$ and a base of a topology $\mathcal{T}$ is any collection $\mathcal{V} \subset \mathcal{T}$ such that every element of $\mathcal{T}$ is a union of some subcollection of $\mathcal{V}$ (thus $\mathcal{V}(\mathcal{U})$ is a base of $\mathcal{T}(\mathcal{U})$).

Elements of a topology are the open sets of a topological space $(X, \mathcal{T})$. Any open set containing a point $x \in X$ is a neighbourhood of x. A subset Y of X is termed closed if its complement (i.e. $X \setminus Y$) is open.

Given a subset Y of X:

(i) the closure of Y, written $\overline{Y}$, is defined to be the intersection of all closed sets containing Y

(ii) the interior of Y, written int Y, is defined to be the union of all open sets contained in Y

(iii) the boundary of Y, written ∂Y, is defined to be $\overline{Y} \setminus \text{int } Y$

Y is a dense subset of X if $Y \subset X \subset \overline{Y}$. A topological space $(X, \mathcal{T})$ is separable if X contains a finite or countable dense subset.

We now introduce the important concept of compactness, but before doing so we give a preliminary definition:

In a topological space $(X, \mathcal{T})$ a subcollection $\mathcal{V}$ of $\mathcal{T}$ is an open covering of a set $B \subset X$ if $B \subset \bigcup_{V \subset \mathcal{V}} V$. If $\mathcal{W}_1 \subset \mathcal{W}$ and both $\mathcal{W}_1$ and $\mathcal{W}$ are both open coverings of a set B, then $\mathcal{W}_1$ is a subcovering of $\mathcal{W}$.

Definition

A set $B \subset X$ is compact if every open covering $\mathcal{V}$ of B has a finite subcovering.

Now suppose that f is a function from X into Y, where X and Y both have topologies defined on them. We say that f is continuous at the point $x_0 \in X$ if for each neighbourhood V of $f(x_0)$ there exists a neighbourhood U of x_0 such that $f(U) \subset V$. We say that f is continuous if it is continuous at all points $x \in X$.

Let $\{x_i\}$ be a sequence of elements in X. We say that the sequence converges to a limit x^* if, given any neighbourhood P of

x^* there exists some integer i_0 such that $x_i \in P$ for all $i \geq i_0$ (for a given sequence x^* may not be unique).

Finally we mention three important classes of topological spaces:

1. A Hausdorff space is a topological space with the property that, given any distinct points x_1, x_2 in the space, there exist disjoint neighbourhoods U_1, U_2 of x_1, x_2 respectively (hence converging sequences in Hausdorff spaces have unique limits).

2. A topological vector space $(X, \mathscr{T})$ is a vector space with a Hausdorff topology such that the functions $(x,y) \rightarrow x+y : X x X \rightarrow X$ and $(\alpha,x) \rightarrow \alpha x : \mathbb{R} x X \rightarrow X$ are continuous

3. A locally convex space is a topological space whose topology is generated by a base composed of convex sets

A.5 <u>Normed Linear Spaces</u>

A normed linear space is a vector space X together with an operator called a norm. The norm is a real-valued function on the linear space and we denote its value at the point x by $\| x \|$. The norm is assumed to satisfy the following axioms:

(i) $\| x \| \geq 0$ for all $x \in X$, $\| x \| = 0$ iff $x = \emptyset$

(ii) $\| x+y \| \leq \| x \| + \| y \|$ for all $x,y \in X$ (the triangle inequality)

(iii) $\| \alpha x \| = | \alpha | \| x \|$ for all scalars α, $x \in X$.

[Note we have only considered real normed linear spaces]

The set $B(x,a) \triangleq \{ y \in X : \|x-y\| < a \}$ is an open ball with centre x and radius a, with $a > o$, and the set $B^C(x,a) \triangleq \{ y \in X : \|x-y\| \leq a \}$ is a closed ball with centre x and radius a, with $a > o$. The topology on X generated by the base consisting of all open balls in X is the usual topology on a normed vector space. To distinguish it from other topologies on X it is called the strong topology. Other topologies which are of interest are the weak topology and the weak* topology, see Luenberger [LU1] or Warga [W3].

Unless otherwise stated we assume for the remainder of this section that X is equipped with its strong topology.

It is thus clear that a set A in X is open if for each $x \in A$ there exists some open sphere S with $x \in S \subset A$; and also if $x_1 \in X$ is disjoint from a closed subset $A_1 \subset X$, there exists some open sphere S_1 disjoint from A_1, with $x_1 \in S$.

A very convenient feature of normed linear spaces is that many of the topological properties introduced in section A4 can be equivalently expressed in terms of sequences. Letting X and Y be normed linear spaces we have:

a. A point x lies in the closure of a subset S of X if there exists some sequence in S converging to the limit x

b. The function $f:X \longrightarrow Y$ is continuous if for every convergent sequence $\{x_i\}$,

$$\lim_{i \to \infty} \{f(x_i)\} = f(x^*) \quad \text{where } x^* = \lim_{i \to \infty} \{x_i\}$$

c. A set S in X is compact iff every sequence in S contains a subsequence converging to some point in S, i.e. if we have an infinite sequence $\{x_i\}$ in S, then there exists a subsequence indexed by $K \subset \{1,2,3,\ldots\}$ and an $x^* \in S$ such that $\lim_{\substack{i \to \infty \\ i \in K}} \{x_i\} = x^*$

We denote this as $x_i \xrightarrow{K} x^*$

A function $f : X \longrightarrow \mathbb{R}$ is upper semicontinuous at x^* if $\limsup_{x \longrightarrow x^*} f(x) = f(x^*)$ and lower semicontinuous at x^* if $\liminf_{x \longrightarrow x^*} f(x) = f(x^*)$. We say that f is upper semicontinuous (lower semicontinuous) if it is upper semicontinuous (lower semicontinuous) at x^* for every $x^* \in X$. f is continuous if it is both upper and lower semicontinuous.

If X is any set, Y a normed linear space, $f_i : X \longrightarrow Y$, $i \in \mathbb{N}$, and $f : X \longrightarrow Y$, then $\lim_i f_i(.) = f(.)$ uniformly or, equivalently $\lim_i f_i(x) = f(x)$ uniformly for $x \in X$, if for every $\varepsilon > 0$ there exists an integer $i(\varepsilon) \in \mathbb{N}$ such that $\| f_i(x) - f(x) \| \leq \varepsilon$ for all $x \in X$ and all $i \geq i(\varepsilon)$.

Two normed linear spaces X,Y are termed isometrically isomorphic if there exists a one-to-one transformation $T : X \longrightarrow Y$ onto Y such that $\|Tx\| = \|x\|$.

A6. Product Spaces

Let X_1, X_2 be sets. Then the product set $X_1 x X_2$ is defined to be the set of all ordered pairs (x_1, x_2) where $x_1 \in X_1$, $x_2 \in X_2$. When X_1, X_2 are topological spaces, there is a standard way of assigning a topology to $X_1 x X_2$:

Let $\mathcal{B}$ be the family of all subsets of $X_1 x X_2$ of the form $T_1 x T_2$ where T_1, T_2 are open sets in X_1, X_2 respectively, then take $\mathcal{T}$ to be the topology generated by $\mathcal{B}$. $\mathcal{T}$ is called the product topology on $X_1 x X_2$.

When X_1 and X_2 are normed linear spaces then the product space $X \triangleq X_1 x X_2$ with the norm defined by $||(x_1, x_2)|| \triangleq ||x_1|| + ||x_2||$ is also a normed linear space.

The product spaces $X = X_1 x X_2 x \ldots x X_n$ of the n topological vector (normed) spaces as well as the product topology (norm) on X are defined in an analogus manner.

A7. <u>Banach Spaces</u>

To state the definition of a Banach space we need to introduce Cauchy sequences. A sequence $\{x_i\}$ in a normed linear space X is called a Cauchy sequence if

$||x_i - x_j|| \longrightarrow 0$ as $i, j \longrightarrow \infty$, i.e. given any $\varepsilon > 0$, there exists an integer i_0 such that $||x_i - x_j|| \leq \varepsilon$ for all $i, j \geq i_0$. Then we say that a normed linear space X is complete if every Cauchy sequence $\{x_i\}$ in X converges to a limit in X. A Banach space is defined to be a complete normed linear space.

It is shown in Sutherland [SU1] that not all normed linear spaces are complete. Some examples of Banach spaces which we shall be using are:

1. The space $C[T; \mathbb{R}]$ of real continuous functions on the interval $T \triangleq [0,1]$ together with the norm,

$$\| x \| = \sup_{0 \le t \le 1} | x(t) |$$

2. The space l_p, $1 \le p \le \infty$

The space l_p, $1 \le p < \infty$ consists of all sequences of scalars $\{a_1, a_2, \ldots\}$ for which

$$\sum_{i=1}^{\infty} | a_i |^p < \infty$$

together with the norm of an element $x = \{a_i\}$ in l_p being defined by

$$\| x \|_p = \left\{ \sum_{i=1}^{\infty} |a_i|^p \right\}^{1/p}$$

The space l_∞ consists of bounded sequences with the norm of an element $x = \{a_i\} \in l_\infty$ being defined by

$$\| x \|_\infty = \text{essential}_i \text{supremum } | a_i |$$

$$= \text{ess}_i \sup | a_i |$$

3. The space $L_p [0,1]$, $1 \le p \le \infty$, (or L_p) where L_p, $1 \le p < \infty$ consists of the space of functions x on $[0,1]$ for which $| x(t) |$ is Lebesgue integrable and the norm is defined by

$$\| x \|_p = \left\{ \int_0^1 | x(t) |^p dt \right\}^{1/p}$$

The space $L_\infty [0,1]$ is the space of all Lebesgue measurable functions on $[0,1]$ which are bounded except on a set of measure zero, and the norm of an element $x(t) \in L_\infty$ is defined as

$$\| x \|_\infty = \text{ess sup } | x(t) |$$

A8. Measures, Measurable Functions and Integrals

To define a measure on an arbitrary set S we need to introduce σ-fields. A family Σ of subsets of S is a σ-field (or algebra) if

(i) the empty set $\emptyset \subset \Sigma$

(ii) $A \subset \Sigma \implies$ its complement in S, i.e. $S \setminus A$ (denoted A^c) $\subset \Sigma$

(iii) $A_i \subset \Sigma$, $i=1,2,3,\ldots$ implies $\bigcup_{i=1}^{\infty} A_i \subset \Sigma$ and $\bigcap_{i=1}^{\infty} A_i \subset \Sigma$

A measure on S is then a function $\mu : \Sigma \longrightarrow \mathbb{R}^+$ satisfying:

a. $\mu(\emptyset) = 0$

b. $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ whenever $A_i \subset \Sigma$ and A_i is disjoint from A_j for $i \neq j$.

We refer to the couple (S, Σ) as a measurable space and to the triple (S, Σ, μ) as a measure space (or just S as a measure space if Σ and μ are understood).

If $\mu(S) = 1$ then μ is called a probability measure. A measure $\mu : \Sigma \longrightarrow \mathbb{R}^+$ is supported on A (or has its support in A) if $\mu(B) = 0$ whenever $A \cap B = \emptyset$ for all $B \subset \Sigma$.

Let (S, Σ), (T, Δ) be measurable spaces. Then a function $f: S \rightarrow T$ is measurable if $f^{-1}(A) \subset \Sigma$ for all $A \subset \Delta$.

We can associate with a measurable function f, a real number, the integral of f over S, written

$$\int_S f \, d\mu$$

The values $\pm\infty$ are allowed for the integral. The functions g for which $\int_S |g| \, d\mu < \infty$ are called integrable. We shall be using Lebesgue integrals where the integration is taken with respect to the Lebesgue measure on the real line.

Due to the lack of space this section is very brief indeed and many results which we use in the main text are omitted. For a more fuller discussion see Kingman and Taylor [K1], Royden [RO1], Rudin [RU1], Warga [W3].

A9. Hilbert Spaces

With the aim to define a Hilbert space we first give a definition of a pre-Hilbert space, which is a linear vector space X together with an operation called an inner product. The inner product is a function from XxX into the real line, whose value at (x_1, x_2) for $x_1, x_2 \in X$, written as $\langle x_1, x_2 \rangle$, satisfies the following axioms:

(i) $\langle x_1, x_2 \rangle = \langle x_2, x_1 \rangle$

(ii) $\langle x_1 + x_2, x_3 \rangle = \langle x_1, x_3 \rangle + \langle x_2, x_3 \rangle$ for all $x_3 \in X$

(iii) $\langle \alpha x_1, x_2 \rangle = \alpha \langle x_1, x_2 \rangle$ for all α real

(iv) $\langle x_1, x_1 \rangle \geq 0$ and $\langle x_1, x_1 \rangle = 0$ iff $x_1 = 0$.

[Note we only consider real linear vector spaces.]

Since the real-valued function $\langle x, x \rangle^{1/2}$ on X has the properties (i)-(iii) in section A5, it is called the induced norm on X and is written $\| x \|$. We can define open spheres, strong topologies, Cauchy sequences, etc. in terms of this norm.

A Hilbert space is a complete pre-Hilbert space. Hence a

Hilbert space is a Banach space equipped with an inner product which induces the norm. $\mathbb{R}^n$, l_2, $L_2[0,1]$ are all Hilbert spaces.

A10. Dual Spaces and Hyperplanes

Let X be a normed linear space. Then the space of all bounded linear functionals on X is called the normed dual of X and is denoted by X*. If $x^* \in X^*$ then we represent its value at $x \in X$ by $\langle x, x^* \rangle$ and the norm of $\| x^* \|$ is defined by

$$\| x^* \| = \sup_{\|x\| \le 1} \langle x, x^* \rangle$$

Thus X* is also a normed linear space.

Some common duals are

(a) $\mathbb{R}^n$ (the n-dimensional Euchidean space) is its own dual

(b) The dual of l_p, $1 \le p < \infty$ is l_q where $1/p + 1/q = 1$, if $p = 1$ take $q = \infty$

(c) The dual of L_p, $1 \le p < \infty$ is L_q where again $1/p + 1/q = 1$

(d) By the Riesz Representation Theorem it is possible to deduce that the dual of $C[T; \mathbb{R}]$ is the space of functions on $[0,1]$ of bounded variation. However, to get a unique representation of the dual we define the normalized space of functions of bounded variation, denoted NBV $[T; \mathbb{R}]$, which consists of all functions of bounded variation on $[0,1]$ which vanish at $t = 0$ and which are continuous from the right on (0,1). The norm of an element ν in this space is $\|\nu\|$ = Total Variation (ν(t))

$$= \text{T.V. } (\nu)$$

(e) The dual of any Hilbert space is itself.

Due to the lack of space we cannot go into more detail about dual spaces but the interested reader may consult the following references for further information, Warga [W3], Dunford and Schwartz [DU1], Luenberger [LU1]. We however state two definitions:

A vector $x^* \in X^*$ is said to be aligned with a vector $x \in X$ if $\langle x, x^* \rangle = \|x^*\| \; \|x\|$ and the vector $x \in X$ and $x^* \in X^*$ are orthogonal if $\langle x, x^* \rangle = 0$ and x^*, x are non zero.

We now turn to the notion of hyperplanes. A hyperplane H in a linear vector space X is a strict linear variety with the property that if V is any linear variety containing H, then either V = X or V = H. It can be shown (see Luenberger [LU1]) that the subset H of a topological vector space X is a closed hyperplane iff

$H = \{x: \langle x, x^* \rangle = c\}$ for some non-zero $x^* \in X^*$ and some real c. In this case x^* is called the normal of the hyperplane H. It is obvious from this that there is a unique correspondence between elements in X^* and closed hyperplanes in X.

The sets

(i) $\{x: \langle x, x^* \rangle < c\}$; (ii) $\{x: \langle x, x^* \rangle > c\}$

(iii) $\{x: \langle x, x^* \rangle \leq c\}$; (iv) $\{x: \langle x, x^* \rangle \geq c\}$

are the half spaces generated by the hyperplane H. The sets (i) and (ii) are open and (iii) and (iv) are closed. (iii) and (iv) are called complementary closed half spaces determined by H.

We now introduce a geometric concept of a hyperplane separating two disjoint nonempty convex sets. First, however, we give a definition:

A closed hyperplane H in a normed space X is said to be a support for a convex set K if K is contained in one of the closed half spaces determined by H, and H contains a point of $\overline{K}$.

Eidelheit Separation Theorem

Let K_1 and K_2 be two convex sets in X such that K_1 has interior points and K_2 contains no interior point of K_1. Then there exists a closed hyperplane H separating K_1 and K_2, i.e. there exists an $x^* \in X^*$ such that

$$\sup_{x \in K_1} \langle x, x^* \rangle \leq \inf_{x \in K_2} \langle x, x^* \rangle$$

This is an important result which is used throughout the fields of optimization and optimal control theory to obtain necessary conditions of optimality.

Section B

B1. Optimization

The concept of optimization is well established as a methodology in the study of many complex decision or allocation problems. For example when tackling a difficult decision problem, which involves selecting several interrelated variables, a sound mathematical principle for analysing and determining its solution is obtained by using optimization techniques. These involve formulating the problem under study into a mathematical setting. A single objective function is constructed to give a measure of the performance depending on a particular decision. To achieve optimality this objective functioin is maximised (or minimised depending on the formulation) subject to certain constraints which restrict the decision making process.

Throughout this thesis we will assume that the problems under study have been formulated into a mathematical model and it is these models with which we concern ourselves.

A vast amount of literature in the general field of optimization is available but the following are of particular significance, [BL1], [C1], [D11], [F1], [HE1], [HES1], [HES2], [HU1], [I1], [J1], [LE1], [L11], [LU1], [LU2], [M1], [N1], [N2], [N4], [P1], [PSH2], [W3], [Y1]. Due to the shortage of space we cannot go into the details of the many different methods of achieving optimality but the interested reader may consult the above references which cover the area quite comprehensively. We only state some of the results which are quite standard and which we shall be requiring in the remainder of the thesis.

First we consider finite dimensional problems (where the optimization is performed over a finite dimensional space) even though most of this thesis considers the infinite dimensional case (commonly referred to as the calculus of variations). These problems are not only of great interest in themselves, but they afford an excellent introduction to more complicated situations that arise when the dimension is infinite.

Consider the problem of minimising a function f defined on an open set $D \subset \mathbb{R}^n$ (this is known as an unconstrained optimization problem). A point $x^* \in D$ is a solution if

$$f(x^*) \leq f(x) \quad \text{for all} \quad x \in D.$$

Now consider a more general problem:

Problem 1: Min $f(x)$

$$\text{s.t} \quad g_i(x) \leq 0 \qquad i = 1,2,\ldots\ldots,m$$

$$h_j(x) = 0 \qquad j = 1,2,\ldots\ldots,r$$

for f, g_i's, h_j's all defined on $D \subset \mathbb{R}^n$ for all i, j. This is referred to as a constrained optimization problem with the g_i's being called the inequality constraints and h_j's the equality constraints. If all f, g_i's, h_j's are linear, Problem 1 is known as a linear programming problem and if any of the functions is nonlinear it is referred to as a nonlinear programming problem. Other terms such as convex, concave, quadratic, etc. also exist if the functions have these properties.

We define the Lagrangian $\mathcal{L}(x,\lambda,\alpha)$ associated with Problem 1 by

$$\mathcal{L}(x,\lambda,\alpha) \triangleq f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{r} \alpha_j h_j(x)$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$. This function plays a very important role in optimization theory as shown in most of the articles cited above. We state a few results from Fiacco and McCormick [F1] applicable to Problem 1:

B2. First Order Necessary Conditions

If the problem functions are once differentiable on D and $x^* \in D$, then the first order necessary conditions for x^* to be a local minimum of Problem 1 are that there exist numbers λ_i^*, α_j^* (called Lagrange multipliers, multipliers, Green functions, etc.) satisfying:

(i) $\lambda_i \geq 0$ for $i = 1,2,\ldots,m$ (1)

(ii) $\nabla \mathscr{L}(x^*, \lambda^*, \alpha^*) = 0$ (2)

(iii) $\lambda_i g_i(x^*) = 0$ for $i = 1,2,\ldots,m$ (3)

where $\nabla \mathscr{L}(.)$ denotes the derivative of the Lagrangian w.r.t. the x argument and $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$, $\alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_r^*)$.

B2 is a "first-order" characterization of local minima in that it only involves the first order partial derivatives of the problem functions. It does not take into account the curvature of the problem, this being measured by the second partial derivatives. Thus B2 just gives the conditions which have to be satisfied by a point for it to be a candidate for a local minimum, it does not say x^* is a local minimum. Conditions which state this result are of second order.

B3. Second Order Necessary and Sufficient Conditions

If the problem functions are twice differentiable on D and $x^* \in D$, then the second order necessary and sufficient conditions for x^* to be an isolated local minimum of Problem 1 is that there exist multipliers λ_i^*, $i=1,2,\ldots,m$, α_j^*, $j=1,2,\ldots,r$ such that (1)-(3) above hold and for every non-zero vector y satisfying

(a) $y^T \nabla g_i(x^*) = 0$ for all $i \in \tilde{I}$, where $\tilde{I}$ is defined as

$$\tilde{I} \triangleq \{i: \lambda_i^* > 0, \ i=1,2,\ldots,m\}$$

(b) $y^T \nabla g_i(x^*) \geq 0$ for all $i \in I-\tilde{I}$ where I is defined as

$$I \triangleq \{i: g_i(x^*) = 0, \ i=1,2,\ldots,m\}$$

(c) $y^T \nabla h_j(x^*) = 0$ for $j=1,2,\ldots,r$ we have that

$$y^T \nabla^2 \mathscr{L}(x^*, \lambda^*, \alpha^*) y > 0$$

where $\nabla^2 \mathscr{L}(.)$ is the second derivative of the Lagrangian w.r.t. x

For more information on finite dimensional optimization, see Fiacco and McCormick [F1], Hestenes [HES2], Mangasarian [M1].

B4. Optimal Control Theory

We now turn to infinite dimensional problems. These arise in situations where the dynamics of the system under consideration have to be added as a constraint in the form of a differential equation. This equation must be satisfied for all time (in some interval of interest), and hence arises the above mentioned infinite dimensional nature. The response of the system depends on the evolution of the state (which defines the internal behaviour of the system), and this in turn depends on the control variable (which is the input to the system). The objective function may be a function of the state or control or both and has to be minimised (or maximised) subject to the system dynamics plus any other constraints present in the problem. This will become clear in the remainder of the text. Throughout this thesis the control function will be denoted by u(.) and the state by x(.). The dimensions of u and x will be $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Also the time interval of interest T will be [0,1].

The dynamics of the system may involve a linear or nonlinear relationship between the state and control. The linear version is commonly written as

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ for a.a. } t \in T$$
$$x(o) = x_o$$

where a.a. denotes almost all, and $A \subset \mathbb{R}^n x \mathbb{R}^n$ and $B \subset \mathbb{R}^n x \mathbb{R}^m$ are matrices which may also be time varying. The nonlinear relationship is usually written as

$$\dot{x}(t) = f(x(t),u(t),t) \quad \text{for a.a.} t \in T$$
$$x(o) = x_o$$

where f: $\mathbb{R}^n x \mathbb{R}^m x T \longrightarrow \mathbb{R}^n$. We will concentrate on the nonlinear case.

Consider a simple optimal control problem of minimising $\int_0^1 l(x(t),u(t),t)dt$ subject to the above nonlinear dynamics. Problems of this sort have been studied extensively in the literature but effective sufficient conditions for optimality have as yet not been obtained except for a few special classes of control problems (quadratic cost and linear dynamics). Necessary conditions for optimality (known as the Maximum Principle or Minimum Principle according to formulation) were, however obtained by Pontryagin and his associates Boltanskii, Gamkrelidze and Mischenko (see Pontryagin et al [PON1]) over twenty years ago. Many other Maximum Principles have appeared over the years addressing various problems and making various assumptions. There are too many of these for us to state them all here but the bibliography at the end of the thesis contains many such results, and the interested reader is invited to consult them. We only present a few of these Maximum Principles which will be of direct use to us in the remainder of this thesis. All of these restrict attention to systems which are governed by difference-differential equations. We first consider the following problem:

B5. Problem P1

$$\text{Min } \int_0^1 l(x(t),x(t-\tau),u(t),t)dt$$

$$\text{s.t. } \dot{x}(t) = f(x(t),x(t-\tau),u(t),t) \text{ for a.a. } t \in T$$

$$x(t) = \phi(t) \quad \text{for all } t \in [-\tau, 0]$$

$$u \in G$$

where τ is a real number, strictly greater than zero and

$G \triangleq \{ u \in L_1^m[0,1] : u(t) \in \Omega$ for a.a. $t \in T \}$ for Ω compact and convex subset of $\mathbb{R}^m$.

Assume the control $u^* \in G$ is optimal for Problem P1 and $x^*(t)$ is the corresponding state trajectory (we refer to such a pair as "(x^*,u^*) is an optimal pair"). Then using results in Huang [HU1], Conner [CON1] or following the methodology in Appendix C, it is straightforward to deduce the following:

Maximum Principle 1

Assume that the following hypothesis hold

(i) The functions $f: \mathbb{R}^n \times \mathbb{R}^n \times \Omega \times T \longrightarrow \mathbb{R}^n$ and $l: \mathbb{R}^n \times \mathbb{R}^n \times \Omega \times T \longrightarrow \mathbb{R}^n$ and their partial derivatives f_x, f_y, f_u and l_x, l_y, l_u (i.e. derivatives w.r.t. $x(t)$, $x(t-\tau)$ and $u(t)$ respectively) exist and are continuous on $\mathbb{R}^n \times \mathbb{R}^n \times \Omega \times T$.

(ii) There exists an $M \in (0, \infty)$ such that $\| f(x,y,u,t) \| \leq M \{ \| x \| + \| y \| + 1\}$ for all $x, y \in \mathbb{R}^n$, all $u \in G$, all $t \in T$

and

$\| f(x^1, y^1, u, t) - f(x^2, y^2, u, t) \| \leq M \{ \| x^1 - x^2 \| + \| y^1 - y^2 \| \}$

for all $x^1, y^1, x^2, y^2 \in \mathbb{R}^n$, all $u \in G$ and all $t \in T$.

(iii) The initial function ϕ is absolutely continuous and bounded on $[-\tau, 0]$.

[Remark:

Throughout this thesis we will be using the argument $y(t)$ in place of $x(t-\tau)$ when it is convenient to do so. Also the dependence on t will not always be shown explicitly, i.e. we will denote $f(x(t), x(t-\tau), u(t), t) = f(x,y,u,t)$. This should cause no confusion.]

If (x^*, u^*) is an optimal pair for Problem P1, then there exists an absolutely continuous function $\lambda(t): T \longrightarrow \mathbb{R}^n$ which is the solution of

$$\begin{aligned}\dot{\lambda}^T(t) = & -\lambda^T(t) f_x(x^*(t), x^*(t-\tau), u^*(t), t) - l_x(x^*(t), x^*(t-\tau), u^*(t), t) \\ & -\lambda^T(t+\tau) f_y(x^*(t+\tau), x^*(t), u^*(t+\tau), t+\tau) - l_y(x^*(t+\tau), x^*(t), \\ & \qquad , u^*(t+\tau), t+\tau)\end{aligned}$$

for a.a. $t \in [0, 1-\tau]$

$$\dot{\lambda}^T(t) = -\lambda^T(t)f_x(x^*(t),x^*(t-\tau),u^*(t),t)-l_x(x^*(t),x^*(t-\tau),u^*(t),t)$$

$$\text{for a.a.} t \in [1-\tau,1]$$

$$\lambda^T(1) = 0$$

such that

$$\int_0^1[\lambda^T(t)f(x^*(t),x^*(t-\tau),u^*(t),t)+l(x^*(t),x^*(t-\tau),u^*(t),t)]\,dt$$

$$= \max_{v\in G} \int_0^1[\lambda^T(t)f(x^*(t),x^*(t-\tau),v(t),t)+l(x^*(t),x^*(t-\tau),v(t),t)]dt$$

where a^T denotes the transpose of a.

This is known as the Maximum Principle in integral form. The last relation given above may be represented more strongly by stating that the maximum is achieved pointwise, i.e. we have

$$\lambda^T(t)f(x^*(t),x^*(t-\tau),u^*(t),t)+l(x^*(t),x^*(t-\tau),u^*(t),t)$$

$$= \max_{v\in G} \{\lambda^T(t)f(x^*(t),x^*(t-\tau),v(t),t)+l(x^*(t),x^*(t-\tau),v(t),t)\}$$

$$\text{a.e. in } T$$

where a.e. denotes almost everywhere. This can be obtained as follows:

Suppose that u^* does not satisfy the Maximum Principle in pointwise form. Then there exists a control $\bar{u}\in G$ such that

$$\lambda^T f(x^*,y^*,u^*,t)+l(x^*,y^*,u^*,t) < \lambda^T f(x^*,y^*,\bar{u},t)+l(x^*,y^*,\bar{u},t)$$

on a measurable subset $E \subset T$ of measure strictly greater than zero. Now define a control $\hat{u}$ as follows

$$\hat{u}(t) = \bar{u}(t) \quad \text{for } t\in E$$

$$= u^*(t) \quad \text{for } t\in T \setminus E$$

This is obviously in G. Then by substituting this in the Maximum Principle in integral form we get

$$\int_0^1 [\lambda^T f(x^*,y^*,u^*,t)+l(x^*,y^*,u^*,t)\ dt$$

$$< \int_0^1 [\lambda^T f(x^*,y^*,\hat{u},t)+l(x^*,y^*,\hat{u},t)\ dt$$

This contradicts the above. Hence the Maximum Principle in pointwise form holds.

We now consider a more general problem in that terminal equality constraints are present.

B6. Problem P2

$$\text{Min} \quad h^o(x(1))$$

$$\text{s.t.} \quad \dot{x}(t) = f(x(t),x(t-\tau),u(t),t) \quad \text{for a.a.} t \in T$$

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau,0]$$

$$h^j(x(1)) = 0 \qquad j=1,2,\ldots,r$$

$$u \in G$$

where G is as for Problem P1. The functions f and ϕ are assumed to satisfy the same hypothesis as above and the functions $h^j: \mathbb{R}^n \to \mathbb{R}$, $j=0,1,\ldots,r$ and their partial derivatives h^j_x exist and are continuous on $\mathbb{R}^n$.

The following result can be deduced using the procedure in Appendix C:

Maximum Principle 2

If (x^*,u^*) is an optimal pair for Problem P2 then there exists an absolutely continuous function $\psi : T \longrightarrow \mathbb{R}^n$ and numbers $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_r$ with $\alpha_0 \leq 0$ such that

$$\dot{\psi}^T(t) = -\psi^T(t)f_x(x^*(t),x^*(t-\tau),u^*(t),t) - \psi^T(t+\tau)f_y(x^*(t+\tau),x^*(t),u^*(t+\tau),t+\tau) \quad \text{for a.a.} t \in [0,1-\tau]$$

$$\dot{\psi}^T(t) = -\psi^T(t)f_x(x^*(t),x^*(t-\tau),u^*(t),t) \quad \text{for a.a.} t \in [1-\tau,1]$$

$$\psi(1) = \sum_{j=0}^{r} \alpha_j h_x^j \, (x^*(1))^T$$

such that

$$\int_0^1 [\psi^T(t)f(x^*(t),x^*(t-\tau),u^*(t),t)]dt = \max_{v \in G} \int_0^1 [\psi^T(t)f(x^*(t),x^*(t-\tau),v(t),t)]dt$$

As in Maximum Principle 1 we can write the above in pointwise form:

$$\psi^T(t)f(x^*,y^*,u^*,t) = \max_{v \in G} \{\psi^T(t)f(x^*,y^*,v,t)\} \quad \text{for a.a.} t \in T$$

Problems P1 and P2 are considered in Chapters 2-4 although the control set is different in Chapters 3 and 4 (this is discussed below). In Chapter 5 we encounter the most difficult problem in this thesis, namely that in which state constraints are present. The Maximum Principle which applies in this situation is stated and derived in detail in Appendix C.

B7. Existence of Minimising Controls

Although the algorithm presented in Chapter 2 is designed to obtain controls satisfying necessary conditions of optimality, i.e. the Maximum Principle (in fact so are all the algorithms in this thesis), it is shown in Warga [W3] that, with the above choice for the control set G (termed ordinary measurable controls in a sense which will become clear later in this chapter), it is not always possible to assure that such optimising controls exist. For example consider the following problem:

Problem A

$$\min_u \; g(x,u) = \int_0^1 \{(x(t))^2 - (u(t))^2\} \, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t) \quad \text{for a.a. } t \in T$$

$$x(o) = o$$

$$\text{with } \Omega = [-1,1] \in \mathbb{R}$$

$$u \in G$$

We shall show that Problem A has no minimising solution. Since $u(t) \in [-1,1]$ for all $t \in T$ hence $(u(t))^2 \leq 1$ for all $u \in G$, $t \in T$ and $g(x,u) = \int_0^1 \{(x(t))^2-(u(t))^2\} \, dt \geq -1$ for all x,u.

If $j \in \mathbb{N}$ (the natural numbers), and we define $u_j(t)$ to be equal to +1 and -1 on alternate successive subintervals of length $1/2j$ of T. Then setting $x_j(t) = \int_0^t u_j(s)ds$ for all $t \in T$ we get $\int_0^t u_j(s)ds \leq 1/2j$ and the cost satisfies

$$(2j)^{-2} -1 \geq g(x_j,u_j) \geq -1$$

Thus $\lim_{j \to \infty} g(x_j,u_j) = -1$ is the minimum cost for Problem A. However, if there exists a $u^* \in G$ and $x^*(t) = \int_0^t u^*(s)ds$ all $t \in T$ such that $g(x^*,u^*) = -1$ then $|u^*(t)| = 1$ a.e. in $[0,1]$, and $x^*(t) = \int_0^t u^*(s)ds = 0$ for all $t \in [0,1]$.

The second relationship yields that $u^*(t) = 0$ a.e. in

[0,1] contradicting the first relation. Thus Problem A does not have its minimising solution.

However, by embedding G in a larger topological space $\underset{\sim}{G}$ of which G is a dense subset and then extending all the functions defining the problem to $\underset{\sim}{G}$, it is possible to guarantee the existence of a minimising relaxed solution $(x^{\underset{\sim}{u}^*}, \underset{\sim}{u}^*)$. The procedure as to how this is done is presented in Warga [W3] and we refer the reader to Warga's book for full details. We will, however, present a brief overview of the essential ideas and the importance of relaxed control problems.

B8. Relaxed Control Problems

We have seen above in Problem A that the minimising control sequence $\{u_j\}$ contains controls of a highly oscillatory nature and in the limit as $j \to \infty$, u_∞ may be thought of as spending equal amounts of time at +1 and -1, i.e. it is "half at +1 and half at -1". In more complicated situations there is no reason why u_j cannot take on several, or infinitely many values in this fashion on any interval of T. Then it can easily be shown that by defining a space of all probability measures on Ω any relaxed control can be represented. For example, in Problem A the limiting control can be represented as

$$\lim_j u_j = \tfrac{1}{2}\delta_1 \oplus \tfrac{1}{2}\delta_{-1}$$

where δ_r is the Dirac measure concentrated at r. This, infact, is the solution to Problem A.

[Remark: We will use the notation $\oplus(\ominus)$ when we mean that two controls (relaxed or ordinary) are added (subtracted) in a relaxed fashion. Also for any control with the symbol "~" we mean it is a relaxed control.]

For more complicated cases it is necessary to integrate w.r.t. the probability measure. Letting $\underset{\sim}{V}$ be the space of all probability measures on Ω, then any relaxed control $\underset{\sim}{u}(t)$ has the representation

$$\underset{\sim}{u}(t) = \int_{\Omega} u(t)\ d(\underset{\sim}{v}(t))(u) \qquad \text{for some probability measures}$$
$$\underset{\sim}{v}(t) \in \underset{\sim}{V} \quad \text{at each } t \in T$$

i.e. a relaxed control is any function $\underset{\sim}{u} : [0,1] \longrightarrow \underset{\sim}{V}$.

Concerning relaxed controls we observe the following (see for example Warga [W3], Young [Y1]):

1. For any continuous function $\phi : \mathbb{R}^n \times \Omega \times T \longrightarrow \mathbb{R}^p$ the corresponding relaxed function $\phi_r : \mathbb{R}^n \times \underset{\sim}{V} \times T \longrightarrow \mathbb{R}^p$ (called the extension of ϕ to the relaxed controls) is defined by

$$\phi_r(x,\underset{\sim}{u},t) = \int_{\Omega} \phi(x,u,t)d\underset{\sim}{v}(u)$$

2. For any function $\chi : \mathbb{R}^n \times \underset{\sim}{V} \longrightarrow \mathbb{R}^s$, relaxed controls act in a "linear" fashion in that if $\underset{\sim}{u}(t) = \alpha \underset{\sim}{u}_1(t) \oplus (1-\alpha)\underset{\sim}{u}_2(t)$ for all $\alpha \in [0,1]$, then

$$\chi(x,\underset{\sim}{u}) = \alpha\, \chi(x,\underset{\sim}{u}_1) + (1-\alpha)\, \chi(x,\underset{\sim}{u}_2) \quad \text{for all } x \in \mathbb{R}^n$$

3. A relaxed control is said to be measurable if for any polynomial $p(u)$ in (the components of) u, the function $\xi : T \longrightarrow \mathbb{R}$ defined by

$$\xi(t) = p(\underset{\sim}{v}(t)) = \int_{\Omega} p(u)d\underset{\sim}{v}(t)(u)$$

is measurable.

We define $\underset{\sim}{G}$ to be the space of all measurable relaxed control functions.

Definition B8.1

We shall say that an infinite sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ of relaxed controls in $\underset{\sim}{G}$ converges to $\underset{\sim}{u}^* \in \underset{\sim}{G}$ in the sense of control measures (i.s.c.m.) if, for every continuous function $\phi: \Omega \times T \to \mathbb{R}$ and every subinterval Δ of T

$$\int_{\Delta} \phi_r(\underset{\sim}{u}_i(t),t)dt \longrightarrow \int_{\Delta}\phi_r(\underset{\sim}{u}^*(t),t)dt$$

as $i \longrightarrow \infty$.

Now it is shown in Warga [W3] that $\underset{\sim}{G}$ is a subset of the space $L^1[T,C(\Omega)]^*$ where $L^1[T,C(\Omega)]$ is the set of all (equivalence classes of) measurable functions $\nu: T \longrightarrow C(\Omega)$ such that $\int_0^1 |\nu(t)|dt < \infty$ and $C(\Omega)$ is the normed vector space of all bounded continuous functions on Ω into the reals with sup norm.

It turns out that the definition of convergence i.s.c.m. is precisely equivalent to the definition of convergence w.r.t. the weak star topology of $L^1[T,C(\Omega)]^*$ (see Warga [W3]).

It is worth noting at this stage that G can be embedded into $\underset{\sim}{G}$ by identifying with each $u \in G$ the Dirac measures $\delta_{u(t)}(.)$, $0 \le t \le 1$, and that G is dense in $\underset{\sim}{G}$. Infact $\underset{\sim}{G}$ is the closure of G. With this property and the fact that $\underset{\sim}{G}$ is compact it is possible to obtain the existence of a relaxed minimising solution (providing of course if "feasible" controls exist). Then, using the convexity of $\underset{\sim}{G}$ it is also possible to derive necessary conditions of optimality for a relaxed solution. These are similar to the ones given above for the case when only ordinary controls are considered.

We now state the Maximum Principle for Problem P1 when G is extended to $\underset{\sim}{G}$ – P1 is then referred to as a relaxed control problem and the Maximum Principle 1 in section B5 becomes the Relaxed Maximum Principle 1. The details are omitted since these are similar to the ones used in Appendix C where the necessary conditions for optimality are obtained for the state constrained optimal control problem.

<u>Relaxed Maximum Principle 1</u>

Assume the same hypothesis as in the Maximum Principle 1 hold and that $(x^{\underset{\sim}{u}^*}, \underset{\sim}{u}^*)$ is an optimal pair for the relaxed Problem P1. Then there exists an absolutely continuous function $\lambda: T \longrightarrow \mathbb{R}^n$ which is the solution of

$$\dot{\lambda}^T(t) = -\lambda^T(t) f_x(x^{\underset{\sim}{u}^*}(t), x^{\underset{\sim}{u}^*}(t-\tau), \underset{\sim}{u}^*(t), t) - l_x(x^{\underset{\sim}{u}^*}(t), x^{\underset{\sim}{u}^*}(t-\tau), \underset{\sim}{u}^*(t), t)$$
$$-\lambda^T(t+\tau) f_y(x^{\underset{\sim}{u}^*}(t+\tau), x^{\underset{\sim}{u}^*}(t), \underset{\sim}{u}^*(t+\tau), t+\tau)$$
$$-l_y(x^{\underset{\sim}{u}^*}(t+\tau), x^{\underset{\sim}{u}^*}(t), \underset{\sim}{u}^*(t+\tau), t+\tau)$$
$$\text{for a.a.} t \in [0, 1-\tau]$$

$$\dot{\lambda}^T(t) = -\lambda^T(t) f_x(x^{\underset{\sim}{u}^*}(t), x^{\underset{\sim}{u}^*}(t-\tau), \underset{\sim}{u}^*(t), t) - l_x(x^{\underset{\sim}{u}^*}(t), x^{\underset{\sim}{u}^*}(t-\tau), \underset{\sim}{u}^*(t), t)$$
$$\text{for a.a.} t \in [1-\tau, 1]$$

$\lambda(1) = 0$

such that

$$\int_0^1 [\lambda^T(t) f(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \underset{\sim}{u}^*, t) + l(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \underset{\sim}{u}^*, t)]dt$$
$$= \max_{\underset{\sim}{v} \in \underset{\sim}{G}} \int_0^1 [\lambda^T(t) f(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \underset{\sim}{v}, t) + l(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \underset{\sim}{v}, t)]dt$$

Note: the subscript "r" has been omitted.

A similar extension to the results presented above also holds for Problem P2.

Pointwise Relaxed Maximum Principles may also be obtained using the fact that G is dense in $\underset{\sim}{G}$ and $\underset{\sim}{u}^*(t)$ is supported on Ω almost everywhere in T. Then the above Relaxed Maximum Principle in integral form may be written as:

$$\lambda^T f(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \underset{\sim}{u}^*, t) + l(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \underset{\sim}{u}^*, t)$$

$$= \max_{\omega \in \Omega} \{\lambda^T f(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \omega, t) + l(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \omega, t)\} \quad \text{a.e. in } T$$

We again stress the point that the concept of relaxed controls has only been described very briefly and for full details and discussions see Warga [W3], Young [Y1], McShane [M^{c}1], [M^{c}2], Lee and Markus [LE1].

CHAPTER 2

A STRONG VARIATIONAL ALGORITHM

2.1 Introduction

When solving optimal control problems using classical methods such as steepest descent, the new control u_α is constructed from the old control u in the following fashion:

$$u_\alpha = u + \alpha s \tag{1.1}$$

where α is the step length and s is the search direction. For small values of step lengths u_α is "close" to u for all time, i.e. as $\alpha \longrightarrow 0$, we have $u_\alpha \longrightarrow u$ uniformly in t. However, in Jacobson and Mayne [J1] a new class of algorithm was presented. This class contained algorithms, now referred to as strong variational algorithms, which generate the new control u_α from the control u using the following formula:

$$\begin{aligned} u_\alpha(t) &= \check{u}(t) \quad \text{for all } t \in I_{\alpha u} \\ &= u(t) \quad \text{otherwise} \end{aligned} \tag{1.2}$$

where $\check{u}$ minimises a Hamiltonian function defined by (2.6), and $I_{\alpha u}$ is a subset of the time interval [0,1] having total length α. It is easy to see that u_α can differ appreciably from u for some t even when α is small, hence the term, "strong variations in control".

These new strong variational algorithms were found to be very effective computationally, hence it was necessary that they be studied theoretically and proofs of convergence obtained. The only procedures available at the time were the ones used in proving convergence of classical algorithms (e.g. steepest descent), where, when the new control u_α defined by (1.1) is used in place of u in the cost function, the estimated change in cost has a linear relationship with α. This makes convergence proofs for these algorithms relatively straightforward. For strong variational algorithms however, this is not the case. When an estimate of the change in cost when the new control u_α defined by (1.2) is used in place of u in the cost function [see (2.11) below] is obtained (using the method described in Jacobson and Mayne [J1]), it is found to be a nonlinear function of α. Hence standard procedures for proving convergence for algorithms based on classical methods cannot be used and a new approach is required.

In [J1], the interval $I_{\alpha u}$ was set to be $[1-\alpha, 1]$, but attempts to prove convergence for this choice failed because a reduction in cost could not be guaranteed for small values of α. Mayne and Polak [MAP1] modified their choice of $I_{\alpha u}$ so that the general convergence theorems of Polak [P1] could be used to ensure convergence. Essentially Mayne and Polak's method was that $I_{\alpha u}$ was chosen so that the estimate in the change of cost $\Delta\hat{V}$, of using u_α in place of u, is bounded above by $\alpha\theta(u)$ where $\theta(u) < 0$ is defined by (2.13). Thus although $\Delta\hat{V}$ is not a linear function of α, it is bounded above by a linear function and this makes a proof of convergence possible.

Mayne and Polak [MAP1] present two algorithms (one conceptual and the other implementable) with convergence proofs for optimal control problems with the control constrained, and in this chapter we extend these results to cover delay systems.

Only the conceptual version of the strong variational algorithm will be presented since it will be obvious from this and Mayne and Polak's implementable algorithm how the implementable version for time delay systems can be obtained. An integral cost as in [MAP1] is considered, but we also assume it to be dependent on $x(t-\tau)$ as well as $x(t)$, $u(t)$ and t, so that it can be shown how a delayed argument in the cost is handled. Mayne and Polak also consider a terminal payoff, but we assume, without loss of generality, that this is absent in our problem.

2.2 Problem Formulation

Unless otherwise stated, all the control problems considered in this thesis will be assumed to be governed by the following nonlinear delay-differential equation:

$$\dot{x}(t) = f(x(t), x(t-\tau), u(t), t) \quad \text{for a.a. } t \in T \tag{2.1}$$

$$x(t) = \phi(t) \quad \text{for all } t \in [-\tau, 0] \tag{2.2}$$

where τ is a positive real number and T is the compact interval $[0,1]$. The function $\phi: [-\tau, 0] \to \mathbb{R}^n$ is assumed to be bounded, continuous and to possess a continuous derivative for all $t \in [-\tau, 0]$. $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control.

We let x^u denote the solution of the delay differential equation (2.1) due to control u and initial condition (2.2), and denote by G the space of admissible controls defined by:

$$G \triangleq \{u \in L_1^m[0,1] \ : \ u(t) \in \Omega \quad \text{for all } t \in T\}$$

where Ω is a compact and convex subset of $\mathbb{R}^m$ with

$$\max \{\|v\| : v \in \Omega\} \leq r$$

The cost functional $V: G \rightarrow \mathbb{R}$ which we want to minimise is defined by

$$V(u) \triangleq \int_0^1 l(x^u(t), x^u(t-\tau), u(t), t)dt \qquad (2.3)$$

We restate the above as our first problem:

Problem P1

$$\min_u \int_0^1 l(x(t), x(t-\tau), u(t), t)dt$$

$$\text{s.t. } \dot{x}(t) = f(x(t), x(t-\tau), u(t), t) \quad \text{for a.a. } t \in T$$

$$x(t) = \phi(t) \quad \text{for all } t \in [-\tau, 0]$$

$$u \in G$$

We make the following assumptions:

Assumption 1

The functions

$$f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times T \longrightarrow \mathbb{R}^n \quad \text{and}$$

$$l : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times T \longrightarrow \mathbb{R}$$

and their partial derivatives f_x, f_{xx}, f_y, f_{yy}, f_u, f_{uu}, f_{xu}, f_{yu}, f_{xy}, l_x, l_{xx}, l_y, l_{yy}, l_u, l_{uu}, l_{xu}, l_{yu} and l_{xy} (i.e. all partial derivatives with respect to x,y,u of all orders upto and including 2) exist and are continuous for all $x,y \in \mathbb{R}^n$, all $u \in G$, all $t \in T$, where y denotes the delayed argument $x(t-\tau)$.

Assumption 2

There exists an $M \in (0, \infty)$ such that

$$\| f(x,y,u,t) \| \leq M \{ \| x \|_1 + \| y \|_1 + 1 \}$$

for all $x,y \in \mathbb{R}^n$, all $u \in G$, all $t \in T$

and

$$\| f(x^1,y^1,u,t) - f(x^2,y^2,u,t) \| \leq M \{ \| x^1 - x^2 \|_1 + \| y^1 - y^2 \|_1 \}$$

for all $x^1, y^1, x^2, y^2 \in \mathbb{R}^n$, all $u \in G$

and for all $t \in T$

Where $\| . \|_1$ is used to denote the L_1 norm defined by

$$\| u \|_1 \triangleq \int_0^1 \| u(t) \| dt$$

[Note: In the sequal L_1 will denote $L_1^m[0,1]$ or $L_1^n[0,1]$ according to context.]

We define the metric $d: G \times G \to \mathbb{R}$ by

$$d(u_1,u_2) \triangleq \int_0^1 \| u_1(t) - u_2(t) \| dt \qquad (2.4)$$

i.e. $d(u_1,u_2)$ is the distance between any two controls $u_1, u_2 \in G$.

Let $I_{u_1,u_2} \subset T$ be defined by

$$I_{u_1,u_2} \triangleq \{ t \in T : u_1(t) \neq u_2(t) \} \qquad (2.5)$$

i.e. I_{u_1,u_2} is the subset of T on which u_1 does not equal to u_2.

Letting $\mu(I)$ denote the Lebesgue measure of interval $I \subset T$ we have, for all $u_1, u_2 \in G$ that

$$d(u_1,u_2) \leq 2 \mu(I_{u_1,u_2}) r$$

Let us define the Hamiltonian function $H: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times T \to \mathbb{R}$ by

$$H(x,y,u,\lambda,t) \triangleq \lambda^T f(x,y,u,t) + l(x,y,u,t) \qquad (2.6)$$

where $\lambda = \lambda^{u}: T \to \mathbb{R}^n$, the costate function, is the solution of

$$-\dot{\lambda}(t) = H_x^T(x^u(t), x^u(t-\tau), u(t), \lambda(t), t) \qquad (2.7)$$

$$+H_y^T(x^u(t+\tau), x^u(t), u(t+\tau), \lambda(t+\tau), t+\tau)$$

$$\text{for a.a. } t \in [0, 1-\tau]$$

$$-\dot{\lambda}(t) = H_x^T(x^u(t), x^u(t-\tau), u(t), \lambda(t), t) \text{ for a.a. } t \in [1-\tau, 1] \qquad (2.8)$$

$$\lambda(1) = 0 \qquad (2.9)$$

Throughout this thesis, when no confusion can arise we will let $x^i = x^{u_i}$, $\lambda^i = \lambda^{u_i}$, $x^* = x^{u^*}$, etc.

It has been shown in Mayne and Polak [MAP1], in the case when delays are absent, that for any u_1, $u_2 \in G$, a first order estimate of

$$\Delta V(u_2, u_1) \triangleq V(u_2) - V(u_1) \qquad (2.10)$$

is obtained by calculating

$$\Delta \hat{V}(u_2, u_1) \triangleq \int_0^1 [H(x^1(t), x^1(t-\tau), u_2(t), \lambda^1(t), t) \qquad (2.11)$$

$$-H(x^1(t), x^1(t-\tau), u_1(t), \lambda^1(t), t)]\, dt$$

The estimate of $\Delta V(u_2, u_1)$ given by (2.11) is valid in the sense that there exists a finite constant C such that

$$\| \Delta V(u_2, u_1) - \Delta \hat{V}(u_2, u_1) \| \leq C[d(u_2, u_1)]^2$$

We prove in Proposition 5.4 that this approximation holds for our problem as well, hence the estimate will be good if u_1 and u_2 are "close".

It will become apparent in the exposition that the following proposition is required to ensure the algorithm is well defined:

Proposition 3

For all $u \in G$ there exists a $\check{u} \in G$ which satisfies

$$\check{u}(t) = \arg\min_{\omega \in \Omega} H(x^u(t), x^u(t-\tau), \omega, \lambda^u(t), t) \quad \text{for a.a.} t \in T$$

In Mayne and Polak [MAP1] the above result was introduced as an assumption, but this was unnecessary since, due to the Weierstrass Theorem (see Luenberger [LU1]), a minimising $\bar{\omega}$ (not necessarily unique) exists for each $t \in T$ because of the compact nature of Ω. Now as a consequence of the McShane-Warfield Halfway Principle [Y1] there exists a measurable control $\check{u} \in G$ which has the desired properties. Since more than one such $\check{u}$ may exist, we denote by $\check{U}(u)$ the set of all controls $v \in G$ which satisfy

$$v(t) = \arg\min_{\omega \in \Omega} H(x^u(t), x^u(t-\tau), \omega, \lambda^u(t), t) \quad \text{for a.a.} t \in T$$

We let $\bar{H}(u,t): G \times T \to \mathbb{R}$ be defined by

$$\bar{H}(u,t) \triangleq \min_{\omega \in \Omega} H(x^u(t), x^u(t-\tau), \omega, \lambda^u(t), t) \quad \text{for a.a.} t \in T \tag{2.12}$$

and we define $\theta: G \to \mathbb{R}$ by

$$\begin{aligned} \theta(u) &\triangleq \Delta\hat{V}(\check{u}, u) \\ &\triangleq \int_0^1 [\bar{H}(u,t) - H(x^u(t), x^u(t-\tau), u(t), \lambda^u(t), t)]\, dt \end{aligned} \tag{2.13}$$

where $\check{u} \in \check{U}(u)$. Since $\check{u}$ is the minimising control, the integrand in (2.13) is nonpositive for all $t \in T$, hence $\theta(u) \leq 0$. $\theta(u)$ is in fact the estimated change of cost if the minimising control $\check{u}$ is used in place of the old control u.

We show that in the event that the algorithm for solving Problem P1 generates an accumulation point $u^* \in G$, u^* satisfies

$$\theta(u^*) = 0 \tag{2.14}$$

Hence we show that u*, if not an optimal control is at least a likely candidate since it satisfies the necessary conditions of optimality in the form of the Maximum Principle (see section B5 in Chapter 1).

2.3 Discussion of Algorithm

We will now briefly discuss Jacobson and Mayne's [J1] method for defining $I_{\alpha u}$ and show why they could not prove convergence. Their new control u_α is defined as in (1.2) and the interval $I_{\alpha u}$ is set to be $[1-\alpha, 1]$. This is shown digramatically in Fig. 3.1.

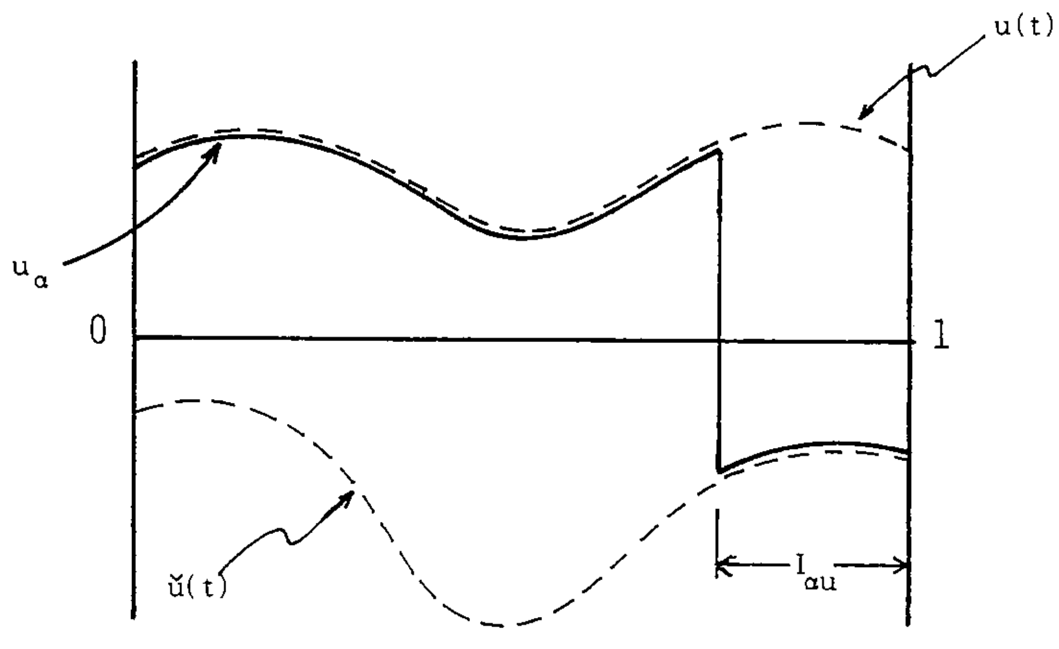


Fig. 3.1

As shown in [J1] the estimate in the change in cost when u_α is used instead of u is given by

$$\hat{\Delta V}(u_\alpha, u) = \int_0^1 [H(x^u(t), x^u(t-\tau), u_\alpha(t), \lambda^u(t), t) - H(x^u(t), x^u(t-\tau), u(t), \lambda^u(t); t)]dt$$

$$= \int_{I_{\alpha u}} [H(x^u(t), x^u(t-\tau), \check{u}(t), \lambda^u(t), t) - H(x^u(t), x^u(t-\tau), u(t), \lambda^u(t), t)]dt$$

To guarantee convergence it must be shown that if the algorithm is at an undesirable control then a reduction in the cost can be obtained for all step lengths $\alpha \in [0,1]$. This is however, not the case for Jacobson and Mayne's method since the following may occur:

Supposing that the algorithm is at a nonoptimal control u, then we must have $\theta(u) < 0$. If $\hat{\Delta V}(u_\alpha, u)$ is plotted against the step length α the situation shown in Fig. 3.2 may occur:

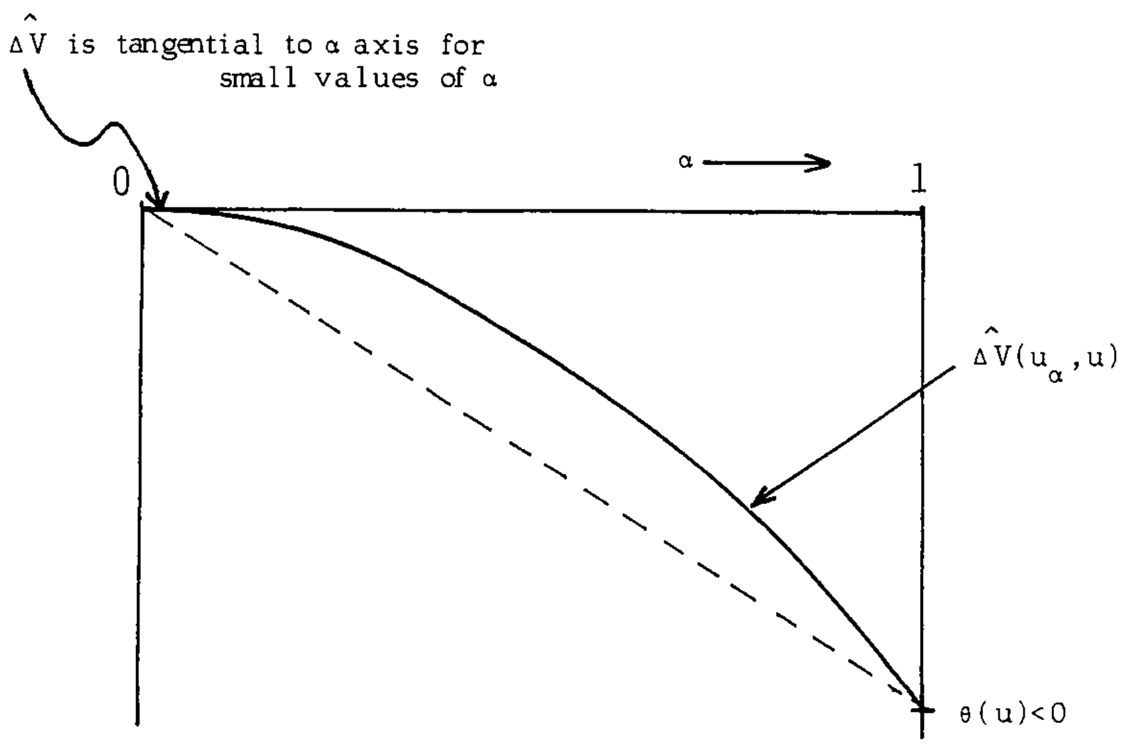


Fig. 3.2

where $\hat{\Delta V}(u_\alpha, u)$ is tangential to the α axis for small step lengths. This means that a reduction in cost cannot be guaranteed for small α, even though the present control is known to be non-optimal. Hence convergence proofs cannot be given for this particular choice of $I_{\alpha u}$. This is because if the above situation arose in an algorithm based on choosing $I_{\alpha u} = [1-\alpha, 1]$, and the algorithm generated a small enough step length α, there would be no reduction in the cost and the algorithm would jam up at an undesirable control.

Mayne and Polak [MAP1] present an alternative method for determining $I_{\alpha u}$ which allows the general convergence theorems of Polak [P1] to be used for proving convergence for their own algorithm. This method will be described in detail here since we will be using it when stating the strong variational algorithm for delay systems.

First, however, we describe an algorithm model proposed by Polak [P1] suitable for solving problem P1. This model makes use of a set valued search function A which maps G into the set of all nonempty subsets of G (we write this as $A:G \to 2^G$) and a stopping rule $V:G \longrightarrow \mathbb{R}$, and is of the following form:

3.0 <u>Algorithm Model 1</u>

Step 0 : $u_0 \in G$

Step 1 : Set $i = 0$

Step 2 : Compute a $v \in A(u_i)$

Step 3 : If $V(v) \geq V(u_i)$ stop

Else continue

Step 4 : Set $u_{i+1} = v$

Set $i = i+1$ and go to Step 2

The following result is established in [P1] for the algorithm model:

Theorem 3.1

Suppose that

(i) $V(.)$ is either continuous at all undesirable controls $u \in G$, or else $V(u)$ is bounded from below for all $u \in G$

(ii) For every $u \in G$ which is not desirable, there exists an $\varepsilon(u) \geq 0$ and a $\delta(u) < 0$ such that

$$V(u_2) - V(u_1) \leq \delta(u) < 0 \ ,$$

for all $u_1 \in G$ such that $||u_1 - u||_G \leq \varepsilon(u)$ and for all $u_2 \in A(u_1)$

Then, either the sequence $\{u_i\}$ constructed by the algorithm model 1 is finite, in which case its last element is desirable, or else it is infinite and every accumulation point of $\{u_i\}$ is desirable.

We provide a proof of Theorem 3.1 for completeness:

If $u^* \in G$ is desirable it follows from above that condition (ii) in Theorem 3.1 does not hold for any $\varepsilon(u)$, $\delta(u)$, u_1, u_2. In particular taking $\varepsilon(u) = 0$, we get $u_1 = u^*$. Hence we have if u^* is desirable that

$$V(u_2) - V(u^*) < 0 \qquad \text{for all } u_2 \in A(u^*)$$

is NOT true. This means we have

$$V(u_2) - V(u^*) \geq 0 \qquad \text{for some } u_2 \in A(u^*)$$

Hence (ii) implies that if $V(u) \geq V(u^*)$ for at least one $u \in A(u^*)$ then u^* is desirable.

We will now prove the theorem.

Suppose firstly that the sequence $\{u_i\}$ is finite, i.e. $\{u_i\} = \{u_0, u_1, \ldots, u_k\}$. Then by Step 3 we must have had

$$V(v) \geq V(u_k) \qquad \text{for } v \in A(u_k)$$

for the algorithm to have stopped. Hence from above we deduce that u_k is desirable.

Now suppose that the sequence $\{u_i\}$ is infinite and that it has a subsequence indexed by $K \subset \{0,1,2,\ldots\}$ which converges to u^*. We assume u^* is not desirable and attempt to obtain a contradiction.

Since u^* is not desirable, there exists an $\varepsilon > 0$ and a $\delta < 0$ and a $k \in K$ such that for all $i \geq k$, $i \in K$

$$\| u_i - u^* \|_G \leq \varepsilon$$

and $V(u) - V(u_i) \leq \delta$ for all $u \in A(u_i)$.

Hence, for any two consecutive elements u_i, u_{i+j} of the subsequence with $i \geq k$, we must have

$$\begin{aligned} V(u_{i+j}) - V(u_i) &= [V(u_{i+j}) - V(u_{i+j-1})] + [V(u_{i+j-1}) - V(u_{i+j-2})] \\ &\quad + \ldots + [V(u_{i+1}) - V(u_i)] \\ &< V(u_{i+1}) - V(u_i) \\ &\leq \delta \quad \text{since } u_{i+1} \in A(u_i) \end{aligned}$$

Now, for $i \in K$, the monotonically decreasing sequence $V(u_i)$ must converge, either because $V(.)$ is continuous at u^*, or else because $V(u)$ is bounded from below on G. But this contradicts the above deduction that $V(u_i)$ is not a Cauchy sequence for $i \in K$, and hence the theorem must be true, i.e. u^* is desirable.

The strong variational algorithm presented in section 2.4 is based on this model and Theorem 3.1 is used to prove its convergence.

We now discuss Mayne and Polak's [MAP1] method for choosing $I_{\alpha u}$ and it will become apparent in the exposition why this choice enables Polak's convergence results to be used.

Suppose the algorithm is at an undesirable control $\ddot{u}$. The control $\check{u}$ which minimises the Hamiltonian is found and $\theta(u)$, the average value of $[\bar{H}(u(t),t)-H(x^u(t),x^u(t-\tau),u(t),\lambda^u(t),t)]$ for $t \in [0,1]$, as defined by (2.13), is calculated. Then the set $I_u \subset T$ defined by

$$I_u \triangleq \{t \in T : \bar{H}(u(t),t)-H(x^u(t),x^u(t-\tau),u(t),\lambda^u(t),t) \leq \theta(u)\} \quad (3.1)$$

is found. I_u is the subset of T for which the integrand in (2.13) has a value which is more negative than the average value $\theta(u)$ [see Fig. 3.3]. Since any closed interval is a countable union of disjoint closed intervals, the set I_u consists of the union of at most a countable number of disjoint intervals.

We now define the subset $I_{\alpha u}$ of T which defines the new control u_α. The total length of $I_{\alpha u}$ is α. If α is less than $\mu(I_u)$ then it is required that $I_{\alpha u}$ be chosen to be a subset of I_u so that the estimate $\Delta\hat{V}(u_\alpha,u)$ of the change of cost due to using u_α in place of u is bounded above by $\alpha\theta(u)$. This can be done by defining $I_{\alpha u}$, as α increases, to be the subset of I_u which covers I_u from the left. Then to define $I_{\alpha u}$ completely, when I_u is

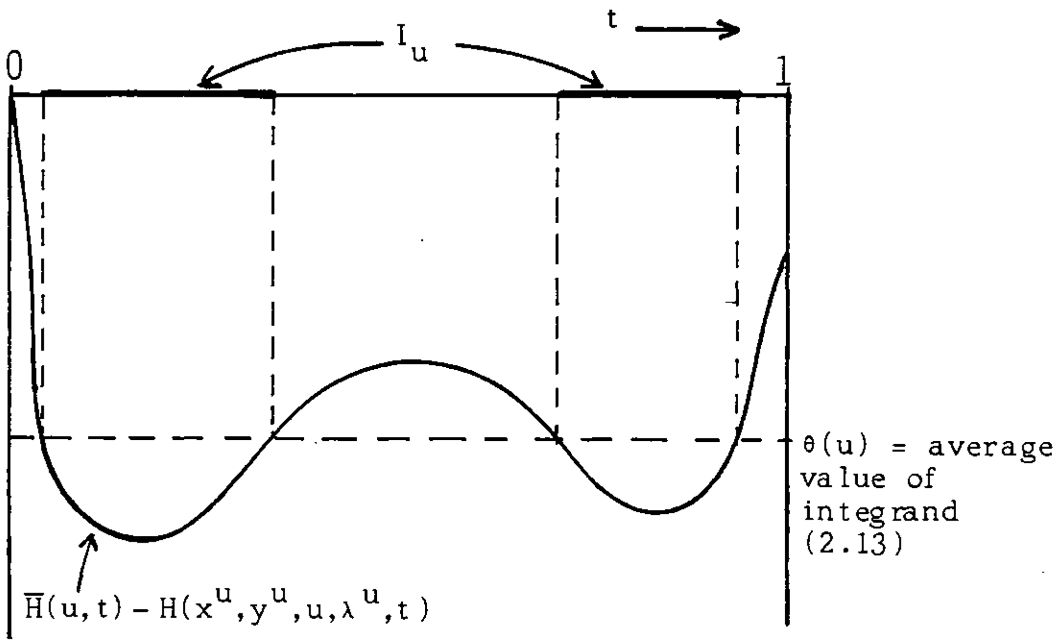


Fig. 3.3

completely covered, $T \setminus I_u$ is covered from the left as well as α is further increased. Thus for $\alpha \in [0,1]$, Mayne and Polak [MAP1] define $I_{\alpha u}$ to be the subset of T having the following properties:

(i) $\mu(I_{\alpha u}) = \alpha$ (3.2)

(ii) if $\alpha \in [0, \mu(I_u)]$, then $I_{\alpha u} \subset I_u$ (3.3)

(iii) if $\alpha \in (\mu(I_u), 1]$, then $I_u \subset I_{\alpha u}$ (3.4)

(iv) for all $\alpha \in [0, \mu(I_u)]$, the following holds

$$\{t \in I_u,\ s \in I_{\alpha u},\ t < s\} \Rightarrow \{t \in I_{\alpha u}\} \quad (3.5)$$

(v) for all $\alpha \in (\mu(I_u), 1]$,

$$\{t \in T,\ s \in I_{\alpha u} \setminus I_u,\ t < s\} \Rightarrow \{t \in I_{\alpha u}\} \quad (3.6)$$

These properties can be seen more clearly in Fig. 3.4.
Hence for any nondesirable control $u \in G$, we have defined

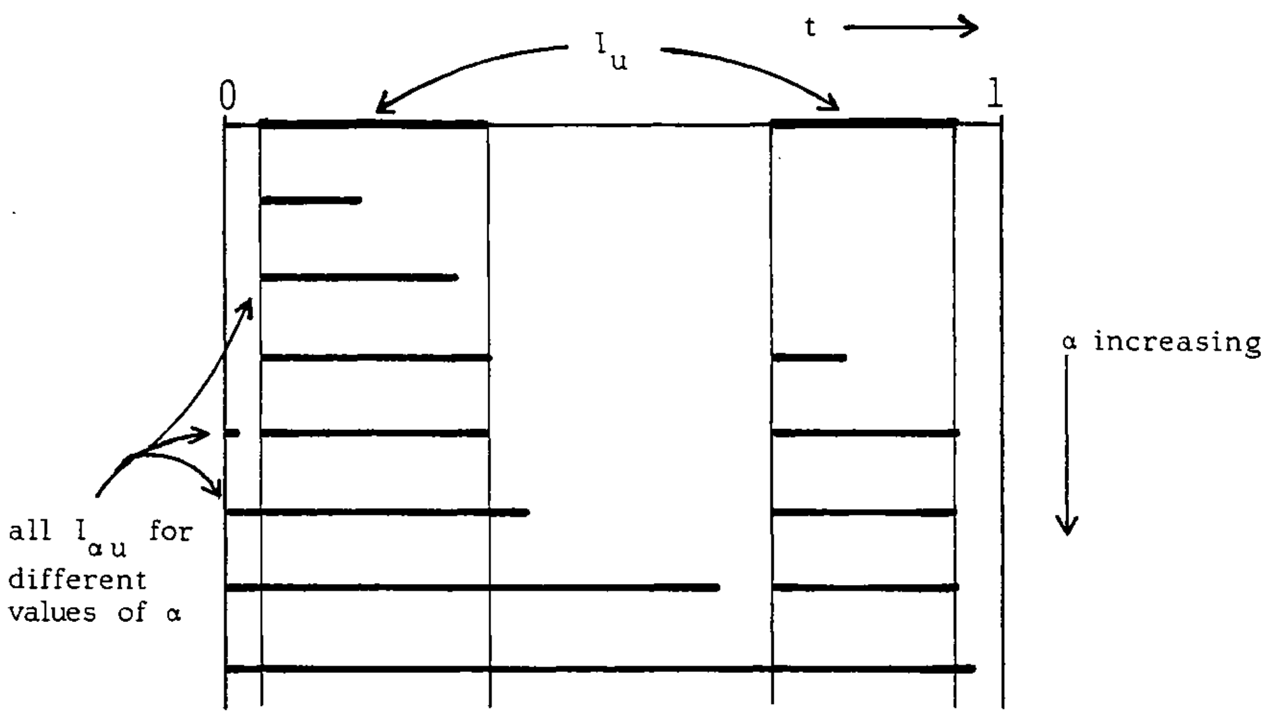


Fig. 3.4

a method for constructing the new control u_α [as defined by (1.2)] for $\alpha \in [0,1]$. To complete the description for determining the new control a rule for choosing the step length α is needed. For this purpose we will make use of a function $\phi : [0,1] \times G \to \mathbb{R}$ defined by

$$\phi(\alpha,u) \triangleq \int_{I_{\alpha u}} [\overline{H}(u(t),t) - H(x^u(t), x^u(t-\tau), u(t), \lambda^u(t), t)]\, dt \quad (3.7)$$

Note that for any u_α defined by (1.2) we have

$$\Delta\hat{V}(u_\alpha, u) = \phi(\alpha, u)$$

Thus $\phi(\alpha,u)$ is the estimated change of cost due to using the new control u_α in place of the old control u.

To find the step length we propose (as in Mayne and Polak [MAP1]) to set it to be the largest $\alpha \in [0,1]$ such that

$$\Delta V(u_\alpha, u) \le \frac{\Delta \hat{V}(u_\alpha, u)}{2}$$

is satisfied. Now with $I_{\alpha u}$ as just described we have that $\Delta\hat{V}(u_\alpha,u) \le \alpha\,\theta\,(u)$ [see Lemma 6.1 below], so for convenience we define the step length $\alpha : G \times G \to \mathbb{R}$ by

$$\alpha(u,\check{u}) \triangleq \max\,\{\alpha \in [\,0,1\,] : \Delta V(u_\alpha, u) \le \alpha\,\theta\,(u)/2\,\} \qquad (3.8)$$

i.e. the step length is a function of not only u but also $\check{u}(t)$ the minimising control in $\check{U}(u)$ that is used in defining u_α.

We are now in a position to present the strong variational algorithm for delay systems.

2.4 Algorithm for Solving Problem P1 (Algorithm 1)

Step 0 : Select a $u_0 \in G$

Step 1 : Set $i = 0$

Step 2 : Compute $x^i(t)$ by solving (2.1) and (2.2)

Step 3 : Compute $\lambda^i(t)$, first over $t \in [1-\tau, 1]$ and then over $[0, 1-\tau]$ by solving (2.7), (2.8) and (2.9)

Step 4 : Compute a $\check{u}_i \in \check{U}(u_i)$

Step 5 : Compute $\theta(u_i) = \Delta\hat{V}(\check{u}_i, u_i)$ using (2.13)

If $\theta(u_i) = 0$ stop

Otherwise continue

Step 6 : Compute the set I_u using (3.1) and define $I_{\alpha u}$ having properties (3.2)-(3.6) for $\alpha \in [0,1]$

Define $u_\alpha(t) = \check{u}(t)$ for $t \in I_{\alpha u}$

$= u(t)$ otherwise

Step 7 : Compute $\alpha_i = \alpha(u_i, \check{u}_i)$ by using (3.8)

Step 8 : Set $u_{i+1} = u_{\alpha_i}$

Set $i = i+1$

Goto Step 2

Algorithm 1 has the following property:

Theorem 1

Suppose Assumptions 1 and 2 are satisfied and that Algorithm 1 generates a sequence $\{u_i\}$. Then this sequence is either finite, in which case the last control is desirable, or it is infinite and every limit point in G, with respect to the metric d, is desirable.

The proof of Theorem 1 is given in section 2.6.

Remark

It should be noted that Theorem 1 does not say that limit points exist. The theorem merely states that if any limit points exist, they will be desirable. To guarantee existence of accumulation points requires additional assumptions (see for example Polak [P1]). One such assumption is that G has a certain compactness property so that for any infinite sequence in G, a convergent subsequence always exists. This will be further discussed in Chapter 3.

2.5 Basic Results

Proposition 5.1

For all $u \in G$, there exists a unique absolutely continuous solution $x^u(.)$ to (2.1) and satisfying (2.2).

Proof

This is a standard result which relies on finding a fixed point in a contraction mapping (this always exists due to Schauder's fixed point theorem), see for example Hale [HAL1], Bellman and Cooke [BC1], Oğuztöreli [OG1].

Proposition 5.2

There exists a $\bar{d} \in (0,\infty)$ such that for all $u \in G$ we have

(i) $\|x^u(t)\| \leq \bar{d}$

(ii) $\|\lambda^u(t)\| \leq \bar{d}$ for all $t \in T$

[Note: Throughout this thesis we will make use of finite constants $M_1, M_2, \ldots, d_1, d_2, \ldots$, etc., which do not depend on the data.]

Proof

(i) Since $x^u(.)$ is the solution of (2.1) and (2.2) we have that

$$x^u(t) = \phi(0)+\int_0^t f(x^u(s),x^u(s-\tau),u(s),s)ds \quad \text{for all } t \in T$$

$$\text{i.e. } \|x^u(t)\| \leq \|\phi(0)\|+\|\int_0^t f(x^u(s),x^u(s-\tau),u(s),s)ds\|$$

$$\leq \|\phi(0)\|+\int_0^t\|f(x^u(s),x^u(s-\tau),u(s),s)\|ds \quad \text{for all } t \in T$$

By Assumption 2 and boundedness of ϕ, we have by letting $\|\phi(0)\| = M_1 \in (0,\infty)$ that

$$\|x^u(t)\| \leq M_1+\int_0^t M\{\|x^u(s)\| + \|x^u(s-\tau)\| +1\}ds \quad \text{for all } t \in T$$

$$\text{i.e. } \|x^u(t)\| \leq M_2 +M \int_0^t\|x^u(s)\|ds+M\int_0^t \|x^u(s-\tau)\|ds \quad \text{for all } t \in T$$

letting $s^1 = s-\tau$ in second integral we get

$$ds^1 = ds, \quad \text{and when } s = t, \; s^1 = t-\tau$$
$$s = 0, \; s^1 = -\tau$$

Therefore

$$\|x^u(t)\| \le M_2 + M \int_0^t \|x^u(s)\| ds + M \int_{-\tau}^{t-\tau} \|x^u(s^1)\| ds^1 \qquad \text{all } t \in T$$

But for $s \in [-\tau, 0]$, $x(s) = \phi(s)$

Hence

$$\|x^u(t)\| \le M_2 + M \int_0^t \|x^u(s)\| ds + M \int_0^{t-\tau} \|x^u(s)\| ds + M \int_{-\tau}^{0} \|\phi(s)\| ds$$
$$\text{for all } t \in T$$

$$\|x^u(t)\| \le M_3 + M \int_0^t \|x^u(s)\| ds + M \int_0^t \|x^u(s)\| ds \qquad \text{for all } t \in T$$

since $\int_0^{t-\tau} \|x^u(s)\| ds \le \int_0^t \|x^u(s)\| ds$ and $\phi(s)$ is bounded for $s \in [-\tau, 0]$.

Hence we have

$$\|x^u(t)\| \le M_3 + M_4 \int_0^t \|x^u(s)\| ds \qquad \text{for all } t \in T$$

Now by an application of Gronwall's inequaltiy (see Halkin [HAK1] or Oǧuztöreli [OG1]) we get

$$\|x^u(t)\| \le M_3 \exp M_4 t \le M_3 \exp M_4 \qquad \text{for all } t \in T$$

Hence $\|x^u(t)\| \le \bar{d}$ for all $t \in T$ where $\bar{d} = M_3 \exp M_4$

(ii) From equations (2.7)-(2.9) we have

$$\lambda^u(t) = \int_t^1 H_x^T(x^u(s), x^u(s-\tau), u(s), \lambda^u(s), s) ds$$
$$\text{for all } t \in [1-\tau, 1]$$

and

$$\lambda^u(t) = \int_t^1 H_x^T(x^u(s), x^u(s-\tau), u(s), \lambda^u(s), s)ds$$

$$+ \int_t^{1-\tau} H_y^T(x^u(s+\tau), x^u(s), u(s+\tau), \lambda^u(s+\tau), s+\tau)ds$$

$$\text{for all } t \in [0, 1-\tau]$$

Using equation (2.6) we have

$$\lambda^u(t) = \int_t^1 [f_x^T(x^u(s), x^u(s-\tau), u(s), s)\lambda^u(s) + l_x^T(x^u(s), x^u(s-\tau), u(s), s)]ds$$

$$\text{for all } t \in [1-\tau, 1]$$

and

$$\lambda^u(t) = \int_t^1 [f_x^T(x^u(s), x^u(s-\tau), u(s), s)\lambda^u(s)$$

$$+ l_x^T(x^u(s), x^u(s-\tau), u(s), s)]ds$$

$$+ \int_t^{1-\tau} [f_y^T(x^u(s+\tau), x^u(s), u(s+\tau), s+\tau)\lambda^u(s+\tau)$$

$$+ l_y^T(x^u(s+\tau), x^u(s), u(s+\tau), s+\tau)]ds$$

$$\text{for all } t \in [0, 1-\tau]$$

Considering only the case $t \in [1-\tau, 1]$ we have

$$\|\lambda^u(t)\| \leq \int_t^1 \|l_x^T(x^u(s), x^u(s-\tau), u(s), s)\|ds$$

$$+ \int_t^1 \|f_x^T(x^u(s), x^u(s-\tau), u(s), s)\| \; \|\lambda^u(s)\|ds$$

$$\text{for all } t \in [1-\tau, 1]$$

Since $\|x^u(t)\| \leq \bar{d}$ for all $u \in G$, all $t \in T$ and l_x and f_x are both continuous on $B_1 x B_1 x \Omega x T$, where B_1 is defined as

$$B_1 = \{ x \in \mathbb{R}^n : \|x\| \leq \bar{d} \},$$

then l_x and f_x are bounded continuous functionals on $B_1 x B_1 x \Omega x T$. Thus there exists constants d_1, $d_2 \in (0, \infty)$ such that

$$\|\lambda^{u}(t)\| \le d_1 + d_2 \int_t^1 \|\lambda^{u}(s)\|\, ds \qquad \text{for all } t \in [1-\tau, 1]$$

By an application of Gronwall's inequality we get the required result for $t \in [1-\tau, 1]$.

The case for $[0, 1-\tau]$ follows similarly. Hence the required results hold.

Proposition 5.3

There exists a $c \in (0, \infty)$ such that for all u_1, $u_2 \in G$ we have

(i) $\|x^1(t) - x^2(t)\| \le cd(u_1, u_2)$

(ii) $\|\lambda^1(t) - \lambda^2(t)\| \le cd(u_1, u_2)$

for all $t \in T$

Proof

(i) Since $x(t)$ is the solution of (2.1) and (2.2) we get

$$x^1(0) = x^2(0) \quad \text{and}$$

$$x^1(t) - x^2(t) = \int_0^t [f(x^1(s), x^1(s-\tau), u_1(s), s) - f(x^2(s), x^2(s-\tau), u_2(s), s)]\, ds \quad \text{for all } t \in T$$

By adding and subtracting terms and taking the modulus we have

$$\begin{aligned}\|x^1(t) - x^2(t)\| \le & \int_0^t \|f(x^1(s), x^1(s-\tau), u_1(s), s) - f(x^2(s), x^1(s-\tau), u_1(s), s)\|\, ds \\ & + \int_0^t \|f(x^2(s), x^1(s-\tau), u_1(s), s) - f(x^2(s), x^2(s-\tau), u_1(s), s)\|\, ds \\ & + \int_0^t \|f(x^2(s), x^2(s-\tau), u_1(s), s) - f(x^2(s), x^2(s-\tau), u_2(s), s)\|\, ds\end{aligned}$$

for all $t \in T$

By the continuity of f_x, f_y and f_u on $B_1 \times B_1 \times \Omega \times T$ there exist finite constants c_1, c_2, c_3 such that

$$\|x^1(t)-x^2(t)\| \leq c_1 \int_0^t \|x^1(s)-x^2(s)\|$$
$$+ c_2 \int_0^t \|x^1(s-\tau)-x^2(s-\tau)\| ds$$
$$+ c_3 \int_0^t \|u_1(s)-u_2(s)\| ds \quad \text{for all } t \in T$$

i.e.

$$\|x^1(t)-x^2(t)\| \leq c_4 \int_0^1 \|x^1(s)-x^2(s)\| ds + c_5 d(u_1,u_2)$$
$$\text{for all } t \in T$$

by the definition of the metric d.

By an application of Gronwall's inequality we get

$$\|x^1(t)-x^2(t)\| \leq c_5 \exp c_4 \; d(u_1,u_2) \text{ for all } t \in T$$

i.e.

$$\|x^1(t)-x^2(t)\| \leq c \; d(u_1,u_2) \quad \text{for all } t \in T \text{ as required}$$

(ii) Using equations (2.7)-(2.9) we have

$$\lambda^2(t)-\lambda^1(t) = -\int_t^1 [H_x^T(x^1(s),x^1(s-\tau),u_1(s),\lambda^1(s),s)$$
$$-H_x^T(x^2(s),x^2(s-\tau),u_2(s),\lambda^2(s),s)] ds$$
$$\text{for all } t \in [1-\tau,1]$$

and

$$\lambda^2(t)-\lambda^1(t) = -\int_t^1 [H_x^T(x^1(s),x^1(s-\tau),u_1(s),\lambda^1(s),s)$$
$$-H_x^T(x^2(s),x^2(s-\tau),u_2(s),\lambda^2(s),s)] ds$$
$$-\int_t^{1-\tau} [H_y^T(x^1(s+\tau),x^1(s),u_1(s+\tau),\lambda^1(s+\tau),s+\tau)$$
$$-H_y^T(x^2(s+\tau),x^2(s),u_2(s+\tau),\lambda^2(s+\tau),s+\tau)] ds$$
$$\text{for all } t \in [0,1-\tau]$$

Consider the case $t \in [1-\tau, 1]$.

As in (i) by adding and subtracting terms and taking the modulus of both sides we have

$$\begin{aligned}\|\lambda^2(t)-\lambda^1(t)\| \le \int_t^1 \| & H_x^T(x^1(s), x^1(s-\tau), u_1(s), \lambda^1(s), s) \\ & - H_x^T(x^2(s), x^1(s-\tau), u_1(s), \lambda^1(s), s)\| \, ds \\ + \int_t^1 \| & H_x^T(x^2(s), x^1(s-\tau), u_1(s), \lambda^1(s), s) \\ & - H_x^T(x^2(s), x^2(s-\tau), u_1(s), \lambda^1(s), s)\| \, ds \\ + \int_t^1 \| & H_x^T(x^2(s), x^2(s-\tau), u_1(s), \lambda^1(s), s) \\ & - H_x^T(x^2(s), x^2(s-\tau), u_2(s), \lambda^1(s), s)\| \, ds \\ + \int_t^1 \| & H_x^T(x^2(s), x^2(s-\tau), u_2(s), \lambda^1(s), s) \\ & - H_x^T(x^2(s), x^2(s-\tau), u_2(s), \lambda^2(s), s)\| \, ds \\ & \text{for all } t \in [1-\tau, 1]\end{aligned}$$

Since the functions H_{xx}, H_{xy}, H_{xu}, $H_{x\lambda}$ are continuous on $B_1 \times B_1 \times \Omega \times B_1 \times T$, there exists finite constants c_1, c_2, c_3, c_4 such that

$$\begin{aligned}\|\lambda^1(t)-\lambda^2(t)\| \le\ & c_1 \int_t^1 \|x^1(s)-x^2(s)\| \, ds + c_2 \int_t^1 \|x^1(s-\tau)-x^2(s-\tau)\| ds \\ & + c_3 \int_t^1 \|\lambda^1(s)-\lambda^2(s)\| \, ds + c_4 \int_t^1 \|u_1(s)-u_2(s)\| \, ds \\ & \text{for all } t \in [1-\tau, 1]\end{aligned}$$

By definition of metric d and part (i) of Proposition 5.3, we deduce that there exists finite c_5, c_6 such that

$$\|\lambda^1(t)-\lambda^2(t)\| \le c_5 d(u_1, u_2) + c_6 \int_t^1 \|\lambda^1(s)-\lambda^2(s)\| \, ds$$

$$\text{for all } t \in [1-\tau, 1]$$

The desired result now follows by an application of Gronwall's inequality.

The case for $t \in [0, 1-\tau]$ follows by a similar procedure.

Proposition 5.4

There exists a $c \in (0, \infty)$ such that, for all u_1, $u_2 \in G$ we have

$$||\Delta V(u_2, u_1) - \hat{\Delta V}(u_2, u_1)|| \leq c\,[\,d(u_1, u_2)]^2$$

Proof

Since $V(u_1) = \int_0^1 l(x^1(t), x^1(t-\tau), u_1(t), t)dt$

Then we have that for any u_1, $u_2 \in G$

$$\Delta V(u_2, u_1) = \int_0^1 [\,l(x^2(t), x^2(t-\tau), u_2(t), t) - l(x^1(t), x^1(t-\tau), u_1(t), t)]\,dt$$

By adding and subtracting terms we get

$$\begin{aligned}\Delta V(u_2, u_1) = &\int_0^1 [l(x^2(t), x^2(t-\tau), u_2(t), t) - l(x^1(t), x^1(t-\tau), u_1(t), t)]\,dt \\ &+ \int_0^1 \lambda^{u_1 T}(t)\,[\,f(x^2(t), x^2(t-\tau), u_2(t), t) - f(x^1(t), x^1(t-\tau), u_1(t), t)\,]\,dt \\ &- \int_0^1 \lambda^{u_1 T}(t)\,[\,\dot{x}^2(t) - \dot{x}^1(t)]\,dt\end{aligned}$$

From (2.6) we have

$$\begin{aligned}H(x^u(t), x^u(t-\tau), u(t), \lambda^u(t), t) = &\lambda^u(t)^T f(x^u(t), x^u(t-\tau), u(t), t) \\ &+ l(x^u(t), x^u(t-\tau), u(t), t)\end{aligned}$$

where $\lambda(.)$ is the solution to (2.7)-(2.9)

Therefore

$$\begin{aligned}\Delta V(u_2, u_1) = \int_0^1 [\,&H(x^2(t), x^2(t-\tau), u_2(t), \lambda^{u_1}(t), t) \\ &- H(x^1(t), x^1(t-\tau), u_1(t), \lambda^{u_1}(t), t)\,]dt\end{aligned}$$

$$- \int_0^1 \lambda^{u_1 T}(t)(\dot{x}^2(t)-\dot{x}^1(t))dt$$

Integrating the second integral by parts we get

$$\int_0^1 \lambda^{u_1 T}(t)(\dot{x}^2(t)-\dot{x}^1(t))dt = [\lambda^{u_1 T}(t)(x^2(t)-x^1(t))]_0^1$$

$$- \int_0^1 \dot{\lambda}^{u_1 T}(t)(x^2(t)-x^1(t))dt$$

$$= - \int_0^1 \dot{\lambda}^{u_1 T}(t)(x^2(t)-x^1(t))dt$$

since $\lambda(1) = 0$ and $x^2(o) = x^1(o)$

Hence the above becomes

$$\Delta V(u_2,u_1) = \int_0^1 [H(x^2,y^2,u_2,\lambda^1,t)-H(x^1,y^1,u_1,\lambda^1,t)]dt$$

$$+ \int_0^1 \dot{\lambda}^{u_1 T}(x^2-x^1)dt$$

where y denotes the delayed argument, i.e. $x(t-\tau) = y(t)$, and the dependence of the functions x, y, u, λ on t is not shown for convenience of notation. This will be done throughout this thesis and should not cause any confusion.

Using Taylor series to expand $H(x^2,y^2,u_2,\lambda^1,t)$ about $H(x^1,y^1,u_2,\lambda^1,t)$ to second approximation with remainder term we have

$$H(x^2,y^2,u_2,\lambda^1,t) = H(x^1,y^1,u_2,\lambda^1,t)+H_x(x^1,y^1,u_2,\lambda^1,t)(x^2-x^1)$$

$$+H_y(x^1,y^1,u_2,\lambda^1,t)(y^2-y^1)+H_{xy}(x^\varepsilon,y^\varepsilon,u_2,\lambda^1,t)\cdot$$

$$\cdot(x^2-x^1)(y^2-y^1)$$

$$+\frac{H_{xx}(x^\varepsilon,y^\varepsilon,u_2,\lambda^1,t)(x^2-x^1)^2}{2}$$

$$+ \frac{H_{yy}(x^\varepsilon,y^\varepsilon,u_2,\lambda^1,t)(y^2-y^1)^2}{2}$$

where $x^{\varepsilon} = (1-\varepsilon)x^1+\varepsilon x^2$ for some $\varepsilon \in (0,1)$ and y^{ε} is similarly defined.

Also since λ^1 is the solution to (2.7)-(2.9) we have

$$\int_0^1 \dot{\lambda}^{1T}(t)(x^2(t)-x^1(t))dt=-\int_0^1 H_x(x^1(t),x^1(t-\tau),u_1(t),\lambda^1(t),t)(x^2(t)-x^1(t))dt$$

$$-\int_0^{1-\tau} H_y(x^1(t+\tau),x^1(t),u_1(t+\tau),\lambda^1(t+\tau),t+\tau)\cdot$$

$$\cdot(x^2(t)-x^1(t))dt$$

$$=-\int_0^1 H_x(x^1(t),x^1(t-\tau),u_1(t),\lambda^1(t),t)(x^2(t)-x^1(t))dt$$

$$-\int_{-\tau}^{1-\tau} H_y(x^1(t+\tau),x^1(t),u_1(t+\tau),\lambda^1(t+\tau),t+\tau)\cdot$$

$$\cdot(x^2(t)-x^1(t))dt$$

since $x^2(t) = x^1(t)$ for $t \in [-\tau,0]$ and second integrand is identically zero. Also using a change of variable we have

$$\int_0^1 \dot{\lambda}^{1T}(x^2-x^1)dt=-\int_0^1[H_x(x^1,y^1,u_1,\lambda^1,t)(x^2-x^1)+H_y(x^1,y^1,u_1,\lambda^1,t)(y^2-y^1)]dt$$

Using this and the definition of $\Delta\hat{V}(u_2,u_1)$ [see (2.11)] we have

$$\Delta V(u_2,u_1)-\Delta\hat{V}(u_2,u_1) = \int_0^1[H_x(x^1,y^1,u_2,\lambda^1,t)-H_x(x^1,y^1,u_1,\lambda^1,t)](x^2-x^1)dt$$

$$+\int_0^1[H_y(x^1,y^1,u_2,\lambda^1,t)-H_y(x^1,y^1,u_1,\lambda^1,t)](y^2-y^1)dt$$

$$+\int_0^1 \Big\{ \frac{H_{xx}(x^{\varepsilon},y^{\varepsilon},u_2,\lambda^1,t)(x^2-x^1)^2}{2}$$

$$+\frac{H_{yy}(x^{\varepsilon},y^{\varepsilon},u_2,\lambda^1,t)(y^2-y^1)^2}{2}$$

$$+H_{xy}(x^{\varepsilon},y^{\varepsilon},u_2,\lambda^1,t)(x^2-x^1)(y^2-y^1)\Big\}dt$$

By the continuity of H_{xu}, H_{yu} in $B_1 x B_1 x \Omega x B_1 x T$ and the boundedness of H_{xx}, H_{yy}, H_{xy} for all $x, y, \lambda \in B_1$, all $u \in G$, for all $t \in T$ there exist finite constants c_1, c_2, c_3, c_4, c_5 such that

$$|| \Delta V(u_2,u_1)-\Delta\hat{V}(u_2,u_1)|| \leq c_1 \int_0^1 || u_2-u_1 || \; || x^2-x^1 || dt$$
$$+c_2 \int_0^1 ||u_2-u_1|| \; ||y^2-y^1|| dt+c_3\int_0^1||x^2-x^1||^2 dt$$
$$+c_4 \int_0^1 || y^2-y^1||^2 dt+c_5 \int_0^1 ||x^2-x^1|| \; ||y^2-y^1|| dt$$

i.e.

$$||\Delta V(u_2,u_1)- \Delta\hat{V}(u_2,u_1)|| \leq c_6 \int_0^1 ||u_2-u_1|| \; ||x^2-x^1||dt+c_7 \int_0^1|| x^2-x^1 ||^2 dt$$

for some c_6, $c_7 \in (0, \infty)$.

Making use of Proposition 5.3 and the definition of the metric d we get

$$|| \Delta V(u_2,u_1)-\Delta\hat{V}(u_2,u_1) || \leq c[d(u_1,u_2)]^2 \qquad \text{as required.}$$

2.6 Proof of Theorem 1

As mentioned previously in the thesis, Theorem 1 will be proved using the convergence properties of the algorithm model (3.0) stated in section 2.3. However, before we can use these results we must show that Algorithm 1 has the properties desired of it, i.e. it satisfies the conditions of Theorem 3.1.

We shall need the following results:

Lemma 6.1

For all $\alpha \in [0,1]$, for all $u \in G$ we have

$$\phi(\alpha,u) \leq \alpha\, \theta(u)$$

Proof

We have by (3.7) and (2.13) that

$$\phi(\alpha,u) = \int_{I_{\alpha u}} [\overline{H}(u(t),t)-H(x^u,y^u,u,\lambda^u,t)]dt$$

and

$$\theta(u) = \int_0^1 [\overline{H}(u(t),t)-H(x^u,y^u,u,\lambda^u,t)]dt$$

Now $\phi(0,u) = 0$ and $\phi(1,u) = \theta(u)$ and $\theta(u)$ is the average value of

$[\overline{H}(u(t),t)-H(x^u(t),x^u(t-\tau),u(t),\lambda^u(t),t)]$. By the definition of I_u [see (3.1)] if $t \in I_u$, we have

$$H(x^u(t),x^u(t-\tau),\check{u}(t),\lambda^u(t),t)-H(x^u(t),x^u(t-\tau),u(t),\lambda^u(t),t) \leq \theta(u)$$

Now by the way $I_{\alpha u}$ is defined, we have that $I_{\alpha u} \subset I_u$ if $\alpha \in [0, \mu(I_u)]$ and so

$$\phi(\alpha,u) \leq \alpha\theta(u) \qquad \text{for } \alpha \in [0,\mu(I_u)]$$

For $\alpha \in (\mu(I_u),1]$ we have that $I_u \subset I_{\alpha u}$ and

$$[\overline{H}(u(t),t)-H(x^u(t),x^u(t-\tau),u(t),\lambda^u(t),t)] > \theta(u) \text{ if } t \in T\backslash I_{\alpha u}$$

Hence

$$\int_{T-I_{\alpha u}} [\overline{H}(u(t),t)-H(x^u,y^u,u,\lambda^u,t)]dt > (1-\alpha)\,\theta(u)$$

Now from (2.13) we have

$$\int_{T-I_{\alpha u}} [\overline{H}(u(t),t)-H(x^u,y^u,u,\lambda^u,t)]dt + \int_{I_{\alpha u}} [\overline{H}(u(t),t)-H(x^u,y^u,u,\lambda^u,t)]dt = \theta(u)$$

Therefore

$$(1-\alpha)\,\theta(u)+\phi(\alpha,u) < \theta(u)$$

and hence $\phi(\alpha,u) < \alpha\theta(u)$ for $\alpha \in (\mu(I_u),1]$.

Combining the above results we get

$$\phi(\alpha,u) \leq \alpha\theta(u) \qquad \text{for all } \alpha \in [0,1].$$

Corollary 6.2

For all $\alpha \in [0,1]$, for all $u \in G$, with u_α as defined in Algorithm 1, we have

$$\Delta V(u_\alpha,u) \leq \alpha\theta(u) + c\alpha^2$$

for some finite c.

Proof

From Proposition 5.4, we have for all u_1, $u_2 \in G$ that

$$\|\Delta V(u_2,u_1) - \hat{\Delta V}(u_2,u_1)\| \leq c_1 [d(u_1,u_2)]^2$$

for some $c_1 \in (0,\infty)$

Hence

$$\|\Delta V(u_\alpha,u) - \hat{\Delta V}(u_\alpha,u)\| \leq c_1[d(u_\alpha,u)]^2$$

Now from Lemma 6.1

$$\hat{\Delta V}(u_\alpha,u) = \phi(\alpha,u) \leq \alpha\,\theta(u) \qquad \text{for all } \alpha \in [0,1]$$

Also $d(u_\alpha,u) \leq 2r\,\mu(I_{\alpha u}) = 2r\alpha$

Therefore

$$\|\Delta V(u_\alpha,u) - \phi(\alpha,u)\| \leq c_1 4r^2\alpha^2$$

i.e. $\Delta V(u_\alpha,u) \leq \alpha\,\theta(u) + c\alpha^2$ as required.

Corollary 6.3

Let $A: G \longrightarrow 2^G$ be the map defined by Algorithm 1 and let $\alpha: GxG \longrightarrow [0,1]$ be as defined in (3.8). Then there exists some $c \in (0,\infty)$ such that

(i) $\Delta V(v,u) \leq \dfrac{-[\theta(u)]^2}{4c}$ for all $v \in A(u)$

(ii) $\alpha \geq \bar{\alpha} \triangleq \dfrac{-\theta(u)}{2c}$ for all $\alpha \in \{\alpha(u,\check{u}) : \check{u} \in G\}$

<u>Proof</u>

From Corollary 6.2 we have

$$\Delta V(u_\alpha,u) \leq \alpha\,\theta(u)+c\,\alpha^2$$

and by (3.8)

$$\alpha(u,\check{u}) = \max\left\{\alpha \in [0,1] \; : \; \Delta V(u_\alpha,u) \leq \frac{\alpha\,\theta(u)}{2}\right\}$$

Therefore the maximum $\alpha \in [0,1]$ such that

$$\alpha\,\theta(u)+c\,\alpha^2 \leq \frac{\alpha\,\theta(u)}{2}$$

occurs at the non zero root of

$$\alpha\,\theta(u) + c\alpha^2 = \frac{\alpha\theta(u)}{2}$$

i.e $\bar{\alpha} = \dfrac{-\theta(u)}{2c}$

and we have that

$$\alpha\,\theta(u)+c\,\alpha^2 \leq \frac{\alpha\,\theta(u)}{2} \quad \text{for all } \alpha \in [0,\bar{\alpha}]$$

Therefore $\alpha(u,\check{u})$ is not smaller than $\bar{\alpha}$ as asserted in (ii).

Using this we have

$$\Delta V(v,u) \leq \alpha(u,\check{u}) \frac{\theta(u)}{2}$$

$$\leq \frac{\bar{\alpha}\,\theta(u)}{2} \qquad \text{for all } v \in A(u)$$

Hence $\Delta V(v,u) \leq \dfrac{-[\theta(u)]^2}{4c}$ for all $v \in A(u)$

This proves (i) .

All the above relations can be seen more clearly in Fig. 6.1.

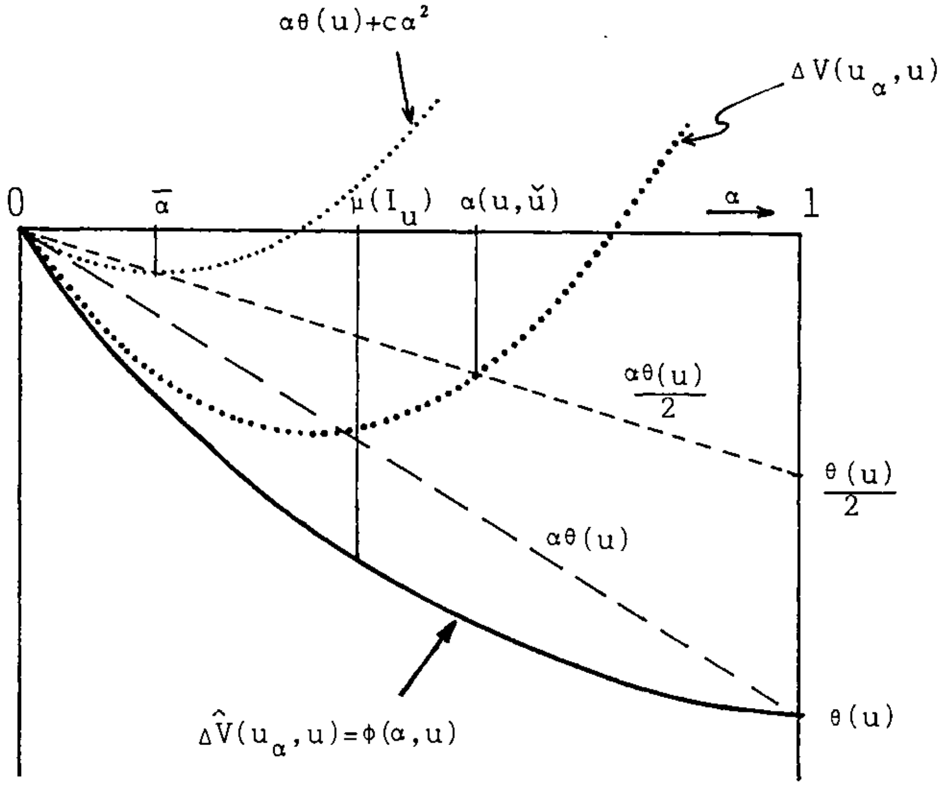


Fig. 6.1

Lemma 6.4

The function $\theta : G \to \mathbb{R}$ is continuous with respect to the metric d.

Proof

From Proposition 5.3, there exists a $c \in (0,\infty)$ such that for all $u,v \in G$ satisfying $d(u,v) \leq \varepsilon$, for some $\varepsilon > 0$, we have

$$\sup_{t \in T} \| x^v(t) - x^u(t) \| \leq c\,\varepsilon$$

$$\sup_{t \in T} \| \lambda^v(t) - \lambda^u(t) \| \leq c\,\varepsilon$$

Let $\eta: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times T \to \mathbb{R}$ be defined by

$$\eta(x,y,\lambda,t) = \min_{\omega \in \Omega} H(x,y,\omega,\lambda,t) \qquad \text{for a.a.} t \in T$$

Note that $\eta(x^u(t), x^u(t-\tau), \lambda^u(t), t) = \overline{H}(u(t),t)$

We recall a theorem from Polak [P1].

Theorem B3.20

Let $\psi(.,.)$ be a continuous function from $\mathbb{R}^p \times \mathbb{R}^q$ into $\mathbb{R}$ and let S be a compact subset of $\mathbb{R}^q$. Then the function

$\chi : \mathbb{R}^p \to \mathbb{R}$ defined by

$\chi(z) = \min \{\psi(z,h) : h \in S\}$

is also continuous.

Using this and the uniform continuity of H on the compact set $B_1 \times B_1 \times \Omega \times B_1 \times T$ we deduce that η is uniformly continuous on $B_1 \times B_1 \times B_1 \times T$.

Returning to the proof of lemma 6.4 we have by the definition of θ that

$$\begin{aligned}\theta(v)-\theta(u) &= \int_0^1 [\eta(x^v(t),x^v(t-\tau),\lambda^v(t),t) - H(x^v(t),x^v(t-\tau),v(t),\lambda^v(t)t)]dt \\ &\quad - \int_0^1 [\eta(x^u(t),x^u(t-\tau),\lambda^u(t),t) - H(x^u(t),x^u(t-\tau),u(t),\lambda^u(t),t)]dt \\ &= \int_0^1 [\eta(x^v,y^v,\lambda^v,t) - \eta(x^u,y^u,\lambda^u,t)]dt \\ &\quad - \int_0^1 [H(x^v,y^v,v,\lambda^v,t) - H(x^u,y^u,u,\lambda^u,t)]dt\end{aligned}$$

By the continuity of η and H the following holds

(i) $\int_0^1 [\eta(x^v, y^v, \lambda^v, t) - \eta(x^u, y^u, \lambda^u, t)]\, dt \leq \varepsilon/2$

for all $u, v \in G$ such that $d(u,v) \leq \delta_1$

This can be shown by adding and subtracting terms as in the proofs of Proposition 5.3 and using Proposition 5.3

(ii) There exists a $\delta \in (0, \delta_1]$ such that for all $u, v \in G$ with $d(u,v) \leq \delta$ we have

$$\int_0^1 [H(x^v, y^v, v, \lambda^v, t) - H(x^u, y^u, u, \lambda^u, t) \leq \varepsilon/2$$

Therefore for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$\| \theta(v) - \theta(u) \| \leq \varepsilon$ for all $u, v \in G$ which satisfy $d(u,v) \leq \delta$.

Hence θ is continuous with respect to the metric d, in fact the proof shows that it is uniformly continuous.

We are now in a position to prove Theorem 1 which we restate here for convenience.

<u>Theorem 1</u>

Suppose Assumptions 1 and 2 are satisfied and that Algorithm 1 generates a sequence $\{u_i\}$. Then this sequence is either finite, in which case the last control is desirable, or it is infinite and every limit point in G, with respect to the metric d, is desirable.

<u>Proof</u>

If Algorithm 1 generates a finite sequence of controls, i.e.

$\{u_i\} = \{u_0, u_1, \ldots, u_k\}$, it is obvious that the algorithm stops at Step 5 when the last element u_k satisfies

$$\theta(u_k) = 0, \quad \text{i.e. } u_k \text{ is desirable}$$

Now assume that Algorithm 1 generates an infinite sequence $\{u_i\}_{i=0}^{\infty}$. We will use Theorem 3.1 to prove Theorem 1.

By our assumptions on 1, V(.) is continuous and bounded from below on G, hence assumption (i) of Theorem 3.1 is satisfied. Now it will be shown that assumption (ii) in Theorem 3.1 is also satisfied. Assume that Algorithm 1 is at an undesirable control $u \in G$, i.e. $\theta(u) = \delta < 0$.

Then by the continuity of θ, there exists an $\varepsilon > 0$ such that for all $v \in G$ satisfying $d(u,v) \leq \varepsilon$ we have

$$-[\theta(v)]^2 \leq -[\theta(u)]^2/2$$

Now since Algorithm 1 produces

$$\Delta V(A(u),u) = \Delta V(u_\alpha,u) \leq -\frac{[\theta(u)]^2}{4c}$$

[using Corollary 6.3], we have that for all $v \in G$ satisfying $d(u,v) \leq \varepsilon$,

$$\Delta V(A(v),v) \leq -\frac{[\theta(v)]^2}{4c} \leq -\frac{[\theta(u)]^2}{8c}$$

i.e. for any undesirable $u \in G$, Algorithm 1 has the property that

$$V(A(v))-V(v) \leq \delta(u) \quad \text{for all } v \in G$$

satisfying $d(u,v) \leq \varepsilon$, where $\delta(u) = \dfrac{-[\theta(u)]^2}{8c} < 0$

This shows that Algorithm 1 satisfies assumption (ii) of Theorem 3.1, therefore we can use it to prove convergence of our algorithm.

Hence from Theorem 3.1 we have that every accumulation point $u^* \in G$ of the sequence $\{u_i\}_{i=0}^{\infty}$ must satisfy

$$V(u)-V(u^*) \geq 0 \qquad \text{for all } u \in A(u^*)$$

But from Corollory 6.3

$$\Delta V(u,u^*) \leq \frac{-[\theta(u^*)]^2}{4c} \qquad \text{for all } u \in A(u^*)$$

This is only possible if $\theta(u^*) = 0$, i.e. if u^* is desirable. This proves Theorem 1.

CHAPTER 3

STEEPEST DESCENT ALGORITHMS WITH RELAXED CONTROLS

3.1 Introduction

In Chapter 2 it was shown that any L_1 accumulation points of a sequence of controls generated by Algorithm 1 were desirable (i.e. satisfied a necessary condition of optimality), although existence of such limit controls could not be guaranteed. In this chapter we present two algorithms for solving a relaxed version of Problem P1 (defined below), both of which circumvent this problem.

The first of these, (Algorithm 2) presents an analysis using a more abstract but also, in a sense, more natural setting than that of Chapter 1. Here our analysis is based on convergence in the sense of control measures. The reason for turning to relaxed controls is that, unlike in L_1, a sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$, of bounded relaxed controls always has accumulation points. Furthermore these accumulation points (in the sense of control measures) satisfy an appropriate optimality condition for the relaxed control problem defined in section 2.

Because of the "linear" nature of relaxed controls the new control $\underset{\sim}{u}_\alpha$ can be constructed as

$$\underset{\sim}{u}_\alpha(t) = (1-\alpha)\underset{\sim}{u}(t) \oplus \alpha \check{\underset{\sim}{u}}(t) \qquad (1.1)$$

where $\oplus$ is in the relaxed sense [see section B8 in Chapter 1], $\underset{\sim}{u}$ is the old control, $\check{\underset{\sim}{u}}$ is a control which minimises a Hamiltonian, and $\alpha \in [0,1]$ is the step length. We will say that $\underset{\sim}{u}_\alpha(t)$ is a relaxed convex combination of the controls $\underset{\sim}{u}(t)$ and $\check{\underset{\sim}{u}}(t)$ when

(1.1) is true.

Both $\underset{\sim}{u}$ and $\underset{\sim}{\check{u}}$ may be relaxed controls, although if $\underset{\sim}{u}(t) = \delta_u(t)$ and $\underset{\sim}{\check{u}}(t) = \delta_{\check{u}}(t)$, i.e. they have their probability measures concentrated at the ordinary controls u(t) and $\check{u}(t)$ respectively, $\underset{\sim}{u}_\alpha$ defined by (1.1) would still be relaxed, having its probability measures concentrated at the two ordinary controls u and $\check{u}$ in the ratio $(1-\alpha):\alpha$ for $\alpha \in (0,1)$. Obviously $\underset{\sim}{u}_\alpha$ reduces to just u or $\check{u}$ when $\alpha = 0$ or 1 respectively.

With $\underset{\sim}{u}_\alpha$ defined as in (1.1), the strong variations in control as discussed in Chapter 2 are meaningless since $\underset{\sim}{u}_\alpha$ has some finite probability measure concentrated at both the controls $\underset{\sim}{u}(t)$ and $\underset{\sim}{\check{u}}(t)$ for all $t \in T$ and for all $\alpha \in (0,1)$. However, it can be seen this definition of $\underset{\sim}{u}_\alpha$ is closer to classical methods of defining new control than the strong variational approach presented in Chapter 2. For example in steepest descent the new control would be defined by

$$\underset{\sim}{u}_\alpha(t) = \underset{\sim}{u}(t) + \alpha \underset{\sim}{s}(t)$$

where $\underset{\sim}{s}(t)$ is the search direction and can be set to $(\underset{\sim}{\check{u}}(t) - \underset{\sim}{u}(t))$, with $\underset{\sim}{\check{u}}(t)$ obtained by, say, minimising a Hamiltonian. Hence $\underset{\sim}{u}_\alpha$ would be defined by

$$\underset{\sim}{u}_\alpha(t) = (1-\alpha)\underset{\sim}{u}(t) + \alpha \underset{\sim}{\check{u}}(t) \qquad (1.2)$$

Comparing (1.1) and (1.2) above, the similarity is very clear, the only difference being that a relaxed convex combination of $\underset{\sim}{u}$ and $\underset{\sim}{\check{u}}$ is taken in (1.1), whereas a standard convex combination is taken in the steepest descent method. Because of this strong similarity we will call this approach (defined by (1.1)) relaxed steepest descent, or, steepest descent with relaxed controls (see Warga [W1]).

This similarity is further strengthened when an estimate of the change in cost when $\underset{\sim}{u}_\alpha$ [as defined by (1.1)] is used in place of the old control u. This estimate [see Lemma 4.1 below], as for ordinary steepest descent methods, is found to be a linear function of the step length α. As will be seen, this makes proving convergence very much simpler than was for the strong variational approach. One of the advantages is that there is no need to define the interval $I_{\alpha u}$ as in Chapter 2, although the relatively straightforward method for choosing the step length can still be used.

The second of the algorithms presented in this chapter (Algorithm 3) is an attempt to make Algorithm 2 implementable in that all the relaxed controls are approximated using ordinary controls to any required degree of accuracy. If this is done at each iteration we show that all limit points satisfy optimality conditions to within "delta". Also we show that if the accuracy of the approximations is increased indefinitely then limit points satisfy optimality conditions "exactly". For both of these cases limit points always exist due to the compactness property of the set of relaxed controls and the fact that the space of ordinary controls is dense in it.

3.2 Relaxed Control Problem

Let us recall Problem P1 in Chapter 2,

Problem P1:

$$\min_u \int_0^1 l(x(t),x(t-\tau),u(t),t)dt$$

$$\text{s.t.}\quad \dot{x}(t) = f(x(t),x(t-\tau),u(t),t) \quad \text{for a.a. } t \in T$$

$$x(t) = \phi(t) \quad \text{for all } t \in [-\tau,0]$$

$$u \in G$$

where $G = \{u \in L_1^m[0,1] : u(t) \in \Omega \text{ for all } t \in T\}$ and all the other objects are the same as in Chapter 2.

To define the relaxed control problem for Problem P1 we first need to state a few definitions and results which are standard in the relaxed control literature [see Section B8 in Chapter 1].

Let $\underset{\sim}{V}$ be the set of probability measures on Ω, so that if $\underset{\sim}{v} \in \underset{\sim}{V}$, then

$\int_{\Omega} d\underset{\sim}{v}(u) = 1$. For any continuous function ϕ: $\mathbb{R}^n \times \Omega \times T \longrightarrow \mathbb{R}^p$, the corresponding relaxed function ϕ_r: $\mathbb{R}^n \times \underset{\sim}{V} \times T \rightarrow \mathbb{R}^p$ is defined by

$$\phi_r(x,\underset{\sim}{v},t) \triangleq \int_{\Omega} \phi\,(x,u,t)\,d\underset{\sim}{v}(u) \tag{2.1}$$

Let $\underset{\sim}{G}$ denote the set of measurable relaxed controls.

Then the relaxed control problem R1 corresponding to problem P1 is

<u>Problem R1</u>

$$\min_{\underset{\sim}{u}} \; V(\underset{\sim}{u}) = \int_0^1 l_r(x(t),x(t-\tau),\underset{\sim}{u}(t),t)\,dt \tag{2.2}$$

$$\text{s.t.}\quad \dot{x}(t) = f_r(x(t),x(t-\tau),\underset{\sim}{u}(t),t) \quad \text{for a.a. } t \in T \tag{2.3}$$

$$x(t) = \phi(t) \quad \text{for all } t \in [-\tau,0] \tag{2.4}$$

$$\underset{\sim}{u} \in \underset{\sim}{G}$$

Remark

For any control with the symbol " $\sim$ " we mean that it is a relaxed control and an element of $\underset{\sim}{G}$. Also unless otherwise stated, for all relaxed functions ϕ_r as defined by (2.1), the subscript r will be omitted since it will be apparent from the text whether it is the original function or its relaxed extension that is being considered.

We define $\lambda^{\underset{\sim}{u}}: T \longrightarrow \mathbb{R}^n$, the extension of the costate function, as the solution of

$$-\dot{\lambda}(t) = H_x^T(x^{\underset{\sim}{u}}(t), x^{\underset{\sim}{u}}(t-\tau), \underset{\sim}{u}(t), \lambda(t), t) \tag{2.5}$$

$$+H_y^T(x^{\underset{\sim}{u}}(t+\tau), x^{\underset{\sim}{u}}(t), \underset{\sim}{u}(t+\tau), \lambda(t+\tau), t+\tau)$$
$$\text{for a.a.} t \in [0, 1-\tau]$$

$$-\dot{\lambda}(t) = H_x^T(x^{\underset{\sim}{u}}(t), x^{\underset{\sim}{u}}(t-\tau), \underset{\sim}{u}(t), \lambda(t), t) \quad \text{for a.a.} t \in [1-\tau, 1] \tag{2.6}$$

$$\lambda(1) = 0 \tag{2.7}$$

where $H: \mathbb{R}^n \times \mathbb{R}^n \times \underset{\sim}{V} \times \mathbb{R}^n \times T \longrightarrow \mathbb{R}$ defined by

$$H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t) = \lambda^{\underset{\sim}{u}T} f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, t) + l(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, t) \tag{2.8}$$

is the extension of the Hamiltonian function defined in Chapter 2 to the set $\underset{\sim}{G}$.

We assume that Assumptions 1 and 2 in Chapter 2 hold in the following exposition.

We will need the following results:

Proposition 2.1

For any $\underset{\sim}{u} \in \underset{\sim}{G}$, there exists absolutely continuous functions $x^{\underset{\sim}{u}}$ and $\lambda^{\underset{\sim}{u}}$, where

(i) $x^{\underset{\sim}{u}}(.)$ is the unique solution of (2.3) and (2.4)

(ii) $\lambda^{\underset{\sim}{u}}(.)$ is the unique solution of (2.5), (2.6) and (2.7)

For a proof of this standard result see Young [Y1], where it is given for the delay-free case. These results can be easily extended using methods in Hale [HAL1] or Bellman and Cooke [BC1] to cover our delayed case.

As in Chapter 2, we can show quite easily that for any two relaxed controls $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2 \in \underset{\sim}{G}$ we have that

$$\Delta \hat{V}(\underset{\sim}{u}_2,\underset{\sim}{u}_1) = \int_0^1 [H(x^{\underset{\sim}{u}_1}(t), x^{\underset{\sim}{u}_1}(t-\tau), \underset{\sim}{u}_2(t), \lambda^{\underset{\sim}{u}_1}(t), t) \tag{2.9}$$

$$-H(x^{\underset{\sim}{u}_1}(t), x^{\underset{\sim}{u}_1}(t-\tau), \underset{\sim}{u}_1(t), \lambda^{\underset{\sim}{u}_1}(t), t)]dt$$

$$= \int_0^1 H(x^{\underset{\sim}{u}_1}(t), x^{\underset{\sim}{u}_1}(t-\tau), \underset{\sim}{u}_2(t) \ominus \underset{\sim}{u}_1(t), \lambda^{\underset{\sim}{u}_1}(t), t)dt$$

is a first order estimate of

$$\Delta V(\underset{\sim}{u}_2,\underset{\sim}{u}_1) \triangleq V(\underset{\sim}{u}_2)-V(\underset{\sim}{u}_1) \tag{2.10}$$

As in Chapter 2 the following proposition which is an extension of Proposition 2.1 in Chapter 2 will be needed for our algorithm to be well defined.

Proposition 2.2

For any $\underset{\sim}{u} \in \underset{\sim}{G}$ there exists a measurable control function $\check{\underset{\sim}{u}} \in \underset{\sim}{G}$ which satisfies

$$\check{\underset{\sim}{u}}(t) = \arg\min_{\underset{\sim}{\omega} \in \underset{\sim}{V}} H(x^{\underset{\sim}{u}}(t), x^{\underset{\sim}{u}}(t-\tau), \underset{\sim}{\omega}, \lambda^{\underset{\sim}{u}}(t), t) \tag{2.11}$$

for a.a. $t \in T$

Note: $\check{\underset{\sim}{u}}$ can always be chosen an ordinary control, see Warga [W3]

Such a $\check{u}$ exists due to similar results as discussed in Chapter 2. We also define by $\check{U}(u)$ the set of all controls $v \in G$ which satisfy

$$v(t) = \arg\min_{\omega \in V} H(x^u(t), x^u(t-\tau), \omega, \lambda^u(t), t) \qquad \text{for a.a.} t \in T$$

As in Chapter 2, we define $\theta_r : G \rightarrow \mathbb{R}$ to be the extension of θ to G by

$$\theta_r(u) = \int_0^1 [H(x^u(t), x^u(t-\tau), \check{u}(t), \lambda^u(t), t) - H(x^u(t), x^u(t-\tau), u(t), \lambda^u(t), t)]dt \tag{2.12}$$

which has the following property:

Proposition 2.3

The relaxed function $\theta : G \rightarrow \mathbb{R}$ (ignoring the subscript r) is sequentially continuous in the sense of control measures (i.s.c.m.).

Proof

Suppose we have an infinite sequence $\{u_i\}_{i=0}^{\infty} \in G$ converging i.s.c.m. to $u^* \in G$, i.e.

$$u_i \xrightarrow[i \to \infty]{} u^* \quad \text{i.s.c.m.}$$

Then we need to show that

$$\theta(u_i) \xrightarrow[i \to \infty]{} \theta(u^*)$$

i.e. $\lim_{i \to \infty} \| \theta(u_i) - \theta(u^*) \| = 0$

From the definition of θ [see (2.12)] we have

$$\theta(\underset{\sim}{u}) = \min_{\underset{\sim}{v} \in \underset{\sim}{G}} \int_0^1 [H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{v}, \lambda^{\underset{\sim}{u}}, t) - H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)]\,dt$$

For all $\underset{\sim}{u} \in \underset{\sim}{G}$ let $\eta(\underset{\sim}{u})$ denote the set of $\underset{\sim}{v} \in \underset{\sim}{G}$ which solves the above equation [note that $\eta(\underset{\sim}{u}) = \check{\underset{\sim}{U}}(\underset{\sim}{u})$], so that

$$\theta(\underset{\sim}{u}) = \int_0^1 [H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t) - H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)]\,dt$$

Hence we have for the above mentioned sequence that

$$\theta(\underset{\sim}{u}_i) = \int_0^1 [H(x^{\underset{\sim}{u}i}, y^{\underset{\sim}{u}i}, \check{\underset{\sim}{u}}_i, \lambda^{\underset{\sim}{u}i}, t) - H(x^{\underset{\sim}{u}i}, y^{\underset{\sim}{u}i}, \underset{\sim}{u}_i, \lambda^{\underset{\sim}{u}i}, t)]\,dt$$

for all $\check{\underset{\sim}{u}}_i \in \eta(\underset{\sim}{u}_i)$, for all i

and

$$\theta(\underset{\sim}{u}^*) = \int_0^1 [H(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \check{\underset{\sim}{u}}^*, \lambda^{\underset{\sim}{u}^*}, t) - H(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \underset{\sim}{u}^*, \lambda^{\underset{\sim}{u}^*}, t)]\,dt$$

for all $\check{\underset{\sim}{u}}^* \in \eta(\underset{\sim}{u}^*)$

Hence we have

$$\theta(\underset{\sim}{u}^*) - \theta(\underset{\sim}{u}_i) \leq \int_0^1 [H(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \check{\underset{\sim}{u}}_i, \lambda^{\underset{\sim}{u}^*}, t) - H(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \underset{\sim}{u}^*, \lambda^{\underset{\sim}{u}^*}, t)]\,dt$$

$$- \int_0^1 [H(x^{\underset{\sim}{u}i}, y^{\underset{\sim}{u}i}, \check{\underset{\sim}{u}}_i, \lambda^{\underset{\sim}{u}i}, t) - H(x^{\underset{\sim}{u}i}, y^{\underset{\sim}{u}i}, \underset{\sim}{u}_i, \lambda^{\underset{\sim}{u}i}, t)]\,dt$$

for all $\check{\underset{\sim}{u}}_i \in \eta(\underset{\sim}{u}_i)$, for all i

and

$$\theta(\underset{\sim}{u}^*) - \theta(\underset{\sim}{u}_i) \geq \int_0^1 [H(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \check{\underset{\sim}{u}}^*, \lambda^{\underset{\sim}{u}^*}, t) - H(x^{\underset{\sim}{u}^*}, y^{\underset{\sim}{u}^*}, \underset{\sim}{u}^*, \lambda^{\underset{\sim}{u}^*}, t)]\,dt$$

$$- \int_0^1 [H(x^{\underset{\sim}{u}i}, y^{\underset{\sim}{u}i}, \check{\underset{\sim}{u}}^*, \lambda^{\underset{\sim}{u}i}, t) - H(x^{\underset{\sim}{u}i}, y^{\underset{\sim}{u}i}, \underset{\sim}{u}_i, \lambda^{\underset{\sim}{u}i}, t)]\,dt$$

for all $\check{\underset{\sim}{u}}^* \in \eta(\underset{\sim}{u}^*)$ for all i

As shown in Appendix A we have

$$x^{\underset{\sim}{u}i}(t) \longrightarrow x^{\underset{\sim}{u}^*}(t) \text{ in } L_\infty \text{ uniformly in } t \in T$$

Also using a similar procedure as in Appendix A we can show that

$$\lambda^{\underset{\sim}{u}_i}(t) \longrightarrow \lambda^{\underset{\sim}{u}^*}(t) \quad \text{in } L_\infty \text{ uniformly in } t \in T$$

Now by making use of arguments similar to those employed in Appendix A, we deduce that:

$$\limsup_{i \to \infty} \| \theta(\underset{\sim}{u}^*) - \theta(\underset{\sim}{u}_i) \| = 0$$

i.e. $\theta(\underset{\sim}{u}_i) \xrightarrow[i \to \infty]{} \theta(\underset{\sim}{u}^*)$, and so θ is sequentially continuous i.s.c.m. This proves the proposition.

The purpose of the algorithms presented in this chapter will be to generate controls $\underset{\sim}{u}^* \in \underset{\sim}{G}$ which satisfy necessary conditions of optimality for the relaxed control problem. It can be shown quite easily from the above discussion that a control $\underset{\sim}{u}^* \in \underset{\sim}{G}$ is optimal for R1 if it satisfies

$$\theta(\underset{\sim}{u}^*) = 0 \qquad (2.13)$$

As in Chapter 2 such a $\underset{\sim}{u}^*$ also satisfies Pontryagin's Maximum Principle stated in Chapter 1.

We will essentially retrace the L_1 analysis in Chapter 2 to show that accumulation points (at least one exists) generated by Algorithm 2 are desirable. The major difference in the two set of results is that we do not possess a metric for $\underset{\sim}{G}$.

We will need the following results which are extensions of results in Section 5 in Chapter 2.

Proposition 2.4

(a) There exists a $d \in (0,\infty)$ such that for all $\underset{\sim}{u} \in \underset{\sim}{G}$

(i) $\| x^{\underset{\sim}{u}}(t) \| \leq d$

(ii) $\| \lambda^{\underset{\sim}{u}}(t) \| \leq d$

for all $t \in T$

(b) If $\{\underset{\sim}{u}_i\}$ is an infinite sequence in $\underset{\sim}{G}$ which converges to $\underset{\sim}{u}^* \in \underset{\sim}{G}$ i.s.c.m. then

(i) $x^{\underset{\sim}{u}_i}(t) \longrightarrow x^{\underset{\sim}{u}^*}(t)$

(ii) $\lambda^{\underset{\sim}{u}_i}(t) \longrightarrow \lambda^{\underset{\sim}{u}^*}(t)$

uniformly in T

Proof

(a)(i) $\| x^{\underset{\sim}{u}}(t) \| \leq d$ can be deduced as in Chapter 2 using the fact that

$$\begin{aligned}\|f_r(x,y,\underset{\sim}{u},t)\| &= \| \int_\Omega f(x,y,u,t)\ d\underset{\sim}{u}(u) \| \\ &\leq \| \ M(\|x\|+\|y\|+1) \int_\Omega d\underset{\sim}{u}(u) \ \| \\ &= M\{ \|x\| + \|y\| + 1 \}\end{aligned}$$

(ii) Using this bound on $x^{\underset{\sim}{u}}(.)$ we can use the same procedure as in Proposition 5.2 in Chapter 2 to show that

$$\| \lambda^{\underset{\sim}{u}}(t) \| \leq d \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{G}, \text{ all } t \in T$$

(b)(i) See Appendix A

Proof of (ii) follows using the same procedure.

Proposition 2.5

For all $\underset{\sim}{u}$, $\underset{\sim}{v} \in \underset{\sim}{G}$, all $\alpha \in [0,1]$ and with $\underset{\sim}{u}_\alpha$ defined as the relaxed convex combination of $\underset{\sim}{u}$ and $\underset{\sim}{v}$, i.e.

$$\underset{\sim}{u}_\alpha(t) = (1-\alpha)\underset{\sim}{u}(t) \oplus \alpha\underset{\sim}{v}(t) \qquad \text{we have}$$

$$\| x^{\underset{\sim}{u}_\alpha}(t) - x^{\underset{\sim}{u}}(t) \| \leq d\alpha \qquad \text{for all } t \in T$$

for some $d \in (0,\infty)$.

Proof

By definition of $x(.)$ we have

$$x^{\underset{\sim}{u}_\alpha}(t)-x^{\underset{\sim}{u}}(t) = \int_0^t [f(x^{\underset{\sim}{u}_\alpha}(s), x^{\underset{\sim}{u}_\alpha}(s-\tau), \underset{\sim}{u}_\alpha(s), s)$$

$$-f(x^{\underset{\sim}{u}}(s), x^{\underset{\sim}{u}}(s-\tau), \underset{\sim}{u}(s), s)]ds \quad \text{for all } t \in T$$

$$= \int_0^1 [(1-\alpha)f(x^{\underset{\sim}{u}_\alpha}, y^{\underset{\sim}{u}_\alpha}, \underset{\sim}{u}, s) + \alpha f(x^{\underset{\sim}{u}_\alpha}, y^{\underset{\sim}{u}_\alpha}, \underset{\sim}{v}, s)$$

$$-f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, s)]\,ds \qquad \text{for all } t \in T$$

Expanding $f(x^{\underset{\sim}{u}_\alpha}, y^{\underset{\sim}{u}_\alpha}, \underset{\sim}{u}, s)$ and $f(x^{\underset{\sim}{u}_\alpha}, y^{\underset{\sim}{u}_\alpha}, \underset{\sim}{v}, s)$ about $x^{\underset{\sim}{u}}(s)$, $x^{\underset{\sim}{u}}(s-\tau)$ we get to "first order" #:

$$x^{\underset{\sim}{u}_\alpha}(t)-x^{\underset{\sim}{u}}(t) = \int_0^t \{\alpha[f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{v}, s) - f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, s)]$$

$$+(1-\alpha)f_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, s)(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}})$$

$$+(1-\alpha)f_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, s)(y^{\underset{\sim}{u}_\alpha} - y^{\underset{\sim}{u}})$$

$$+\alpha f_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{v}, s)(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}})$$

$$+\alpha f_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{v}, s)(y^{\underset{\sim}{u}_\alpha} - y^{\underset{\sim}{u}})\}ds \qquad \text{for all } t \in T$$

By Assumption 2 and boundedness of f, f_x, f_y on $B_1 \times B_1 \times \underset{\sim}{V} \times T$,

where $$B_1 = \{x \in \mathbb{R}^n : \|x\| \leq d\},$$

The development here is not entirely rigorous since we have omitted the "remainder term"; it can be made so by arguments similar to those used in Proposition 5.4 in Chapter 2.

there exist $d_1, d_2 \in (0, \infty)$ such that

$$\| x^{u_\alpha}(t) - x^{u}(t) \| \leq \int_0^t d_1 \alpha ds + d_2 \int_0^t \| x^{u_\alpha}(s) - x^{u}(s) \| ds$$

By an application of Gronwall's inequality we get

$$\| x^{u_\alpha}(t) - x^{u}(t) \| \leq d\alpha \qquad \text{for all } t \in T$$

as required.

We now present Algorithm 2 which is also based on the algorithm model in section 3 in Chapter 2.

3.3 Algorithm 2 (For Solving Problem R1)

Step 0 : Select a $u_0 \in G$

Step 1 : Set i=0

Step 2 : Compute $x^{u_i}(.)$ by solving (2.3) and (2.4)

Step 3 : Compute $\lambda^{u_i}(.)$ by solving (2.5)-(2.7)

Step 4 : Compute a $\check{u}_i \in \check{U}(u_i)$

Step 5 : Compute $\theta(u_i)$ using (2.12)

If $\theta(u_i) = 0$ Stop,

Else continue

Step 6 : Define $u_\alpha(.)$ as

$u_\alpha(t) \triangleq (1-\alpha)u_i(t) \oplus \alpha\check{u}_i(t)$

Compute $\alpha_i \in [0,1]$ as the largest number which satisfies

$$\Delta V(u_{\alpha_i}, u_i) \leq \frac{\alpha_i \theta(u_i)}{2}$$

Step 7 : Set $u_{i+1} = u_{\alpha_i}$

Set i = i+1

Goto Step 2

Remark

1. The step length α_i defined in Step 6 is a function of $\underset{\sim}{u}_i$ and $\check{\underset{\sim}{u}}_i$ as in Chapter 2 but we do not state this explicitly for convenience

2. The interval $I_{\alpha u}$ in Chapter 2 is not needed since the linear nature of relaxed controls allow a descent property at each nondesirable $\underset{\sim}{u}$.

   This will become apparent in the following Theorem which is the main result in this section and states the convergence properties of Algorithm 2.

Theorem 2

Suppose Assumptions 1 and 2 in Chapter 2 hold and that Algorithm 2 generates a sequence of (relaxed) controls $\{\underset{\sim}{u}_i\}$. This sequence is either finite, in which case the last control is desirable, or it is infinite and every limit point, $\underset{\sim}{u}^*$ in the sense of control measures (at least one exists), is desirable.

3.4 Proof of Theorem 2

The proof of Theorem 2 will be obtained using the same procedure as in proving Theorem 1 in Chapter 2. We will need the following:

Lemma 4.1

For all $\alpha \in [0,1]$, for all $\underset{\sim}{u}, \check{\underset{\sim}{u}} \in \underset{\sim}{G}$ such that $\check{\underset{\sim}{u}} \in \check{\underset{\sim}{U}}(\underset{\sim}{u})$ we have

$$\Delta \hat{V}(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) = \alpha \theta(\underset{\sim}{u})$$

where $\underset{\sim}{u}_\alpha$ is as defined above.

Proof

By definition [see (1.1) and (2.9)] we have

$$u_\alpha(t) = (1-\alpha)u(t) \oplus \alpha\check{u}(t) \qquad \text{for } \alpha \in [0,1], \text{ all } t \in T$$

and

$$\Delta\hat{V}(u_\alpha, u) = \int_0^1 [H(x^u, y^u, u_\alpha, \lambda^u, t) - H(x^u, y^u, u, \lambda^u, t)]\,dt$$

i.e. $$\Delta\hat{V}(u_\alpha, u) = \int_0^1 [(1-\alpha)H(x^u, y^u, u, \lambda^u, t) + \alpha H(x^u, y^u, \check{u}, \lambda^u, t)$$

$$-H(x^u, y^u, u, \lambda^u, t)]\,dt$$

Therefore

$$\Delta\hat{V}(u_\alpha, u) = \alpha\int_0^1 [H(x^u, y^u, \check{u}, \lambda^u, t) - H(x^u, y^u, u, \lambda^u, t)]\,dt$$

$$\Delta\hat{V}(u_\alpha, u) = \alpha\theta(u)$$

by (2.12) as required.

Remark

Note that as in classical methods the estimate of the change in cost when the old control is replaced by the new control is a linear function of the step length α.

Lemma 4.2

For all $\alpha \in [0,1]$, for all $u, \check{u} \in G$ such that $\check{u} \in \check{U}(u)$ we have

$$\| \Delta V(u_\alpha, u) - \Delta\hat{V}(u_\alpha, u) \| \leq d\alpha^2$$

for some $d \in (0, \infty)$.

Proof

By definition

$$\Delta V(u_\alpha, u) = \int_0^1 [l(x^{u_\alpha}(t), x^{u_\alpha}(t-\tau), u_\alpha(t), t) - l(x^u(t), x^u(t-\tau), u(t), t)]\,dt$$

Adding and subtracting terms we get by the definition of the Hamiltonian [see (2.8)]

$$\Delta V(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) = \int_0^1 [H(x^{\underset{\sim}{u}_\alpha}, y^{\underset{\sim}{u}_\alpha}, \underset{\sim}{u}_\alpha, \lambda^{\underset{\sim}{u}}, t) - H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)]dt$$

$$- \int_0^1 \lambda^{\underset{\sim}{u}}(t)^T [\dot{x}^{\underset{\sim}{u}_\alpha}(t) - \dot{x}^{\underset{\sim}{u}}(t)]\, dt$$

Integrating last term by parts we get the following

$$\int_0^1 \lambda^{\underset{\sim}{u}T}(\dot{x}^{\underset{\sim}{u}_\alpha} - \dot{x}^{\underset{\sim}{u}})dt = [\lambda^{\underset{\sim}{u}T}(t)(x^{\underset{\sim}{u}_\alpha}(t) - x^{\underset{\sim}{u}}(t))]_0^1 - \int_0^1 \dot{\lambda}^{\underset{\sim}{u}T}(t)(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}})dt$$

$\lambda(1) = 0$ and $x^{\underset{\sim}{u}_\alpha}(0) = x^{\underset{\sim}{u}}(0)$

Also using Taylor series to expand the Hamiltonian to "first approximation" we get (see footnote on page 89):

$$\Delta V(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) = \int_0^1 [H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}_\alpha, \lambda^{\underset{\sim}{u}}, t) + H_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}_\alpha, \lambda^{\underset{\sim}{u}}, t)(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}})$$

$$+ H_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}_\alpha, \lambda^{\underset{\sim}{u}}, t)(y^{\underset{\sim}{u}_\alpha} - y^{\underset{\sim}{u}}) - H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)$$

$$+ \dot{\lambda}^{\underset{\sim}{u}T}(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}})]dt$$

Using $\underset{\sim}{u}_\alpha = (1-\alpha)\underset{\sim}{u} \oplus \alpha\check{\underset{\sim}{u}}$ and the definition of $\lambda^{\underset{\sim}{u}}$ we have that

$$\Delta V(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) = \int_0^1 \{(1-\alpha)H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t) + \alpha H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t)$$

$$- H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)$$

$$+ (1-\alpha)H_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}}) + \alpha H_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t) \cdot$$

$$\cdot\, (x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}})$$

$$+ (1-\alpha)H_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)(y^{\underset{\sim}{u}_\alpha} - y^{\underset{\sim}{u}}) + \alpha H_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t) \cdot$$

$$\cdot\, (y^{\underset{\sim}{u}_\alpha} - y^{\underset{\sim}{u}})$$

$$- H_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}})\}dt$$

$$- \int_0^{1-\tau} H_y(x^{\underset{\sim}{u}}(t+\tau), x^{\underset{\sim}{u}}(t), \underset{\sim}{u}(t+\tau), \lambda^{\underset{\sim}{u}}(t+\tau), t+\tau)(x^{\underset{\sim}{u}_\alpha}(t) - x^{\underset{\sim}{u}}(t))dt$$

As in the proof of Proposition 5.4 in Chapter 2 we have

$$\int_0^{1-\tau} H_y(x^{\underset{\sim}{u}}(t+\tau), x^{\underset{\sim}{u}}(t), \underset{\sim}{u}(t+\tau), \lambda^{\underset{\sim}{u}}(t+\tau), t+\tau)(x^{\underset{\sim}{u}\alpha}(t)-x^{\underset{\sim}{u}}(t))dt$$

$$= \int_0^1 H_y(x^{\underset{\sim}{u}}(t), x^{\underset{\sim}{u}}(t-\tau), \underset{\sim}{u}(t), \lambda^{\underset{\sim}{u}}(t), t)(x^{\underset{\sim}{u}\alpha}(t-\tau)-x^{\underset{\sim}{u}}(t-\tau))dt$$

Hence we get

$$\begin{aligned}\Delta V(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) = {} & \alpha \int_0^1 [H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t) - H(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)]\,dt \\ & + \alpha \int_0^1 [H_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t) - H_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)](x^{\underset{\sim}{u}\alpha} - x^{\underset{\sim}{u}})dt \\ & + \alpha \int_0^1 [H_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t) - H_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)](y^{\underset{\sim}{u}\alpha} - y^{\underset{\sim}{u}})dt\end{aligned}$$

Using the definition of θ we have

$$\begin{aligned}||\Delta V(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) - \alpha\theta(\underset{\sim}{u})|| \le {} & \alpha \int_0^1 ||H_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t) - H_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)|| \cdot \\ & \cdot ||x^{\underset{\sim}{u}\alpha} - x^{\underset{\sim}{u}}||\,dt \\ & + \alpha \int_0^1 ||H_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t) - H_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t)||\,||y^{\underset{\sim}{u}\alpha} - y^{\underset{\sim}{u}}||\,dt\end{aligned}$$

Using Lemma 4.1 we get

$$\begin{aligned}||\Delta V(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) - \widehat{\Delta V}(\underset{\sim}{u}_\alpha, \underset{\sim}{u})|| \le {} & \alpha \int_0^1 \{||H_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t)|| + ||H_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t||\} \cdot \\ & \cdot ||x^{\underset{\sim}{u}\alpha} - x^{\underset{\sim}{u}}||\,dt \\ & + \alpha \int_0^1 \{||H_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}}, \lambda^{\underset{\sim}{u}}, t)|| + ||H_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, \lambda^{\underset{\sim}{u}}, t||\} \cdot \\ & \cdot ||y^{\underset{\sim}{u}\alpha} - y^{\underset{\sim}{u}}||\,dt\end{aligned}$$

Since H_x and H_y are bounded on the compact set $B_1 x B_1 x \underset{\sim}{V} x T$, there exist constants $d_1, d_2 \in (0, \infty)$ such that

$$\begin{aligned}||\Delta V(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) - \widehat{\Delta V}(\underset{\sim}{u}_\alpha, \underset{\sim}{u})|| \le {} & 2\alpha d_1 \int_0^1 ||x^{\underset{\sim}{u}\alpha}(t) - x^{\underset{\sim}{u}}(t)||\,dt \\ & + 2\alpha d_2 \int_0^1 ||x^{\underset{\sim}{u}\alpha}(t-\tau) - x^{\underset{\sim}{u}}(t-\tau)||\,dt\end{aligned}$$

i.e. we have

$$\| \Delta V(\underset{\sim}{u}_{\alpha},\underset{\sim}{u})-\Delta \hat{V}(\underset{\sim}{u}_{\alpha},\underset{\sim}{u}) \| \leq d_3 \ \alpha \int_0^1 \| x^{\underset{\sim}{u}_{\alpha}}(t)-x^{\underset{\sim}{u}}(t) \| \, dt$$

for some finite d_3

From Proposition 2.5 we have

$$\| x^{\underset{\sim}{u}_{\alpha}}(t) - x^{\underset{\sim}{u}}(t) \| \leq d_4 \alpha \qquad \text{for all } t \in T$$

for some $d_4 \in (0, \infty)$

Hence we deduce that

$$\| \Delta V(\underset{\sim}{u}_{\alpha},\underset{\sim}{u})-\Delta \hat{V}(\underset{\sim}{u}_{\alpha},\underset{\sim}{u}) \| \leq d \alpha^2$$

as required.

Note

Again using Lemma 4.1 this gives us

$$\Delta V(\underset{\sim}{u}_{\alpha},\underset{\sim}{u}) \leq \alpha \theta (\underset{\sim}{u}) + d \alpha^2$$

This is precisely the inequality obtained in Chapter 2, where a complicated procedure was needed to define the special interval $I_{\alpha u}$ which guaranteed convergence of Algorithm 1.

Using the same procedures as in Chapter 2 we can deduce the following result:

Corollary 4.3

Let $\underset{\sim}{A}:\underset{\sim}{G} \longrightarrow 2^{\underset{\sim}{G}}$ be the map defined by Algorithm 2, then for all $\underset{\sim}{u} \in \underset{\sim}{G}$ there exists a $c \in (0,\infty)$ such that:

(i) $\Delta V(\underset{\sim}{v},\underset{\sim}{u}) \leq -[\theta(\underset{\sim}{u})]^2/4c$ for all $\underset{\sim}{v} \in \underset{\sim}{A}(\underset{\sim}{u})$

(ii) $\alpha_i \geq -\dfrac{\theta(u)}{2c}$ where α_i is the step length defined in Step 6 of Algorithm 2

Remark

From Corollary 4.3 we deduce that Algorithm 2 is well defined in that if it is at an undesirable control $\underset{\sim}{u}$ (i.e. $\theta(\underset{\sim}{u}) < 0$) a descent property exists and it can move towards a "better" control.

We now prove Theorem 2, which we restate here for convenience.

Theorem 2

Suppose Assumptions 1 and 2 in Chapter 2 hold and that Algorithm 2 generates a sequence of (relaxed) controls $\{\underset{\sim}{u}_i\}$. This sequence is either finite, in which case the last element is desirable, or it is infinite and every limit point, $\underset{\sim}{u}^*$ in the sense of control measures (at least one exists), is desirable.

Proof

If Algorithm 2 generates a finite sequence of controls it is trivally seen from Step 5 of the algorithm that the last element, $\underset{\sim}{u}_k$, satisfies $\theta(\underset{\sim}{u}_k) = 0$ since algorithm terminates. Hence $\underset{\sim}{u}_k$ is desirable.

Now assume that Algorithm 2 generates an infinite sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$. By our results in Appendix A we have that this sequence has at least one accumulation point $\underset{\sim}{u}^* \in \underset{\sim}{G}$; i.e. there exists a subsequence indexed by $K \in \{0,1,2,\ldots\}$ such that

$$\underset{\sim}{u}_i \xrightarrow{K} \underset{\sim}{u}^* \qquad \text{i.s.c.m}$$

By the sequential continuity of θ we have

$$\theta(\underset{\sim}{u}_i) \xrightarrow{K} \theta(\underset{\sim}{u}^*)$$

Assume to the contrary that $\theta(\underset{\sim}{u}^*) = -\delta < 0$. Then there exists an i_0 such that

$$\theta(\underset{\sim}{u}_i) \leq \frac{\theta(\underset{\sim}{u}^*)}{2} = -\delta/_2 \qquad \text{for all } i \geq i_0, \quad i \in K$$

By Corollary 4.3 we have

$$V(\underset{\sim}{u}^*) - V(\underset{\sim}{u}_{i_0}) = \sum_{\substack{i \in K \\ i \geq i_0}} V(\underset{\sim}{u}_{i+1}) - V(\underset{\sim}{u}_i)$$

$$+ \sum_{\substack{i \notin K \\ i \geq i_0}} V(\underset{\sim}{u}_{i+1}) - V(\underset{\sim}{u}_i)$$

i.e. $$V(\underset{\sim}{u}^*) - V(\underset{\sim}{u}_{i_0}) \leq \sum_{\substack{i \in K \\ i \geq i_0}} V(\underset{\sim}{u}_{i+1}) - V(\underset{\sim}{u}_i)$$

$$= \sum_{\substack{i \in K \\ i \geq i_0}} \Delta V(\underset{\sim}{u}_{\alpha_i}, \underset{\sim}{u}_i)$$

$$\leq \sum_{\substack{i \in K \\ i \geq i_0}} - \frac{\delta^2}{16c} \cdot i$$

$$\longrightarrow -\infty$$

but $\{V(\underset{\sim}{u}_i)\}_i$ is a bounded monotonically decreasing sequence which converges to $V(\underset{\sim}{u}^*)$. This contradicts the above result. Hence assumption $\theta(\underset{\sim}{u}^*) < 0$ is false and so we must have $\theta(\underset{\sim}{u}^*) = 0$, i.e. $\underset{\sim}{u}^*$ is desirable.

3.5 Approximation to Relaxed Control Problem

The main objection to Algorithms 1 and 2 is that they require exact minimization of a Hamiltonian at an infinite number of points at each iteration. This can be overcome using the methods in Mayne and Polak [MAP1] to derive an implementable algorithm for our delay cases. However, in Algorithm 2 a further complication is quite evident, viz., relaxed controls have to be programmed, and these by there nature (measures at different controls for all $t \in T$) can be very expensive to represent on a computer. We propose here to approximate the relaxed controls (to any required accuracy) using ordinary controls, which are much easier to simulate, and hence make Algorithm 2 more implementable. This approximation will be performed using the method proposed by Gamkrelidze [G1] where the time interval is partitioned into disjoint segments I_i, $i = 1,2,3,....$ and assigning the new control to be the old control u for part of the interval I_i for each i, and for the rest letting it be the minimising control $\check{u}$. It will become obvious that in this case the new control $u_\alpha(\varepsilon,t)$ is not only dependent on the step length α, as before, but also on the accuracy of the approximation ε, i.e. the partition size. The ratio as to how each interval I_i is subdivided is dependent on the step length.

Approximating the relaxed controls, together with a slightly different method for choosing the step length will give Algorithm 3. A different method is used to determine α so that it can be seen that a choice as to how certain operations are performed does exist as long as the overall algorithm fits the model on which it is based.

We will now describe how the new control $u_\alpha(\varepsilon,t)$ is generated from an old control u, which we assume is not optimal.

<u>Note</u>

We can assume, without loss in generality, that the old control u(t) is an ordinary control. This is because the space of ordinary controls G is dense in the space of relaxed controls $\underset{\sim}{G}$. Therefore any control $\underset{\sim}{u}(t) \in \underset{\sim}{G}$ can be approximated to any degree of accuracy using an ordinary control $u(t) \in G$. [See Warga [W3]].

The control $\check{\underset{\sim}{u}} \in \underset{\sim}{G}$ which solves

$$\check{\underset{\sim}{u}}(t) = \min_{\underset{\sim}{w} \in \underset{\sim}{V}} H(x^u(t), x^u(t-\tau), \underset{\sim}{w}, \lambda^u(t), t) \qquad \text{for a.a. } t \in T$$

is found. This can be approximated by $\check{u} \in G$ to give $\delta_{\check{u}}(t)$ which minimises the Hamiltonian, where $\delta_{\check{u}}(t)$ is the Dirac measures concentrated at $\check{u} \in G$ for all $t \in T$. The relaxed control $\underset{\sim}{u}_\alpha$ for $\alpha \in [0,1]$ is defined as the relaxed convex combination of the controls u and $\check{u}$, i.e.

$$\underset{\sim}{u}_\alpha(t) = (1-\alpha)u(t) \oplus \alpha\check{u}(t) \tag{5.1}$$

Then the step length $\alpha(u,\check{u}) \in [0,1]$ is determined by minimising the cost function

$$V(\underset{\sim}{u}_\alpha) = \int_0^1 l(x^{\underset{\sim}{u}_\alpha}(t), x^{\underset{\sim}{u}_\alpha}(t-\tau), \underset{\sim}{u}_\alpha(t), t)dt$$

over α.

Where $x^{\underset{\sim}{u}_\alpha}: T \to \mathbb{R}^n$ is the solution of the delay-differential equation:

$$\dot{x}(t) = (1-\alpha)f(x(t), x(t-\tau), u(t), t) + \alpha f(x(t), x(t-\tau), \check{u}(t), t) \qquad \text{for a.a. } t \in T$$

$$x(t) = \phi(t) \qquad \text{for } t \in [-\tau, 0]$$

Thus having obtained the relaxed control $\underset{\sim}{u}_\alpha(u,\check{u})$ (written $\underset{\sim}{u}_\alpha$ for convenience), an approximation (to any degree of accuracy) to it is made incorporating only ordinary controls, Gamkrelidze [G1].

We now present the method (subprocedure A) which describes how the relaxed control is approximated.

Subprocedure A

A number $\varepsilon > 0$ is chosen depending on the required accuracy.

Now for any $\varepsilon > 0$, an integer N is defined which equals the smallest number of disjoint intervals of T such that conditions (5.3)-(5.4) stated below hold. For further details see also Lemma (7.1).

Once N has been defined, partition the time interval into disjoint segments I_i, i=1,2,.....,N. Then further subdivide each segment I_i into the ratio $(1-\alpha):\alpha$ and denote the respective sections as $I_{i,1}$ and $I_{i,2}$ for i=1,2,.....,N.

Now $\mu(I_{i,1}) = \int_{I_i}(1-\alpha)dt = (1-\alpha)\,|I_i|$

and $\mu(I_{i,2}) = \int_{I_i}\alpha dt = \alpha\,|I_i|$

Then define the approximation, $u_\alpha(\varepsilon)$, to the relaxed control $\underset{\sim}{u}_\alpha$ by

$$u_\alpha(\varepsilon,t) = u(t) \quad \text{for } t \in I_{i,1} \tag{5.2}$$

$$= \check{u}(t) \quad \text{for } t \in I_{i,2}$$

for i=1,2,.....,N

[Note that the approximation is dependent on the chosen accuracy.]

This is a valid approximation if the trajectories $x^{u_\alpha(\varepsilon)}(t)$ and $x^{\underset{\sim}{u}_\alpha}(t)$, due to the ordinary approximating control and the actual relaxed control respectively, approach one another uniformly over all $u, \check{u} \in G$, for all $t \in T$ and for all $\alpha \in [0,1]$ as the partition fineness is increased towards infinity. This is proved to be true in Lemma 7.1.

A method for determining the integer N (the number of partitions necessary to obtain the accuracy required) is still needed. This can be obtained using the following method:

Given an $\varepsilon > 0$ the time interval T is divided into disjoint segments I_i, $i = 1,2,3,\dots,N$ such that

$$\| f(x,y,u,t^1) - f(x,y,u,t^2) \| \leq \varepsilon \tag{5.3}$$

for all t^1, t^2 in the same interval I_i and for all $x,y \in B_1$ and for all $u \in G$,

and

$$\int_{I_i} M \{ \| x \| + \| y \| + 1 \} dt \leq \varepsilon \tag{5.4}$$

From Assumptions 1 and 2 the above are easily seen to hold, and N can be easily determined (see also Lemma 7.1).

This completes the description of subprocedure A, and we are now in a position to present Algorithm 3. Before doing so, however, it is worth mentioning that if the above approximating method is used in any algorithm, two cases of interest arise straight away, i.e.

1. Once N has been found (depending on the ε chosen) it can be kept constant throughout the implementation of the algorithm. If this is done, the best one can hope for is that any accumulation points generated will satisfy optimality conditions to within "delta", where δ will be dependent on the ε chosen.

2. For the second case, N can be initially set at N_0 as for Case 1, but it is increased at each iteration of the algorithm and is therefore a monotonically increasing sequence $\{N_j\}_{j=0}^{\infty}$ (e.g. such a sequence can be generated by $N_{j+1} = N_j + w$, $w > 0$ or $N_{j+1} = wN_j$, $w > 1$). This means as the algorithm proceeds the approximations to the relaxed controls become more and more refined, and hence epsilon approaches zero, i.e. as $N \longrightarrow \infty$, the partition size becomes infinitely small and $\varepsilon \longrightarrow 0$.

   In this case one would hope that the limit points satisfy optimality conditions "exactly", i.e. any limit control u^* should satisfy $\theta(u^*) = 0$ in the limit as $N \longrightarrow \infty$.

Both of the above cases will be studied and it is shown that the above conjectures do in fact hold.

3.6 Algorithm 3

Step 0 : Select a $u_0 \in G$, $\varepsilon > 0$

Step 1 : Set $i = 0$

Step 2 : Compute $x^{u_i}(.)$ by solving (2.1) and (2.2) in Chapter 2

Step 3 : Compute $\lambda^{u_i}(.)$ by solving (2.7)-(2.9) in Chapter 2

Step 4 : Compute a $\delta_{\check{u}_i}(t) \in \check{\underset{\sim}{U}}(u_i(t))$ [see text]

Step 5 : Compute $\theta(u_i)$ using (2.13) in Chapter 2

If $\theta(u_i) \geq -\varepsilon$ Stop

Else continue

Step 6 : Define for $\alpha \in [0,1]$ the control $\underset{\sim}{u}_\alpha$ by

$\underset{\sim}{u}_\alpha(t) = (1-\alpha)u_i(t) \oplus \alpha \check{u}_i(t)$

Compute $\alpha_i \in [0,1]$ which minimises $V(\underset{\sim}{u}_\alpha)$ and calculate the new relaxed control $\underset{\sim}{u}_{\alpha_i}$.

Step 7 : Compute the ordinary control $u_{\alpha_i}(\varepsilon,t)$ which approximates $\underset{\sim}{u}_{\alpha_i}$ by using subprocedure A with the degree of accuracy set at ε.

Step 8 : Set $u_{i+1} = u_{\alpha_i}(\varepsilon)$

Set $i = i+1$

Goto Step 2

Algorithm 3, above is for case 1 described in the preceding section where the partition mesh is kept constant at a chosen value. The algorithm can be modified very easily to incorporate case 2. We will briefly state the changes which are required to make this modification.

In Step 0 instead of needing a $\varepsilon > 0$, a monotone decreasing sequence $\{\varepsilon_i\}_{i=0}^{\infty}$ is required, where $\varepsilon_0 > 0$, $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 \ldots\ldots$ and $\lim_i \varepsilon_i = 0$.

The stopping condition in Step 5 should be modified to

Stop if $\theta(u_i) = 0$, Else continue.

In Step 7 the degree of accuracy will be dependent on ε_i (i.e. the number of the partitioning increases as the algorithm proceeds), and the ε_i has to be updated to ε_{i+1} in Step 8 at each iteration.

The convergence properties of Algorithm 3 will now be stated. We will consider case 1 and case 2 separately and show that Algorithm 3 has the following properties.

Theorem 3

Let Assumptions 1 and 2 in Chapter 2 be satisfied and suppose Algorithm 3 generates a sequence $\{u_i\}$ of (ordinary) controls. Then we have:

Case 1

Given any $\delta>0$, N (and hence $\varepsilon>0$) may be chosen such that the sequence is either finite, in which case the last element is desirable to within "delta", or it is infinite and every accumulation point u* generated by the algorithm satisfies an optimality condition to within "delta", i.e. $\theta(u^*) \geq -\delta$.

Case 2

That the sequence is either finite, in which case the last element is desirable (exactly), or it is infinite and every accumulation point u* satisfies optimality conditions "exactly", i.e. $\theta(u^*) = 0$.

Remark

For both cases, since the optimization is still being done over the space of relaxed controls the existence of accumulation points is guaranteed.

3.7 Proof of Theorem 3

Before we can attempt to prove any property that Algorithm

3 might have we must show that the approximation described by subprocedure A is valid in some sense. For this reason we present the following:

Lemma 7.1

The trajectories $x^{u_\alpha(\varepsilon)}(t)$ and $x^{\underset{\sim}{u}_\alpha}(t)$, due to the ordinary control $u_\alpha(\varepsilon,t)$ [as defined by (5.2)] and the relaxed control $\underset{\sim}{u}_\alpha$ [as defined by (5.1)] respectively, approach one another uniformly over all $u,\check{u} \in G$, for all $t \in T$ and all $\alpha \in [0,1]$ as the partition fineness is increased towards infinity, i.e.

$$\sup_{\substack{u,\check{u} \in G \\ \alpha \in [0,1]}} \| x^{u_\alpha(\varepsilon)}(t) - x^{\underset{\sim}{u}_\alpha}(t) \| \longrightarrow 0$$

uniformly in $t \in T$ as $\varepsilon \to 0$

Proof

Since $x^{u_\alpha(\varepsilon)}(.)$ is the solution of the delay-differential equation

$$\dot{x}(t) = f(x(t),x(t-\tau),\ u_\alpha(\varepsilon,t),t) \qquad \text{for a.a.} t \in T$$

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau,0]$$

we get

$$x^{u_\alpha(\varepsilon)}(t) = \phi(0) + \int_0^t f(x^{u_\alpha(\varepsilon)}(s),x^{u_\alpha(\varepsilon)}(s-\tau),u_\alpha(\varepsilon,s),s)ds$$

for all $t \in T$

Similarly we get

$$x^{\underset{\sim}{u}_\alpha}(t) = \phi(0) + \int_0^t f(x^{\underset{\sim}{u}_\alpha}(s),x^{\underset{\sim}{u}_\alpha}(s-\tau),\underset{\sim}{u}_\alpha(s),s)\ ds \text{ for all } t \in T$$

Therefore we get

$$x^{u_\alpha(\varepsilon)}(t)-x^{\underset{\sim}{u}_\alpha}(t) = \int_0^t [f(x^{u_\alpha(\varepsilon)}(s),x^{u_\alpha(\varepsilon)}(s-\tau),u_\alpha(\varepsilon,s),s)$$

$$-f(x^{\underset{\sim}{u}_\alpha}(s),x^{\underset{\sim}{u}_\alpha}(s-\tau),\underset{\sim}{u}_\alpha(s),s)]ds \text{ for all } t \in T$$

i.e

$$\| x^{u_\alpha(\varepsilon)}(t)-x^{\underset{\sim}{u}_\alpha}(t) \| \le \| \int_0^t [f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), u_\alpha(\varepsilon,s),s)$$

$$-f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), \underset{\sim}{u}_\alpha(s),s)]ds \|$$

$$+ \| \int_0^t [f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), \underset{\sim}{u}_\alpha(s),s)$$

$$-f(x^{\underset{\sim}{u}_\alpha}(s), x^{\underset{\sim}{u}_\alpha}(s-\tau), \underset{\sim}{u}_\alpha(s),s)]ds \|$$

Now by Assumption 2 we have

$$\| \int_0^t [f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), \underset{\sim}{u}_\alpha(s),s)$$

$$-f(x^{\underset{\sim}{u}_\alpha}(s), x^{\underset{\sim}{u}_\alpha}(s-\tau), \underset{\sim}{u}_\alpha(s),s)]ds \|$$

$$\le M \int_0^t \{ \|x^{u_\alpha(\varepsilon)}(s)-x^{\underset{\sim}{u}_\alpha}(s)\| + \| x^{u_\alpha(\varepsilon)}(s-\tau)$$

$$-x^{\underset{\sim}{u}_\alpha}(s-\tau) \| \} ds$$

$$\le d_1 \int_0^t \|x^{u_\alpha(\varepsilon)}(s)-x^{\underset{\sim}{u}_\alpha}(s)\| ds \qquad \text{for all } t \in T$$

for some $d_1 \in (0, \infty)$.

We will now estimate the term

$$\| \int_0^t [f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), u_\alpha(\varepsilon,s),s)$$
$$-f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), \underset{\sim}{u}_\alpha(s),s)] ds \|$$

using arguments essentially due to Gamkrelidze [G1] .

By Assumption 1 and 2, we have that for a given $\varepsilon > 0$, there exists an integer N, equal to the smallest number of disjoint intervals I_i, i=1,2,...,N, which is a subdivision of the time interval [0,1], such that

$$\| f(x,y,u,t') - f(x,y,u,t'') \| \leq \varepsilon$$

$$\| f(x,y,\check{u},t_1) - f(x,y,\check{u},t_2) \| \leq \varepsilon$$

for all t', t'' and all t_1, t_2 in the same interval I_i for all $u, \check{u} \in G$ and for all $x, y \in B_1$ and

$$\int_{I_i} f(x,y,u,t)dt \leq \int_{I_i} M\{\|x\| + \|y\| + 1\} dt \leq \varepsilon$$

We will start by estimating

$$\sum_{i=1}^{L} \| \int_{I_i} [f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), u_\alpha(\varepsilon, s), s)$$

$$-f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), \underset{\sim}{u}_\alpha(s), s)]ds \|$$

for some integer L

We will use the fact that

$$\int_{I_i} f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), u_\alpha(\varepsilon, s), s)ds$$

$$= \int_{I_{i,1}} f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), u(s), s)ds$$

$$+ \int_{I_{i,2}} f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), \check{u}(s), s)ds$$

for each i

Therefore we get

$$\| \int_{I_i} [f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), u_\alpha(\varepsilon, s), s)$$

$$-f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), \underset{\sim}{u}_\alpha(s), s)]ds \|$$

$$= \| \int_{I_{i,1}} f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), u(s), s)ds$$

$$+ \int_{I_{i,2}} f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), \check{u}(s), s)ds$$

$$- \int_{I_i} [(1-\alpha) f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), u(s), s)$$

$$+ \alpha f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), \check{u}(s), s)] ds \; ||$$

Let $f_1(x,y,s) = f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), u(s), s)$

$f_2(x,y,s) = f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), \check{u}(s), s)$

$\alpha_1 = (1-\alpha)$

and $\alpha_2 = \alpha$

Then the above becomes

$$|| \int_{I_{i,1}} f_1(x,y,s) ds + \int_{I_{i,2}} f_2(x,y,s) ds$$

$$- \int_{I_i} [\alpha_1 f_1(x,y,s) + \alpha_2 f_2(x,y,s)] ds \; ||$$

$$= || \sum_{j=1}^{2} \int_{I_{i,j}} f_j(x,y,s) ds - \int_{I_i} \sum_{j=1}^{2} \alpha_j f_j(x,y,s) ds \; ||$$

Let t_i be any point in I_i, $i=1,2,\ldots,N$ and denote $f_j(x,y,t_i)$ by $f_{ij}(x,y)$

Then

$$\int_{I_i} \sum_{j=1}^{2} \alpha_j f_{ij}(x,y) ds = \sum_{j=1}^{2} f_{ij}(x,y) \int_{I_i} \alpha_j \, ds$$

$$= \sum_{j=1}^{2} f_{ij}(x,y) \, \mu(I_{i,j})$$

$$= \sum_{j=1}^{2} \int_{I_{i,j}} f_{ij}(x,y) ds$$

Hence we have

$$|| \sum_{j=1}^{2} \int_{I_{i,j}} [f_j(x,y,s) - f_{ij}(x,y)] ds + \int_{I_i} \sum_{j=1}^{2} \alpha_j [f_{ij}(x,y) - f_j(x,y,s)] ds \; ||$$

$$\leq \sum_{j=1}^{2} \int_{I_{i,j}} \| f_j(x,y,s) - f_{ij}(x,y) \| ds$$

$$+ \int_{I_i} \sum_{j=1}^{2} \alpha_j \| f_{ij}(x,y) - f_j(x,y,s) \| ds$$

$$\leq \sum_{j=1}^{2} \varepsilon \int_{I_{i,j}} ds + \varepsilon \int_{I_i} \sum_{j=1}^{2} \alpha_j \, ds \qquad \text{from above}$$

$$= 2\varepsilon\, \mu(I_i)$$

Therefore we have that

$$\sum_{i=1}^{L} \Big\| \sum_{j=1}^{2} \int_{I_{i,j}} f_j(x,y,s)ds - \int_{I_j} \sum_{j=1}^{2} \alpha_j f_j(x,y,s)ds \Big\|$$

$$\leq \sum_{i=1}^{L} 2\varepsilon\, \mu(I_i)$$

Therefore for every $t \in T$, there exists an integer L_0 and some I_β (the interval left at end of segment $[0,t] \subset T$ in the partitioning procedure) such that

$$\Big\| \int_0^t [f(x,y,s) - \sum_{j=1}^{2} \alpha_j f_j(x,y,s)]ds \Big\|$$

$$\leq \sum_{i=1}^{L_0} \Big\| \int_{I_i} [f(x,y,s) - \sum_{j=1}^{2} \alpha_j f_j(x,y,s)]ds \Big\|$$

$$+ \Big\| \int_{I_\beta} [f(x,y,s) - \sum_{j=1}^{2} \alpha_j f_j(x,y,s)]ds \Big\|$$

where we use

$$f(x,y,s) = f(x^{u_\alpha(\varepsilon)}(s), x^{u_\alpha(\varepsilon)}(s-\tau), u_\alpha(\varepsilon,s), s)$$

This gives

$$\left\| \int_0^t \left[f(x,y,s) - \sum_{j=1}^{2} \alpha_j f_j(x,y,s) \right] ds \right\| \leq 2\varepsilon \sum_{i=1}^{L_0} \mu(I_i)$$

$$+ 2 \int_{I_{I_\beta}} M\{ \|x\| + \|y\| + 1 \} ds$$

$$\leq 2\varepsilon \sum_{i=1}^{L_0} \mu(I_i) + 2\varepsilon$$

$$\leq 4\varepsilon$$

since $\sum_{i=1}^{L_0} \mu(I_i) \leq \mu(T) = 1$

Hence using the above results we get

$$\| x^{u_\alpha(\varepsilon)}(t) - x^{\underset{\sim}{u}\alpha}(t) \| \leq d_1 \int_0^t \| x^{u_\alpha(\varepsilon)}(s) - x^{\underset{\sim}{u}\alpha}(s) \| ds + 4\varepsilon$$

for all $t \in T$

By an application of Gronwall's Inequality we get

$$\| x^{u_\alpha(\varepsilon)}(t) - x^{\underset{\sim}{u}\alpha}(t) \| \leq 4\varepsilon \exp \int_0^t d_1 ds$$

i.e.

$$\| x^{u_\alpha(\varepsilon)}(t) - x^{\underset{\sim}{u}\alpha}(t) \| \leq 4\varepsilon \exp(d_1 t) \quad \text{for all } t \in T$$

Now as the integer N is increased the partition gets finer and as $N \to \infty$, $\mu(I_i) \to 0$ for $i=1,2,\ldots,N$ and $\varepsilon \to 0$.

This means that in the limit (i.e. as $N \to \infty$) we have

$$\lim_{N\to\infty} \sup_{\substack{u,\check{u} \in G \\ \alpha \in [0,1]}} \| x^{u_\alpha(\varepsilon)}(t) - x^{\underset{\sim}{u}\alpha}(t) \| = 0 \text{ uniformly for all } t \in T$$

This proves the lemma and hence the approximation is valid.

Most of the results which were given to prove convergence of Algorithms 1 and 2 also hold for Algorithm 3 as well, with a few obvious modifications. However, some results are quite different, and so we present them here since they will be needed in studying the convergence properties of Algorithm 3.

<u>Proposition 7.2</u>

There exists a $d \in (0,\infty)$ such that for all $u,\check{u} \in G$, all $\alpha \in [0,1]$, given $\varepsilon \geq 0$ and with $u_\alpha(\varepsilon,t)$ as defined in subprocedure A, then:

$$\| x^{u_\alpha(\varepsilon)}(t) - x^u(t) \| \leq d\alpha \qquad \text{for all } t \in T$$

<u>Proof</u>

By definition of x(.) we have

$$x^{u_\alpha(\varepsilon)}(t)-x^u(t) = \int_0^t [f(x^{u_\alpha(\varepsilon)}(s),x^{u_\alpha(\varepsilon)}(s-\tau),u_\alpha(\varepsilon,s),s)$$

$$-f(x^u(s),x^u(s-\tau),u(s),s)]ds \qquad \text{for every } t \in T$$

Using Taylor expansion on first term in integrand we get to "first approximation" (see footnote on page 89)

$$f(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},u_\alpha(\varepsilon),s) = f(x^u,y^u,u,s)+f_x(x^u,y^u,u,s)(x^{u_\alpha(\varepsilon)}-x^u)$$

$$+f_y(x^u,y^u,u,s)(y^{u_\alpha(\varepsilon)}-y^u)$$

$$+f_u(x^u,y^u,u,s)(u_\alpha(\varepsilon)-u)$$

Therefore we get

$$\| x^{u_\alpha(\varepsilon)}(t)-x^u(t) \| \leq \int_0^t \| f_x(x^u,y^u,u,s)\| \; \| x^{u_\alpha(\varepsilon)}-x^u\| \, ds$$

$$+ \int_0^t \|f_y(x^u,y^u,u,s\| \; \| y^{u_\alpha(\varepsilon)}-y^u\| \, ds$$

$$+ \int_0^t \|f_u(x^u,y^u,u,s)\| \; \|u_\alpha(\varepsilon)-u\| \, ds \qquad \text{for all } t \in T$$

By the boundedness of f_x, f_y, f_u on $B_1 \times B_1 \times \Omega \times T$ there exist finite d_1, d_2, d_3 such that

$$\| x^{u_\alpha(\varepsilon)}(t) - x^u(t) \| \le d_1 \int_0^t \| x^{u_\alpha(\varepsilon)}(s) - x^u(s) \| \, ds$$

$$+ d_2 \int_0^1 \| x^{u_\alpha(\varepsilon)}(s-\tau) - x^u(s-\tau) \| \, ds$$

$$+ d_3 \int_0^t \| u_\alpha(\varepsilon, s) - u(s) \| \, ds \quad \text{for all } t \in T$$

$$\le d_4 \int_0^t \| x^{u_\alpha(\varepsilon)}(s) - x^u(s) \| \, ds$$

$$+ d_3 \int_0^t \| u_\alpha(\varepsilon, s) - u(s) \| \, ds \quad \text{for all } t \in T$$

for some $d_4 \in (0, \infty)$.

By the definition of $u_\alpha(\varepsilon, s)$ [see Subprocedure A] we have

$$\int_0^t \| u_\alpha(\varepsilon, s) - u(s) \| \, ds \le \int_0^1 \| u_\alpha(\varepsilon, s) - u(s) \| \, ds$$

$$= \sum_{i=1}^{N} \int_{I_i} \| u_\alpha(\varepsilon, s) - u(s) \| ds$$

$$= \sum_{i=1}^{N} \int_{I_{i,2}} \| \check{u}(t) - u(t) \| dt$$

since $u_\alpha(\varepsilon, t) = u(t)$ for $t \in I_{i,1}$ for each i.

By the boundedness of Ω (see Chapter 2) we have

$$\| \check{u}(t) - u(t) \| \le \| \check{u}(t) \| + \| u(t) \| \le 2r$$

Hence

$$\int_0^t \| u_\alpha(\varepsilon, s) - u(s) \| ds \le 2r \sum_{i=1}^{N} \int_{I_{i,2}} dt$$

$$= 2r \sum_{i=1}^{N} \mu(I_{i,2})$$

Assume for convenience that when the partitioning is done that the time interval is divided into N equal subdivisions. Hence $\mu(I_i) = {}^1/_N$.

Therefore

$$\int_0^t \| u_\alpha(\varepsilon, s) - u(s) \| \, ds \leq 2r\alpha$$

because of the ratios of $I_{i,1}$ to $I_{i,2}$ is $(1-\alpha):\alpha$ substituting in above we get

$$\| x^{u_\alpha(\varepsilon)}(t) - x^u(t) \| \leq d_4 \int_0^t \| x^{u_\alpha(\varepsilon)}(s) - x^u(s) \| \, ds + 2rd_3\alpha$$

for all $t \in T$

By an application of Gronwall's Inequality we deduce that

$$\| x^{u_\alpha(\varepsilon)}(t) - x^u(t) \| \leq 2rd_3\alpha \exp d_4 t \qquad \text{for all } t \in T$$

i.e.

$$\| x^{u_\alpha(\varepsilon)}(t) - x^u(t) \| \leq d\alpha \qquad \text{for all } t \in T$$

as required.

<u>Lemma 7.3</u>

For all $\alpha \in [0,1]$, for all $u \in G$, given $\varepsilon \geq 0$, there exists an integer N as defined in subprocedure A such that

$$\Delta\hat{V}(u_\alpha(\varepsilon), u) \leq \alpha\theta(u) + \varepsilon$$

[Note: The same problem as in Jacobson and Mayne [J1] (see Chapter 1) of not being able to guarantee a descent property for some cases may occur if the approximation is not fine enough. This can be seen as follows:

If the approximation is very coarse, i.e. ε is quite large and for

$$\Delta\hat{V}(u_\alpha(\varepsilon), u) \leq \alpha\theta(u) + \varepsilon$$

the right hand side may be greater than zero for small α. Hence the partition size might have to be increased as algorithm progresses if a small step length is generated and the above mentioned situation arises so that a descent property is maintained.]

<u>Proof of Lemma 7.3</u>

By definition we have

$$\Delta\hat{V}(u_\alpha(\varepsilon), u) = \int_0^1 [H(x^u(t), x^u(t-\tau), u_\alpha(\varepsilon, t), \lambda^u(t), t)$$
$$- H(x^u(t), x^u(t-\tau), u(t), \lambda^u(t), t)]\, dt$$

where $\quad u_\alpha(\varepsilon, t) = u(t) \qquad \text{for } t \in I_{i,1}$

$\qquad\qquad\qquad = \check{u}(t) \qquad \text{for } t \in I_{i,2}$

$\qquad\qquad\qquad\qquad \text{for } i = 1, 2, \ldots, N$

Therefore

$$\Delta\hat{V}(u_\alpha(\varepsilon), u) = \int_0^1 [H(x^u, y^u, u_\alpha(\varepsilon), \lambda^u, t) - H(x^u, y^u, \underset{\sim}{u}_\alpha, \lambda^u, t)]\, dt$$
$$+ \int_0^1 [H(x^u, y^u, \underset{\sim}{u}_\alpha, \lambda^u, t) - H(x^u, y^u, u, \lambda^u, t)]\, dt$$

where $\underset{\sim}{u}_\alpha = (1-\alpha)u \oplus \alpha\check{u}$ as before

i.e.

$$\Delta\hat{V}(u_\alpha(\varepsilon), u) = \sum_{i=1}^{N} \{ \int_{I_{i,1}} H(x^u, y^u, u, \lambda^u, t)dt + \int_{I_{i,2}} H(x^u, y^u, \check{u}, \lambda^u, t)dt$$
$$- \int_{I_i} [(1-\alpha)H(x^u, y^u, u, \lambda^u, t) + \alpha H(x^u, y^u, \check{u}, \lambda^u, t)]\, dt \}$$

$$+\int_0^1 [(1-\alpha)H(x^u,y^u,u,\lambda^u,t)+\alpha H(x^u,y^u,\check{u},\lambda^u,t)$$
$$-H(x^u,y^u,u,\lambda^u,t)]dt$$

Letting $H_1(x,y,\lambda,t) = H(x^u,y^u,u,\lambda^u,t)$

$H_2(x,y,\lambda,t) = H(x^u,y^u,\check{u},\lambda^u,t)$

$\alpha_1 = (1-\alpha)$

and $\alpha_2 = \alpha$

Then we get

$$\hat{\Delta V}(u_\alpha(\varepsilon),u) = \sum_{i=1}^{N} \{ \sum_{j=1}^{2} \int_{I_{i,j}} H_j(x,y,\lambda,t)dt - \int_{I_i} \sum_{j=1}^{2} \alpha_j H_j(x,y,\lambda,t)dt \}$$
$$+ \alpha \int_0^1 [H(x^u,y^u,\check{u},\lambda,t)-H(x^u,y^u,u,\lambda^u,t)]dt$$

By using a similar procedure as in the proof of Lemma 7.1 we can show that given $\varepsilon \geq 0$, there exists a partition N of [0,1] such that

$$\sum_{i=1}^{N} \| \sum_{j=1}^{2} \int_{I_{i,j}} H_j(x,y,\lambda,t)dt - \int_{I_i} \sum_{j=1}^{2} \alpha_j H_j(x,y,\lambda,t)dt \| \leq \varepsilon$$

This together with the definition of θ gives

$$\hat{\Delta V}(u_\alpha(\varepsilon),u) \leq \alpha\theta(u) + \varepsilon$$

as required.

Lemma 7.4

For all $\alpha \in [0,1]$, for all $u \in G$, given $\varepsilon \geq 0$, there exists an integer N (partition fineness) as before such that

$$\Delta V(u_\alpha(\varepsilon),u) \leq \alpha\theta(u) + d\alpha^2 + \varepsilon$$

for some finite d.

Proof

By definition

$$\Delta V(u_\alpha(\varepsilon),u) = V(u_\alpha(\varepsilon))-V(u)$$

$$= \int_0^1 [l(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},u_\alpha(\varepsilon),t)-l(x^u,y^u,u,t)]dt$$

$$+\int_0^1 \lambda^{u\,T}[f(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},u_\alpha(\varepsilon),t)-f(x^u,y^u,u,t)]dt$$

$$- \int_0^1 \lambda^{u\,T}(\dot{x}^{u_\alpha(\varepsilon)}-\dot{x}^u)dt$$

Using the same procedure as before we get

$$\Delta V(u_\alpha(\varepsilon),u)= \int_0^1 [H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},u_\alpha(\varepsilon),\lambda^u,t)-H(x^u,y^u,u,\lambda^u,t)]dt$$

$$+ \int_0^1 \dot{\lambda}^{u\,T}[x^{u_\alpha(\varepsilon)}-x^u]\,dt$$

Adding and subtracting terms again we get

$$\Delta V(u_\alpha(\varepsilon),u) = \int_0^1 [H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},u_\alpha(\varepsilon),\lambda^u,t)$$

$$-H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},\underset{\sim}{u}_\alpha,\lambda^u,t)]dt$$

$$+ \int_0^1 [H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},\underset{\sim}{u}_\alpha,\lambda^u,t)-H(x^u,y^u,u,\lambda^u,t)]dt$$

$$+ \int_0^1 \dot{\lambda}^{u\,T}(x^{u_\alpha(\varepsilon)}-x^u)dt$$

Substituting for λ and using Taylor expansion to "first order" we get (see footnote on page 89)

$$\Delta V(u_\alpha(\varepsilon),u) = \int_0^1 [H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},u_\alpha(\varepsilon),\lambda^u,t)$$

$$-H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},\underset{\sim}{u}_\alpha,\lambda^u,t)]dt$$

$$+ \int_0^1 [H(x^u,y^u,\underset{\sim}{u}_\alpha,\lambda^u,t)+H_x(x^u,y^u,\underset{\sim}{u}_\alpha,\lambda^u,t)(x^{u_\alpha(\varepsilon)}-x^u)$$

$$+H_y(x^u,y^u,\underset{\sim}{u}_\alpha,\lambda^u,t)(y^{u_\alpha(\varepsilon)}-y^u)-H(x^u,y^u,u,\lambda^u,t)]dt$$

$$- \int_0^1 [H_x(x^u,y^u,u,\lambda^u,t)(x^{u_\alpha(\varepsilon)}-x^u)$$

$$+H_y(x^u,y^u,u,\lambda^u,t)(y^{u_\alpha(\varepsilon)}-y^u)]dt$$

By definition of $\underset{\sim}{u}_\alpha$ and θ we get

$$\Delta V(u_\alpha(\varepsilon),u) = \int_0^1 [H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},u_\alpha(\varepsilon),\lambda^u,t) - H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},\underset{\sim}{u}_\alpha,\lambda^u,t)]\,dt$$

$$+\alpha\theta(u)$$

$$+\int_0^1 [H_x(x^u,y^u,\underset{\sim}{u}_\alpha,\lambda^u,t)-H_x(x^u,y^u,u,\lambda^u,t)]\cdot (x^{u_\alpha(\varepsilon)}-x^u)\,dt$$

$$+\int_0^1 [H_y(x^u,y^u,\underset{\sim}{u}_\alpha,\lambda^u,t)-H_y(x^u,y^u,u,\lambda^u,t)]\cdot (y^{u_\alpha(\varepsilon)}-y^u)\,dt$$

$$\| \Delta V(u_\alpha(\varepsilon),u)-\alpha\theta(u)\| \le \| \int_0^1 [H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},u_\alpha(\varepsilon),\lambda^u,t) - H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},\underset{\sim}{u}_\alpha,\lambda^u,t)]\,dt \|$$

$$+\alpha\int_0^1 \|H_x(x^u,y^u,\check{u},\lambda^u,t)-H_x(x^u,y^u,u,\lambda^u,t)\| \cdot \| x^{u_\alpha(\varepsilon)}-x^u \|\,dt$$

$$+\alpha\int_0^1 \|H_y(x^u,y^u,\check{u},\lambda^u,t)-H_y(x^u,y^u,u,\lambda^u,t)\| \cdot \| y^{u_\alpha(\varepsilon)}-y^u \|\,dt$$

Using the same procedure as in Lemma 7.1 we can deduce that

$$\| \int_0^1 [H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},u_\alpha(\varepsilon),\lambda^u,t) - H(x^{u_\alpha(\varepsilon)},y^{u_\alpha(\varepsilon)},\underset{\sim}{u}_\alpha,\lambda^u,t]\,dt \| \le \varepsilon$$

for N large enough.

Also by the boundedness of H_x and H_y on $B_1 x B_1 x \Omega x B_1 x T$, there exist constants $d_1, d_2 \in (0,\infty)$ such that

$$\| \Delta V(u_\alpha(\varepsilon),u)-\alpha\theta(u) \| \le \varepsilon + 2d_1 \alpha \int_0^1 \| x^{u_\alpha(\varepsilon)}(t)-x^u(t) \| dt$$

$$+2d_2 \alpha \int_0^1 \| x^{u_\alpha(\varepsilon)}(t-\tau)-x^u(t-\tau) \| dt$$

$$\le \varepsilon + d_3 \alpha \int_0^1 \| x^{u_\alpha(\varepsilon)}(t)-x^u(t) \| dt$$

for some finite d_3 .

Now by Proposition 7.2, there exists a $d_4 \in (0,\infty)$ such that

$$\| x^{u_\alpha(\varepsilon)}(t)-x^u(t) \| \le d_4 \alpha \qquad \text{for all } t \in T$$

Hence

$$\| \Delta V(u_\alpha(\varepsilon),u)-\alpha\theta(u) \| \le \varepsilon + d_3 d_4 \alpha^2$$

i.e.

$$\Delta V(u_\alpha(\varepsilon),u) \le \alpha\theta(u)+d\alpha^2+\varepsilon$$

as required.

Lemma 7.5

For all $u \in G$, the step length $\tilde{\alpha}$ found by Algorithm 3 satisfies

$$\tilde{\alpha} \ge -\frac{\theta(u)}{d}$$

for some finite d.

Proof

In the statement of the algorithm, the step length $\tilde{\alpha}$ is found by minimising the cost function $V(\underset{\sim}{u}_\alpha)$ at each iteration, where $\underset{\sim}{u}_\alpha$ is defined by $\underset{\sim}{u}_\alpha = (1-\alpha)u \oplus \alpha\check{u}$, i.e. $\tilde{\alpha} \in [0,1]$ is found such that

$$V(\underset{\sim}{u}_\alpha) = \int_0^1 [(1-\alpha)l(x^{\underset{\sim}{u}_\alpha},y^{\underset{\sim}{u}_\alpha},u,t)+\alpha l(x^{\underset{\sim}{u}_\alpha},y^{\underset{\sim}{u}_\alpha},\check{u},t)]dt$$

is minimised. It is quite straightforward to see that the above function is continuous with respect to α.

Hence differentiating with respect to α we get

$$\frac{d}{d\alpha} V(\underset{\sim}{u}_\alpha) = \int_0^1 \{(1-\alpha)[l_x(x^u,y^u,u,t)\,\delta x(u,\alpha,t)+l_y(x^u,y^u,u,t)\delta y(u,\alpha,t)]$$

$$+\alpha[l_x(x^u,y^u,\check{u},t)\,\delta x(u,\alpha,t)+l_y(x^u,y^u,\check{u},t)\delta y(u,\alpha,t)]$$

$$+l(x^u,y^u,\check{u},t)-l(x^u,y^u,u,t)\}\,dt \qquad \text{(A)}$$

for $\alpha \in (0,1)$

where $\delta y(u,\alpha,t) = \delta x(u,\alpha,t-\tau)$, and $\delta x(u,\alpha,t)$ is determined as follows:

$x^{\underset{\sim}{u}\alpha}(t) = x(u,\alpha,t)$ is the solution of

$\dot{x}(t) = f(x(t),x(t-\tau),\underset{\sim}{u}_\alpha,t)$ for a.a.$t \in T$

$x(t) = \phi(t)$ for all $t \in [-\tau,0]$

i.e. $\dot{x}(u,\alpha,t) = (1-\alpha)f(x(u,\alpha,t),x(u,\alpha,t-\tau),u(t),t)$

$+\alpha f(x(u,\alpha,t),x(u,\alpha,t-\tau),\check{u}(t),t)$ for a.a.$t \in T$

$x(t) = \phi(t)$ for all $t \in [-\tau,0]$

Differentiating this w.r.t α we get

$$\delta\dot{x}(u,\alpha,t)=(1-\alpha)[f_x(x^u,y^u,u,t)\,\delta x(u,\alpha,t)+f_y(x^u,y^u,u,t)\,\delta y(u,\alpha,t)]$$

$$+\alpha[f_x(x^u,y^u,\check{u},t)\,\delta x(u,\alpha,t)+f_y(x^u,y^u,\check{u},t)\delta y(u,\alpha,t)]$$

$$+f(x^u,y^u,\check{u},t)-f(x^u,y^u,u,t) \qquad \text{(B)}$$

$\delta x(t) = 0 \qquad t < 0$

which has as its solution $\delta x(u,\alpha,t)$

Therefore to minimise $V(\underset{\sim}{u}_\alpha)$ at each iteration equate (A) to zero.

By adding and subtracting terms we get

$$\int_0^1 \{l(x^u,y^u,\check{u},t)-l(x^u,y^u,u,t)$$

$$+(1-\alpha)\,[l_x(x^u,y^u,u,t)\,\delta x(u,\alpha,t)+l_y(x^u,y^u,u,t)\,\delta y(u,\alpha,t)\,]$$

$$+\alpha[l_x(x^u,y^u,\check{u},t)\,\delta x(u,\alpha,t)+l_y(x^u,y^u,\check{u},t)\delta y(u,\alpha,t)\,]$$

$$+\lambda^{u\,T}\{f(x^u,y^u,\check{u},t)-f(x^u,y^u,u,t)$$

$$+(1-\alpha)\,[f_x(x^u,y^u,u,t)\delta x(u,\alpha,t)+f_y(x^u,y^u,u,t)\delta y(u,\alpha,t)\,]$$

$$+\alpha\,[f_x(x^u,y^u,\check{u},t)\delta x(u,\alpha,t)+f_y(x^u,y^u,\check{u},t)\delta y(u,\alpha,t)\,]\}$$

$$-\lambda^{u\,T}\delta\dot{x}(u,\alpha,t)\,\}\,dt \;=0$$

By the definition of the Hamiltonian we have

$$\int_0^1 [H(x^u,y^u,\check{u},\lambda^u,t)-H(x^u,y^u,u,\lambda,t)\,]dt$$

$$+\int_0^1 (1-\alpha)\,[H_x(x^u,y^u,u,\lambda^u,t)\,\delta x(u,\alpha,t)+H_y(x^u,y^u,u,\lambda^u,t)\delta y\,(u,\alpha,t)]\,dt$$

$$+\int_0^1 \alpha\,[H_x(x^u,y^u,\check{u},\lambda^u,t)\,\delta x(u,\alpha,t)+H_y(x^u,y^u,\check{u},\lambda^u,t)\delta y\,(u,\alpha,t)]\,dt$$

$$-\int_0^1 \lambda^{u\,T}(t)\delta\dot{x}(u,\alpha,t)dt \;=\;0$$

Integrating last term by parts we get

$$\int_0^1 \lambda^{u\,T}\delta\dot{x}(u,\alpha,t)dt \;=\; [\,\lambda^{u\,T}(t)\,\delta x(u,\alpha,t)\,]_0^1-\int_0^1\dot{\lambda}^{u\,T}(t)\delta x(u,\alpha,t)dt$$

$$=\;-\int_0^1\dot{\lambda}^{u\,T}(t)\delta x(u,\alpha,t)dt$$

since $\lambda(1)\;=\;0$ and $\delta x(u,\alpha,0)\;=\;0$.

Using this and substituting for $\dot{\lambda}^u$ from (2.7)-(2.9) in Chapter 2, and using the definition of θ we get that

$$\theta\,(u)+\int_0^1 (1-\alpha)\,[H_x(x^u,y^u,u,\lambda^u,t)\,\delta x\,(u,\alpha,t)+H_y(x^u,y^u,u,\lambda^u,t)\cdot$$

$$\cdot\,\delta y(u,\alpha,t)\,]dt$$

$$+\int_0^1 \alpha[H_x(x^u,y^u,\check{u},\lambda^u,t)\,\delta x(u,\alpha,t)+H_y(x^u,y^u,\check{u},\lambda^u,t)\delta y\,(u,\alpha,t)]\,dt$$

$$-\int_0^1 [H_x(x^u,y^u,u,\lambda^u,t)\,\delta x(u,\alpha,t)+H_y(x^u,y^u,u,\lambda,t)\delta y\,(u,\alpha,t)]\,dt$$

$$= 0$$

i.e. we have

$$\theta(u)+\alpha\int_0^1 [H_x(x^u,y^u,\check{u},\lambda^u,t)-H_x(x^u,y^u,u,\lambda^u,t)]\,\delta x\,(u,\alpha,t)dt$$

$$+\alpha\int_0^1 [H_y(x^u,y^u,\check{u},\lambda^u,t)-H_y(x^u,y^u,u,\lambda^u,t)]\,\delta y\,(u,\alpha,t)dt$$

$$= 0$$

Now $[H_x(x^u,y^u,\check{u},\lambda^u,t)-H_x(x^u,y^u,u,\lambda^u,t)]$ is not identically equal to zero if $u(t)\neq \check{u}(t)$ and the function H_x is bounded on $B_1 x B_1 x\,\Omega x B_1 x T$. Similar properties hold for $[H_y(x^u, y^u, \check{u}, \lambda^u, t) -H(x^u, y^u, u, \lambda^u, t)]$.

Also, $\delta x(u, \alpha, t)$ is the solution of (B) which exists and is unique. It is not identically zero if $u(t)\neq \check{u}(t)$ and is bounded.

Hence there exists some finite d, so that the above equation becomes

$$\theta(u) + d\,\alpha \geq 0$$

i.e. $\tilde{\alpha} \geq \dfrac{-\theta(u)}{d}$ as required

We are now in a position to prove Theorem 3 which we restate here for convenience.

Theorem 3

Let Assumptions 1 and 2 in Chapter 2 be satisfied and suppose that Algorithm 3 generates a sequence $\{u_i\}$ of (ordinary) controls. Then we have:

Case 1

Given any $\delta > 0$, N (and hence $\varepsilon > 0$) may be chosen such that the sequence is either finite, in which case the last element is desirable to within delta, or it is infinite and every accumulation point u* generated by the algorithm satisfies an optimality condition to within delta, i.e. $\theta(u^*) \geq -\delta$.

Case 2

That the sequence is either finite, in which case the last element is desirable (exactly), or it is infinite and every accumulation point u* satisfies optimality conditions exactly, i.e. $\theta(u^*) = 0$.

Proof

For both cases if the sequence is finite, the last element u_k trivally satisfies the optimality conditions required, hence we ony need to consider the case where Algorithm 3 generates an infinite sequence of controls.

Case 1

Assume that an infinite sequence $\{u_i\}_{i=0}^{\infty}$ in G is generated by Algrorithm 3, and suppose that u* is an accumulation control of this sequence,

i.e. $$u_i \xrightarrow[i]{} u^*$$

By Lemma 7.4 we have that

$$\Delta V(u_\alpha(\varepsilon), u) \leq \alpha\theta(u) + d\alpha^2 + \varepsilon$$

i.e. $$\Delta V(u_{i+1}, u_i) \leq \alpha\theta(u_i) + d\alpha^2 + \varepsilon \qquad \text{for all } i$$

Since the cost functional is assumed to be bounded on the compact convex set $\underset{\sim}{G}$ we must have that

$$\lim_i \Delta V(u_{i+1},u_i) = 0$$

that is $\Delta V(u^*_\alpha(\varepsilon),u^*) = 0$

Therefore we get from above that

$$\alpha\theta(u^*) + d\alpha^2 + \varepsilon \geq 0$$

From Lemma 7.5 we have that the step length $\tilde{\alpha} \geq -\dfrac{\theta(u^*)}{d_1}$

Hence we have

$$-\frac{\theta^2(u^*)}{d_1} + d\,\frac{\theta^2(u^*)}{d_1^2} + \varepsilon \geq 0$$

$$\frac{(d-d_1)}{d_1^2}\,\theta^2(u^*) \geq -\varepsilon$$

i.e. $(d_1-d)\,\theta^2(u^*) \leq d_1^2\,\varepsilon$

$$\Rightarrow \theta^2(u^*) \leq \frac{d^2\,\varepsilon}{d_1-d}$$

Hence $\theta(u^*) \geq -\delta$

where $\delta = \sqrt{\dfrac{d_1^2\,\varepsilon}{d_1-d}}$

Since θ is negative semi definite.

Hence the limit point satisfies optimality conditions to within delta as required.

Case 2

Here the fineness of the partitions, denoted by the integer N, is increased to infinity. The proof is exactly along the lines of Case 1 except that $\mu(I_i)$, the measure of the intervals I_i, i=1,2,...N, approaches zero and $\varepsilon \longrightarrow 0$ as $N \longrightarrow \infty$. Hence the inequality

$$\Delta V(u^*_\alpha(\varepsilon), u^*) \leq \alpha\,\theta(u^*) + d\alpha^2 + \varepsilon$$

above in Case 1 reduces to

$$\Delta V(u^*_\alpha(0), u^*) \leq \alpha\,\theta(u^*) + d\alpha^2$$

Therefore $\Delta V(u^*_\alpha(0), u^*) = 0 \leq \alpha\,\theta(u^*) + d\alpha^2$ and we deduce that

$$\theta^2(u^*) \geq 0$$

i.e. $\theta(u^*) = 0$ as required.

CHAPTER 4

AN EXACT PENALTY FUNCTION ALGORITHM FOR TERMINAL EQUALITY CONSTRAINED CONTROL PROBLEMS

4.1 Introduction

So far in this thesis, problems involving only control constraints have been investigated and it has been shown how these can be dealt with, i.e. by optimising a subproblem over the permissible controls to obtain a descent property. In this chapter we study problems which are more complex in that they also include a finite number of terminal constraints. To be specific the problems under consideration will be "Optimal Control Problems with Control and Terminal Equality Constraints" (see Mayne and Polak [MAP2] and Problem P2 below). Mayne and Polak [MAP2] present an algorithm which solves these problems using an exact penalty function and we extend their results to cover delay systems. This extension also includes some crucial differences from the approach taken in [MAP2]. One of these is that we construct a sequence of relaxed controls, rather than of ordinary controls as in [MAP2]. Both our algorithm and that in MAP2 generate a (possibly) relaxed control in the limit, satisfying first order optimality conditions. However the advantage of our approach appears to be that the construction of the sequence of relaxed controls is much simpler than that of the ordinary controls. The price paid is that we must introduce the sophisticated notion of 'relaxed controls' but Mayne and Polak have to do this anyway to study the convergence properties of their algorithm.

Another distinction is that the new control, $\underset{\sim}{u}_{new}$, constructed by our algorithm is a relaxed convex combination of the old control $\underset{\sim}{u}$ and another control $\underset{\sim}{v}$ found by solving a subproblem, i.e. $\underset{\sim}{u}_{new} = (1-\alpha)\underset{\sim}{u} \oplus \alpha\underset{\sim}{v}$, where α may be thought of as the step length. This is in contrast to the classical method used in [MAP2] where the new control is determined by the usual search direction, step length method. Our different method of defining the new control gives a further important difference from the approach taken by Mayne and Polak - they linearize their intermediate problem about the control variable u as well as the state variable x, whereas we only need to linearize about x. This linearization in u is required by Mayne and Polak to guarantee certain results needed in proving convergence of their algorithm. However we show below that, because of the way we define our new control, these results can be obtained using other methods, namely via the linear nature of relaxed controls (see section B8 in Chapter 1), and hence the differentiation w.r.t u is not required. Because of this our procedure presented in this chapter gives, in a sense, better approximations (to the nonlinear problem at each iteration) than does the method by Mayne and Polak. It therefore seems plausible to expect our procedure to perform better than Mayne and Polak's when the two schemes are implemented.

Apart from these significant differences the procedure for solving Problem P2 (the delayed problem) and the scheme in [MAP2] is quite similar and we present it here because it forms a convenient stepping stone to solving one of the most difficult problems encountered in optimal control. This is known as "The State Constrained Control Problem" and we consider it in Chapter

5 where it is solved using an exact penalty function method.

Mayne and Polak present two algorithms for solving their problem, one conceptual and the other implementable. We, however, will only present the conceptual algorithm.

4.2 Problem Statement

We will briefly state the problem which will be under investigation in this chapter using the same notation and terminology as in earlier chapters. The optimal control problem under consideration will be the following:

$$\text{Min } \{ g^o(\underset{\sim}{u}) \; : \; g^j(\underset{\sim}{u}) = 0 \quad , \quad j=1,2,\ldots\ldots,r, \; \underset{\sim}{u} \in \underset{\sim}{G} \}$$

where $g^j(\underset{\sim}{u})=h^j(x^{\underset{\sim}{u}}(1))$ for $j=0,1,2,\ldots\ldots,r$ and $x^{\underset{\sim}{u}}:T \to \mathbb{R}^n$ is the solution of the delay-differential equation

$$\dot{x}(t) = f(x(t),x(t-\tau), \underset{\sim}{u}(t),t) \qquad \text{for a.a.} t \in T$$

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau,0]$$

as in Chapter 3, and $\underset{\sim}{G}$ is the space of measurable relaxed controls. The following hypothesis is assumed to hold:

Assumption 1

The function $f : \mathbb{R}^n \mathrm{x}\, \mathbb{R}^n \mathrm{x} \underset{\sim}{V} \mathrm{x} T \longrightarrow \mathbb{R}^n$ and its partial derivatives f_x, f_y, f_{xx}, f_{yy}, f_{xy} and the functions $h^j : \mathbb{R}^n \longrightarrow \mathbb{R}$, $j=0,1,2,\ldots\ldots,r$ and their partial derivatives h^j_x, h^j_{xx} exist and are continuous (on their domains) [Note the differentiability w.r.t. u, hypothesized in [MAP2], is not required].

Assumption 2

There exists an $M \in (0,\infty)$ such that

$$\| f(x,y,\underset{\sim}{u},t) \| \le M \{ \|x\| + \|y\| + 1 \} \quad \text{for all } x,y \in \mathbb{R}^n,$$

all $\underset{\sim}{u} \in \underset{\sim}{G}$, all $t \in T$

and

$$\| f(x^1,y^1,\underset{\sim}{u},t) - f(x^2,y^2,\underset{\sim}{u},t) \| \le M\{\|x^1-x^2\| + \|y^1-y^2\| \}$$

for all x^1, y^1, x^2, $y^2 \in \mathbb{R}^n$, all $\underset{\sim}{u} \in \underset{\sim}{G}$, all $t \in T$.

We restate the above as Problem P2:

$$\min_{\underset{\sim}{u}} \; h^0(x(1)) \tag{2.1}$$

$$\text{s.t} \quad \dot{x}(t) = f(x(t), x(t-\tau), \underset{\sim}{u}(t), t) \quad \text{for a.a } t \in T \tag{2.2}$$

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau, 0] \tag{2.3}$$

$$h^j(x(1)) = 0 \qquad j=1,2,\ldots\ldots,r \tag{2.4}$$

$$\underset{\sim}{u} \in \underset{\sim}{G} \tag{2.5}$$

Note

An objective functional of the form (2.1) is called a terminal payoff since it only depends on the final state $x(1)$. Problems of this kind are more general than they appear at first glance. Let us suppose that instead of (2.1), we have the optimal control problem with integral cost of the form

$$\min_{\underset{\sim}{u}} \; \int_0^1 l(x(t), \underset{\sim}{u}(t), t)\, dt \tag{$\dagger$}$$

subject to the constraints (2.2)-(2.5). This can be easily transformed to the form (2.1) by defining a new state variable $x_0(t)$, say, by the following relationship

$$\dot{x}_0(t) = l(x(t),\underset{\sim}{u}(t),t) \qquad \text{for a.a } t \in T$$

$$x_0(o) = 0$$

Then (†) can be written as

$$\underset{\underset{\sim}{u}}{\text{Min}} \quad x_0(1)$$

which is a simple terminal payoff. Hence terminal cost problems are equivalent to integral cost problems.

Let $R \triangleq \{x^{\underset{\sim}{u}}(1):\underset{\sim}{u} \in \underset{\sim}{G}\}$ denote the reachable set of our system. Before we discuss our penalty function approach for solving Problem P2 we present some basic results which we will make use of:

For all $\underset{\sim}{u}$, $\underset{\sim}{v} \in \underset{\sim}{G}$, let $z^{\underset{\sim}{u},\underset{\sim}{v}}:T \to \mathbb{R}^n$ denote the solution of

$$\dot{z}(t) = A^{\underset{\sim}{u}}(t)z(t)+B^{\underset{\sim}{u}}(t)z(t-\tau)+ \Delta f(\underset{\sim}{v},\underset{\sim}{u}) \tag{2.6}$$

$$\text{for a.a } t \in T$$

$$z(t) = 0 \qquad \text{for } t \in [-\tau,0] \tag{2.7}$$

where

$$\left.\begin{aligned} A^{\underset{\sim}{u}}(t) &= f_x(x^{\underset{\sim}{u}}(t),\ x^{\underset{\sim}{u}}(t-\tau),\ \underset{\sim}{u}(t),t) \\ B^{\underset{\sim}{u}}(t) &= f_y(x^{\underset{\sim}{u}}(t),\ x^{\underset{\sim}{u}}(t-\tau),\ \underset{\sim}{u}(t),t) \end{aligned}\right\} \tag{2.8}$$

and

$$\begin{aligned} \Delta f(\underset{\sim}{v},\underset{\sim}{u}) &= f(x^{\underset{\sim}{u}}(t),\ x^{\underset{\sim}{u}}(t-\tau),\ \underset{\sim}{v}(t),t)-f(x^{\underset{\sim}{u}}(t),x^{\underset{\sim}{u}}(t-\tau),\underset{\sim}{u}(t),t) \\ &= f(x^{\underset{\sim}{u}}(t),x^{\underset{\sim}{u}}(t-\tau),\underset{\sim}{v}(t) \ominus \underset{\sim}{u}(t),t) \end{aligned} \tag{2.9}$$

$z^{\underset{\sim}{u},\underset{\sim}{v}}$ may be regarded as a first order estimate of $x^{\underset{\sim}{v}}-x^{\underset{\sim}{u}}$ for $\underset{\sim}{u},\underset{\sim}{v} \in \underset{\sim}{G}$, i.e. (2.6) and (2.7) is in effect obtained by linearizing the nonlinear system defined by (2.2) and (2.3) about the control $\underset{\sim}{u}$.

For all $\underset{\sim}{u} \in \underset{\sim}{G}$ let $R(\underset{\sim}{u})$ denote the reachable set of this linearized system, i.e.

$$R(\underset{\sim}{u}) = \{z^{\underset{\sim}{u},\underset{\sim}{v}}(1) : \underset{\sim}{v} \in \underset{\sim}{G}\}$$

Then we have the following result:

Proposition 2.1

For all $\underset{\sim}{u} \in \underset{\sim}{G}$, the set $R(\underset{\sim}{u})$ is convex and compact.

Proof

Easily deduced from Corollary 37.9 in Young [Y1] using results from Bellman and Cooke [BC1], Oğuztöreli [OG1].

The adjoint functions $\lambda : T \longrightarrow \mathbb{R}^n$ are defined as follows:

Definition 2.2

For all $\underset{\sim}{u} \in \underset{\sim}{G}$, $\lambda_j^{\underset{\sim}{u}}(t)$, $j=0,1,\ldots,r$ is the solution of

$$-\dot{\lambda}_j^T(t) = \lambda_j^T(t)A^{\underset{\sim}{u}}(t) + \lambda_j^T(t+\tau)B^{\underset{\sim}{u}}(t+\tau) \qquad (2.10)$$

$$\text{for a.a } t \in [0, 1-\tau]$$

$$-\dot{\lambda}_j^T(t) = \lambda_j^T(t)A^{\underset{\sim}{u}}(t) \qquad \text{for a.a } t \in [1-\tau, 1] \qquad (2.11)$$

$$\lambda_j(1) = h_x^j(x^{\underset{\sim}{u}}(1))^T \qquad (2.12)$$

where $A^{\underset{\sim}{u}}$, $B^{\underset{\sim}{u}}$ are defined in (2.8).

Proposition 2.3

For all $\underset{\sim}{u} \in \underset{\sim}{G}$, there exist absolutely continuous functions $x^{\underset{\sim}{u}}$, $\lambda_j^{\underset{\sim}{u}}$, $j=0,1,2,\ldots,r$ which are unique solutions of (2.2), (2.3) and (2.10)-(2.12) respectively.

Furthermore for all $\underset{\sim}{u},\underset{\sim}{v} \in \underset{\sim}{G}$ there exists an absolutely continuous function $z^{\underset{\sim}{u},\underset{\sim}{v}}$ which is the unique solution of (2.6), (2.7).

For a proof of this standard result see Young [Y1] where it is given for the delay free case. This result can be easily extended using methods in Hale [HAL1], Bellman and Cooke [BC1] and Oǧuztöreli [OG1] to cover our delay case.

Proposition 2.4

For all $\underset{\sim}{u} \in \underset{\sim}{G}$, for all $t \in T$,

(i) $\| x^{\underset{\sim}{u}}(t) \| \leq d$

(ii) $\| \lambda_j^{\underset{\sim}{u}}(t) \| \leq d$ for $j=0,1,2,\ldots,r$

for some finite d.

The proof of this is a straightforward application of the Gronwall Inequality as in Chapters 2 and 3.

Since $\| x^{\underset{\sim}{u}} \|_\infty \leq d$ for all $\underset{\sim}{u} \in \underset{\sim}{G}$ we need only consider $(x,y,\underset{\sim}{u},t)$ lying in the compact set $B_1 x B_1 x \underset{\sim}{G} x T$ where we define B_1 (as before) by

$$B_1 \triangleq \{x \in \mathbb{R}^n : \| x \| \leq d \}$$

Hence terms like f, f_x, f_{xx}, etc. are uniformly continuous on $B_1 x B_1 x \underset{\sim}{G} x T$.

Now using the properties possessed by relaxed controls (see sections B7 and B8 in Chapter 1) we can prove the following:

Proposition 2.5

For all sequences $\{\underset{\sim}{u}_i\}_{i=0}^{\infty} \in \underset{\sim}{G}$ converging i.s.c.m. to $\underset{\sim}{u}^* \in \underset{\sim}{G}$ we have $z^{\underset{\sim}{u}_i,\underset{\sim}{v}} \longrightarrow z^{\underset{\sim}{u}^*,\underset{\sim}{v}}$ in L_∞ uniformly in $\underset{\sim}{v} \in \underset{\sim}{G}$.

Proof

For all $\underset{\sim}{u}$, $\underset{\sim}{v} \in \underset{\sim}{G}$ we have from (2.6), (2.7), $z^{\underset{\sim}{u},\underset{\sim}{v}}$ is the solution of

$$\dot{z}(t)=A^{\underset{\sim}{u}}(t)z(t)+B^{\underset{\sim}{u}}(t)z(t-\tau)+\Delta f(\underset{\sim}{v},\underset{\sim}{u}) \qquad \text{for a.a.} t \in T$$

$$z(t)=0 \qquad \text{for } t \in [-\tau, 0]$$

As in Appendix C we have

$$z^{\underset{\sim}{u},\underset{\sim}{v}}(t) = \int_0^t \phi^{\underset{\sim}{u}}(s,t)\ \Delta f(\underset{\sim}{v},\underset{\sim}{u})ds \qquad \text{for all } t \in T$$

where $\phi^{\underset{\sim}{u}}$ is the state transition matrix which satisfies

$$\frac{\delta\phi(s,t)}{\delta s} = -\phi(s,t)A^{\underset{\sim}{u}}(s)-\phi(s+\tau,t)B^{\underset{\sim}{u}}(s+\tau) \qquad \text{for a.a.} s \in [0,t]$$

$$\phi(t,t) = I$$

and $\phi(s,t) = 0$ for $s > t$

For any sequence $\underset{\sim}{u}_i \longrightarrow \underset{\sim}{u}^*$ i.s.c.m. it can be shown using a similar procedure as in Appendix A that $\phi^i(s,t)$ converges to $\phi^*(s,t)$ uniformly in $s,t \in T$, where ϕ^i and ϕ^* satisfy the above adjoint relations with $\underset{\sim}{u}$ replaced by $\underset{\sim}{u}_i$ and $\underset{\sim}{u}^*$ respectively.

Then for all $\underset{\sim}{u}_i, \underset{\sim}{v} \in \underset{\sim}{G}$

$$z^{\underset{\sim}{u}_i,\underset{\sim}{v}}(t) = \int_0^t \phi^i(s,t)\ \Delta f(\underset{\sim}{v},\underset{\sim}{u}_i)ds \qquad \text{for all } t \in T$$

Adding and subtracting terms we get

$$z^{\underset{\sim}{u}_i,\underset{\sim}{v}}(t) = \int_0^t \{\phi^*(s,t)\ \Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)-\phi^*(s,t)\ \Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)$$
$$+\phi^i(s,t)\ \Delta f(\underset{\sim}{v},\underset{\sim}{u}_i) + \phi^*(s,t)\Delta f(\underset{\sim}{v},\underset{\sim}{u}_i)$$
$$-\phi^*(s,t)\Delta f(\underset{\sim}{v},\underset{\sim}{u}_i)\}\ ds \qquad \text{for all } t \in T$$

Then we have

$$z^{\underset{\sim}{u}_i,\underset{\sim}{v}}(t) = \int_0^t \phi^*(s,t)\Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)ds$$

$$+ \int_0^t \phi^*(s,t)\,[\Delta f(\underset{\sim}{v},\underset{\sim}{u}_i)-\Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)]\,ds$$

$$+ \int_0^t [\phi^i(s,t)-\phi^*(s,t)]\,\Delta f(\underset{\sim}{v},\underset{\sim}{u}_i)ds \quad \text{for all } t \in [0,T]$$

By the uniform continuity of f and ϕ, we deduce that as $\underset{\sim}{u}_i \longrightarrow \underset{\sim}{u}^*$ i.s.c.m, $z^{\underset{\sim}{u}_i,\underset{\sim}{v}}(t) \longrightarrow z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t)$ uniformly in $t \in T$ and $\underset{\sim}{v} \in \underset{\sim}{G}$, where

$$z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t) = \int_0^t \phi^*(s,t)\,\Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)ds \qquad \text{for all } t \in T$$

This proves the Proposition.

Proposition 2.6

For all $\alpha \in [0,1]$, all $\underset{\sim}{u},\underset{\sim}{v} \in \underset{\sim}{G}$ we have

$$\alpha\, z^{\underset{\sim}{u},\underset{\sim}{v}}(t) = z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t) \qquad \text{for all } t \in T$$

where $\underset{\sim}{u}_\alpha = (1-\alpha)\,\underset{\sim}{u} \oplus \alpha\underset{\sim}{v}$.

Proof

From (2.6) $z^{\underset{\sim}{u},\underset{\sim}{v}}$ is the solution of

$$\dot{z}(t) = A^{\underset{\sim}{u}}(t)z(t)+B^{\underset{\sim}{u}}(t)z(t-\tau)+f(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\ \underset{\sim}{v} \ominus \underset{\sim}{u},t) \qquad \text{for a.a. } t \in T$$

i.e. we have

$$\dot{z}^{\underset{\sim}{u},\underset{\sim}{v}}(t) = A^{\underset{\sim}{u}}(t)z^{\underset{\sim}{u},\underset{\sim}{v}}(t)+B^{\underset{\sim}{u}}(t)z^{\underset{\sim}{u},\underset{\sim}{v}}(t-\tau)+f(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\ \underset{\sim}{v} \ominus \underset{\sim}{u},t) \qquad \text{a.a. } t \in T$$

Multiplying throughout by $\alpha \in [0,1]$ we get

$$\alpha\dot{z}^{\underset{\sim}{u},\underset{\sim}{v}}(t)=A^{\underset{\sim}{u}}(t).\alpha z^{\underset{\sim}{u},\underset{\sim}{v}}(t)+B^{\underset{\sim}{u}}(t).\alpha z^{\underset{\sim}{u},\underset{\sim}{v}}(t-\tau)+\alpha f(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\ \underset{\sim}{v} \ominus \underset{\sim}{u},t) \qquad \text{for a.a. } t \in T \quad (A)$$

Similarly $z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}$ is the solution of

$$\dot{z}(t)=A^{\underset{\sim}{u}}(t)z(t)+B^{\underset{\sim}{u}}(t)z(t-\tau)+f(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{u}_\alpha \ominus \underset{\sim}{u},t) \quad \text{a.e in } T$$

Substituting for $\underset{\sim}{u}_\alpha$ and using the "linear" nature of relaxed controls we get

$$\dot{z}(t)=A^{\underset{\sim}{u}}(t)z(t)+B^{\underset{\sim}{u}}(t)z(t-\tau)+\alpha f(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{v} \ominus \underset{\sim}{u},t) \quad \text{for a.a. } t \in T$$

i.e.

$$\dot{z}^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t)=A^{\underset{\sim}{u}}(t)z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t)+B^{\underset{\sim}{u}}(t)z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t-\tau)+\alpha f(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{v} \ominus \underset{\sim}{u},t)$$

$$\text{for a.a } t \in T \qquad \text{(B)}$$

Comparing the two delay-differential equations (A) and (B) for $\alpha z^{\underset{\sim}{u},\underset{\sim}{v}}$ and $z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}$ we deduce the required result.

<u>Proposition 2.7</u>

For all $\underset{\sim}{u}, \underset{\sim}{v} \in \underset{\sim}{G}$, $\alpha \in [0,1]$ we have

$$\| (x^{\underset{\sim}{u}_\alpha}(t)-x^{\underset{\sim}{u}}(t)) - z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t) \| \leq d\alpha^2$$

for some $d \in (0,\infty)$ where $\underset{\sim}{u}_\alpha$ is as in Proposition 2.6.

<u>Proof</u>

Let $e(t)=(x^{\underset{\sim}{u}_\alpha}(t)-x^{\underset{\sim}{u}}(t))-z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t)$

Then we have that

$$e(t)= \int_0^t \{f(x^{\underset{\sim}{u}_\alpha},y^{\underset{\sim}{u}_\alpha},\underset{\sim}{u}_\alpha,s)-f(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{u},s)$$
$$-A^{\underset{\sim}{u}}(s)z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(s)-B^{\underset{\sim}{u}}(s)z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(s-\tau)-\Delta f(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) \}ds \quad \text{for all } t \in T$$

Expanding $f(x^{\underset{\sim}{u}_\alpha},y^{\underset{\sim}{u}_\alpha},\underset{\sim}{u}_\alpha,s)$ using Taylor series to second order with remainder term we get

$$e(t)=\int_0^t\{f(x^{u},y^{u},u_\alpha,s)+f_x(x^{u},y^{u},u_\alpha,s)(x^{u_\alpha}-x^{u})$$

$$+f_y(x^{u},y^{u},u_\alpha,s)(y^{u_\alpha}-y^{u})+f_{xy}(x^{u\varepsilon},y^{u\varepsilon},u_\alpha,s)(x^{u_\alpha}-x^{u})(y^{u_\alpha}-y^{u})$$

$$+\frac{f_{xx}(x^{u\varepsilon},y^{u\varepsilon},u_\alpha,s)(x^{u_\alpha}-x^{u})^2}{2}+\frac{f_{yy}(x^{u\varepsilon},y^{u\varepsilon},u_\alpha,s)(y^{u_\alpha}-y^{u})^2}{2}$$

$$-f(x^{u},y^{u},u,s)$$

$$-f_x(x^{u},y^{u},u,s)z^{u,u_\alpha}(s)-f_y(x^{u},y^{u},u,s)z^{u,u_\alpha}(s-\tau)$$

$$-f(x^{u},y^{u},u_\alpha\ominus u,s)\ \}\,ds \qquad \text{for all } t\in T$$

where $x^{u\varepsilon} \triangleq (1-\varepsilon)x^{u} + \varepsilon x^{u_\alpha}$ for some $\varepsilon \in (0,1)$ and $y^{u\varepsilon}$ is similiarly defined.

Remark

Throughout this thesis we write expressions involving multiplications between matrices, vectors, etc. in a form that is most convenient. For example, in the above equation we have written

$$\frac{(x^{u_\alpha}-x^{u})^T f_{xx}(x^{u\varepsilon},y^{u\varepsilon},u_\alpha,s)(x^{u_\alpha}-x^{u})}{2}$$

as

$$\frac{f_{xx}(x^{u\varepsilon},y^{u\varepsilon},u_\alpha,s)(x^{u_\alpha}-x^{u})^2}{2}$$

etc. These two expressions are equivalent when their modulii is taken. This should cause no confusion.

Adding and subtracting terms

$$e(t)=\int_0^t \{f_x(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{u},s)e(s)+f_y(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{u},s)e(s-\tau)$$

$$+[f_x(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{u}_\alpha,s)-f_x(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{u},s)](x^{\underset{\sim}{u}\alpha}-x^{\underset{\sim}{u}})$$

$$+[f_y(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{u}_\alpha,s)-f_y(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{u},s)](y^{\underset{\sim}{u}\alpha}-y^{\underset{\sim}{u}})$$

$$+f_{xy}(x^{\underset{\sim}{u}\epsilon},y^{\underset{\sim}{u}\epsilon},\underset{\sim}{u}_\alpha,s)(x^{\underset{\sim}{u}\alpha}-x^{\underset{\sim}{u}})(y^{\underset{\sim}{u}\alpha}-y^{\underset{\sim}{u}})$$

$$+\frac{f_{xx}(x^{\underset{\sim}{u}\epsilon},y^{\underset{\sim}{u}\epsilon},\underset{\sim}{u}_\alpha,s)(x^{\underset{\sim}{u}\alpha}-x^{\underset{\sim}{u}})^2}{2}+\frac{f_{yy}(x^{\underset{\sim}{u}\epsilon},y^{\underset{\sim}{u}\epsilon},\underset{\sim}{u}_\alpha,s)(y^{\underset{\sim}{u}\alpha}-y^{\underset{\sim}{u}})^2}{2}\}ds$$

for all $t \in T$

Since $\underset{\sim}{u}_\alpha = (1-\alpha)\underset{\sim}{u} \oplus \alpha\underset{\sim}{v}$ we get

$$e(t)=\int_0^t \{f_x(\underset{\sim}{u})e(s)+f_y(\underset{\sim}{u})e(s-\tau)$$

$$+\alpha f_x(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{v}\ominus\underset{\sim}{u},s)(x^{\underset{\sim}{u}\alpha}-x^{\underset{\sim}{u}})$$

$$+\alpha f_y(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{v}\ominus\underset{\sim}{u},s)(y^{\underset{\sim}{u}\alpha}-y^{\underset{\sim}{u}})$$

$$+f_{xy}(x^{\underset{\sim}{u}\epsilon},y^{\underset{\sim}{u}\epsilon},\underset{\sim}{u}_\alpha,s)(x^{\underset{\sim}{u}\alpha}-x^{\underset{\sim}{u}})(y^{\underset{\sim}{u}\alpha}-y^{\underset{\sim}{u}})$$

$$+\frac{f_{xx}(x^{\underset{\sim}{u}\epsilon},y^{\underset{\sim}{u}\epsilon},\underset{\sim}{u}_\alpha,s)(x^{\underset{\sim}{u}\alpha}-x^{\underset{\sim}{u}})^2}{2}+\frac{f_{yy}(x^{\underset{\sim}{u}\epsilon},y^{\underset{\sim}{u}\epsilon},\underset{\sim}{u}_\alpha,s)(y^{\underset{\sim}{u}\alpha}-y^{\underset{\sim}{u}})^2}{2}\}ds$$

where $f_x(\underset{\sim}{u}) = f_x(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{u},s)$

and $f_y(\underset{\sim}{u}) = f_y(x^{\underset{\sim}{u}},y^{\underset{\sim}{u}},\underset{\sim}{u},s)$

By the uniform boundedness of f_x, f_{xx}, f_y, f_{yy}, f_{xy}, there exist finite constants d_1, d_2, d_3, d_4, d_5, d_6, d_7 such that

$$\| e(t) \| \le d_1\int_0^t \| e(s) \| ds+d_2\int_0^t \| e(s-\tau) \| ds$$

$$+\alpha d_3\int_0^t \| x^{\underset{\sim}{u}\alpha}(s)-x^{\underset{\sim}{u}}(s) \| +\alpha d_4\int_0^t \| x^{\underset{\sim}{u}\alpha}(s-\tau)-x^{\underset{\sim}{u}}(s-\tau) \| ds$$

$$+d_5\int_0^t \| x^{\underset{\sim}{u}\alpha}(s)-x^{\underset{\sim}{u}}(s) \| \, \| x^{\underset{\sim}{u}\alpha}(s-\tau)-x^{\underset{\sim}{u}}(s-\tau) \| ds$$

$$+d_6\int_0^t \| x^{\underset{\sim}{u}\alpha}(s)-x^{\underset{\sim}{u}}(s) \|^2 ds+d_7\int_0^t \| x^{\underset{\sim}{u}\alpha}(s-\tau)-x^{\underset{\sim}{u}}(s-\tau) \|^2 ds$$

for all $t \in T$

i.e. we have

$$\| e(t) \| \leq d_8 \int_0^t \| e(s) \| \, ds + d_9 \alpha \int_0^t \| x^{\underset{\sim}{u}_\alpha}(s) - x^{\underset{\sim}{u}}(s) \| \, ds$$

$$d_{10} \int_0^t \| x^{\underset{\sim}{u}_\alpha}(s) - x^{\underset{\sim}{u}}(s) \|^2 ds \qquad \text{for all } t \in T$$

for some d_8, d_9, $d_{10} \in (0, \infty)$.

Using Proposition 2.5 in Chapter 3 we have that

$$\| x^{\underset{\sim}{u}_\alpha}(t) - x^{\underset{\sim}{u}}(t) \| \leq d_{11} \alpha \qquad \text{for all } t \in T$$

for some finite d_{11}.

Hence the above becomes

$$\| e(t) \| \leq d_8 \int_0^1 \| e(s) \| \, ds + d_{12} \alpha^2 \qquad \text{for all } t \in T$$

for some finite d_{12}.

By Gronwall inequality we have

$$\| e(t) \| \leq d_{12} \alpha^2 \exp d_8 \qquad \text{for all } t \in T$$

Hence we have

$$\| (x^{\underset{\sim}{u}_\alpha}(t) - x^{\underset{\sim}{u}}(t)) - z^{\underset{\sim}{u}, \underset{\sim}{u}_\alpha}(t) \| \leq d \, \alpha^2 \qquad \text{for all } t \in T$$

as required.

4.3 The Exact Penalty Function

The two types of contraints, (2.4) and (2.5), make it very difficult to obtain efficient algorithms. Of course, algorithms of the standard penalty type are easily developed, but these are usually computationally expensive and lead to ill-conditioning as the penalty term approaches infinity (see Appendix B). The non-differentiability of the control constraints (in the space of control functions) make it very difficult to extend finite dimensional algorithms (e.g. gradient projection algorithms or multiplier methods) to cover our infinite dimensional case.

Therefore we present a different approach for solving Problem P2. We propose, as in Mayne and Polak [MAP2], to solve P2 by solving an equivalent unconstrained problem $P2_c$ defined below. This method uses an exact penalty function to handle the cost and terminal constraints and uses the control constraints to define the space of permissible search directions. The penalty parameter c will be adjusted automatically to ensure equivalence of the two problems. A finite c will achieve this equivalence.

We will now formulate the equivalent unconstrained problem $P2_c$ which will be easier to solve than P2. For this purpose we define $\gamma: \underset{\sim}{G} \longrightarrow \mathbb{R}$ as follows

$$\gamma(\underset{\sim}{u}) = \max_{j=1-r} \{ |g^j(\underset{\sim}{u})| \}$$

where j=1-r denotes $j \in \{1,2,\ldots,r\}$

i.e. $$\gamma(\underset{\sim}{u}) = \max_{j=1-2r} \{g^j(\underset{\sim}{u})\} \qquad (3.1)$$

where $g^j(\underset{\sim}{u})$ for $j=1-r$ is defined by

$$g^{j+r}(\underset{\sim}{u}) = -g^j(\underset{\sim}{u})$$

For all c>0, we define $\tilde{\gamma}_c: \underset{\sim}{G} \longrightarrow \mathbb{R}$ by

$$\tilde{\gamma}_c(\underset{\sim}{u}) = \frac{h^0(x^{\underset{\sim}{u}}(1))}{c} + \gamma(\underset{\sim}{u}) \qquad (3.2)$$

Then we define Problem $P2_c$ by:

Problem $P2_c$

$$\underset{\underset{\sim}{u}}{\text{Min}} \quad \tilde{\gamma}_c(\underset{\sim}{u})$$

$$\text{s.t.} \quad \underset{\sim}{u} \in \underset{\sim}{G}$$

A suitable, finite value of c (which guarantees equivalence) will be determined by Algorithm 4.

Problem $P2_c$ could have been defined using a different form, e.g. $\tilde{\gamma}_c(\underset{\sim}{u})=g^{\circ}(\underset{\sim}{u})+c\,\gamma(\underset{\sim}{u})$ instead of (3.2) as discussed in Appendix B, but we use (3.2) to follow the methodology proposed by Mayne and Polak. The different method will be used to solve the state constrained control problem in Chapter 5.

To solve problem $P2_c$ we need to know whether γ and/or $\tilde{\gamma}_c$ can be reduced at each $\underset{\sim}{u} \in \underset{\sim}{G}$. For this purpose we define $\theta:\underset{\sim}{G} \longrightarrow \mathbb{R}$ and $\tilde{\theta}_c:\underset{\sim}{G} \longrightarrow \mathbb{R}$ (which may be regarded as estimates of the maximum reduction in γ and $\tilde{\gamma}_c$ respectively) by:

$$\theta(\underset{\sim}{u})=\min_{\underset{\sim}{v}\in\underset{\sim}{G}} \; \max_{j=1-2r} \{h^j(x^{\underset{\sim}{u}}(1))+\langle h_x^j(x^{\underset{\sim}{u}}(1)),z^{\underset{\sim}{u},\underset{\sim}{v}}(1)\rangle\}-\gamma(\underset{\sim}{u}) \tag{3.3}$$

$$\tilde{\theta}_c(\underset{\sim}{u})=\min_{\underset{\sim}{v}\in\underset{\sim}{G}} \; \max_{j=1-2r} \{h^j(x^{\underset{\sim}{u}}(1))+\langle \frac{h_x^{\circ}(x^{\underset{\sim}{u}}(1))}{c} + h_x^j(x^{\underset{\sim}{u}}(1)),z^{\underset{\sim}{u},\underset{\sim}{v}}(1)\rangle\}$$

$$-\gamma(\underset{\sim}{u}) \tag{3.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product, i.e. $\langle x,y\rangle=x^Ty$.

Since $\underset{\sim}{G}$ is compact, $\theta(\underset{\sim}{u})$ and $\tilde{\theta}_c(\underset{\sim}{u})$ and their corresponding minimising controls in $\underset{\sim}{G}$ exist for all $\underset{\sim}{u} \in \underset{\sim}{G}$.

We will need the following properties for the functions θ and $\tilde{\theta}_c$.

Proposition 3.1

The functions θ, $\tilde{\theta}_c$ (for all $c \in (0,\infty)$) are sequentially continuous i.s.c.m.

Proof

Consider an infinite sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty} \in \underset{\sim}{G}$ converging i.s.c.m. to $\underset{\sim}{u}^* \in \underset{\sim}{G}$, i.e. we have

$$\underset{\sim}{u}_i \xrightarrow[i \to \infty]{} \underset{\sim}{u}^* \quad \text{i.s.c.m.}$$

Then we need to show that as $i \longrightarrow \infty$, we have

$$\tilde{\theta}_c(\underset{\sim}{u}_i) \longrightarrow \tilde{\theta}_c(\underset{\sim}{u}^*)$$

and $$\theta(\underset{\sim}{u}_i) \longrightarrow \theta(\underset{\sim}{u}^*)$$

Consider $\tilde{\theta}_c$.

For all $c > 0$ we define $\psi_c : \underset{\sim}{G} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\psi_c(\underset{\sim}{u},z) = \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}}(1)) + < \frac{h_x^0(x^{\underset{\sim}{u}}(1))}{c} + h_x^j(x^{\underset{\sim}{u}}(1)), z > \} - \gamma(\underset{\sim}{u})$$

Since h^j and $\underset{\sim}{u} \longrightarrow h_x^j(x^{\underset{\sim}{u}}(1))$, $j=0,1,\ldots,2r$ are sequentially continuous i.s.c.m., and γ is the maximum of continuous functions, we must have that ψ_c is continuous in $\underset{\sim}{u}$, z.

Hence $$\tilde{\theta}_c(\underset{\sim}{u}) = \min_{\underset{\sim}{v} \in \underset{\sim}{G}} \psi_c(\underset{\sim}{u}, z^{\underset{\sim}{u},\underset{\sim}{v}}(1))$$

For each $\underset{\sim}{u} \in \underset{\sim}{G}$, let $\eta(\underset{\sim}{u})$ denote the set of $\underset{\sim}{v} \in \underset{\sim}{G}$ which solve the above equation, (the set $\eta(\underset{\sim}{u})$ is non-empty) so that

$$\tilde{\theta}_c(\underset{\sim}{u}_i) = \psi_c(\underset{\sim}{u}_i, z^{\underset{\sim}{u}_i, \check{\underset{\sim}{u}}_i}(1)) \qquad \text{for all } \check{\underset{\sim}{u}}_i \in \eta(\underset{\sim}{u}_i)$$

and similarly we have

$$\tilde{\theta}_c(\underset{\sim}{u}^*) = \psi_c(\underset{\sim}{u}^*, z^{\underset{\sim}{u}^*,\check{\underset{\sim}{u}}^*}(1)) \qquad \text{for all } \check{\underset{\sim}{u}}^* \in \eta(\underset{\sim}{u}^*)$$

Hence we have that

$$\tilde{\theta}_c(\underset{\sim}{u}^*)-\tilde{\theta}_c(\underset{\sim}{u}_i) \leq \psi_c(\underset{\sim}{u}^*, z^{\underset{\sim}{u}^*,\check{\underset{\sim}{u}}_i}(1))-\psi_c(\underset{\sim}{u}_i, z^{\underset{\sim}{u}_i,\check{\underset{\sim}{u}}_i}(1))$$

$$\text{for all } \check{\underset{\sim}{u}}_i \in \eta(\underset{\sim}{u}_i)$$

and

$$\tilde{\theta}_c(\underset{\sim}{u}^*)-\tilde{\theta}_c(\underset{\sim}{u}_i) \geq \psi_c(\underset{\sim}{u}^*, z^{\underset{\sim}{u}^*,\check{\underset{\sim}{u}}^*}(1))- \psi_c(\underset{\sim}{u}_i, z^{\underset{\sim}{u}_i,\check{\underset{\sim}{u}}^*})$$

$$\text{for all } \check{\underset{\sim}{u}}^* \in \eta(\underset{\sim}{u}^*)$$

Now since $\underset{\sim}{u}_i \longrightarrow \underset{\sim}{u}^*$ i.s.c.m. we have by Proposition 2.5 that

$$z^{\underset{\sim}{u}_i,\underset{\sim}{v}} \longrightarrow z^{\underset{\sim}{u}^*,\underset{\sim}{v}} \quad \text{in } L_\infty \text{ uniformly in } v \in G.$$

Hence $\| \psi_c(\underset{\sim}{u}^*, z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(1))-\psi_c(\underset{\sim}{u}_i, z^{\underset{\sim}{u}_i,\underset{\sim}{v}}(1))\| \longrightarrow 0$ uniformly in $\underset{\sim}{v} \in \underset{\sim}{G}$ as $i \longrightarrow \infty$.

Hence $\| \tilde{\theta}_c(\underset{\sim}{u}^*)-\tilde{\theta}_c(\underset{\sim}{u}_i) \| \longrightarrow 0$ as $i \longrightarrow \infty$.

Hence $\tilde{\theta}_c(\underset{\sim}{u}_i) \longrightarrow \tilde{\theta}_c(\underset{\sim}{u}^*)$.

Therefore $\tilde{\theta}_c$ is sequentially continuous i.s.c.m.

The sequential continuity of θ can be obtained in a similar fashion.

4.4 The Constraint Qualification

Before proceeding any further it is necessary to make further assumptions concerning Problem P2. These extra assumptions, which constitute a constraint qualification, are needed, firstly, to ensure certain conditions of optimality and secondly to ensure that the algorithm does not jam up at undesirable points.

This can be further explained by considering the following nonlinear programming problem:

$$\text{Min} \{ g^0(x) \;:\; g^j(x) = 0, \quad j=1,2,\ldots,s \}.$$

It is well known (see Chapter 1) that a first order necessary condition for x^* to be optimal is that there exist multipliers λ^j, $j=0,1,..,s$, not all zero, such that $\sum_{j=0}^{s} \lambda^j g_x^j(x^*)=0$. The assumption that $\{ g_x^j(x^*),\ j = 1,2,\ldots,s\}$ be a set of linearly independent vectors ensures that $\lambda^0 \neq 0$ in which case it may be normalized to unity. A further assumption that the set $\{g_x^j(x),\ j=1,2,\ldots,s \}$ be linearly independent for all x ensures that given any non-feasible x, it is possible to generate a new point "closer" to the feasible region.

We will now state the constraint qualifications in a form applicable to Problem P2. To ensure that the algorithm generates desirable limit points, we need that for each $\underset{\sim}{u} \in \underset{\sim}{G}$ which is not feasible for P2 (i.e. $\gamma(\underset{\sim}{u})>0$), we can reduce $\gamma(\underset{\sim}{u})$. Now $\gamma(\underset{\sim}{u})$ can be reduced if $\theta(\underset{\sim}{u})<0$. Therefore our first constraint qualification is that $\theta(\underset{\sim}{u})<0$ for all $\underset{\sim}{u} \in \underset{\sim}{G}$ such that $\gamma(\underset{\sim}{u})>0$.

Another property required is that for any $\underset{\sim}{u}^*$ which is a global or local minima for Problem P2, then there exists a finite c such that $\underset{\sim}{u}^*$ is also a solution for $P2_c$ (i.e. which makes P2 and $P2_c$ equivalent). As stated in Mayne and Polak [MAP2], the essence of this is that for each $\underset{\sim}{u} \in \underset{\sim}{G}$ such that $\gamma(\underset{\sim}{u})=0$, the point $0 \in \mathbb{R}^r$ is in the interior of the set

$$\{ (\langle h_x^1(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1) \rangle,\ \langle h_x^2(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1) \rangle,\ \ldots.. $$
$$\ldots.,\ \langle h_x^r(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1) \rangle) : \underset{\sim}{v} \in \underset{\sim}{G} \ \}$$

To formulate this into a constraint qualification we will use the methodology in [MAP2]. Although this procedure is quite long and complicated, we use it here because we want to deviate as little as possible from Mayne and Polak's original method for solving the delay free version of Problem P2. However in Chapter 5 we present a much neater method where we only consider "calm" problems.

For the case of only one terminal equality constraint the constraint qualification may be stated as

$$\min_{\underset{\sim}{v} \in \underset{\sim}{G}} < h_x^1(x^{\underset{\sim}{u}}(1)),\ z^{\underset{\sim}{u},\underset{\sim}{v}}(1) > \ < 0$$

and $$\min_{\underset{\sim}{v} \in \underset{\sim}{G}} < h_x^2(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1) > = \min_{\underset{\sim}{v} \in \underset{\sim}{G}} \{ - < h_x^1(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1) > \} < 0$$

for all $\underset{\sim}{u} \in \underset{\sim}{G}$ with $\gamma(\underset{\sim}{u}) = 0$ (this means $0 \in \mathbb{R}$ is in the interior of the set

$$\{ < h_x^1(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1) > : \underset{\sim}{v} \in \underset{\sim}{G} \}).$$

To deal with the general case (r > 1, terminal equality constraints) we need to define a few terms. Let $I_1, I_2, \ldots, I_{2^r}$ denote the following sets

$$\begin{aligned} I_1 &= \{1,2,3,\ldots\ldots\ldots, r-2, r-1, r\} \\ I_2 &= \{1,2,3,\ldots\ldots\ldots, r-2, r-1, 2r\} \\ I_3 &= \{1,2,3,\ldots\ldots\ldots, r-2, 2r-1, r\} \\ I_4 &= \{1,2,3,\ldots\ldots\ldots, r-2, 2r-1, 2r\} \qquad (4.1) \\ &\ \ \vdots \qquad\qquad \vdots \\ I_{2^r} &= \{r+1, r+2, r+3, \ldots\ldots\ldots 2r-2, 2r-1, 2r\} \end{aligned}$$

These sets have the property that if $j \in I_i$, $j<r$ then $j+r \notin I_i$. Define J by

$$J \triangleq \{ I_i : i = 1,2,\ldots,2^r \}$$

Then for each $I \in J$ let $\phi^I : G \longrightarrow \mathbb{R}$ be defined by

$$\phi^I(\underset{\sim}{u}) \triangleq \min_{\underset{\sim}{v} \in \underset{\sim}{G}} \max_{j \in I} \langle h^j_x(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1) \rangle \tag{4.2}$$

Proposition 4.1

For each $I \subset J$, the function ϕ^I is sequentially continuous i.s.c.m.

The proof is similar to the proof of sequential continuity i.s.c.m. of $\tilde{\theta}_c$ and θ.

We now state our constraint qualification.

Assumption 3

(a) If $\gamma(\underset{\sim}{u})>0$, $\underset{\sim}{u} \in \underset{\sim}{G}$, then $\theta(\underset{\sim}{u})<0$

(b) If $\gamma(\underset{\sim}{u}) = 0$, $\underset{\sim}{u} \in \underset{\sim}{G}$, then $\phi^I(\underset{\sim}{u}) < 0$ for all $I \subset J$

We are now in a position to state some consequences of Assumption 3. The first is that the set $W(\underset{\sim}{u})$ defined by

$$W(\underset{\sim}{u}) \triangleq \{(h^0(x^{\underset{\sim}{u}}(1)), h^1(x^{\underset{\sim}{u}}(1)), \ldots, h^r(x^{\underset{\sim}{u}}(1))) : \underset{\sim}{u} \in \underset{\sim}{G}\} \subset \mathbb{R}^{r+1}$$

is not tangential to the cost axis at $\underset{\sim}{u}^*$ (a minimising solution), see Fig. 4.1 where the case of one terminal equality constraint is shown.

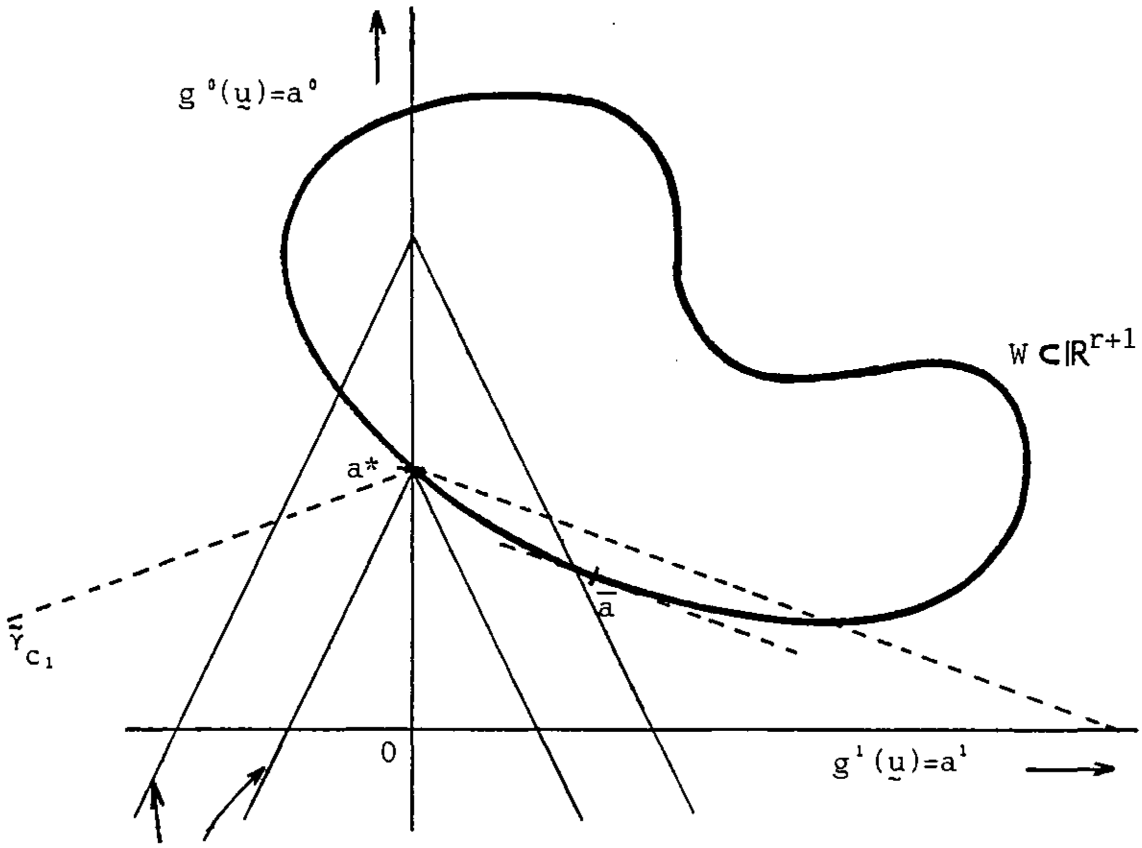


Fig. 4.1

This shows the set W of values attained by $(g^0(\underset{\sim}{u}), g^1(\underset{\sim}{u}))$ as $\underset{\sim}{u}$ ranges over the constraint set. Obviously $a^* = (a_0^*, a_1^*) = (g^0(\underset{\sim}{u}^*), g^1(\underset{\sim}{u}^*)) = (g^0(\underset{\sim}{u}^*), 0)$ is the optimal point and so $\underset{\sim}{u}^*$ is the solution to P2. Two sets of constant cost contours of $\tilde{\gamma}_c$ are shown for $c=c_1$ and c_2, $c_2 > c_1$. Clearly the solution of P_{c_2} :

$$\min \; \{ \tilde{\gamma}_{c_2} = \frac{a_0}{c_2} + a_1 \; : \; a \in W \}$$

is also the solution of P2. On the other hand, $\min \{ \tilde{\gamma}_{c_1} = \frac{a_0}{c_1} + a_1 : a \in W \}$ occurs at $\bar{a}$ which does not satisfy the equality constraint. Clearly the minimum value of c required to guarantee equivalence is given by the slope of the supporting hyperplane to W passing through a^*. This is obviously finite if W is not tangential to the cost axis at $\underset{\sim}{u}^*$ as stated above.

The second consequence of Assumption 3 gives us a necessary condition of optimality and is stated in the following proposition:

Proposition 4.2

Let $\underset{\sim}{u}^* \in \underset{\sim}{G}$ be optimal for P2. Then there exist multipliers $\psi^1, \psi^2, \ldots, \psi^r \in \mathbb{R}$ such that

$$\langle h_x^0(x^{\underset{\sim}{u}^*}(1)) + \sum_{j=1}^{r} \psi^j h_x^j (x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*, \underset{\sim}{v}}(1) \rangle \geq 0 \qquad (4.3)$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

Proof

We define the cone $C(\underset{\sim}{u}^*)$ as follows

$$C(\underset{\sim}{u}^*) \triangleq \{ \alpha z(1) : z \in R(\underset{\sim}{u}^*), \alpha \geq 0 \}$$
$$\equiv \{ z^{\underset{\sim}{u}^*, \underset{\sim}{u}_\alpha}(1) : \underset{\sim}{v} \in \underset{\sim}{G}, \alpha \geq 0 \}$$

for $\underset{\sim}{u}_\alpha = (1-\alpha)\underset{\sim}{u}^* \oplus \alpha \underset{\sim}{v}$

Since $R(\underset{\sim}{u}^*) = \{ z^{\underset{\sim}{u}^*, \underset{\sim}{v}}(1) : \underset{\sim}{v} \in \underset{\sim}{G} \}$

we have $R(\underset{\sim}{u}^*) \subset C(\underset{\sim}{u}^*)$.

Let $\tilde{R}_c(\underset{\sim}{u}^*)$ denote the compact set

$$\{ x^{\underset{\sim}{v}}(1) - x^{\underset{\sim}{u}^*}(1) : \underset{\sim}{v} \in \underset{\sim}{G} \}.$$

We intend to show that $C(\underset{\sim}{u}^*)$ is a canonical approximation of the second kind (defined below) to $\tilde{R}_c$ at the origin and then use Theorem 12 in Canon, Cullum and Polak [C1] to invoke the Proposition. However before we do so we present for the sake of completeness a definition given in Canon et al [C1]:

Definition 4

Given a subset $\Omega \subset \mathbb{R}^n$, then a convex cone $c(\check{z}, \Omega) \subset \mathbb{R}^n$ is called a canonical approximation of the second kind to the set Ω at $z \in \Omega$ if for any collection $\{ \delta z_1, \delta z_2, \ldots, \delta z_k \}$ of linearly independent vectors in $c(\hat{z}, \Omega)$ there exists an $\varepsilon > 0$, possibly depending on $\hat{z}$, δz_1, $\delta z_2, \ldots, \delta z_k$, and a continuous map $\xi(.)$ from $co\{\hat{z}, \hat{z}+\varepsilon\delta z_1, \hat{z}+\varepsilon\delta z_2, \ldots, \hat{z}+\varepsilon\delta z_k\}$ into Ω such that

$$\xi(\hat{z}+\delta z) = \hat{z}+\delta z+0(\delta z)$$

where
$$\lim_{||z||\to 0} \frac{||0(\delta z)||}{||\delta z||} = 0$$

Let z_1, $z_2, \ldots, z_q$ be any finite collection of linearly independent vectors in $C(\underset{\sim}{u}^*)$ and let $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_q \in \underset{\sim}{G}$ be the corresponding set of generating controls, i.e. we have

$$z_i = z^{\underset{\sim}{u}^*, \underset{\sim}{v}_i}(1) \qquad i=1,2,\ldots,q$$

Then consider any point $z=\sum_{i=1}^{q} \mu_i(z)z_i$ in the convex hull of $\{0,z_1,z_2,\ldots,z_q\}$, so that $\mu_i(z)\geq 0$, $i=1,2,\ldots,q$ and $\sum_{i=1}^{q} \mu_i(z)\leq 1$. z and its barycentric co-ordinates $\mu_i(z)$, $i=1,2,\ldots,q$ are related by $z= \mathbf{Z}\,\mu(z)$ where $\mathbf{Z}$ denotes the non-singular matrix with columns $z_1,z_2,\ldots,z_q$ and $\mu(z)=(\mu_1(z),\mu_2(z),\ldots,\mu_q(z))^T$. As $z\longrightarrow 0$, so does $\sigma(z)=\sum_{i=1}^{q}\mu_i(z)$ and $\sigma(z)\in[0,1]$ for all $z \in co\{0,z_1,z_2,\ldots,z_q\}$. Now any $z \in co\{0,z_1,z_2,\ldots,z_q\}$ may be written in the form $z=\sum_{i=1}^{q} \chi_i(z)\sigma(z)z_i = \sigma(z)\,\mathbf{Z}\,\chi(z)$ where $\chi_i(z)=\dfrac{\mu_i(z)}{\sigma(z)}$ $i=1,2,\ldots,q$, so that

$\chi_i(z)\geq 0$ for all $i=1,2,\ldots,q$ and $\sum_{i=1}^{q}\chi_i(z)=1$

Now define the map δx: $co\{0,z_1,\ldots,z_q\}\longrightarrow \tilde{R}_c(\underset{\sim}{u}^*)$ by

$$\delta x(z) = x^{\underset{\sim}{v}^*(z)}(1)-x^{\underset{\sim}{u}^*}(1)$$

where $\underset{\sim}{v}^*(z) = (1-\sigma(z))\underset{\sim}{u}^* \oplus \sigma(z)\underset{\sim}{v}(z)$

and $\underset{\sim}{v}(z) = \chi_1(z)\underset{\sim}{v}_1 \oplus \chi_2(z)\underset{\sim}{v}_2 \oplus \ldots \oplus \chi_q(z)\underset{\sim}{v}_q$

Since the map $z \to \mu(z)$, $co\{0,z_1,\dots,z_q\} \to \mathbb{R}^q$ is continuous so are the maps $z \to \sigma(z)$, $co\{0,z_1,\dots,z_q\} \to \mathbb{R}$ and $z \to \chi(z)$, $co\{0,z_1,\dots,z_q\} \to \mathbb{R}^q$. Hence the map $z \to \underset{\sim}{v}(z), co\{0,z_1,\dots,z_q\} \to \underset{\sim}{G}$ is sequentially continuous i.s.c.m., and so is the map $z \to \delta x(z)$, $co\{0,z_1,\dots,z_q\} \to \tilde{R}_c(\underset{\sim}{u}^*)$.

Now any point z in co $\{0,z_1,z_2,\dots,z_q\}$ may be expressed as

$$z = \sum_{i=1}^{q} \chi_i(z)\sigma(z)z_i$$

$$= \sum_{i=1}^{q} \chi_i(z)\sigma(z)z^{\underset{\sim}{u}^*,\underset{\sim}{v}i}(1)$$

Now from Proposition 2.6 this becomes

$$z = z^{\underset{\sim}{u}^*,\underset{\sim}{v}^*(z)}$$

where $\underset{\sim}{v}^*(z) = (1-\sum_{i=1}^{q} \chi_i(z)\sigma(z))\underset{\sim}{u}^* \oplus \chi_1(z)\sigma(z)\underset{\sim}{v}_1 \oplus$

$$\oplus \chi_2(z)\sigma(z)\underset{\sim}{v}_2 \oplus \dots \oplus \chi_q(z)\sigma(z)\underset{\sim}{v}_q$$

i.e. $\underset{\sim}{v}^*(z) = (1-\sigma(z))\underset{\sim}{u}^* \oplus \sigma(z)\underset{\sim}{v}(z)$ where

$$\underset{\sim}{v}(z) = \chi_1(z)\underset{\sim}{v}_1 \oplus \chi_2(z)\underset{\sim}{v}_2 \oplus \dots\dots \oplus \chi_q(z)\underset{\sim}{v}_q$$

From Proposition 2.7 we have

$$\| (x^{\underset{\sim}{u}\alpha}-x^{\underset{\sim}{u}})-z^{\underset{\sim}{u},\underset{\sim}{u}\,\alpha} \| \leq d_1\alpha^2 \qquad \text{for some } d_1 \in (0,\infty)$$

Hence from above, for any $z \in co\{0,z_1,\dots,z_q\}$ we get

$$\| x(z)-z \| \leq d_1 [\sigma(z)]^2$$

Since $\mu(z) = Z^{-1}z$, it follows from above that $\sigma(z) = \|\mu(z)\|_1 \leq d_2 \|z\|$ for some $d_2 \in (0,\infty)$, and therefore

$$\| \delta x(z)-z \| = 0(\|z\|)$$

where $\dfrac{0(\|z\|)}{z} \longrightarrow 0$ as $z \longrightarrow 0$

Hence it follows from Definition 4 (by setting $\hat{z}=0$, $\delta z_i = z_i$, $i=1,2,\ldots,q$, $\varepsilon=1$) that $C(\underset{\sim}{u}^*)$ is a canonical approximation of the second kind to $\tilde{R}_c$ at the origin. Hence by Canon, Cullum and Polak [C 1] (Theorem 12 on page 27) we have that if $\underset{\sim}{u}^* \in \underset{\sim}{G}$ is optimal for Problem P2, then there exist multipliers $\psi_0, \psi_1, \ldots, \psi_r \in \mathbb{R}$, not all zero such that

$$< \psi_0 h^0_x(x^{\underset{\sim}{u}^*}(1)) + \sum_{j=1}^{r} \psi_j h^j_x(x^{\underset{\sim}{u}^*}(1)), z > \quad \geq 0 \qquad \text{for all } z \in C(\underset{\sim}{u}^*)$$

and therefore for all $z \in R(\underset{\sim}{u}^*)$

i.e.

$$< \psi_0 h^0_x(x^{\underset{\sim}{u}^*}(1)) + \sum_{j=1}^{r} \psi_j h^j_x (x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*, \underset{\sim}{v}}(1) > \quad \geq 0 \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{G}$$

Assume that $\psi_0 = 0$, then not all the multipliers $\psi_1, \psi_2, \ldots, \psi_r$ are zero and

$$\sum_{j=1}^{r} \psi_j h^j_x (x^{\underset{\sim}{u}^*}(1))^T z^{\underset{\sim}{u}^*, \underset{\sim}{v}}(1) \geq 0 \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{G} \tag{1}$$

But by Assumption 3, since $\underset{\sim}{u}^*$ is optimal for P2 we have $\gamma(\underset{\sim}{u}^*)=0$, and

$$\phi^I(\underset{\sim}{u}^*) = \max_{j \in I} < h^j_x(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*, \bar{\underset{\sim}{v}}} >$$
$$< 0 \quad \text{for all } I \subset J$$

where $\bar{\underset{\sim}{v}} \in \underset{\sim}{G}$ is any minimising control for $\phi^I(\underset{\sim}{u}^*)$.

Hence there exists an $I \subset J$ such that

$$\max_{j \in 1,2,..r} < \psi_j h_x^j(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*,\bar{\underset{\sim}{v}}}(1) > \quad < 0$$

If $\max_{j=1-r} < \psi_j h_x^j(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*,\bar{\underset{\sim}{v}}}(1) > < 0$, then all

$$< \psi_j h_x^j(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*,\bar{\underset{\sim}{v}}}(1) > \quad < 0$$

This contradicts (1). Hence $\psi_0 \neq 0$ and may be normalised to unity.

Remark

The results of Proposition 4.2 can be stated in the (equivalent) form of a Maximum Principle (see Chapter 1 and Appendix C).

As a consequence of Proposition 4.2 we have the following corollaries whose proofs are obvious:

Corollary 4.3

If $\underset{\sim}{u}^* \in \underset{\sim}{G}$ is optimal for P2 then

$$\min \{ <h_x^0(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(1) > : <h_x^j(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(1)> = 0, j=1,2,\dots,r, \ \underset{\sim}{v} \in \underset{\sim}{G} \}$$

$$= 0$$

Corollary 4.4

If $\underset{\sim}{u}^* \in \underset{\sim}{G}$ is optimal for P2, then the ray $\{\sigma(-1,0,0,\dots 0): \sigma > 0\} \subset \mathbb{R}^{r+1}$ and the set $W(\underset{\sim}{u}^*) \triangleq \{(<h_x^0(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(1)>, \dots \dots, <h_x^r(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(1)>) : \underset{\sim}{v} \in \underset{\sim}{G}\}$ in $\mathbb{R}^{r+1}$ are strictly linearly separated.

We are now in a position to define the desirable sets for

Problems P2 and $P2_c$. It is obvious from the text that the desirable set, Δ, for P2 is defined by

$$\Delta \triangleq \{\underset{\sim}{u}^* \in \underset{\sim}{G} : \gamma(\underset{\sim}{u}^*) = 0 \text{ and (4.3) is satisfied } \} \tag{4.4}$$

We assume that the set Δ is not empty. This essentially means that there exists at least one feasible control $\underset{\sim}{u} \in \underset{\sim}{G}$, i.e.

$$h^j(x^{\underset{\sim}{u}}(1)) = 0 \qquad j=1,2,\ldots,r.$$

For Problem $P2_c$ we have by a straightforward generalization of Theorem (2.1) in Dem'yanov and Malozemov [DEM1], that if $\underset{\sim}{u}^*$ is optimal for $P2_c$ then

$$\max_{j=1-2r} < \frac{h_x^0(x^{\underset{\sim}{u}^*}(1)}{c} + h_x^j(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*, \underset{\sim}{v}}(1) > \; \geq 0$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

From the definition of $\tilde{\theta}_c$ we obtain:

<u>Proposition 4.5</u>

Suppose $\underset{\sim}{u}^*$ is optimal for problem $P2_c$, $c>0$, then $\tilde{\theta}_c(\underset{\sim}{u}^*)=0$.

Hence we define the set Δ_c of desirable points for Problem $P2_c$, $c>0$ by:

$$\Delta_c \triangleq \{ \underset{\sim}{u}^* \in \underset{\sim}{G} : \tilde{\theta}_c(\underset{\sim}{u}^*) = 0 \} \tag{4.5}$$

We need the following result to make the two problems equivalent:

<u>Proposition 4.6</u>

Suppose that $\underset{\sim}{u}^* \in \Delta$ (i.e. $\underset{\sim}{u}^*$ is desirable for P2), then there exists a $c^*>0$ such that $\tilde{\theta}_c(\underset{\sim}{u}^*)=0$ for all $c \geq c^*$.

Proof

Since $\underset{\sim}{u}^* \in \Delta$, then the ray $L = \{\sigma(-1,0,\dots 0): \sigma > 0\} \subset \mathbb{R}^{r+1}$ and the set $W(\underset{\sim}{u}^*) = \{(\langle h_x^0(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(1)\rangle, \dots, \langle h_x^r(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(1)\rangle : \underset{\sim}{v} \in \underset{\sim}{G}\} \subset \mathbb{R}^{r+1}$ are strictly linearly separated, i.e. there exists a $\psi = (1,\psi^1,\psi^2,\dots,\psi^r) \in \mathbb{R}^{r+1}$ such that

$$\langle \psi, y\rangle \geq 0 \qquad \text{for all } y \in W(\underset{\sim}{u}^*) \text{ and}$$

$$\langle \psi, y\rangle < 0 \qquad \text{for all } y \in L$$

Consider now $W_c \subset \mathbb{R}^{r+1}$, c>0 defined by $W_c \triangleq \{(y^0, y^1, \dots, y^r): \frac{y^0}{c} + y^j < 0, \frac{y^0}{c} - y^j < 0, \ j=1,2,\dots,r\}$. W_c is a convex cone with its apex at 0 and having L in its interior. As c increases, W_c becomes more and more acute and there exists a $c^* < \infty$ such that for all $c \geq c^*$, $W_c \cap W(\underset{\sim}{u}^*) = \emptyset$. Indeed for all $y \in W_c$, $|y^j| < |\frac{y^0}{c}|$ for all j=1–r, so if we set $c^* = 2 \sum_{j=1}^{r} |\psi^j| \geq 0$, then for all $c > c^*$, $y \in W_c$ implies $\langle\psi, y\rangle < 0$, where $\psi = (1,\psi^1,\dots,\psi^r)$ is specified in Proposition 4.2. This is because for $y \in W_c$ we have

$$\begin{aligned} \langle\psi,y\rangle &= y^0 + \sum_{j=1}^{r} \psi^j y^j \\ &\leq y^0 + \sum_{j=1}^{r} |\psi^j|\,|y^j| \\ &< y^0 + \sum_{j=1}^{r} |\psi^j|\,\left|\frac{y^0}{c}\right| \end{aligned}$$

since $|y^j| < |\frac{y^0}{c}|$ for all $j \in 1,2,\dots,r$, i.e.

$$\begin{aligned} \langle\psi,y\rangle &< y^0 + \frac{1}{c}|y^0| \sum_{j=1}^{r} |\psi^j| \\ &\leq y^0 + \left|\frac{y^0}{2}\right| \qquad \text{for all } c \geq c^* \\ &\leq \frac{y^0}{2} < 0 \end{aligned}$$

i.e. $\langle\psi, y\rangle < 0 \qquad$ for all $c \geq c^*$.

However for $y \in W(\underset{\sim}{u}^*)$ we have from Proposition 4.2 that $\langle \psi, y\rangle \geq 0$, so we have $W(\underset{\sim}{u}^*)$ and W_c are disjoint for $c \geq c^*$.

We now use this to show that $\underset{\sim}{u}^*$ is desirable for $P2_c$ for $c \geq c^*$. Suppose contrary to what is to be proven that $\underset{\sim}{u}^* \notin \Delta_c$, i.e. $\tilde{\theta}_c(\underset{\sim}{u}^*) = -\delta < 0$.

Now because

$$\tilde{\theta}_c(\underset{\sim}{u}^*) = \min_{\underset{\sim}{v} \in \underset{\sim}{G}} \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}^*}(1)) + \langle \frac{h_x^0(x^{\underset{\sim}{u}^*}(1))}{c} + h_x^j(x^{\underset{\sim}{u}^*}(1)), z^{\underset{\sim}{u}^*, \underset{\sim}{v}}(1) \rangle \} - \gamma(\underset{\sim}{u}^*)$$

we have since $\underset{\sim}{u}^* \in \Delta$ that $\gamma(\underset{\sim}{u}^*)=0$, therefore $h^j(x^{\underset{\sim}{u}^*}(1))=0$ for $j=1-2r$, hence

$$\tilde{\theta}_c(\underset{\sim}{u}^*) = \min_{\underset{\sim}{v} \in \underset{\sim}{G}} \max_{j=1-2r} \langle \frac{h_x^0(x^{\underset{\sim}{u}^*}(1))}{c} + h_x^j(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u}^*, \underset{\sim}{v}}(1) \rangle$$

$$= \min_{y \in W(\underset{\sim}{u}^*)} \max_{j=1-r} \{ \frac{y^0}{c} + y^j, \ \frac{y^0}{c} - y^j \}$$

But $\tilde{\theta}_c(\underset{\sim}{u}^*) = -\delta < 0$ implies there exists a $y \in W(\underset{\sim}{u}^*)$ such that $\frac{y^0}{c} + y^j < 0$ and $\frac{y^0}{c} - y^j < 0$ for all $j \in 1,2,\ldots,r$, which contradicts disjointness of $W(\underset{\sim}{u}^*)$ and W_c for $c \geq c^*$. Hence $\tilde{\theta}_c(\underset{\sim}{u}^*)=0$ for all $c \geq c^*$.

4.5 The Algorithm Model

The basic procedure we plan to undertake is to first solve Problem $P2_c$ and then find a finite c which makes it equal to Problem P2, thus solving P2. Finding such a c is quite a difficult task and is somewhat neglected in the literature. It is not sufficient to establish the existence of a finite c to guarantee equivalence, a means to increase c to a suitable value is necessary. An ad hoc procedure of increasing c at each iteration is not satisfactory as it would get excessively large and lead to computational difficulties that occur in ordinary penalty methods. Hence a method for increasing c if some test, depending on the control $\underset{\sim}{u}$, is satisfied is obviously required. This test may be of the form $t_c(\underset{\sim}{u}) > 0$.

An algorithm model which uses a test function in this way is stated in Polak [P2] and we base our algorithm for solving Problem P2 on a generalised version of it. The model consists of two main components, which are:

(i) an algorithm (defined by a point-to-set map $A_c : \underset{\sim}{G} \longrightarrow 2^{\underset{\sim}{G}}$) for solving Problem $P2_c$ for all $c>0$.

(ii) a test function $t_c : \underset{\sim}{G} \longrightarrow \mathbb{R}$ for deciding when c should be increased.

A monotonically increasing sequence $\{c_j\}_{j=0}^{\infty}$ is also required for the model to be well defined. We now present the model on which Algorithm 4 (for solving P2) will be based:

<u>Algorithm Model 2</u>

Step 0 : Select $\underset{\sim}{u}_0 \in \underset{\sim}{G}$, $\{c_j\}_{j=0}^{\infty}$

Step 1 : Set i=0

Set j=0

Step 2 : If $t_{c_j}(\underset{\sim}{u}_i)>0$, set j=j+1 and repeat Step 2

Else proceed to Step 3

Step 3 : If $\underset{\sim}{u}_i \in \Delta_{c_j}$ stop

Else compute a $\underset{\sim}{u} \in A_{c_j}(\underset{\sim}{u}_i)$

Step 4 : Set $\underset{\sim}{u}_{i+1}=\underset{\sim}{u}$

Set $i=i+1$

Goto Step 2

The following result is presented in Polak [P2] for the Algorithm Model 2:

Theorem 5.1

If

(i) For each j, A_{c_j} is such that any accumulation point $\underset{\sim}{u}^* \in \underset{\sim}{G}$ i.s.c.m. of an infinite sequence $\{\underset{\sim}{u}_i\}$ in $\underset{\sim}{G}$ with $\underset{\sim}{u}_{i+1} \in A_{c_j}(\underset{\sim}{u}_i)$ for all i, satisfies $\underset{\sim}{u}^* \in \Delta_{c_j}$.

(ii) For each j, if $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ is any sequence in $\underset{\sim}{G}$ converging to $\underset{\sim}{u}^*$ i.s.c.m., then $t_{c_j}(\underset{\sim}{u}_i)$ converges to $t_{c_j}(\underset{\sim}{u}^*)$, i.e. t_{c_j} is sequentially continuous i.s.c.m.

(iii) For each j, if $\underset{\sim}{u}^* \in \Delta_{c_j}$ and $t_{c_j}(\underset{\sim}{u}^*) \leq 0$, then $\underset{\sim}{u}^* \in \Delta$

(iv) For every $\underset{\sim}{u}^* \in \underset{\sim}{G}$, there exists an integer j^* such that if $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ is a sequence in $\underset{\sim}{G}$ converging i.s.c.m. to $\underset{\sim}{u}^*$, then there exists an integer i_0 such that

$$t_{c_j}(\underset{\sim}{u}_i) \leq 0 \qquad \text{for all } i \geq i_0, \quad \text{all } j \geq j^*$$

Then the algorithm either constructs a finite sequence, in which case the last control is desirable for Problem P2, or it constructs an infinite sequence and every limit point (at least one exists) is desirable for P2.

The proof is given in Polak [P2] but we also give it here for completeness.

If the algorithm generates a finite sequence $\{\underset{\sim}{u}_0,\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_k\}$ then it is obvious from Step 2 and 3 that $t_{c_j}(\underset{\sim}{u}_k)\leq 0$ and $\tilde{\theta}_{c_j}(\underset{\sim}{u}_k)=0$ for the procedure to stop. By the definition of Δ_c, then from (iii) in Theorem 5.1, $\underset{\sim}{u}_k \in \Delta$.

Now suppose that the algorithm generates an infinite sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$. By compactness of the relaxed control problem, there exists a subsequence indexed by $K \subset \{0,1,2,\ldots\}$ and a $\underset{\sim}{u}^* \in \underset{\sim}{G}$ such that

$$\underset{\sim}{u}_i \xrightarrow{K} \underset{\sim}{u}^* \quad \text{i.s.c.m.}$$

Condition (ii) in Theorem 5.1 implies $t_{c_j}(\underset{\sim}{u}_i) \xrightarrow{K} t_{c_j}(\underset{\sim}{u}^*)$ conditions (i), (iii) and (iv) imply that there exists a c^* such that $t_c(\underset{\sim}{u}^*)\leq 0$ and that $\underset{\sim}{u}^* \in \Delta_c$ for all $c\geq c^*$. Hence we again have that $\underset{\sim}{u}^* \in \Delta$, i.e. $\underset{\sim}{u}^*$ is desirable for P2.

We will now construct Algorithm 4 using the above model. To do this we will need to construct the map A_c and the test function t_c so that they have the properties required of them.

4.6 Construction of Algorithm 4

We will first present an algorithm for solving Problem $P2_c$ and establish its convergence, i.e. show that condition (i) of Theorem 5.1 is satisfied. In the subalgorithm described below the map A_c is defined in Steps 2 to 5.

Subalgorithm for Solving $P2_c$

Step 0 : Select a $\underset{\sim}{u}_0 \in \underset{\sim}{G}$, $c>0$

Step 1 : Set $i=0$

Step 2 : Compute $x^{\underset{\sim}{u}_i}$ by solving (2.2), (2.3)

Step 3 : Compute $\tilde{\theta}_c(\underset{\sim}{u}_i)$ and find a control $\check{\underset{\sim}{u}}_i \in \underset{\sim}{G}$ which achieves the minimum

Step 4 : If $\tilde{\theta}_c(\underset{\sim}{u}_i)=0$ stop

Else define $\underset{\sim}{u}_{\alpha_i} = (1-\alpha_i)\underset{\sim}{u}_i \oplus \alpha_i \check{\underset{\sim}{u}}_i$ where $\alpha_i \in [0,1]$ is the largest number which satisfies

$$\tilde{\gamma}_c(\underset{\sim}{u}_{\alpha_i}) - \tilde{\gamma}_c(\underset{\sim}{u}_i) \leq \frac{\alpha_i \tilde{\theta}_c(\underset{\sim}{u}_i)}{2}$$

Step 5 : Set $\underset{\sim}{u}_{i+1} = \underset{\sim}{u}_{\alpha_i}$

Set $i=i+1$

Goto Step 2

Remark

$\tilde{\theta}_c(\underset{\sim}{u}_i)$ in Step 3 may be computed using a method similar to Procedure W in Chapter 5 or to the one used by Mayne and Polak [MAP2].

If the algorithm is at a non-optimal control for $P2_c$ (i.e. $\tilde{\theta}_c(\underset{\sim}{u})<0$) we will need, for the subalgorithm to be well defined, the step length α_i to be strictly greater than zero so that a descent property exists. This will be deduced when proving Theorem 6.1 which states the convergence properties of the subalgorithm.

Theorem 6.1

Suppose all the assumptions stated in the text are satisfied, then the subalgorithm for solving $P2_c$ either generates a finite sequence of controls, in which case the last element is desirable, or it generates an infinite sequence and every accumulation point i.s.c.m. is in Δ_c.

Before we prove Theorem 6.1, we state a few results which we will need:

Proposition 6.2

For all $\underset{\sim}{u} \in \underset{\sim}{G}$, all $\alpha \in [0,1]$, all $c>0$, we have

$$\Delta\hat{\gamma}_c(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) \leq \alpha\tilde{\theta}_c(\underset{\sim}{u})$$

where $\underset{\sim}{u}_\alpha$ is defined in Step 4 of subalgorithm and $\Delta\hat{\gamma}_c(\underset{\sim}{v},\underset{\sim}{u})$ is the first order estimate of the change in the penalised cost resulting from using the control $\underset{\sim}{v}$ in place of $\underset{\sim}{u}$ (this approximation will be made more precise in the following discussion).

Proof

By definition we have

$$\Delta\hat{\gamma}_c(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) \triangleq \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}}(1)) + < \frac{h^0_x(x^{\underset{\sim}{u}}(1))}{c} + h^j_x(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(1) > \} - \gamma(\underset{\sim}{u})$$

where $\underset{\sim}{u}_\alpha = (1-\alpha)\underset{\sim}{u} \oplus \alpha\check{\underset{\sim}{u}}$

and $z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(1)$ is the solution of

$$z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(1) = \int_0^1 [A^{\underset{\sim}{u}}(t) z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t) + B^{\underset{\sim}{u}}(t) z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t-\tau) + f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}_\alpha \ominus \underset{\sim}{u}, t)]dt$$

i.e. $z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(1) = \alpha \int_0^1 \Phi(1,t) f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}} \ominus \underset{\sim}{u}, t)dt$

where $\Phi(1,t)$ is the transition matrix (see section 5.1 in Appendix C). Now using this and the definition of λ^j we get

$$<h^j_x(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(1)> = \alpha \int_0^1 <\lambda^j(1), \Phi(1,t) f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}} \ominus \underset{\sim}{u}, t)> dt$$

$$= \alpha \int_0^1 [\lambda^j(t)^T] f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}} \ominus \underset{\sim}{u}, t)dt$$

where $\lambda^j : T \longrightarrow \mathbb{R}^n$, $j=0,1,\ldots,2r$ is the adjoint variable which satisfies (2.10)-(2.12). Hence using this in the above we get

$$\Delta\hat{\gamma}_c(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) = \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}}(1)) + \alpha\int_0^1 (\frac{\lambda^0(t)^T}{c} + \lambda^j(t)^T) f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}} \ominus \underset{\sim}{u}, t)dt \} - \gamma(\underset{\sim}{u})$$

$$= \max_{j=1-2r} \{ (h^j(x^{\underset{\sim}{u}}(1)) - \gamma(\underset{\sim}{u})) + \alpha\int_0^1 (\frac{\lambda^0(t)^T}{c} + \lambda^j(t)^T) f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}} \ominus \underset{\sim}{u}, t)dt \}$$

Now since $\gamma(\underset{\sim}{u}) \geq h^j(x^{\underset{\sim}{u}}(1))$ $j=1,2,\ldots,2r$ we have $h^j(x^{\underset{\sim}{u}}(1)) - \gamma(\underset{\sim}{u}) \leq 0$ for all $j \in 1,2,\ldots,2r$. Hence for all $\alpha \in [0,1]$ we have

$$\alpha(h^j(x^{\underset{\sim}{u}}(1)) - \gamma(\underset{\sim}{u})) \geq h^j(x^{\underset{\sim}{u}}(1)) - \gamma(\underset{\sim}{u})$$

Hence we have by using this in above

$$\Delta\hat{\gamma}_c(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) \leq \max_{j=1-2r} \{ \alpha[h^j(x^{\underset{\sim}{u}}(1)) - \gamma(\underset{\sim}{u})] + \alpha\int_0^1 (\frac{\lambda^0(t)^T}{c} + \lambda^j(t)^T) f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}} \ominus \underset{\sim}{u}, t)dt \}$$

Now by definition $\tilde{\theta}_c(\underset{\sim}{u})$ equals

$$\tilde{\theta}_c(\underset{\sim}{u}) = \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}}(1)) + \langle \frac{h_x^0(x^{\underset{\sim}{u}}(1))}{c} + h_x^j(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\check{\underset{\sim}{u}}}(1) \rangle \} - \gamma(\underset{\sim}{u})$$

where $\check{\underset{\sim}{u}} \in \underset{\sim}{G}$ is a minimising control for $\tilde{\theta}_c(\underset{\sim}{u})$.

Using the same procedure as above we deduce that

$$\tilde{\theta}_c(\underset{\sim}{u}) = \max_{j=1-2r} \{ [h^j(x^{\underset{\sim}{u}}(1)) - \gamma(\underset{\sim}{u})] + \int_0^1 (\frac{\lambda^0(t)^T}{c} + \lambda^j(t)^T) f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \check{\underset{\sim}{u}} \ominus \underset{\sim}{u}, t)dt \}$$

Therefore substituting for $\tilde{\theta}_c(\underset{\sim}{u})$ above we get

$$\Delta\hat{\gamma}_c(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) \le \alpha\,\tilde{\theta}_c(\underset{\sim}{u})$$

as required.

Proposition 6.3

For all $c>0$, $\alpha \in [0,1]$, for all $\underset{\sim}{u} \in \underset{\sim}{G}$ there exists a finite d such that

$$\| \Delta\tilde{\gamma}_c(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) - \Delta\hat{\gamma}_c(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) \| \le d\,\alpha^2$$

where $\underset{\sim}{u}_\alpha$ is again as in Step 4 of subalgorithm and $\Delta\tilde{\gamma}_c(\underset{\sim}{v},\underset{\sim}{u})$ is defined to be the change in the cost function $\tilde{\gamma}_c$ when the control $\underset{\sim}{u}$ is replaced by $\underset{\sim}{v}$ and $\Delta\hat{\gamma}_c$ is as in Proposition 6.2

Proof

By definition (3.2) we have

$$\tilde{\gamma}_c(\underset{\sim}{u}_\alpha) - \tilde{\gamma}_c(\underset{\sim}{u}) = \frac{h^0(x^{\underset{\sim}{u}_\alpha}(1))}{c} - \frac{h^0(x^{\underset{\sim}{u}}(1))}{c}$$

$$+ \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}_\alpha}(1)) \} \quad - \gamma(\underset{\sim}{u})$$

Expanding $h^0(x^{\underset{\sim}{u}_\alpha}(1))$ and $h^j(x^{\underset{\sim}{u}_\alpha}(1))$ by Taylor series we get

$$\Delta\tilde{\gamma}_c(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) = \frac{h^0(x^{\underset{\sim}{u}}(1))}{c} + \frac{h_x(x^{\underset{\sim}{u}}(1))^T(x^{\underset{\sim}{u}_\alpha}(1) - x^{\underset{\sim}{u}}(1))}{c}$$

$$+ \frac{h^0_{xx}(x^{\underset{\sim}{u}_\varepsilon}(1))(x^{\underset{\sim}{u}_\alpha}(1) - x^{\underset{\sim}{u}}(1))^2}{2c} - \frac{h^0(x^{\underset{\sim}{u}}(1))}{c}$$

$$+ \max_{j=1-2r} \Big\{ h^j(x^{\underset{\sim}{u}}(1)) + h^j_x(x^{\underset{\sim}{u}}(1))(x^{\underset{\sim}{u}_\alpha}(1) - x^{\underset{\sim}{u}}(1))$$

$$+ \frac{h^j_{xx}(x^{\underset{\sim}{u}_\varepsilon}(1))(x^{\underset{\sim}{u}_\alpha}(1) - x^{\underset{\sim}{u}}(1))^2}{2} \Big\}$$

$$- \gamma(\underset{\sim}{u})$$

where $x^{\underset{\sim}{u}\varepsilon} \triangleq (1-\varepsilon)x^{\underset{\sim}{u}} + \varepsilon x^{\underset{\sim}{u}\alpha}$ for some $\varepsilon \in (0,1)$.

Adding and subtracting $\Delta\hat{\gamma}_c(\underset{\sim}{u}_\alpha, \underset{\sim}{u})$ we get

$$\tilde{\Delta}\gamma_c(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) = \hat{\Delta}\gamma_c(\underset{\sim}{u}_\alpha, \underset{\sim}{u})$$

$$- \max_{j=1-2r} \{h^j(x^{\underset{\sim}{u}}(1)) + \langle \frac{h^0_x(x^{\underset{\sim}{u}}(1))}{c} + h^j_x(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{u}\alpha}(1)\rangle\}$$

$$+ \frac{h^0_x(x^{\underset{\sim}{u}}(1))^T(x^{\underset{\sim}{u}\alpha}(1)-x^{\underset{\sim}{u}}(1))}{c} + \frac{h^0_{xx}(x^{\underset{\sim}{u}\varepsilon}(1))(x^{\underset{\sim}{u}\alpha}(1)-x^{\underset{\sim}{u}}(1))^2}{2c}$$

$$+ \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}}(1)) + h^j_x(x^{\underset{\sim}{u}}(1))^T(x^{\underset{\sim}{u}\alpha}(1)-x^{\underset{\sim}{u}}(1))$$

$$+ \frac{h^j_{xx}(x^{\underset{\sim}{u}\varepsilon}(1))(x^{\underset{\sim}{u}\alpha}(1)-x^{\underset{\sim}{u}}(1))^2}{2} \}$$

i.e.

$$\tilde{\Delta\gamma}_c(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) \leq \hat{\Delta\gamma}_c(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) + \frac{h^0_x(x^{\underset{\sim}{u}}(1))^T(x^{\underset{\sim}{u}\alpha}(1)-x^{\underset{\sim}{u}}(1)-z^{\underset{\sim}{u},\underset{\sim}{u}\alpha}(1))}{c}$$

$$+ \max_{j=1-2r} h^j_x(x^{\underset{\sim}{u}}(1))^T(x^{\underset{\sim}{u}\alpha}(1)-x^{\underset{\sim}{u}}(1)-z^{\underset{\sim}{u},\underset{\sim}{u}\alpha}(1))$$

$$+ \frac{h^0_{xx}(x^{\underset{\sim}{u}\varepsilon}(1))(x^{\underset{\sim}{u}\alpha}(1)-x^{\underset{\sim}{u}}(1))^2}{2c}$$

$$+ \max_{j=1-2r} \frac{h^j_{xx}(x^{\underset{\sim}{u}\varepsilon}(1))(x^{\underset{\sim}{u}\alpha}(1)-x^{\underset{\sim}{u}}(1))^2}{2}$$

By the boundedness of h^j_x, h^j_{xx} $j=0,1,\ldots,2r$ for all $\underset{\sim}{u} \in \underset{\sim}{G}$ and all finite $c>0$, there exist finite constants d_1, d_2 such that

$$\| \tilde{\Delta\gamma}_c(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) - \hat{\Delta\gamma}_c(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) \| \leq d_1 \| (x^{\underset{\sim}{u}\alpha} - x^{\underset{\sim}{u}}) - z^{\underset{\sim}{u},\underset{\sim}{u}\alpha} \|$$

$$+ d_2 \| x^{\underset{\sim}{u}\alpha} - x^{\underset{\sim}{u}} \|^2$$

By Proposition 2.7 in section 2

$\| (x^{u_\alpha}-x^{u})-z^{u,u_\alpha}\| \leq d_3\alpha^2$ for some finite d_3, and by Proposition 2.5 in Chapter 3 $\| x^{u_\alpha}-x^{u} \| \leq d_4\alpha$ for some $d_4 \in (0,\infty)$. Hence we deduce that

$$\| \Delta\tilde{\gamma}_c(u_\alpha,u) - \Delta\hat{\gamma}_c(u_\alpha,u) \| \leq d\,\alpha^2$$

as required.

Note that from Proposition 6.2

$$\Delta\hat{\gamma}_c(u_\alpha,u) \leq \alpha\tilde{\theta}_c(u)$$

Hence $\Delta\tilde{\gamma}_c(u_\alpha,u) \leq \alpha\tilde{\theta}_c(u)+d\alpha^2$ for all $\alpha \in [0,1]$.

Proposition 6.4

For all $u \in G$ which are not optimal for $P2_c$, i.e. $\tilde{\theta}_c(u)<0$, we have that the step length α determined in Step 4 of the subalgorithm is strictly greater than zero.

Proof

From above we have

$$\Delta\tilde{\gamma}_c(u_\alpha,u) \leq \alpha\tilde{\theta}_c(u) + d\,\alpha^2$$

Hence for all $u \in G$ such that $\tilde{\theta}_c(u)<0$ we have that

$$\tilde{\gamma}_c(u_\alpha) - \tilde{\gamma}_c(u) \leq \frac{\alpha\,\tilde{\theta}_c(u)}{2}$$

for all $\alpha \leq \min\{1, -\frac{\tilde{\theta}_c(u)}{2d}\}$ as shown in Mayne and Polak [MAP2]. Therefore the α chosen by the subalgorithm is the maximum α which satisfies the above relation i.e. $\max \alpha \geq \min \{1, -\frac{\tilde{\theta}_c(\underset{\sim}{u})}{2d}\} > 0$ when $\tilde{\theta}_c(\underset{\sim}{u}) < 0$.

Hence α found in Step 4 of subalgorithm is strictly greater than zero when at a non-optimal control. Hence the subalgorithm is well defined.

Proof of Theorem 6.1

If a finite sequence is constructed the last control $\underset{\sim}{u}_k$ trivially satisfies $u_k \in \Delta_c$ since the subalgorithm terminated.

Now suppose that the subalgorithm generates an infinite sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ in $\underset{\sim}{G}$. Then there exists (by the results in Appendix A) a convergent subsequence indexed by $K \subset \{0,1,2,...\}$ and a $\underset{\sim}{u}^* \in \underset{\sim}{G}$ such that $\underset{\sim}{u}_i \xrightarrow{K} \underset{\sim}{u}^*$ i.s.c.m.

We need to prove that $\tilde{\theta}_c(\underset{\sim}{u}^*)=0$. Assume contrary to what is to be proven that $\tilde{\theta}_c(\underset{\sim}{u}^*)=-\delta<0$.

From Step 4 of the subalgorithm we have $\tilde{\gamma}_c(\underset{\sim}{u}_{i+1}) - \tilde{\gamma}_c(\underset{\sim}{u}_i) \leq \frac{\alpha_i \tilde{\theta}_c(\underset{\sim}{u}_i)}{2}$ for all i for all c>0.

As $\underset{\sim}{u}_i \xrightarrow{K} \underset{\sim}{u}^*$ i.s.c.m. we have by the sequential continuity i.s.c.m. of $\tilde{\theta}_c$ that

$$\tilde{\theta}_c(\underset{\sim}{u}_i) \xrightarrow{K} \tilde{\theta}_c(\underset{\sim}{u}^*) = -\delta$$

Hence there exists an integer i_0 such that

$$\tilde{\theta}_c(\underset{\sim}{u}_i) \leq \frac{\tilde{\theta}(\underset{\sim}{u}^*)}{2} = -\frac{\delta}{2} \qquad \text{for all } i \geq i_0, \quad i \in K$$

Now from Proposition 6.4

$$\alpha_i \geq \min \{1, -\frac{\tilde{\theta}_c(\underset{\sim}{u}_i)}{2d}\} \quad \text{for all } i$$

and hence $\alpha_i \geq \alpha(\underset{\sim}{u}^*) = \min\ \{1, -\frac{\tilde{\theta}_c(\underset{\sim}{u}^*)}{2d}\} > 0$

for all $i \geq i_0$, $i \in K$.

Hence $\tilde{\gamma}_c(\underset{\sim}{u}_{i+1}) - \tilde{\gamma}_c(\underset{\sim}{u}_i) \leq \alpha(\underset{\sim}{u}^*)\ \frac{\tilde{\theta}_c(\underset{\sim}{u}^*)}{4}$

for all $i \geq i_0$, $i \in K$

Therefore we have

$$\tilde{\gamma}_c(\underset{\sim}{u}^*) - \tilde{\gamma}_c(\underset{\sim}{u}_{i_0}) = \sum_{\substack{i \in K \\ i \geq i_0}} \tilde{\gamma}_c(\underset{\sim}{u}_{i+1}) - \tilde{\gamma}_c(\underset{\sim}{u}_i)$$

$$+ \sum_{\substack{i \notin K \\ i \geq i_0}} \tilde{\gamma}_c(\underset{\sim}{u}_{i+1}) - \tilde{\gamma}_c(\underset{\sim}{u}_i)$$

$$\leq \sum_{\substack{i \in K \\ i \geq i_0}} \tilde{\gamma}_c(\underset{\sim}{u}_{i+1}) - \tilde{\gamma}_c(\underset{\sim}{u}_i)$$

$$\leq \sum_{\substack{i \in K \\ i \geq i_0}} \alpha(\underset{\sim}{u}^*)\ \frac{\tilde{\theta}_c(\underset{\sim}{u}^*)}{4}\ .\ i$$

$$\longrightarrow -\infty$$

But $\{\tilde{\gamma}_c(\underset{\sim}{u}_i)\}$ is a bounded monotonically decreasing sequence which converges to $\tilde{\gamma}_c(\underset{\sim}{u}^*)$ since $\tilde{\gamma}_c(\underset{\sim}{u}_i) \xrightarrow{K} \tilde{\gamma}_c(\underset{\sim}{u}^*)$. This contradicts above. Hence assumption $\tilde{\theta}_c(\underset{\sim}{u}^*) < 0$ is false and so we must have $\tilde{\theta}_c(\underset{\sim}{u}^*) = 0$, i.e. $\underset{\sim}{u}^* \in \Delta_c$.

This proves Theorem 6.1 and shows that the map A_c has the properties desired of it in Theorem 5.1 (i).

<u>The Test Function t_c</u>

We turn our attention now to finding a suitable test function which will satisfy conditions (ii)-(iv) of Theorem 5.1. Finding such functions is quite a difficult task as mentioned by

Mayne and Polak [MAP2], but we are fortunate that the test function used in [MAP2] is also valid for our delay case. We propose for our test function $t_c : \underset{\sim}{G} \longrightarrow \mathbb{R}$ the following

$$t_c(\underset{\sim}{u}) = \tilde{\theta}_c(\underset{\sim}{u}) + \frac{\gamma(\underset{\sim}{u})}{c} \tag{6.1}$$

The following lemmas show that this choice for t_c does indeed satisfy the conditions (ii)-(iv) of Theorem 5.1.

Lemma 6.5

For all $c>0$, t_c is sequentially continuous i.s.c.m.

Proof

From Proposition 3.1, $\tilde{\theta}_c$ is sequentially continuous i.s.c.m. The sequential continuity i.s.c.m. of

$$\gamma(\underset{\sim}{u}) = \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}}(1)) \}$$

follows from the sequential continuity of the map $\underset{\sim}{u} \longrightarrow x^{\underset{\sim}{u}}(1)$ and the continuity of h^j, $j=1,2,\ldots,2r$.

Lemma 6.6

For all $c>0$, if $\underset{\sim}{u} \in \Delta_c$ and $t_c(\underset{\sim}{u}) \le 0$, then $\underset{\sim}{u} \in \Delta$.

Proof

Now since $\underset{\sim}{u} \in \Delta_c$, we have $\tilde{\theta}_c(\underset{\sim}{u})=0$ and since $t_c(\underset{\sim}{u}) = \tilde{\theta}_c(\underset{\sim}{u}) + \frac{\gamma(\underset{\sim}{u})}{c}$, therefore if $t_c(\underset{\sim}{u}) \le 0$ and $\tilde{\theta}_c(\underset{\sim}{u})=0$ we have $\gamma(\underset{\sim}{u}) \le 0$. But $\gamma(\underset{\sim}{u})$ is positive definite, hence $\gamma(\underset{\sim}{u})=0$ for all $c>0$, i.e. $\underset{\sim}{u}$ is feasible for Problem P2.

Now we must show that 4.3 is true for $\underset{\sim}{u}$ to be in Δ, i.e. we must show that there exist multipliers $\psi^1, \psi^2, \ldots, \psi^r \in \mathbb{R}$ such that

$$<h_x^0(x^{\underset{\sim}{u}}(1)) + \sum_{j=1}^{r} \psi^j h_x^j(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1)> \quad \geq 0 \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{G}$$

Assume contrary to what is to be proven that $\underset{\sim}{u} \notin \Delta$, then from the definition of Δ, we have, for each non zero $\psi^1, \psi^2, \ldots, \psi^r \in \mathbb{R}$

$$<h_x^0(x^{\underset{\sim}{u}}(1)) + \sum_{j=1}^{r} \psi^j h_x^j(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1)> \quad < 0$$

for some $\underset{\sim}{v} \in \underset{\sim}{G}$

Hence for all $c>0$ we have

$$< \frac{h_x^0(x^{\underset{\sim}{u}}(1))}{c} + h_x^j(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1)> \quad < 0$$

for some $\underset{\sim}{v} \in \underset{\sim}{G}$, for all $j=1,2,\ldots,2r$.

But from the definition of $\tilde{\theta}_c$ we have

$$\tilde{\theta}_c(\underset{\sim}{u}) = \min_{\underset{\sim}{v} \in \underset{\sim}{G}} \ \max_{j=1\text{-}2r} \{< \frac{h_x^0(x^{\underset{\sim}{u}}(1))}{c} + h_x^j(x^{\underset{\sim}{u}}(1)), z^{\underset{\sim}{u},\underset{\sim}{v}}(1) > \}$$

$$= 0 \qquad \text{from above.}$$

This gives a contradiction, and therefore we must have $\underset{\sim}{u} \in \Delta$ as required.

<u>Lemma 6.7</u>

For any $\underset{\sim}{u}^* \in \underset{\sim}{G}$, there exists a $c^* > 0$, such that for any infinite sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ in $\underset{\sim}{G}$ converging to $\underset{\sim}{u}^*$ i.s.c.m., there exists an integer i_0 such that

$t_c(\underset{\sim}{u}_i)\leq 0$ for all $i\geq i_0$

for all $c\geq c^*$

Proof

There are two cases to be considered, which are

(i) $\gamma(\underset{\sim}{u}^*)>0$

(ii) $\gamma(\underset{\sim}{u}^*)=0$

Case (i) : $\gamma(\underset{\sim}{u}^*)>0$

Here the limit point is not feasible for P2. By Assumption 3, we have

$\theta(\underset{\sim}{u}^*)=-\delta<0$. Now by the definition of $\tilde{\theta}_c$ (see (3.4)) we have that

$$\tilde{\theta}_c(\underset{\sim}{u}_i) = \min_{z} \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}i}(1)) + < \frac{h^0_x(x^{\underset{\sim}{u}i}(1))}{c} + h^j_x(x^{\underset{\sim}{u}i}(1)), z > \} - \gamma(\underset{\sim}{u}_i)$$

for all i

Now by the boundedness of h^0_x and $z^{\underset{\sim}{u},\underset{\sim}{v}}(1)$ for all $\underset{\sim}{u},\underset{\sim}{v}\in\underset{\sim}{G}$ there exists a finite d_1 such that

$$\tilde{\theta}_c(\underset{\sim}{u}_i) \leq \frac{d_1}{c} + \min_{\underset{\sim}{v}\in\underset{\sim}{G}} \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}i}(1)) + < h^j_x(x^{\underset{\sim}{u}i}(1)), z^{\underset{\sim}{u}i,\underset{\sim}{v}}(1) > \} - \gamma(\underset{\sim}{u}_i)$$

Using the definition of θ (see (3.3)) we get

$$\tilde{\theta}_c(\underset{\sim}{u}_i) \leq \frac{d_1}{c} + \theta(\underset{\sim}{u}_i)$$

Also by the boundedness of γ in $\underset{\sim}{G}$ we have $\gamma(\underset{\sim}{u})\leq d_2$ for all $\underset{\sim}{u}\in\underset{\sim}{G}$, for some finite d_2. Let $d=\max\{d_1,d_2\}$.

Then since $t_c(\underset{\sim}{u}_i) = \tilde{\theta}_c(\underset{\sim}{u}_i) + \dfrac{\gamma(\underset{\sim}{u}_i)}{c}$ for all $c>0$, all i

we get by substituting for $\tilde{\theta}_c$ and using the bound on γ

$$t_c(\underset{\sim}{u}_i) \le \theta(\underset{\sim}{u}_i) + \frac{2d}{c}$$

Now since $\theta(\underset{\sim}{u}^*) = -\delta < 0$, we have by the sequential continuity of θ that there exists an integer i_0 such that

$$\theta(\underset{\sim}{u}_i) \le \frac{\theta(\underset{\sim}{u}^*)}{2} = -\frac{\delta}{2} \qquad \text{for all } i \ge i_0$$

Hence $t_c(\underset{\sim}{u}_i) \le -\dfrac{\delta}{2} + \dfrac{2d}{c}$ for all $i \ge i_0$

Let $c^* = \dfrac{\delta d_1}{c}$, then we have

$$t_c(\underset{\sim}{u}_i) \le -\frac{\delta}{2} + \frac{\delta}{4} \qquad \text{for all } i \ge i_0$$
$$\text{for all } c \ge c^*$$

Hence $t_c(\underset{\sim}{u}_i) \le -\dfrac{\delta}{4} < 0$ for all $i \ge i_0$, $c \ge c^*$.

<u>Case (ii)</u> : $\gamma(\underset{\sim}{u}^*) = 0$

By Assumption 3 we have

$$\phi^I(\underset{\sim}{u}^*) = -\delta < 0 \qquad \text{for all } I \subset J.$$

From equation (4.2) we recall that for any $I \subset J$

$$\phi^I(\underset{\sim}{u}^*) = \min_{z \in R(\underset{\sim}{u}^*)} \max_{j \in I} \langle h_x^j(x^{\underset{\sim}{u}^*}(1)), z \rangle$$

We define $\tilde{\phi}_c^I : \underset{\sim}{G} \longrightarrow \mathbb{R}$ for all $c>0$, all $I \subset J$ by

$$\tilde{\phi}_c^I(\underset{\sim}{u}) \triangleq \min_{z \in R(\underset{\sim}{u})} \max_{j \in I} \{ \langle \frac{h_x^0(x^{\underset{\sim}{u}}(1))}{c} + h_x^j(x^{\underset{\sim}{u}}(1)), z \rangle \} \tag{6.2}$$

From the boundedness of h_x^0 and $z^{\underset{\sim}{u},\underset{\sim}{v}}(1)$ for all $\underset{\sim}{u},\underset{\sim}{v} \in \underset{\sim}{G}$, there exists a constant $M \in (0,\infty)$ such that

$$\tilde{\phi}_c^I(\underset{\sim}{u}) \leq \frac{M}{c} + \min_{z \in R(\underset{\sim}{u})} \max_{j \in I} < h_x^j(x^{\underset{\sim}{u}}(1)), z >$$

Using definition of ϕ^I we have

$$\tilde{\phi}_c^I(\underset{\sim}{u}) \leq \frac{M}{c} + \phi^I(\underset{\sim}{u}) \qquad \text{for all } \underset{\sim}{u} \in \underset{\sim}{G}, \text{ all } c>0, \text{ and for all } I \subset J$$

Since $\underset{\sim}{u}_i \longrightarrow \underset{\sim}{u}^*$ i.s.c.m., we have by the sequential continuity of ϕ^I that

$$\phi^I(\underset{\sim}{u}_i) \longrightarrow \phi^I(\underset{\sim}{u}^*) \qquad \text{for all } I \subset J$$

Then there exists an integer i_0 such that

$$\phi^I(\underset{\sim}{u}_i) \leq \frac{\phi^I(\underset{\sim}{u}^*)}{2} = -\frac{\delta}{2} \qquad \text{for all } i \geq i_0$$

Using this in above we get

$$\tilde{\phi}_c^I(\underset{\sim}{u}_i) \leq \frac{M}{c} - \frac{\delta}{2} \qquad \text{for all } i \geq i_0$$

Let c_1 satisfy $c_1 = \frac{4M}{\delta}$

Then we have

$$\tilde{\phi}_c^I(\underset{\sim}{u}_i) \leq -\frac{\delta}{4} \qquad \text{for all } i \geq i_0, \text{ all } c \geq c_1$$

Now for all $c>0$ we have

$$\tilde{\theta}_c(\underset{\sim}{u}_i) = \min_z \max_{j=1-2r} \{ h^j(x^{\underset{\sim}{u}_i}(1)) + < \frac{h_x^0(x^{\underset{\sim}{u}_i}(1))}{c} + h_x^j(x^{\underset{\sim}{u}_i}(1)), z > \} - \gamma(\underset{\sim}{u}_i)$$

$$= \min_z \max \{ \max_{j \in I} [h^j(x^{\underset{\sim}{u}_i}(1)) - \gamma(\underset{\sim}{u}_i) + < \frac{h_x^0(x^{\underset{\sim}{u}_i}(1))}{c} + h_x^j(x^{\underset{\sim}{u}_i}(1)), z >],$$

$$, \max_{j\in I^c} [h^j(x^{\underset{\sim}{u}_i}(1)) - \gamma(\underset{\sim}{u}_i) + < \frac{h^0_x(x^{\underset{\sim}{u}_i}(1))}{c} + h^j_x(x^{\underset{\sim}{u}_i}(1)), z >] \}$$

for all $I \subset J$

where I^c denotes the complement of I in $\{1,2,\ldots,2r\}$.

Let $z_i \in R(\underset{\sim}{u}_i)$ denote any minimising z in the definition of $\tilde{\phi}^I_c(\underset{\sim}{u}_i)$ (see (6.2)).

Since $R(\underset{\sim}{u}_i)$ is convex and compact (and contains the origin in $\mathbb{R}^n$) we have that $\varepsilon z_i \in R(\underset{\sim}{u}_i)$ for all $\varepsilon \in [0,1]$. Substituting εz_i for z in the above equation we get, recalling the definition of $\tilde{\phi}^I_c$ and noting that $h^j(x^{\underset{\sim}{u}_i}(1)) - \gamma(\underset{\sim}{u}_i) \le 0$ for all $j=1,2,\ldots,2r$ that

$$\tilde{\theta}_c(\underset{\sim}{u}_i) \le \min_{\varepsilon\in[0,1]} \max \{\varepsilon\tilde{\phi}^I_c(\underset{\sim}{u}_i), \max_{j\in I^c} [h^j(x^{\underset{\sim}{u}_i}(1)) - \gamma(\underset{\sim}{u}_i) + \varepsilon b] \} \quad (6.3)$$

where $b \in (0,\infty)$ is defined by

$$b = \sup \{ [\frac{\|h^0_x(x^{\underset{\sim}{u}_i}(1))\|}{c} + \|h^j_x(x^{\underset{\sim}{u}_i}(1))\|] . \|z_i\| : \underset{\sim}{u}_i \in \underset{\sim}{G}$$

$$c \ge c_1 , \ j=1,2,\ldots,2r \}$$

Now for any $\underset{\sim}{u} \in \underset{\sim}{G}$ let $J(\underset{\sim}{u}) \subset J$ denote the class of sets $\{I \subset J : h^j(x^{\underset{\sim}{u}}(1)) \ge 0, \ j \in I\}$. Hence if $I \subset J(\underset{\sim}{u})$ then $h^j(x^{\underset{\sim}{u}}(1)) \le 0$ for all $j \in I^c$. Since (6.3) holds for all $I \subset J$, it also holds for all $I \subset J(\underset{\sim}{u}_i)$, so that

$$\tilde{\theta}_c(\underset{\sim}{u}_i) \le \min_{\varepsilon\in[0,1]} \max \{ \varepsilon\tilde{\phi}^I_c(\underset{\sim}{u}_i), -\gamma(\underset{\sim}{u}_i) + \varepsilon b \}$$

since $\max_{j\in I^c} h^j(x^{\underset{\sim}{u}_i}(1)) = 0$ for $I \subset J(\underset{\sim}{u}_i)$

This holds for all $c \ge c_1$, $I \subset J(\underset{\sim}{u}_i)$.

Hence from the above we have

$$\tilde{\theta}_c(\underset{\sim}{u}_i) \le \min_{\varepsilon \in [0,1]} \max \{ -\frac{\varepsilon\delta}{4}, -\gamma(\underset{\sim}{u}_i)+\varepsilon b \}$$

for all $i \ge i_0$, $c \ge c_1$

Since $t_c(\underset{\sim}{u}) = \tilde{\theta}_c(\underset{\sim}{u}) + \frac{\gamma(\underset{\sim}{u})}{c}$ we get

$$t_c(\underset{\sim}{u}_i) \le \min_{\varepsilon \in [0,1]} \max \{ \frac{\gamma(\underset{\sim}{u}_i)}{c} - \frac{\varepsilon\delta}{4}, -\gamma(\underset{\sim}{u}_i)(1-\frac{1}{c})+\varepsilon b \} \tag{6.4}$$

for all $i \ge i_0$, $c \ge c_1$

The first term on the right side of (6.4) is negative if $\varepsilon > \varepsilon_1(\underset{\sim}{u}_i) = \frac{4\gamma(\underset{\sim}{u}_i)}{c\delta}$ and the second is negative if $\varepsilon < \varepsilon_2(\underset{\sim}{u}_i) = \frac{\gamma(\underset{\sim}{u}_i)(1-1/c)}{b}$

Clearly $\varepsilon_1(\underset{\sim}{u}_i) \le \varepsilon_2(\underset{\sim}{u}_i)$ if

$$\frac{4}{c\delta} \le \frac{(1-\frac{1}{c})}{b}$$

i.e. if $c \ge 1 + \frac{4b}{\delta}$

Hence $t_c(\underset{\sim}{u}_i) \le 0$ for all $i \ge i_0$ and for all $c \ge c^* \triangleq \max\{c_1, 1+\frac{4b}{\delta}\}$.

Since all the conditions in Theorem 5.1 are satisfied with the above choices for the map A_c and the test function t_c, we can now present the complete algorithm for solving Problem P2 via an exact penalty function method.

4.7 Algorithm 4

Data : $\underset{\sim}{u}_0 \in \underset{\sim}{G}$, $0 < c_0 < c_1 < \ldots$ $\lim_j c_j = \infty$

Step 0 : Set i=0

Set j=0

Step 1 : Compute $x^{\underset{\sim}{u}_i}$ by solving (2.2), (2.3)

Compute $\lambda_j^{\underset{\sim}{u}_i}$ by solving (2.10)-(2.12) for j=0,1,2,...,2r

Compute $h^{j}(x^{u_i}(1))$, $h_x^{j}(x^{u_i}(1))$ for $j=0,1,2,\ldots.2r$, and $\gamma(\underset{\sim}{u}_i)$

Step 2 : Compute $\tilde{\theta}_{c_j}(\underset{\sim}{u}_i)$ (see (3.4)) and the minimising control $\check{\underset{\sim}{u}}_i \in \underset{\sim}{G}$

Step 3 : Compute $t_{c_j}(\underset{\sim}{u}_i)$

If $t_{c_j}(\underset{\sim}{u}_i) > 0$ set $c_j = c_{j+1}$

set $j = j=1$

and goto Step 2

Else proceed to Step 4

Step 4 : If $\tilde{\theta}_{c_j}(\underset{\sim}{u}_i)=0$ stop

Else proceed to Step 5

Step 5 : Define $\underset{\sim}{u}_{\alpha_i} =(1-\alpha_i)\underset{\sim}{u}_i \oplus \alpha_i \check{\underset{\sim}{u}}_i$ where $\alpha_i \in [0,1]$ is the largest number which satisfies

$$\tilde{\gamma}_{c_j}(\underset{\sim}{u}_{\alpha_i})-\tilde{\gamma}_{c_j}(\underset{\sim}{u}_i) \leq \frac{\alpha_i\tilde{\theta}_{c_j}(\underset{\sim}{u}_i)}{2}$$

Step 6 : Set $\underset{\sim}{u}_{i+1} = \underset{\sim}{u}_{\alpha_i}$

Set $i=i+1$

Goto Step 1

Since Algorithm 4 satisfies all the hypothesis in Theorem 5.1 we deduce the following convergence properties for it:

Theorem 4

Suppose the assumptions stated in the text are satisfied and that Algorithm 4 generates a sequence of controls $\{\underset{\sim}{u}_i\}$. Then

(i) This sequence is finite $\{\underset{\sim}{u}_i\}_{i=0}^{k}$, in which case c_j is increased only a finite number of times as well, and the last control is desirable for P2, i.e. $\underset{\sim}{u}_k \in \Delta$.

(ii) The sequence is infinite $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ and there exists a $j^* < \infty$ such that $j \leq j^*$ throughout the computation (i.e. c_j is still only increased a finite number of times and then remains constant at c_{j^*}) and every limit point $\underset{\sim}{u}^*$ i.s.c.m. of this sequence (at least one always exists) is desirable for Problem P2.

Remark

In this chapter we reduce a certain class of "smooth" control problems to problems in which the data is not continuously differentiable in the x variable by introduction of exact penalty functions. It is possible that the reduced problem can be tackled by means of recently developed algorithms for non-smooth mathematical programming problems (see for example Mifflin [MI1], [MI2]). However, our more ad hoc approach in which we devise local approximations to the special non-smooth functions which we encounter are quite adequate for our purposes.

CHAPTER 5

THE STATE CONSTRAINED CONTROL PROBLEM

5.1 Introduction

In this chapter we deal with the most difficult problem considered in this thesis. This problem is known as the state constrained control problem (SCCP) and the reasons as to why it is so difficult to solve (and analyse!) will become quite apparent during the course of this chapter. The need for investigation of SCCP's is very important since it is well known that most practical control problems do have the state constrained in some way - for example in management problems, inventories cannot be negative, in forestery we cannot harvest non-existent trees and in engineering problems, operating conditions of various devices need to be kept in certain regions for correct functioning or to prevent damage. Hence it seems reasonable that when a control problem is being formulated these constraints on the state be included and handled in some desirable way.

A typical state constrained control problem is stated below

$$\begin{aligned} &\text{Min} \quad J(x,\underset{\sim}{u}) \\ &\text{s.t} \quad \dot{x}(t) = f(x,\underset{\sim}{u},t) \qquad \text{for a.a.} t \in T \\ &\qquad x(o) = x_0 \\ &\qquad g(x(t),t) \leq 0 \qquad \text{for every } t \\ &\qquad \underset{\sim}{u} \in \underset{\sim}{G} \end{aligned}$$

where $g: \mathbb{R}^n \times T \longrightarrow \mathbb{R}$ specifies the state constraint. Since this constraint must be specified for each $t \in T$ we essentially have an infinite dimensional problem (due to the system dynamics) subject to an infinite number of constraints. It seems quite a formidable task to tackle such a problem at first sight, and indeed, as shown in numerous articles, e.g. Neustadt [N2], Gamkrelidze and Kharatishvili [GK1], Ioffe and Tihomirov [I1], Warga [W3], that trying to find solutions to SCCP's leads to certain difficulties. These can be explained by considering what happens to the state when it strikes the constraint boundary. After the point of impact the state is not free to proceed in its previous course (unless of course if it approaches the constraint tangentially), and hence there is an abrupt change in the state velocity (it is useful to think of the physical analogy of a ball following a trajectory that strikes a wall). When the Maximum Principle is formulated for SCCP's it has been shown (see for example [N2], [I1]) that these abrupt changes cause the adjoint variable to have discontinuities (jumps), and it is working with such nonsmooth functions that cause the above mentioned difficulties.

5.2 Literature Review on SCCP

State constrained control problems have been investigated thoroughly from a theoretical viewpoint and certain necessary conditions of optimality have been obtained, e.g. Ioffe and Tihomirov [I1], Neustadt [N1], [N2], [N4], Gamkrelidze and Kharatishvili [GK1], Warga [W3]. The conditions are in the form of a Maximum Principle (or Minimum Principle depending on formulation) where certain approximations are made. These are obtained by approximating non-convex sets by convex ones and non-linearized functions by linear ones. Then by showing disjointness of two convex sets (one which has non-empty interior) it is possible to assure the existence of a hyperplane which separates the two sets. In state constrained control problems these sets turn out to be in the space of continuous functions, and so the hyperplane which defines the multipliers and hence the Maximum Principle, is in the dual of such spaces, namely in the space of all functions of bounded variation.

Appendix C shows one particular method for obtaining necessary conditions of optimality and it is obvious to see there that the procedure is quite complicated.

Now although these conditions of optimality reveal the structure of optimal controls, these are of little or no use at all for actually solving a practical problem. What is required is a computational procedure for numerically solving SCCP's. Because of the above mentioned difficulties (notably occurrence of jumps in the adjoint variable) these are few and far between. The few methods that are available are quite restrictive to the type of problem they address or the assumptions made are quite severe.

One of the first algorithmic methods for solving a SCCP was presented by Hager [H1], where a quadratic cost, linear system dynamics and linear state constraints were considered (the added complexity of linear control constraints is also included). By defining a dual function and then a dual problem, which is solved using the Ritz-Trefftz method (where the dual variables are approximated by finite element subspaces), Hager arrives at a solution for the original problem. The same method has been extended to cover a general convex problem, see Hager [H2].

A semi-dual method has also been developed by Hager (see Hager and Ianculescu [HA1], [IA1]) in which only the dynamics of the system are included in the dual functional and the state constraints are handled explicitly. The error estimates of the solutions found at each iteration, however, rely on regularity results for the full dual and so the two methods are closely related. Both the dual and semi-dual methods have no extension to solve the general nonlinear SCCP since the methods require that the dual functional be convex in its arguments. This is not the case when nonlinear dynamics are considered since the reachable set is generally not convex.

Lasiecka in [L1] considers a SCCP with delay very similar to our own [Problem P3 defined below]. She formulates a discrete control problem using finite difference methods and approximates the feasible set in such a way that the discrete control obtained from solving the finite dimensional approximation produces feasible trajectories for the original problem. No method is given for solving either problem but the error between solutions of the continuous and discrete problems is estimated.

Warga [W2] has recently developed an interative scheme which can handle nonlinear SCCP's. He proposes a feasible directions method which extends Mayne and Polak's [MAP3]. procedure to cover an infinite number of constraints. The method requires a feasible initial point and all subsequent points generated by the algorithm remain feasible. The procedure can be programmed quite easily but the necessity for a feasible starting point is quite restrictive since no method of obtaining such a point is given.

A feasible control may be obtained by a standard technique of operating on the constraint and introducing slack variables, see for example Polak [P1], Zountendijk [Z1], Luenberger [LU2], Canon, Cullum and Polak [C1]. The precise method is as follows:

Introduce the slack variable $\alpha \geq 0$ and consider the following problem in $\mathbb{R} \times \underset{\sim}{G}$

$$\underset{\alpha, \underset{\sim}{u}}{\text{Min}} \quad \alpha$$

$$\text{s.t.} \quad -\alpha.1 + g(x^{\underset{\sim}{u}}(t), t) \leq 0 \qquad \text{for all } t$$

where $g: \mathbb{R}^n \times T \longrightarrow \mathbb{R}$ is the state constraint and $\alpha.1$ denotes a constant function of value α for all t.

This can be solved using Warga's iterative method. An initial feasible point for this problem $(\alpha_0, \underset{\sim}{u}_0)$ may be chosen by selecting any $\underset{\sim}{u}_0 \in \underset{\sim}{G}$ and setting $\alpha_0 = \max_{0 \leq t \leq 1} \{ g(x^{\underset{\sim}{u}_0}, t), 0 \}$.

If $\alpha_0 = 0$ then we already have a feasible control for the original problem and hence Warga's scheme may be applied to solve it.

If $\alpha_0 > 0$, then Warga's scheme produces a sequence $\{ \alpha_i, \underset{\sim}{u}_i \}$ when solving the above problem in $\mathbb{R} \times \underset{\sim}{G}$. Because of the convergence properties proved in [W2] we must have $\alpha_i \longrightarrow 0$ and $\underset{\sim}{u}_i \longrightarrow \bar{\underset{\sim}{u}} \in \underset{\sim}{G}$, with $g(x^{\bar{\underset{\sim}{u}}}, t) \leq 0$ for all t, i.e. $\bar{\underset{\sim}{u}}$ is feasible for the original problem.

However, the main disadvantage of this scheme is that in general, a feasible point is guaranteed only after an infinite number of iterations. This is somewhat of a drawback since it may be required to perform an infinite number of iterations before an attempt to solving the original problem can be made.

We propose to solve the state contrained control problem [Problem P3 defined below] using an Exact Penalty Function method for which the foundations have been laid in Chapter 4, although a more usual penalty term will be used as discussed in Appendix B. Our method, we believe, improves the approach taken by Warga [W2] since it will function for any initial point regardless of being feasible or not. Also the procedure allows for points outside of the feasible region to be generated as long as a descent in the objective functional justifies it.

Before we proceed any further with our discussion we formulate our SCCP and state the basic hypothesis assumed.

5.3 Problem Statement

In this chapter we will be considering the following non-linear state constrained optimal control problem with delay:

<u>Problem P3</u>

$$\min_{\underset{\sim}{u}} \int_0^1 l(x(t),\underset{\sim}{u}(t),t)dt \tag{3.1}$$

$$\text{s.t. } \dot{x}(t) = f(x(t),x(t-\tau),\underset{\sim}{u}(t),t) \quad \text{for a.a.} t \in T \tag{3.2}$$

$$x(t) = \phi(t) \quad \text{for all } t \in [-\tau,0] \tag{3.3}$$

$$g(x(t),t) \leq 0 \quad \text{for every } t \in T \tag{3.4}$$

$$\underset{\sim}{u} \in \underset{\sim}{G}$$

where $\underset{\sim}{G}$ is the space of measurable relaxed controls. ϕ is as before and the same terminology and notation as in previous chapters is adhered to.

We make the following assumptions on the functions l,f and g:

Assumption 1

The functions

$f : \mathbb{R}^n x \mathbb{R}^n x \mathbb{R}^m xT \longrightarrow \mathbb{R}^n$

$l : \mathbb{R}^n x \mathbb{R}^m xT \longrightarrow \mathbb{R}$

$g : \mathbb{R}^n xT \longrightarrow \mathbb{R}$

and their partial derivatives $f_x, f_{xx}, f_y, f_{yy}, f_{xy}, l_x, l_{xx}, g_x, g_x$ exist and are continuous on their respective domains.

Assumption 2

There exists an $M \in (0,\infty)$ such that

$$\| f(x,y,\underset{\sim}{u},t) \| \leq M \{ \|x\| + \|y\| + 1 \}$$

for all $x,y \in \mathbb{R}^n$, all $\underset{\sim}{u} \in \underset{\sim}{G}$ and for all $t \in T$

and

$$\| f(x^1,y^1,\underset{\sim}{u},t) - f(x^2,y^2,\underset{\sim}{u},t) \| \leq M \{ \| x^1 - x^2 \| + \| y^1 - y^2 \| \}$$

for all $x^1, y^1, x^2, y^2 \in \mathbb{R}^n$, for all $\underset{\sim}{u} \in \underset{\sim}{G}$ and for all $t \in T$.

5.4.1 Penalised Problem and Intermediate Problem

As in the last chapter we will use a penalty function to formulate an equivalent problem. To do this we define $\gamma : \underset{\sim}{G} \longrightarrow \mathbb{R}$ by

$$\gamma(\underset{\sim}{u}) \triangleq \max_{0 \leq t \leq 1} \{ g(x^{\underset{\sim}{u}}(t),t), 0 \} \tag{4.1}$$

and for all $K \geq 0$ we define $\gamma_K : \underset{\sim}{G} \longrightarrow \mathbb{R}$ by

$$\gamma_K(\underset{\sim}{u}) \triangleq \int_0^1 l(x^{\underset{\sim}{u}}(t),\underset{\sim}{u}(t),t)dt + K\gamma(\underset{\sim}{u}) \tag{4.2}$$

Now we define the following problem for all $K \geq 0$,

$\underline{\text{Problem } P3_K}$: Min $\{\gamma_K(\underset{\sim}{u}) : \underset{\sim}{u} \in \underset{\sim}{G}\}$

This will be referred to as the penalised problem.

Certain assumptions have to be made to ensure that Problem P3 is equivalent to Problem $P3_K$ (for some finite K). These will be fully discussed later but for the moment we will assume that the two problems are equivalent and try to solve $P3_K$ using a similar method as in Chapter 4 (with some modifications so that the state constraint can be handled). For this reason we define $\tilde{\theta}_K : \underset{\sim}{G} \longrightarrow \mathbb{R}$ by

$$\tilde{\theta}_K(\underset{\sim}{u}) \triangleq \min_{\underset{\sim}{v} \in \underset{\sim}{G}} \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v},\underset{\sim}{u})]dt + K \max_{0 \le t \le 1} \{g(x^{\underset{\sim}{u}},t) + g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}(t), 0\} - K\gamma(\underset{\sim}{u}) \qquad (4.3)$$

where $l_x(\underset{\sim}{u}) = l_x(x^{\underset{\sim}{u}}, \underset{\sim}{u}, t)$

$$\Delta l(\underset{\sim}{v},\underset{\sim}{u}) = l(x^{\underset{\sim}{u}}, \underset{\sim}{v}, t) - l(x^{\underset{\sim}{u}}, \underset{\sim}{u}, t)$$

and $z^{\underset{\sim}{u},\underset{\sim}{v}} : T \longrightarrow \mathbb{R}^n$ is the solution of

$$\dot{z}(t) = A^{\underset{\sim}{u}}(t)z(t) + B^{\underset{\sim}{u}}(t)z(t-\tau) + \Delta f(\underset{\sim}{v},\underset{\sim}{u}) \qquad \text{for a.a. } t \in T \qquad (4.4)$$

$$z(t) = 0 \qquad \text{for all } t \in [-\tau, 0] \qquad (4.5)$$

where $A^{\underset{\sim}{u}}(t) = f_x(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, t)$ (4.6)

$$B^{\underset{\sim}{u}}(t) = f_y(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, t) \qquad (4.7)$$

$$\Delta f(\underset{\sim}{v},\underset{\sim}{u}) = f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{v}, t) - f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \underset{\sim}{u}, t) \qquad (4.8)$$

As shown in previous chapters $z^{\underset{\sim}{u},\underset{\sim}{v}}$ is a first order estimate of $x^{\underset{\sim}{v}} - x^{\underset{\sim}{u}}$, and $\tilde{\theta}_K(\underset{\sim}{u})$ can be thought of as being an estimate of the maximum reduction in $\gamma_K(\underset{\sim}{u})$.

When solving $P3_K$, $\tilde{\theta}_K$ has to be determined at each iteration. For this reason we refer to the problem of obtaining $\tilde{\theta}_K$ as the intermediate problem for $P3_K$, denoted as Problem $P3K_{INT}$.

Now a method for solving $P3K_{INT}$ at each iteration is required if $P3_K$ (and hence the original problem) is to be solved. $P3K_{INT}$ in the form described by (4.3) is very difficult to solve because of the maximum being taken over an infinite number of points. Therefore we propose to reformulate it using the method proposed by Warga [W4], into the following equivalent form:

Problem $P3_K(\underset{\sim}{u})$:

$$\min_{\substack{\underset{\sim}{v} \in \tilde{G} \\ \beta \in \mathbb{R}}} \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v},\underset{\sim}{u})]dt + K\beta - K\gamma(\underset{\sim}{u})$$

$$\text{s.t.} \quad g(x^{\underset{\sim}{u}},t) + g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) - \beta.1 \leq 0 \qquad \text{for every } t \in T$$

$$\beta \geq 0$$

where $\beta.1$ means a constant function of value β for all $t \in T$.

This is similar to the linearized version of the original problem P3, (which Warga [W2] considers and solves), but because of the penalty term $K\beta$, our method does not require the necessity to start (or stay) in the feasible region. This is the essential difference in the approach taken by Warga and ours. However, we will use a method very similar to Warga's to solve $P3_K(\underset{\sim}{u})$.

5.4.2 Equivalence of Problems $P3K_{INT}$ and $P3_K(\underset{\sim}{u})$

Before we proceed any further with presenting our method for solving $P3_K(\underset{\sim}{u})$ we prove that Problems $P3K_{INT}$ and $P3_K(\underset{\sim}{u})$ are equivalent as stated.

The two problems can be written as follows:

$$P3K_{INT} : \min_{\underset{\sim}{v} \in \underset{\sim}{G}} L(\underset{\sim}{v}) + K\psi(\underset{\sim}{v}) - K\gamma(\underset{\sim}{u})$$

$$P3_K(\underset{\sim}{u}) : \min_{\substack{\underset{\sim}{v} \in \underset{\sim}{G} \\ \beta \in \mathbb{R}}} L(\underset{\sim}{v}) + K\beta - K\gamma(\underset{\sim}{u})$$

$$\text{s.t.} \quad \psi(\underset{\sim}{v}) \le \beta$$

where $L(\underset{\sim}{v}) = \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v},\underset{\sim}{u})]dt$

and $\psi(\underset{\sim}{v}) = \max_{0 \le t \le 1} \{g(x^{\underset{\sim}{u}},t) + g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}(t), 0\}$

Then we have for each $\underset{\sim}{v} \in \underset{\sim}{G}$

$$L(\underset{\sim}{v}) + K\psi(\underset{\sim}{v}) \le L(\underset{\sim}{v}) + K\beta$$

if $\beta \ge \psi(\underset{\sim}{v})$. Therefore the solution of $P3K_{INT}$, written inf $(P3K_{INT})$, is less (or equal) to the solution of $P3_K(\underset{\sim}{u})$ [denoted as inf $(P3_K(\underset{\sim}{u}))$].

$$\text{i.e.} \quad \inf (P3K_{INT}) \le \inf (P3_K(\underset{\sim}{u})) \qquad \text{(A)}$$

We also have for all $\underset{\sim}{v} \in \underset{\sim}{G}$

$$L(\underset{\sim}{v}) + K\psi(\underset{\sim}{v}) = L(\underset{\sim}{v}) + K\bar{\beta}$$

where $\bar{\beta}(\ge \psi(\underset{\sim}{v}))$ is defined by $\bar{\beta} = \psi(\underset{\sim}{v})$

This implies that

$$\inf (P3_K(\underset{\sim}{u})) \le \inf (P3K_{INT}) \qquad \text{(B)}$$

From (A) and (B) we deduce that the two problems have the same solution, and are therefore equivalent.

5.5 Solution of Problem $P3_K(\underset{\sim}{u})$

5.5.1 Conceptual Procedure for Solving $P3_K(\underset{\sim}{u})$

Our method for solving Problem $P3_K(\underset{\sim}{u})$ is based closely on the method proposed by Warga [W2] and we will discuss this procedure fully.

Solving $P3_K(\underset{\sim}{u})$ essentially involves determining the $(\beta^*,\underset{\sim}{v}^*) \in \mathbb{R} \times \underset{\sim}{G}$ which minimizes

$$\int_0^1 [\, l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v},\underset{\sim}{u})\,] dt + K\beta - K\gamma(\underset{\sim}{u})$$

over the set $S(\underset{\sim}{u})$, where we define $S(\underset{\sim}{u})$ by

$$S(\underset{\sim}{u}) = \{(\beta,\underset{\sim}{v}) : \beta \geq 0, \quad \beta \in \mathbb{R},$$
$$g(x^{\underset{\sim}{u}},t) + g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) - \beta.1 \leq 0 \text{ all } t, \underset{\sim}{v} \in \underset{\sim}{G} \}$$

This as we show below is a special case of problem II in Warga [W2] which we restate here for convenience:

Problem II in [W2]

Let H be a real Hilbert space, $\mathscr{H} = \mathbb{R} \times H$, the corresponding Hilbert space with inner product

$$(\alpha,x).(\beta,y) = \alpha\beta + x.y$$

and let A be a closed, convex bounded subset of $\mathscr{H}$. Then Problem II is

$$\text{Min} \quad \{\beta : (\beta,\emptyset) \in A\}$$

We now show that Problem $P3_K(\underset{\sim}{u})$ is a special case of the above problem:

Let $T \triangleq [0,1]$ and let $\|.\|_2$ be the usual norm of the Hilbert space $L^2[T, \mathbb{R}]$. It is easily seen that any compact subset P of $(C[T; \mathbb{R}], \|\cdot\|_{sup})$ is also a compact subset of $(L^2[T, \mathbb{R}], \|\cdot\|_2)$ and that in P, $\|\cdot\|_{sup}$-convergence and $\|\cdot\|_2$-convergence are equivalent.

We also have by our assumptions of boundedness that there exists a finite d such that

$$\int_0^1 [l_x(\underset{\sim}{u})z^{\underset{\sim}{u},\underset{\sim}{v}}+\Delta l(\underset{\sim}{v},\underset{\sim}{u})]dt+K\beta-\gamma(\underset{\sim}{u}) \geq -d$$

$$g(x^{\underset{\sim}{u}},t)+g_x(x^{\underset{\sim}{u}},t)z^{\underset{\sim}{u},\underset{\sim}{v}}-\beta.1 \geq -d \qquad \text{all } t, \text{for all } \underset{\sim}{u},\underset{\sim}{v} \in \underset{\sim}{G}, \text{ and finite } \beta$$

This bound d will be dependent on the control $\underset{\sim}{u}$ about which the problem has been linearized.

Now for any $\underset{\sim}{u} \in \underset{\sim}{G}$ we define the following set $A(\underset{\sim}{u})$ by

$$A(\underset{\sim}{u}) = \{(\int_0^1 [l_x(\underset{\sim}{u})z^{\underset{\sim}{u},\underset{\sim}{v}}+\Delta l(\underset{\sim}{v},\underset{\sim}{u})]dt+K\beta-K\gamma(\underset{\sim}{u}),$$
$$, g(x^{\underset{\sim}{u}},t)+g_x(x^{\underset{\sim}{u}},t)z^{\underset{\sim}{u},\underset{\sim}{v}}-\beta.1-\nu(t)):$$
$$\underset{\sim}{v} \in \underset{\sim}{G}, \ \nu \in L^2[T, \mathbb{R}], \nu \in [0,d(\underset{\sim}{u})] \text{a.e. in } T,$$
$$\beta \in [0,d(u)] \}$$

Then $A(\underset{\sim}{u})$ is a convex, closed and bounded subset of the Hilbert space $\mathbb{R} \times L^2[T, \mathbb{R}]$. It is clear (c.f. Warga [W2]) that

$$\underset{\underset{\sim}{v},\beta}{\text{Min}} \{\int_0^1 [l_x(\underset{\sim}{u})z^{\underset{\sim}{u},\underset{\sim}{v}}+\Delta l(\underset{\sim}{v},\underset{\sim}{u})]dt+K\beta-K\gamma(\underset{\sim}{u}) : (\beta,\underset{\sim}{v}) \in S(\underset{\sim}{u})\}$$

$$\equiv \min (\alpha : (\alpha,\emptyset) \in A(\underset{\sim}{u}))$$

where $\emptyset$ is the function which is identically zero. Thus $P3_K(\underset{\sim}{u})$

is a special case of Problem II above, and hence any method which solves II may be applied to solve $P3_K(\underset{\sim}{u})$. Solving Problem $P3_K(\underset{\sim}{u})$ involves determining the point $(\alpha^*, \emptyset) \in \mathbb{R} \times L^2[T, \mathbb{R}]$ as shown in Fig 5.1.

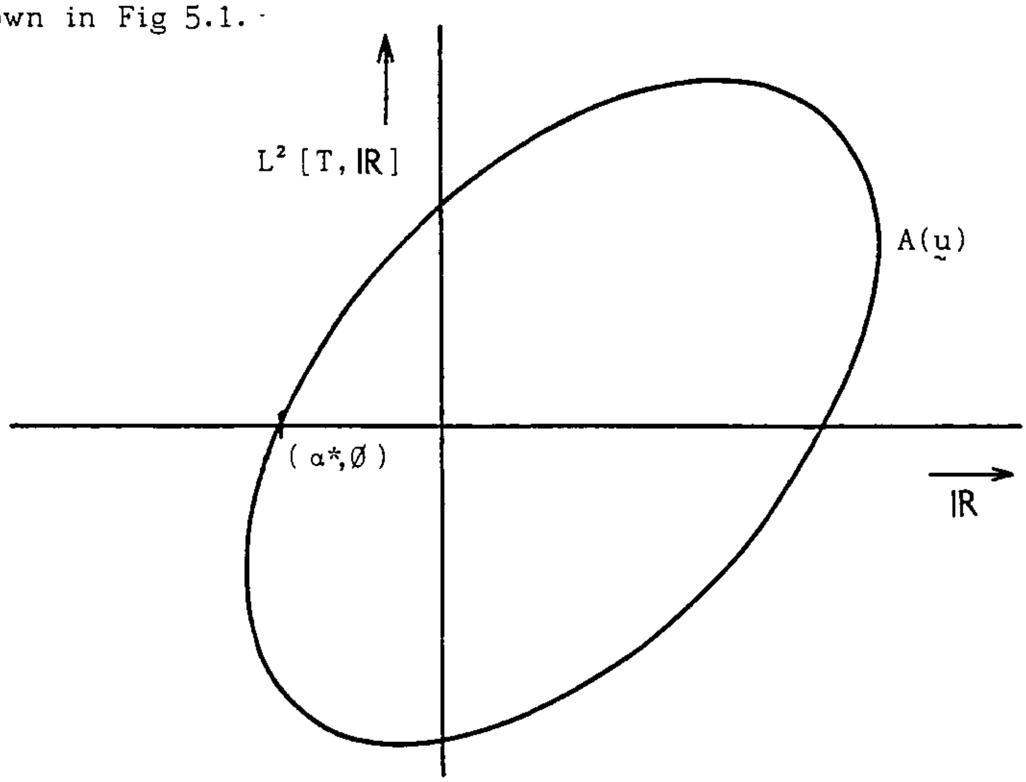


Fig 5.1

Problem $P3_K(\underset{\sim}{u})$ always has a solution since $\underset{\sim}{v} = \underset{\sim}{u}$ and $\beta = \gamma(\underset{\sim}{u}) \in S(\underset{\sim}{u})$, in which case $z \equiv 0$ for all t, setting $\nu(t) = g(x^{\underset{\sim}{u}}, t)$ all t we obtain $(0, \emptyset) \in A(\underset{\sim}{u})$ for all $\underset{\sim}{u} \in \underset{\sim}{G}$.

We propose to use an iterative scheme similar to that presented in Warga [W2], but modified in such a way so that it can be applied to solve $P3_K(\underset{\sim}{u})$. This procedure which we refer to as Procedure W will require finding a point nearest to the convex bounded set over which the problem is defined (i.e. the set $A(\underset{\sim}{u})$) at each stage. We will determine such a point by an application of another iterative scheme (referred to as Procedure Y) which is nested inside Procedure W. The precise method for solving $P3_K(\underset{\sim}{u})$ will now be presented.

Procedure W

Step 0 : A point $x=(\alpha,\chi(.)) \in A(\underset{\sim}{u})$ is required (this may be chosen as $(0, \emptyset)$ as discussed above), as well as a $\sigma_0 < \alpha^*$, α^* is the optimal cost for $P3_K(\underset{\sim}{u})$

Step 1 : Set $a_0 = (\sigma_0, \emptyset) \in \mathbb{R} \times L^2[T, \mathbb{R}]$

Set $i = 0$

Step 2 : Compute the point $s(a_i) = (s^0(a_i), s^1(a_i))$ in $A(\underset{\sim}{u})$ which minimises the distance to a_i, together with the control $\bar{\underset{\sim}{v}}_i$ and $\bar{\beta}_i$ which achieve it, i.e.

$$s^0(a_i) = \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\bar{\underset{\sim}{v}}_i} + \Delta l(\bar{\underset{\sim}{v}}_i, \underset{\sim}{u})]dt + K\bar{\beta}_i - K\gamma(\underset{\sim}{u})$$

and $$s^1(a_i)(t) = g(x^{\underset{\sim}{u}}, t) + g_x(x^{\underset{\sim}{u}}, t) z^{\underset{\sim}{u},\bar{\underset{\sim}{v}}_i} - \bar{\beta}_i . 1$$

This is done by using the following Procedure Y:

Procedure Y

Step A : Set $y_0 = x$

Set $j = 0$

Step B: Compute $r_j = (h_j, \eta_j(t)) \in A(\underset{\sim}{u})$ which satisfies

$(y_j - a_i).r_j \leq (y_j - a_i).r$ for all $r \in A(\underset{\sim}{u})$

together with the $\underset{\sim}{v}_j$ and β_j which achieve it. The symbol "." denotes the inner product in $\mathbb{R} \times L^2[T, \mathbb{R}]$ by

$$(a_1, b_1(t)).(a_2, b_2(t)) = a_1 a_2 + \int_0^1 b_1(t) b_2(t)\, dt$$

for all $a_1, a_2 \in \mathbb{R}$ and $b_1, b_2 \in L^2[T, \mathbb{R}]$

i.e. $(y_j - a_i)$ is an inner normal to $A(\underset{\sim}{u})$ at r_j (see Warga [W2])

If $r_j = y_j$, set $y_l = y_j$, $\underset{\sim}{v}_l = \underset{\sim}{v}_j$ and $\beta_l = \beta_j$ for $l > j$ and return to Procedure W. Else continue

Step C: Choose A_j as any closed convex subset of $A(\underset{\sim}{u})$ containing y_j and r_j (let A_j be the segment joining y_j and r_j)

Step D: Compute q_j as the point in A_j which minimises the distance to a_i

Step E: Set $y_{j+1} = q_j$

Set $j = j+1$

Goto Step B

Having obtained $y_\infty = s(a_i)$, $\overline{\underset{\sim}{v}}_i = \underset{\sim}{v}_\infty$, $\overline{\beta}_i = \beta_\infty$ return to Procedure W

Step 3 : If $s^\circ(a_i) - \sigma_i \le 0$, set $\sigma_l = \sigma_i$, $\overline{\underset{\sim}{v}}_l = \overline{\underset{\sim}{v}}_i$, $\overline{\beta}_l = \overline{\beta}_i$ for all $l > i$ and terminate Procedure W. Else proceed to Step 4

Step 4 : Set $\sigma_{i+1} = s^\circ(a_i) + (s^\circ(a_i) - \sigma_i)^{-1} \int_0^1 \| s^1(a_i)(t) \|^2 dt$

Set $a_{i+1} = (\sigma_{i+1}, \emptyset)$

Set $i = i+1$

Goto Step 2

This algorithm is illustrated in Figs 5.2 and 5.3. Fig 5.2 shows how the shortest distance from a_i to $A(\underset{\sim}{u})$ is determined at each stage and Fig 5.3 shows how the solution to Problem $P3_K(\underset{\sim}{u})$ is found.

It is proved in Warga [W2] that if Procedure Y constructs points $y_0, y_1, y_2, \ldots$, then $\lim_{j \to \infty} y_j = s(a_i)$ for each i.

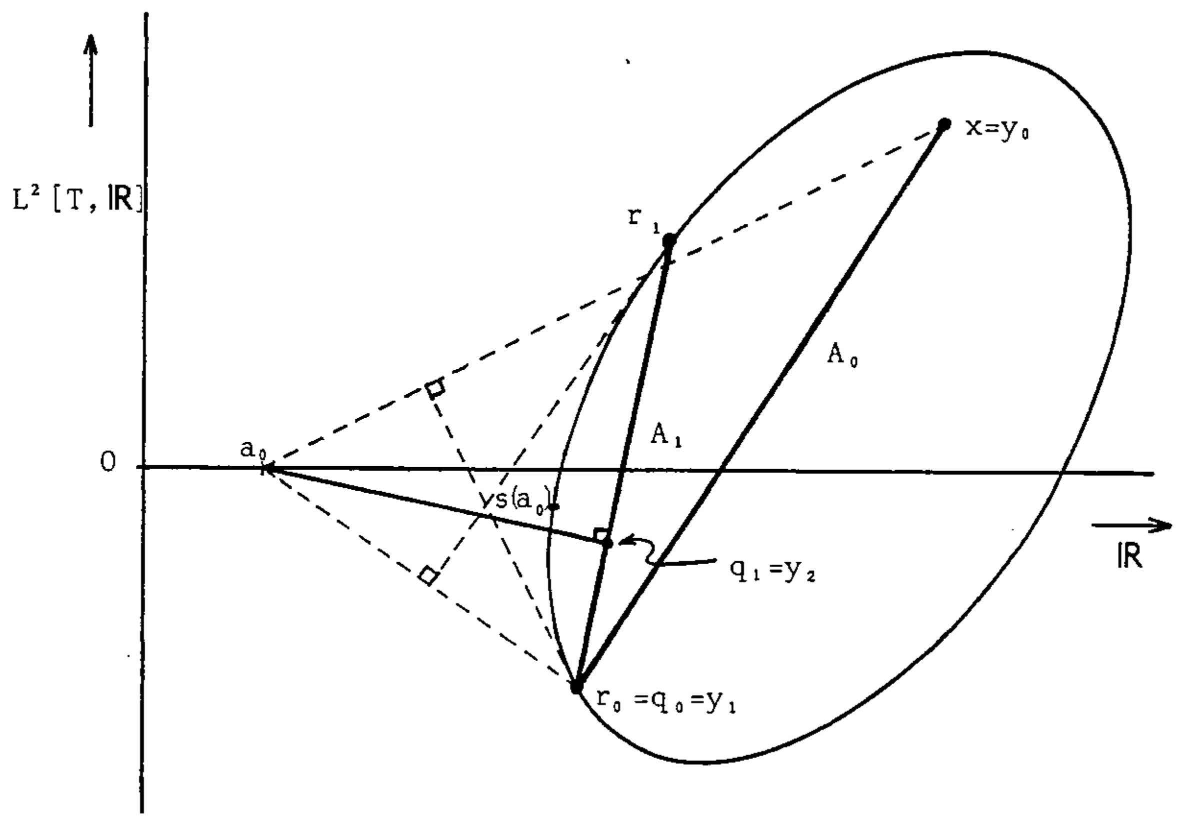


Fig. 5.2 Procedure Y

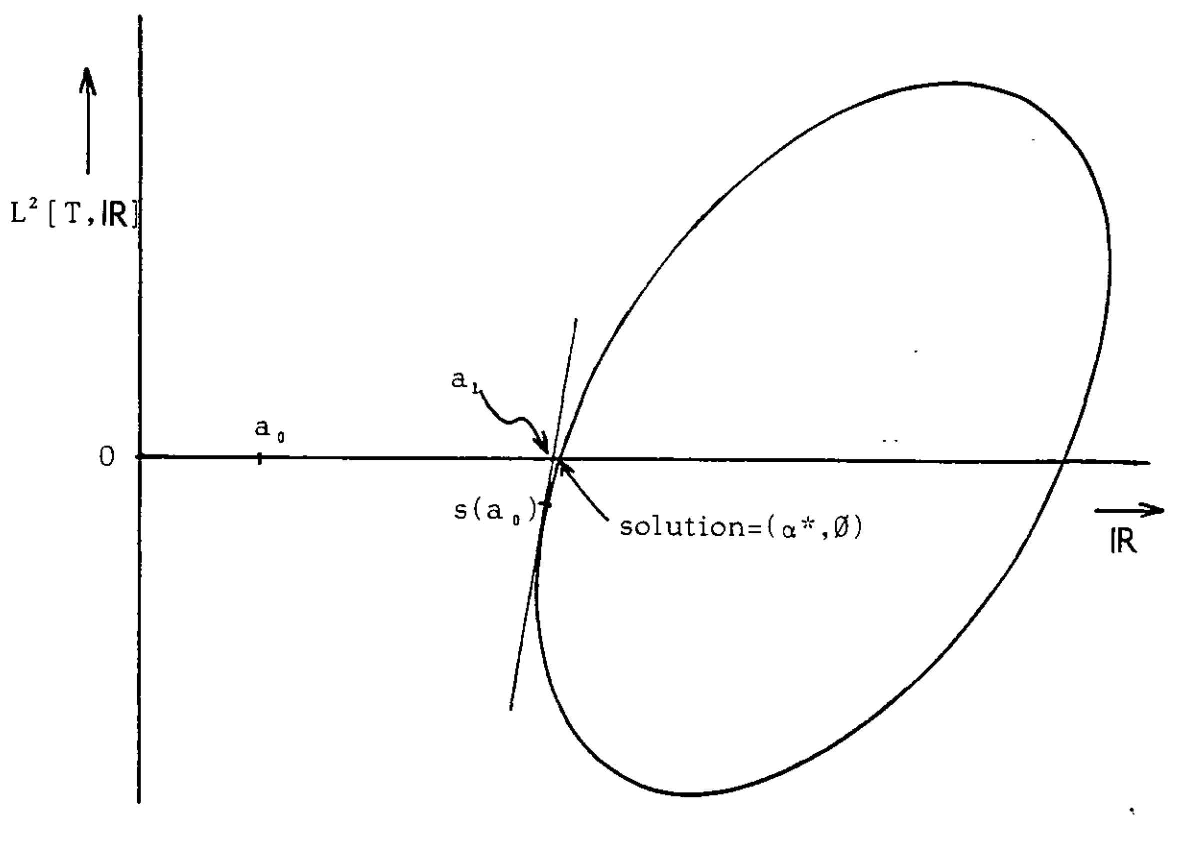


Fig.5.3 Procedure W

It is not difficult now to show that $\underset{\sim}{y}_j \longrightarrow \bar{\underset{\sim}{y}}_i$ i.s.c.m. and $\beta_j \longrightarrow \bar{\beta}_i$ are the values of $(\underset{\sim}{y}, \beta)$ which achieve $s(a_i)$.

Also, procedure W has the following convergence properties (again proved in [W2]).

<u>Theorem 5.1</u>

Let $\sigma_1, \sigma_2, \sigma_3, \ldots.$ be constructed by Procedure W. Then

$$\sigma_i \leq \sigma_{i+1} \leq \alpha^* \qquad\qquad i=0,1,2,\ldots$$

and
$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} s(a_i) = (\alpha^*, \emptyset)$$

However, before Procedures W and Y can actually be applied to obtain the solution to $P3_K(\underset{\sim}{u})$ a number of difficulties must still be overcome. These are:

1. The bound $d(\underset{\sim}{u}) \in (0,\infty)$ which satisfies

$$\int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v},\underset{\sim}{u})]\,dt + K\beta - K\gamma(\underset{\sim}{u}) \geq -d(\underset{\sim}{u})$$

$$g(x^{\underset{\sim}{u}},t) + g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) - \beta.1 \geq -d(\underset{\sim}{u}) \qquad \text{all } t \in T$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$ is required.

2. The starting point σ_0 which is less or equal to α^* is required, i.e. the starting point must be to the left of the set $A(\underset{\sim}{u})$ on the cost axis.

3. At step B of Procedure Y we must find a point $r_j \in A(\underset{\sim}{u})$ which satisfies $(y_j - a_i).r_j \leq (y_j - a_i).r$ for all $r \in A(\underset{\sim}{u})$ for a given y_j, a_i. This means for a given $p=(p^1, p^2(t)) \in \mathbb{R} \times L^2[T, \mathbb{R}]$ we need to find $r(p) \in A(\underset{\sim}{u})$ such that

$$p.r(p) \leq p.r \qquad\qquad \text{for all } r \in A(\underset{\sim}{u})$$

4. At step D the point $q_j \in A_j$ which minimises the distance to a_i is required.

We will now show how these problems are dealt with.

a. The constant $d(\underset{\sim}{u})$ may be set to be

$$d(\underset{\sim}{u}) = -\inf\{K\beta - K\gamma(\underset{\sim}{u}), g(x^{\underset{\sim}{u}},t) - \beta.1, \quad t \in T$$
$$\beta \in [0, d_1(\underset{\sim}{u})]\}$$
$$-\inf\{d_2(\underset{\sim}{u}), d_3(t); \ t \in T\}$$

where $d_1(\underset{\sim}{u}) = \max\limits_{\substack{0 \le t \le 1 \\ \underset{\sim}{v} \in \underset{\sim}{G}}} g(x^{\underset{\sim}{u}},t) + g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}(t), 0\}$

There is no need to consider β greater than this since the larger the β the greater the contribution to the cost.

Also $d_2(\underset{\sim}{u}) = \int_0^1 \min\limits_{\underset{\sim}{v} \in \underset{\sim}{G}} [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v},\underset{\sim}{u})] dt$

$d_3(t) = \min\limits_{\underset{\sim}{v} \in \underset{\sim}{G}} g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) \qquad t \in T$

This just gives the lowest value that any point in $A(\underset{\sim}{u})$ can take.

b. σ_0 may be chosen as $-d(\underset{\sim}{u})$, hence it can at most be on the boundary of $A(\underset{\sim}{u})$.

c. For any $p = (p^1, p^2(t)) \in \mathbb{R} \times L^2[T, \mathbb{R}]$, we need to find $r(p) \in A(\underset{\sim}{u})$ which solves $\min\limits_{r \in A(\underset{\sim}{u})} p.r$

Now since

$$A(\underset{\sim}{u}) = \{(\int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v},\underset{\sim}{u})] dt + K\beta - K\gamma(\underset{\sim}{u}), g(x^{\underset{\sim}{u}},t)$$
$$+ g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}} - \beta.1 + \nu(t)) :$$

$$: \underset{\sim}{v} \in \underset{\sim}{G},\ \beta \in [0, d_1(\underset{\sim}{u})],\ \nu \in L^2[T, \mathbb{R}],$$

$$\nu(t) \in [0, d(\underset{\sim}{u})] \text{ a.e. in } T \}$$

the above problem reduces to

$$\min_{\substack{\underset{\sim}{v} \in \underset{\sim}{G} \\ \beta \in [0, d_1(\underset{\sim}{u})] \\ \nu(t) \in [0, \tilde{d}(\underset{\sim}{u})] \text{ a.e. in } T}} (p^1, p^2(t)) . \left(\int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v}, \underset{\sim}{u})] dt + K\beta - K\gamma(\underset{\sim}{u}),\ g(x^{\underset{\sim}{u}}, t) + g_x(x^{\underset{\sim}{u}}, t) z^{\underset{\sim}{u},\underset{\sim}{v}} - \beta . 1 + \nu(t) \right)$$

$$= \min_{\underset{\sim}{v}, \beta, \nu} p^1 \{ \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v}, \underset{\sim}{u})] dt + K\beta - K\gamma(\underset{\sim}{u}) \}$$
$$+ \int_0^1 p^2(t) . (g(x^{\underset{\sim}{u}}, t) + g_x(x^{\underset{\sim}{u}}, t) z^{\underset{\sim}{u},\underset{\sim}{v}} - \beta . 1 + \nu(t)) dt$$

$$= \min_{\underset{\sim}{v}, \beta, \nu} (K p^1 - \int_0^1 p^2(t) dt) \beta + \int_0^1 p^2(t)\, \nu(t) dt$$
$$+ p^1 \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v}, \underset{\sim}{u})] dt - p^1 K\gamma(\underset{\sim}{u})$$
$$+ \int_0^1 p^2(t) (g(x^{\underset{\sim}{u}}, t) + g_x(x^{\underset{\sim}{u}}, t) z^{\underset{\sim}{u},\underset{\sim}{v}}) dt$$

This gives the solutions to β and ν to be

$$\beta(p) = 0 \text{ or } d_1(\underset{\sim}{u})$$

$$\nu(p)(t) = 0 \text{ or } d(\underset{\sim}{u}) \qquad \text{for every } t \in T$$

depending on the sign of their respective coefficients.

Hence it remains to solve

$$\min_{\underset{\sim}{v}} \quad p^1 \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v}, \underset{\sim}{u})] dt + \int_0^1 p^2(t) g_x(x^{\underset{\sim}{u}}, t) z^{\underset{\sim}{u},\underset{\sim}{v}}\, dt$$

where $z^{\underset{\sim}{u},\underset{\sim}{v}}$ is the solution to the difference-differential equation defined by (4.4),(4.5).

Using the Maximum Principle for delay systems and the properties of relaxed controls (see Chapter 1) it is quite easy to show that a minimising control $\underset{\sim}{v}(p) \in \underset{\sim}{G}$ can be obtained by setting $\underset{\sim}{v}(p) = \delta_\omega$ (measure concentrated at a single ordinary control

$\omega(t)$ where for each $t \in T$, $v(t)=\omega(t)$ maximises

$$\psi(t)^T f(x^{\underset{\sim}{u}}, y^{\underset{\sim}{u}}, \delta_v \Theta \underset{\sim}{u}(t), t) + p^1 l(x^{\underset{\sim}{u}}, \delta_v \Theta \underset{\sim}{u}(t), t)$$

and $\psi : T \longrightarrow \mathbb{R}^n$ is the solution of

$$\psi^T(t) = \int_t^1 \psi^T(s) A^{\underset{\sim}{u}}(s) + \psi^T(s+\tau) B^{\underset{\sim}{u}}(s+\tau)$$

$$+ \int_t^1 [p^1 l_x(\underset{\sim}{u}) + p^2(s) g_x(x^{\underset{\sim}{u}}, s)] ds$$

$$\text{for all } s \in [0, 1-\tau]$$

$$\psi^T(t) = \int_t^1 \psi^T(s) A^{\underset{\sim}{u}}(s) \, ds$$

$$+ \int_t^1 [p^1 l_x(\underset{\sim}{u}) + p^2(s) g_x(x^{\underset{\sim}{u}}, s)] ds$$

$$\text{for all } t \in [1-\tau, 1]$$

$$\psi(1) = 0$$

Thus the necessary point $r(p) \in A(\underset{\sim}{u})$ is given by

$$r(p) = \{\int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u}, \underset{\sim}{v}(p)}(t) + \Delta l(\underset{\sim}{v}(p), \underset{\sim}{u})] dt + K\beta(p) - K\gamma(\underset{\sim}{u}),$$

$$g(x^{\underset{\sim}{u}}, t) + g_x(x^{\underset{\sim}{u}}, t) z^{\underset{\sim}{u}, \underset{\sim}{v}(p)}(t) - \beta(p).1 + \nu(p)(t)\}$$

d. The point $q_j \in A_j$ which minimises the distance to a_i can be obtained as follows:

Defining A_j to be the segment joining y_j and r_j we have the situation as shown in Fig. 5.4.

Now any point in the segment RY (i.e. the set A_j) can be expressed as

$$q = (1-\delta) y_j + \delta r_j \qquad \text{for } \delta \in [0,1]$$

and $q = y_j$ when $\delta = 0$

$= r_j$ when $\delta = 1$

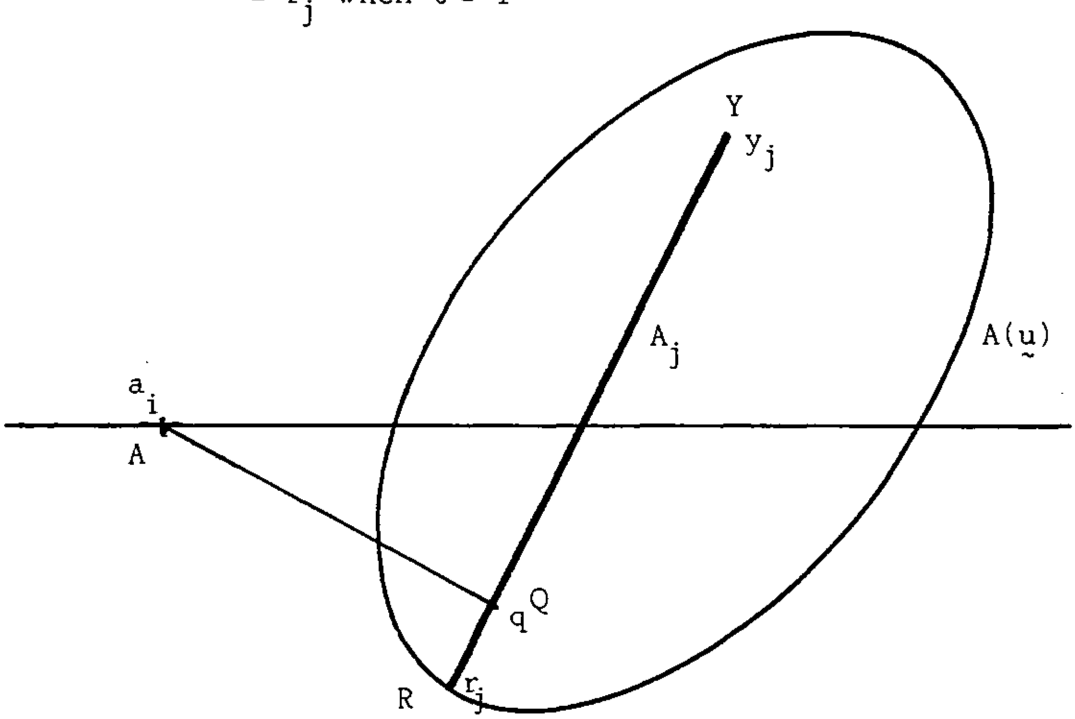


Fig 5.4

We want to calculate $\delta = \delta_{min}$ so that q minimises the distance AQ . By the projection theorem, see Luenberger [LU1], this occurs when the inner product of AQ is orthogonal to RY, i.e. $\bar{\delta}$ is determined by solving the following:

$$(a_i - [(1-\bar{\delta})y_j + \bar{\delta}r_j]).(r_j - q_j) = 0$$

Now if $a_i = (\sigma, \emptyset)$, $y_j = (\alpha, \chi(t))$ and $r_j = (h, \eta(t))$ we get

$$((\sigma,\emptyset) - \{(1-\bar{\delta})(\alpha,\chi(t)) + \bar{\delta}(h,\eta(t))\}).((h,\eta(t)) - (\alpha,\chi(t))) = 0$$

i.e.

$$([\sigma - (1-\bar{\delta})\alpha - \bar{\delta}h], (-1+\bar{\delta})\chi(t) - \bar{\delta}\eta(t)).(h-\alpha, \eta(t) - \chi(t)) = 0$$

Using the inner product in the Hilbert space $\mathbb{R} \times L^2[T, \mathbb{R}]$ as defined above we get

$$[\sigma-(1-\bar{\delta})\alpha-\bar{\delta}h](h-\alpha)-((1-\bar{\delta})\chi(t)+\bar{\delta}\eta(t)).(\eta(t)-\chi(t))=0$$

i.e.

$$[\sigma-\alpha-\bar{\delta}(h-\alpha)](h-\alpha)-\int_0^1(\chi(t)+\bar{\delta}(\eta(t)-\chi(t)))(\eta(t)-\chi(t))dt=0$$

$$(\sigma-\alpha)(h-\alpha)-\bar{\delta}(h-\alpha)^2-\int_0^1\chi(t)(\eta(t)-\chi(t))dt$$

$$-\bar{\delta}\int_0^1(\eta(t)-\chi(t))^2dt=0$$

Therefore

$$\bar{\delta} = \frac{(\sigma-\alpha)(h-\alpha)-\int_0^1\chi(t)(\eta(t)-\chi(t))dt}{(h-\alpha)^2+\int_0^1(\eta(t)-\chi(t))^2dt}$$

Thus we get

$$\begin{aligned}\delta_{min} &= 0 && \text{if } \bar{\delta} \le 0\\ &= \bar{\delta} && \text{if } \bar{\delta} \in (0,1)\\ &= 1 && \text{if } \bar{\delta} \ge 1\end{aligned}$$

This determines the required point q_j at each stage.

5.5.2 Implementable Procedure for Solving Problem $P3_K(\underset{\sim}{u})$

Although Procedure W can be used to solve Problem $P3_K(\underset{\sim}{u})$, it is quite obvious that the method is conceptual since it may require an infinite number of iterations to obtain $s(a_i)$ for each i. We therefore present now an implementable version (Procedure W_1) of the scheme where the (infinite) Procedure Y is truncated after a finite number of steps according to a specific test. Before we present the implementable procedure we need to say a few words about the terminology which we will be using. For any point $a \in \mathbb{R} \times L^2[T, \mathbb{R}]$, Procedure Y, starting from any point $y \in A(\underset{\sim}{u})$, constructs a sequence $\{\xi_i(a,y)\} \in A(\underset{\sim}{u})$ converging to $s(a)$, and when stating Procedure W_1 we shall refer to such sequences directly.

Procedure W_1

Step 1 : The following data is required:

A point $x_0 = (\alpha_0, \chi_0(.)) \in A(\underset{\sim}{u})$ together with the $\underset{\sim}{v}_0$, β_0 which achieve it, $\sigma_0 \leq \alpha^*$, a number $M \in (0,1)$

Step 1 : Set $a_0 = (\sigma_0, \emptyset) \in \mathbb{R} \times L^2[T, \mathbb{R}]$

Set $i=0$

Step 2 : Define $\lambda_i = (\lambda_i^0, \lambda_i^1(t)) = x_i - a_i$

If $\lambda_i^0 < 0$, set $p_i = (\lambda_i^0)^{-1} \lambda_i$ and compute $b_i = (b_i^0, b_i^1(t)) \in A(\underset{\sim}{u})$ which satisfies

$$p_i . b_i \leq p_i . b_i \qquad \text{for all } b \in A(\underset{\sim}{u})$$

together with the $\underset{\sim}{v}(p_i)$, $\beta(p_i)$ which achieve it (this can be obtained by the method described in the last section). Then goto Step 3.

Otherwise proceed to Step 4.

Step 3 : If $\lambda_i^0 < 0$ and $p_i . b_i \geq \sigma_i + M \mid \lambda_i \mid$

set $\sigma_{i+1} = p_i . b_i$, $a_{i+1} = (\sigma_{i+1}, \emptyset)$

$x_{i+1} = x_i$

$\underset{\sim}{v}_{i+1} = \underset{\sim}{v}_i$

$\beta_{i+1} = \beta_i$

set $i = i+1$ and goto Step 2

Otherwise proceed to Step 4.

Step 4 : Determine $\xi_j(a_i, x_i)$ for $j=1,2,\ldots.$

If for some integer j_0 the conditions in Step 3 are satisfied with x_i replaced by $\xi_{j_0}(a_i, x_i)$ terminate this internal iteration and set $\sigma_{i+1} = \sigma_i$

$$x_{i+1} = \xi_{j_0}(a_i, x_i)$$

set $\underset{\sim}{v}_{i+1}$ and β_{i+1} as those that achieve

$\xi_{j_0}(a_i, x_i)$ in $A(\underset{\sim}{u})$

set $i = i+1$ and goto Step 2

We shall say that Procedure W_1 stops at N if Step 4 applies for i=N and a j_0 does not exist.

It is shown in Warga [W2] that Procedure W has the following properties.

Theorem 5.2

Since Problem $P3_K(\underset{\sim}{u})$ has a solution Procedure W_1 does not stop at any j and $\lim_i \sigma_i = \alpha^*$ and $\lim_i x_i = (\alpha^*, \emptyset)$

Remark 5.3

The above theorem shows that since $x_i = (\alpha_i, \chi_i(t))$ converges to $(\alpha^*, \emptyset)$ we must have that for any $\delta > 0$, there exists an i_0 such that

$$\| \alpha_i - \alpha^* \| \leq \delta$$

and $\| \chi_i(t) \|_{L_2} \leq \delta$ for all $i \geq i_0$.

This property will be referred to directly when we present our implementable procedure for solving the original Problem P3.

Hence using either one of the two Procedures W or W_1 we can obtain $\underset{\sim}{v}^*$, $\underset{\sim}{\beta}^*$ which solve the intermediate problem $P3_K(\underset{\sim}{u})$ together with the optimal cost $\alpha^* = \tilde{\theta}_K(\underset{\sim}{u})$.

5.6 Equivalence of Problem P3 and $P3_K$

Before we can proceed any further to solve P3 we need to present some hypothesis which will make our procedure well defined. These are analogous to the "constraint qualifications" in Chapter 4, which, in our opinion, are somewhat awkward even in problems with a finite number of constraints.

We propose a different line of attack which will guarantee that Problems P3 and $P3_K$ are equivalent for all $K \geq K^*$ for some K^*. This is achieved by imposing a certain calmness condition on P3. This hypothesis although very simple at first sight has profound consequences as we shall shortly show.

First, however, for purposes of motivation, we consider a finite dimensional optimization problem subject to an inequality constraint, i.e. consider

Problem F
$$\min_{x \in X} g^0(x)$$
$$\text{s.t. } g^1(x) \leq 0$$

where $X \subset \mathbb{R}^n$ is closed and $g^j: x \longrightarrow \mathbb{R}$, j=0,1.

We define a family of optimization problems by

$$\eta(\alpha) \triangleq \min\{g^0(x) : g^1(x) \leq \alpha\} \qquad (6.1)$$

It is obvious from the definition of η that it is monotone decreasing with increasing α, i.e. $\eta(0) \geq \eta(\alpha)$ for $\alpha \geq 0$.

Suppose $\eta(0)$ is finite and that there exists a finite $\bar{K} > 0$ such that

$$\eta(\alpha) \geq \eta(0) - \bar{K}\alpha \qquad \text{for all } \alpha \qquad (6.2)$$

This is true if η is bounded from below for all α, see Fig 6.1.

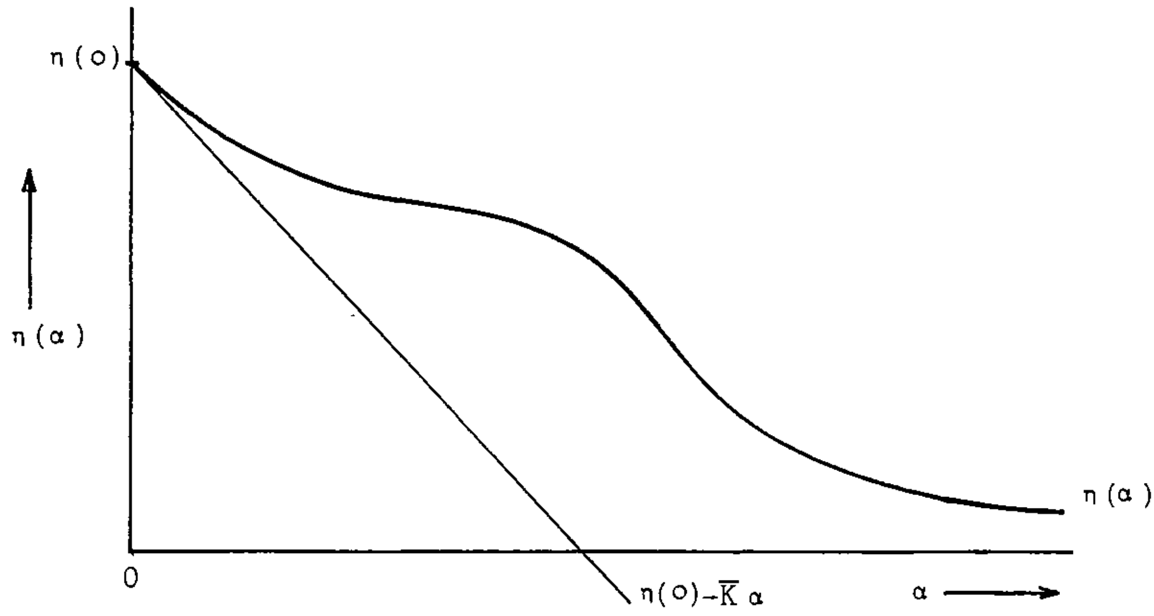


Fig 6.1

Now we consider the following problem:

Problem F_K

$$\text{Min} \quad g^0(x) + K\,\gamma(x) \qquad \text{for } K \geq 0$$

where $\gamma(x) = \max\{g^1(x), 0\}$.

Now choose a $\tilde{K} > \bar{K}$ and let $\bar{x}$ minimise

$$x \longrightarrow g^0(x) + \tilde{K}\,\gamma(x)$$

The following inequality holds

$$g^0(\bar{x}) + \tilde{K}\gamma(\bar{x}) \geq \min \{g^0(x) : g^1(x) \leq \gamma(\bar{x})\} + \tilde{K}\gamma(\bar{x})$$

This is because we have

$$g^0(\bar{x}) \geq \min \{g^0(x) : g^1(x) \leq \gamma(\bar{x})\}$$

from the fact that for $x=\bar{x}$ equality is achieved. Hence the minimum on the right hand side is at most equal to $g^0(\bar{x})$, and another $x_1 \in X$ may exist which does minimise $g^0(x)$ and also satisfies the constraint $g^1(x_1) \leq \gamma(\bar{x})$. Hence we have

$$g^0(\bar{x})+\tilde{K}\gamma(\bar{x}) \geq \eta(\gamma(\bar{x}))+\tilde{K}\gamma(\bar{x})$$

From above we have $\eta(\alpha) \geq \eta(0)-\bar{K}\alpha$ for all α, i.e. we have

$$\eta(\gamma(\bar{x})) \geq \eta(0)-\bar{K}\gamma(\bar{x})$$

Therefore

$$g^0(\bar{x})+\tilde{K}\gamma(\bar{x}) \geq \eta(0)+(\tilde{K}-\bar{K})\gamma(\bar{x}) \tag{6.3}$$

Also since $\bar{x}$ minimises $x \longrightarrow g^0(x)+\tilde{K}\gamma(x)$ we have

$$\eta(0) = \min \{g^0(x) : g^1(x) \leq 0\} \geq g^0(\bar{x})+\tilde{K}\gamma(\bar{x}) \tag{6.4}$$

This is because for any feasible x on the left hand side of inequality, i.e. any x such that $g^1(x) \leq 0$, we have

$$g^0(x) = g^0(x)+\tilde{K}\gamma(x) \geq g^0(\bar{x})+\tilde{K}\gamma(\bar{x}) \quad \text{for all } x \text{ s.t. } g^1(x) \leq 0$$

Hence we have

$$\eta(0) \geq g^0(\bar{x})+\tilde{K}\gamma(\bar{x})$$

But from (6.3) we have

$$g^0(\bar{x})+\tilde{K}\gamma(\bar{x}) \geq \eta(0)+(\tilde{K}-\bar{K})\gamma(\bar{x})$$

Hence we deduce that

$$\eta(0) \geq \eta(0) + (\tilde{K}-K)\gamma(\bar{x})$$

i.e. $(\tilde{K}-K)\gamma(\bar{x}) \leq 0$

Since $\tilde{K} > K$ we must have $\gamma(\bar{x}) \leq 0$, but γ is non-negative by definition and hence $\gamma(\bar{x})=0$, i.e. $\bar{x}$ is feasible for problem F.

Therefore from (6.3) we get, by letting $\gamma(\bar{x}) = 0$ that

$$g^0(\bar{x}) \geq \eta(0)$$

But (6.4) gives $\eta(0) \geq g^0(\bar{x})$

Hence $g^0(\bar{x}) = \eta(0)$

i.e. $\bar{x}$ is the solution for problem F.

Therefore we deduce that by choosing $\tilde{K}$ sufficiently large ($\tilde{K} > \bar{K}$ as shown above), we can solve problem F_K by solving the unconstrained problem F_K, i.e. F_K is equivalent to F for $K > \bar{K}$.

Now we consider the real problem in question and derive conditions to impose on it so that $P3_K$ is equivalent to P3. Problem P3 is

$$\min_{\underset{\sim}{u} \in \underset{\sim}{G}} \int_0^1 l(x^{\underset{\sim}{u}}, \underset{\sim}{u}, t)dt$$
$$\text{s.t.} \quad g(x^{\underset{\sim}{u}}, t) \leq 0 \qquad \text{for every } t$$

where $x^{\underset{\sim}{u}}: T \longrightarrow \mathbb{R}^n$ is the solution to the delay-differential equation

$$\dot{x}(t) = f(x, y, \underset{\sim}{u}, t) \qquad \text{a.a.} t \in T$$
$$\dot{x}(t) = \phi(t) \qquad \text{for every } t \in [-\tau, 0]$$

We start by defining the following family of optimal control problems:

$$\chi(\alpha) \triangleq \min_{\underset{\sim}{u} \in \underset{\sim}{G}} \{\int_0^1 l(x^{\underset{\sim}{u}}, \underset{\sim}{u}, t)dt \ : \ g(x^{\underset{\sim}{u}}, t) \leq \alpha, t \in T\}$$

Obviously problem P3 is embedded in this family and solving P3 may be regarded as determining the value $\chi(0)$

$$\text{Letting } l(\underset{\sim}{u}) \triangleq \int_0^1 l(x^{\underset{\sim}{u}}, \underset{\sim}{u}, t)dt$$
$$\text{and } \gamma(\underset{\sim}{u}) \triangleq \max_{0 \leq t \leq 1} \{g(x^{\underset{\sim}{u}}, t), 0\}$$

we have

$$\chi(\alpha) = \min_{\underset{\sim}{u} \in \underset{\sim}{G}} \{ l(\underset{\sim}{u}) : \gamma(\underset{\sim}{u}) \leq \alpha \}$$

The following assumptions are made on χ.

Assumption 3

(i) $\liminf_{\alpha \downarrow 0} \left\{ \frac{\chi(\alpha)-\chi(0)}{\alpha} \right\} = -K_0$ for some $K_0 \in (0,\infty)$ (6.5)

(ii) $\chi(0)$ is finite (6.6)

This assumption is known as calmness as defined by Clarke [CL1] and basically means that the rate of change of minimum cost at $\alpha=0$ is not infinite for the problem in which $g(x(t),t) \leq \alpha$ for all $t \in T$ replaces $g(x(t),t) \leq 0$ for all $t \in T$.

As proposed in the text we intend to solve P3 by solving Problem $P3_K$ defined by

$$\min_{\underset{\sim}{u} \in \underset{\sim}{G}} \{ l(\underset{\sim}{u})+K\gamma(\underset{\sim}{u}) \} \qquad \text{for } K > 0$$

Hence it is necessary to select K sufficiently large so that a solution that minimises $\underset{\sim}{u} \longrightarrow l(\underset{\sim}{u})+K\gamma(\underset{\sim}{u})$ is also a solution to

$$\chi(0) = \min \{ l(\underset{\sim}{u}) : \gamma(\underset{\sim}{u}) \leq 0 \}$$

We will first show that the family of problems χ defined above need only be considered for $\alpha \in (0, \alpha_0)$ for some $\alpha_0 > 0$ for K sufficiently large. For example consider a sequence of problems:

$$P_{K_j} : \min_{\underset{\sim}{u}} \{ l(\underset{\sim}{u}) + K_j \gamma(\underset{\sim}{u}) \}$$

where $K_j \longrightarrow \infty$ as $j \longrightarrow \infty$. Then we claim that the penalty $\gamma(\underset{\sim}{u}_j) \longrightarrow 0$ as $j \longrightarrow \infty$ for the minimising control $\underset{\sim}{u}_j$ for each j. To prove this, we assume to the contrary that as $K_j \longrightarrow \infty$, $\gamma(\underset{\sim}{u}_j) \longrightarrow \delta > 0$ then

$$\liminf_{j \to \infty} \{ l(\underset{\sim}{u}) + K_j \gamma(\underset{\sim}{u}) \} = \infty$$

Now from (6.4) by using the same arguments as for Problems F and F_K we deduce that

$$\chi(0) \geq l(\underset{\sim}{u}_j) + K_j \gamma(\underset{\sim}{u}_j) \qquad \text{for all } j$$

Taking limits of both sides we get

$$\chi(0) \geq \infty$$

But $\chi(0)$ is finite from (6.6), hence we get a contradiction and so we must have $\gamma(\underset{\sim}{u}_j) \longrightarrow 0$ as $K_j \longrightarrow \infty$, and attention can be limited to arbitrary small α when studying the family χ provided K is large enough. Now we show that for $K=\overline{K} > K_0$, there exists an $\alpha_0 > 0$ such that

$$\frac{\chi(\alpha) - \chi(0)}{\alpha} \geq -\overline{K} \qquad \text{for all } \alpha \in (0, \alpha_0] \tag{6.7}$$

To prove this we assume this is not true and obtain a contradiction, i.e. we have sequences $\{K_j\}$, $\{\alpha_j\}$, $K_j > K_0$ all j and $K_j \longrightarrow \infty$, $\alpha_j \longrightarrow 0$ as $j \longrightarrow \infty$ such that

$$\frac{\chi(\alpha) - \chi(0)}{\alpha} < -K_j \qquad \text{for all } \alpha \in (0, \alpha_j]$$

Taking infimal limits of both sides we get

$$\liminf_{j \to \infty} \left\{ \frac{\chi(\alpha) - \chi(0)}{\alpha} \right\} < \liminf_{j \to \infty} (-K_j) = -\infty$$

This gives

$$\liminf_{\alpha \downarrow 0} \left\{ \frac{\chi(\alpha) - \chi(0)}{\alpha} \right\} < -\infty$$

This contradicts (6.5), hence hypothesis (6.7) must hold.

Now as we have shown that by choosing K sufficiently large we can restrict attention to small enough α and exactly the same procedure as for Problems F and F_K gives the required result that solving $P3_K$ gives a solution for P3. The only difference in the arguments is that for Problems F, F_K

$\eta(\alpha) \geq \eta(0) - \bar{K}\alpha$ holds for all α but under our assumption (6.5) for the control problem P3 we only have

$$\chi(\alpha) - \chi(0) \geq -\bar{K}\alpha \text{ holding for } \alpha \in (0, \alpha_0]$$

This does not change the results since for $\bar{K}$ large enough then $\gamma(\underset{\sim}{u}_j)$ is arbitrarly small and the same formulae hold for both cases. This is shown diagrammatically in Fig 6.2. The essence of the diagram is that from (6.5) the slope of χ approaches $-K_0$ as $\alpha \downarrow 0$ (i.e. α decreases to zero). Therefore if a line of greater slope ($K > K_0$) is drawn at $\chi(0)$ then the graph χ will lie above $\chi(0) - K\alpha$ for $\alpha \in (0, \alpha_0]$ for some $\alpha_0 > 0$. This is clear from Fig 6.2.

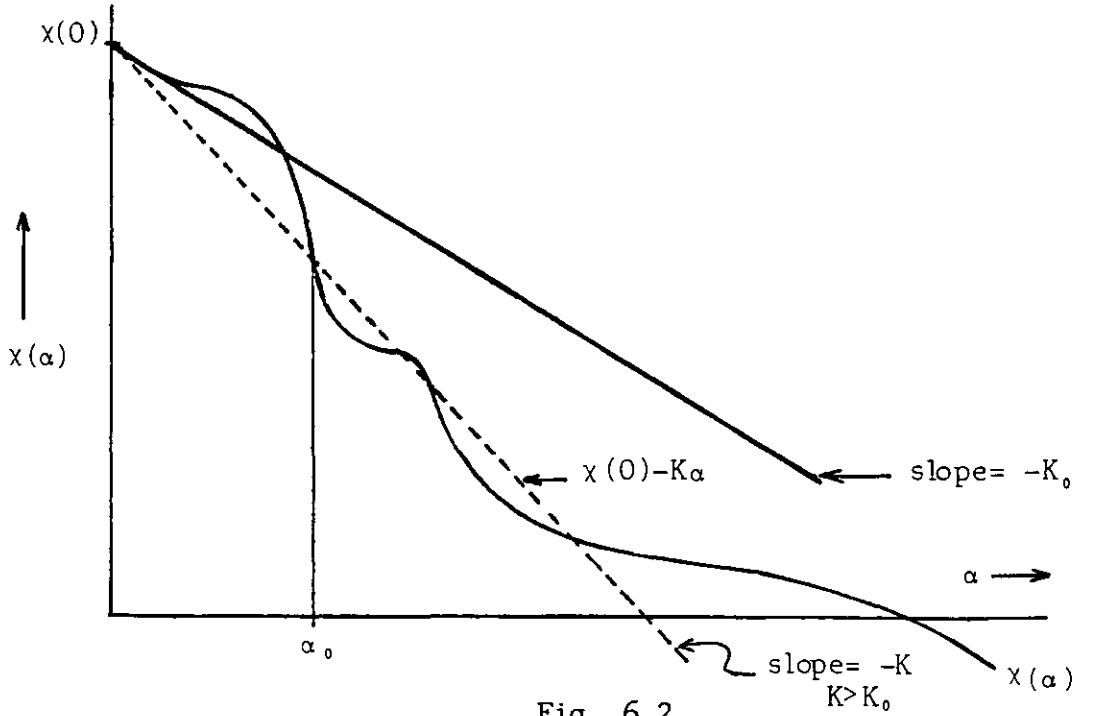


Fig 6.2

Therefore Assumption 3 guarantees the equivalence of Problems P3 and $P3_K$ for $K \geq K^*$ for some $K^* \in (0, \infty)$.

However the problem of obtaining K^* still remains. K^* can be determined in principle by obtaining necessary conditions of optimality for Problems P3 and $P3_K$ and comparing the results. This is done in Appendix D.

5.7 Desirable Sets for Problems P3 and $P3_K$

Our algorithm for solving Problem P3 will find controls $\underset{\sim}{u}^* \in \underset{\sim}{G}$ satisfying $\gamma(\underset{\sim}{u}^*)=0$ and the Maximum Principle in Appendix C. This states that if $\underset{\sim}{u}^* \in \underset{\sim}{G}$ is optimal for P3, then there exists a scaler $\lambda_0 \leq 0$ and a function $\lambda \in NBV[T; \mathbb{R}]$, not both zero such that

$$\lambda_0 \int_0^1 [l_x(\underset{\sim}{u}^*) z^{\underset{\sim}{u}^*, \underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v}, \underset{\sim}{u}^*)]dt + \int_0^1 g_x(x^*, t) z^{\underset{\sim}{u}^*, \underset{\sim}{v}}(t) d\lambda(t) \leq 0$$

$$\text{for all } \underset{\sim}{v} \in \underset{\sim}{G}$$

with λ nonincreasing and constant for $t \in I$, where $I \triangleq \{t \in T: g(x^*, t) < 0\}$. We will only consider normal problems, i.e. ones with $\lambda_0 < 0$ in which case it may be normalized to -1. This can be done by making the following hypothesis:

<u>Assumption 4</u>

For all $\underset{\sim}{u} \in \underset{\sim}{G}$ there exists a control $\underset{\sim}{v} \in \underset{\sim}{G}$ such that

$$g(x^{\underset{\sim}{u}}, t) + g_x(x^{\underset{\sim}{u}}, t) z^{\underset{\sim}{u}, \underset{\sim}{v}}(t) < 0 \qquad \text{for all } t \in T$$

We prove that with this assumption $\lambda_0 \neq 0$. Assume to the contrary that $\lambda_0 = 0$. Then λ is nonzero and we have

$$\int_0^1 g_x(x^*,t) z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t)\, d\lambda(t) \leq 0 \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{G}$$

Since λ is constant for $t \in I$ we have

$$\int_{\tilde{I}} g_x(x^*,t) z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t)\, d\lambda(t) \leq 0 \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{G}$$

where $\tilde{I} = T \setminus I$.

Let $\underset{\sim}{v}^*$ be the control that satisfies

$$g(x^*,t) + g_x(x^*,t) z^{\underset{\sim}{u}^*,\underset{\sim}{v}^*}(t) < 0 \qquad \text{for all } t \in T$$

such a $\underset{\sim}{v}^*$ exists due to Assumption 4. Then for $t \in \tilde{I}$ (i.e. when $g(x^*,t) = 0$) we have

$$g_x(x^*,t) z^{\underset{\sim}{u}^*,\underset{\sim}{v}^*}(t) < 0$$

Hence since λ is nonincreasing we must have

$$\int_{\tilde{I}} g_x(x^*,t) z^{\underset{\sim}{u}^*,\underset{\sim}{v}^*}(t)\, d\lambda(t) > 0$$

This contradicts the Maximum Principle. Therefore $\lambda_0 \neq 0$ and may be normalized to -1.

In view of the above discussion we define the desirable set of Problem P3 as follows

$$\Delta = \{ \underset{\sim}{u}^* \in \underset{\sim}{G} : \gamma(\underset{\sim}{u}^*) = 0 \text{ and } \underset{\sim}{u}^* \text{ satisfies the State Constrained Maximum Principle} \} \qquad (7.1)$$

As in Chapter 4 we assume Δ is nonempty.

Similarly for the family of Problems $P3_K$, $K \geq 0$ we define the corresponding desirable sets by

$$\Delta_K \triangleq \{ \hat{\underset{\sim}{u}} \in \underset{\sim}{G} : \tilde{\theta}_K(\hat{\underset{\sim}{u}}) = 0 \} \qquad (7.2)$$

From Appendix D if $\hat{\underset{\sim}{u}} \in \underset{\sim}{G}$ is optimal for $P3_K$ then

$$\lambda_0 \int_0^1 [\, l_x(\hat{x},\hat{\underset{\sim}{u}},t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v},\hat{\underset{\sim}{u}})\,] dt + \int_0^1 g_x(\hat{x}(t),t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}} d\lambda(t) \leq 0$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

where $T.V(\lambda) \leq -\lambda_0 K$

and λ is non increasing

Let $J(\underset{\sim}{v}) \subset T$ be defined by $J(\underset{\sim}{v}) \triangleq \{ t \in T : g_x(\hat{x},t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}}(t) < 0 \}$ and let $\tilde{J}(\underset{\sim}{v}) \triangleq T \backslash J(\underset{\sim}{v})$

i.e. $\tilde{J}(\underset{\sim}{v}) \triangleq \{ t \in T : g_x(\hat{x},t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}}(t) \geq 0 \}$

Then the above becomes

$$\lambda_0 \int_0^1 [\, l_x(\hat{\underset{\sim}{u}}) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v},\hat{\underset{\sim}{u}})\,] dt + \int_{T \cap \tilde{J}} g_x(\hat{x},t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}}(t) d\lambda(t)$$

$$+ \int_{T \cap J} g_x(\hat{x},t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}}(t) d\lambda(t) \leq 0 \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{G}$$

Since λ is non increasing

$$\int_{T \cap J(\underset{\sim}{v})} g_x(\hat{x},t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}}(t) d\lambda(t) \geq 0$$

Therefore we get

$$\lambda_0 \int_0^1 [\, l_x(\hat{\underset{\sim}{u}}) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v},\hat{\underset{\sim}{u}})\,] dt + \int_{T \cap \tilde{J}} g_x(\hat{x},t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}}(t) d\lambda(t) \leq 0$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

This can be written as

$$\lambda_0 \int_0^1 [\, l_x(\hat{\underset{\sim}{u}}) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v},\hat{\underset{\sim}{u}})\,] dt + \int_0^1 \max_t \{ g_x(\hat{x},t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}}(t), 0 \} d\lambda(t) \leq 0$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

Now $\int_0^1 \max_t \{ g_x(\hat{x},t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}}(t), 0 \} d\lambda(t)$

$$\leq \max_{0 \leq t \leq 1} \{ g_x(\hat{x},t) z^{\hat{\underset{\sim}{u}},\underset{\sim}{v}}(t), 0 \} \int_0^1 |d\lambda(t)|$$

It follows from this inequality that

$$\lambda_0 \int_0^1 [\, l_x(\hat{u}) z^{\hat{u},v} + l(v,\hat{u})\,]dt - \max_{0\le t\le 1} \{\, g_x(\hat{x},t) z^{\hat{u},v}(t), 0 \,\}\, T.V(\lambda) \le 0$$

for all $v \in G$

Substituting for $T.V(\lambda)$ and using the fact that $\lambda_0 = -1$ for normal problems we get

$$\int_0^1 [\, l_x(\hat{u}) z^{\hat{u},v} + \Delta l(v,\hat{u})\,]\, dt + K \max_{0\le t\le 1} \{\, g_x(\hat{x},t) z^{\hat{u},v}(t), 0 \,\} \ge 0$$

for all $v \in G$ (7.3)

where equality holds for $v = \hat{u}$

Now

$$\tilde{\theta}_K(\hat{u}) = \min_{v \in G} \int_0^1 [\, l_x(\hat{u}) z^{\hat{u},v}(t) + \Delta l(v,\hat{u})\,]\, dt$$

$$+ K \max_{0\le t\le 1} \{\, g(\hat{x},t) + g_x(\hat{x},t) z^{\hat{u},v}, 0 \,\} - K\gamma(\hat{u})$$

$$\le \min_{v \in G} \int_0^1 [\, l_x(\hat{u}) z^{\hat{u},v}(t) + \Delta l(v,\hat{u})\,]\, dt$$

$$+ K \max_{0\le t\le 1} \{\, g_x(\hat{x},t) z^{\hat{u},v}(t), 0 \,\}$$

$= 0$ by(7.3)

It is thus clear that $\hat{u} \in \Delta_K$.

We also have the following result.

Proposition 7.1

Suppose $u^* \in \Delta$ then there exists a $K^* \in [\, 0,\infty)$ such that $\tilde{\theta}_K(u^*) = 0$ for all $K \ge K^*$.

Proof

Since $u^* \in \Delta$, we must have $\gamma(u^*) = 0$ and there exist λ_0, λ in the Maximum principle such that

$$\lambda_0 \int_0^1 [\, l_x(u^*) z^{u^*,v} + \Delta l(v,u^*)\,]dt + \int_0^1 g_x(x^*,t) z^{u^*,v}(t)\, d\lambda(t) \le 0$$

for all $v \in G$

Since $\int_0^1 g(x^*,t)d\lambda(t)=0$ we have

$$\lambda_0 \int_0^1 [l_x(\underset{\sim}{u}^*)z^{\underset{\sim}{u}^*,\underset{\sim}{v}}+\Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)]dt+\int_0^1[g(x^*,t)+g_x(x^*,t)z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t)]d\lambda(t)\leq 0$$

$$\text{for all } \underset{\sim}{v} \in \underset{\sim}{G}$$

Using a similar procedure as above we deduce that

$$\int_0^1[l_x(\underset{\sim}{u}^*)z^{\underset{\sim}{u}^*,\underset{\sim}{v}}+\Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)]dt+T.V(\lambda).\max_{0\leq t\leq 1}\{g(x^*,t)+g_x(x^*,t)z^{\underset{\sim}{u}^*,\underset{\sim}{v}},0\}$$

$$\geq 0 \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{G} \quad (7.4)$$

Now by definition we have

$$\tilde{\theta}_K(\underset{\sim}{u}^*) = \min_{\underset{\sim}{v}\in\underset{\sim}{G}} \int_0^1[l_x(\underset{\sim}{u}^*)z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t)+\Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)]dt$$

$$+ K \max_{0\leq t\leq 1} \{ g(x^*,t)+g_x(x^*,t)z^{\underset{\sim}{u}^*,\underset{\sim}{v}},0\} \quad (7.5)$$

since $\gamma(\underset{\sim}{u}^*)=0$. It is clear from (7.4), (7.5) and the results in Appendix D that, if $K>T.V(\lambda)$, then

$$\tilde{\theta}_K(\underset{\sim}{u}^*) = 0$$

i.e. for $K^* = T.V(\lambda)$ we have that

$$\underset{\sim}{u}^* \in \Delta_K \qquad \text{for all } K>K^* \text{ as required.}$$

5.8 Algorithms for Solving $P3_K$ and P3

An algorithm for solving $P3_K$ based on the Algorithm Model 1 in Chapter 2 will be presented in this section. After proving convergence we shall modify it so that it can be used to solve Problem P3. This will be done by adding a rule which increases the penalty parameter K if certain conditions are satisfied.

5.8.1 Conceptual Algorithm for Solving $P3_K$

Step 0 : Select a $\underset{\sim}{u}_0 \in \underset{\sim}{G}, K > 0$

Step 1 : Set i=0

Step 2 : Formulate Problem $P3_K(\underset{\sim}{u}_i)$ and solve it using procedure W to obtain its solution $(\beta_i, \underset{\sim}{v}_i) \in S(\underset{\sim}{u}_i)$ together with the cost η_i

Step 3 : If $\eta_i=0$, set $\underset{\sim}{u}_j=\underset{\sim}{u}_i$ for $j>i$ and stop
Else continue

Step 4 : If $\eta_i<0$, define for $\alpha \in [0,1]$ the control $\underset{\sim}{u}_\alpha=(1-\alpha)\underset{\sim}{u}_i \oplus \alpha\underset{\sim}{v}_i$

Step 5 : Compute $\alpha_i \in [0,1]$ which is the largest number which satisfies

$$\gamma_K(\underset{\sim}{u}_{\alpha_i}) - \gamma_K(\underset{\sim}{u}_i) \leq \frac{\alpha_i \eta_i}{2}$$

Step 6 : Set $\underset{\sim}{u}_{i+1} = \underset{\sim}{u}_{\alpha_i}$
Set i = i+1
Goto Step 2

The above algorithm has the following convergence properties.

Theorem 8.1

Assume all the hypothesis stated above are satisfied, then Algorithm 5.8.1 generates a finite sequence of controls, in which case the last element is desirable, or it generates an infinite sequence and every accumulation point i.s.c.m. is in Δ_K.

The proof is similar to the ones given for earlier algorithms in this thesis so we do not go into too much detail when presenting it below.

To prove Theorem 8.1 we will require the following results:

Proposition 8.2

The function $\tilde{\theta}_K$ (for all $K>0$) is sequentially continuous i.s.c.m.

Proof

Consider an infinite sequence $\{u_i\}_{i=0}^{\infty} \in \underset{\sim}{G}$ converging i.s.c.m. to $\underset{\sim}{u}^* \in \underset{\sim}{G}$, i.e. we have

$$\underset{\sim}{u}_i \xrightarrow[i \to \infty]{} \underset{\sim}{u}^* \text{ i.s.c.m.}$$

Then we need to show that as $i \to \infty$

$$\tilde{\theta}_K(\underset{\sim}{u}_i) \longrightarrow \tilde{\theta}_K(\underset{\sim}{u}^*)$$

Define for all $K>0$, $\psi_K : \underset{\sim}{G} \times \underset{\sim}{G} \longrightarrow \mathbb{R}$ by

$$\psi_K(\underset{\sim}{u},\underset{\sim}{v}) = \int_0^1 [\, l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v},\underset{\sim}{u})\,]dt$$

$$+ K \max_{0 \le t \le 1} \{ g(x^{\underset{\sim}{u}},t) + g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}(t), 0 \} - K\gamma(\underset{\sim}{u})$$

Using results in Chapter 4 and Appendix A we deduce that $\underset{\sim}{u} \longrightarrow x^{\underset{\sim}{u}}$ is sequentially continuous, hence l, l_x, g, g_x is sequentially continuous in $\underset{\sim}{u}$ and z is sequentially continuous in $(\underset{\sim}{u},\underset{\sim}{v})$. Therefore $\psi_K(\underset{\sim}{u},\underset{\sim}{v})$ is sequentially continuous (in $(\underset{\sim}{u},\underset{\sim}{v})$).

But $$\tilde{\theta}_K(\underset{\sim}{u}) = \min_{\underset{\sim}{v} \in \underset{\sim}{G}} \psi_K(\underset{\sim}{u},\underset{\sim}{v})$$

The rest of the proof that $\tilde{\theta}_K(\underset{\sim}{u})$ is sequentially continuous i.s.c.m. follows exactly as in Proposition 3.1 in Chapter 4.

Proposition 8.3

For all $\underset{\sim}{u} \in \underset{\sim}{G}$, all $\alpha \in [0,1]$, $K>0$, with $\underset{\sim}{u}_\alpha$ as defined in Step 4 of algorithm 5.8.1 we have

$$\Delta \hat{\gamma}_K(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) \le \alpha\, \tilde{\theta}_K(\underset{\sim}{u})$$

where we define $\Delta\hat{\gamma}_K : \underset{\sim}{G} \times \underset{\sim}{G} \longrightarrow \mathbb{R}$ by

$$\Delta\hat{\gamma}_K(\underset{\sim}{v},\underset{\sim}{u}) \triangleq \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v},\underset{\sim}{u})]dt + K \max_{0 \le t \le 1} \{ g(x^{\underset{\sim}{u}},t) + g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}, 0 \} - K\gamma(\underset{\sim}{u})$$

<u>Proof</u>

By definition we have

$$\Delta\hat{\gamma}_K(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) = \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t) + \Delta l(\underset{\sim}{u}_\alpha,\underset{\sim}{u})] dt$$

$$+ K \max_{0 \le t \le 1} \{ g(x^{\underset{\sim}{u}},t) + g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t), 0 \} - K\gamma(\underset{\sim}{u})$$

Now from Proposition 2.6 in Chapter 4

$$\alpha z^{\underset{\sim}{u},\underset{\sim}{v}}(t) = z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t) \qquad \text{for all } t \in T$$

where $\underset{\sim}{u}_\alpha = (1-\alpha)\,\underset{\sim}{u} \oplus \alpha\,\underset{\sim}{v}$

Hence we have

$$\Delta\hat{\gamma}_K(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) = \alpha \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}} + \Delta l(\underset{\sim}{v},\underset{\sim}{u})] dt$$

$$+ K \max_{0 \le t \le 1} \{ g(x^{\underset{\sim}{u}},t) - \gamma(u) + \alpha g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}(t), -\gamma(u) \}$$

Now for all $\alpha \in [0,1]$

$$\alpha[g(x^{\underset{\sim}{u}},t) - \gamma(\underset{\sim}{u})] \ge g(x^{\underset{\sim}{u}},t) - \gamma(\underset{\sim}{u})$$

$$\text{and} \qquad -\alpha\gamma(\underset{\sim}{u}) \ge -\gamma(\underset{\sim}{u})$$

Using this and substituting for $\tilde{\theta}_K(\underset{\sim}{u})$ we get

$$\Delta\hat{\gamma}_K(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) \le \alpha\tilde{\theta}_K(\underset{\sim}{u})$$

as required.

Proposition 8.4

For all $\underset{\sim}{u}$, $\underset{\sim}{v} \in \underset{\sim}{G}$, all $\alpha \in [0,1]$, $K > 0$, such that $\underset{\sim}{u}_\alpha = (1-\alpha)\underset{\sim}{u} \oplus \alpha \underset{\sim}{v}$ we have

$$\| \Delta\tilde{\gamma}_K(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) - \Delta\hat{\gamma}_K(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) \| \leq d\alpha^2$$

for some $d \in (0,\infty)$

where $\Delta\tilde{\gamma}_K : \underset{\sim}{G} \times \underset{\sim}{G} \longrightarrow \mathbb{R}$ is defined by

$$\Delta\tilde{\gamma}_K(\underset{\sim}{v}, \underset{\sim}{u}) \triangleq \gamma_K(\underset{\sim}{v}) - \gamma_K(\underset{\sim}{u})$$

Proof

$$\Delta\tilde{\gamma}_K(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) = \int_0^1 [\, l(x^{\underset{\sim}{u}_\alpha}, \underset{\sim}{u}_\alpha, t) - l(x^{\underset{\sim}{u}}, \underset{\sim}{u}, t)\,]dt$$

$$+ K \max_{0 \leq t \leq 1} \{ g(x^{\underset{\sim}{u}_\alpha}, t), 0 \} - K\gamma(\underset{\sim}{u})$$

Adding and subtracting $\Delta\hat{\gamma}_K(\underset{\sim}{u}_\alpha, \underset{\sim}{u})$ on right hand side and expanding $l(x^{\underset{\sim}{u}_\alpha}, \underset{\sim}{u}_\alpha, t)$ and $g(x^{\underset{\sim}{u}_\alpha}, t)$ by Taylor series about $x^{\underset{\sim}{u}}$ and rearranging we get

$$\Delta\tilde{\gamma}_K(\underset{\sim}{u}_\alpha, \underset{\sim}{u}) \leq \Delta\hat{\gamma}_K(\underset{\sim}{u}_\alpha, \underset{\sim}{u})$$

$$+ \int_0^1 \{ l_x(\underset{\sim}{u})(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}} - z^{\underset{\sim}{u}, \underset{\sim}{u}_\alpha}) + \frac{l_{xx}(x^{\underset{\sim}{u}_\epsilon}, \underset{\sim}{u}_\alpha, t)(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}})^2}{2}$$

$$+ \alpha[l_x(x^{\underset{\sim}{u}}, \underset{\sim}{v}, t) - l_x(x^{\underset{\sim}{u}}, \underset{\sim}{u}, t)]\ (x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}})\}\, dt$$

$$+ K \max_{0 \leq t \leq 1} \{ g_x(x^{\underset{\sim}{u}}, t)(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}} - z^{\underset{\sim}{u}, \underset{\sim}{u}_\alpha}), 0 \}$$

$$+ K \max_{0 \leq t \leq 1} \{ \frac{g_{xx}(x^{\underset{\sim}{u}_\epsilon}, t)(x^{\underset{\sim}{u}_\alpha} - x^{\underset{\sim}{u}})^2}{2}, 0 \}$$

where $x^{\underset{\sim}{u}_\epsilon} \triangleq (1-\epsilon)x^{\underset{\sim}{u}} + \epsilon x^{\underset{\sim}{u}_\alpha}$ for some $\epsilon \in (0,1)$.

By boundedness of l_x, l_{xx}, g_x, g_{xx} for all $\underset{\sim}{u}, \underset{\sim}{v} \in \underset{\sim}{G}$, all finite x we have that there exist finite constants d_1, d_2, d_3, d_4, d_5 such that

$$\| \Delta\tilde{\gamma}_K(\underset{\sim}{u}_\alpha,\underset{\sim}{u}_\alpha)-\Delta\hat{\gamma}_K(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) \| \leq \int_0^1 \{ d_1 \| x^{\underset{\sim}{u}_\alpha}-x^{\underset{\sim}{u}}-z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha} \|$$

$$+d_2 \| x^{\underset{\sim}{u}_\alpha}-x^{\underset{\sim}{u}} \|^2+\alpha d_3 \| x^{\underset{\sim}{u}_\alpha}-x^{\underset{\sim}{u}} \| \} \, dt$$

$$+ K \max_{0\leq t\leq 1} d_4 \| x^{\underset{\sim}{u}_\alpha}-x^{\underset{\sim}{u}}-z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha} \|$$

$$+ K \max_{0\leq t\leq 1} d_5 \| x^{\underset{\sim}{u}_\alpha}-x^{\underset{\sim}{u}} \|^2$$

Now from Proposition 2.5 in Chapter 3 and Proposition 2.7 in Chapter 4,

$$\| x^{\underset{\sim}{u}_\alpha}(t)-x^{\underset{\sim}{u}}(t) \| \leq d_6 \alpha$$

$$\| x^{\underset{\sim}{u}_\alpha}(t)-x^{\underset{\sim}{u}}(t)-z^{\underset{\sim}{u},\underset{\sim}{u}_\alpha}(t) \| \leq d_7 \alpha^2$$

for all $\underset{\sim}{u}, \underset{\sim}{v} \in \underset{\sim}{G}$, $\alpha \in [0,1]$ for all $t \in T$

for some finite d_6, d_7.

Hence we deduce

$$\| \Delta\tilde{\gamma}_K(\underset{\sim}{u}_\alpha,\underset{\sim}{u})-\Delta\hat{\gamma}_K(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) \| \leq d\,\alpha^2 \qquad \text{for some } d \in (0,\infty)$$

Note using Proposition 8.3, if $\underset{\sim}{v}$ is a minimising control in $\tilde{\theta}_K$ (as in algorithm) we have

$$\Delta\tilde{\gamma}_K(\underset{\sim}{u}_\alpha,\underset{\sim}{u}) \leq \alpha\,\tilde{\theta}_K(\underset{\sim}{u})+d\,\alpha^2 \qquad \text{for all } \alpha \in [0,1]$$

Proposition 8.5

For all $\underset{\sim}{u} \in \underset{\sim}{G}$ which are not optimal for $P3_K$ (i.e. $\tilde{\theta}_K(\underset{\sim}{u})<0$) the step length α_i defined in Step 5 of the algorithm is strictly greater than zero.

The proof of this is exactly the same as for Proposition 6.4 in Chapter 4.

We deduce Theorem 8.1 from Propositions 8.4 and 8.5 by the procedure used in proving Theorem 6.1 in Chapter 4.

We see that Algorithm 5.8.1 produces controls $\underset{\sim}{u}^* \in \underset{\sim}{G}$ which are desirable for problem $P3_K$, i.e. $\tilde{\theta}_K(\underset{\sim}{u}^*)=0$. We modify this algorithm now so that it may be applied to solve the original SCCP P3.

5.8.2 Algorithm 5: Conceptual Algorithm for Solving Problem P3

Data : $\underset{\sim}{u}_0 \in \underset{\sim}{G}$, $0<K_0<K_1<\ldots\ldots \lim_j K_j = \infty$

Step 0 : Set i=0

Set j=0

Step 1 : Formulate Problem $P3_{K_j}(\underset{\sim}{u}_i)$ and solve it using Procedure W to obtain its solution $(\beta_i, \underset{\sim}{v}_i) \in S(\underset{\sim}{u}_i)$ together with the optimal cost $\tilde{\theta}_{K_j}(\underset{\sim}{u}_i)$

Step 2 : (I) If $\tilde{\theta}_{K_j}(\underset{\sim}{u}_i) \geq 0$ and $\gamma(\underset{\sim}{u}_i)=0$

set $\underset{\sim}{u}_l = \underset{\sim}{u}_i$ for $l>i$ and stop

Else continue

(II) If $\tilde{\theta}_{K_j}(\underset{\sim}{u}_i) \geq 0$ and $\gamma(\underset{\sim}{u}_i) > 0$

set $K_j = K_{j+1}$

set $j = j+1$

and goto Step 1

Else continue

Step 3 : Define for $\alpha \in [0,1]$ the control

$$\underset{\sim}{u}_{\alpha} = (1-\alpha)\underset{\sim}{u}_i \oplus \alpha \underset{\sim}{v}_i$$

Step 4 : Compute α_i as the largest number in $[0,1]$ which satisfies

$$\gamma_{K_j}(\underset{\sim}{u}_{\alpha_i}) - \gamma_{K_j}(\underset{\sim}{u}_i) \leq \alpha_i \tilde{\theta}_{K_j}(\underset{\sim}{u}_i)/2$$

Step 5 : Set $\underset{\sim}{u}_{i+1} = \underset{\sim}{u}_{\alpha_i}$

Set $i = i+1$

Goto Step 1

We now come to Theorem 5 which states the convergence properties of Algorithm 5 and is, we judge, the central result in this thesis.

Theorem 5

Suppose all the assumptions in the text are satisfied and that every "linearized problem" is calm# (c.f. Assumption 3 and succeeding remarks) with a common constant K_0. If Algorithm 5 generates a sequence of controls $\{\underset{\sim}{u}_i\}$ in $\underset{\sim}{G}$ then we have the following: either

By the "linearized problem" we mean Problem P3 linearized about some control $\underset{\sim}{u} \in \underset{\sim}{G}$, and is the following:

$$\min_{\underset{\sim}{v} \in \underset{\sim}{G}} \int_0^1 [l_x(\underset{\sim}{u}) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v},\underset{\sim}{u})]\, dt$$

$$\text{s.t} \quad g(x^{\underset{\sim}{u}},t) + g_x(x^{\underset{\sim}{u}},t) z^{\underset{\sim}{u},\underset{\sim}{v}}(t) \leq 0 \quad \text{for every } t \in T$$

where $z^{\underset{\sim}{u},\underset{\sim}{v}}$ is the solution of (4.4), (4.5) and everything else is as in the text.

The assumption that this problem is calm for all $\underset{\sim}{u} \in G$ essentially means that the problem $P3_K(\underset{\sim}{u})$ has $\beta = 0$ as its solution for $K > K_0$, K finite.

a. The sequence is finite (in which case the sequence $\{K_j\}$ is also finite, and the penalty parameter remains finite) and the last control is desirable for problem P3.

or

b. The sequence $\{\underset{\sim}{u}_i\}$ is infinite and the following holds:

(i) The algorithm constructs only a finite sequence $\{K_j\}_{j=0}^{N}$ i.e. K_j is only increased a finite number of times after which it remains constant at $K_N<\infty$.

(ii) Every accumulation point $\underset{\sim}{u}^* \in \underset{\sim}{G}$ i.s.c.m. of the sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ (at least one exists due to results in Appendix A) is desirable for Problem P3, i.e. $\underset{\sim}{u}^* \in \Delta$.

Before we present the proof of Theorem 5 we state a result we will need:

Lemma 8.6

For all $K \geq 0$ if $\underset{\sim}{u}^* \in \Delta_K$ and $\gamma(\underset{\sim}{u}^*)=0$ then $\underset{\sim}{u}^* \in \Delta$.

Proof

Since $\gamma(\underset{\sim}{u}^*)=0$ it satisfies the first condition for $\underset{\sim}{u}^*$ to be in Δ. Now since we also have $\underset{\sim}{u}^* \in \Delta_K$, i.e. $\underset{\sim}{u}^*$ is desirable for Problem $P3_K \Longrightarrow \tilde{\theta}_K(\underset{\sim}{u}^*)=0$.

Then from Appendix D (the Maximum Principle for Problem $P3_K$) there exist $\lambda_0 \in \mathbb{R}$, $\lambda \in NBV[T, \mathbb{R}]$ such that

$$\lambda_0 \int_0^1 [\, l_x(\underset{\sim}{u}^*) z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)]dt + \int_0^1 g_x(x^*,t) z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t) d\lambda(t) \leq 0$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

where $\lambda_0 \leq 0$, λ non increasing and constant on $t \in I \triangleq \{t \in T: g(x^*,t) < 0\}$.

These are precisely the necessary conditions of optimality for Problem P3 (as mentioned in Appendix D). The only difference is that here there is no statement which says $\underset{\sim}{u}^*$ is feasible for P3. But from the hypothesis in Lemma 8.6 this is assumed to be true.

Therefore $\underset{\sim}{u}^* \in \Delta$.

<u>Proof of Theorem 5</u>

a. Suppose Algorithm 5 generates a finite sequence of controls $\{\underset{\sim}{u}_i\}_{i=0}^{m}$. Then the iterative procedure can only terminate if condition I in Step 2 is satisfied, i.e. $\tilde{\theta}_K(\underset{\sim}{u}_m)=0$ and $\gamma(\underset{\sim}{u}_m)=0$. This means $\underset{\sim}{u}_m$ is feasible for P3 and by Lemma 8.6 we have that it is also desirable for P3.

Now we assume that Algorithm 5 generates an infinite sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty} \in \underset{\sim}{G}$. Then by the compactness results in Appendix A there exists controls $\underset{\sim}{u}^* \in \underset{\sim}{G}$ (at least one exists) and a subsequence of $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ indexed by $M \subset \{0,1,2,\ldots\}$ such that

$$\underset{\sim}{u}_i \xrightarrow{\quad M \quad} \underset{\sim}{u}^* \quad \text{i.s.c.m.}$$

We will prove that all such limit points are desirable for Problem P3, but first we show that the sequence $\{K_j\}$ of penalty parameters remains finite, i.e. K_j is only increased a finite number of times. To do this we only need to show that there exist some finite $K^* < \infty$ such that for all $K \geq K^*$ the condition which requires K_j to be increased is never satisfied, i.e. in Step 2II of algorithm.

The penalty parameter needs to be increased if the $\underset{\sim}{u}_i$'s are converging to a control which is not feasible for Problem P3. This just implies that the K_j is not large enough to satisfy the calmness hypothesis.

What is required at any point where $\tilde{\theta}_{K_j}(\underset{\sim}{u}_i) \geq 0$ and $\gamma(\underset{\sim}{u}_i) > 0$ is that a descent direction exists, i.e. the value of the penalised cost can be reduced. We show that this is indeed the case.

Assume $(\beta_i, \underset{\sim}{v}_i) \in S(\underset{\sim}{u}_i)$ is the solution to $\tilde{\theta}_{K_j}(\underset{\sim}{u}_i)$ and $\gamma(\underset{\sim}{u}_i) = \delta > 0$. We can assume that $\beta_i = 0$ since it can be made to be so by choosing K_j to be large enough (see footnote on page 216). Then we have

$$\int_0^1 [\, l_x(\underset{\sim}{u}_i) z^{\underset{\sim}{u}_i, \underset{\sim}{v}_i} + \Delta l(\underset{\sim}{v}_i, \underset{\sim}{u}_i)]\, dt \; - \; K_j \delta \geq 0$$

Let $\int_0^1 [\, l_x(\underset{\sim}{u}_i) z^{\underset{\sim}{u}_i, \underset{\sim}{v}_i} + \Delta l(\underset{\sim}{v}_i, \underset{\sim}{u}_i)]\, dt = d_1$

and suppose we have

$$d_1 - K_j \delta = d_2 \qquad \text{for } d_2 \geq 0$$

Then for $K > \dfrac{(d_1 - d_2)}{\delta}$ we have $\tilde{\theta}_K(\underset{\sim}{u}_i) < 0$.

Hence since $\tilde{\theta}_K$ is an estimate of the maximum reduction of the penalised cost, a descent direction exists if the penalty parameter is made large enough.

We have just shown that Algorithm 5 generates a finite sequence $\{K_j\}_{j=0}^{N}$ after which stage β_i =0 and condition 2II is never satisfied — so K_j is never updated. Then Algorithm 5 reduces to the subalgorithm for solving Problem $P3_{K_N}$. This we have shown converges to a control $\tilde{u}^* \in \tilde{G}$ satisfying $\tilde{\theta}_{K_N}(\tilde{u}^*)=0$. Since as mentioned above we never have $\tilde{\theta}_{K_N}(\tilde{u}_i)=0$ and $\gamma(\tilde{u}_i)>0$, we must have $\gamma(\tilde{u}^*)=0$.

Therefore again using Lemma 8.6 we deduce that this limit control $\tilde{u}^*$ is desirable for P3, i.e. $\tilde{u}^* \in \Delta$.

5.8.3 Implementable Algorithm for Solving Problem P3

It is quite obvious that Algorithm 5 cannot be implemented on any computer since each iteration requires exact solution of an intermediate problem. This in turn may require an infinite number of iterations to compute and hence some modifications to Algorithm 5 are required if the method is to be used to solve any problem numerically.

In this section we present such modifications and obtain an implementable version of Algorithm 5 where the intermediate problems are only solved approximately. These approximations are increased indefinitely to ensure convergence to desirable points for P3. The procedure obtained is still conceptual since exact integration of delay-differential equations is assumed but the computation is reduced drastically since a finite number of iterations are required to solve the intermediate problem rather than an infinite number.

Algorithm 6

Data : $\tilde{u}_0 \in \tilde{G}$, $0<K_0<K_1\ldots\ldots$ $\lim_j K_j = \infty$

$$\delta_0 > \delta_1 > \ldots\ldots \lim_i \delta_i = 0$$

Step 0 : Set $i = 0$

Set $j = 0$

Step 1 : Formulate Problem $P3_{K_j}(\underset{\sim}{u}_i)$ and solve it using procedure W_1 to obtain its approximate solution $(\beta_i, \underset{\sim}{v}_i) \in S(\underset{\sim}{u}_i)$ and approximate cost μ_i in the sense that

$$\| \mu_i - \tilde{\theta}_{K_j}(\underset{\sim}{u}_i) \| \leq \delta_i$$

$$\| \chi_i(t) \| \leq \delta_i$$

where $\chi_i(t) = g(x^i,t) + g_x(x^i,t) z^{\underset{\sim}{u}_i, \underset{\sim}{v}_i} - \beta_i . 1 + \nu_i(t)$

(This as shown in section 5.5.2 is achieved after a finite number of iterations.)

Step 2 : If $\mu_i \geq -\delta_i$ and $\gamma(\underset{\sim}{u}_i) > 0$

Set $K_j = K_{j+1}$

Set $j = j+1$

and goto Step 1

Else continue

Step 3 : Define for $\alpha \in [0,1]$ the control

$$\underset{\sim}{u}_\alpha = (1-\alpha)\underset{\sim}{u}_i \oplus \alpha \underset{\sim}{v}_i$$

Step 4 : Compute α_i as the largest number in $[0,1]$ which satisfies

$$\gamma_{K_j}(\underset{\sim}{u}_{\alpha_i}) - \gamma_{K_j}(\underset{\sim}{u}_i) \leq \frac{\alpha_i \mu_i}{2}$$

Step 5 : Set $\underset{\sim}{u}_{i+1} = \underset{\sim}{u}_{\alpha_i}$

Set $\delta_i = \delta_{i+1}$

Set $i = i+1$

Goto Step 1

It is quite obvious that Algorithm 6 is a general case of Algorithm 5. We assume that Algorithm 6 generates an infinite sequence of controls $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ and approximate costs $\{\mu_i\}_{i=0}^{\infty}$ (filling in with the terminating point if need be). Then it can easily be proved that $\lim_{i} \mu_i = 0$ using a method similar to the ones used in proving convergence of algorithms presented earlier in the thesis.

Also as for Algorithm 5, K_j is increased only a finite number of times and then remains fixed. Hence as for Algorithm 5 we must have all limit points of the sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ to be desirable for Problem P3.

Proof of the above properties for Algorithm 6 is very similar to the proofs for Algorithm 5. The only difference is that Algorithm 6 solves the intermediate problems to an approximate degree defined by δ_i which decreases to 0 as $i \longrightarrow \infty$ whereas Algorithm 5 solves these problems exactly throughout the scheme. Hence in the limit as $i \longrightarrow \infty$ the two methods are identical and the same results as for Algorithm 5 (for case b) hold for Algorithm 6 as well.

5.9 Numerical Experience

A number of state constrained examples were programmed using the procedure presented in this chapter to show how the method behaves on actual test problems. The numerical results for these problems are presented in this section.

The test problems considered are restricted to systems governed by ordinary-differential equations because it was felt there was no advantage in considering more difficult cases - all that is required is that the state be calculated by solving the dynamical equation and this can be done in any problem, delayed or otherwise. So although the theory presented in this chapter does cover the more general case our examples do not mainly for making the task of obtaining the state easier to compute. Also only scaler problems are considered - again this is so that the computing is easier and operations such as inversion of matrices, etc. do not have to be performed. Obviously multivariable examples can also be solved by incorporating the extra software.

All the following results were obtained by partitioning the time interval into 50 disjoint segments.

Problem 1

The first problem considered was the following:

$$\underset{\underset{\sim}{u}}{\text{Min}} \quad - \int_0^1 x(t)\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = \underset{\sim}{u}(t) \qquad \text{a.a. } t \in T$$

$$x(0) = -\tfrac{1}{2}$$

$$x(t) \leq 0 \qquad \text{for every } t$$

$$\Omega = [-1,1] \in \mathbb{R}$$

$$\underset{\sim}{u} \in \underset{\sim}{G}$$

This is a very simple example where the solution can be obtained very easily by hand. By inspection the solution is the following:

$$u^*(t) = +1 \quad \text{for all } t \in [0, \tfrac{1}{2})$$

$$= 0 \quad \text{for all } t \in [\tfrac{1}{2}, 1]$$

$$x^*(t) = -\tfrac{1}{2}+t \quad \text{for all } t \in [0, \tfrac{1}{2}]$$

$$= 0 \quad \text{for all } t \in [\tfrac{1}{2}, 1]$$

and the optimal cost is 0.125.

Since Problem 1 is already linear there is no need to linearize it. Hence when it is formulated into the form $P3_K(\underset{\sim}{u})$ it will infact be the penalised version of Problem 1 - a solution at Step 1 will give a solution to Problem 1, i.e. the solution will be obtained in one iteration of the algorithm.

The numerical results are shown in Figs 8.1-8.4 and compare well with the ones given above. It should be noted that Fig 8.1-8.4 show the stages to finding the solution to $P3_K(\underset{\sim}{u})$ which is an internal iterative process. Once this has been done a solution to Problem 1 has been obtained. The primal functional for Problem 1 is also shown in Fig 8.5 to show that the calmness hypothesis is satisfied.

<u>Problem 2</u>

The second problem considered was obtained from Ianculescu and Hager [IA1] and so our results will be compared with the ones given by them. The problem is a little bit more difficult than Problem 1 in that it involves a quadratic cost but the dynamics are still linear. In fact the problem is the following:

$$\min_{\underset{\sim}{u}} \quad \tfrac{1}{2} \int_0^1 \{ x^2(t) + \underset{\sim}{u}^2(t) \}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = \underset{\sim}{u}(t) \qquad \text{for a.a.} t \in T$$

$$x(0) = -\frac{(5e+3)}{4(e-1)}$$

$$x(t) + h \leq 0 \quad \text{for every } t, \qquad h = \frac{2\sqrt{e}}{(e-1)}$$

$$\Omega = [-1,1]$$

$$\underset{\sim}{u} \in \underset{\sim}{G}$$

Because of the nonlinear nature of the cost functional when solving $P3_K$ it has to be linearized and solutions to $P3_K(\underset{\sim}{u}_i)$ found. This gives a descent property for the original problem and a new (better) control can be found. Repetition of the procedure yields a sequence of controls with accumulation points that are "desirable" controls. Figs 8.6–8.9 show the results obtained numerically and they are seen to be very close to the ones presented by Ianculescu and Hager.

Again, by changing the state constraint the primal functional was obtained. This is shown in Fig 8.10 and calmness is again satisfied at $h = \frac{2\sqrt{e}}{(e-1)}$

Problem 3

The third problem considered is the following

$$\min_{\underset{\sim}{u}} \quad \int_0^1 \{x^2(t) + \underset{\sim}{u}^2(t)\}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = x(t)\underset{\sim}{u}(t) \qquad \text{a.a.} t \in T$$

$$x(0) = -3$$

$$t - x^2(t) + h \leq 0 \qquad \text{for every } t, \quad h=2$$

$$\Omega = [-1,1]$$

$$\underset{\sim}{u} \in \underset{\sim}{G}$$

A theoretical solution was not available for comparison with the numerical results obtained. However the limit control generated by the algorithm is feasible and, from an intuitive viewpoint, a plausible candidate to be the optimal solution of Problem 3. Numerical results for Problem 3 are shown in Figs 8.11-8.14.

SCALAR LINEAR PROBLEM 1 : CONTROL FUNCTIONS

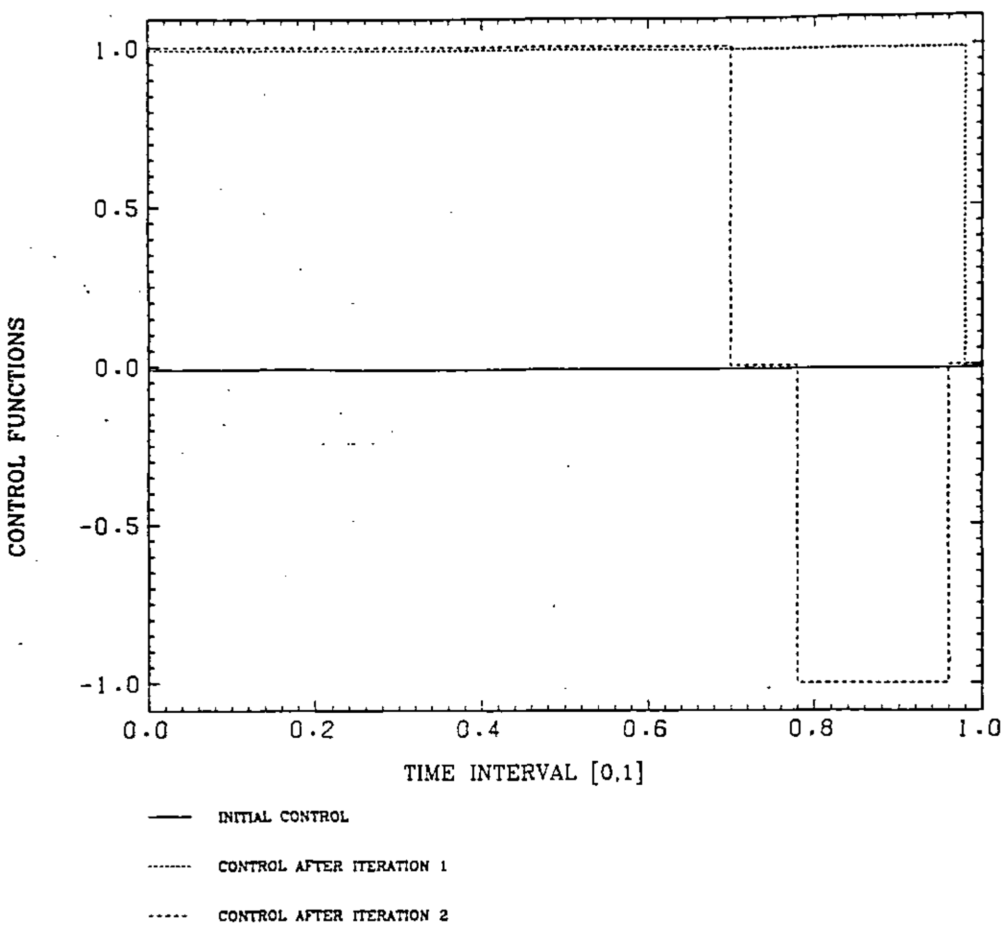


Fig 8.1

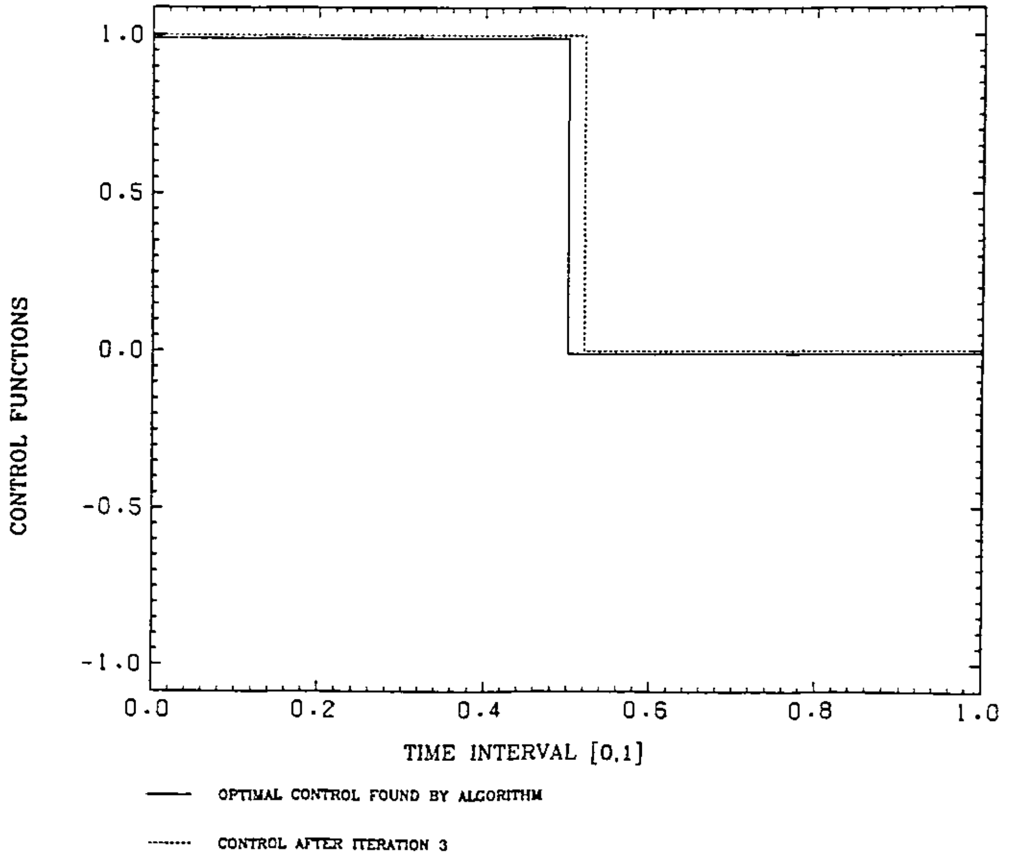


Fig 8.2

SCALAR LINEAR PROBLEM 1 : STATE TRAJECTORIES

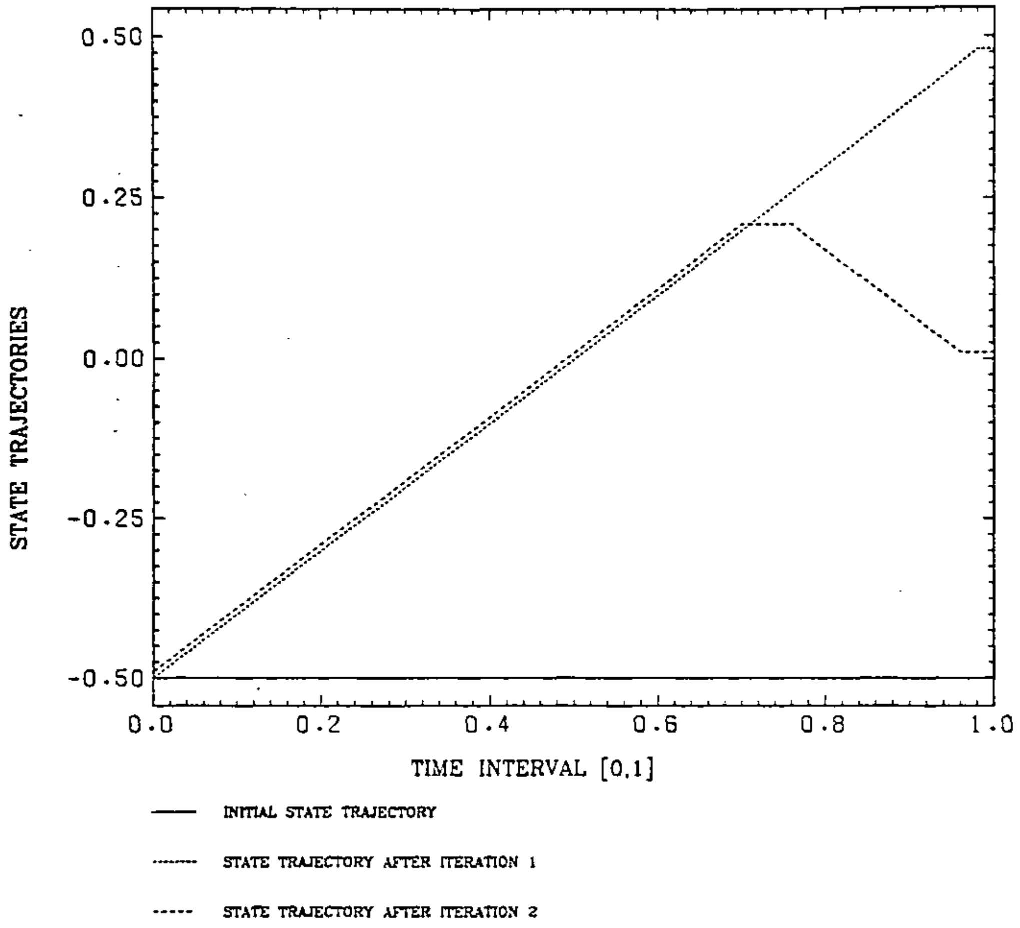


Fig 8.3

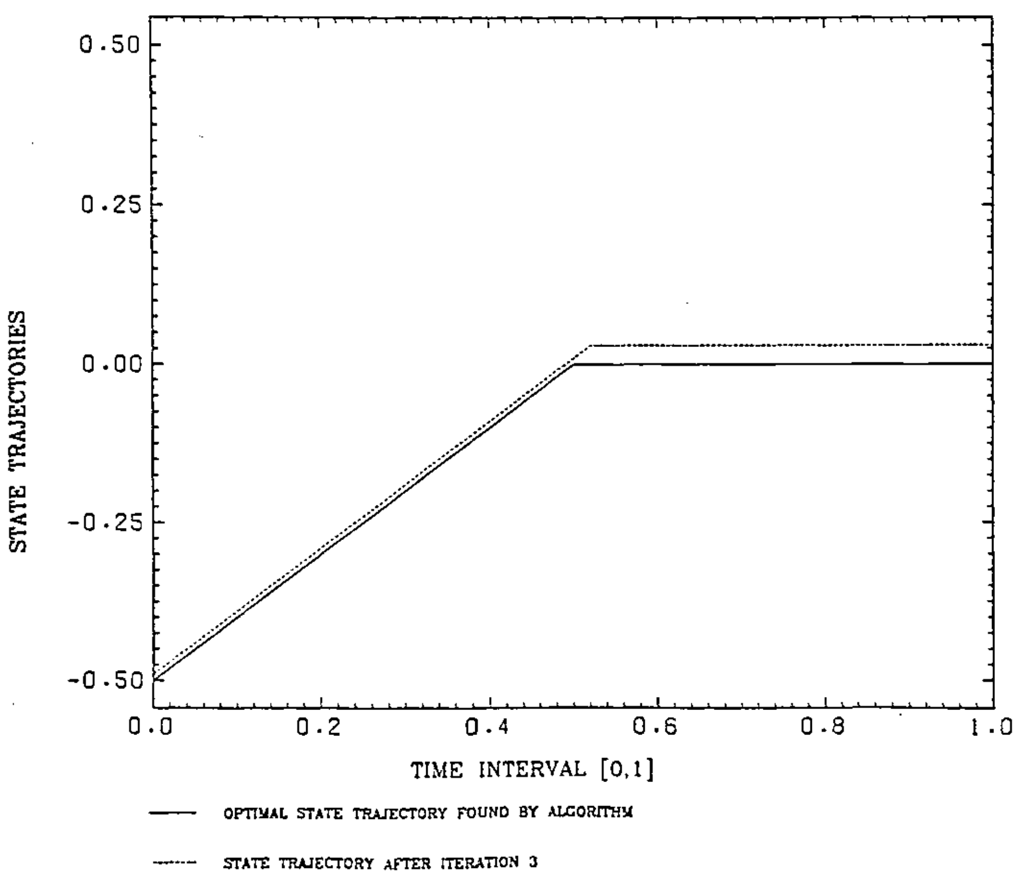
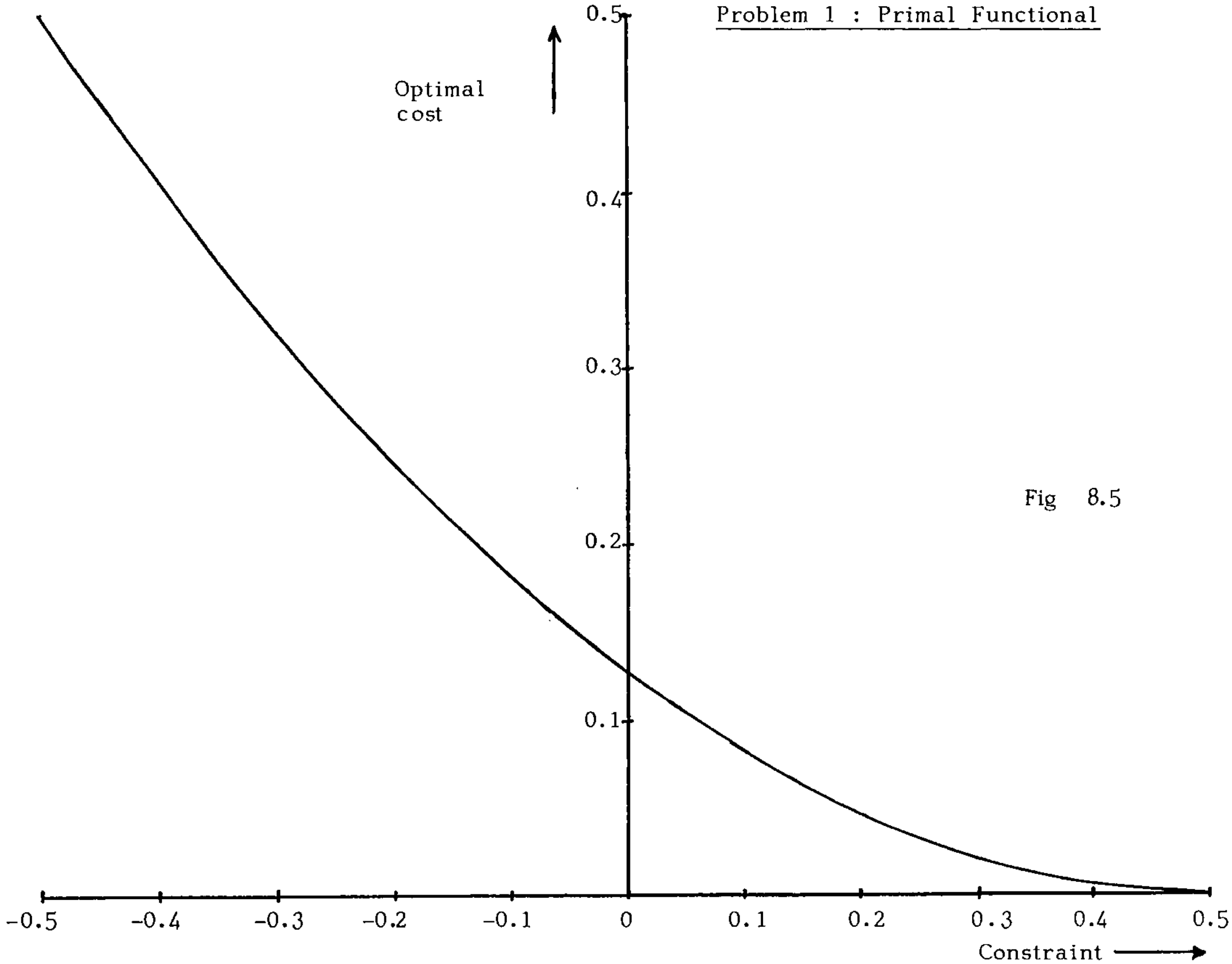


Fig 8.4



Problem 1 : Primal Functional

Fig 8.5

PROBLEM 2 : CONTROL FUNCTIONS

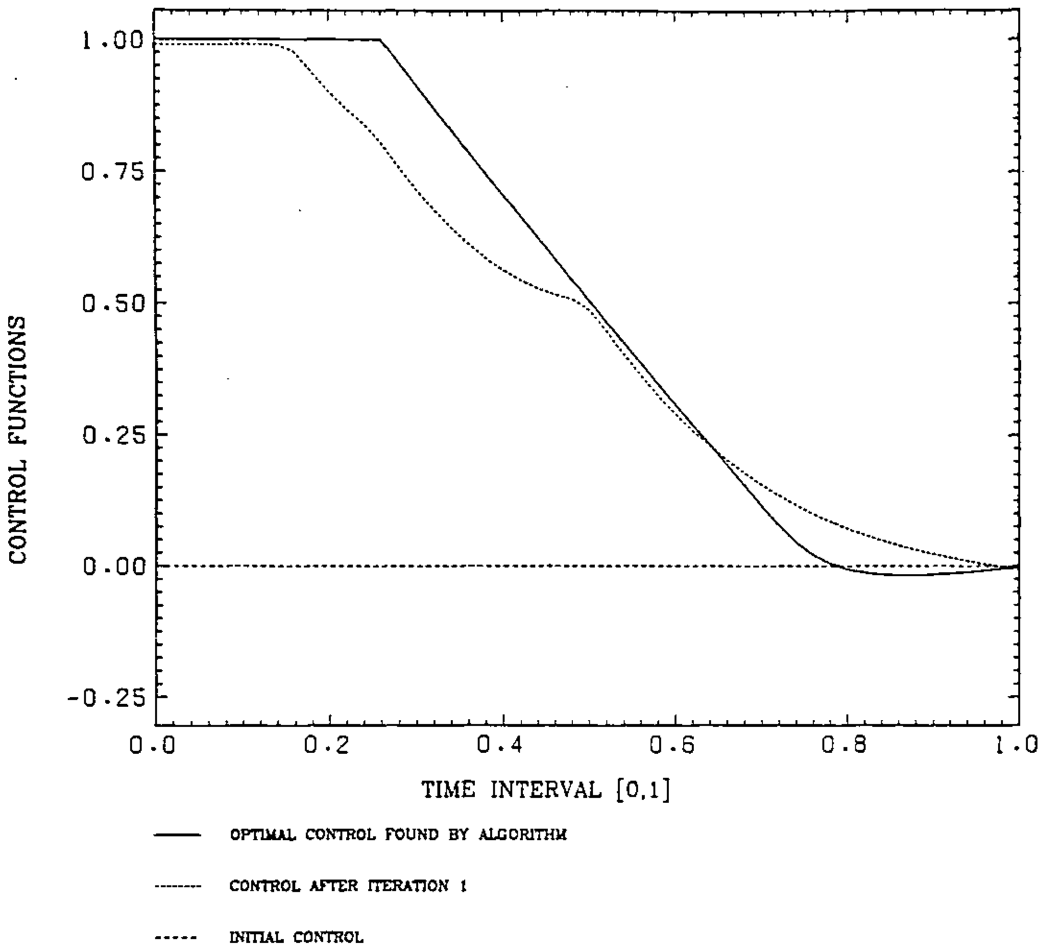


Fig 8.6

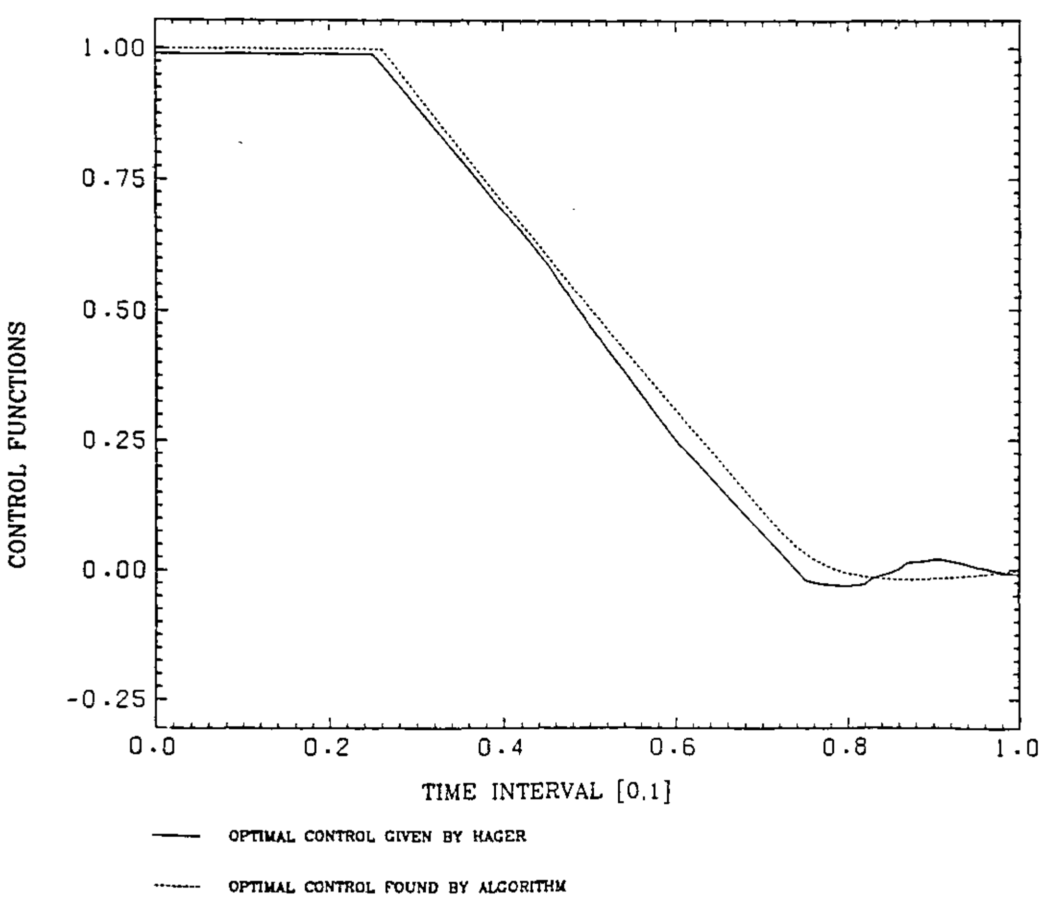


Fig 8.7

PROBLEM 2 : STATE TRAJECTORIES

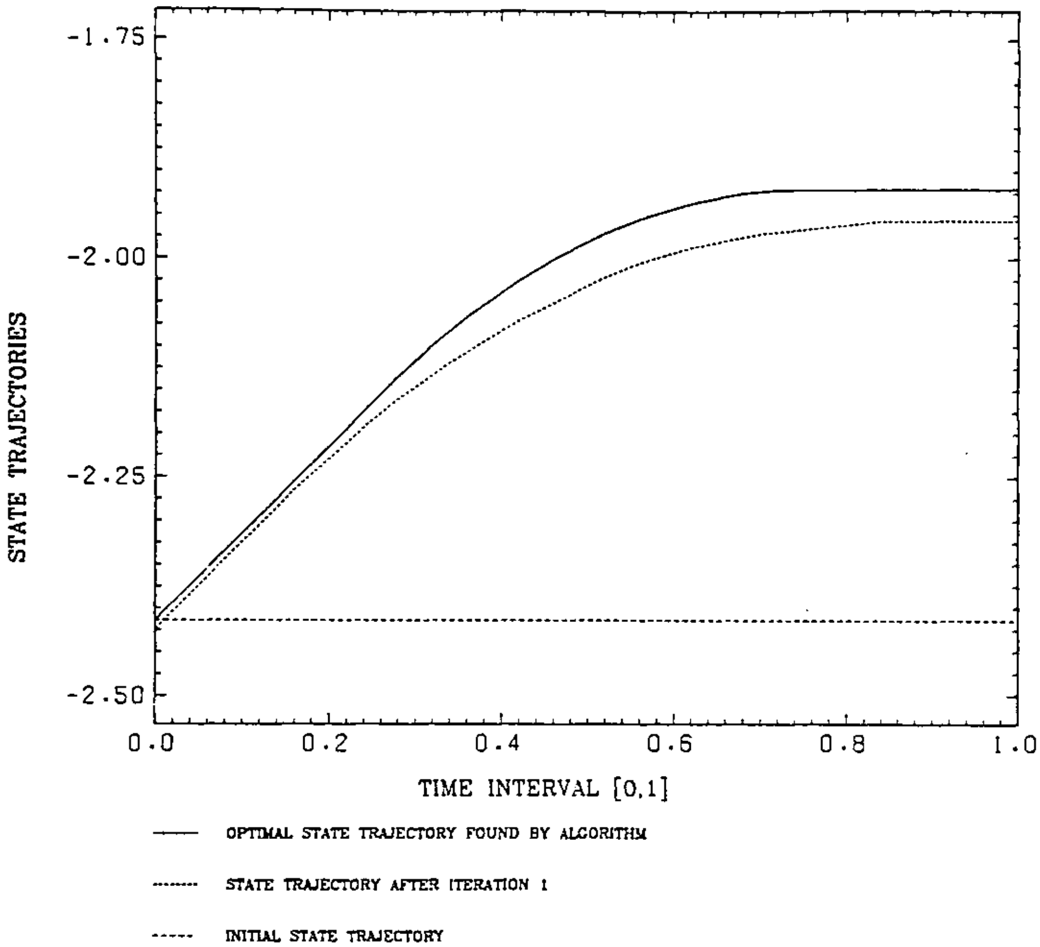


Fig 8.8

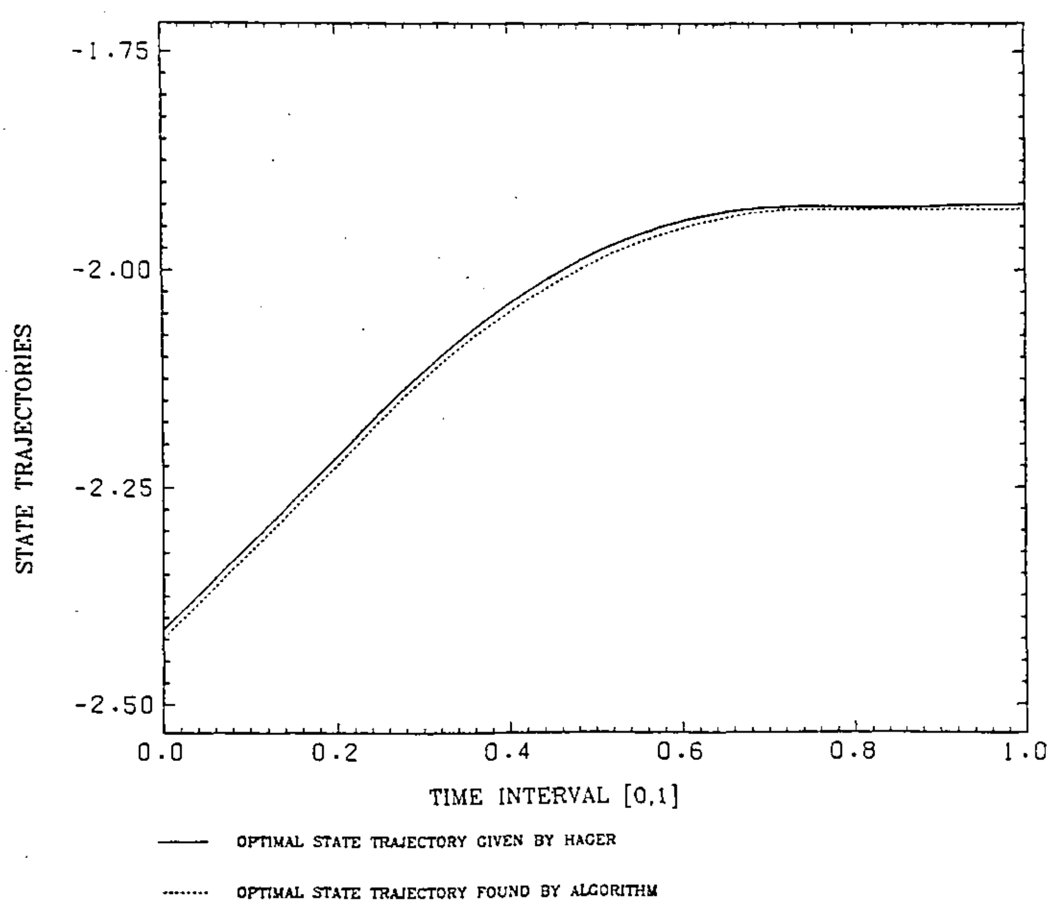


Fig 8.9

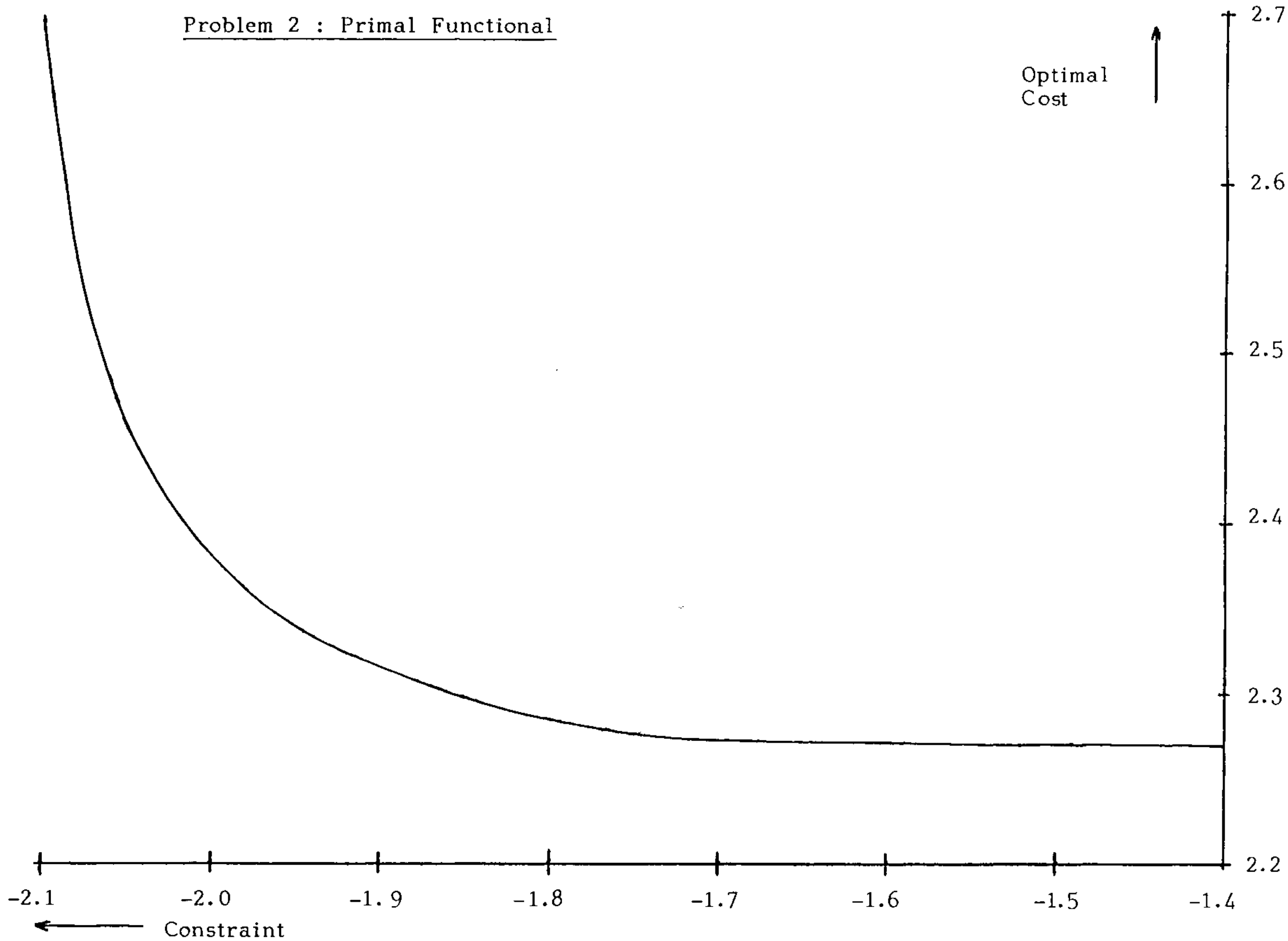

Problem 2 : Primal Functional
Optimal Cost
2.7
2.6
2.5
2.4
2.3
2.2
-2.1
-2.0
-1.9
-1.8
-1.7
-1.6
-1.5
-1.4
Constraint

Fig. 8.10

NONLINEAR PROBLEM 3 : CONTROL FUNCTIONS

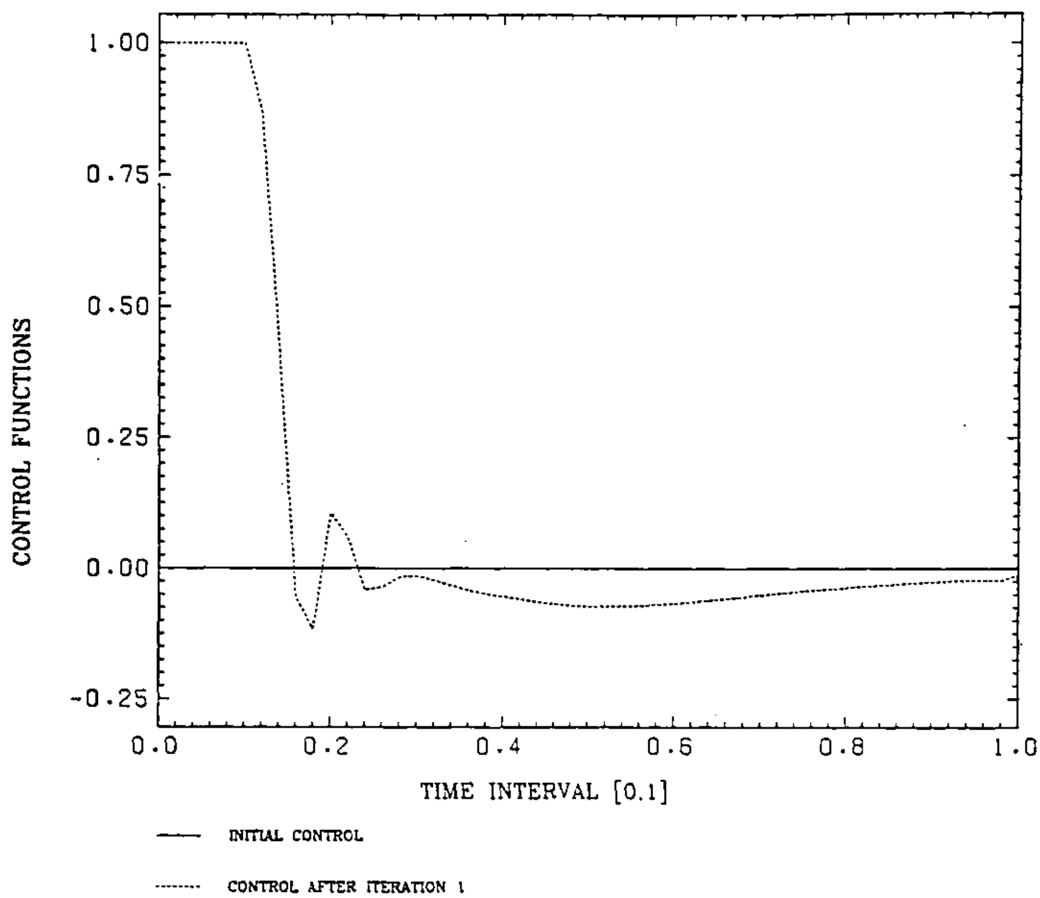


Fig 8.11

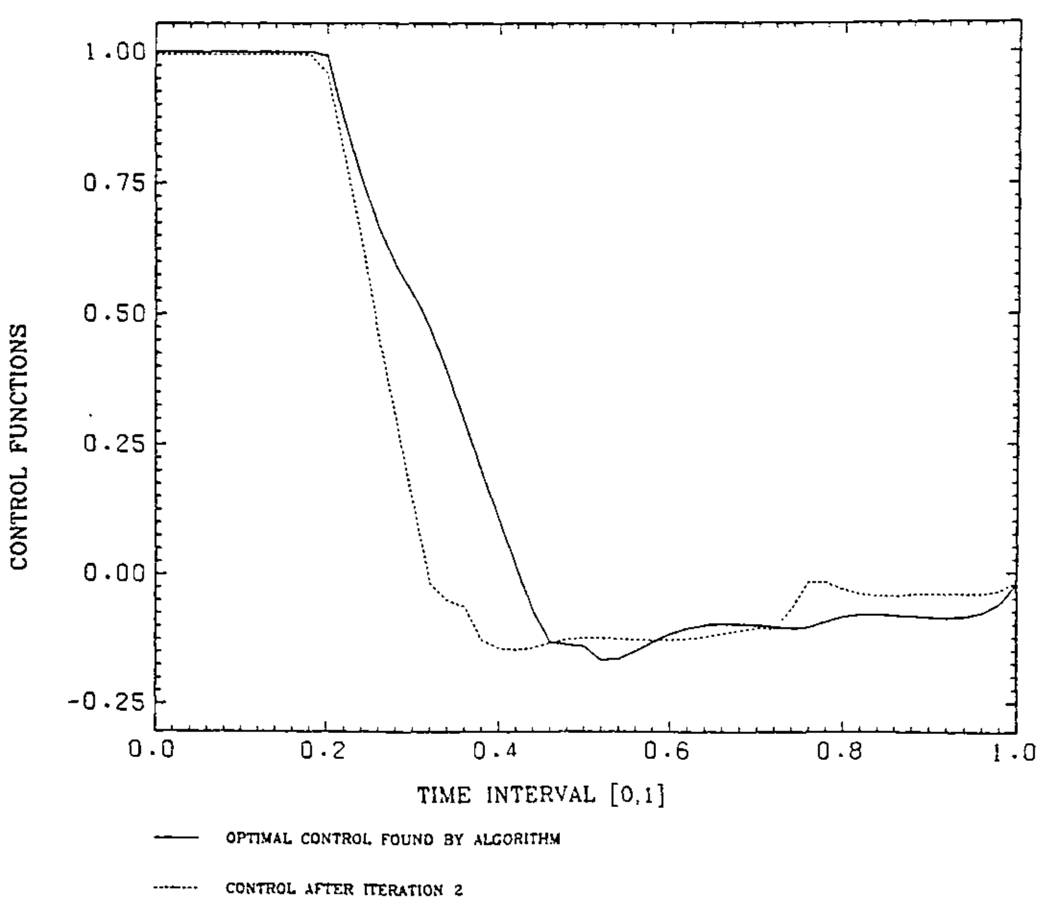


Fig 8.12

NONLINEAR PROBLEM 3 : STATE TRAJECTORIES

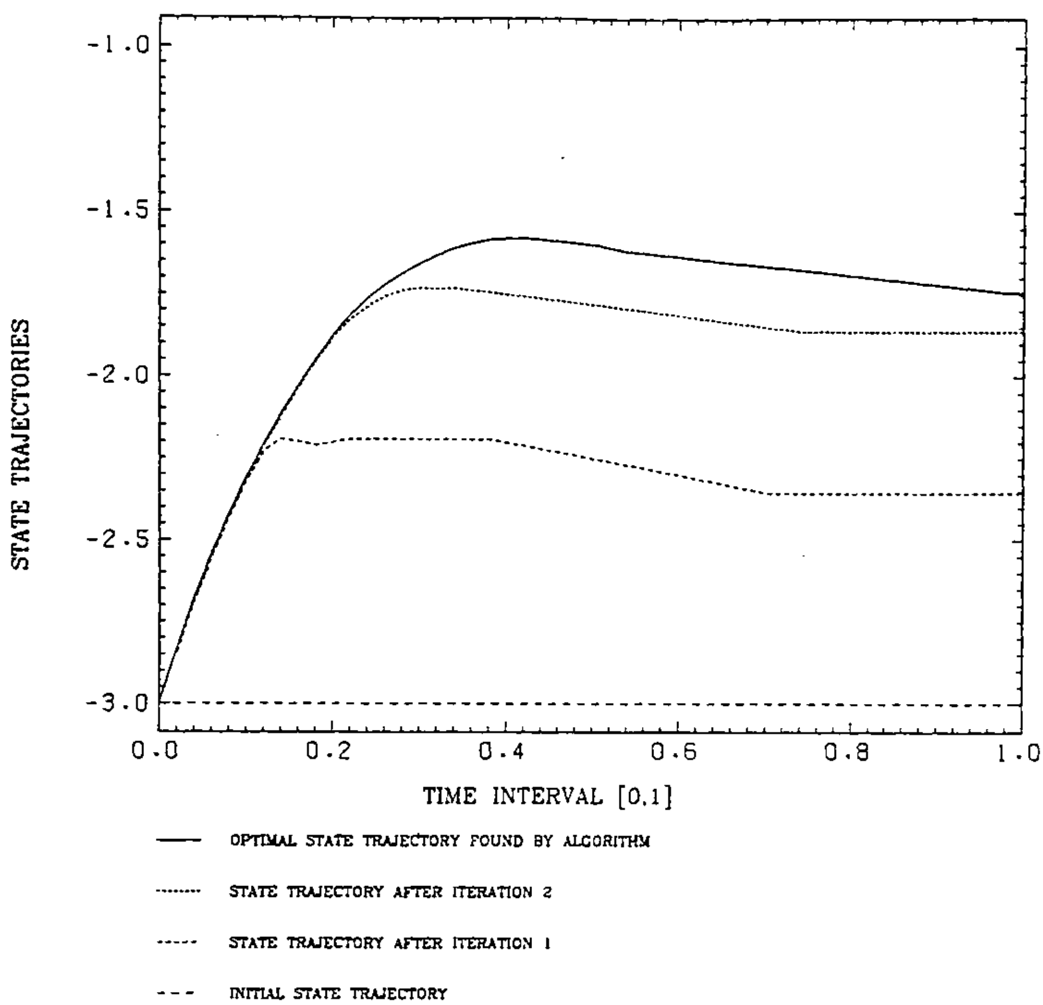


Fig 8.13

NONLINEAR PROBLEM 3 : CONSTRAINT FUNCTIONS

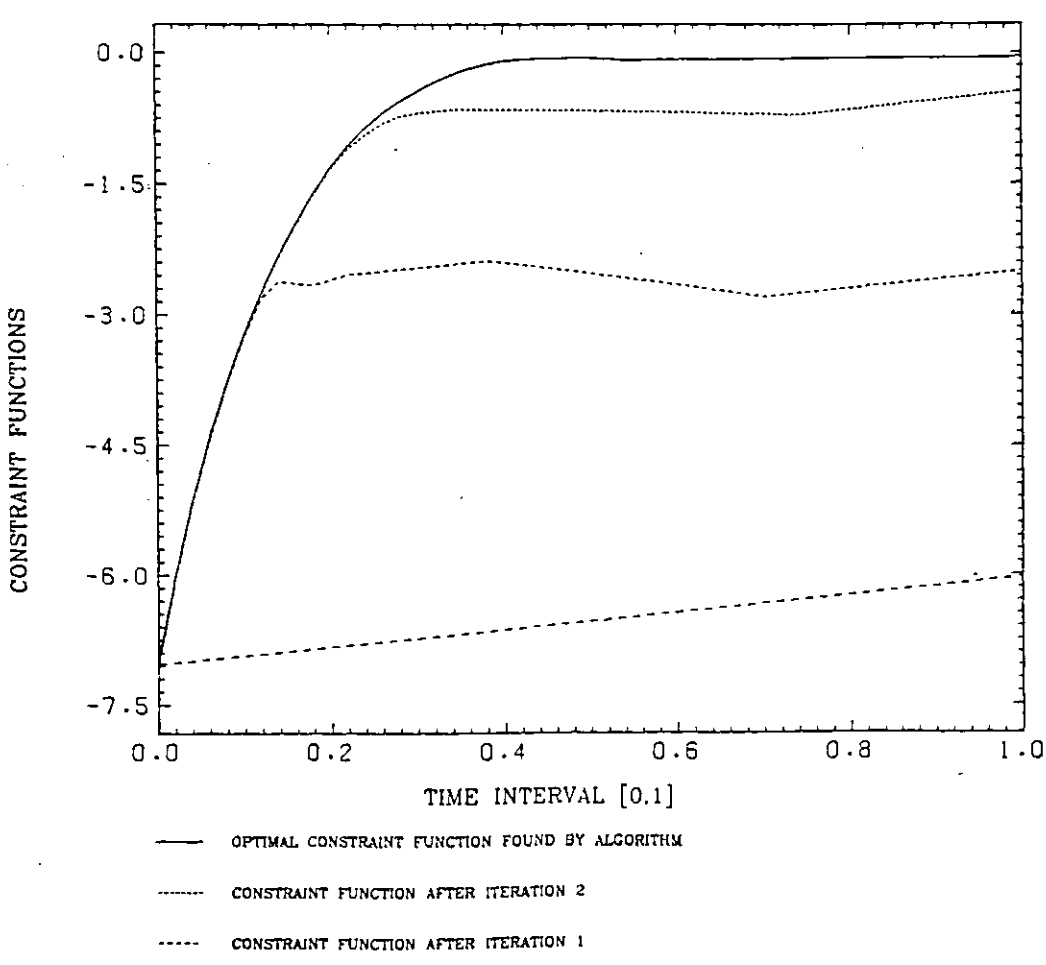


Fig 8.14

CHAPTER 6

CONCLUSIONS

In this chapter we state the main contributions of this thesis and the further work it suggests. It is believed that each chapter (with the exception of Chapter 1) is self-contained and can stand on its own merit, but also that the whole thesis has a sense of continuity in that the problems tackled in it get progressively more complicated and difficult to solve with each chapter.

We now go through the thesis in a little more detail and outline what we believe are the main contributions.

Section A in Chapter 1 contains no new material as it is meant to be of an introductory nature. Section B presents necessary conditions of optimality relating directly to the problems considered in Chapters 2-4. These extend results obtained by Huang to cover relaxed control problems.

Chapter 2 addresses an optimal control problem with control constraints only (referred to as Problem P1). An algorithm encorporating strong variations in the controls together with its convergence results is presented. This is an extension of Mayne and Polak's [MAP1] work to delay systems. Although the results are similar to the ones in [MAP1] the approach needed in obtaining them is quite different. This is because the Differential Dynamic Programming techniques on which Mayne and Polak base their approach are not applicable to delay systems.

Although in Chapter 2 we present an algorithm which constructs a sequence with all limit points satisfying the necessary conditions of optimality, it is not assured that such accumulation points exist. Because of this we turn to relaxed controls which possess a certain compactness property (see Appendix A) and hence limit points are guaranteed. An algorithm (Algorithm 2) which solves Problem P1 over the space of relaxed controls together with its convergence properties is presented in Chapter 3. It is shown that each limit point (at least one exists) satisfies the necessary conditions of optimality. A further, quite novel, algorithm (Algorithm 3) where all relaxed controls are approximated using ordinary controls is also given. This algorithm is an attempt to make Algorithm 2 implementable since the approximating ordinary controls require much less computer time to simulate than a measure value control (a relaxed control). It is shown that if these approximations are made at each iteration then the algorithm produces limit points (controls) which satisfy optimality conditions to within "delta". Also if the accuracy of the approximations is increased indefinitely then limit points satisfy optimality conditions "exactly". Algorithm 3 is a new method for which minimising solutions are guaranteed as well as convergence to these solutions.

In Chapter 4 an optimal control problem with terminal equality and control constraints, (Problem P2) is considered. An algorithm (Algorithm 4), using an Exact Penalty Function method (see Appendix B), which solves P2 together with its convergence results is presented. This work is an extension, together with some modifications and improvements, of the work done by Mayne and Polak [MAP2] to delay systems. The problem is not linearized w.r.t the control argument as done in [MAP2], therefore we can expect our approximations to be better than those of Mayne and Polak. However we still deduce similar results as in

[MAP2] but we do so by a completely different method, namely by using the "linear" nature of relaxed controls.

In Chapter 5 a general nonlinear state constrained control problem with delay, (Problem P3), is examined and solved using an Exact Penalty Function method. The existence of a finite penalty parameter K which makes Problem P3 and its penalised counterpart (Problem $P3_K$) equivalent is deduced by imposing a calmness hypothesis. This is a new method of guaranteeing exactness of a penalty method and is much neater than existing schemes (see for example Mayne and Polak [MAP2]). Theoretical conditions for finding such a K also exist and these are derived in Appendix D.

A conceptual algorith (Algorithm 5) which solves Problem P3 together with its convergence results is presented. This is basically a method which solves Problem $P3_K$ and includes a rule for increasing K whenever necessary to guarantee equivalence. Solving Problem $P3_K$ involves solving intermediate problems $P3_K(\underset{\sim}{u})$, which are linearized versions of $P3_K$ at every stage. These are solved using an internal procedure, in the main algorithm, based on the method presented by Warga [W2].

In an attempt to make Algorithm 5 implementable we also present Algorithm 6 which only solves the intermediate problems approximately. This is done by truncating the internal procedure for solving $P3_K(\underset{\sim}{u})$ when the required approximation is reached. Obviously the accuracy of the approximations is increased to ensure convergence.

It is shown that both algorithms generate controls which satisfy necessary conditions for optimality for Problem P3 (these are presented in Appendix C). Again the minimisation is done over the space of relaxed controls to guarantee existence of minimising solutions.

Except for the few test examples stated in Chapter 5, we have not had the opportunity to investigate the performance of Algorithms 1-6 for a large number of practical problems. It would be useful to do this and compare the numerical results with existing methods.

A problem worth investigating is to see if the procedure of approximating relaxed controls presented in Chapter 3 can be used in Algorithms 4-6 (or in any general algorithm which uses relaxed controls), i.e. so that limit points satisfy optimality conditions to "delta" and $\delta \longrightarrow 0$ as the accuracy is increased indefinitely. It seems quite plausible this is indeed the case. If so, it would be possible to save much computer time if these approximations are used when the algorithms are implemented.

In Chapters 4 and 5 we reduce "smooth" problems to "nonsmooth" ones and use the structure of the problems to obtain local approximations and hence their solutions. It would be useful to investigate if the nonsmooth problems can be solved directly by the methods proposed by Mifflin [MI2] and, if so, how the two methods compare practically.

Appendix A

Compactness Results for the Relaxed Control Problem with Delay

Most optimal control algorithms construct a sequence of controls whose corresponding costs form a monotonically decreasing, converging sequence. Because of this it suffices to require that the sequence of controls have at least one accumulation point and that any limit point of this sequence satisfies an optimality condition, rather than to require that it converges. If the sequence of controls is constructed so that it remains in a compact subset of $\mathbb{R}^n$, existence of an accumulation point is guaranteed. However, many optimal control algorithms are constructed to generate a sequence of controls which remain in L_∞- bounded sets and to show that any L_2- accumulation point satisfies some necessary condition of optimality. Unfortunately, there is no mathematical basis for assuming that a sequence of controls in an L_∞-bounded set has an L_2-accumulation point. This has been the main reason for investigation of the relaxed control problem.

In this Appendix we present a theory which extends the compactness results of Williamson and Polak [WIL1] for the Relaxed Control Problem to cover delay systems.

The problems under consideration will be of the following form although extra constraints (state, terminal, etc.) may be added without changing the results presented below (the same notation and terminology as in the text is used).

$$\text{Problem } P \qquad \min_u \; h(x^u(1)) \tag{1}$$

$$\text{subject to } \dot{x}(t) = f(x(t),x(t-\tau),u(t),t) \text{ for a.a } t \in T \tag{2}$$

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau,0] \tag{3}$$

$$u \in G \tag{4}$$

Problem P will be termed original in a sense which will become clear later on in our discussion.

The following assumptions are made.

Assumption A

The function $f\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times T \longrightarrow \mathbb{R}^n$ and its partial derivatives f_x, f_y exist and are continuous on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times T$. The function $h\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ and its derivative h_x exist and are continuous on $\mathbb{R}^n$.

Assumption B

There exists an $M \in (0,\infty)$ such that

$$\| f(x,y,u,t) \| \leq M \{ \| x \| + \| y \| +1 \}$$

$$\text{for all } x,y \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ t \in T$$

and

$$\| f(x_1,y_1,u,t)-f(x_2,y_2,u,t) \| \leq M \{ \| x_1-x_2 \| + \| y_1-y_2 \| \}$$

for all $x_1,x_2,y_1,y_2 \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, all $t \in T$.

Everything else is exactly the same as defined in the text. With $\underset{\sim}{G}$ denoting the space of measurable relaxed controls we define the relaxed control Problem $\underset{\sim}{P}$ by extending h and f to $\underset{\sim}{G}$ as follows:

Problem $\underset{\sim}{P}$ $\qquad \min_{\underset{\sim}{u}} \quad h(x^{\underset{\sim}{u}}(1))$ (5)

s.t. $\dot{x}(t) = f(x(t), x(t-\tau), \underset{\sim}{u}(t), t)$ for a.a. $t \in T$ (6)

$x(t) = \phi(t)$ for all $t \in [-\tau, 0]$ (7)

$\underset{\sim}{u} \in \underset{\sim}{G}$ (8)

The existence and uniqueness of an absolutely continuous function x(t) which satisfies (2), (3) or (6), (7) [i.e. when the control is an ordinary measurable one, or a relaxed measurable one] follows by writing $f(x(t),x(t-\tau),t)$ for $f(x(t),x(t-\tau),u(t),t)$ or $f(x(t),x(t-\tau),\underset{\sim}{u}(t),t)$ respectively and appealing to the following standard result:

Lemma 1

Suppose that Assumptions A and B are satisfied. Then for any ordinary control $u \in G$ or any relaxed control $\underset{\sim}{u} \in \underset{\sim}{G}$ and any initial condition $\phi(t)$ for all $t \in [-\tau, 0]$, there exists an absolutely continuous function $x^u(t)$ or $x^{\underset{\sim}{u}}(t)$ for all $t \in T$, that is the unique solution to (2) or (6) respectively.

The proof of Lemma 1 is a standard result - see for example Hale [HAL1], Bellman and Cooke [BC1], or Oǧuztöreli [OG1].

We now recall a definition from Chapter 1.

Definition B8.1

A sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ of relaxed controls in $\underset{\sim}{G}$ converges to $\underset{\sim}{u}^* \in \underset{\sim}{G}$ in the sense of control measures (i.s.c.m.) if, for every continuous function $\phi : \Omega \times T \longrightarrow \mathbb{R}$ and every interval Δ of T

$$\int_\Delta \phi(\underset{\sim}{u}_i(t),t)dt \longrightarrow \int_\Delta \phi(\underset{\sim}{u}^*(t),t)dt$$

as $i \longrightarrow \infty$

We are now in a position to state our first compactness results, whose proof is given in Young [Y1].

Theorem 1

Let $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ be an infinite sequence in $\underset{\sim}{G}$. Then $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ has an accumulation point $\underset{\sim}{u}^* \in \underset{\sim}{G}$ i.s.c.m, i.e. there exists an infinite subsequence indexed by $K \subset \{0,1,2,....\}$ such that

$$\underset{\sim}{u}_i \xrightarrow{K} \underset{\sim}{u}^* \quad \text{i.s.c.m.}$$

Our second compactness result will be established as a consequence of the following Lemmas.

Lemma 2

Let B_1, Ω be compact sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively (as in text). Let $x^i(t)$ and $x^*(t)$ be continuous functions from T to B_1 such that $x^i(t)$ converges to $x^*(t)$ uniformly in T. Also, let $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ be a sequence of relaxed controls that converge i.s.c.m. to $\underset{\sim}{u}^*$. Then for each subinterval Δ of T we have

$$\int_\Delta f(x^i(t),x^i(t-\tau),\underset{\sim}{u}_i(t),t)dt \longrightarrow \int_\Delta f(x^*(t),x^*(t-\tau),\underset{\sim}{u}^*(t),t)dt$$

Proof

This follows directly from Definition B8.1 and the uniform continuity of f on $B_1 x B_1 x \Omega x T$.

Before the next Lemma can be presented we need to say a few words about the terminology we will be using. When we wish to stress the dependence on the function $f(x(t), x(t-\tau), \underset{\sim}{u}(t), t)$ subject to Assumptions A and B of the trajectories determined by (6), (7) we shall term them f-trajectories. Then we shall compare them with r-trajectories which will be similarly determined when $f(x, y, \underset{\sim}{u}, t)$ is replaced by a continuous function of the form $r(\underset{\sim}{u}(t), t)$ (for a "fixed" x and y which has been absorbed into the t-dependence of r). This will be done using the next key Lemma stated in Young [Y1] for ordinary differential equations, and which we extend to cover our delay case.

Lemma 3

Let $r(\underset{\sim}{u}(t), t) = f(x^{\underset{\sim}{u}^*}(t), x^{\underset{\sim}{u}^*}(t-\tau), \underset{\sim}{u}(t), t)$ where $x^{\underset{\sim}{u}^*}(t)$ is continuous on $T^1 \subset T$ and for i=1,2,3,.... let $x^i(t)$ and $\xi^i(t)$, for $t \in T^1$ denote relaxed f- and r-trajectories determined by the same initial condition $\phi(t)$ for all $t \in [-\tau, 0]$ and the same relaxed controls $\underset{\sim}{u}_i(t)$. Then as $i \longrightarrow \infty$, $x^i(t)$ tends uniformly in T^1 to $x^{\underset{\sim}{u}^*}(t)$ if and only if $\xi^i(t)$ does so.

Proof

We may suppose by subdivision that the length $|T^1|$ of T^1 is $\leq 1/2M_1$ where M_1 is the finite constant defined below.

We write

$$a_i = \sup_{t \in T^1} \| x^i(t) - x^{\underset{\sim}{u}^*}(t) \|$$

$$b_i = \sup_{t \in T^1} \| \xi^i - x^{\underset{\sim}{u}^*}(t) \|$$

By our assumptions we have that

$$\| f(x^i(t),x^i(t-\tau),\underset{\sim}{u}_i(t),t)-f(x^{\underset{\sim}{u}^*}(t),x^{\underset{\sim}{u}^*}(t-\tau),\underset{\sim}{u}_i(t),t) \|$$

$$\leq M \sup_{t \in T^1} \{ \| x^i(t)-x^{\underset{\sim}{u}^*}(t) \| + \| x^i(t-\tau)-x^{\underset{\sim}{u}^*}(t-\tau) \| \}$$

$$\leq M_1 \sup_{t \in T^1} \| x^i(t)-x^{u^*}(t) \| \quad \text{for some finite } M_1$$

$$= M_1 \; a_i \quad \text{for all } t \in T^1$$

Now since

$$x^i(t)-\xi^i(t) = \int_0^t [f(x^i(s),x^i(s-\tau),\underset{\sim}{u}_i(s),s)$$

$$-f(x^{\underset{\sim}{u}^*}(s),x^{\underset{\sim}{u}^*}(s-\tau),\underset{\sim}{u}_i(s),s)] \, ds$$

$$\text{for all } t \in T^1$$

It follows quite easily that

$$\| x^i(t)-\xi^i(t) \| \leq M \int_0^t \{ \| x^i(s)-x^{\underset{\sim}{u}^*}(s) \| + \| x^i(s-\tau)-x^{\underset{\sim}{u}^*}(s-\tau) \| \} \, ds$$

$$\leq M_1 \sup_{t \in T^1} \| x^i(t)-x^{\underset{\sim}{u}^*}(t) \| \int_{T^1} ds$$

$$\text{for all } t \in T^1$$

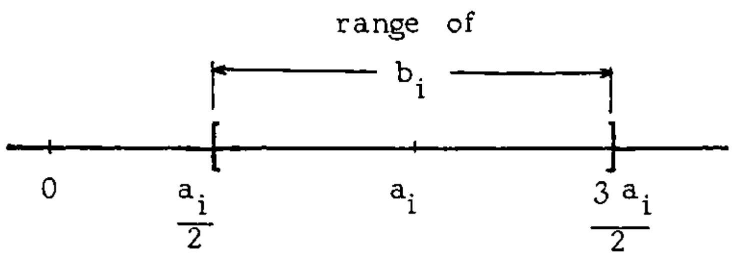
i.e. $\| x^i(t)-\xi^i(t) \| \leq M_1 \; a_i \; \| T^1 \| \leq \dfrac{a_i}{2}$ for all $t \in T^1$

so that we have

$$\| a_i-b_i \| \leq \sup_{t \in T^1} \| x^i(t)-\xi^i(t) \| \leq \frac{a_i}{2}$$

i.e. $^1/_2 \; a_i \leq b_i \leq {}^3/_2 . \, a_i$

This situation is shown in the diagram overleaf.



Hence if $b_i \longrightarrow 0$, i.e. $\xi^i(t)$ tends uniformly to $x^{u^*}(t)$ in T', we must have $a_i \longrightarrow 0$, i.e. $x^i(t)$ also tends uniformly to $x^{u^*}(t)$ in T'.

Using Lemmas 2 and 3 we will now deduce our second compactness result.

Theorem 2

Let B_1 and Ω be arbitrary compact sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. If $\{\underset{\sim}{u}_i, x^i\}_{i=0}$ is an infinite sequence of relaxed controls and their corresponding trajectories such that $\{\underset{\sim}{u}_i\} \in \underset{\sim}{G}$ and $\{x^i\} \in B_1$ with $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ converging to $\underset{\sim}{u}^* \in \underset{\sim}{G}$ i.s.c.m. and $\{x^i\}_{i=0}^{\infty}$ converging to x^* uniformly, then $x^*(t) = x^{\underset{\sim}{u}^*}(t)$, i.e. the limiting trajectory is the trajectory due to the limiting control.

Furthermore, given a sequence $\{\underset{\sim}{u}_i, x^i\}_{i=0}^{\infty}$ such that $\{\underset{\sim}{u}_i\} \in \underset{\sim}{G}$ then there always exists a subsequence that satisfies the above hypothesis and conclusions.

That is we have

$$x^i(t) \longrightarrow x^{\underset{\sim}{u}^*}(t) \text{ uniformly in } T \text{ as } \underset{\sim}{u}_i \longrightarrow \underset{\sim}{u}^* \text{ i.s.c.m.}$$

Proof

By Theorem 1 there always exists a subsequence indexed by $K \subset \{0,1,2,....\}$ of an infinite sequence $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$ of relaxed controls in $\underset{\sim}{G}$ such that $\underset{\sim}{u}_i \xrightarrow{K} \underset{\sim}{u}^*$ i.s.c.m. for some $\underset{\sim}{u}^* \in \underset{\sim}{G}$.

Let the corresponding trajectories be $x^i(t)$. Since $\underset{\sim}{u}^* \in \underset{\sim}{G}$, then by Lemma 1 there exists a unique solution $x^{\underset{\sim}{u}^*}(t)$ which satisfies the delay-differential equation

$$\dot{x}(t) = f(x(t), x(t-\tau), \underset{\sim}{u}^*(t), t) \qquad \text{for a.a.} \; t \in T$$

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau, 0]$$

As in Lemma 3, let $\xi^i(t)$ be the r-trajectory for control $\underset{\sim}{u}_i$ where $r(\underset{\sim}{u}_i(t),t) = f(x^{\underset{\sim}{u}^*}(t), x^{\underset{\sim}{u}^*}(t-\tau), \underset{\sim}{u}_i(t), t)$ and let $x^i(t)$ be the f-trajectory for the same control $\underset{\sim}{u}_i$ and the same initial conditions, i.e. we have

$$\xi^i(t) = \phi(0) + \int_0^t f(x^{\underset{\sim}{u}^*}(s), x^{\underset{\sim}{u}^*}(s-\tau), \underset{\sim}{u}_i(s), s)\, ds$$

and $x^i(t) = \phi(0) + \int_0^t f(x^i(s), x^i(s-\tau), \underset{\sim}{u}_i(s), s)\, ds$ for all $t \in T$

Now as $\underset{\sim}{u}_i \xrightarrow{K} \underset{\sim}{u}^*$ i.s.c.m. suppose that $\xi^i(t) \xrightarrow{K} \xi^*(t)$ uniformly in T. Then using Lemma 2 we get

$$\xi^*(t) = \phi(0) + \int_0^t f(x^{\underset{\sim}{u}^*}(s), x^{\underset{\sim}{u}^*}(s-\tau), \underset{\sim}{u}^*(s), s)\, ds \qquad \text{for all } t \in T$$

$$= x^{\underset{\sim}{u}^*}(t)$$

i.e. we have $\xi^i(t) \xrightarrow{K} x^{\underset{\sim}{u}^*}(t)$ uniformly in T as $\underset{\sim}{u}_i \xrightarrow{K} \underset{\sim}{u}^*$ i.s.c.m.

Hence using Lemma 3 we get that $x^i(t) \xrightarrow{K} x^{\underset{\sim}{u}^*}(t)$ uniformly in T as well. This proves the Theorem.

Appendix B

Penalty Function Methods

1. Introduction

In this appendix we present a brief overview on penalty methods and how they are used in solving very general optimization problems. These methods are quite classical in that one such method was presented as early as 1943 by Courant [CO1] who solved a difficult problem by solving a related (but simpler) problem.

Penalty methods are basically procedures for approximating constrained optimization problems by unconstrained problems. These approximations are accomplished by adding to the objective function a term which prescribes a high cost for violation of the constraints. In the usual implementation of the scheme, the solution to the constrained optimization problem is obtained only when the penalty parameter approaches infinity. For example, consider the following nonlinear programming problem:

Problem B1: $\min_{x} \; f(x)$

$$\text{s.t. } h_i(x) = 0 \;, \; i=1,2,\ldots,m$$

where the functions f, h_i's are real valued continuously differentiable functions on $\mathbb{R}^n$. A penalty function method of solving this is to solve the following unconstrained problem for $K>0$,

Problem $B1_K$: $\min_{x} \; \gamma_K(x)$

where we define $\gamma_K : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\gamma_K(x) \triangleq f(x) + K \sum_{i=1}^{m} \{h_i(x)\}^2$$

In this formulation K is called the penalty parameter and the term $K \sum_{i=1}^{m} \{h_i(x)\}^2$ the penalty term. Obviously a number of different forms of penalty term with correspondingly different characteristics exist. These do however have one common feature, namely that the larger the penalty parameter K, the closer is the approximation.

For this reason it seems desirable, when applying this method practically, to select K as large as is possible. However this is not the complete story since when calculating gradients (for obtaining search directions, etc.), K is required to be fairly small so that the penalty term does not completely swamp out the original cost functional. A common technique for evading this conflict is to solve a sequence of unconstrained problems $B1_{K_j}$ with K_j increasing to infinity. If this is done and a sequence of points $\{x_i\}_{i=0}^{\infty}$ is obtained, it is proved in Luenberger [LU2] that any limit point of this sequence does indeed solve the original problem.

Inequality constraints such as $g_i(x) \leq 0$, $i=1,2,\ldots,r$ may be handled using a penalty term of the form $K \sum_{i=1}^{r} \{g_i^+(x)\}^2$ where $g_i^+(x) \triangleq \max_i \{ g_i(x), 0 \}$.

2. Connection with Geometric Methods

At first sight the penalty function method may seem a somewhat crude method for solving constrained optimization problems, but it has been shown (see for example Luenberger [LU1]) to be closely connected with geometric ideas and the concept of dual problems. For example, consider the following problem (which is clearly equivalent to Problem B1):

Problem B2: $\quad \text{Min} \quad f(x)$

$$\text{s.t} \quad \sum_{i=1}^{m} h_i^2(x) \le 0$$

[Note that there is no x such that $\sum_i^m h_i^2(x) < 0$, i.e. the constraint is not regular as defined in Luenberger [LU1]]

Then define the following family of problems:

$$\chi(\alpha) = \inf \{ f(x) : \sum_{i=1}^{m} h_i^2(x) \le \alpha \}$$

This is called the primal functional for Problem B2 and its relationship with $\alpha \ge 0$ is of the form shown in Fig.1.

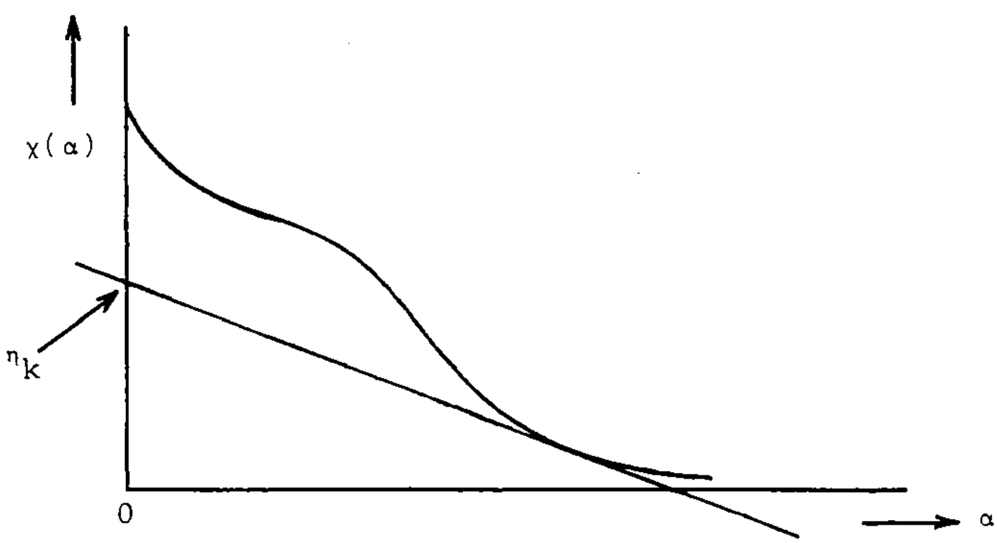


Fig 1

It is easy to see that χ is nonincreasing with increasing α and $\alpha = 0$ is a boundary point. The solution to Problem B1 is equal to the maximum intercept with the vertical axis of all closed hyperplanes (which in this case are just straight lines) that support χ. This maximum intercept is of course given by the Lagrange multipliers of the problem and may be infinite.

To state this more mathematically we introduce the dual functional for Problem B2 to be defined on the positive real line as

$$\eta(c) = \inf_{x} \{ f(x) + c \sum_{i=1}^{m} h_i^2(x) \}$$

Then the dual problem is defined as

$$\max_{c \geq 0} \eta(c)$$

It is easy from this interpretation that provided $\eta(c^*)$ is finite for some $c^* \geq 0$,

$$\max_{c \geq 0} \eta(c) \leq \min_{\alpha \leq 0} \chi(\alpha)$$

and hence the dual functional always serves as a lower bound to the value of the primal problem (Luenberger [LU1]).

Hence by choosing a $K > 0$ and minimising $f(x)+K \sum_{i=1}^{m} h_i^2(x)$ determines as shown in Fig 1 a supporting hyperplane to χ (defined by this K) and a value η_K for the dual functional. Provided χ is continuous it is obvious that as K is increased η_K will increase monotonically toward $\chi(0)$. Since $\alpha = 0$ is a boundary point of the region of definition of X, a (perhaps vertical) support hyperplane always exists.

3. Advantages and Disadvantages

The main advantage of using penalty methods is that many difficult problems, incorporating constraints which cannot be handled by any other method, can be transformed into much simpler unconstrained ones. These can then be solved using any one of a vast array of efficient methods which already exist and are widely available (e.g. Newton, steepest descent, conjugate gradient, etc. see Polak [P1]). In the normal application of the method the penalty term is constructed to have similar properties as the functions defining the original problem, i.e. differentiability, etc. so that normal gradient techniques can be applied for solving the penalised problem. However for obtaining a solution to the original problem requires solving a sequence of problems such as

$$\text{Min } \gamma_{K_j}(x) = f(x) + K_j \sum_{i=1}^{m} h_i^2(x)$$

for K_j going to infinity. This, as shown by several authors e.g. Luenberger [LU1], Lootsma [L01], Ryan [RY1], leads to certain difficulties which can be explained by considering the behaviour of γ_K as K gets large. A typical form of penalty term is shown in Fig. 2 where it is obvious to note that as K increases the penalty term has steeper slopes outside the feasible region.

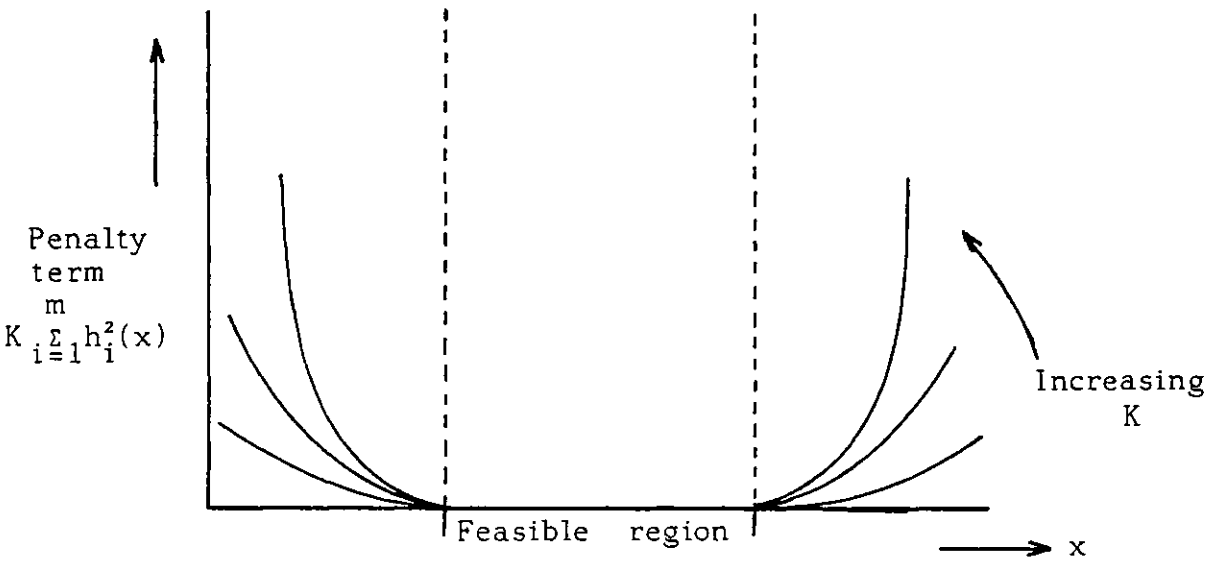


Fig. 2

This implies that as K increases γ_K becomes a function whose minima lies in a steep sided "valley" with the "valley" slopes getting steeper. It is this which causes the above mentioned problems since the Hessian (the second derivative of γ_K w.r.t. x) becomes increasingly ill-conditioned in the sense that its magnitude tends to infinity as $K \longrightarrow \infty$. This, as one might expect, causes difficulties in obtaining solutions numerically and convergence, if it occurs is likely to be slow.

These and other properties of penalty methods are fully discussed in Ryan [RY1] and the interested reader is advised to consult it for full details.

4. The Exact Penalty Function Method

Because of the above mentioned difficulties a similar method, referred to as the Exact Penalty Function method, has emerged which also adds a penalty term to the objective function, but it does not need K to be increased to infinity for assuring equivalence.

The method can be explained by considering Problem B1 again. If the m constraints are formulated into the following equivalent single constraint,

$$\sum_{i=1}^{m} | h_i(x) | \leq 0$$

then the penalty term $K \sum_{i=1}^{m} | h_i(x) |$ can be used in defining $\gamma_K(x)$. Then if the h_i's have nonvanishing first derivatives at the solution, then the primal functional will have a finite slope at $\alpha = 0$, and hence a finite K will yield a support hyperplane. This fact is attractive from a computational viewpoint since the sequence of unconstrained problems of minimising $\gamma_K(x)$ need only be solved upto some finite K^*, and therefore the Hessian is prevented from becoming ill-conditioned.

However, a slight problem does arise with this method, namely that normal optimization techniques which involve calculating gradients cannot be employed to solve $\gamma_K(x)$ since the penalty term is non-differentiable. It turns out (see Bertsekas [BER2]) that, except for trivial cases, this non-differentiability is a necessary evil if our penalty method is to be exact.

Although ordinary gradient methods cannot be used to solve $\gamma_K(x)$ there are modified procedures which are applicable to the non-smooth problem in question. Some of these are discussed in Pietzykowski [PI1], Mifflin [MI1] [MI2] and the interested reader is referred there.

With reference to the above discussion we can deduce that the exact penalty method is a very good procedure for solving problems involving complex constraints numerically. Indeed it is for this reason that we apply the method when dealing with the constrained optimal control problems in Chapters 4 and 5. The state constrained problem considered in Chapter 5, in particular is a very good example of how such a complicated problem can be handled using an exact penalty function.

Although we have only discussed penalty function methods for finite dimensional optimization, the extension to infinite dimensional problems is fairly obvious and so we omit it here, see for example Czap [CZ1], Polak [P1].

Appendix C

The Maximum Principle for State Constrained Control Problems with Delay

1. Introduction

In [N2] Neustadt introduces an extremal theory which may be used to obtain necessary optimality conditions for a broad class of problems. We use this theory here to derive the Maximum Principle for the State Constrained nonlinear optimal control problem with delay (i.e. Problem P3 in text). This theory relies heavily on replacing non-convex sets by "quasiconvex" approximations and using "linearized" approximations for the nonlinear functions defining the problem.

We will, for completeness, derive an abstract Maximum Principle for a broad class of problems following the methodology of Neustadt and then use this theory to obtain the Maximum Principle for the State Constrained Control Problem with delay. These results have been proved in Huang [HU1] for a more general delayed control problem than the one considered by us - Huang considers k delayed arguments instead of only one. We give our results here mainly to obtain necessary conditions of optimality in a form which is directly applicable to the problem in question, i.e. Problem P3.

2. Concept of Extremality

Here we introduce the concept of an extremal and how optimization problems can be reformulated as extremal problems. Let $\mathscr{E}$ be an arbitrary set and let $\phi_0, \phi_1, \phi_2, \ldots, \phi_S$ be real-valued functions on $\mathscr{E}$. Now consider the following optimization problem:

Problem (OP)
$$\min_{e \in \mathscr{E}} \quad \phi_0(e)$$
$$\text{s.t.} \quad \phi_i(e) \leq 0 \quad , \; i=1,2,\ldots,s$$

Assume that this has $e_0 \in \mathscr{E}$ as its solution.

In order to state this in terms of extremals we need to define the following:

1. An arbitrary set $\mathscr{Y}$, and a subset $\mathscr{E}$ of $\mathscr{Y}$
2. A normed linear topological vector space $\mathscr{L}$
3. An open, convex cone Z in $\mathscr{L}$
4. A function $\phi: \mathscr{E} \longrightarrow \mathscr{L}$

Definition 2.1

A set $Z \subset \mathscr{L}$ (linear space) is a convex cone if it is not empty and if we have

$$\alpha Z + \beta Z \subset Z \qquad , \text{ all } \quad \alpha > 0, \; \beta \geq 0$$

Some examples of convex cones are:

$\{ x \in \mathbb{R}^n : x_i < 0 \text{ all } i \}$ is an open convex cone in $\mathbb{R}^n$

$\{x \in C : x(t) < 0 \text{ all } t \}$ is an open convex cone in C, where C is the space of continuous functions $t \longrightarrow x(t)$ on a given time interval with sup norm.

We are now in a position to define the notion of extremality:

Definition 2.2

An element $e_0 \in \mathcal{E}$ is called a (ϕ, Z)-extremal if

(i) $\phi(e_0) \in \bar{Z}$ (i.e. the closure of Z)

(ii) the set $\{e : e \in \mathcal{E}, \phi(e) \in Z\}$ is empty

This can be highlighted by considering Problem (OP). Since e_0 is the solution, we have if we set $\mathcal{L}$, Z, ϕ as follows,

$$\mathcal{L} = \mathbb{R}^{s+1}$$

$$Z = \{y=(y^0,y^1,\ldots,y^s) : y \in \mathbb{R}^{s+1}, y^i<0 \ , \ i=0,1,2,\ldots,s\},$$

i.e. the negative orthant in $\mathbb{R}^{s+1}$

$$\phi(e) = (\phi_0(e)-\phi_0(e_0)), \phi_1(e), \phi_2(e),\ldots, \phi_s(e))$$

then it is immediately seen that e_0 is also a (ϕ, Z)-extremal.

3. The Abstract Maximum Principle

Before we state the abstract Maximum Principle, we present some motivation. Consider the optimization problem (OP) with $\mathcal{E}$ a closed, nonempty convex subset of $\mathbb{R}^k$, and let us suppose that ϕ_i, $i=0,1,\ldots,s$ are convex and everywhere defined on $\mathcal{E}$. Then the well known Kuhn-Tucker Theorem (e.g. Pshenichniy [PSH1]) gives the following necessary condition for an element to be optimal:

If e_0 solves (OP) $\Longrightarrow$ There exists a non-zero vector $l = (l_0, l_1, \ldots, l_s) \in \mathbb{R}^{s+1}$ such that

$$l_0 \phi_0(e_0) + \sum_{i=1}^{s} l_i \phi_i(e_0) \leq l_0 \phi(e) + \sum_{i=1}^{s} l_i \phi_i(e) \qquad \text{for all } e \in \mathcal{E}$$

with $l_i \geq 0$, $i=1,2,\ldots,s$.

With stronger assumptions (e.g. Slater conditions) it can also be assumed that $l_0 > 0$, and in this case it may be normalized to unity.

Now in terms of extremality it can easily be seen that this translates into:

e_0 is a (ϕ, Z)-extremal only if there exists a non-zero $h = (h_0, h_1, \ldots, h_s) \in \mathbb{R}^{s+1}$ such that

$$h \cdot [\phi(e) - \phi(e_0)] \leq 0 \quad \text{for all } e \in \mathcal{E}$$

and $h \cdot y \geq 0$ for all $y \in Z$

(where "." denotes the normal Euclidean inner product).

Our abstract Maximum Principle is nothing but an extension of the Kuhn-Tucker Theorem when most of the convexity assumptions do not hold. Also, to develop the necessary extremal theory, we will use the same tools as required in proving the Kuhn-Tucker theorem - namely the separation principle for convex sets, after justifying replacing the non-convex problem by a suitable convex approximation.This will become quite apparent in the following discussion. However, to be able to formulate such a convex problem which approximates the original optimization problem we need to make a few definitions.

Definition 3.1

Let $\mathscr{E}$ and $\mathscr{M}$ be subsets of a normed linear space $\mathscr{Y}$. Then $\mathscr{M}$ is a first order convex approximation to $\mathscr{E}$ at $e_0 \in \mathscr{Y}$ if:

(i) $\mathscr{M}$ is convex and contains 0

(ii) For every finite set $\{y_1, y_2, \ldots, y_\nu\} \subset \mathscr{M}$, every $\eta > 0$ there exists some $\varepsilon_1 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1]$ there exists a continuous linear map

$$\gamma_\varepsilon : \text{co}\,\{y_1, y_2, \ldots, y_\nu\} \longrightarrow \mathscr{E} \quad \text{with the following property}$$

$$\left\| \frac{(\gamma_\varepsilon(y) - e_0)}{\varepsilon} - y \right\|_{\mathscr{Y}} < \eta \quad \text{for all } y \in \text{co}\,\{0, y_1, \ldots, y_\nu\}$$

This definition states that, given any simplex in $\mathscr{M}$ containing 0, it can be mapped into $\mathscr{E} - e_0$ by a "slight continuous distortion"

$$y \longmapsto \frac{\gamma_\varepsilon(y) - e_0}{\varepsilon} = \xi_\varepsilon(y)$$

followed by a "shrinking"

$$\xi_\varepsilon(y) \longmapsto \varepsilon\, \xi_\varepsilon(y) = \gamma_\varepsilon(y) - e_0$$

Definition 3.2

Let $\mathscr{Y}$, $\mathscr{L}$ be normed linear spaces. We say that the functional $\psi : \mathscr{Y} \to \mathscr{L}$ is mildly differentiable at $e_0 \in \mathscr{Y}$, if there exists a continuous linear function $h : \mathscr{Y} \to \mathscr{L}$ such that

$$\frac{[\psi(e_0+\varepsilon y)-\psi(e_0)]}{\varepsilon} \xrightarrow[\substack{\varepsilon \to 0^+ \\ y \to x \ (\text{in } \mathcal{Y}\text{-norm})}]{} h(x) \quad (\text{in } \mathcal{L}\text{-norm})$$

In this case $h(x)$ is called the mild differential of ψ at e_0.

Note:

Using the definitions of Frechet and Gateaux derivatives (see Luenberger [LU1]) we have that

Frechet Differentiability	$\Longrightarrow$	Mild Differentiability	$\Longrightarrow$	Gateaux Differentiability

(at $e_0 \in \mathcal{Y}$)

We now state our Abstract Maximum Principle.

4. The Abstract Maximum Principle

Theorem 4.1

Let the following be given:

(i) $\mathcal{Y}$, a normed linear space

(ii) Two subsets $\mathcal{E}, \mathcal{M} \subset \mathcal{Y}$ and $\mathcal{M}$ is a first order convex approximation to $\mathcal{E}$ at e_0

(iii) $\mathcal{L}$, a normed linear space with $Z \subset \mathcal{L}$ being an open convex cone

(iv) $\phi: \mathcal{Y} \to \mathcal{L}$ a continuous function with mild differential $y \longrightarrow h(y)$

(v) $e_0 \in \mathcal{E}$ is a (ϕ, Z)-extremal

(These are all as defined before for Problem (OP).)

Then, there exists a linear functional $l \in \mathcal{L}^*$ (the topological dual of $\mathcal{L}$), not equal to zero such that

(a) $l \circ h(y) \leq 0$ for all $y \in \mathcal{M}$

(b) $l(z) \geq 0$ for all $z \in Z$

(c) $l \circ \phi(e_0) = 0$

(where $l \circ h$ denotes composition of functions)

This theorem, as mentioned before, is an outgrowth of the Kuhn-Tucker theorem of the last section, since when ϕ is linear and $\mathcal{E}$ is convex, the theorem implies the earlier result with $\mathcal{E}$ serving as its own convex approximation and h coinciding with $\phi - \phi(e_0)$.

The proof of Theorem 4.1 is given in Neustadt [N2] and the interested reader is referred there.

We will now apply the above extremal theory to derive the Maximum Principle for the State Constrained Control Problem with delay.

5. State Constrained Control Problem (SCCP) with Delay

For convenience we will restate the SCCP with delay which is under consideration:

Let τ be a positive real number and $T \triangleq [0,1]$ be a compact interval. Then our problem is

Problem P3

$$\min_{\underset{\sim}{u}} \int_0^1 l(x(t), \underset{\sim}{u}(t), t)dt$$

s.t. $\dot{x}(t) = f(x(t), x(t-\tau), \underset{\sim}{u}(t), t)$ for a.a. $t \in T$

$x(t) = \phi(t)$ for all $t \in [-\tau, 0]$

$g(x(t), t) \leq 0$ for every $t \in T$

$\underset{\sim}{u} \in \underset{\sim}{G}$

where the notation and terminology is the same as in the text. The assumptions which we make are also given in Chapter 5.

Assume that Problem P3 has $\underset{\sim}{u}^*(t) \in \underset{\sim}{G}$ as its optimal control and $x^*(t)$ be the corresponding optimal trajectory.

We will need to reformulate Problem P3 into a form to which the extremal theory discussed above may be applied. To do this we propose the following:

The dynamic constraint and the control constraints be handled implicitly by setting $\mathscr{E}$ to be the composite space of (all the admissable trajectories arising from admissable controls $\underset{\sim}{u} \in \underset{\sim}{G}$) x (the admissible control set $\underset{\sim}{G}$), while the state constraint be dealt with explicitly by incorporating it in the inequality constraint $\phi(.) \leq 0$ [c.f. Problem (OP)].

i.e. let

$$D = \{ x \in \mathscr{A} \mid \; x(t) \in B_1 \qquad \text{for all } t \in T$$
$$\dot{x}(t) = f(x(t), x(t-\tau), \underset{\sim}{u}(t), t) \qquad \text{for a.a.} t \in T$$
$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau, 0]$$
$$\underset{\sim}{u} \in \underset{\sim}{G} \; \}$$

where B_1 is as defined earlier in text and $\mathscr{A}$ is the class of n-dimensional absolutely continuous functions. Then we define $\mathscr{E}$ by

$$\mathscr{E} \triangleq D \times \underset{\sim}{G}$$

This we can embed in the composite space of all continuous functions $T \to \mathbb{R}^n$ with sup norm (written $C(T; \mathbb{R}^n)$) x the control set $\underset{\sim}{G}$, so that we have $\mathscr{Y}$ as

$$\mathscr{Y} = C(T; \mathbb{R}^n) \times \underset{\sim}{G}$$

$\mathscr{L}$ can be set to be $\mathbb{R} \times C(T; \mathbb{R})$ and let Z be defined by

$$Z = \{\xi : \xi \in \mathbb{R}, \xi < 0\} \times \{y \in C(T; \mathbb{R}) : y(t) < 0 \quad \text{for all } t\}$$

[It is trivial to show that Z is an open convex cone in $\mathscr{L}$.]

Let $\phi : \mathscr{Y} \longrightarrow \mathscr{L}$ be defined by

$$\phi(x, \underset{\sim}{u}) \triangleq \left(\int_0^1 [l(x^{\underset{\sim}{u}}, \underset{\sim}{u}, t) - l(x^*, \underset{\sim}{u}^*, t)]\, dt,\ g(x^{\underset{\sim}{u}}(t), t)\right).$$

Then it is quite obvious from Definition (2.2) that $(x^*, \underset{\sim}{u}^*)$ is a (ϕ, Z)-extremal.

Because of the continuity assumptions we have that

$$x \longmapsto \left(\int_0^1 l(x, \underset{\sim}{u}, t)\, dt,\ \{t \longrightarrow g(x(t), t)\}\right) : C(T; \mathbb{R}^n) \longrightarrow \mathbb{R} \times C(T : \mathbb{R})$$

is Frechet differentiable at x^*, and ϕ is linear with respect to the relaxed control $\underset{\sim}{u}$. Therefore the derivative w.r.t. $\underset{\sim}{u}$ at $\underset{\sim}{u}^*$ coincides with $\phi - \phi(\underset{\sim}{u}^*)$, i.e.

$$\delta\phi((x^*, \underset{\sim}{u}^*); (z, \underset{\sim}{v})) = \left(\int_0^1 [l_x(x^*, \underset{\sim}{u}^*, t)\, z(t) + l(x^*, \underset{\sim}{v}, t) - l(x^*, \underset{\sim}{u}^*, t)]\, dt,\right.$$

$$\left., g_x(x^*(t), t)\, z(t)\right)$$

Since Frechet differentiability implies mild differentiability our abstract Maximum Principle is applicable (with the above choices for ϕ, Z, $\mathscr{E}$, $\mathscr{Y}$, $\mathscr{L}$, etc.) provided we can find a suitable first order convex approximation to $\mathscr{E}$ at $(x^*, \underset{\sim}{u}^*)$. As $\mathscr{E} = D \times \underset{\sim}{G}$, it is well known that for nonlinear systems D need not be convex (examples can be easily constructed). $\underset{\sim}{G}$ on the other hand is convex and compact, (see for sample Warga [W3]). Therefore to find a first order convex approximation to $\mathscr{E}$, we can have $\underset{\sim}{G}$ serving as its own convex approximation and thus we only need to find a first order convex approximation to D. This will shortly be constructed, but first we present a general solution for a linear differential difference equation which will be needed in the sequel.

5.1 Linear Differential - Difference Equation

Let T and τ be as before and consider the following linear differential difference equation

$$\dot{x}(t) = A_1(t)x(t)+A_2(t)x(t-\tau) + p(\underset{\sim}{u}(t),t) \quad \text{for a.a.} t \in T \qquad (5.1.1)$$

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau,0] \qquad (5.1.2)$$

where A_1 and A_2 are nxn matrix valued functions defined and continuous on T and $p(\underset{\sim}{u}(t),t)$ is a given n-vector valued function of the control $\underset{\sim}{u}$ and time $t \in T$. Let $\Phi(s,t)$ be the unique matrix function defined for $t \in T$, $s \in [0,\infty)$, which for each $t \in T$ is an absolutely continuous function of $s \in [0,t]$ and which satisfies the differential-difference equation

$$\frac{\delta \Phi(s,t)}{\delta s} = -\Phi(s,t)A_1(s)-\Phi(s+\tau,t)A_2(s+\tau) \qquad \text{for a.a.} s \in [0,t] \; (5.1.3)$$

and $\Phi(t,t) = I$ (the identity matrix) (5.1.4)

and which for s>t is given by

$$\Phi(s,t) = 0 \qquad (5.1.5)$$

Equation (5.1.3) together with the boundary condition (5.1.4) and relation (5.1.5) is the adjoint system to the homogeneous part of (5.1.1) and $\Phi(s,t)$ is the state transition matrix. Now using a very similar procedure as in the paper by Huang [HU1] we can deduce the following result:

Theorem 5.1

If all the above hypothesis hold, then the solution of (5.1.1), with given initial condition (5.1.2), is given by

$$x(t) = \Phi(0,t)\,\phi(0) + \int_{-\tau}^{0} \Phi(s+\tau,t)A_2(s+\tau)\,\phi(s)ds$$

$$+ \int_0^t \Phi(s,t)\; p(\underset{\sim}{u}(s),s)ds \qquad \text{for all } t \in T$$

5.2 First Order Convex Approximation

We will now find a first order convex approximation to D defined above for the SCCP with delay. Let $\Phi(s,t)$ be the matrix valued function defined above with

$$A_1(t) = f_x(x^*(t),x^*(t-\tau),\underset{\sim}{u}^*(t),t)$$
$$A_2(t) = f_y(x^*(t),x^*(t-\tau),\underset{\sim}{u}^*(t),t)$$

for all $t \in T$, where f_x is the derivative of f w.r.t. the first argument (i.e. x(t)) and f_y is the derivative w.r.t. its second argument (i.e. $x(t-\tau)$).

Also let

$$p(\underset{\sim}{v}(t),t) = f(x^*(t),x^*(t-\tau),\underset{\sim}{v}(t),t)-f(x^*(t),x^*(t-\tau),\underset{\sim}{u}^*(t),t) \qquad \text{for all } t \in T$$

$$\triangleq \Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)$$

and $\phi(t) \equiv 0$ for all $t \in [-\tau,0]$.

Then the solution to the delay differential equation

$$\dot{x}(t) = A_1(t)x(t) + A_2(t)x(t-\tau)+p(\underset{\sim}{v}(t),t) \qquad \text{for a.a. } t \in T$$

is $z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(.)$ where $z^{\underset{\sim}{u}^*,\underset{\sim}{v}}$ is the first order estimate of $x^{\underset{\sim}{v}}-x^{\underset{\sim}{u}^*}$ (as proved in text).

Therefore from Theorem 5.1 we have that for any $\underset{\sim}{v} \in \underset{\sim}{G}$,

$$z^{\underset{\sim}{u}*,\underset{\sim}{v}}(t) = \int_0^t \Phi(s,t)\ \Delta f(\underset{\sim}{v},\underset{\sim}{u}*)ds \qquad \text{for all } s \in T$$

Next we define Q to be the following subset of $C(T; \mathbb{R}^n)$

$$Q = \{ z^{\underset{\sim}{u}*,\underset{\sim}{v}}(t) \quad ; \quad \text{for all } t \in T,\ \underset{\sim}{v} \in \underset{\sim}{G} \}$$

Then we have the following result:

Theorem 5.2

Q is a first order convex approximation to D at x*.

Proof

(i) $0 \in Q$, since if $\underset{\sim}{v} = \underset{\sim}{u}* \in \underset{\sim}{G}$ then the solution $z^{\underset{\sim}{u}*,\underset{\sim}{u}*}(t)$ is identically zero for all $t \in T$.

(ii) Q is convex.

For any positive integer l let P^l denote the following subset of $\mathbb{R}^l$:

$$P^l = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) : \alpha_i \geq 0,\ i=1,2,\ldots,l,\ \sum_{i=1}^{l} \alpha_i = 1 \}$$

Let $z_i(t) = z^{\underset{\sim}{u}*,\underset{\sim}{v}_i}(t)$, $i=1,2,\ldots,l$ be elements of Q, where $\underset{\sim}{v}_i \in \underset{\sim}{G}$, $i=1,2,\ldots,l$. Then since

$$z\,(t) = \int_0^1 \Phi(s,t)\ \Delta f(\underset{\sim}{v},\underset{\sim}{u}*)ds$$

we have for all $t \in T$,

$$\sum_{i=1}^{l} \alpha_i \; z_i(t) = \sum_{i=1}^{l} \alpha_i \; z^{\underset{\sim}{u}^*,\underset{\sim}{v}_i}(t)$$

$$= \sum_{i=1}^{l} \int_0^t \Phi(s,t)\alpha_i \Delta f(\underset{\sim}{v}_i, \underset{\sim}{u}^*)ds$$

$$= \int_0^t \Phi(s,t)\Delta f(\sum_{i=1}^{l} \alpha_i \underset{\sim}{v}_i, \underset{\sim}{u}^*)ds$$

since $\sum_{i=1}^{l} \alpha_i \underset{\sim}{u}^* = \underset{\sim}{u}^*$

Now $\sum_{i=1}^{l} \alpha_i \underset{\sim}{v}_i \in \underset{\sim}{G}$ since $\underset{\sim}{G}$ is convex

Hence $\sum_{i=1}^{l} \alpha_i z_i \in Q$ and therefore Q is convex.

(iii) The remainder of the proof is to verify that if $\{z_1, z_2, \ldots, z_l\}$ is any finite subset of Q and if $\eta > 0$, there exists some $\varepsilon_1 > 0$, such that for any $\varepsilon \in (0, \varepsilon_1]$, there is a continuous map $\gamma_\varepsilon : co\{z_1, z_2, \ldots, z_l\} \longrightarrow D$ such that

$$\left\| \frac{[\gamma_\varepsilon(z) - x^*(t)]}{\varepsilon} - z \right\| < \eta \text{ for all } z \in co\{0, z_1, \ldots, z_l\}$$

This is actually proved in Huang [HU1] where weaker assumptions are made and the number of delayed arguments is k compared to only one in this thesis. The proof is rather laborious but quite straightforward and will not be presented here. The interested reader is referred to Huang's paper for full details of the proof.

Hence since we have just shown that Q is a first order approximation to D we may define $\mathcal{M}$ by

$\mathcal{M} = Q \times \underset{\sim}{G}$, and then $\mathcal{M}$ is a first order convex approximation to $\mathcal{E}$ at $(x^*, \underset{\sim}{u}^*)$.

5.3 Maximum Principle for the SCCP with Delay

Let ϕ, Z, $\mathcal{E}$, $\mathcal{Y}$, $\mathcal{L}$, $\mathcal{M}$, $\delta\phi(.)$ be as defined above. Then we have from our Abstract Maximum principle the following:

Theorem 5.3

If $(x^*, \underset{\sim}{u}^*)$ is an (ϕ, Z)-extremal for Problem P3 then there exists a non-zero $l \in \mathcal{L}^*$ such that

(i)	$l \circ \delta\phi((x^*, \underset{\sim}{u}^*);(y, \underset{\sim}{v})) \leq 0$	for all $(y, \underset{\sim}{v}) \in \mathcal{M}$
(ii)	$l(\sigma) \geq 0$	for all $\sigma \in \overline{Z}$
(iii)	$l \circ \phi(x^*, \underset{\sim}{u}^*) = 0$	

Recalling that any bounded linear functional on $C[T; \mathbb{R}]$ has the representation (see for example Dunford and Schwartz [DU1] or Luenberger [LU1])

$$x \longmapsto \int_0^1 x(t)\, d\lambda(t)$$

for some $\lambda(t) \in NBV[T; \mathbb{R}]$, where $NBV[T; \mathbb{R}]$ is the space of functions of bounded variation, continuous from the right in $[0,1)$ with $\lambda(1)=0$.

Since $\mathcal{L} = \mathbb{R} \times C[T ; \mathbb{R}]$, we have as its dual $\mathcal{L}^*$ is $\mathbb{R} \times NBV[T; \mathbb{R}]$. Then $l \in \mathcal{L}^*$ in Theorem 5.3 is of the form

$(\lambda_0, \lambda(t))$ where $\lambda_0 \in \mathbb{R}$ and $\lambda \in NBV[T; \mathbb{R}]$.

Also since

$$\delta\Phi((x^*,\underset{\sim}{u}^*);(z,\underset{\sim}{v})) = \left(\int_0^1 [\, l_x(x^*,\underset{\sim}{u}^*,t)z + \Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)]\, dt,\right.$$

$g_x(x^*,t)z(t))$ where $\Delta l(\underset{\sim}{v},\underset{\sim}{u}^*) = l(x^*,\underset{\sim}{v},t) - l(x^*,\underset{\sim}{u}^*,t)$ for all t.

Then it is immediate that the following holds:

Theorem 5.4

If $(x^*,\underset{\sim}{u}^*)$ is an (Φ, Z)-extremal for Problem P3, then there exists a $\lambda_0 \in \mathbb{R}$, $\lambda \in NBV[T;\mathbb{R}]$, not both zero such that

(i) $$\lambda_0 \int_0^1 [\, l_x(x^*,\underset{\sim}{u}^*,t)\, z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)]\, dt$$

$$+ \int_0^1 g_x(x^*(t),t)\, z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t)\, d\lambda(t) \leq 0$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

(by definition of $\mathcal{M}$)

(ii) $\lambda_0 \leq 0$ and λ is non increasing

(iii) λ is constant for $t \in I \subset T$ where I is defined by:

$$I = \{\, t \in T : g(x^*(t),t) < 0 \,\}$$

Since $z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t) = \int_0^t \Phi(s,t)\, \Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)ds$ for all $t \in T$ we have from (i) in Theorem 5.4 that

$$\lambda_0 \int_0^1 l_x(x^*\underset{\sim}{u}^*,t) \int_0^t \Phi(s,t)\Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)ds dt + \lambda_0 \int_0^1 \Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)dt$$

$$+ \int_0^1 g_x(x^*,t) \int_0^t \Phi(s,t)\Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)ds\; d\lambda(t) \leq 0$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

Interchanging order of double integrations we get

$$\lambda_0 \int_0^1 \int_s^1 l_x(x^*,\underset{\sim}{u}^*,t)\Phi(s,t)dt\, \Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)ds + \lambda_0 \int_0^1 \Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)dt$$

$$+ \int_0^1 \int_s^1 g_x(x^*,t)\Phi(s,t)d\lambda(t)\Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)ds \leq 0$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

Let $\psi^T(s) = \lambda_0 \int_s^1 l_x(x^*,\underset{\sim}{u}^*,t)\Phi(s,t)dt + \int_s^1 g_x(x^*,t)\,\Phi(s,t)d\,\lambda(t)$

for all $s \in T$

Hence we have

$$\int_0^1 \{\int_s^1 \lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\Phi(s,t)dt + \int_s^1 g_x(x^*,t)\Phi(s,t)d\lambda(t)\}\,\Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)ds$$

$$+ \lambda_0\int_0^1 \Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)dt \leq 0 \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{G}$$

Substituting for $\psi(s)$ we have

$$\int_0^1 \psi^T(s)\ \Delta f(\underset{\sim}{v},\underset{\sim}{u}^*)ds + \lambda_0\int_0^1 \Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)dt \leq 0 \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{G}$$

Recalling that $\Delta f(\underset{\sim}{v},\underset{\sim}{u}^*) = f(x^*,y^*,\underset{\sim}{v},s) - f(x^*,y^*,\underset{\sim}{u}^*,s)$ all $s \in T$

and $\Delta l(\underset{\sim}{v},\underset{\sim}{u}^*) = l(x^*,\underset{\sim}{v},s) - l(x^*,\underset{\sim}{u}^*,s)$ all $s \in T$

Hence we have

$$\int_0^1 [\psi^T(s)f(x^*(s),x^*(s-\tau),\underset{\sim}{v}(s),s) + \lambda_0 l(x^*(s),\underset{\sim}{v}(s),s)]ds$$

$$\leq \int_0^1 [\psi^T(s)f(x^*(s),x^*(s-\tau),u^*(s),s) + \lambda_0 l(x^*(s),\underset{\sim}{u}^*(s),s)]ds$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

This is the Maximum principle for SCCP with delay.

We now derive an expression for the costate function ψ. Since by definition we have

$$\psi^T(s) = \int_s^1 \lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\Phi(s,t)dt + \int_s^1 g_x(x^*,t)\Phi(s,t)d\lambda(t)$$

Also $\Phi(s,t)$ is the solution of

$$\frac{\delta\Phi(s,t)}{\delta s} = -\Phi(s,t)A_1(s) - \Phi(s+\tau,t)A_2(s+\tau) \qquad \text{for all } s \in [0,t]$$

$$\Phi(t,t) = I \qquad \text{and} \qquad \Phi(s,t) = 0 \qquad \text{for } s > t$$

i.e. we have

$$\Phi(s,t) = I + \int_s^t [\Phi(\sigma,t)A_1(\sigma)+\Phi(\sigma+\tau,t)A_2(\sigma+\tau)]\,d\sigma$$

Substituting this in equation for ψ we get

$$\psi^T(s) = \int_s^1 \lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\,\{ I+\int_s^t[\Phi(\sigma,t)A_1(\sigma)+\Phi(\sigma+\tau,t)A_2(\sigma+\tau)]\,d\sigma\}dt$$

$$+ \int_s^1 g_x(x^*,t)\,\{ I+ \int_s^t [\Phi(\sigma,t)A_1(\sigma)+\Phi(\sigma+\tau,t)A_2(\sigma+\tau)]\,d\sigma\}d\lambda(t)$$

Interchanging order of double integrations

$$\psi^T(s) = \int_s^1 \int_\sigma^1 \lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\Phi(\sigma,t)dt\; A_1(\sigma)d\sigma$$

$$+ \int_s^1\int_\sigma^1 \lambda_0\, l_x(x^*,\underset{\sim}{u}^*,t)\Phi(\sigma+\tau,t)dt\; A_2(\sigma+\tau)d\sigma$$

$$+ \int_s^1 \int_\sigma^1 g_x(x^*,t)\,\Phi(\sigma,t)d\lambda(t)\; A_1(\sigma)d\sigma$$

$$+ \int_s^1\int_\sigma^1 g_x(x^*,t)\Phi(\sigma+\tau,t)d\lambda(t)\; A_2(\sigma+\tau)d\sigma$$

$$+ \int_s^1 \lambda_0\, l_x(x^*,\underset{\sim}{u}^*,t)dt+\int_s^1 g_x(x^*,t)d\lambda(t)$$

i.e.

$$\psi^T(s) = \int_s^1 \{\int_\sigma^1 \lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\Phi(\sigma,t)dt+ \int_\sigma^1 g_x(x^*,t)\Phi(\sigma,t)d\lambda(t)\}\,\cdot$$

$$\cdot\; A_1(\sigma)d\sigma$$

$$+ \int_s^1\{\int_\sigma^1 \lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\,\Phi(\sigma+\tau,t)dt+\int_\sigma^1 g_x(x^*,t)\,\Phi(\sigma+\tau,t)\,\cdot$$

$$\cdot\, d\lambda(t)\}\; A_2(\sigma+\tau)d\sigma$$

$$+ \int_s^1 \lambda_0\; l_x(x^*,\underset{\sim}{u}^*,t)dt+\int_s^1 g_x(x^*,t)d\lambda(t)$$

Now by definition

$$\psi^T(\sigma) = \int_\sigma^1 \lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\Phi(\sigma,t)dt+\int_\sigma^1 g_x(x^*,t)\Phi(\sigma,t)d\lambda(t)$$

Hence we have

$$\int_{\sigma}^{1} \lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\,\Phi(\sigma+\tau,t)\,dt+\int_{\sigma}^{1} g_x(x^*,t)\,\Phi(\sigma+\tau,t)\,d\lambda(t)$$

$$= \int_{\sigma+\tau}^{1}\lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\,\Phi(\sigma+\tau,t)\,dt+\int_{\sigma+\tau}^{1} g_x(x^*,t)\,\Phi(\sigma+\tau,t)\,d\lambda(t)$$

since $\Phi(\sigma,t) = 0$ for $\sigma > t$

$$= \psi^T(\sigma+\tau)$$

Hence using this in above we get

$$\psi^T(s) = \int_s^{1-\tau} \psi^T(\sigma)A_1(\sigma)+ \psi^T(\sigma+\tau)A_2(\sigma+\tau)\,d\sigma$$

$$+ \int_s^1 \lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\,dt+\int_s^1 g_x(x^*,t)\,d\lambda(t) \qquad \text{for } s \in [0,1-\tau]$$

$$\psi^T(s) = \int_s^1 \psi^T(\sigma)A_1(\sigma)$$

$$+ \int_s^1 \lambda_0 l_x(x^*,\underset{\sim}{u}^*,t)\,dt+ \int_s^1 g_x(x^*,t)\,d\lambda(t) \quad \text{for } s \in [1-\tau,1]$$

Combining all the above results, we have just proved the following, which is the Maximum Principle for the State Constrained Control Problem with delay:

Theorem 5.5

Under the above mentioned assumptions on Problem P3, and if $(x^*,\underset{\sim}{u}^*)$ is an optimal pair for P3, there exists a real number λ_0 and a $\lambda \in NBV[T;\mathbb{R}]$, not both zero such that

(i) $\lambda_0 \leq 0$ and λ is nonincreasing

(ii) λ is constant for $I \subset T$ (as defined in Theorem 5.4)

(iii) $\int_0^1 [\psi^T(t)f(x^*,y^*,\underset{\sim}{u}^*,t)+ \lambda_0 l(x^*,\underset{\sim}{u}^*,t)]\,dt$

$$= \max_{\underset{\sim}{v} \in \underset{\sim}{G}} \int_0^1 [\psi^T(t)f(x^*,y^*,\underset{\sim}{v},t)+\lambda_0 l(x^*,\underset{\sim}{v},t)]\,dt$$

where $\psi : T \longrightarrow \mathbb{R}^n$ is the solution of

$$\psi^T(t) = \int_t^1 \psi^T(s) f_x(x^*(s), x^*(s-\tau), u^*(s), s)\, ds$$

$$+ \int_t^1 \psi^T(s+\tau) f_y(x^*(s+\tau), x^*(s), u^*(s+\tau), s+\tau)\, ds$$

$$+ \int_t^1 \lambda_0 l_x(x^*(s), u^*(s), s)\, ds + \int_t^1 g_x(x^*(s), s)\, d\lambda(s)$$

$$\text{for all } t \in [0, 1-\tau]$$

$$\psi^T(t) = \int_t^1 \psi^T(s) f_x(x^*(s), x^*(s-\tau), u^*(s), s)\, ds$$

$$+ \int_t^1 \lambda_0 l_x(x^*(s), u^*(s), s)\, ds + \int_t^1 g_x(x^*(s), s)\, d\lambda(s)$$

$$\text{for all } t \in [1-\tau, 1]$$

$$\psi(1) = 0$$

As in section B8 in Chapter 1, the Maximum Principle may be stated in "pointwise" form as

$$\psi^T f(x^*, y^*, u^*, t) + \lambda_0 l(x^*, u^*, t)$$

$$= \max_{w \in \Omega} \{ \psi^T f(x^*, y^*, w, t) + \lambda_0 l(x^*, w, t) \}$$

$$\text{for a.a. } t \in T$$

APPENDIX D

RELATIONS BETWEEN PROBLEMS P3 AND $P3_K$

In this appendix we obtain necessary conditions for both problems P3 and $P3_K$, and in doing so we deduce certain relations between the two problems. The two problems in question are:

<u>Problem P3</u>

$$\min_{\underset{\sim}{u}} \int_0^1 l(x(t),\underset{\sim}{u}(t),t)dt$$

$$\text{s.t. } \dot{x}(t) = f(x(t),x(t-\tau),\underset{\sim}{u}(t),t) \qquad \text{for a.a.} t \in T$$

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau,0]$$

$$g(x(t),t) \leq 0 \qquad \text{for every } t \in T$$

$$\underset{\sim}{u} \in \underset{\sim}{G}$$

<u>Problem $P3_K$</u> (for $K > 0$)

$$\min_{\underset{\sim}{u}} \int_0^1 l(x(t),\underset{\sim}{u}(t),t)dt + K \max_{0 \leq t \leq 1} \{g(x(t),t),0\}$$

$$\text{s.t } \dot{x}(t) = f(x(t),x(t-\tau),\underset{\sim}{u}(t),t) \qquad \text{for a.a.} t \in T$$

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau,0]$$

$$\underset{\sim}{u} \in \underset{\sim}{G}$$

Assume for the moment that the two problems are equivalent and that $(x^*,\underset{\sim}{u}^*)$ is an optimal pair for P3 and $P3_K$.

We will assume that the hypotheses stated in Chapter 5 holds here as well. Then it is shown in Appendix C that the Maximum Principle for Problem P3 is

1. There exists a real number λ_0 and a $\lambda(t) \in NBV[T;\mathbb{R}]$ not both zero such that

(i) $\lambda_0 \leq 0$ and λ is nonincreasing

(ii) λ is constant for $I \subset T$ where I is defined by

$$I \triangleq \{t \in T \ : \ g(x^*(t),t) < 0\}$$

(iii) $$\int_0^1[\psi^T(t)f(x^*,y^*,\underset{\sim}{u}^*,t) + \lambda_0 l(x^*,\underset{\sim}{u}^*,t)]\,dt$$

$$= \max_{\underset{\sim}{v} \in \underset{\sim}{G}} \int_0^1[\psi^T(t)f(x^*,y^*,\underset{\sim}{v},t)+l(x^*,\underset{\sim}{v},t)]\,dt$$

where $\psi(t) : T \to \mathbb{R}^n$ is the solution of

$$\psi^T(t) = \int_t^1 \psi^T(s)f_x(x^*(s),x^*(s-\tau),\underset{\sim}{u}^*(s),s)\,ds$$
$$+\int_t^1 \psi^T(s+\tau)f_y(x^*(s+\tau),x^*(s),\underset{\sim}{u}^*(s+\tau),s+\tau)ds$$
$$+\int_t^1 \lambda_0 l_x(x^*(s),\underset{\sim}{u}^*(s),s)ds+ \int_t^1 g_x(x^*(s),s)d\lambda(s)$$
$$\text{for all } t \in [0,1-\tau)$$

$$\psi^T(t) = \int_t^1 \psi^T(s)f_x(x^*(s),x^*(s-\tau),\underset{\sim}{u}^*(s),s)\,ds$$
$$+\int_t^1 \lambda_0 l_x(x^*(s),\underset{\sim}{u}^*(s),s)ds+\int_t^1 g_x(x^*(s),s)d\lambda(s)$$
$$\text{for all } t \in [1-\tau,1]$$

$$\psi(1) = 0$$

We now consider Problem $P3_K$. Warga [W4] has shown how this can be reduced to a standard problem ($P3_W$ below). For the sake of completeness we describe this reduction, and describe the Maximum Principle for the reduced problem here. To define $P3_W$ we need to introduce the following:

$$\dot{p}(t) = l(x(t),\underset{\sim}{u}(t),t) \qquad \text{for a.a.} t \in T$$

$$p(0) = 0$$

and $$r(t) = \max_{0 \le s \le 1} \{g(x(s),s),0\} \qquad \text{for } t \in T$$

Then we define $P3_w$ by

<u>Problem $P3_w$</u>

$$\text{Min } p(1)+Kr(1)$$

$$\begin{aligned}
\text{s.t} \quad \dot{x}(t) &= f(x,y,\underset{\sim}{u},t) && \text{for a.a.} t \in T \\
x(t) &= \phi(t) && \text{for all } t \in [-\tau,0] \\
\dot{p}(t) &= l(x,\underset{\sim}{u},t) && \text{for a.a } t \in T \\
p(0) &= 0 \\
\dot{r}(t) &= 0 && \text{for a.a } t \in T \\
r(0) &\ge 0 \\
g(x,t)-r(t) &\le 0 && \text{for every } t \in T \\
\underset{\sim}{u} &\in \underset{\sim}{G}
\end{aligned}$$

Let x^*, $p^*(1)$, $r^*(1)$ be optimal for $P3_w$ and the optimal control being $\underset{\sim}{u}^* \in \underset{\sim}{G}$. We will use the extremal theory stated in Appendix C to derive the Maximum Principle for $P3_w$. This is done as follows:

$$\text{Set } D \triangleq \{x \in \mathcal{A} : \begin{array}{ll} \dot{x}(t) = f(x,y,\underset{\sim}{u},t) & \text{a.a } t \in T \\ x(t) = \phi(t) & \text{for all } t \in [-\tau,0] \\ \underset{\sim}{u} \in \underset{\sim}{G} \;\} \end{array}$$

as in Appendix C and define P, $\tilde{R} \in \mathbb{R}$ by

$$P = \{p(1) : p(1) = \int_0^1 l(x,\underset{\sim}{u},t)dt, \; x \in D, \; \underset{\sim}{u} \in \underset{\sim}{G}\}$$

$$\tilde{R} = \{r \quad : r \ge 0\}$$

Using the same notation as in Appendix C we define $\mathscr{E} \triangleq D \times P \times \tilde{R}$.

Embed $\mathscr{E}$ in $\mathscr{Y} \triangleq C(T; \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}$

$\mathscr{L}$ is set to be $\mathbb{R} \times C[T; \mathbb{R}]$ and define the open cone Z in $\mathscr{L}$ by

$$Z \triangleq \{\xi : \xi \in \mathbb{R}, \xi < 0\} \times \{y \in C[T; \mathbb{R}] : y(t) < 0 \text{ all } t\}$$

Letting $\phi : \mathscr{Y} \to \mathscr{L}$ be defined by

$$\phi(x,p,r) \triangleq (p(1)+Kr(1)-p^*(1)-Kr^*(1), g(x,t)-r(t))$$

it is easy to see that x^*, p^*, r^* is a (ϕ, Z)-extremal for Problem $P3_w$.

We have that

$$\delta\phi((x^*,p^*,r^*);(z,h,s)) = (h(1)+Ks-p^*(1)-Kr^*, g_x(x^*,t)z+s)$$

and $Q \triangleq \{ z^{\underset{\sim}{u}^*, \underset{\sim}{v}}(t) \; ; \text{ for all } t \in T, \underset{\sim}{v} \in \underset{\sim}{G} \}$ is a first order convex approximation to D at x^* (as in Appendix C). Using a similar method we can show that

$$H \triangleq \{h(1) : h(1) = \int_0^1 [\, l_x(x^*, \underset{\sim}{u}^*, t) z(t) + l(x^*, \underset{\sim}{v}, t)\,] dt$$

$$z \in Q, \; \underset{\sim}{v} \in \underset{\sim}{G} \;\}$$

is a first order convex approximation to P at x^*. Then defining $\mathscr{M}$ by $\mathscr{M} \triangleq Q \times H \times \tilde{R}$, all the conditions in Appendix C are satisfied. Therefore we can deduce that the following is true for the above choices of ϕ, Z , $\mathscr{E}$, $\mathscr{L}$, $\mathscr{Y}$, $\mathscr{M}$, $\delta\phi$:

There exists a nonzero $l \in \mathscr{L}^*$ such that

(i) $\quad l \circ \delta\phi((x^*,p^*,r^*) \; ; \; (z,h,s)) \leq 0$

for all $(z,h,s) \in \mathscr{M}$

(ii) $\quad l(\sigma) \geq 0 \qquad$ for all $\sigma \in \bar{Z}$

(iii) $\quad l \circ \Phi(x^*,p^*,r^*) = 0$

Consider (i). Since $\mathscr{L}^* = \mathbb{R} \times NBV[T;\mathbb{R}]$ l is of the form (λ_0, λ) and therefore (i) becomes

$$\lambda_0(h(1)+Ks-p^*(1)-Kr^*)+\int_0^1(g_x(x^*,t)z+s)d\lambda(t)\leq 0$$

for all $z \in Q$, all $h \in H$, all $s \in \tilde{R}$

Since this holds for all $(z,h,s) \in \mathscr{M}$ we must have

(a) $\quad \lambda_0 h(1) \leq 0 \quad$ for all $h(1) \in H$

(b) $\quad \lambda_0 Ks + \int_0^1 s d\lambda(t) \leq 0 \quad$ for all $s \in \tilde{R}$

(c) $\quad \int_0^1 g_x(x^*,t)zd\lambda(t) \leq 0 \quad$ for all $z \in Q$

From (b) we get $\lambda_0 K + \int_0^1 d\lambda(t) \leq 0$

i.e. $\quad T.V(\lambda) \leq -\lambda_0 K$

Now $r^*=0$ since $r^* = \max_{0\leq t\leq 1} \{g(x^*,t),0\}$ and x^* is the solution, therefore $g(x^*,t)\leq 0$ for all $t \in T$.

Then the above equation becomes

$$\lambda_0(h(1)-p^*(1))+\int_0^1 g_x(x^*,t)zd\lambda(t)\leq 0$$

for all $h \in H$, $z \in Q$

By definition of H, and using the form of $h(1)$ for all $h \in H$ we get

$$\lambda_0\{\int_0^1[l_x(x^*,\underset{\sim}{u}^*,t)z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t)+l(x^*,\underset{\sim}{v},t)]dt-p^*(1)\}$$

$$+\int_0^1 g_x(x^*,t)z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t)d\lambda(t)\leq 0$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

Now $p^*(1) = \int_0^1 l(x^*,\underset{\sim}{u}^*,t)dt$ (solution of $P3_w$).

Letting $\Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)=l(x^*,\underset{\sim}{v},t)-l(x^*,\underset{\sim}{u}^*,t)$

and $\quad l_x(\underset{\sim}{u}^*) = l_x(x^*,\underset{\sim}{u}^*,t)$ $\quad$ we get

$$\lambda_0 \int_0^1 [\, l_x(\underset{\sim}{u}^*) z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t) + \Delta l(\underset{\sim}{v},\underset{\sim}{u}^*)\,]dt + \int_0^1 g_x(x^*,t) z^{\underset{\sim}{u}^*,\underset{\sim}{v}}(t)\, d\lambda(t) \leq 0$$

for all $\underset{\sim}{v} \in \underset{\sim}{G}$

The rest of the proof for deriving the Maximum Principle for $P3_W$ follows exactly along the lines for P3 shown in Appendix C. Using this procedure we arrive at a Maximum Principle which is the same as for Problem P3 but with one extra condition, which is that $T.V(\lambda) \leq -\lambda_0 K$. From (ii) above we can deduce that $\lambda_0 \leq 0$ and for normal problems we can normalize λ_0 to -1, therefore we get $T.V(\lambda) \leq K$, i.e. the multiplier associated with the state constraint has its variation bounded by K . Hence if K is chosen so that $K \geq T.V(\lambda)$ where λ is the multiplier in the Maximum Principle for P3 we can deduce that the necessary conditions for optimality for P3 and $P3_K$ are equivalent.

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