ACOUSTIC TRANSMISSION AND RADIATION FOR TWO-DIMENSIONAL DUCTS WITH FLOW

by

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ABSTRACT

Two topics in the theory of sound are covered.

In Part I the generation and transmission of sound in two dimensional ducts is investigated. The ducts contain a basic uniform background flow. In one case the duct is taken to have parallel plane walls, and in another case the duct contains a finite section of greater width. The sound is generated by a piston vibrating sinusoidally, set in one of the duct walls, perpendicular to the flow. The sound field is found and the force on the piston and the net powerflow into the duct are also calculated. The calculations are undertaken with no viscous effects being taken into account.

In Part II ducts with elastic walls or with semi-infinite elastic - semi-infinite rigid walls are considered. The sound is generated by a point source or an incoming wave from the rigid part of the duct, respectively. Calculations are made for the acoustic field at points inside the duct and also for the radiating sound field outside the duct. Interesting effects, amongst them "Leaky waves", are found in the radiating field. These results are for the limit of light fluid loading. Extensive use is made of the Wiener-Hopf technique.

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CONTENTS

INTRODUCTION

In many acoustical problems of practical interest the presence of a background flow is significant. Obvious examples are those of aircraft in flight and the flows of gases and liquids through pipes.

Previous work such as that by MUNGUR & GLADWELL (24), MORFEY (22), FFOWCS WILLIAMS & LOVELY (12), DOWLING (11), TAYLOR (26) and lastly LEPPINGTON & LEVINE (17) has revealed some surprising effects due directly to the presence of flow. In order to extend our basic understanding of flow phenomena it is desirable to study in detail some of the fundamental and relatively simple model problems that are amenable to mathematical analysis.

This thesis is concerned with the effects of steady background flow on duct acoustics. It falls naturally into two separate parts.

In Part I steady flow down a two-dimensional duct is disturbed by time-harmonic transverse vibrations of a piston set in one of its sidewalls.

Chapter 1 deals with the simplest case where the duct has parallel walls. A linearised boundary condition, to account for the piston motion, leads to an approximate solution throughout most of the flow region but near the ends there are essential nonlinear effects which have to be dealt with by detailed local analysis. This leads to the more interesting geometry in Chapter 2. Here the abrupt widening and narrowing of the duct generates further reflection and resonance phenomenon.

The acoustic fluctuations in these problems have been caused by the motions of a side panel vibrating like a piston. A more realistic model for panel vibration has to consider its flexibility and this is taken up in Part II of the thesis which consists of Chapters 3 and 4. These chapters analyse the interaction between flow and bounding elastic surfaces.

The simplest prototype duct problem is that where both sidewalls of a two-dimensional duct are elastic. This is studied in detail in Chapter 3. In order to analyse the interaction between flows and an elastic surface with an adjoining rigid surface a semi-infinite rigid/semi-infinite elastic duct is considered in Chapter 4. The complexity of the problem is such that tractable exact solutions have not been found, so in both cases emphasis was placed on a limit of specific interest, namely that of light fluid loading.

Explicit expressions are obtained for reflection and transmission coefficients and for interesting beaming effects that persist to relatively large distances from the duct joint.

PART I

THE GENERATION OF SOUND BY PISTONS IN DUCTS CONTAINING FLOW

INTRODUCTION

The generation of sound in ducts has been studied by many authors, MÖHRING (27) and MORFEY (22) are amongst the few who have considered the presence of a basic uniform background flow. It is the generation and transmission of sound in the presence of a basic uniform background flow that is investigated in the first part of this thesis.

It is the aim of this work to undertake some detailed analysis of one of the most basic problems of this sort in order that we might understand more fully the results of problems of a more complex nature. To this end,in Chapters 1 and 2,we consider two-dimensional ducts that have walls that are both rigid and parallel. Sound is produced by the time harmonic oscillations of a piston set in a sidewall of the duct.

In Chapter 1 we study the simplest case of the piston set in an infinite duct that has parallel walls and contains flow. The acoustic disturbances upstream and downstream of the piston are identified and these results used to evaluate the force on the piston and the total power flow into the duct. Either the Green's function method or Fourier transforms can be used for this analysis along with the method of Matched Asymptotic Expansions. The inner regions are those around the piston ends and the outer region being the rest of the duct.

In Chapter 2 the analysis of Chapter 1 is extended to a more complex geometry - that of an expansion chamber -

still with a basic background flow. The flow around the piston is analysed in exactly the same way as before but Matched Asymptotic Expansions are also used to analyse the flow across the duct expansions.

The acoustic disturbances are found in the expansion chamber and in the duct leading to and from it. Once again the force on the piston and the power flow into the expansion chamber and hence into the duct are found from these results.

CHAPTER 1 : The generation of sound by a piston in a duct containing flow

1.1 INTRODUCTION AND SUMMARY

In this chapter the velocity field of, the force on, and the work done by, an oscillating piston set in a side wall of a two-dimensional duct containing a basic uniform flow are modelled. In the linearised model, with inviscid and compressible fluid, the piston is represented by an oscillating line source at each end with an oscillating line source distribution in between. This linearisation of the piston effect is, however, not adequate for the calculation of the force near the piston ends. In these regions the shape of the piston has to be taken into account by a local transformation of the basic problem followed by a match with the first model.

In section 1.2 a rigorous definition of the non-linear problem is given. This is then linearised to give the model problem to be investigated. The boundary conditions, with particular reference to the 'top hat' piston profile, are also discussed here.

In section 1.3 one of the two possible methods of attack (the Green's function method) is pursued. The other method, Fourier transforms, is investigated in Appendix 1. In section 1.4 the theory for the work done (in the linearised theory) is developed. In section 1.5 the velocity potential calculated in section 1.3 is used along with the theory of section 1.4 to find the work done by the piston.

In section 1.6 inner and outer regions, near the piston ends, are set up. The inner solution is found in section 1.7 and then used in section 1.8, along with the outer solutions from section 1.3, to find an approximation to the force on the piston.

Discussion of the results takes place in section 1.9.

1.2 PROBLEM DEFINITION AND LINEARISATION

Inviscid, compressible fluid flows adiabatically in an infinite two-dimensional duct that has rigid parallel walls. In cartesian co-ordinates (x,y) the walls are at y = 0 and y = d. The basic uniform flow, parallel to the walls and of constant magnitude U with velocity potential Ux, is perturbed by the small, time-harmonic, oscillations of a piston that is forced to vibrate about its mean position in the duct wall. As shown in Figure 1 the piston is set in the lower baffle between x = -a and x = a and its displaced surface is given by

$$h(x,t) = \mathrm{Re}\{h(x)e^{-i\omega t}\} \quad . \tag{1.2.1}$$

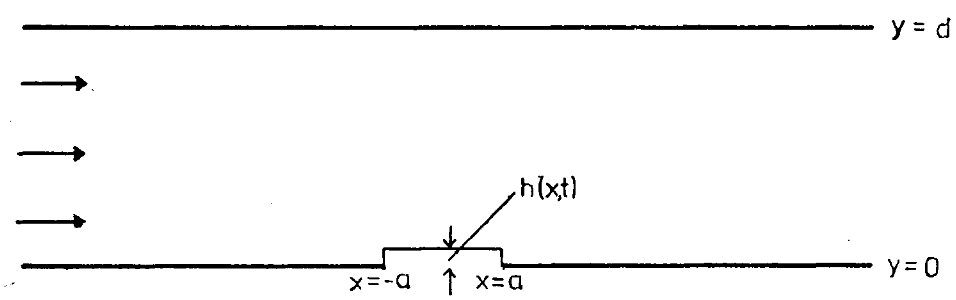

Figure 1

If ϕ is the total velocity potential for the fluid in the duct, then writing

$$\phi = Ux + \phi_1 \ , \tag{1.2.2}$$

ϕ_1 is the velocity potential of the disturbance, taken to be small compared to the basic flow. We now let

$$\phi_1 = \mathrm{Re}\{\phi_2 \ e^{-i\omega t}\} \tag{1.2.3}$$

to take into account the periodic response of the disturbance to the piston.

LIGHTHILL [(19) equation (61)] gives, for the nonlinear equation of motion of the fluid ,

$$c^2\nabla^2\phi = \frac{\partial^2\phi}{\partial t^2} + 2\,\frac{\partial\phi}{\partial x_i}\,\frac{\partial^2\phi}{\partial x_i\partial t} + \frac{\partial\phi}{\partial x_i}\,\frac{\partial\phi}{\partial x_j}\,\frac{\partial^2\phi}{\partial x_i\partial x_j} \qquad (1.2.4)$$

where c is the local sound speed and ∇^2 the Laplacian operator.

The boundary condition on the duct walls is that there is no flow through them;

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi_1}{\partial y} = \frac{\partial\phi_2}{\partial y} = 0 \quad \text{for all } x; y=d; \quad \text{for } |x| > a,\ y = 0\,. \qquad (1.2.5)$$

The corresponding condition on the piston surface is that the surface specification's material derivative is zero, that is

$$\frac{D}{Dt}\{y-h(x,t)\} = 0 \qquad \text{on} \qquad y = h(x,t)\,. \qquad (1.2.6)$$

The velocity potential is calculated, to first order in h(x,t) (taken to be small compared to υ) by linearising the equation of motion, (1.2.4), and the boundary condition on y = h(x,t) (equation (1.2.6)) to one on y = 0 by neglecting terms of order $(\phi_1)^2$.

After such linearisation equation (1.2.4) becomes

$$\left[\nabla^2 + \left(k + im\,\frac{\partial}{\partial x}\right)^2\right]\phi_2 = 0 \qquad (1.2.7)$$

where $k = \frac{\omega}{c_o}$, a wave number, and $m = \frac{U}{c_o}$ the Mach number. The total velocity $\underline{U}_T$, the basic flow $\underline{U}$, and the perturbations $\underline{u}$ are given by

$$\underline{U}_T = \left[U + \frac{\partial\phi_1}{\partial x}, \frac{\partial\phi_1}{\partial y} \right], \ \underline{U} = (U,o), \ \underline{u} = \left[\frac{\partial\phi_1}{\partial x}, \frac{\partial\phi_1}{\partial y} \right],$$

with c_o a constant (let $c_o = c$). (1.2.8)

Linearising equation (1.2.6), assuming h, $\frac{\partial\phi_1}{\partial x}$ and $\frac{\partial h}{\partial x}$ are all small gives

$$\frac{\partial\phi_1}{\partial y} = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \quad \text{on} \quad y = 0 \ . \tag{1.2.9}$$

When expressed in stretched coordinates $x_1 = x(1-m^2)^{-\frac{1}{2}}$, $y_1 = y$, $K = k(1-m^2)^{-\frac{1}{2}}$, and with

$$\phi_2(x,y) = \phi_3(x_1,y_1)\ \exp\{-imKx_1\} \tag{1.2.10}$$

equations (1.2.5), (1.2.7) and (1.2.9) become

$$\frac{\partial\phi_3}{\partial y_1} = 0 \quad \text{for all } x_1, y_1 = d; |x_1| > \frac{a}{(1-m^2)^{\frac{1}{2}}} ; \quad y_1 = 0 \tag{1.2.11}$$

$$(\nabla_1^2 + K^2)\phi_3 = 0 \tag{1.2.12}$$

$$\frac{\partial}{\partial y_1}\{\phi_3\ e^{-imKx_1}\} = \frac{\partial h}{\partial t} + \frac{U}{(1-m^2)^{\frac{1}{2}}}\ \frac{\partial h}{\partial x_1} \tag{1.2.13}$$

where ∇_1^2 is the Laplacian with respect to the new coordinates (x_1,y_1) and

$$\phi_1(x,y) = \text{Re}\{\phi_3(x(1-m^2)^{-\frac{1}{2}},y)\ \exp\{-i(\omega t + \frac{mkx}{1-m^2})\}\} \tag{1.2.14}$$

For some piston profiles, in particular the 'top hat' as described below, the solution breaks down at the piston ends because the perturbation is no longer small compared to the basic flow. In these regions the velocity potential will be calculated using matched asymptotic expansions based on the piston profile h(x,t) and the Mach number m being small.

The 'top hat' profile is defined as

$$h(x,t) = h_0\{H(x+a)-H(x-a)\}\cos\omega t \tag{1.2.15}$$

where H is the Heaviside step function, whereas the general piston profile has a general h(x,t) such that $h(x,t) = h(x)\cos\omega t$ and $\max_x |h(x)| = h_0$.

For the 'top hat' profile differentiation with respect to t gives no special problems but formal differentiation with respect to x gives

$$\frac{\partial}{\partial x} h(x,t) = h_0\{\delta(x+a)-\delta(x-a)\}\cos\omega t \tag{1.2.16}$$

where $\delta(x)$ is the Dirac delta function. This violates the condition that $\frac{\partial h}{\partial x}$ is small and the potential calculated by this procedure is regarded as an 'outer expansion' in the language of matched asymptotic expansions and equation (1.2.8) becomes

$$\frac{\partial\phi_1}{\partial y} = \mathrm{Re}\{c\,h_0\,[m\{\delta(x+a)-\delta(x-a)\}-ik\{H(x+a)-H(x-a)\}]e^{-i\omega t}\} \tag{1.2.17}$$

Although this has singularities at x = a and x = -a these infinite slope singularities are integrable and in the calculations of the work done by the piston they only appear inside integrals so they can be accepted and used in sections (1.4) and (1.5). However in order to calculate the force on the piston, the flow in the small regions around the piston ends, the inner approximation, will have to be investigated further.

The latter half of expression (1.2.17) represents a distribution of line sources between x = -a and x = a. These sources, of constant strength proportional to ωh_0, represent the flux of the piston's volume forced into and then pulled out of the flow, linearised to its mean position on the baffle.

The first half of expression (1.2.17), two pulsating line sources (at x = -a and x = a), represents the blocking effect that the piston's ends would have were they to actually be present in the flow. Again linearisation back to the mean position on the baffle has taken place.

From here it is possible to proceed in one of two ways.

Firstly we could find the Green's function for the problem and use that to find the linearised velocity potential. Alternatively we could take Fourier transforms of the equation, solving the resulting inversion for the velocity potential.

The first method, applied to the 'top hat' profile, will be detailed in the main text whereas the second method, for a generalised piston profile, is given in Appendix 1.

1.3 GREEN'S FUNCTION METHOD

In order to employ the Green's function method we must first find the Green's function $G(x_1,y_1,X_0,Y_0)$ for the problem. The function $G(x_1,y_1,X_0,Y_0)$ satisfies the following equations and boundary conditions :—

$$(\nabla_1^2+K^2)G = \delta(x_1-X_0)\ \delta(y_1-Y_0), \tag{1.3.1}$$

an outgoing wave form at infinity,

$$\frac{\partial G}{\partial y_1} = 0 \quad \text{on the duct walls.}$$

Then ϕ_3 is given by

$$\phi_3 = \int_{s_0} G \frac{\partial\phi}{\partial n}\, ds_0 \tag{1.3.2}$$

where s_0 is all the duct surfaces (X_0,Y_0) and n is the normal there.

G can be found, by a superposition of separable solutions, to be

$$G(x_1,y_1,X_0,Y_0) = \frac{1}{2iKd}\exp\{-iK|x_1-X_0|\} - \frac{1}{d}\sum_{n=1}^{\infty}\frac{1}{\gamma_n} \exp\{-\gamma_n|x_1-X_0|\}\cos(\frac{n\pi}{d}y_1)\cos(\frac{n\pi}{d}Y_0) \tag{1.3.3}$$

where

$$\gamma_n = ((\frac{n\pi}{d})^2-K^2)^{\frac{1}{2}} \quad \text{for } \frac{n\pi}{d} > K, \quad -i(K^2-(\frac{n\pi}{d})^2)^{\frac{1}{2}} \quad \text{for } \frac{n\pi}{d} < K \tag{1.3.4}$$

and it is assumed that K is not a multiple of $\frac{n\pi}{d}$.

The velocity potential ϕ_1 can then be found by taking the boundary conditions from equation (1.2.17) (after transforming into stretched coordinates) with G from equation (1.3.3) and substituting into equation (1.3.2) to give after integration:-

For $-a < x < a$

$$\phi_1 = -\frac{\omega h_0}{k^2 d}\{\sin\omega t\ [\ \cos(\frac{k(x-am)}{1-m^2})\sin(\frac{k(a-mx)}{1-m^2} - \omega t)\}$$

$$+ \sum_{n=1}^{N} \frac{h_0}{dB_n(1-m^2)^{\frac{1}{2}}}\{(\frac{\omega}{F_n} + U)\sin(F_n(x-a) + \omega t)$$

$$- (\frac{\omega}{E_n} + U)\sin(E_n(x+a)+\omega t)+\omega(\frac{1}{E_n} - \frac{1}{F_n})\sin\omega t\}\cos\frac{n\pi y}{d}$$

$$+ \sum_{N+1}^{\infty} \mathrm{Re}\{\frac{h_0}{dB_n(1-m^2)^{\frac{1}{2}}}[(\frac{\omega}{F_n} + U)\ \exp\{-iF_n(x-a)\} - (\frac{\omega}{E_n}+ U)\ \exp\{-iE_n(x+a)\}$$

$$+ \omega(\frac{1}{E_n} - \frac{1}{F_n})]e^{-i\omega t}\ \cos\frac{n\pi y}{d} \qquad (1.3.5)$$

where

$$F_n = \frac{mk}{1-m^2} + \frac{B_n}{(1-m^2)^{\frac{1}{2}}} \quad ; \quad E_n = \frac{mk}{1-m^2} - \frac{B_n}{(1-m^2)^{\frac{1}{2}}} \quad \text{for} \quad n \le N$$

$$F_n = \frac{mk}{1-m^2} + \frac{iB_n}{(1-m^2)^{\frac{1}{2}}} \quad ; \quad E_n = \frac{mk}{1-m^2} - \frac{iB_n}{(1-m^2)^{\frac{1}{2}}} \quad \text{for} \quad n \ge N$$

$$N = [\ \frac{kd}{(1-m^2)^{\frac{1}{2}}\pi}] \quad ; \quad B_n = |K^2 - (\frac{n\pi}{d})^2|^{\frac{1}{2}} \qquad (1.3.6)$$

Here we must exclude the resonance condition $K = \frac{n\pi}{d}$, since in such circumstances the assumption that a time-periodic solution exists is invalid. For this case one would have to study the initial value problem, and this is not pursued here.

For $x > a$

$$\phi_1 = -\frac{c^2 h_0}{\omega d}\ \sin(\frac{ka}{1+m})\ \cos\{\omega t - \frac{kx}{1+m}\} \qquad (1.3.7)$$

$$- \sum_{1}^{N} \frac{2h_0}{dB_n(1-m^2)^{\frac{1}{2}}}\{\frac{\omega}{E_n} + U\}\ \cos(\frac{n\pi y}{d})\ \sin(aE_n)\cos(\omega t+xE_n)$$

$$+ \sum_{N+1}^{\infty} \exp\{-xB_n\}(1-m^2)^{-\frac{1}{2}}\ \cos(\frac{n\pi y}{d})\{\text{constant}\times\cos(\frac{mkx}{1-m^2} + \omega t)$$

$$+ \text{constant}\times\sin(\frac{mkx}{1-m^2} + \omega t)\}$$

ϕ_1 for $x < -a$ is similar to (1.3.7), the main difference being that E_n is replaced by F_n.

The far field downstream disturbance potential thus consists of a y-independent travelling wave, the first term of expression (1.3.7), a set (the sum up to N) of y-dependent travelling waves, and an infinite set (the sum from N+1 to ∞) of terms that decay exponentially with distance from the origin. The upstream potential, similar in nature to the downstream potential, contains the same types of terms in the same numbers.

Having found the velocity potential in the vicinity of the piston as well as upstream and downstream of it, we now calculate the rate of work done by the piston on the fluid, that is the amount of work needed to be done to force the piston to sustain this periodic motion.

1.4 RATE OF WORK DONE BY PISTON: THEORY

We will now calculate the net, time-averaged, energy flux into the duct which is the average rate of work done by the piston (W).

$$W = \left\langle \int_{s_1} p \frac{\partial h}{\partial t} ds_1 \right\rangle \tag{1.4.1}$$

where <> indicates time average, p is the pressure on the piston and s_1 is thepiston's surface linearised to y = 0.

If ρ_0 is the fluid'suniform density at infinity then Bernoulli's equation gives

$$p = \text{constant} - \rho_0 \left\{\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i}\right)\right\}. \tag{1.4.2}$$

Taking the reference pressure at infinity to be zero and denoting the excess pressure by pressure p we have

$$p = \rho_0 \left\{\frac{U^2}{2} - \frac{\partial \phi}{\partial t} - \frac{1}{2}\left(\frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i}\right)\right\}. \tag{1.4.3}$$

This then gives, after linearisation with respect to the perturbation magnitude,

$$p = -\rho_0 \left\{\frac{\partial \phi_1}{\partial t} + U \frac{\partial \phi_1}{\partial x}\right\}. \tag{1.4.4}$$

LIGHTHILL[(19) equation 65] tells us that

$$\frac{\partial E^*}{\partial t} + \frac{\partial N_i^*}{\partial x_i} = 0, \tag{1.4.5}$$

where

$$\underline{N}^* = \left(\underline{u}.\underline{U} + \frac{c^2}{\rho_0}\rho\right)\left(\rho_0 \underline{u} + \underline{U}\rho\right) \tag{1.4.6}$$

and

$$E^{*} = \frac{1}{2}\rho_0 u^2 + \frac{1}{2}\frac{c^2}{\rho_0}\rho^2 + (\underline{u}.\underline{U})\rho \qquad (1.4.7)$$

and ρ is the excess density.

One can think of E^* and $\underline{N}^*$ as giving a measure of the excess energy and energy flux respectively; such an interpretation requires some caution, however, since second order terms in the acoustic fluctuation have been ignored. As E^* and $\underline{N}^*$ are of second order, if a true measure of the excess energy and energy flux were required (which here they are not) the second order terms would have to be included.

Now to order $(h(x,t))^2$, that is to first order, equation (1.4.1) gives

$$W = \left< \int_{s_1} N_i^* \, dx_i \right> \qquad (1.4.8)$$

which after substitution for ρ,p,U,$\underline{u}$ becomes

$$W = \left< -\int_{s_1} \rho_0 \frac{\partial\phi_1}{\partial t}\left(\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}\right)dx \right> + O(h)^3 \quad . \qquad (1.4.9)$$

It can be shown that $\left< \int_{s_1} N_i^* dx_i \right>$ is path independent. That is to say $\left< \int_s \underline{N}^* \underline{ds} \right>$ over any path s from x = -a to x = a in or bounding the fluid will give the same result.

Integration of (1.4.5) over any closed fluid volume v, with surface s yields

$$\int_v \frac{\partial N_i^*}{\partial x_i} dv = \int_s N_i^* ds_i = \frac{\partial}{\partial t}\int_v E^* \, dv \qquad (1.4.10)$$

but as the time average of the right hand side of equation (1.4.10) is zero it follows that

$$< \int_s N_i{*}ds_i > = 0.$$

The integral (1.4.8) can thus be evaluated over any surface S from A to B (Figure 2). In particular, since $\underline{N}^*$ has zero normal component on the rigid duct walls

$$W = \int_{FE} + \int_{CD} \tag{1.4.13}$$

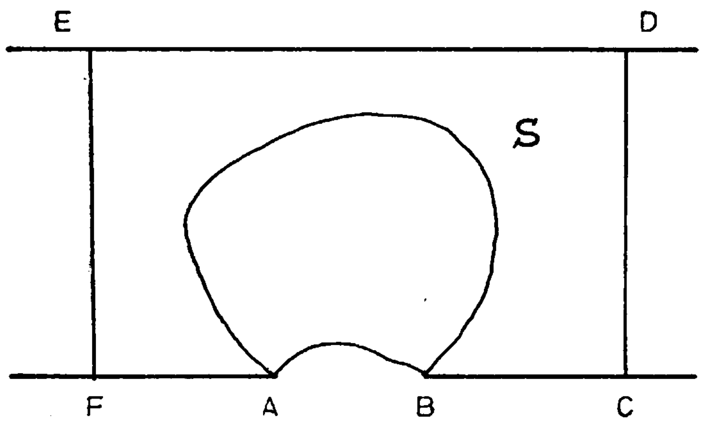

Figure 2

In general the rate of work done by the piston, W, is given

$$W = < \int_S N_i{*}ds_i > \tag{1.4.14}$$

and although $\underline{N}^*$ and E* are not the true energy flux or excess energy their integrated values give the true power flow.

Proceeding in a different way, since other second order terms have to be properly accounted for, CANTRELL and HART (4) showed, by considering the equations of motion up to and including second order terms, that the net, time-averaged, contribution of these other terms in the velocity to the integral was zero. Such a powerful and algebraically complex proof, involving all second order terms, is not required here.

1.5 RATE OF WORK DONE BY PISTON: CALCULATION

Taking the integral form of the perturbation velocity potential (A1.6) and substituting into equation (1.4.9) yields for the net, time-averaged work done by the piston

$$W = \frac{\rho_0\omega}{16\pi^3} \mathrm{Re}\left|\iiiint_{-\infty}^{+\infty} \int_L \frac{i \cosh\gamma d}{\gamma \sinh\gamma d} (U\tau+\omega)(Ur-\omega)\tilde{H}(r)\tilde{H}(\tau) \times \right.$$

$$\left. \times \exp i\{-\tau y+sy-sx-rx\} d\tau dr dy dx ds \right| . \qquad (1.5.1)$$

The x and y and the r and τ integrations can be undertaken without specific knowledge of the profile transform $\tilde{H}$ (the Fourier transform of h) to give

$$W = -\frac{\rho_0\omega}{4\pi} \mathrm{Re} \int_L (Us+\omega)^2 \tilde{H}(s)\tilde{H}(-s) \frac{i \cosh \gamma d}{\gamma \sinh \gamma d} ds . \qquad (1.5.2)$$

$\tilde{H}(s)\tilde{H}(-s)$ is an even real function and if the piston is finite $\tilde{H}(s)$ is also analytic. The path of integration, L, is along the real axis with indentations in the upper half plane for poles to the left of the origin and in the lower half plane for poles to the right. The integrand is wholly imaginary for real s so W can be evaluated on sight to be πi times the sum of the residues (times the appropriate sign):-

$$W = \frac{\rho_0\omega}{4d(1-m^2)^{\frac{1}{2}}} \left\{\frac{(1-m^2)^{\frac{1}{2}}}{2k} \left| \left(\frac{\omega}{1+m}\right)^2 \tilde{H}\left(\frac{k}{1+m}\right) \tilde{H}\left(\frac{-k}{1+m}\right) + \left(\frac{\omega}{1-m}\right)^2 H\left(\frac{k}{1-m}\right) H\left(\frac{-k}{1-m}\right) \right| \right.$$

$$\left. + \sum_1^N \frac{1}{B_n} \left| (UE_n+\omega)^2 \tilde{H}(E_n) \tilde{H}(-E_n) + (UF_n+\omega)^2 \tilde{H}(F_n) \tilde{H}(-F_n) \right| \right\} .$$

$$(1.5.3)$$

For the 'top-hat' profile where $h(x) = h_0[H(x+a)-H(x-a)]$

$$W = \frac{h_0^2 \rho c}{2d} \{\sin^2(\frac{ak}{1+m}) + \sin^2(\frac{ak}{1-m})\}$$

$$+ \frac{h_0^2 \rho_0 \omega}{d(1-m^2)^{\frac{1}{2}}} \{ \sum_1^N \frac{1}{B_n} |(\frac{\omega}{E_n} + U)^2 \sin^2 aE_n + (\frac{\omega}{F_n} + U)^2 \sin^2 aF_n|\}. \quad (1.5.4)$$

Alternatively these results could be obtained by substituting the expansion for ϕ_1 into either integral for W and then integrating each individual term of the expansion separately. The results can also be obtained, because of the relationships documented in section 1.4, by calculating W as in (1.4.13) using either the Fourier transform integral or the series expansion for ϕ_1 and then integrating over any two surfaces x = constant across the duct (one upstream and one downstream).

We see in equations (1.5.3) and (1.5.4) that the decaying modes of the potential do not contribute to the work done. The surfaces of integration do not have to be sufficiently far up or down the duct for the decaying modes to be small as their contribution is identically zero. The contributions from the travelling modes are easily identifiable as to which mode they come from. The first terms of expressions (1.5.3) and (1.5.4) come from the plane waves which do not depend for their existence on the size of k relative to the duct width d and are always present even for very small d.

If U = 0 (no flow) then expression (1.5.3) gives the elementary result for a piston in a baffle in a duct namely

$$W = \frac{\rho_0 \omega^3}{2d} \{\frac{1}{2k} \tilde{H}(k) \tilde{H}(-k) + \sum_{n=1}^N \overline{B}_n^{-1} \tilde{H}(\overline{B}_n) \tilde{H}(-\overline{B}_n)\} \quad (1.5.5)$$

where $\overline{B}_n = |k^2 - (\frac{n\pi}{d})^2|^{\frac{1}{2}}$

and, more accurately, expression (1.5.4) for small m becomes

$$W = \frac{h_0^2 \rho_0}{d} \{c^3 (\sin ka)^2 + 2\omega^3 \sum_{n=1}^{N} \overline{B}_n^{-3} (\sin a\,\overline{B}_n)^2\}$$

$$+ \; m^2 \rho_0 c^2 h_0^2 \omega d^{-1} \{ka^2 \cos 2ka + a \sin 2ka$$

$$+ \sum_1^N \overline{B}_n^{-1} | \tfrac{1}{2} (\tfrac{n\pi}{d})^2 (2(\tfrac{n\pi}{d})^2 + k^2)\overline{B}_n^{-4} + ak^2 \overline{B}_n^{-3} (2k^2 - 5(\tfrac{n\pi}{d})^2) \sin 2a\overline{B}_n$$

$$- \overline{B}_n^{-2} (\tfrac{1}{2} (\tfrac{n\pi}{d})^2 (2(\tfrac{n\pi}{d})^2 + k^2)\overline{B}_n^{-2} - 2a^2 k^4) \cos 2a\overline{B}_n \; |\} + 0(m^4) .$$

(1.5.6)

The difference between expressions (1.5.3) and (1.5.5) and (1.5.4) and (1.5.6) (with m = 0) is the extra amount of work done by the piston against the flow whilst maintaining its periodic motion. In expression (1.5.6) the effect of the flow, for small m, is much clearer in that it is represented by the additional terms.

In the limits $ka \to 0$ and $d \to \infty$ this problem reduces to the elementary one of a compact piston in an infinite baffle bounding a semi-infinite fluid region. For this problem the work rate, W_a, is known to be, HARDING-PAYNE (13)

$$W_a = \rho_0 h_0^2 a^2 \omega^3 (1 - m^2)^{-5/2} \; [1 + \frac{m^2}{2}] + O((ka)^4) \qquad (1.5.7)$$

and indeed expression (1.5.4) does reduce to this in the appropriate limits.

1.6 INNER AND OUTER REGIONS

An approximation to the force, F, on the piston will now be calculated. F is the integrated pressure across the piston surface and is therefore given by

$$F = \int_s p\,ds \quad . \tag{1.6.1}$$

In the linearized theory of sections 1.2 and 1.3 the velocity potential has a singularity proportional to log r (where r is the distance from either piston end) so that p has a non-integrable singularity of order $(x \pm a)^{-1}$ as $x \to \pm a$. This arises from the linearisation of the boundary conditions, which is clearly invalid near the piston ends as already mentioned. We thus regard the previously calculated potential as an "outer expansion" that should only hold for $r \gg h_0$. In the small inner region at each end of the piston, $r \ll a$, we take the full boundary conditions and calculate an approximation to the potential, based on the local dominant small length scale h_0, using the method of matched asymptotic expansions.

Consider first the flow around the right hand piston end x = a. Rescaling the flow with respect to h(t) such that

$$X = \frac{x-a}{h(t)} \quad ; \quad Y = y/h(t) \tag{1.6.2}$$

gives a new problem, flow over a stationary step as shown in Figure 3. The problem with flow has thus been replaced by a new static one

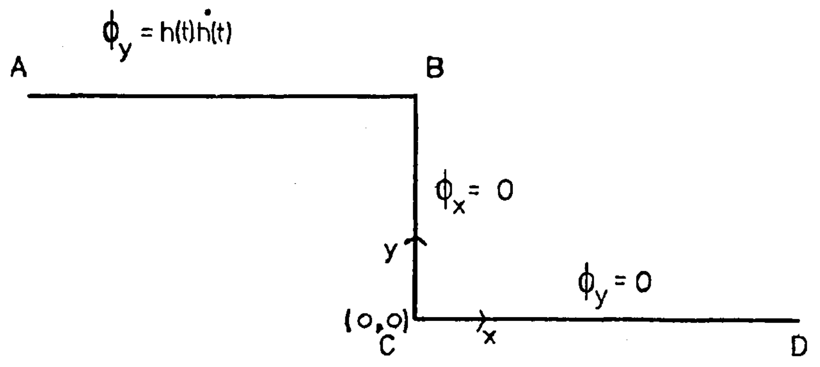

Figure 3

The boundary conditions in this new system are

$$\frac{\partial\phi}{\partial Y} = \begin{cases} 0 & \text{on } Y = 0, \quad X > 0 \\ h(t)\dot{h}(t) & \text{on } Y = 1, \quad X < 0 \end{cases} \tag{1.6.3}$$

$$\frac{\partial\phi}{\partial X} = 0 \quad \text{on} \quad X = 0, \quad 0 \le Y \le 1 \quad . \tag{1.6.4}$$

However the limit of ϕ as $|X^2+Y^2| \to \infty$ is not yet known and will be determined by matching the inner solution, expanded in powers of the small parameters $h(t)$ and m, to the outer solution expanded in a similar way using the method of VAN DYKE (27).

In Appendix 2 the limit as $|X^2+Y^2| \to \infty$ of $\phi(X,Y)$ is found by this method to satisfy

$$\phi = U\, h(t)\tilde{\phi} + \text{constant} + \text{higher order terms} \tag{1.6.5}$$

where $\tilde{\phi}$ is $O(1)$ but unknown.

Substituting expression (1.6.5) into equation of motion (1.2.6) shows that

$$\nabla^2\tilde{\phi} = O(h(t)^2,\ h(t)m, m^2) \text{ in the inner region } (X,Y) \tag{1.6.6}$$

so

$$\nabla^2\phi = O(h(t)^3 m,\ h(t)^2 m^2,\ h(t)m^3) \text{ in the inner region } (X,Y) \qquad (1.6.7)$$

To solve for ϕ let ϕ be denoted by ϕ_i in the inner region and ϕ_o in the outer region. In the inner region the equation of motion (1.2.4) has been reduced to Laplace's equation to at least the first few orders thus making the flow not only quasi-static but almost incompressible as well. First we find the inner solution ϕ_i.

1.7 INNER SOLUTION

The inner solution ϕ_i satisfies

$$\nabla^2\phi_i = 0 \quad \text{to order (parameter)}^4 \tag{1.7.1}$$

$$\left.\begin{aligned} \frac{\partial\phi_i}{\partial X} &= 0 \quad \text{on } X=0,\ 0\le Y\le 1 \\ \frac{\partial\phi_i}{\partial Y} &= \begin{cases} 0 & \text{on } Y=0,\ X>0 \\ h(t)\dot{h}(t) & \text{on } Y=1,\ X<0 \end{cases} \end{aligned}\right\} \tag{1.7.2}$$

$$\phi_i \text{ matches with } \phi_o \text{ as } |X^2+Y^2| \to \infty \tag{1.7.3}$$

Geometry as in Figure 3.

In order to solve for ϕ_i we consider a transformation that maps the step into an easily analysed straight line.

Consider the Schwartz-Christoffel transformation

$$z = \frac{1}{\pi}\{(\zeta^2-1)^{\frac{1}{2}} + \cosh^{-1}\xi\} \tag{1.7.4}$$

where $z = X+iY$, $\zeta = \xi+i\eta$. This maps the step ABCD of Figure 3 onto the straight line A'B'C'D' of Figure 4.

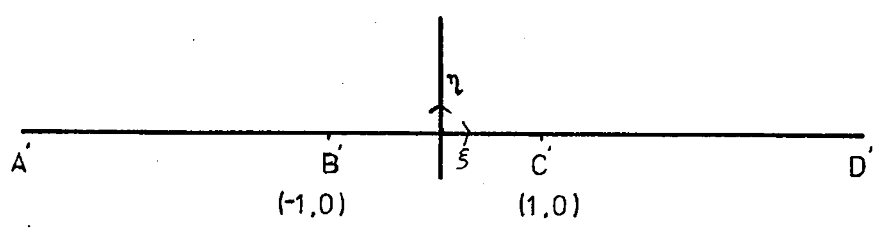

Figure 4: ζ-plane

If ψ is the potential in the ζ-plane then the equations (1.7.1), (1.7.2) and (1.7.3) become, on putting $\zeta_1 = \zeta+1$

$$\frac{\partial\psi}{\partial\eta_1} = \begin{cases} 0 \text{ on } \eta_1 = 0 \ , \ \xi_1 > 0 \\ h(t)\dot{h}(t)\{\frac{\xi_1}{\xi_1 - 2}\} \text{ on } \eta_1 = 0, \ \xi_1 < 0 \end{cases} \tag{1.7.5}$$

$$\nabla_1^2\psi = 0 \tag{1.7.6}$$

$$\phi(X,Y) = \text{Re}\{W_z(z)\} = \text{Re}\{W_{\zeta_1}(\zeta_1(z))\} \tag{1.7.7}$$

where $W_z(\ )$ denotes the complex potential in the z-plane.

The general solution to this problem can be found; it is

$$\phi_1 = \text{Re}\{-\frac{h(t)\dot{h}(t)}{\pi^2}[\zeta_1\log\zeta_1 + \frac{1}{2}(\log\zeta_1)^2 + F_1\zeta_1 + F_2]$$
$$+ \phi_{iR} + \frac{Uh(t)}{\pi}[\zeta_1 + F_3] + Ua\} \tag{1.7.8}$$

where ϕ_{iR} satisfies

$$\nabla_1^2\phi_{iR} = 0 \tag{1.7.9}$$

$$\frac{\partial\phi_{iR}}{\partial\eta_1} = \begin{cases} 0 \quad \text{on} \quad \xi_1 > 0 \quad \eta_1 = 0 \\ \frac{h(t)\dot{h}(t)}{\pi}\{(\frac{\xi_1}{\xi_1 - 2})^{\frac{1}{2}} - 1 - \frac{1}{\xi_1}\} \text{ on } \xi_1 < 0 \quad \eta_1 = 0 \end{cases} \tag{1.7.10}$$

and F_i, i=1,2,3 are constants as yet undetermined. They can be found by matching with the outer expansion. This is not done here since these constants prove to be not relevant in the calculation of the force. The first part of expression (1.7.8) comes into the solution due to the flux through the upper half of the step, the second half comes from the eigensolution ψ of $\nabla_1^2\psi = 0$, $\frac{\partial\psi}{\partial\eta_1} = 0$ on all boundaries.

A similar calculation yields the potential for the flow around the other end of the piston.

1.8 THE FORCE ON THE PISTON

The inner solution of section 1.7 holds in

$$|(x\pm a)^2 + y^2|^{\frac{1}{2}} << a \tag{1.8.1}$$

whereas the outer solutions hold in

$$|(x\pm a)^2 + y^2|^{\frac{1}{2}} >> |h(t)| \ , \ |X^2+Y^2| >> 1. \tag{1.8.2}$$

Both solutions hold when

$$|h(t)| << |(x\pm a)^2 + y^2|^{\frac{1}{2}} << a, \quad |x\pm a| \to 0 \quad \text{and} \quad |X| \to \infty \tag{1.8.3}$$

We can write equation (1.6.1) as

$$F = \int_{-a}^{-a+\lambda} p\,dx + \int_{-a+\lambda}^{a-\mu} p\,dx + \int_{a-\mu}^{a} p\,dx \tag{1.8.4}$$

where λ and μ are arbitrary except for the stipulation that

$$|h(t)| << \lambda,\mu << a \tag{1.8.5}$$

that is both λ and μ lie in the regions in which both the inner and outer solutions are valid.

For the second term of expression (1.8.4) the outer solution is valid and expression (1.4.4), linearised to $y = 0$, can be used for the pressure.

For the first and last terms of expression (1.8.4) the outer solution is no longer valid and the inner solutions, with expression (1.4.3) for the pressure, must be used instead.

Substituting the required formulae into these integrals and expanding in powers of m and h(t), rejecting all terms wholly dependent on λ and μ (because they are arbitrary so cannot appear in the answer) gives for the force on the piston

$$F = 2\rho_0 \omega^2 \frac{h_0}{d} \{ \frac{1}{k_3} (ka \cos\omega t - \sin ka \cos(ka-\omega t))$$

$$+ \sum_{n=1}^{N} \bar{B}_n^{-3} [2a\bar{B}_n \cos\omega t - \sin\omega t - \sin(2a\bar{B}_n - \omega t)]$$

$$+ \sum_{n=N+1}^{\infty} \bar{B}_n^{-3} [1 - 2a\bar{B}_n - \exp\{-2a\bar{B}_n\}] \cos\omega t \}$$

$$+ 2\rho_0 U^2 \frac{h_0}{\pi} \cos\omega t \log_e |h_0 \cos\omega t| + 0(U^2 h(t)) \quad , \qquad (1.8.6)$$

The first term in this expression for F comes from that part of the potential that has no singularity at the piston ends. It could be found by letting λ and μ tend to zero in the first term in the power series expansion, in h(t) and m, of the second integral of expression (1.8.4).

The second term of expression (1.8.6) is that contribution to the force that comes from the discontinuities at the piston ends.

1.9. DISCUSSION

The most significant fact in this calculation is that the singularities in the model problem, caused by the piston ends, do not have to be accounted for when calculating the power flow W. This is because the time-average of the singularities in this part of the problem is integrable.

However, in the calculations of the force F(t), where no time average is taken, full account of the end contributions must be taken. This leads to a large fluid-loading, of order $h(t)\log|h(t)|$, which is larger than the linearised quantities of order h(t).

We should note that the power flow W and the force on the piston F(t) are not independent, since

$$W = \langle -\omega h_0 \sin\omega t\, F(t) \rangle \tag{1.9.1}$$

If this time averaged quantity is evaluated using expression (1.8.6) for F(t) it is found that the log term has zero average. The leading term of this agrees with the leading term of expression (1.5.6). Thus

$$W = \frac{h_0^{\,2}\rho_0}{d}\{c^3(\sin ka)^2 + 2\omega^3 \sum_{n=1}^{N} \bar{B}_n^{\,-3}(\sin a\bar{B}_n)^2\} + O(h(t)m^2) \tag{1.9.2}$$

Note that there are no terms of order $h(t)m$ in equation (1.8.6) and therefore no terms of order $h(t)^2 m$ in expression (1.9.2). This is not surprising since reversing the flow in this symmetric problem (changing the sign of m) should not affect the total force on the piston or the work rate

Linearisation of Lighthill's equation lead to a standard problem but the model broke down at the piston ends. When the behaviour of the fluid at the piston ends was properly treated, however, we found that this inviscid model gave a finite force on the piston. Viscosity need not be taken into account to make the force finite and the inviscid solution made a sound first approximation to the force and the work done.

CHAPTER 2 : The generation of sound by a piston set in an expansion chamber in a duct containing flow.

2.1 INTRODUCTION AND SUMMARY

In this chapter the calculations of the velocity potential for the force on, and the work done by, a piston are extended to a more complicated and thus a more physically realistic geometry. The same linearised model for the piston as was used previously is employed.

The expansion chamber is a wider region of duct for which the inlet to and outlet from are not necessarily of the same width. They contain a steady background flow.

The solution from Chapter 1 is employed and as before it breaks down in the vicinity of the piston ends. Details of the local flow are just as before so their calculation will not be repeated here.

The solution also breaks down near the changes in duct width and the method of matched asymptotic expansions is used along with a Schwartz-Christoffel transformation to bridge this region.

The solution for the velocity potential can then be used to calculate the force on the piston and the work done by it in sustaining the motion.

In section 2.2 the nonlinear problem is rigorously defined and the nature of the steady background flow is discussed. In section 2.3 the linearised problem for all the straight sections of duct is considered and in section 2.4 the potentials are expanded for the method of matched asymptotic expansions. In section 2.5 the solution in one of the regions joining the straight sections is found and expanded in preparation for sections 2.6 to 2.8 in which the Van Dyke matching is done. The rate of work done by the piston and the force on it are calculated in sections 2.9 and 2.10 respectively. Section 2.11 contains discussion of the results.

2.2 PROBLEM DEFINITION

Inviscid compressible fluid flows adiabatically through a two-dimensional duct whose geometry abruptly changes from width $2d_1$ to $2d_3$ and then to $2d_5$, with $d_3 > d_1$, $d_3 > d_5$.

Acoustic fluctuations to the steady stream are produced by the small amplitude, time-harmonic, transverse vibrations of a piston set in a side wall in the widest part of the duct, see Figure 1. The problem is to find the velocity potential when the piston amplitude is small compared with the other length scales in the problem and when the Mach number of the steady duct flow is small.

Using cartesian coordinates (x,y) the walls of the duct are at

$$y = \begin{cases} \pm d_1 & \text{for } x \leq -L_1 \\ \pm d_3 & \text{for } -L_1 \leq x \leq L_2 \\ \pm d_5 & \text{for } L_2 \leq x \end{cases} \tag{2.2.1}$$

The magnitude of the basic flow for $x \ll -L_1$ is U and it follows from conservation of mass that it must have the asymptotic value $\frac{Ud_1}{d_5}$ as $x \to \infty$. Furthermore, as the width $d_3 \ll L_1 + L_2$ the steady flow within the expansion chamber will be nearly parallel (away from the ends) and its speed will be $\frac{Ud_1}{d_3}$.

It can be seen from Figure 1 that the piston is set between $x = -a$ and $x = a$ on $y = -d_3$ and it has the displaced surface given by $h(x,t) = \text{Re}\{h_0 e^{-i\omega t}\}$.

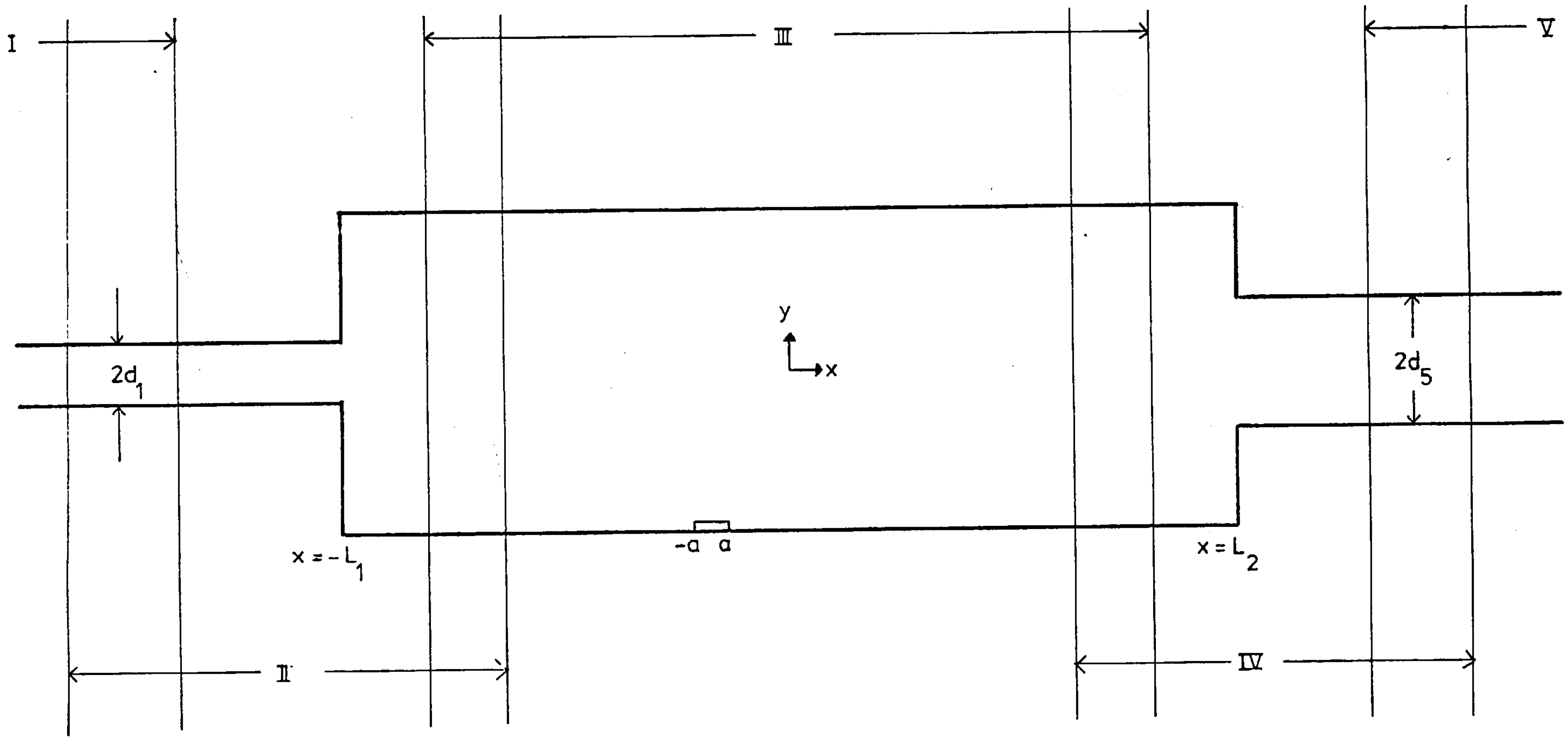

FIGURE 1: THE REGIONS

The regions shown in Figure 1 are as follows. Region 1 is from minus infinity to many duct widths to the left of $x = -L_1$ and Region V is from infinity to many duct widths to the right of $x = L_2$. Region III lies between many duct widths to the right of $x = -L_1$ and many duct widths to the left of $x = L_2$. Regions II and IV overlap and join the other three as shown.

The basic steady velocity potentials Φ_j are:

in Region I, $\Phi_1 = Ux + c_1 = Ux \; (c_1 = 0, \text{ say})$ (2.2.2a)

in Region III, $\Phi_3 = \frac{Ud_1}{d_3} x + c_3 = \alpha Ux + c_3 = U_2 x + c_3 \left(\alpha = \frac{d_1}{d_3}\right)$ (2.2.2b)

in Region V, $\Phi_5 = \frac{Ud_1}{d_5} x + c_5 = \beta Ux + c_5 = U_3 x + c_5 \left(\beta = \frac{d_1}{d_5}\right)$ (2.2.2c)

The velocity potential, with arbitrary multiplying factors, is calculated to first order in $h(x,t)$ (small when compared with U) by linearising the equation of motion and the boundary condition on $y = -d_3 + h(x,t)$ to one on $y = -d_3$ as in the previous chapter.

The approximations Φ_j, $j = 1,3,5$, breakdown near $x = -L_1$ and $x = L_2$ because the flow there is no longer uniform. Matched asymptotic expansions will be used to find the velocity potentials in these regions, assuming that the parameters kd_3 and m (a mach number) are $\ll 1$.

2.3 REGIONS I, III and V

As in the previous chapter the equation of motion of the fluid is given by LIGHTHILL [(19) equation 61] to be

$$c^2\nabla^2\phi = \frac{\partial^2\phi}{\partial t^2} + \frac{\partial\phi}{\partial x_m}\frac{\partial^2\phi}{\partial x_m\partial t} + \frac{\partial\phi}{\partial x_m}\frac{\partial\phi}{\partial x_n}\frac{\partial^2\phi}{\partial x_m\partial x_n} \qquad m,n = 1,2, \tag{2.3.1}$$

where ϕ is the total velocity potential, c the local sound speed and ∇^2 the Laplacian operator, $\phi = \hat{\Phi}_j$ for j = 1,2,3,4,5. For j = 1,3,5 we set $\hat{\Phi}_j = \Phi_j + \hat{\phi}_j$ where $\hat{\phi}_j$ are the velocity potentials of the linearised perturbations assuming $\hat{\phi}_j$ is small compared to the basic flow and neglecting terms of order $(\hat{\phi}_j)^2$. This then gives

$$\underline{U}_{jT} = (\frac{\partial\Phi_j}{\partial x} + \frac{\partial\hat{\phi}_j}{\partial x}\ ,\ \frac{\partial\hat{\phi}_j}{\partial y}) \tag{2.3.2a}$$

$$\underline{u}_j = (\frac{\partial\hat{\phi}_j}{\partial x}\ ,\ \frac{\partial\hat{\phi}_j}{\partial y}) \tag{2.3.2b}$$

$$\underline{U}_j = (U_j, 0) \tag{2.3.2c}$$

where $\underline{U}_{jT}$, $\underline{u}_j$ and $\underline{U}_j$ are the total velocity field, the perturbed velocity field and the basic velocity field respectively.

$\hat{\phi}_j$ satisfies the linearised equation of motion

$$c_o^2\nabla^2\hat{\phi}_j = \frac{\partial^2\hat{\phi}_j}{\partial t^2} + 2U_j\frac{\partial^2\hat{\phi}_j}{\partial x\partial t} + U_j^{\,2}\frac{\partial^2\hat{\phi}_j}{\partial x^2}\ . \tag{2.3.3}$$

Substituting $\hat{\phi}_j = \mathrm{Re}\{\phi_j\ e^{-i\omega t}\}$ gives the convected form of Helmholtz's equation in each of the three regions

$$\nabla^2\phi_j = -\ \{k + im_j\frac{\partial}{\partial x}\}^2\ \phi_j\ \ ,\qquad j = 1,3,5, \tag{2.3.4}$$

where $k = \frac{\omega}{c}$ is a wave number, $m_1 = m = \frac{U}{c}$ a Mach number and $m_3 = \alpha m$, $m_5 = \beta m$ are modified Mach numbers.

Expressed in stretched coordinates $\hat{x}_j$, $\hat{y}_j$ where $\hat{y}_j = y$, $\hat{x}_j = x(1-m_j^2)^{-\frac{1}{2}}$ with $K_j = k\frac{\hat{x}_j}{x}$ and

$$\phi_j(x,y) = \psi_j(\hat{x}_j,\hat{y}_j)\ \exp\{-im_jK_j\hat{x}_j\} \tag{2.3.5}$$

equation (2.3.4) becomes

$$(\hat{\nabla}_j^2+K_j^2)\psi_j = 0 \tag{2.3.6}$$

where $\hat{\nabla}_j^2$ is the Laplacian with respect to stretched coordinates.

So

$$\hat{\phi}_j = U_jx + c_j + \mathrm{Re}\{\psi_j(x(1-m_j^2)^{-\frac{1}{2}},y)\exp[-i\omega t-im_jkx(1-m_j^2)^{-1}]\} \tag{2.3.7}$$

Region I

Equation (2.3.6) with j = 1 gives

$$(\hat{\nabla}_1^2+K_1^2)\psi_1=0 \tag{2.3.8}$$

with boundary conditions

$$\frac{\partial\psi_1}{\partial y} = 0 \quad \text{on } y = \pm d_1 \tag{2.3.9}$$

ϕ_1 an outgoing wave at minus infinity

ϕ_1 must match the potential in Region II as $x \to -L_1-0$.

Region V

Equation (2.3.6) with j = 5 gives

$$(\hat{\nabla}_5^2+K_5^2)\psi_5=0. \tag{2.3.10}$$

with boundary conditions

$$\frac{\partial\psi_5}{\partial y} = 0 \quad \text{on} \quad y = \pm d_5 \tag{2.3.11}$$

ϕ_5 an outgoing wave at infinity

ϕ_5 must match the potential in Region IV as $x \to L_2 + 0$.

Region III

Letting $\hat{\phi}_3 = \hat{\phi}_{3c} + \hat{\phi}_{3p}$ where $\hat{\phi}_{3p}$ is the velocity potential due to a piston in a duct of width $2d_3$ with basic flow U_3x, $\hat{\phi}_{3c}$ is thus the velocity potential representing the perturbations travelling along the expansion chamber , confined by reflection at each end.

The function $\hat{\phi}_{3p}$ is given in Chapter 1 and behaves as

$$\hat{\phi}_{3p} \sim -\frac{ch_0}{2kd_3}\sin\frac{ka}{1+m_3}\,\mathrm{Re}\{\exp\{-i\omega t + \frac{ikx}{1+m_3}\}\} \text{ for } x >> a \tag{2.3.12}$$

and

$$\hat{\phi}_{3p} \sim -\frac{ch_0}{2kd_3}\sin\frac{ka}{1-m_3}\,\mathrm{Re}\{\exp\{-i\omega t - \frac{ikx}{1-m_3}\}\} \text{ for } x << -a \tag{2.3.13}$$

Equation (2.3.6) with j = 3 then gives

$$(\hat{\nabla}_3{}^2 + K_3{}^2)\psi_{3c} = 0 \tag{2.3.14}$$

with the boundary conditions

$$\frac{\partial\psi_{3c}}{\partial y} = 0 \quad \text{on} \quad y = \pm d_3 \tag{2.3.15}$$

$\phi_3 = \phi_{3c} + \phi_{3p}$ must match to the solutions in both Regions II and IV as $x \to -L_1+$ and L_2- respectively.

The general solution to

$$(\hat{\nabla}_j^{\,2} + K_j^{\,2})\psi_j = 0 \qquad (2.3.16)$$

$$\frac{\partial\psi_j}{\partial y_j} = 0 \quad \text{on} \quad y_j = \pm d_j \qquad (2.3.17)$$

is found in Appendix 3.

The forms of the solution appropriate here are

$$\hat{\phi}_1 + Ux + \text{Re}\{\hat{A}_1 \exp\{-i\omega t - \frac{ikx}{1-m}\}\} + \text{e.s.t.} \qquad (2.3.18a)$$

$$\hat{\phi}_5 = U\beta x + \text{Re}\{\hat{B}_5 \exp\{-i\omega t + \frac{ikx}{1+\beta m}\}\} + c_5 + \text{e.s.t.} \qquad (2.3.18b)$$

$$\hat{\phi}_3 = U\alpha x + \text{Re}\{e^{-i\omega t}[\hat{A}_3 \exp\{-\frac{ikx}{1-\alpha m}\} + \hat{B}_3 \exp\{\frac{ikx}{1+\alpha m}\}]\}$$
$$+ c_3 + \hat{\phi}_{3p} + \text{e.s.t.} \qquad (2.3.18c)$$

where $\hat{A}_1$, $\hat{A}_3$, $\hat{B}_3$, $\hat{B}_5$ are constants as yet not evaluated and e.s.t. denotes exponentially small terms (as $|x| \to \infty$ for other parameters fixed).

Considering the factors in $\hat{\phi}_{3p}$ and $\hat{\phi}_3$ it is expedient to put

$$\hat{A}_j = \frac{ch_0}{2d_3} A_j \quad j=1,3; \qquad \hat{B}_j = \frac{ch_0}{2d_3} B_j \quad j=3,5, \quad \text{and} \quad \phi_j = \frac{ch_0}{2d_3} \tilde{\phi}_j \qquad (2.3.19)$$

The expansion of the three velocity potentials, in matching coordinates, as the joining regions are approached, will now be considered.

2.4. EXPANSION OF POTENTIALS

Consider first the matching from Region I to Region II and from Region III to Region II.

In Regions I and III let the non-dimensional variables characterised by the local length scale $\lambda = k^{-1}$, known from now on as 'outer variables', be

$$x_1 = kx + kL_1 = kx + \ell_1; \quad y_1 = ky \tag{2.4.1}$$

In Region II let the local non-dimensional variables based on the local dominant length scale d_3, known as the 'inner variables', be

$$x_2 = \frac{x_1}{kd_3} \quad ; \quad y_2 = \frac{y_1}{kd_3} \tag{2.4.2}$$

In Region I the velocity potential $\tilde{\phi}_1$ in terms of outer variables is

$$\tilde{\phi}_1 = A_1 \exp\{\frac{i\ell_1}{1-m} - \frac{ix_1}{1-m}\} \tag{2.4.3}$$

Letting $A_1 = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots$ where, $\varepsilon = kd_3 << 1$ is the small matching parameter, gives, on expanding in this small parameter,

$$\tilde{\phi}_1 = (a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots) \exp\{\frac{i\ell_1}{1-m} - \frac{ix_1}{1-m}\} \tag{2.4.4}$$

$\tilde{\phi}_j^{\,n}$, in the notation of matched asymptotic expansions, denotes expansion up to the n^{th} power of ε in the first (outer or inner) variable and $\tilde{\phi}_j^{\,nm}$ denotes the expansion up to the m^{th} power of ε of $\tilde{\phi}_j^{\,n}$ after it has been rewritten in the second (inner or outer) variable. So corresponding forms in Region I for potential $\tilde{\phi}_1$ are

$$\tilde{\phi}_1^{\ 00} = a_0 \exp\{\frac{i\ell_1}{1-m}\} \ , \qquad (2.4.5a)$$

$$\tilde{\phi}_1^{\ 01} = a_0\{1 - \frac{i\varepsilon x_2}{1-m}\} \exp\{\frac{i\ell_1}{1-m}\} , \qquad (2.4.5b)$$

$$\tilde{\phi}_1^{\ 11} = \{a_0 + \varepsilon a_1 - \frac{i\varepsilon x_2 a_0}{1-m}\} \ \exp\{\frac{i\ell_1}{1-m}\} , \qquad (2.4.5c)$$

$$\tilde{\phi}_1^{\ 22} = \{a_{00} + ma_{01} + \varepsilon[a_{10} + ma_{11} - ix_2(1+m)(a_{20} + a_{21})]$$
$$+ \varepsilon^2[a_{20} + ma_{21} - \frac{1}{2}(a_{00} + ma_{01})(1+2m)x_2^{\ 2} - ix_2(1+m)(a_{10} + ma_{11})]\} \times$$
$$\times \exp\{\frac{i\ell_1}{1-m}\} \qquad (2.4.5d)$$

where (2.4.5d) has also been expanded for small m (up to O(m)) and where

$$a_i = a_{i0} + ma_{i1} + m^2 a_{i2} + \ldots . \qquad (2.4.6)$$

In Region III the velocity potential $\tilde{\phi}_3$ in terms of the outer variable is

$$\tilde{\phi}_3 = \{A_3 - \frac{1}{k}\sin_1\frac{ka}{1-\alpha m}\} \exp\{\frac{i\ell_1}{1-\alpha m} - \frac{ix_i}{1-\alpha m}\} + B_3 \exp\{\frac{-i\ell_1}{1+\alpha m} + \frac{ix_1}{1+\alpha m}\} , \qquad (2.4.7)$$

Letting

$$A_3 = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \ldots .$$
$$B_3 = \hat{d}_0 + \varepsilon\hat{d}_1 + \varepsilon^2\hat{d}_2 + \ldots .$$

and expanding as before in ε gives

$$\tilde{\phi}_3 = \{c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \ldots - \frac{a}{1-\alpha m} + \frac{\varepsilon^2 a^3}{6(1-\alpha m)^3 d_3^{\ 2}} + \} \times$$
$$\times \exp\{\frac{i\ell_1}{1-\alpha m} - \frac{ix_1}{1-\alpha m}\} + \{\hat{d}_0 + \varepsilon\hat{d}_1 + \varepsilon^2\hat{d}_2 + \ldots\} \exp\{-\frac{i\ell_1}{1+\alpha m} + \frac{ix_1}{1+\alpha m}\} , \qquad (2.4.8)$$

For the matching expansions this then gives

$$\tilde{\phi}_3^{00} = \{c_0 - \frac{a}{1-\alpha m}\} \exp\{\frac{i\ell_1}{1-\alpha m}\} + \hat{d}_0 \exp\{-\frac{i\ell_1}{1+\alpha m}\}, \tag{2.4.9a}$$

$$\tilde{\phi}_3^{01} = \{c_0 - \frac{a}{1-\alpha m}\} \{1 - \frac{i\varepsilon x_2}{1-\alpha m}\} \exp\{\frac{i\ell_1}{1-\alpha m}\} + \hat{d}_0\{1 + \frac{i\varepsilon x_2}{1+\alpha m}\} \exp\{-\frac{i\ell_1}{1+\alpha m}\}, \tag{2.4.9b}$$

$$\tilde{\phi}_3^{11} = \{c_0 - \frac{a}{1-\alpha m} + \varepsilon c_1 - \frac{i\varepsilon x_2}{1-\alpha m}(c_0 - \frac{a}{1-\alpha m})\} \exp\{\frac{i\ell_1}{1-\alpha m}\} + \{\hat{d}_0 + \varepsilon \hat{d}_1 + \frac{i\hat{d}_0 x_2 \varepsilon}{1+\alpha m}\} \exp\{-\frac{i\ell_1}{1+\alpha m}\}, \tag{2.4.9c}$$

$$\tilde{\phi}_3^{22} = \{c_0 - \frac{a}{1-\alpha m} + \varepsilon[c_1 - \frac{ix_2}{1-\alpha m}(c_0 - \frac{a}{1-\alpha m})] + \varepsilon^2[c_2 + \frac{a^3}{6(1-\alpha m)^3 d_3^{\,2}} - \frac{ix_2 c_1}{1-\alpha m} - \tfrac{1}{2}(c_0 - \frac{a}{1-\alpha m})\frac{x_2^{\,2}}{(1-\alpha m)^2}]\} \exp\{\frac{i\ell_1}{1-\alpha m}\} + \{\hat{d}_0 + \varepsilon[\hat{d}_1 + \frac{ix_2\hat{d}_0}{1+\alpha m}] + \varepsilon^2[\hat{d}_2 + \frac{ix_2\hat{d}_1}{1+\alpha m} - \frac{x_2^{\,2}\hat{d}_0}{2(1+\alpha m)^2}]\} \exp\{-\frac{i\ell_1}{1+\alpha m}\}. \tag{2.4.9d}$$

2.5 REGION II

Figure 2 shows the Regions VI and VII across which the matching has to be achieved. In Regions VI and VII the solutions of both Region I and Region II and of Regions II and III are valid so matching can be undertaken. First the solution in Region II must be considered.

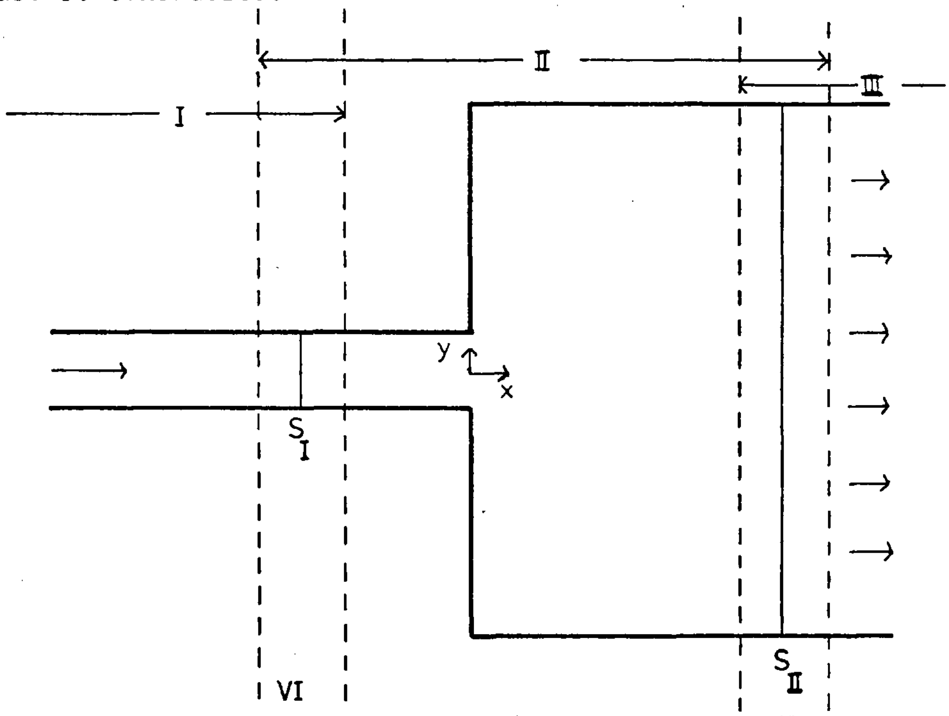

Figure 2

In Region II the basic time-independent part of the flow, Φ_2, satisfies, from equation (2.3.1)

$$c^2\nabla^2\Phi_2 = \sum_{\substack{X_n=x_1y_1 \\ X_m=x_1y_1}}\sum \frac{\partial\Phi_2}{\partial X_n}\frac{\partial^2\Phi_2}{\partial X_m\partial X_n}\frac{\partial\Phi_2}{\partial X_m} \quad . \tag{2.5.1}$$

In inner variables (x_2,y_2), with ∇_2^2 denoting the Laplacian with respect to those variables this becomes

$$d_3{}^2c^2\nabla_2{}^2\Phi_2 = \sum_{X_j=x_2,y_2} \sum_{X_i=x_2y_2} \frac{\partial\Phi_2}{\partial X_j} \frac{\partial^2\Phi_2}{\partial X_i \partial X_j} \frac{\partial\Phi_2}{\partial X_i} . \qquad (2.5.2)$$

We know, from matching leading orders, that

$$\Phi_2(x_2,y_2) \to \begin{cases} Ud_3x_2 - UL_1 & \text{as} \quad x_2 \to -\infty \\ U\alpha d_3x_2 - U\alpha L_1 + c_3 & \text{as} \quad x_2 \to \infty . \end{cases} \qquad (2.5.3)$$

Putting $\Phi_2(x_2,y_2) = Ud_3\Psi_2 - UL_1$ gives

$$\Psi_2(x_2,y_2) \to \begin{cases} x_2 & \text{as} \quad x_2 \to -\infty \\ \alpha x_2 + c_3 & \text{as} \quad x_2 \to \infty \end{cases} \qquad (2.5.4)$$

and

$$\nabla_2{}^2\Psi_2 = m^2 \sum_{X_i=x_2y_2} \sum_{X_j=x_2y_2} \frac{\partial\Psi_2}{\partial X_j} \frac{\partial^2\Psi_2}{\partial X_i \partial X_j} \frac{\partial\Psi_2}{\partial X_i} . \qquad (2.5.5)$$

Expanding ψ_2 in powers of m,thus

$$\Psi_2 = \Psi_{20} + m\Psi_{21} + m^2\Psi_{22} + \ldots \quad , \qquad (2.5.6)$$

yields

$$\nabla_2{}^2\Psi_{20} = 0 \quad \text{and} \quad \Psi_{20} \to \begin{cases} x_2 & \text{as} \quad x_2 \to -\infty \\ \alpha x_2 + c_3 & \text{as} \quad x_2 \to \infty . \end{cases} \qquad (2.5.7)$$

Putting $\hat{\Phi}_2 = Ud_3\Psi_2 - UL_1 + \hat{\phi}_2 = Ud_3\Psi_2 - UL_1 + \mathrm{Re}\{\phi_2 e^{-i\omega t}\}$ (2.5.8)

and substituting into equation (2.3.1) gives, after linearisation,

$$\nabla_2{}^2\phi_2 = -\varepsilon^2\phi_2 - 2i\varepsilon m \frac{\partial\Psi_2}{\partial x_i} \frac{\partial\phi_2}{\partial x_i}$$
$$+ m^2\{2 \frac{\partial\Psi_2}{\partial x_j} \frac{\partial^2\Psi_2}{\partial x_i \partial x_j} \frac{\partial\phi_2}{\partial x_i} + \frac{\partial\Psi_2}{\partial x_i} \frac{\partial\Psi_2}{\partial x_j} \frac{\partial^2\phi_2}{\partial x_i \partial x_j}\} ,$$
(2.5.9)

So $\nabla_2^2 \phi_2 = O(\varepsilon^2, \varepsilon m, m^2)$ assuming all derivatives of potentials on the right hand side are bounded.

Note: This is not strictly valid for the right angles in the duct. We need to assume that these corners are slightly rounded off to ensure boundedness without affecting the factors in the following Schwartz-Christoffel transformation.

If we put $\phi_2 = \frac{ch}{2d_3} \tilde{\phi}_2$ as before and expand $\tilde{\phi}_2$ in powers of ε

$$\tilde{\phi}_2 = \phi_{20} + \varepsilon\phi_{21} + \varepsilon^2\phi_{22} + \dots \quad \text{for} \quad \varepsilon \ll 1 , \tag{2.5.10}$$

then we have from equation (2.5.9) the following equations governing the inner expansion of $\tilde{\phi}_2$;

$$\nabla_2^2 \phi_{20} = O(m^2) \; ; \quad \nabla_2^2 \phi_{21} = O(m) \tag{2.5.11}$$

Expanding ϕ_{20} in powers of m and then deducing the properties of each part of the expansion gives

$$\phi_{20} = \phi_{200} + m\phi_{201} + m^2\phi_{202} + \dots \tag{2.5.12}$$

and

$$\phi_{21} = \phi_{210} + m\phi_{211} + m^2\phi_{212} + \dots \tag{2.5.13}$$

for $m \ll 1$ giving

$$\nabla_2^2 \phi_{200} = 0 \; ; \quad \nabla_2^2 \phi_{201} = 0 \; ; \quad \nabla_2^2 \phi_{210} = 0. \tag{2.5.14}$$

2.6 VAN DYKE MATCHING-ORDER ZERO

Matching, in the style of VAN DYKE (27), in the Regions VI and VII gives

$$\tilde{\phi}_1^{00} = \tilde{\phi}_2^{00} \text{ as } x_2 \to -\infty \tag{2.6.1}$$

and thus

$$\lim_{x_2 \to -\infty} \tilde{\phi}_2 = a_0 \exp\{\frac{i\ell_1}{1-m}\} \tag{2.6.2}$$

$$\tilde{\phi}_3^{00} = \tilde{\phi}_2^{00} \text{ as } x_2 \to +\infty \tag{2.6.3}$$

and thus

$$\lim_{x_2 \to +\infty} \tilde{\phi}_2 = \{c_0 - \frac{a}{1-\alpha m}\} \exp\{\frac{i\ell_1}{1-\alpha m}\} + \hat{d}_0 \exp\{-\frac{i\ell_1}{1+\alpha m}\} . \tag{2.6.4}$$

Consider now $\tilde{\phi}_2$ in terms of the outer variables:

$$\begin{aligned}\tilde{\phi}_2^{0} &= \phi_{20}(x_2,y_2) = \phi_{200}(x_2,y_2) + m\phi_{201}(x_2,y_2) + m^2\phi_{202}(x_2,y_2) \\ &= \phi_{200}(\frac{x_1}{\varepsilon}, \frac{y_1}{\varepsilon}) + m\phi_{201}(\frac{x_1}{\varepsilon}, \frac{y_1}{\varepsilon}) + m^2\phi_{202}(\frac{x_1}{\varepsilon}, \frac{y_1}{\varepsilon}) + \ldots\end{aligned} \tag{2.6.5}$$

ϕ_{200} and ϕ_{201} both satisfy Laplace's equation, so there can be no terms of lower order than zero in ε, (because if there does exist such a term $\varepsilon^{p/q}$ then it must come from $x_2^{-p/q}$ which is not a solution of Laplace's equation and has a singularity at the origin). Therefore we have for $\tilde{\phi}_2^{00}$

$$\tilde{\phi}_2^{00} = \tilde{\phi}_{200} + m\phi_{201} + 0(m^2) \tag{2.6.6}$$

So

$$\tilde{\phi}_2^{00} = a_0 \exp\{\frac{i\ell_1}{1-m}\} \text{ to order } m^2 \tag{2.6.7a}$$

by matching in Region VI and

$$\tilde{\phi}_2^{\,00} = \{c_0 - \frac{a}{1-\alpha m}\} \exp\{\frac{i\ell_1}{1-\alpha m}\} + \hat{d}_0 \exp\{-\frac{i\ell_1}{1+\alpha m}\} \text{ to } O(m^2) \tag{2.6.7b}$$

by matching in Region VII. Also

$$\nabla_2^{\,2}\tilde{\phi}_2^{\,00} = 0 \text{ to } O(m^2) \tag{2.6.8}$$

Now $\tilde{\phi}_2^{\,00}$ matches to two constants to order m^2 and satisfies Laplace's equation so $\tilde{\phi}_2^{\,00}$ to order m^2 is a constant. Substituting this result into the general equation (2.5.9) yields similar results for ϕ_{20j}, j = 2,3,4, etc., $\phi_2^{\,00}$ is therefore identically a constant at all orders of m so

$$\tilde{\phi}_2^{\,00} = \tilde{\phi}_2^{\,0} = \tilde{\phi}_{20} \ .$$

Equations (2.6.2) and (2.6.4) thus yield

$$a_0 \exp\{\frac{i\ell_1}{1-m}\} = (c_0 - \frac{a}{1-\alpha m}) \exp\{\frac{i\ell_1}{1-\alpha m}\} + \hat{d}_0 \exp\{-\frac{i\ell_1}{1+\alpha m}\} \ . \tag{2.6.9}$$

Matching also gives

$$\begin{aligned} \tilde{\phi}_1^{\,01} &= \tilde{\phi}_2^{\,10} \quad \text{as} \quad x_2 \to -\infty \ , \\ \tilde{\phi}_3^{\,01} &= \tilde{\phi}_2^{\,10} \quad \text{as} \quad x_2 \to +\infty \ . \end{aligned} \tag{2.6.10}$$

Now we have already shown that

$$\tilde{\phi}_1^{\,01} = \tilde{\phi}_1^{\,00} - \frac{i\varepsilon x_2}{1-m} \ \phi_1^{\,00} \tag{2.6.11}$$

and

$$\tilde{\phi}_3^{\,01} = \tilde{\phi}_3^{\,00} - \frac{i\varepsilon x_2}{1-\alpha m} (c_0 - \frac{a}{1-\alpha m}) \exp\{\frac{i\ell_1}{1-\alpha m}\} + \frac{i\varepsilon x_2}{1+\alpha m} \ \hat{d}_0 \exp\{-\frac{i\ell_1}{1+\alpha m}\} \tag{2.6.12}$$

[from equations (2.4.5b) and (2.4.9b)].

However it is also known that

$$\tilde{\phi}_2 = \text{constant} + \varepsilon\tilde{\phi}_{21}(x_2 y_2) \tag{2.6.13}$$

and substituting this into equation (2.5.9) gives

$$\varepsilon\nabla_2^{\,2}\phi_{21} = m^2\varepsilon\{2\,\frac{\partial\psi_2}{\partial x_j}\,\frac{\partial^2\psi_2}{\partial x_i\partial x_j}\,\frac{\partial\phi_{21}}{\partial x_i} + \frac{\partial\psi_2}{\partial x_j}\,\frac{\partial\psi_2}{\partial x_i}\,\frac{\partial^2\phi_{21}}{\partial x_i\partial x_j}\} \tag{2.6.14}$$

The substitution of expression (2.5.13) into equation (2.6.14) tells us that

$$\nabla_2^{\,2}\phi_{210} = 0\ ; \quad \nabla_2^{\,2}\phi_{211} = 0 \tag{2.6.15}$$

assuming once again that the derivatives of ψ_2 are bounded.

Using an argument similar to that used for ϕ_{200} and ϕ_{201} we can show that

$$\phi_{21}(x_2,y_2) = \text{constant} + \tilde{\phi}_{210}(\frac{x_1}{\varepsilon}, \frac{y_1}{\varepsilon}) + m\tilde{\phi}_{211}(\frac{x_1}{\varepsilon}, \frac{y_1}{\varepsilon}) + 0(m^2) \tag{2.6.16}$$

where both $\tilde{\phi}_{210}$ and $\tilde{\phi}_{211}$ are to lowest order ε^{-1}.

This gives for the matching expansion

$$\tilde{\phi}_2^{\;10} = \tilde{\phi}_1^{\;00} + \varepsilon\{\tilde{\phi}_{210} + m\tilde{\phi}_{211}\} + 0(m^2)\ , \tag{2.6.17}$$

If $\quad a_p = \sum_{n=0}^{\infty} a_{pn}m^n \qquad p = 0,1,2,3,\ldots$

and if similar expansions are chosen for b_p's and c_p's and d_p's (with $\hat{d}_{pn} = d_{pn}$) then the expansions of equations (2.6.11) and (2.6.12) in powers of m are

$$\tilde{\phi}_1^{01} = \tilde{\phi}_1^{00} - i\epsilon x_2\{a_{00}+m[a_{01}+a_{00}(1+i\ell_1)]\}\, e^{i\ell_1} + O(m^2) \text{ as } x_2 \to -\infty , \tag{2.6.18}$$

$$\tilde{\phi}_3^{01} = \tilde{\phi}_1^{00} - i\epsilon x_2\{c_{00}-a+m[c_{01}-\alpha a+(c_{00}-a)(1+i\ell_1)\alpha]\}\, e^{i\ell_1} + i\epsilon x_2\{d_{00}+m[d_{01}+\alpha d_{00}(i\ell_1-1)]\}\, e^{-i\ell_1} \text{ as } x_2 \to \infty , \tag{2.6.19}$$

So matching tells us that

$$\tilde{\phi}_{210} + m\tilde{\phi}_{211} \sim \begin{cases} -ix_2\{a_{00}+m[a_{01}+a_{00}(1+i\ell_1)]\}\, e^{i\ell_1} & \text{as } x_2 \to -\infty , \quad (2.6.20a) \\ -ix_2\{c_{00}-a+m[c_{01}-\alpha a+(c_{00}-a)(1+i\ell_1)\alpha]\}e^{i\ell_1} & \\ +ix_2\{d_{00}+m[d_{01}+\alpha d_{00}(i\ell_1-1)]\}e^{-i\ell_1} & \text{as } x_2 \to \infty . \quad (2.6.20b) \end{cases}$$

Conservation of energy gives the flux across S_I equal to the flux across S_{II} (see Figure 2). In integral form that is

$$\int_{S_I} \frac{\partial}{\partial x_2} (\tilde{\phi}_{210}+m\tilde{\phi}_{211})dy_2 = \int_{S_{II}} \frac{\partial}{\partial x_2} (\tilde{\phi}_{210}+m\tilde{\phi}_{211})dy_2 \tag{2.6.21}$$

since $\tilde{\phi}_{210}$ and $\tilde{\phi}_{211}$ satisfy Laplace's equation. This equation gives us two more relationships between the constants in which we are interested. The $O(1)$ term of expression (2.6.21) is

$$-a_{00}d_1e^{i\ell_1} = d_3\{d_{00}e^{-i\ell_1} - (c_{00}-a)e^{i\ell_1}\} \tag{2.6.22}$$

and the $O(\epsilon)$ term is

$$-d_1[a_{01}+a_{00}(1+i\ell_1)]e^{i\ell_1} = d_3\{[d_{01}+\alpha d_{00}(i\ell_1-1)]e^{-i\ell_1} - [c_{01}-\alpha a+(c_{00}-a)(1+i\ell_1)\alpha]e^{i\ell_1}\} , \tag{2.6.23}$$

2.7 MATCHING-ORDER ONE

Equations (2.4.5c) and (2.4.9c) partly expanded for small m give

$$\tilde{\phi}_1^{11} = \tilde{\phi}_1^{00} + \varepsilon\{a_{10}+m[a_{11}+i\ell a_{10}]-ix_2(a_{00}+m[a_{01}+a_{00}(1+i\ell_1)])\}e^{i\ell_1} + O(m^2) \text{ as } x_2 \to -\infty \, , \qquad (2.7.1)$$

$$\tilde{\phi}_3^{11} = \tilde{\phi}_1^{00} + \varepsilon\{c_{10}+m[c_{11}+i\ell_1\alpha c_{10}]-ix_2(c_{00}-a+m[c_{01}-a\alpha+(c_{00}-a)(1+i\ell_1)\alpha])\}e^{i\ell_1} + \varepsilon\{d_{10}+m[d_{11}+i\ell_1\alpha d_{10}]+ix_2(d_{00}+m[d_{01}+d_{00}\alpha(i\ell_1-1)]\}e^{-i\ell_1} + O(m^2) \text{ as } x_2 \to +\infty \, , \qquad (2.7.2)$$

for the outer expansions.

Considering the inner solution it is known that

$$\tilde{\phi}_2^{1} = \tilde{\phi}_1^{00} + \varepsilon\phi_{210} + m\varepsilon\phi_{211} + O(m^2)$$
$$= \tilde{\phi}_1^{00} + \varepsilon\{\text{constant} + \tilde{\phi}_{210}+m\tilde{\phi}_{211}\} + O(m^2) \qquad (2.7.3)$$

and therefore

$$\tilde{\phi}_2^{11} = \tilde{\phi}_1^{00} + \varepsilon\{\text{constant} + \tilde{\phi}_{210}+m\tilde{\phi}_{211}\} + O(m^2) \qquad (2.7.3a)$$

because as expression (2.7.3) is identically $\tilde{\phi}_2^{10}$ it must also be $\tilde{\phi}_2^{11}$. From the zero order matching it is known that $\tilde{\phi}_2$ behaves, to $O(m^2)$, like a linear function of x_2 on moving away from the sharp change of width in the duct into the matching regions. If ϕ_p is a particular solution then

$$\phi_p \sim \begin{cases} x_2 & \text{as} \quad x_2 \to +\infty \\ \frac{d_3}{d_1} x_2 + J_{13} & \text{as} \quad x_2 \to -\infty \end{cases} \tag{2.7.4}$$

where d_3/d_1 is determined by the mass flux condition as before and J_{13} by the geometry via the Schwartz-Christoffel transformation - see Appendix 4. This is also the solution of the basic flow in equation (2.5.3) giving

$$c_3 = -UJ_{13}\alpha d_3 = -UJ_{13}d_1 \; ; \quad c_5 = -Ud_1(J_{13}+J_{53}) . \tag{2.7.5}$$

The general solution for $\tilde{\phi}_2$ is thus given by

$$\tilde{\phi}_2 \sim \begin{cases} \tilde{\phi}_1^{\,00} + \varepsilon\{\lambda_1 x_2+\lambda_2\} \text{ as } x_2 \to \infty \\ \tilde{\phi}_1^{\,00} + \varepsilon\{\lambda_1(\frac{x_2}{\alpha} + J_{13})+ \lambda_2\} \quad \text{as } x_2 \to -\infty , \end{cases} \tag{2.7.6}$$

where λ_1 and λ_2 are constants to be determined by matching.

Van Dyke matching gives

$$\begin{aligned} \tilde{\phi}_1^{\,11} &= \tilde{\phi}_2^{\,11} \quad \text{as} \quad x_2 \to -\infty , \\ \tilde{\phi}_2^{\,11} &= \tilde{\phi}_3^{\,11} \quad \text{as} \quad x_2 \to \infty , \end{aligned} \tag{2.7.7}$$

that is

$$\lambda_1 = -i\alpha(a_{00}+m[a_{01}+a_{00}(1+i\ell_1)])e^{i\ell_1} \tag{2.7.8}$$

$$\lambda_2 = iJ_{13}\alpha(a_{00}+m[a_{01}+ a_{00}(1+i\ell_1)])e^{i\ell_1}+(a_{10}+m[a_{11}+i\ell_1 a_{10}])e^{i\ell_1} \tag{2.7.9}$$

and

$$\begin{aligned} \lambda_1 = &-i(c_{00}-a+m[c_{01}-a\alpha+(c_{00}-a)(1+i\ell_1)\alpha])e^{i\ell_1} \\ &+ i(d_{00}+m[d_{01}+d_{00}\alpha(i\ell_1-1)])e^{-i\ell_1} \end{aligned} \tag{2.7.10}$$

$$\lambda_2 = (c_{10}+m[c_{11}+i\ell_1\alpha\, c_{10}])e^{i\ell_1} + (d_{10}+m[d_{11}+i\ell_1\alpha d_{10}])e^{-i\ell_1}. \qquad (2.7.11)$$

Equating equations (2.7.8) and (2.7.10) and also (2.7.9) and (2.7.11) gives two equations. We are interested in one of them, namely,

$$a_{00}iJ_{13}ae^{i\ell_1} + a_{10}e^{i\ell_1} = c_{10}e^{i\ell_1} + d_{10}e^{-i\ell_1} . \qquad (2.7.12)$$

The other equation involves a_{11}, c_{11} and d_{11} which we will not attempt to evaluate here.

2.8 MATCHING-ORDER TWO

Equations (2.4.5d) and (2.4.8d) give

$$\tilde{\phi}_1^{\,22} = \{a_{20} - \frac{a_{00}}{2} x_2^{\,2} - i a_{10} x_2\} e^{i\ell_1} + 0(m) \tag{2.8.1}$$

$$\tilde{\phi}_3^{\,22} = \{c_{20} + \frac{a^3}{6d_3^{\,2}} - (c_{00}-a)\frac{x_2^{\,2}}{2} - i x_2 c_{10}\} e^{i\ell_1} + \{d_{20} - d_{00}\frac{x_2^{\,2}}{2} + d_{10} i x_2\} e^{-i\ell_1} + 0(m) \; . \tag{2.8.2}$$

Equation (2.5.9) with the expansion (2.5.10) along with the similar expansion

$$\phi_{22} = \phi_{220} + m\phi_{221} + \ldots \tag{2.8.3}$$

yields

$$\nabla_2^{\,2}\phi_{220} = -\phi_{220} = -\,a_{00} \tag{2.8.4}$$

giving

$$\phi_{220} = -\frac{a_{00}}{2} x_2^{\,2} + \psi_{220} \quad \text{where } \nabla_2^{\,2}\psi_{220} = 0 \; , \tag{2.8.5}$$

Using a similar argument for ψ_{220} as was used for equations (2.7.8) to (2.7.11) yields

$$\phi_{200} \sim \begin{cases} -\dfrac{a_{00}}{2} x_2^{\,2} + \lambda_3 x_2 + \lambda_4 & \text{as } x_2 \to +\infty \\[2ex] -\dfrac{a_{00}}{2} x_2^{\,2} + \lambda_3\{x_2/\alpha + J_{13}\} + \lambda_4 & \text{as } x_2 \to -\infty \; , \end{cases} \tag{2.8.6}$$

The ε^2 term of $\tilde{\phi}_2^{\,22}$ is $\varepsilon^2\phi_{220}$; thus equating the $0(\varepsilon^2)$ terms of $\tilde{\phi}_1^{\,22}$ and $\tilde{\phi}_2^{\,22}$ as $x \to -\infty$ and of $\tilde{\phi}_3^{\,22}$ and $\tilde{\phi}_2^{\,22}$ as $x \to +\infty$ gives

$$-\alpha a_{10} e^{i\ell_1} = -c_{10} e^{i\ell_1} + d_{10} e^{-i\ell_1} \quad . \tag{2.8.9}$$

Now the O(1) and O(m) equations from expanding expression (2.6.9) plus equations (2.6.22), (2.6.23), (2.7.12) and (2.8.9) give a system that is by no means complete. However if the entire matching procedure employed in Region II is repeated in Region IV then the following equations can be obtained.

$$b_0 \exp\{\frac{i\ell_5}{1+\beta m}\} = c_0 \exp\{-\frac{i\ell_5}{1-\alpha m}\} + \{ d_0 - \frac{a}{1+\alpha m}\} \exp\{\frac{i\ell_5}{1+\alpha m}\} , \qquad (2.8.10)$$

$$d_5 b_{00}\, e^{i\ell_5} = d_3\{-c_{00}e^{-i\ell_5} + (d_{00}-a)e^{i\ell_5}\} , \qquad (2.8.11)$$

$$d_5\{b_{01}-\beta b_{00}-i\ell_5 b_{00}\}e^{i\ell_5} = - d_3\{c_{01}+\alpha c_{00}(1-i\ell_5)\}e^{-i\ell_5} + d_3\{d_{01}-a\alpha-(1+i\ell_5)(d_{00}-a)\alpha\}e^{i\ell_5} , \qquad (2.8.12)$$

$$b_{10}e^{i\ell_5} + \overline{K}_{53}\, \alpha/\beta\; i b_{00}e^{i\ell_5} = c_{10}e^{-i\ell_5} + d_{10}e^{i\ell_5}, \qquad (2.8.13)$$

$$\alpha/\beta\; b_{10}e^{i\ell_5} = -c_{10}e^{-i\ell_5} + d_{10}e^{i\ell_5} . \qquad (2.8.14)$$

These twelve equations do make a closed system that can be solved to give a_{00}, a_{01}, a_{10}, b_{00}, b_{01}, b_{10}, c_{00}, c_{01}, c_{10}, d_{00}, d_{01}, and d_{10}. These are listed in Appendix 5.

Finally the evaluation of these constants means that we have calculated the velocity potentials $\hat{\Phi}_j$, j = 1,3,5 as listed in Appendix 5.

2.9 RATE OF WORK DONE BY PISTON

Let us now consider the net, time averaged, energy flux into the duct, that is the average rate of work done by the piston.

The work by CANTRELL & HART (4) and LEPPINGTON & LEVINE (17) (as well as that in the previous chapter, section 1.4) shows that the time-averaged power flow W across the piston is the same, to order h_0, as the net power flow from the expansion chamber into Regions I and V.

Equation (1.4.9) for the power flux in a duct of width $2d_3$ with basic uniform flow αU gives

$$W = < - \int_{S_I} \rho_0 \frac{\partial \hat{\phi}_3}{\partial t} \left(\frac{\partial h}{\partial t} + \alpha U \frac{\partial h}{\partial x}\right) dx > + O(h^3) \qquad (2.9.1)$$

where once again < > denotes the time-average, ρ_0 the density at infinity and s_I is the piston's surface.

If $\hat{\phi}_3$ is expressed as the sum of two components, $\hat{\phi}_3 = \hat{\phi}_{3P} + \hat{\phi}_{3C}$, then contributions from $\hat{\phi}_{3P}$ to the work done, W_P, are given by expressions (1.5.3) and (1.5.4) as

$$W_P = \frac{h_0^2 \rho_0 c^3}{4d_3} \left\{\left(\sin \frac{ak}{1+\alpha m}\right)^2 + \left(\sin \frac{ak}{1-\alpha m}\right)^2\right\} \qquad (2.9.2)$$

$$= \frac{h_0^2 \rho_0 c^3}{2d_3} \left\{(\sin ak)^2 + \alpha^2 m^2 ak(ka\cos 2ka + \sin ka)\right\} + O(m^4) \, , \qquad (2.9.2a)$$

In order to calculate W_C, the contribution from $\hat{\phi}_{3C}$, we must consider equation (2.3.18c),

$$\hat{\phi}_{3C} = \text{Re}\ \{\frac{ch_0}{2d_3}\ e^{-i\omega t}\ \{A_3\ \exp\{-\frac{ikx}{1-\alpha m}\} + B_3\ \exp\{\frac{ikx}{1+\alpha m}\}\}\} \tag{2.9.3}$$

and substituting this into equation (2.9.1) yields

$$W_C = -\frac{h_0{}^2\rho_0 c^3 k}{2d_3}\ \text{Re}\ \{A_3 \sin\frac{ak}{1-\alpha m} + B_3 \sin\frac{ak}{1+\alpha m}\}\ . \tag{2.9.4}$$

Substituting in the expansions for the constants A_3 and B_3 and using the constants from Appendix 5, gives, after addition, for the total work done by the piston

$$\begin{aligned} W = {} & \frac{4h_0{}^2\rho_0 c^3 a^2 k^2 \alpha}{d_3}\ \{\beta[\cos^2\ell_1 + \alpha^2\sin^2\ell_1] + \alpha^2\sin^2\ell_5 + \beta^2\cos^2\ell_5\}\overline{XX} \\ & + \frac{k^3 h_0{}^2\rho_0 c^3\alpha^2 i}{8}\ \{J_{13}[a_{00}{}^2 e^{2i\ell_1} - \bar{a}_{00}{}^2 e^{-2\ell_1}] \\ & \qquad + J_{53/\beta^2}\ [b_{00}{}^2 e^{2i\ell_5} - \overline{b_{00}}\ e^{-2i\ell_5}]\} \end{aligned} \tag{2.9.5}$$

+ higher order terms

where $\overline{}$ denotes the complex conjugate.

We notice here that there is no term of order k^2m indicating that a reversal of the flow would not alter the time-averaged work done by the piston. As in the previous calculation W is also equal to

$$W = \langle \int_S N_1{}^* dS_i \rangle \tag{2.9.6}$$

where S is any surface from one end of the piston to the other lying in or on the duct. It is a trivial matter to show that this is true for S_{II} and S_{III} of Figure 3, but the same results can be obtained for S_I and S_{IV}. This is not shown here due to the complexity of the algebra. However this does afford a check on the validity of the matching as S_I and S_{IV} lie in Regions I and V which lie beyond Regions II and IV across which the solutions have been matched.

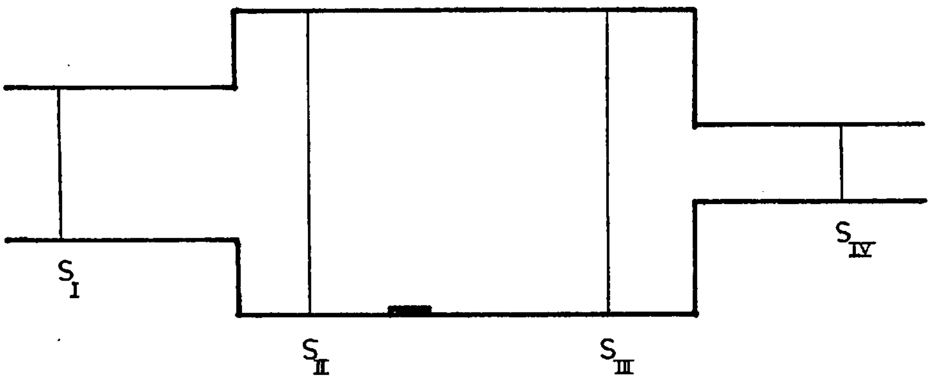

Figure 3

2.10 FORCE ON THE PISTON

The force on the piston, F, needed to sustain the piston's periodic motion (the pressure integrated across the piston surface) is given by equation (1.6.1) as

$$F_T = \int_{s_1} p\,ds_1 \quad , \tag{2.10.1}$$

The contribution from the $\hat{\phi}_{3P}$ part of $\hat{\phi}_3$, F_P, is given by the previous chapter, for small ε, as

$$F_P = -\rho_0 \frac{c^2k^2h_0}{d_3^2}\varepsilon(\sin\omega t - \frac{2a\varepsilon}{3d_3}\cos\omega t)$$
$$+ 8c^2h_0\rho_0\varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{(n\pi)^3}\{1 - \frac{an\pi}{d_3} - \exp(-\frac{an\pi}{d_3})\}\cos\omega t$$
$$+ 2\rho_0 \frac{\alpha^2c^2m^2h_0}{\pi}\cos\omega t \log_e|(h_0\cos\omega t)| + O(\varepsilon m^2; m^3; \varepsilon^3, \varepsilon^2 m) \, , \tag{2.10.2}$$

From $F_c = \int_{s_1} p_c ds_1$ where p_c is the pressure contribution from the $\hat{\phi}_{3C}$ part of the velocity potential, with

$$p_c = -\rho_0 \{\frac{\partial\hat{\phi}_{3c}}{\partial t} + \alpha U \frac{\partial\hat{\phi}_{3c}}{\partial x}\} \, , \tag{2.10.3}$$

we have

$$F_c \doteq \frac{\rho_0 c^2 h_0}{d_3} \mathrm{Re}\{ie^{-i\omega t}[A_3 \sin\frac{ka}{1-\alpha m} + B_3 \sin\frac{ka}{1+\alpha m}]\} \, , \tag{2.10.4}$$

Substituting for A_3 and B_3 from the constants in Appendix 5 and adding to F_P gives for F_T, the total force on the piston:

$$F_T = \frac{2\rho_0\alpha^2c^2m^2h_0}{\pi} \cos\omega t \log_e |h_0 \cos\omega t|$$

$$- \frac{8\rho_0 c^2 a^2 h\alpha}{d_3^2} \{\beta[\cos^2\ell_1+\alpha^2\sin^2\ell_1] + \alpha^2\sin^2\ell_5+\beta^2\cos^2\ell_5\} \times$$

$$\times \chi\bar{\chi}\varepsilon\sin\omega t$$

$$- \varepsilon\cos\omega t - \frac{2\rho_0 c^2 h_0 a^2}{d_3^2} \chi\bar{\chi}[\alpha^2\beta^2(1+\alpha^2)\sin 2\ell_5+(1-\alpha^2)(\alpha^2-\beta^2)$$

$$\sin(2\ell_1+2\ell_5) -(1-\alpha^2)(\alpha^2-\beta^2) \sin 2\ell_1]$$

$$+ \varepsilon^2\cos\omega t \frac{\rho_0 c^2 h_0 a}{d_3^2} [\frac{2a^2}{3d_3} + \frac{\alpha^2}{4a} (J_{13}(a_{00}{}^2 e^{2i\ell_1} + \overline{a_{00}}{}^2 e^{-2i\ell_1})$$

$$+ J_{53}/\beta^2 (b_{00}{}^2 e^{2i\ell_5} + \overline{b_{00}}{}^2 e^{-2i\ell_5}))]$$

$$- \varepsilon^2\sin\omega t \frac{\rho_0 c^2 h_0 \alpha^2}{4d_3} i\{J_{13}(a_{00}{}^2 e^{2i\ell_1} - \overline{a_{00}}{}^2 e^{-2i\ell_1})$$

$$+ J_{53}/\beta^2 (b_{00}{}^2 e^{2i\ell_5} - \overline{b_{00}}{}^2 e^{-2i\ell_5})\}$$

$$+ \frac{8c^2 h_0 \rho_0}{\pi^3} \varepsilon^2 \sum_{n=1}^{\infty} 1/n^3\{1- \frac{an\pi}{d_3} - \exp\{- \frac{an\pi}{d_3}\}\}\cos\omega t \qquad (2.10.5)$$

The first term comes from that part of the velocity potential that has singularities at the piston ends, the rest coming from the main body of the piston. Once again we notice that there is no term of order εm.

2.11 DISCUSSION

In this problem, as previously, the singularities in the model problem caused by the piston ends, did not have to be taken into account when calculating the power flow W.

It was noted in Chapter 1 that there were not any terms of the form k^2m in W and this was not surprising given the symmetry of the problem with respect to the flow and thus Mach number m. In the calculation in this chapter however the downstream duct is of a different width to the upstream duct and the piston is not set in the centre, yet there is still no term of the form k^2m. This does suggest some inbuilt features of such problems that preclude the presence of such terms.

In the calculations of the force F(t) the singularities of course did have to be taken into account again, giving a large fluid loading, of order $h(t) \log|h(t)|$, which is larger than the other linearised quantities of order h(t).

So in this chapter the method developed for forming a linearised model of a piston set in a duct wall, developed in the previous chapter, has been successfully applied to the more complex situation of a piston in an expansion chamber enabling the power flow from and the force on such a piston to be calculated.

APPENDIX 1

FOURIER TRANSFORM METHOD

With this method we take Fourier transforms of the linearised equation of motion and then substitute in the transforms of the boundary conditions. This gives an expression for the transform of the potential which can then be inverted.

If the general piston profile is $Re\{h(x)e^{-i\omega t}\}$, where its time independent transform H(s) is given by

$$\tilde{H}(s) = \int_{-\infty}^{+\infty} h(x)e^{isx}dx \qquad (A1.1)$$

and its inverse by

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{H}(s)e^{-isx}ds , \qquad (A1.2)$$

then the transform Φ of ϕ_2 and its inverse are given by

$$\Phi(x,y) = \int_{-\infty}^{+\infty} \phi_2(x,y)e^{isx}dx, \quad \phi_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi(s,y)e^{-isx}ds . \qquad (A1.3)$$

Taking the transform of equation (1.2.7) and substituting in the transforms of the boundary conditions, equations (1.2.5) and (1.2.9), gives

$$\Phi(s,y) = \frac{i \cosh \gamma(s)(y-d)}{2\pi\ \gamma(s)\sinh\gamma(s)d} \iint_{-\infty}^{+\infty} \tilde{H}(\tau)(\omega+U\tau)e^{-i\tau x+isx}d\tau dx \qquad (A1.4)$$

where

$$\gamma(s) = \begin{cases} (s^2(1-m^2)-2mks-k^2)^{\frac{1}{2}} , & |s| \geq k(1-m^2)^{-\frac{1}{2}} \\ -i(k^2+2mks-s^2(1-m^2))^{\frac{1}{2}} , & |s| \leq k(1-m^2)^{-\frac{1}{2}} \end{cases} . \qquad (A1.5)$$

Substituting into the inversion formula gives

$$\phi_2(x,y) = \frac{i}{4\pi^2} \iiint_{-\infty}^{+\infty} \frac{\cosh\gamma(s)(y-d)}{\gamma(s)\sinh\gamma(s)d} \tilde{H}(\tau)(\omega+U\tau) \exp\{-i\tau r+isr-isx\}d\tau drds \ . \tag{A1.6}$$

When h(x) is specified $\tilde{H}(\tau)$ is known. This can be substituted into equation (A1.6) and then an inversion or an approximation to an inversion can be obtained. In the case of the 'top-hat' profile the inversion can be done exactly to give the results of section 1.3.

APPENDIX 2

INNER LIMIT OF THE OUTER SOLUTION

If we take expression (A1.6) for ϕ_2 and substitute in the Fourier transform of the 'top-hat' profile and do the r and τ integrals we have

$$\phi_1(x,y) = \mathrm{Re}\{ \frac{i\omega h_0}{\pi} \int_{-\infty}^{+\infty} \frac{\cosh\gamma(s)(y-d)}{\gamma(s)\sinh\gamma(s)d} \frac{\sin as}{s} (1+\frac{ms}{k}) e^{-isx-i\omega t} ds\} \tag{A2.1}$$

Expanding this for small m and h(t), considering only that part centred on (a,0), and finding the limit as $s \to \infty$ of each term, we find, from LIGHTHILL (18) p. 43 , that the limit as $|(x-a)^2 + y^2| \to 0$ of ϕ_1, is such that

$$\phi_1 \sim Ux - \frac{Uh(t)}{\pi} \log r + Uh(t)A_1 \tag{A2.2}$$

$$- \frac{\dot{h}(t)}{\pi} ((x-a)\log r - y\theta + A_2 x) + \text{h.o.t.}$$

where $r^2 = [(x-a)^2+y^2]$; $\theta = \tan^{-1} y/x-a$ and where the constants A_1 and A_2 are determinate but not calculated here.

Expressed in terms of inner variables and expanded again expression (A2.2) gives

$$\phi_1 \sim Uh(t)X + Ua - \frac{Uh(t)}{\pi} \log Rh(t) + Uh(t)A_1 \tag{A2.3}$$

$$- \frac{1}{\pi} \dot{h}(t)\ h(t)X \log h(t)R + \frac{1}{\pi} \dot{h}(t)h(t)Y\theta - \frac{1}{\pi} \dot{h}(t)h(t)A_2 X$$

where $R = (X^2+Y^2)^{\frac{1}{2}}$.

We can now see from expression (A2.3) that as $R \to \infty$ the inner potential ϕ_1 is such that

$$\phi_i \sim Uh(t)\phi_{i_1} + h(t)\dot{h}(t)\phi_{i_2} + \text{constant } (Ua)$$

where ϕ_{i_1} and ϕ_{i_2} are O(1).

APPENDIX 3

GENERAL SOLUTION FOR ψ_j

The problem is to find the general solution to

$$(\nabla_i^2 + K_i^2)\psi_i = 0 \qquad \text{(A3.1)}$$

with

$$\frac{\partial \psi_i}{\partial y_i} = 0 \text{ on } y_i = \pm d_i \,. \qquad \text{(A3.2)}$$

Equation (A3.1) can also be written as

$$\frac{\partial^2 \psi_i}{\partial x_i^2} + \frac{\partial^2 \psi_i}{\partial y_i^2} + K_i^2 \psi_i = 0 \qquad \text{(A3.3)}$$

and this can be solved using the method of separation of variables.

Putting $\psi_i(x_i y_i) = F_i(x_i) G_i(y_i)$ (A3.4)

and substituting into equation (A3.3) gives

$$\frac{F_i''}{F_i} + K_i^2 = -\frac{G_i''}{G_i} = \gamma^2 \qquad \text{(A3.5)}$$

where γ is the separation constant.

Equations (A3.5) can be rewritten

$$G_i'' + \gamma^2 G_i = 0 \text{ and } F_i'' + (K_i^2 - \gamma^2) F_i = 0 \qquad \text{(A3.6a)} \; \text{(A3.6b)}$$

Now the general solutions to (A3.6a) and (A3.6b) give as a general solution for ψ_i

$$\psi_i = [A\cos\gamma y_i + B\sin\gamma y_i][C\exp(-i\mu_i x_i) + D\exp(i\mu_i x_i)] \qquad \text{(A3.7)}$$

where $\mu_i^2 = K_i^2 - \gamma^2$, unless μ_i or $\gamma = 0$ when F_i or G_i are polynomials of order one.

Application of boundary conditions (A3.2) requires that

$$-A \sin \gamma d_i + B \cos \gamma d_i = 0$$

and (A3.8)

$$A \sin \gamma d_i + B \cos \gamma d_i = 0 \,.$$

That is either $B = 0$ and $\sin \gamma d_i = 0$ (A3.9a)

or $A = 0$ and $\cos \gamma d_i = 0$. (A3.10a)

Condition (A3.9a) $\rightarrow \gamma d_i = n\pi$ so $\gamma = \frac{n\pi}{d_i}$ and $B_n = 0$, (A3.9b)

Condition (A3.10a) $\rightarrow \gamma d_i = (n+\tfrac{1}{2})\pi$ so $\gamma = (n+\tfrac{1}{2})\frac{\pi}{d_i}$ and $A_n = 0$, (A3.10b)

Both conditions give the same result when applied to the general solution (A3.7)

$$\psi_i(x_i,y_i) = \sum_{n=0}^{\infty} \{C_n \exp(-i\mu_{ni}x_i) + D_n \exp(i\mu_{ni}x_i)\} \cos\frac{n\pi}{2d_i}(y_i-d_i) \qquad \text{(A3.11)}$$

where $\mu_{ni} = \left(\left(\frac{n\pi}{2d_i}\right)^2 - K_i^2\right)^{\frac{1}{2}}$ $n = 1,2,3,\ldots$ (A3.12)

since $K_i d_i \ll 1$ and $\mu_{ni} = iK_i$ for $n = 0$.

Expression (A3.11) gives for ψ_i

$$\psi_i(x_i y_i) = C_0\{\exp -iK_i x_i\} + D_0\{\exp iK_i x_i\} + \sum_{n=1}^{\infty} \{C_n \exp\{-i\mu_{ni}x_i\} + D_n \exp\{i\mu_{ni}x_i\}\} \cos\frac{n\pi}{2d_i}(y_i-d_i) \qquad \text{(A3.13)}$$

where the constants $C_n D_n$ will be determined by other conditions on the function.

APPENDIX 4

SCHWARTZ-CHRISTOFFEL TRANSFORMATIONS

We wish to evaluate the constants J_{ij} such that the velocity potential ψ satisfies

$$\nabla^2\psi = 0 \tag{A4.1}$$

$$\psi \to x \quad \text{as} \quad x \to +\infty \tag{A4.2}$$

$$\psi \to \frac{d_j}{d_i} x + J_{ij} \text{ as } x \to -\infty \ (d_i < d_j) \tag{A4.3}$$

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on duct walls} \tag{A4.4}$$

in the interior of the duct shown in Figure 4.1

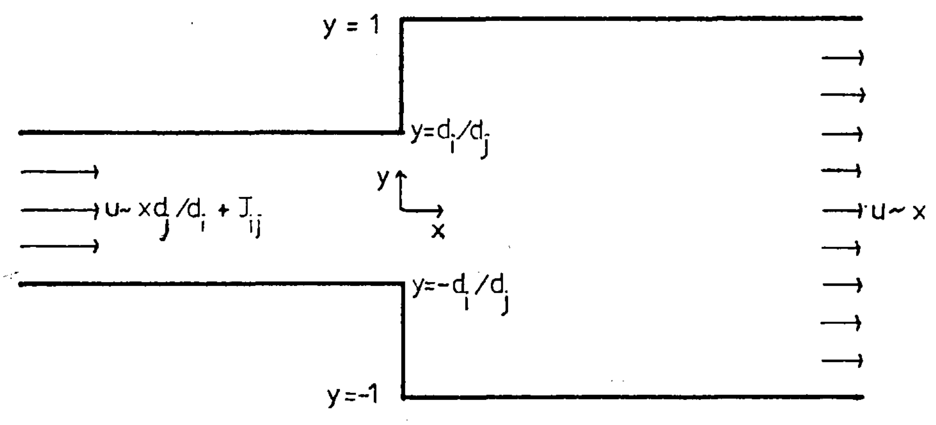

Figure 4.1: Z-plane, Z = x + iy

As the problem is symmetric about the line y = 0 the easier half-plane problem, to which it is equivalent, will be solved instead. That is ψ satisfies (A4.1), (A4.2), (A4.3) and (A4.4) for the duct geometry in Figure 4.2.

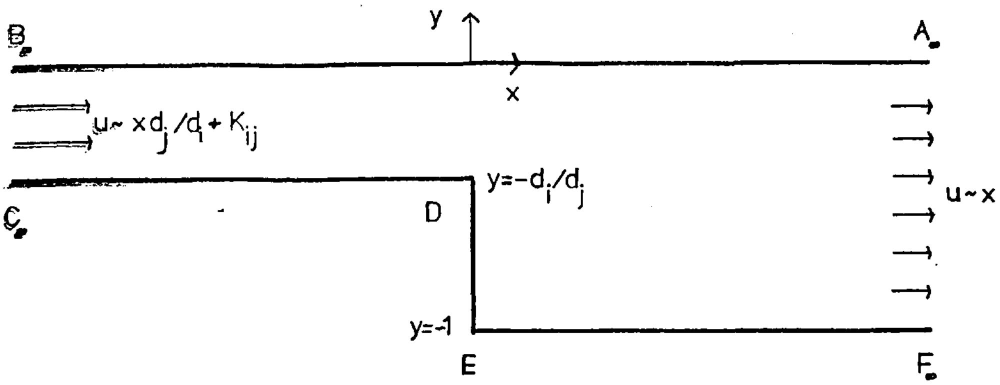

Figure 4.2: Z-plane

Consider the transformation given by KOBER [++section 12.8] which maps this half-duct and its interior onto the real line and the half-plane above it as in Figure 4.3:

$$t = f(z) \; ; \; \frac{dz}{dt} = \frac{(t-1)^{\frac{1}{2}}}{\pi t(t-a)} \qquad a = d_j^{\,2}/d_i^{\,2} \, , \tag{A4.5}$$

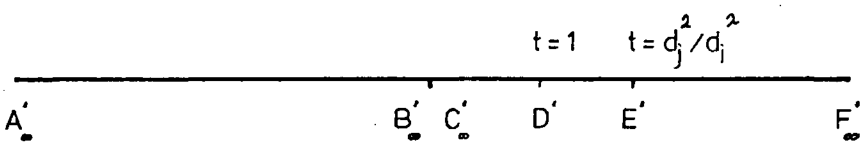

Figure 4.3: t-plane, $t = t_1 + it_2$.

Note the mapping takes the points A_∞, B_∞, C_∞, D, E, F_∞ in Figure 4.2 onto their corresponding primes in Figure 4.3.

The inverse of this mapping is

$$z = \frac{1}{\pi}\cosh^{-1}\left(\frac{2t-a-1}{a-1}\right) - \frac{d_i}{d_j \pi}\cosh^{-1}\frac{(a+1)t-2a}{(a-1)t} - i \, . \tag{A4.6}$$

The velocity potentials as $x \to \pm\infty$ in the Z-plane correspond to those as $t \to \infty$ and $t \to 0$ in the t-plane.

++DICTIONARY OF CONFORMAL REPRESENTATIONS, KOBER, H. 1952 DOVER

(i) Consider $t \to -\infty$, that is $x \to \infty$.

From equation (A4.6) in this limit

$$z \sim \frac{1}{\pi} \log\left(\frac{4td_i^2}{d_j^2-d_i^2}\right) - \frac{d_i}{\pi d_j} \cosh^{-1}\left(\frac{d_j^2+d_i^2}{d_j^2-d_i^2}\right) - i \qquad \text{(A4.7)}$$

Now if $W(Z) = \psi + i\psi'$ is the complex velocity potential for this incompressible irrotational flow in 2-D and ψ' is the stream function then

$$\psi = \text{Re}\{W(t=f(z))\} . \qquad \text{(A4.8)}$$

Expression (A4.7) gives

$$W(z) \sim z \sim \frac{1}{\pi} \log t + \frac{1}{\pi} \log\left(\frac{4d_i^2}{d_j^2-d_i^2}\right) - \frac{d_i}{\pi d_j} \cosh^{-1}\left(\frac{d_j^2+d_i^2}{d_j^2-d_i^2}\right) - i$$

(A4.9)

or $$W \sim \frac{1}{\pi} \log t + \text{constant } C_1 . \qquad \text{(A4.9a)}$$

(ii) Consider $t \to 0$, that is $x \to -\infty$.

Now

$$\frac{2t-a-1}{a-1} \to -\left(\frac{a+1}{a-1}\right) = -\left(\frac{d_j^2+d_i^2}{d_j^2-d_i^2}\right) \text{ as } t \to 0 , \qquad \text{(A4.10)}$$

(assuming $d_j > d_i$); this gives for the first $\cosh^{-1}$ term in expression (A4.6)

$$\cosh^{-1}\left(\frac{2t-a-1}{a-1}\right) \sim \cosh^{-1}\left(\frac{d_j^2+d_i^2}{d_j^2-d_i^2}\right) + i\pi , \qquad \text{(A4.11)}$$

also

$$\frac{(a+1)t-2a}{(a-1)t} \sim -\frac{2a}{(a-1)t} \text{ as } t \to 0 \qquad \text{(A4.12)}$$

thus giving for the second $\cosh^{-1}$ term

$$\cosh^{-1}\left(\frac{(a+1)t-2a}{(a-1)t}\right) \sim \log t + \log \frac{4d_j^2}{d_j^2-d_i^2} + i\pi \qquad \text{(A4.13)}$$

and hence for the whole expression

$$z \sim \frac{1}{\pi}\{\cosh^{-1}\left(\frac{d_j^2+d_i^2}{d_j^2-d_i^2}\right) + i\pi\} - \frac{d_i}{\pi d_j}\ \log\left(\frac{4d_j^2}{d_j^2-d_i^2}\right) + i\pi\}$$

$$+ \frac{d_i}{\pi d_j}\log t - i \quad \text{as} \quad t \to 0. \qquad \text{(A4.14)}$$

However from equation (A4.3)

$$W(z) \sim d_j z/d_i \ \text{ as } z \to -\infty \qquad \text{(A4.15)}$$

so

$$W(z) \sim \frac{1}{\pi}\log t - \frac{1}{\pi}\log\{\frac{4d_j^2}{d_j^2-d_i^2})\} - i + \frac{d_i}{\pi d_j}\cosh^{-1}\left(\frac{d_j^2+d_i^2}{d_j^2-d_i^2}\right) \qquad \text{(A4.16)}$$

and so for small t

$$W(t) \sim \frac{1}{\pi}\log t + \text{constant } C_2. \qquad \text{(A4.17)}$$

The upstream section of the duct corresponds to $t = 0$ so in the t-plane the velocity potential is that due to a point source at $t = 0$. Such a source has a complex potential:

$$W = \frac{1}{\pi}\log t + \text{constant } C_3 \quad . \qquad \text{(A4.18)}$$

Choosing $C_3=C_1$ we have

$$\text{at } z = \infty \qquad W(z) \sim z - C_1+C_3 = Z \qquad \text{(A4.19)}$$

$$\text{at } z = -\infty \qquad W(z) \sim \frac{d_j}{d_i}z - C_2+C_1$$

$$= \frac{d_j}{d_i}z + J_{ij} \qquad \text{(A4.20)}$$

where $$J_{ij} = C_1 - C_2 = \frac{2}{\pi}\log\left(\frac{4d_i d_j}{d_j^{\,2} - d_i^{\,2}}\right) - \frac{1}{\pi}\,\frac{d_j^{\,2} + d_i^{\,2}}{d_i d_j}\cosh^{-1}\left(\frac{d_j^{\,2} + d_i^{\,2}}{d_j^{\,2} - d_i^{\,2}}\right) . \tag{A4.21}$$

APPENDIX 5

CONSTANTS AND POTENTIALS

If $\chi = [(\alpha-\beta)(1-\alpha)\exp\{i(\ell_1+\ell_5)\} + (\alpha+\beta)(1+\alpha)\exp\{-i(\ell_1+\ell_5)\}]^{-1}$

then the constants are

$$a_{00} = 2ae^{-i\ell_1}\{(\alpha-\beta)e^{i\ell_5} - (\alpha+\beta)e^{-i\ell_5}\}\chi \tag{A5.1}$$

$$b_{00} = -2ae^{-i\ell_5}\{(1+\alpha)e^{-i\ell_1} + (1-\alpha)e^{i\ell_1}\}\chi \tag{A5.2}$$

$$c_{00} = a(\alpha-\beta)e^{-i\ell_5}\{(1+\alpha)e^{-i\ell_1} + (1-\alpha)e^{i\ell_1}\}\chi \tag{A5.3}$$

$$d_{00} = a(1-\alpha)e^{i\ell_1}\{(\alpha-\beta)e^{i\ell_5} - (\alpha+\beta)e^{-i\ell_5}\}\chi \tag{A5.4}$$

$$a_{01} = -2ae^{-i\ell_1}\{[\alpha+i\ell_1(1-\alpha)](\alpha-\beta)e^{i\ell_5} + [\alpha-i\ell_1(1-\alpha)](\alpha+\beta)e^{i\ell_5}\}\chi \tag{A5.5}$$

$$b_{01} = 2a\beta e^{-i\ell_5}\{[\alpha+i\ell_5(\alpha-\beta)](1+\alpha)e^{-i\ell_1} - [\alpha-i\ell_5(\alpha-\beta)](1-\alpha)e^{i\ell_1}\}\chi \tag{A5.6}$$

$$c_{01} = a\alpha(\beta-\alpha)e^{i\ell_5}[(1+\alpha)e^{-i\ell_1} - (1-\alpha)e^{i\ell_1}]\chi \tag{A5.7}$$

$$d_{01} = -a\alpha(1-\alpha)e^{i\ell_1}[(\alpha-\beta)e^{i\ell_5} + (\alpha+\beta)e^{-i\ell_5}]\chi \tag{A5.8}$$

$$a_{10} = 2i\alpha\{a_{00}J_{13}[i\beta\sin(\ell_1+\ell_5)-\alpha\cos(\ell_1+\ell_5)] + b_{00}J_{53}\alpha/\beta e^{-i(\ell_1+\ell_5)}\}\chi \tag{A5.9}$$

$$b_{10} = 2i\beta\alpha\{a_{00}J_{13}\alpha e^{i(\ell_1-\ell_5)} + \frac{b_{00}}{\beta}J_{53}[i\sin(\ell_1+\ell_5)-\alpha\cos(\ell_1+\ell_5)]\}\chi \tag{A5.10}$$

$$c_{10} = i\alpha^2/\beta\, e^{i\ell_5}[i\beta a_{00}(\beta-\alpha)J_{13}e^{i\ell_1} + b_{00}(1+\alpha)J_{53}e^{-i\ell_1}]\chi \tag{A5.11}$$

$$d_{10} = \frac{i\alpha^2}{\beta}e^{i\ell_1}\{\beta(\alpha+\beta)a_{00}J_{13}e^{-i\ell_5} + (1-\alpha)b_{00}J_{53}e^{i\ell_5}\}\chi. \tag{A5.12}$$

The velocity potentials are given by

$$\hat{\Phi}_1 \sim Ux + \frac{ch_0}{2d_3} \mathrm{Re}\{(a_{00}+ma_{10}+\varepsilon a_{10})\exp(-i\omega t - \frac{ikx}{1-m})\}, \qquad (A5.13)$$

$$\hat{\Phi}_3 \sim U\alpha x - UJ_{13}d_1 + \tilde{\phi}_{3P}$$
$$+ \mathrm{Re}\{e^{-i\omega t} [(c_{00}+mc_{01}+\varepsilon c_{10}) \exp\{-\frac{ikx}{1-\alpha m}\}$$
$$+ (d_{00}+md_{01}+\varepsilon d_{10}) \exp\{\frac{ikx}{1+\alpha m}\}]\}, \qquad (A5.14)$$

$$\hat{\Phi}_5 \sim U\beta x - Ud_1(J_{13}+J_{53})$$
$$+ \mathrm{Re}\ (b_{00}+mb_{01}+\varepsilon b_{10}) \exp\{-i\omega t + \frac{ikx}{1+\alpha m}\}\}, \qquad (A5.15)$$

where

$$\tilde{\phi}_{3P} \sim -\frac{ch_0}{2kd_3} \sin\frac{ka}{1+\alpha m} \mathrm{Re}\{\exp\{-i\omega t + \frac{ikx}{1+\alpha m}\}\} \quad \text{for } x >> a$$

$$\tilde{\phi}_{3P} \sim -\frac{ch_0}{2kd_3} \sin\frac{ka}{1-\alpha m} \mathrm{Re}\{\exp\{-i\omega t - \frac{ikx}{1-\alpha m}\}\} \quad \text{for } x << a,$$

(A5.16)

PART II

TRANSMISSION IN AND RADIATION FROM WHOLLY OR PARTLY ELASTIC WALLED DUCTS WITH FLOW

INTRODUCTION

Many authors have considered the interaction between sound and infinite or semi-infinite wave-bearing surfaces. Usually these surfaces have been taken to be alone in an infinite fluid so that their responses are not affected by the presence of any other restrictive surfaces, passive or active. LAMB (16) in 1959 was first to consider the diffraction problem of a semi-infinite elastic plate, but his analysis of the fluid-plate coupling system is incomplete. MORSE & INGARD (23) in 1968 dealt with coupled fluid-plate systems in some detail, their analysis failing to describe the systems in the regions in which were contained the most interesting phenomenon; consequently they identified spurious beaming effects to infinity. LYAMSHEV (20) includes the vital feature of a basic flow behind the plate, but his analysis, being similar to Lamb's, is lacking in the same way. The most valuable background to this work is that of CRIGHTON (7),(8), CRIGHTON & LEPPINGTON (9), and CANNELL (2), (3). In 1971 and 1979 Crighton successfully analysed the waves on a fluid loaded elastic plate and identified the special regions in which a beaming effect (Leaky waves) could be identified for moderately large distances and light fluid loading. CRIGHTON & LEPPINGTON (9) and CANNELL (2), (3) took the important step of considering semi-infinite geometries and employing the Wiener-Hopf technique. Although here we will restrict ourselves to solutions for the light fluid loading limit we will have the added complexity of not only a second wave-bearing surface, parallel to the first, but we will also consider the case of the fluid in the duct created by these two parallel surfaces being in constant uniform motion.

In Chapter 3 we study the interaction between an acoustic source, in an elastic walled duct containing flow, and the duct wall. The waves that propagate up and down stream in the duct, as well as those in the duct walls, are examined along with the disturbances present outside the duct, caused by the transmission through the duct walls. The effect of the confinement of the source in a duct is noted, as is the effect of the flow within the duct on the velocity potentials inside and outside as well as the waves on the duct surface.

In Chapter 4 we study the interaction of an acoustic plane wave travelling down an inhomogeneous duct, from the rigid walled section to the elastic walled section, with the wall discontinuity at the join and with the elastic parts of the wall. Here we examine the waves transmitted or reflected by the duct discontinuity, the waves present in the wall and also those present in the otherwise stationary fluid outside the duct.

CHAPTER 3: The transmission of sound from a point source in an elastic walled duct containing flow.

3.1 INTRODUCTION AND SUMMARY

In this chapter we investigate the two-dimensional problem of a point source in an infinite duct that has parallel elastic walls and contains uniform flow. This is the simplest model of a duct system that contains almost all the important features - wave-bearing surfaces - fluid flow - a disturbance (a point source). The only important effect, apart from the third dimension, that is not incorporated is that of geometric complexity.

The choice of the thin elastic plate as a model of a wave-bearing surface is, of course, open to criticism. The TIMOSHENKO-MINDLIN plate equation is a better model of, say, underwater systems if the frequency is high but in the aerodynamic context the thin elastic plate equation is adequate. The adoption of the Timoshenko-Mindlin plate would result in considerable additional algebraic complexity.

In section 3.2 a complete definition of the system under examination is given. The model boundary value problem for small disturbances is formulated in section 3.3 by linearising the equations and boundary conditions, as set out in the previous section, by assuming that all disturbances are small compared with the basic uniform constant flow. The possible free wave solutions that the system can support are found in section 3.4 and the nature and properties of each solution are discussed. This then gives a good idea of the possible wave solutions and also locates the possible poles of any integrand of the problem, which will be discussed in detail in section 3.7. A Fourier transform analysis of the problem is done in section 3.5 giving Fourier inversion integrals for the velocity potentials and the wall displacement. In section 3.6 the duct's velocity potential is briefly discussed and its major features mentioned. A coordinate and variable of integration transformation given in section 3.7 allows the first estimate, for most angles θ, of the external potential to be found in section 3.8. Sections3.9 and 3.10 deal with the special asymptotics needed when θ is near 'Mach Angles' and it is at these that we find a beaming effect at moderate distances (Leaky waves). The behaviour of this external velocity potential is summarised in section 3.11 and the wall displacement is discussed in section 3.12. Section 3.13 consists of observations on and a summary of the analysis.

3.2 PROBLEM DEFINITION

Inviscid, compressible fluid flows adiabatically in a two-dimensional infinite duct which has parallel elastic walls with identical still fluid outside the duct. The walls are a distance 2d apart at $y = \pm d$ where (x,y) are cartesian coordinates. The basic interior flow has uniform constant velocity with potential $\phi = Ux$.

Sound is produced by pulsations of a line source at $(x,y) = (0,0)$ of strength $\mathrm{Re}\{2\lambda e^{-i\omega t}\}$ (ω is the frequency, t the time variable). The time-harmonic oscillations of this source cause perturbations in the uniform flow which in turn, via the elastic walls of the duct, cause perturbations outside the duct ,

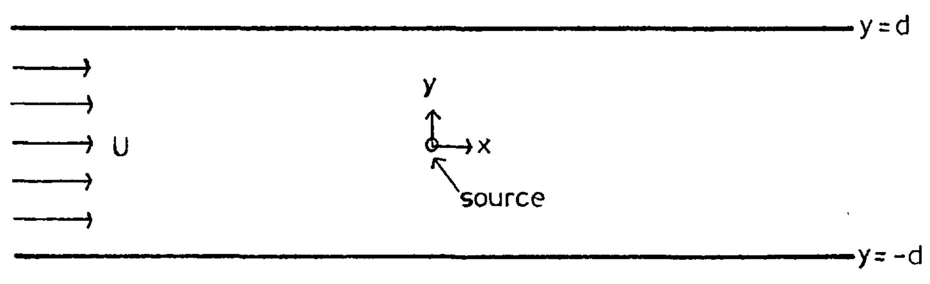

Figure 1

As shown in Figure 1 the problem is symmetric about the x-axis, so the equivalent half-space problem will be solved.

The half-space problem consists of a basic uniform flow U down a duct of width d with the plane $y = 0$ forming an infinite barrier and a thin elastic plate at $y = d$. The plane $y = 0$ consists of a rigid baffle filling the half-planes $y = 0 \quad x > 0$ and $y = 0 \quad x < 0$

with a time-periodic source of strength $\mathrm{Re}\{\lambda e^{-i\omega t}\}$ oscillating into the duct at y = 0, x = 0, as shown in Figure 2.

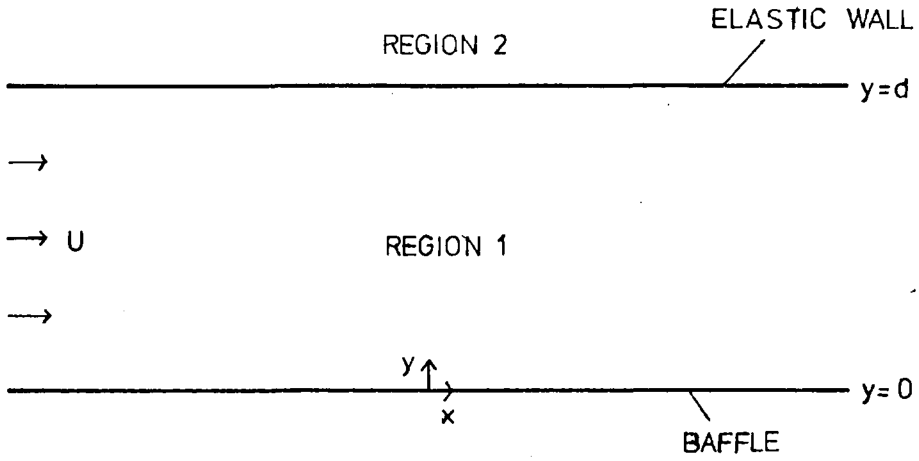

Figure 2

The small amplitude vertical displacement, $\eta(x,t)$, of the elastic plate above the plane y = d is governed by the equation

$$D \frac{\partial^4}{\partial x^4} \eta(x,t) + 2Mh \frac{\partial^2}{\partial t^2} \eta(x,t) = - [p]_-^+ \quad \text{on } y = d + \eta(x,t) \tag{3.2.1}$$

see MORSE & INGARD (23) where

$D = \dfrac{2Eh^3}{3(1-\sigma_p^{\,2})}$ is the wall's boundary stiffness

E is the Young's modulus for the wall

2h is the wall's thickness

σ_p is Poisson's ratio

M is the wall's volume density

$[p]_-^+ \equiv p(x,d+0)-p(x,d-0)$ is the fluid pressure difference across the wall.

If Φ is the total velocity potential write

$$\Phi = \begin{cases} Ux + \hat{\phi}_1 & \text{in Region 1} \\ \hat{\phi}_2 & \text{in Region 2} \end{cases} \qquad (3.2.2)$$

where $\hat{\phi}_1$ and $\hat{\phi}_2$ are the velocity potentials of the disturbance in Regions 1 and 2 respectively.

If we now write

$$\hat{\phi}_j = \text{Re}\{\phi_j e^{-i\omega t}\} \text{ and } \qquad \Phi = \Phi_j \qquad j = 1,2 \qquad (3.2.3)$$

to take into account the periodic response of the disturbance to the source then we have

$$\Phi_1 = Ux + \text{Re}\{\phi_1 e^{-i\omega t}\}$$
$$\Phi_2 = \text{Re}\{\phi_2 e^{-i\omega t}\} \qquad (3.2.4)$$

We now wish to obtain solutions for ϕ_1, ϕ_2 and η. The velocity potentials of the disturbance are calculated by assuming that the disturbance is small compared with the basic flow so that the equations of motion can be linearised with respect to small ϕ_i, by neglecting terms of order $(\phi_i)^2$, and by replacing the boundary conditions on $y = d + \eta(x,t)$ with ones on $y = d$. Fourier transforms can then be taken and integral representations found for ϕ_1 and ϕ_2. Contour integration can then be used to find the exact solution formally and then the leading order terms for the far field $(kr \to \infty)$ in the light fluid loading limit.

3.3 THE MODEL PROBLEM FROM LINEARISATION

We now formulate the boundary value problem for ϕ_1, ϕ_2 and η.

Equation (3.2.1) governs the motion of the elastic wall and if the parameters k, μ, $\hat{\varepsilon}$, ε are defined by

$$\frac{2Mh\omega^2}{D} = \mu^4 \; ; \qquad \hat{\varepsilon} = \frac{\rho_0}{k} \cdot \frac{1}{Mh} \; ; \qquad \varepsilon = \frac{\rho_0 \omega^2}{D} \; ;$$

$$\varepsilon = \frac{\hat{\varepsilon} k \mu^4}{2} \; ; \qquad k = \frac{\omega}{c} \tag{3.3.1}$$

where ρ_0 is the density of the fluid in the undisturbed state and c is the sound speed in the fluid, then k is a wave number for the acoustic source and $\hat{\varepsilon}$ is a non-dimensional fluid loading parameter which is a measure of the inertia of a column of fluid one acoustic wavelength in depth relative to the inertia of the wall beneath it.

The linearised version of (3.2.1) is

$$\frac{\partial^4}{\partial x^4} \eta(x,t) + \frac{\mu^4}{\omega^2} \frac{\partial^2}{\partial t^2} \eta(x,t) = - \frac{\varepsilon}{\rho_0 \omega^2} [p]_-^+ \quad \text{on } y = d, \tag{3.3.2}$$

At a later stage we will consider $\hat{\varepsilon} \ll 1$ ($\varepsilon \ll 1$ for fixed k and μ) that is when the coupling between the wall and the fluid is small. However we will not consider an expansion in small ε of relevant quantities such as potentials at this stage as much information, particularly the nature of the waves present in the disturbance, depends on the $O(\varepsilon^2)$ terms and expansion to this order here would result in too much algebraic complexity.

The boundary condition on both sides of the wall between the wall and the fluid is that the material derivative is zero

$$\frac{D}{Dt}\{y-\eta(x,t)\} = 0 \qquad \text{on} \quad y = d + \eta(x,t) \quad , \tag{3.3.3}$$

After linearisation this yields, on the Region 2 side

$$\frac{\partial\hat{\phi}_2}{\partial y} = \frac{\partial}{\partial t}\eta(x,t) \qquad \text{on } y = d \quad , \tag{3.3.4}$$

and on the Region 1 side the convected equivalent

$$\frac{\partial\hat{\phi}_1}{\partial y} = \frac{\partial}{\partial t}\eta(x,t) + U\frac{\partial}{\partial x}\eta(x,t) \qquad \text{on} \quad y = d \tag{3.3.5}$$

thus ensuring that the normal displacement across the plate is continuous. The extra term in (3.3.5) is the convected term present solely due to the flow in the duct.

Writing $\eta(x,t) = \text{Re}\{\eta(x)e^{-i\omega t}\}$ to take into account the response of the plate to the harmonic excitation of the source gives for equations (3.3.4) and (3.3.5)

$$\frac{\partial\phi_2}{\partial y} = -\,i\omega\eta(x) \qquad \text{on } y = d \tag{3.3.6}$$

$$\frac{\partial\phi_1}{\partial y} = -\,i\omega\eta(x) + U\frac{\partial}{\partial x}\eta(x) \qquad \text{on } y = d \; . \tag{3.3.7}$$

The boundary condition on the baffle y = 0 is

$$\frac{\partial\phi_1}{\partial y} = \lambda\delta(x) \tag{3.3.8}$$

where the Dirac function $\delta(x)$ allows for the presence of the source.

Additional conditions are that $\hat{\phi}_1$ and $\hat{\phi}_2$ represent outgoing waves at infinity.

LIGHTHILL[(19) equation (61)] gives for the equation of motion of the fluid

$$c^2\nabla^2\Phi = \frac{\partial^2\Phi}{\partial t^2} + 2\frac{\partial\Phi}{\partial x_i}\frac{\partial^2\Phi}{\partial x_i\partial t} + \frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\frac{\partial^2\Phi}{\partial x_i\partial x_j} \qquad (3.3.9)$$

where ∇^2 is the Laplacian operator.

Substituting (3.2.4) into (3.3.9) and linearising with respect to small ϕ_1 by neglecting $O(\phi_1{}^2)$ terms gives a convected Helmholtz equation for ϕ_1

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(k+im\frac{\partial}{\partial x}\right)^2\right]\phi_1 = 0 \qquad 0 \le y \le d \qquad (3.3.10)$$

where $m = \frac{U}{c}$ is the Mach number for the flow.

A similar substitution for ϕ_2 yields the classic Helmholtz equation for ϕ_2

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right]\phi_2 = 0 \qquad (3.3.11)$$

Lastly the pressure difference $p]_-^+$ across the plate is

$$p]_-^+ = p_2 - p_1 \qquad (3.3.12)$$

where the excess pressures $(\hat{p}_1, \hat{p}_2)$ on either side of the plate are given by

$$\hat{p}_j = \mathrm{Re}\{p_j\, e^{-i\omega t}\} \qquad j = 1,2 \qquad (3.3.13)$$

$$p_1 = \rho_0\{i\omega\phi_1 - U\frac{\partial\phi_1}{\partial x}\} \quad ; \qquad p_2 = i\omega\rho_0\phi_2 \ . \qquad (3.3.14)$$

3.4 POSSIBLE WAVE SOLUTIONS

Before undertaking a formal analysis of this problem by taking Fourier Transforms it is instructive to consider what waves produced by the coupling of the fluid and the plate it is possible for the half-space system to sustain.

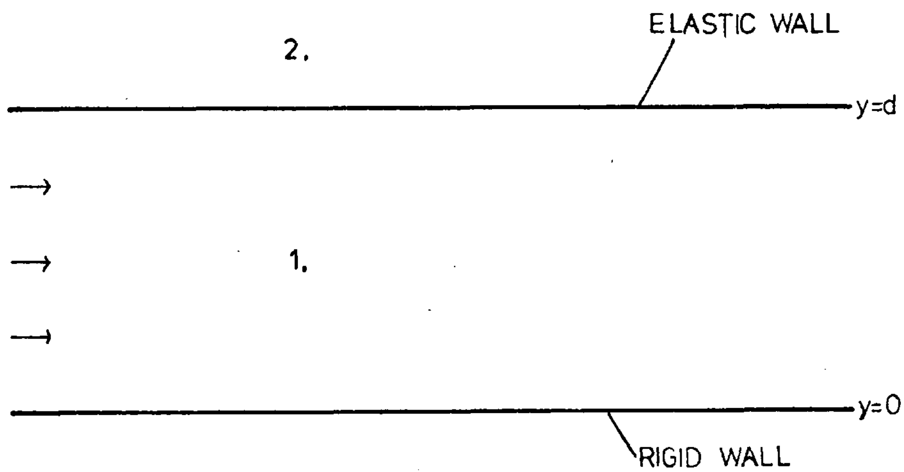

Figure 3

We wish to determine what free wave solutions of equations (3.3.2), (3.3.6), (3.3.7), (3.3.8), (3.3.9), (3.3.10), (3.3.11) of the form

$$\phi_1 = F_1(y)e^{-isx} \quad ; \quad \phi_2 = F_2(y)e^{-isx} \tag{3.4.1}$$

exist.

Substitution into equation (3.3.10) for ϕ_1 in (3.4.1) gives

$$F_1''(y)-(s^2(1-m^2) - 2mks-k^2)\ F_1(y) = 0 \ . \tag{3.4.2}$$

So

$$F_1(y) = Ae^{\gamma(s)y} + Be^{-\gamma(s)y} \tag{3.4.3}$$

where A and B are undetermined constants and where

$$\gamma(s) = \gamma = (s^2(1-m^2) - 2mks-k^2)^{\frac{1}{2}} \qquad (3.4.4)$$

has branch cuts from -k/1+m to -∞ and from k/1-m to ∞ along the real axis such that $\gamma(0) = -ik$ (see Figure 4).

The conditions of zero normal derivative on y = 0 and normalisation yield

$$F_1(y) = \cosh\gamma(s)y; \quad \phi_1(x,y) = \cosh\gamma(s)y\ e^{-isx} . \qquad (3.4.5)$$

Substitution into equation (3.3.11) for ϕ_2 in (3.4.1) gives

$$F_2''(y) - (s^2-k^2)\ F_2(y) = 0 \qquad (3.4.6)$$

so

$$F_2(y) = \hat{A}e^{\hat{\gamma}(s)y} + \hat{B}^{-\hat{\gamma}(s)y} \qquad (3.4.7)$$

where $\hat{A}$ and $\hat{B}$ are also undetermined constants and $\hat{\gamma}(s)$ is the non-convected form of $\gamma(s)$ i.e. $\hat{\gamma}(s) = \gamma(s)\big|_{m=0} = \hat{\gamma}$ has branch cuts from ±k to ±∞ (see Figure 4).

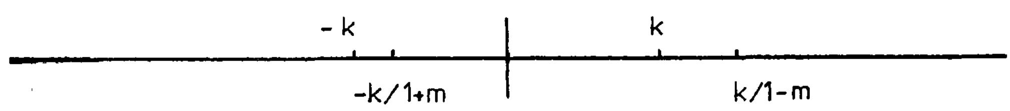

Figure 4

Thus for real s (with s below the cut from k to ∞ and above the cut from $\frac{-k}{1-m}$ to ∞ , $Re(\gamma) \geq 0$ and $Re(\hat{\gamma}) \geq 0$.

An outgoing finite waves at infinity requires

$$\phi_2 = Be^{-\hat{\gamma}(s)y-isx} \quad . \tag{3.4.8}$$

Applying boundary conditions (3.3.6), (3.3.7) and (3.3.8) on $y = d$ via $\eta(x)$ gives

$$B = -e^{-\hat{\gamma}(s)d} \gamma\sinh\gamma d \;\; \hat{\gamma}^{-1}(1 + \frac{ms}{k})^{-1} \quad . \tag{3.4.9}$$

The substitution of these expressions for ϕ_1 and ϕ_2 along with those for $\eta(x)$ and the pressure excesses p_1 and p_2 into the elastic plate equation (3.3.2) gives as the condition for the possible values of s the roots of

$$F(s) = (s^4-\mu^4)\,\hat{\gamma}(s)\gamma(s)\,\sinh\gamma(s)d - \varepsilon[\gamma(s)\sinh\gamma(s)d$$

$$+ (1 + \frac{ms}{k})^2\,\hat{\gamma}(s)\cosh\gamma(s)d] = 0 \quad . \tag{3.4.10}$$

In Appendix 1 the method for finding the zeros of F(s), for small ε, is outlined and the zeros and their nature are listed.

Now the possible behaviour of ϕ_2 when ε is small will be considered. The cases $\mu < k$ and $\mu > k$ will again be dealt with separately and it is the exponent in which we have the most interest. The exponentials of the potentials are of the form

$$\exp\{ax+by\} \equiv \exp\{(a_1+ia_2)x + (b_1+ib_2)y\} \tag{3.4.11}$$

where a_1, a_2, b_1 and b_2 are all real. The real part a_1 of the complex coefficient a in the exponent indicates either growth or decay in x. Each of these coefficients depends on the small parameter ε. If a_1 is $O(\varepsilon)$, for example, this will be referred to as an $O(\varepsilon)$ decay in x. The imaginary part of a, a_2, in the exponent indicates a travelling wave in x.

If a_2 is $O(\varepsilon)$, for example, this will be referred to as an $O(\varepsilon)$ wave in x. The coefficients b of y will be described in a similar way.

3.4.1 BELOW COINCIDENCE $k < \mu$

The solutions associated with S_1 and S_2 are waves in the negative and positive x directions respectively and have exponential decay in y; thus they are subsonic surface waves (assuming decay and not growth) with exact wave numbers S_1 and S_2. The solutions from S_3 and S_4 are waves at $O(1)$ in y but have an $O(\varepsilon)$ decay in y and an $O(1)$ decay in x. The S_5 and S_6 solutions are waves in the x-plane with $O(\varepsilon)$ decay in y; these too then are subsonic surface waves on the plate. The S_7 contribution has, in both x and y, waves at $O(1)$ and at $O(\varepsilon)$ but decay at $O(\varepsilon^2)$. S_8 produces a subsonic surface wave in a similar way to S_1, $S_{\pm n}$ for $n \le N$ give contributions similar to S_7 and $S_{\pm n}$ $(n>N)$ give contributions similar to S_3 and S_4.

3.4.2 ABOVE COINCIDENCE $k > \mu$

The nature of the contributions from S_3, S_4, S_7, S_8 and $S_{\pm n}$ is unchanged. S_1 and S_2 give waves at $O(1)$ but decay at $O(\varepsilon)$. S_5 and S_6 give an $O(1)$ wave in x but $O(\varepsilon)$ decay in both x and y.

Although, as mentioned above, it is possible for the system to be satisfied by potentials which exhibit exponential growth in some directions, in practice such solutions are seldom triggered off in regions where they are not ultimately decaying at large distances; they might be present for instance in the case of unstable motion e.g. vortex sheets.*1

By looking at this system we have determined the possible free-wave solutions and their behaviour. Now Fourier transforms of the problem will be taken.

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*1

THERE ARE CASES IN WHICH SPACIAL AND TEMPORAL INSTABILITIES HAVE BEEN CONFIRMED FOR UNIFORM FLOW BUT FOR THE PARAMETER RANGE CONSIDERED HERE IT HAS BEEN SHOWN, BY USE OF ROUCHÉ'S THEOREM IN APPENDIX 1, THAT THIS IS NOT THE CASE AND THAT THE POLES DO NOT CROSS THE PATH OF INTEGRATION IN ACCORDANCE WITH CAUSALITY REQUIREMENTS.

3.5 FOURIER ANALYSIS

Defining the Fourier transform $\bar{\phi}_j$ of ϕ_j by

$$\bar{\phi}_j = \int_{-\infty}^{+\infty} \phi_j e^{isx}\, dx \quad , \qquad j = 1,2 \quad , \tag{3.5.1}$$

its inverse is given by

$$\phi_j = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{\phi}_j \, e^{-isx}\, ds \, , \quad j = 1,2 \quad . \tag{3.5.2}$$

Taking transforms of (3.3.10) and (3.3.11) yields

$$-(1-m^2)s^2\bar{\phi}_1 + \frac{d^2}{dy^2}\bar{\phi}_1 + 2mks\bar{\phi}_1 + k^2\bar{\phi}_1 = 0 \tag{3.5.3}$$

and

$$\frac{d^2}{dy^2}\bar{\phi}_2 - (s^2-k^2)\bar{\phi}_2 = 0 \quad . \tag{3.5.4}$$

In the physical problem k and μ are real but for mathematical convenience, to improve the convergence of the transform integrals, k and μ are taken to be slightly complex, $k = k_1 + ik_2$, $\mu = \mu_1 + i\mu_2$ (k_1, k_2, μ_1, μ_2 real) for small positive k_2, μ_2. Later on we let $k_2, \mu_2 \to 0$ to give the result.

The general solutions to (3.5.3) and (3.5.4) are

$$\bar{\phi}_1 = A_1 e^{\gamma(s)y} + B_1 e^{-\gamma(s)y} \tag{3.5.5}$$

$$\bar{\phi}_2 = A_2 e^{\hat{\gamma}(s)y} + B_2 e^{-\hat{\gamma}(s)y} \tag{3.5.6}$$

where $\gamma(s)$ and $\hat{\gamma}(s)$ are as given by equation (3.4.4) but now have branch cuts from $-k/(1+m)$ to $-\infty$ and $k/(1-m)$ to ∞ and from $\pm k$ to $\pm\infty$ along straight lines of constant argument. (see Figure 5). So for real s; $Re(\gamma(s)) \geq 0$ and $Re(\hat{\gamma}(s)) \geq 0$.

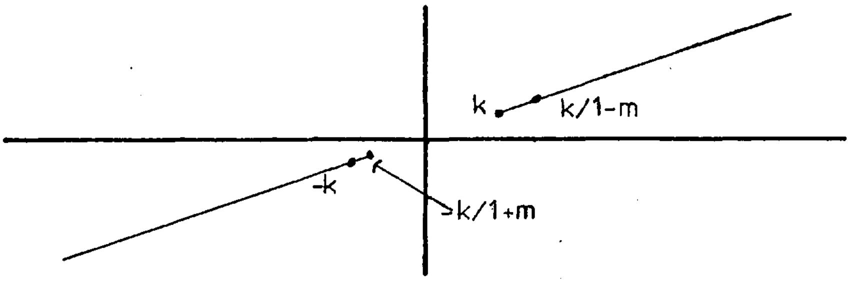

Figure 5: S-plane

Let $\bar{\eta}$ be the transform of $\eta(x)$.

Taking transforms of (3.3.6) and (3.3.7), substituting for $\bar{\phi}_1$ and $\bar{\phi}_2$ from (3.5.5) and (3.5.6), applying the transform of (3.3.8) and the outgoing condition at infinity yields

$$\bar{\phi}_1 = -[i\bar{\eta}(\omega+Us)\cosh\gamma y + \lambda\cosh\gamma(y-d)]\gamma^{-1}\operatorname{cosech}\gamma d \qquad (3.5.7)$$

$$\bar{\phi}_2 = ick\hat{\gamma}^{-1}\bar{\eta}\exp\{\hat{\gamma}(d-y)\} \quad . \qquad (3.5.8)$$

Taking transforms of equations (3.3.14) gives

$$\bar{p}_1 = i\rho_0(\omega+Us)\bar{\phi}_1 \quad ; \quad \bar{p}_2 = i\omega\rho_0\bar{\phi}_2 \quad . \qquad (3.5.9)$$

Now substituting into (3.3.2) and taking transforms gives

$$(s^4-\mu^4)\bar{\eta} = -\frac{i\varepsilon}{\omega}\{\bar{\phi}_2 - (1+\frac{ms}{k})\bar{\phi}_1\} \quad \text{on} \quad y = d \ , \qquad (3.5.10)$$

On substituting for the $\bar{\phi}_j$'s,

$$\bar{\eta} = -i(1+\frac{ms}{k})\ \rho_0\ \frac{\omega}{D}\ \lambda\hat{\gamma}\ F(s)^{-1} \qquad (3.5.11)$$

where F(s) is given by equation (3.4.10), thus

$$\bar{\phi}_1 = -\lambda\gamma^{-1} \operatorname{cosech}\gamma d[\cosh\gamma(y-d) + \varepsilon\hat{\gamma}(1+ \frac{ms}{k})^2 \cosh\gamma y\ F(s)^{-1}] , \tag{3.5.12}$$

$$\bar{\phi}_2 = \varepsilon(1+ \frac{ms}{k})\lambda\ F(s)^{-1} \exp\{\hat{\gamma}(d-y)\}. \tag{3.5.13}$$

Inverting these transforms gives

$$\phi_1 = -\frac{\lambda}{2\pi} \int_{\Gamma_1} \gamma^{-1}\operatorname{cosech}\gamma d[\cosh\gamma(y-d) + \varepsilon\hat{\gamma}(1+ \frac{ms}{k})^2 \cosh\gamma y\ F(s)^{-1}]e^{-isx}\, ds , \tag{3.5.14}$$

$$\phi_2 = \frac{\lambda\varepsilon}{2\pi} \int_{\Gamma_1} (1+ \frac{ms}{k}) \exp\{\hat{\gamma}(d-y)-isx\}F(s)^{-1}ds , \tag{3.5.15}$$

$$\eta(x) = -\frac{i\varepsilon\lambda}{2\pi\omega} \int_{\Gamma_1} (1+ \frac{ms}{k})\hat{\gamma}\ F(s)^{-1}e^{-isx}ds \tag{3.5.16}$$

where here we have let $k_2 \to 0$, that is k is now real again and Γ_1 is the path in the S-plane shown in Figure 6.

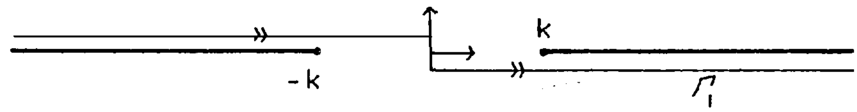

Figure 6: S-plane

Γ_1 is 'above' the real axis for Re(s) < 0 and 'below' the real axis for Re(s) > 0. This ensures an outgoing wave at infinity for ϕ_2. Notice that the cuts in the plane due to $\gamma(s)$ are not in evidence because in F(s), as well as the rest of the integrands, $\gamma(s)$ combines with other such terms to be regular in S. Notice also that these expressions hold for all ε.

We now look briefly at ϕ_1 and then go onto examine the behaviour of ϕ_2 in some detail. The analysis involved in this will then be of use in examining $\eta(x)$.

3.6 THE DUCT POTENTIAL ϕ_1

Equation (3.5.14) gives for the duct potential ϕ_1

$$\phi_1 = -\frac{\lambda}{2\pi} \int_{\Gamma_1} \frac{\cosh\gamma(y-d)e^{-isx}}{\gamma\sinh\gamma d}\, ds - \frac{\lambda\varepsilon}{2\pi} \int_{\Gamma_1} \hat{\gamma}\left(1+\frac{ms}{k}\right)^2 \cosh\gamma y\; F(s)^{-1} e^{-isx} ds \qquad (3.6.1)$$

The first of these two integrals represents the potential due to a point source in an infinite rigid walled duct with flow. Inversion of the integral gives the potential as an infinite sum of modes, some travelling up and some down the duct, and some exhibiting exponential decay away from the source.

The second term is the correction term due to the presence of elasticity in the duct walls. So putting $\varepsilon = 0$ and thus making the duct rigid walled makes this integral identically zero. Of course the presence of the duct is reflected in this integral too by the hyperbolic functions.

It is very interesting to note that the velocity potential in the duct can be split into these two very distinct parts, one due to the duct, one due to the elasticity of the walls, even though no assumptions have been made about the size of the fluid loading parameter ε. However a little further examination, of the integrand of the second term, reveals that the poles of the integrand - required if any inversion is going to be undertaken, are not to be found simply unless $\varepsilon \ll 1$. This is because it is only in this case that we can say that the zeros of F(s) are perturbations away from those of f(s). If ε were considered large then the zeros of g(s) would need to be known exactly.

3.7 THE EXTERNAL VELOCITY POTENTIAL ϕ_2

To study ϕ_2 a new coordinate system (r,θ) is introduced.

$$x = r\cos\theta \ ; \quad y-d = r\sin\theta \tag{3.7.1}$$

Region 2 now becomes

$$0 \le r < \infty \ ; \quad -\pi \le \theta < \pi \tag{3.7.2}$$

and (3.5.15) becomes

$$\phi_2(r,\theta) = \frac{\lambda\varepsilon}{2\pi} \int_{\Gamma_1} (1+\frac{ms}{k}) \ F(s)^{-1} \ \exp\{-isr\cos\theta - \hat{\gamma}(s)r\sin\theta\}ds \ . \tag{3.7.3}$$

Remember there are branch cuts in the S-plane from $\pm k$ to $\pm\infty$ due to the branches of $(s^2-k^2)^{\frac{1}{2}}$.

Using a substitution due to CLEMMOW (6), $s = k\cos\Theta$ where $\Theta = \alpha + i\beta$, α and β real, thus maps the S-plane onto the strip $0 \le \alpha \le \pi$, $-\infty < \beta < \infty$ in the complex Θ plane such that $\hat{\gamma}(s) = -ik\sin\Theta$.

It follows that

$$\phi_2(r,\theta) = -\frac{\varepsilon\lambda k}{2\pi} \int_{\Gamma_2} (1+m\cos\Theta)\tilde{F}(\Theta)^{-1} \ \exp\{-ikr\cos(\theta+\Theta)\}\sin\Theta d\Theta \tag{3.7.4}$$

where

$$\tilde{F}(\Theta) = F(k\cos\Theta) \tag{3.7.5}$$

and Γ_2 is as shown in Figure 7.

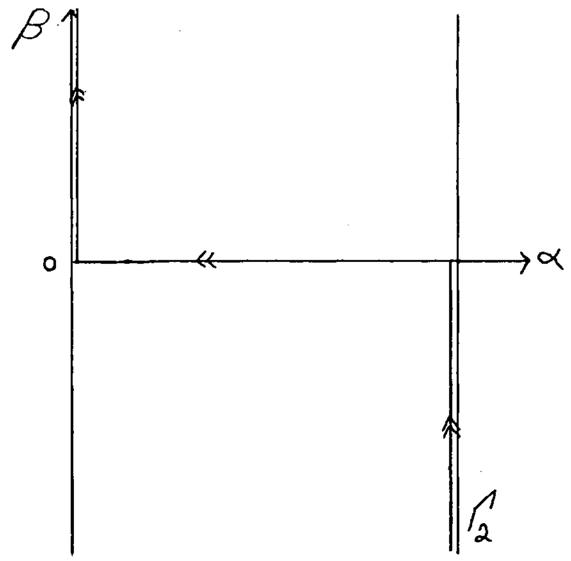

Figure 7: Θ-plane

Notice that unlike the S-plane there are no branch cuts in the Θ-plane.

To obtain the far field behaviour of ϕ_2 the 'method of steepest descents' will be used to find the asymptotic behaviour in the limit $kr \to \infty$ for small ε.

Now the exponent term $-ikr\cos(\theta+\Theta)$ has a saddle point at $\Theta = \pi - \theta$ and the path of steepest descent is on that branch of $\cos(\alpha+\theta)\cosh\beta = -1$ which behaves, near the saddle point, like $\alpha + \beta = \pi - \theta$. This path is called Γ_3 (Figure 8).

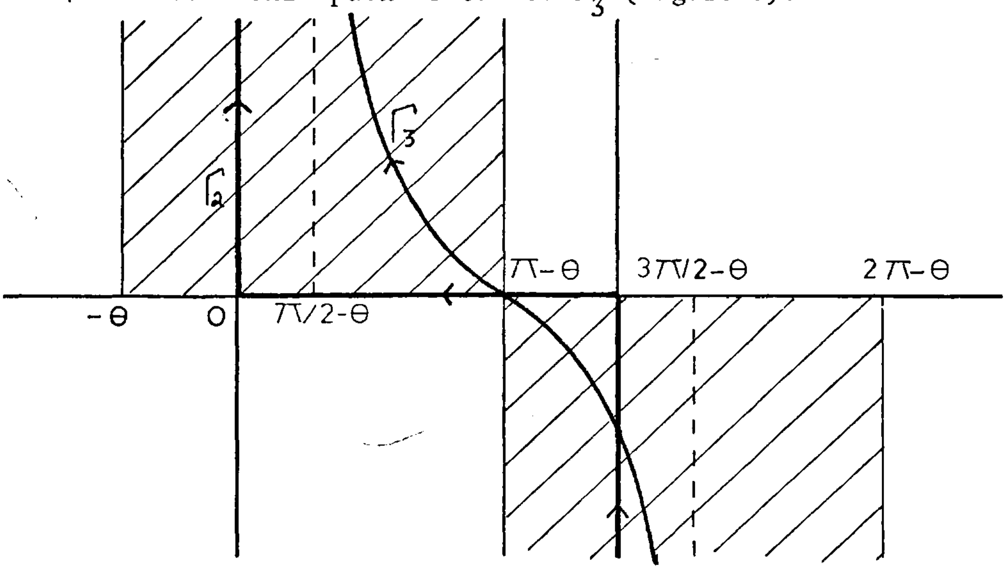

Figure 8: Θ-plane

If Γ_2 is deformed onto Γ_3 then there is no contribution from the connecting sections at infinity but Cauchy's Residue theorem must be employed to account for any poles picked up in the process.

Notice (Figure 8) that most of Γ_3 lies well inside the shaded regions which show where the integrand is exponentially small as $kr \to \infty$ (the far field). Γ_3 may be further deformed anywhere well inside this region, to avoid passing through or near most poles, as such deformations contribute only exponentially small terms [see DAVIS & LEPPINGTON (10)].

This then gives

$$\phi_2(r,\theta) = \int_{\Gamma_2} (\quad) d\Theta = \phi_S + 2\pi i \sum (\pm \text{ Residues captured}) \qquad (3.7.6)$$

where

$$\phi_S = \int_{\Gamma_3} (\quad) d\Theta \ , \qquad (3.7.7)$$

In order to determine an approximation for ϕ_2 we need to know the positions of the poles and zeros of the integrand. We will then be in a position to say if the poles are captured or not on deformation of Γ_2 onto Γ_3 and also if their contribution, after capture, is significant; this depends on their nature. If a captured pole lies well inside the shaded region, its contribution is exponentially small for sufficiently large kr. Thus captured poles are significant only if they are near the edges of the shaded regions, where the exponential decay is slow (or else outside the shaded regions - which never occurs here).

The poles of the integrand correspond to the zeros of F(s) that were calculated in Appendix 1 for section 3.4.

The positions of these poles, relative to the paths of integration, on both the S and Θ planes are shown in Figures 9 to 14, the cases $\mu > k$ and $\mu < k$ being treated independently.

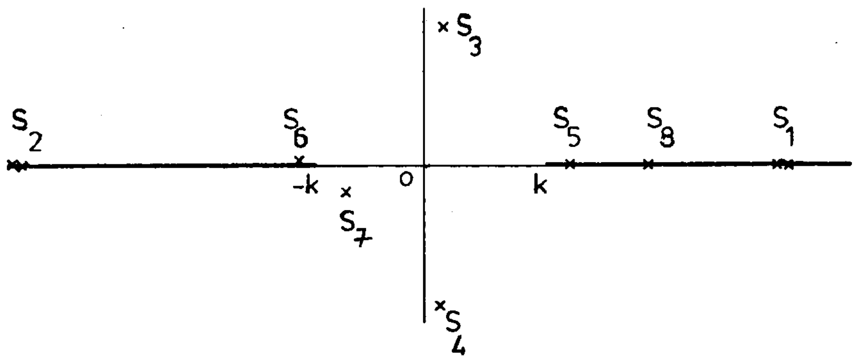

Figure 9: S-plane

Figure 9 shows the position of the poles of the integrand corresponding to the zeros $S_1,\ldots,S_8$ for $\mu > k$ and Figure 10 shows their position in the Θ-plane.

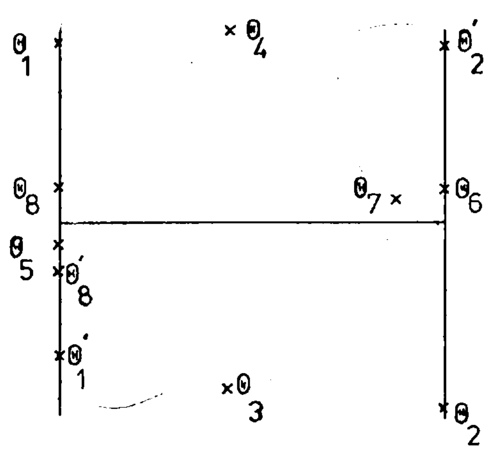

Figure 10: Θ-plane

Figures 11 and 12 show similar positions for $S_{\pm n}$ and $\Theta_{\pm n}$, they apply for $\mu > k$: $\mu < k$

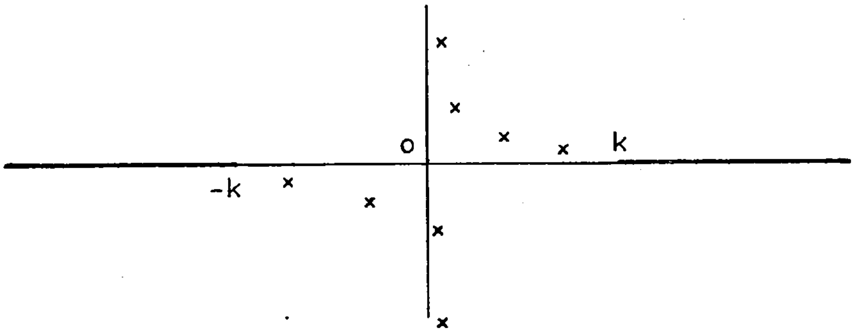

Figure 11: S-plane, x denotes $S_{\pm n}$

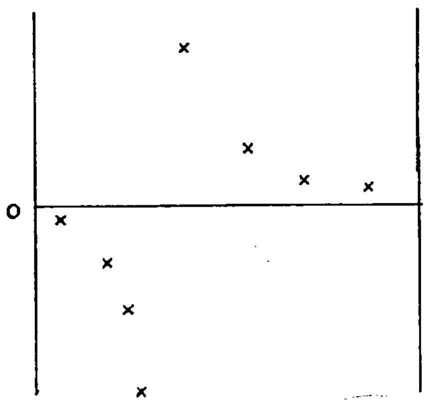

Figure 12: Θ-plane, x denotes $\Theta_{\pm n}$

Figures 13 and 14 are similar to 9 and 10 but are for the case $\mu < k$.

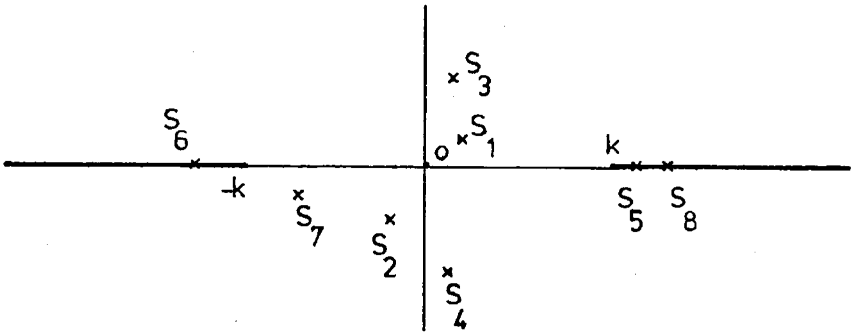

Figure 13: S-plane

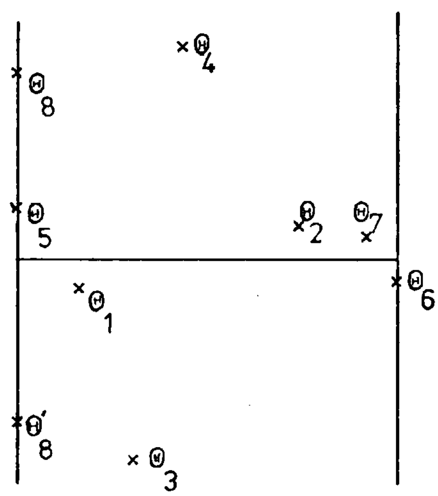

Figure 14: Θ-plane

Before proceeding it is instructive to make some observaticns on the positions of the poles relative to the path of integration.

BELOW COINCIDENCE $k < \mu$

It can be seen from Figures 8 and 10 that Θ_5 and Θ_6 can never be captured by the deformation of Γ_2 to Γ_3 nor can Θ_1', Θ_2' and Θ_8'. All the rest must be included in the following analysis, as must Θ_5 and Θ_6. For although they cannot be captured they can be near the saddle point.

ABOVE COINCIDENCE $\mu < k$

It can be seen from Figures 8 and 12 that all except Θ_8' can be captured by deformation.

In Appendix 2 the method for finding the residues of $F(s)^{-1}$ at the points in the S-plane is outlined and the resulting residues are listed. The potentials ψ_{11}, ψ_{21} etc. corresponding to the residues of the whole integrand are also listed.

From equation (3.7.6)

$$\phi_2(r,\theta) \sim \phi_s + \sum_j \pm 2\pi i \psi_{\ell j} \qquad \ell = 1,2 \qquad (3.7.8)$$

where j is summed over those poles that are captured, and give significant contributions. First the case of the saddle point not near any poles or zeros is dealt with.

3.8 SADDLE POINT NOT NEAR A POLE OR ZERO

When the saddle point of Γ_3 is not in the immediate vicinity of a pole of $F(s)^{-1}$ then using the deformations as shown in Figure 15 for $0 < \theta \leq \pi/2$ and in Figure 16 for $\pi/2 < \theta < \pi$ all other poles in the Θ plane can be avoided [see DAVIS & LEPPINGTON (10)]. This means that it does not matter if these poles are captured or not since their contribution on capture is only of the same order and magnitude as the error in the potential, that is to say they are of higher order than that to which the approximation is being taken.

ϕ_S can then be approximated to by the 'method of steepest descents' on

$$\phi_S = -\frac{\lambda k\varepsilon}{2\pi}\int_{\Gamma_3}(1+m\cos\Theta)\tilde{F}(\Theta)^{-1}\exp\{-ikr\cos(\theta+\Theta)\}\sin\Theta d\Theta \qquad (3.8.1)$$

This then gives for the leading term in ϕ_S, ϕ_S' for $kr >> 1$

$$\phi_S' \sim \frac{i\lambda\varepsilon}{(2\pi kr)^{\frac{1}{2}}}\exp\{ikr-i\pi/4\}(1-m\cos\theta)[(k^4\cos^4\theta-\mu^4)\gamma(-k\cos\theta)\sinh\gamma(-k\cos\theta)d]^{-1} \qquad (3.8.2)$$

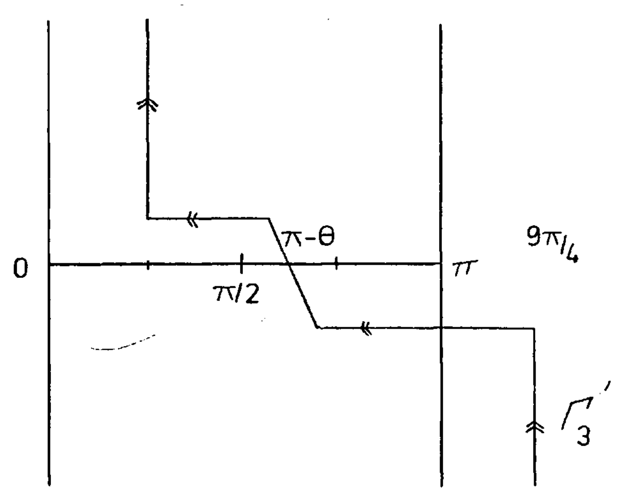

Figure 15: Θ-plane

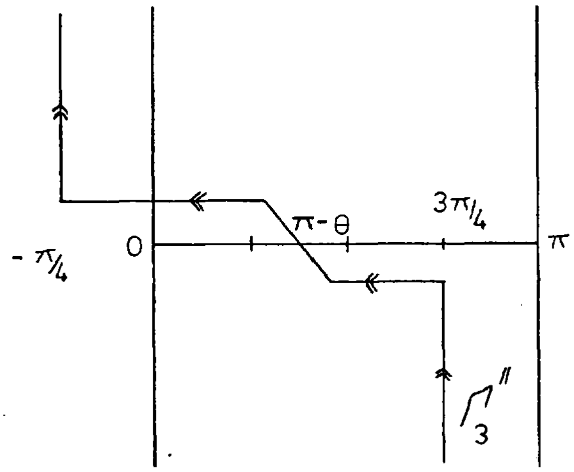

Figure 16: Θ-plane

BELOW COINCIDENCE $\mu > k$

From Figure 10 it can be seen that

$$\phi_2(r,\theta) \sim \phi_S' - 2\pi i\psi_{12} \quad \text{to leading order for } 0 < \theta < \frac{\pi}{2} \tag{3.8.3}$$

if Γ_3 is deformed as in Figure 15 and

$$\phi_2(r,\theta) \sim \phi_S' + 2\pi i\psi_{11} + 2\pi i\psi_{18} \quad \text{to leading order for } \frac{\pi}{2} < \theta < \pi \tag{3.8.4}$$

if Γ_3 is deformed as in Figure 16.

The poles at S_7 and $S_{\pm n}$ $(n \leq N)$ do not contribute significantly, even when captured, if they are not near the saddle point.

ABOVE COINCIDENCE $k > \mu$

From Figure 14 it can be seen that

$$\phi_2(r,\theta) \sim \phi_S' \quad \text{to leading order for } 0 < \theta < \frac{\pi}{2} \tag{3.8.5}$$

if Γ_3 is deformed as in Figure 15.

It can also be seen that

$$\phi_2(r,\theta) \sim \phi_S' + 2\pi i\psi_{28} \text{ to leading order for } \pi/2 < \theta < \pi \tag{3.8.6}$$

if Γ_3 is deformed as in Figure 16.

For this case the poles at S_1, S_2, S_5, S_6, S_7 and $S_{\pm n}$ $(n \leq N)$ do not contribute significantly even when captured unless they are near the saddle point.

3.9 SADDLE POINT NEAR A POLE BUT NOT A ZERO

When one of the poles of $F(s)^{-1}$ is near the saddle point then the method of steepest descents (as employed on equation (3.8.1) to give equation (3.8.2)) is no longer applicable. A different approximation to replace ϕ_S', taking into account the presence of the pole, must be found. The potential is then given by equations (3.8.3) to (3.8.6) with ϕ_S' replaced by ϕ_{SP} which is now calculated.

Following the work of CRIGHTON (7) and (8), equation (3.8.1) for ϕ_S can be recast in the form of a FRESNEL integral [see CLEMMOW (6)]. The appropriate expansion as formed in Appendix 3 along with Cauchy's Residue theorem can then be used to calculate ϕ_{SP}. So $\phi_{SP} = \phi_S$ (when pole not captured)

$\phi_{SP} = \phi_S + 2\pi i \times$ Residue (when pole captured) .

(3.9.1)

Equation (3.8.1) is of the form

$$\phi_S = \int_{\Gamma_3} \hat{F}(\Theta)\exp\{-ikr\cos(\theta+\Theta)\}d\Theta \tag{3.9.2}$$

where

$$\hat{F}(\Theta) = -\frac{\lambda\varepsilon k}{2\pi}(1+m\cos\Theta)\sin\Theta\,\tilde{F}(\Theta)^{-1} \quad . \tag{3.9.3}$$

If the case of the saddle point at $\Theta = \pi - \theta$ being near the pole at $\Theta = \Theta_j$ is considered then $F_j(\Theta)$ can be defined by

$$F_j(\Theta) = \sin\tfrac{1}{2}(\Theta-\Theta_j)\,\hat{F}(\Theta) \tag{3.9.4}$$

where $F_j(\Theta)$ has no pole at Θ_j and

$$\phi_S = \int_{\Gamma_3} F_j(\Theta)\left\{\sin\frac{\Theta-\Theta_j}{2}\right\}^{-1}\exp\{-ikr\cos(\theta+\Theta)\}d\Theta \tag{3.9.5}$$

Although the integrand still has a pole in the vicinity of the saddle point $F_j(\Theta)$ is regular in this region and the major contribution to the integral still comes from this region so a good approximation to $F_j(\Theta)$ is $F_j(\pi-\theta)$ and to ϕ_S is thus

$$\phi_S \sim F_j(\pi-\theta) \int_{\Gamma_3} \{\sin \frac{\Theta-\Theta_j}{2}\}^{-1} \exp\{-ikr\cos(\theta+\Theta)\}d\Theta \quad . \qquad (3.9.6)$$

Applying the shift of origin transformation of the integration variable

$$\Omega = \Theta - (\pi-\theta) \qquad (3.9.7)$$

the path Γ_3 becomes the path Γ_4 in the Ω-plane, see Figure 17.

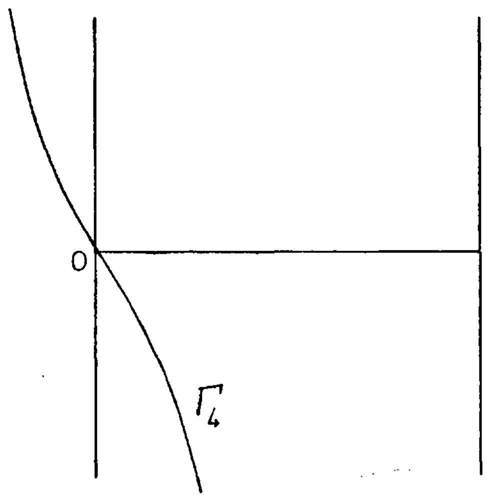

Figure 17: Ω-plane

ϕ_S is then given by

$$\phi_S \sim F_j(\pi-\theta) \int_{\Gamma_4} \{\cos \frac{\Omega-\theta-\Theta_j}{2}\}^{-1} \exp\{ikr\cos\Omega\}d\Omega \quad . \qquad (3.9.8)$$

Putting $\Omega = -\Omega$ in expression (3.9.8) and taking half the sum of this and expression (3.9.8.) gives

$$\phi_S \sim 2F_j(\pi-\theta)\cos\left(\frac{\theta+\Theta_j}{2}\right)\int_{\Gamma_4}\frac{\cos\Omega/2}{\cos\Omega+\cos(\theta+\Theta_j)}\exp\{ikr\cos\Omega\}d\Omega \ . \qquad (3.9.9)$$

The complex variable of integration Ω can be replaced by a real one, τ, via

$$ikr\cos\Omega = ikr - kr\tau^2 \quad . \qquad (3.9.10)$$

The integral then contains a Fresnel integral (see CLEMMOW (6)) because

$$\phi_S \sim 2F_j(\pi-\theta)e^{i\pi/4 + ikr}\int_{-\infty}^{+\infty}\frac{b}{\tau^2+ib^2}\exp\{-kr\tau^2\}d\tau \qquad (3.9.11)$$

where $b = i\sqrt{2}\cos\left(\frac{\theta-\Theta_j}{2}\right)$ (3.9.12)

and $\displaystyle\int_{-\infty}^{+\infty}\frac{b}{\tau^2+ib^2}\exp\{-kr\tau^2\}d\tau = \pm 2\sqrt{\pi}\,F(\pm b\sqrt{kr})$ (3.9.13)

where $F(a) = e^{ia^2}\displaystyle\int_a^{\infty}e^{-i\tau^2}d\tau$ is the Fresnel integral . (3.9.14)

The properties of the Fresnel integral are listed in Appendix 3.

There are two types of contribution that must be considered. First there are those due to Θ_1 and Θ_2 for $\mu < k$. It is only for these two that the imaginary part of Θ_j is $O(\varepsilon)$. Otherwise the imaginary part is $O(\varepsilon^2)$.

The 'Mach Angle' for each pole is defined by

$$k\cos\Theta_{jm} = \mathrm{Re}(S_j) + O(\varepsilon^2) \quad . \qquad (3.9.15)$$

In the (r,θ) plane this gives a 'Mach Angle' in real space of

$$\theta_{jm} = \pi - \Theta_{jm} \quad . \tag{3.9.16}$$

If the 'Capture Angle', Θ_j' , for each pole is defined by

$$\Theta_j' = \Theta_{jm} + i(k^2-\hat{S}_j^{\,2})^{-\frac{1}{2}} \; Im(S_j) + O(\varepsilon^2) \tag{3.9.17}$$

and a corresponding θ_j' in the (r,θ) plane by

$$\theta_j' = \pi - \Theta_j' \tag{3.9.18}$$

then

(i) for a pole just above the real axis (Θ_2), $\theta < \theta_j'$ corresponds to the capture of the pole and $\pi/4 < \arg b < \frac{5\pi}{4}$, whereas $\theta > \theta_j'$ corresponds to the pole not being captured and $-3\pi/4 < \arg b < \pi/4$.

(ii) for a pole just below the real axis (Θ_1), $\theta > \theta_j'$ corresponds to the capture of the pole and $-3\pi/4 < \arg b < \frac{\pi}{4}$, whereas $\theta < \theta_j'$ corresponds to the pole not being captured and $\pi/4 < \arg b < 5\pi/4$.

Let us consider first the potential contribution when θ is near θ_2'.

For $-3\pi/4 < \arg b < \pi/4$ we have $\theta > \theta_2'$ and

(i) for $|\theta-\theta_2'|$ of $O(1)$ and $kr \gg 1$, $b\sqrt{kr}$ is large and expression (A3.9) holds so

$$\phi_{SP} \sim -\sqrt{\frac{2\pi}{kr}} \; \hat{F}(\pi-\theta) e^{-i\pi/4 + ikr} + O(kr)^{-3/2} \quad ; \tag{3.9.19}$$

(ii) for $|\theta-\theta_2'|$ of $O(\varepsilon)$ and $kr \gg \varepsilon^{-2}$, $b\sqrt{kr}$ is still large and equation (3.9.19) is still valid ;

(iii) for $|\theta-\theta_2'|$ of $O(\varepsilon)$ and $1 \ll kr \ll \varepsilon^{-2}$ $b\sqrt{kr}$ is small and expression (A3.10) holds ,

$$\phi_{SP} \sim -2\pi i \; F_2(\pi-\theta) \; \exp\{-ikr\cos(\theta+\Theta_2)\} \quad . \tag{3.9.20}$$

For $\pi/4 < \arg b < \frac{5\pi}{4}$ we have $\theta < \theta_2'$ and

(iv) for $|\theta-\theta_2'|$ of $O(1)$ and $kr \gg 1$, $b\sqrt{kr}$ is large and the contribution from the residue combines with (A3.11) to give ϕ_{SP} which satisfies equation (3.9.19);

(v) for $|\theta - \theta_2'|$ of $O(\varepsilon)$ and $kr \gg \varepsilon^{-2}$, $b\sqrt{kr}$ is still large so equation (3.9.19) still holds;

(iv) for $|\theta-\theta_2'|$ of $O(\varepsilon)$ and $1 \ll kr \ll \varepsilon^{-2}$, $b\sqrt{kr}$ is small giving equation (3.9.20) as the expression ϕ_{SP}.

For the contribution when θ is near θ_1', similar results can be obtained as follows :—

(i) when $|\theta-\theta_1'|$ is of $O(1)$ and $kr \gg 1$ or
when $|\theta-\theta_1'|$ is of $O(\varepsilon)$ and $kr \gg \varepsilon^{-2}$

$$\phi_{SP} \sim - \sqrt{\frac{2\pi}{kr}}\ \hat{F}(\pi-\theta)e^{-i\pi/4 + ikr} \quad ; \tag{3.9.21}$$

(ii) when $|\theta-\theta_1'|$ is of $O(\varepsilon)$ and $1 \ll kr \ll \varepsilon^{-2}$

$$\phi_{SP} \sim 2\pi i\ F_1(\pi-\theta)\ \exp\{-ikr\cos(\theta+\Theta_1)\} \quad . \tag{3.9.22}$$

The second sort of contribution, from Θ_7 and $\Theta_{\pm n}$, occurs where the imaginary term is $O(\varepsilon^2)$. We define a 'Modified Mach Angle' for each pole by

$$k\cos\Theta_{jm} = \mathrm{Re}(S_j) + O(\varepsilon^2) \tag{3.9.23}$$

and a 'Modified Capture Angle' by

$$\Theta_j' = \Theta_{jm} + i(k^2-\hat{S}_j^{\,2})^{-\frac{1}{2}}\ \mathrm{Im}(S_j) + O(\varepsilon^3) \quad . \tag{3.9.24}$$

The corresponding angles in the (r,θ) plane are then given by

$$\theta_{jm} = \pi - \Theta_{jm} \tag{3.9.25}$$

$$\theta_j' = \pi - \Theta_j' \tag{3.9.26}$$

The cases of poles being above or below the real axis are different just as before. The contributions in the two cases differ and are set out below with no further elaboration as the analysis is similar to that of the previous case.

A POLE ABOVE THE REAL AXIS

(i) When $|\theta-\theta_j'|$ is of $O(1)$ and $kr \gg 1$ or (3.9.27)
when $|\theta-\theta_j'|$ is of $O(\varepsilon)$ and $kr \gg \varepsilon^{-2}$ or
when $|\theta-\theta_j'|$ is of $O(\varepsilon^2)$ and $kr \gg \varepsilon^{-4}$,

$$\phi_{SP} \sim \sqrt{\frac{2\pi}{kr}}\ \hat{F}(\pi-\theta)e^{-i\pi/4 + ikr} + O(kr)^{-3/2} \quad . \tag{3.9.28}$$

(ii) When $|\theta-\theta_j'|$ is of $O(\varepsilon^2)$ and $1 \ll kr \ll \varepsilon^{-4}$ or
when $|\theta-\theta_j'|$ is of $O(\varepsilon)$ and $1 \ll kr \ll \varepsilon^{-2}$ (3.9.29)

$$\phi_{SP} \sim -\ 2\pi i\ F_j(\pi-\theta)\ \exp\{-ikr\cos(\theta+\Theta_j)\} \quad . \tag{3.9.30}$$

A POLE BELOW THE REAL AXIS

(i) When $|\theta-\theta_j'|$ and kr satisfy conditions (3.9.27) then ϕ_{SP} is given by equation (3.9.28), whereas when $|\theta-\theta_j'|$ and kr satisfy conditions (3.9.29), ϕ_{SP} is given by

$$\phi_{SP} \sim 2\pi i\ F_j(\pi-\theta)\exp\{-ikr\cos(\theta+\Theta_j)\} \quad . \tag{3.9.31}$$

Thus when θ is near a 'Mach Angle', θ_{jm}, the potential exhibits a beaming effect of intermediate range, centred on the capture angle, known as a Leaky wave (or Modified Leaky wave in the case of an $O(\varepsilon^2)$ imaginary term). At large enough kr the usual $(kr)^{-\frac{1}{2}}$ decaying potential is all that is present. At intermediate distances near the Mach angle, $1 \ll kr \ll \varepsilon^{-2}$ in the case of Leaky waves and $1 \ll kr \ll \varepsilon^{-4}$ in the cases of Modified Leaky waves, a much stronger wave, stronger by $(kr)^{\frac{1}{2}}$ than the far field potential but of the same form, with just half the amplitude of the poles residue contribution, is present.

Notice that the Modified Leaky waves differ from those of CRIGHTON (7) and (8) in that they are narrower but are present over a much greater range.

3.10 SADDLE POINT NEAR A POLE AND A ZERO

Lastly we consider the poles at Θ_5 and Θ_6. Here the saddle point of the integrand is near one of its zeros. In the "below-coincidence" case and the second term in the steepest descents expansion, of order $(kr)^{-3/2}$, combines with the term already found to give the dominant contribution.

The more interesting case is the "above-coincidence" one ($\mu < k$). Here the saddle point of the integrand is near one of its poles as well as one of its zeros and for some θ the pole can be captured. The previous approximation method fails here since the integrand is approximated by its value at the saddle point and this becomes small and then zero as θ is near and then equal to 0 or π.

There are two possible methods for estimating ϕ_S in this case.

The first method is to isolate the pole of the integrand in such a manner that we can subsequently approximate it. Putting

$$\hat{F}(\Theta) = Q_j(\Theta) + q_j \sec \frac{1}{2}(\Theta-\Theta_j); \quad q_j = \frac{\lambda\varepsilon}{4\pi} R_j \ , \tag{3.10.1}$$

so q_j is half the residue of $\hat{F}(\Theta)$ at Θ_j, is an excellent method of doing this. The function $Q_j(\Theta)$ has no poles or zeros near the saddle point so the standard steepest descents method can be employed to approximate its integral. The integral of the second term can be recast in the form of a Fresnel integral as was done previously. This method is not employed since the steepest descent analysis can be algebraically complicated and the results are difficult to interpret.

The second method, the one used here, is to rewrite (3.9.2) in the form

$$\phi_S = \int_{\Gamma_3} \frac{\hat{F}(\Theta)}{G_j(\Theta)} G_j(\Theta) \exp\{-ikr \cos(\theta+\Theta)\}d\Theta \tag{3.10.2}$$

where the function $G_j(\Theta)$ has a pole and zero in the same places as $\hat{F}(\Theta)$ but can be split into the sum of two manageable parts.

For the saddle point near Θ_5 we choose

$$G_5(\Theta) = \sin\frac{\Theta}{2} \cos \frac{i\varepsilon}{2k(k^4-\mu^4)} \left[\sin \frac{1}{2}(\Theta - \frac{i\varepsilon}{k(k^4-\mu^4)})\right]^{-1} \tag{3.10.3}$$

and for near Θ_6 we choose

$$G_6(\Theta) = \cos\frac{\Theta}{2} \cos \frac{i\varepsilon}{2k(k^4-\mu^4)} \left[\cos \frac{1}{2}(\Theta + \frac{i\varepsilon}{k(k^4-\mu^4)})\right]^{-1} \tag{3.10.4}$$

because they can be split into

$$1 + \sin \frac{i\varepsilon}{2k(k^4-\mu^4)} \cos\frac{\Theta}{2} \left[\sin \frac{1}{2} (\Theta - \frac{i\varepsilon}{k(k^4-\mu^4)})\right]^{-1} \tag{3.10.5}$$

and

$$1 + \sin \frac{i\varepsilon}{2k(k^4-\mu^4)} \sin\frac{\Theta}{2} \left[\cos \frac{1}{2} (\Theta + \frac{i\varepsilon}{k(k^4-\mu^4)})\right]^{-1} \tag{3.10.6}$$

respectively.

We will consider first the region near $\theta=\pi$, that is, the saddle point is near $\Theta_5 - \frac{i\varepsilon}{k(k^4-\mu^4)}$. The integral in equation (3.10.2), after substituting, equation (3.10.5) can be split up to give

$$\phi_S = \int_{\Gamma_3} \frac{\hat{F}(\Theta)}{G_5(\Theta)} \exp\{-ikr\cos(\theta+\Theta)\}d\Theta$$

$$+ \int_{\Gamma_3} \hat{F}(\Theta) \sin\frac{i\varepsilon}{2k(k^4-\mu^4)} \cos\frac{\Theta}{2} \left[G_5(\Theta)\sin\frac{1}{2}\left(\Theta - \frac{i\varepsilon}{k(k^4-\mu^4)}\right)\right]^{-1}$$

$$\exp\{-ikr\cos(\theta+\Theta)\}d\Theta = I_1 + I_2 \quad . \tag{3.10.7}$$

The first of these two integrals has an integrand which is well behaved in the region of the saddle point so the method of steepest descents can be employed to give, for the far field, $kr >> 1$,

$$I_1 = \int_{\Gamma_3} \frac{\hat{F}(\Theta)}{G_5(\Theta)} \exp\{-ikr\cos(\theta+\Theta)\}d\Theta$$

$$\sim -\sqrt{\frac{2\pi}{kr}} \frac{\hat{F}(\pi-\theta)}{G_5(\pi-\theta)} e^{-ikr-i\pi/4} + O(kr)^{-3/2} \text{ for } kr >> 1 \ . \tag{3.10.8}$$

For the second integral, as the major contribution still comes from near the saddle point, most of the integrand can be approximated by the value at the saddle point, giving

$$I_2 \sim \frac{\hat{F}(\pi-\theta)}{G_5(\pi-\theta)} \sin\frac{i\varepsilon}{2k(k^4-\mu^4)} \cos\left(\frac{\pi-\theta}{2}\right) \int_{\Gamma_3} \left[\sin\frac{1}{2}\left(\Theta - \frac{i\varepsilon}{k(k^4-\mu^4)}\right)\right]^{-1}$$

$$\exp\{-ikr\cos(\theta+\Theta)\}d\Theta \tag{3.10.9}$$

$$= \pm \frac{\hat{F}(\pi-\theta)}{G_5(\pi-\theta)} \sin\frac{i\varepsilon}{2k(k^4-\mu^4)} \sin\frac{\Theta}{2}\, 4\sqrt{\pi}\, e^{-i\pi/4 + ikr}$$

$$F\left(\pm i\sqrt{2kr}\cos\frac{1}{2}\left(\Theta + \frac{i\varepsilon}{k(k^4-\mu^4)}\right)\right) \tag{3.10.10}$$

with F and $(\pm)$ signs as before.

Now we know that for $\theta < \theta_5'$, $\phi_{SP} = \phi_S$, and that for $\theta > \theta_5'$, $\phi_{SP} = \phi_S + 2\pi i \times$ Residue. Employing the expansions that we have for the Fresnel integral gives the following

(i) When $|\theta-\theta_5'|$ is of $O(1)$ and $kr \gg 1$ and
when $|\theta-\theta_5'|$ is of $O(\varepsilon)$ and $kr \gg \varepsilon^{-2}$,

$$\phi_{SP} \sim - \sqrt{\frac{2\pi}{kr}}\ \hat{F}(\pi-\theta)e^{-i\pi/4 + ikr} + O(kr)^{-3/2} + 2\pi i\psi_{25}\,. \tag{3.10.11}$$

(ii) When $|\theta-\theta_5'|$ is of $O(\varepsilon)$ and $1 \ll kr \ll \varepsilon^{-2}$,

$$\phi_{SP} \sim - \sqrt{\frac{2\pi}{kr}}\ \frac{F(\pi-\theta)}{G_5(\pi-\theta)}\ e^{-i\pi/4 + ikr} + O(kr)^{-3/2} + 2\pi i\psi_{25}\,.$$

$$- 2\pi i \sin\frac{i\varepsilon}{2k(k^4-\mu^4)} \sin\frac{\theta}{2}\,\frac{\hat{F}(\pi-\theta)}{G_5(\pi-\theta)}\, e^{ikr}, \tag{3.10.12}$$

which is a Leaky wave plus the pole contribution.

So when θ is $O(1)$ away from θ_5' the potential ϕ_{SP} consists of just the radiating field, to first order like $(kr)^{-1/2}$ for $kr \gg 1$. However when θ is within $O(\varepsilon)$ of θ_5' the far field is not detected until $kr \gg \varepsilon^{-2}$ where as θ approaches π the amplitude of the $(kr)^{-1/2}$ terms decays and the $(kr)^{-3/2}$ term dominates it. Present here also is the residue contribution due to the pole which, as θ approaches π does not decay until further and further out, until, finally, on $\pi=\theta$, it becomes a marginally subsonic weak surface wave.

The analysis can be repeated for θ near 0, that is the saddle point is near $\theta_6 \sim \pi - \dfrac{i\varepsilon}{k(k^4-\mu^4)}$. The same behaviour, including a Leaky wave, but with the marginally subsonic weak surface wave travelling in the opposite direction, results.

3.11 THE EXTERNAL VELOCITY POTENTIAL : SUMMARY

The major contributions to the far field of the potential ϕ_2 are thus as follows:—

3.11.1 Below Coincidence ($k < \mu$)

The potential ϕ_2 consists of the radiating far field, of order $(kr)^{-\frac{1}{2}}$ or $(kr)^{-3/2}$ near the walls, plus (in the positive x-direction) a subsonic surface wave $-2\pi i\psi_{12}$ and (in the negative x-direction) two subsonic surface waves $2\pi i\psi_{11}$ and $2\pi i\psi_{18}$. The Modified Leaky waves are also present.

3.11.2 Above Coincidence ($\mu < k$)

The potential ϕ_2 again consists of the radiating far field, of order $(kr)^{-\frac{1}{2}}$ or $(kr)^{-3/2}$ near the walls, plus a marginally subsonic weak surface wave in each direction, an ordinary subsonic wave $2\pi i\psi_{28}$ in the negative x-direction, very weak Leaky waves and Modified Leaky waves as described.

Thus a typical sketch of maximum amplitude, A, against angle θ, at a constant r, with $1 << kr << \varepsilon^{-2}$ would be of the form Figure 18.

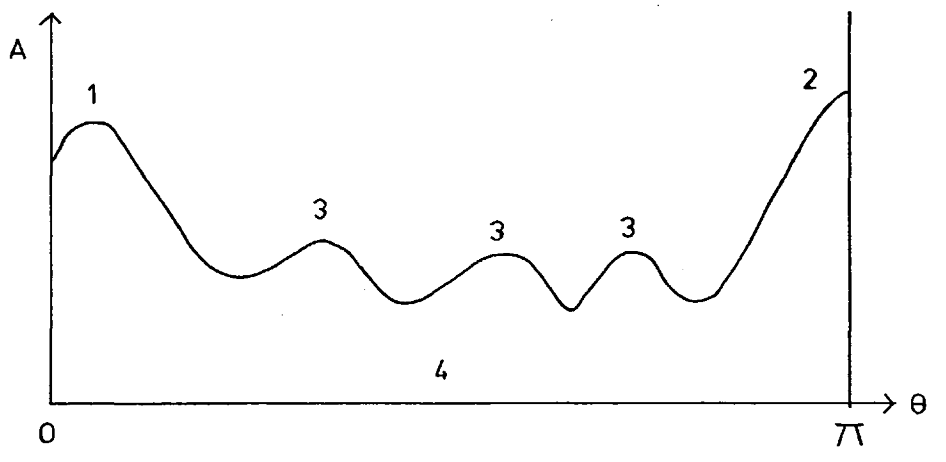

Figure 18: Diagramatic only

1 is a weak subsonic surface wave
2 is a subsonic surface wave
3 are Leaky or Modified Leaky wave peaks
4 is the far field radiation $(kr)^{-\frac{1}{2}}$

3.12 THE WALL DISPLACEMENT $\eta(x)$

Lastly we consider the wall displacement $\eta(x)$. Equation (3.5.16) gives us

$$\eta(x) = -\frac{i\varepsilon\lambda}{2\pi\omega}\int_{\Gamma_1}(1+\frac{ms}{k})\hat{\gamma}(S)F(S)^{-1}\exp\{-isx\}ds \qquad (3.12.1)$$

Putting $s = k\cos\Theta$ gives

$$\eta(x) = \frac{\varepsilon\lambda k}{2\pi c}\int_{\Gamma_2}(1+m\cos\Theta)\sin^2\Theta\hat{F}(\Theta)^{-1}\exp\{-ikx\cos\Theta\}d\Theta \quad . \qquad (3.12.2)$$

We will again use the method of steepest descents through the saddle point for finding η_S. For $x < 0$ this is through $\Theta = 0$ so the paths Γ_3 and Γ_4 coincide. For $x > 0$ the path is through $\Theta = \pi$ so the transformation has to be used to map Γ_3 onto Γ_4. This then gives for the saddle point contribution in the far field

$$\eta_{S1}(x) = -\frac{\lambda k(1-m)}{\sqrt{2\pi}\, c\ \gamma(-k)\sinh\gamma(-k)d\gamma(kx)^{3/2}}e^{ikx-i\pi/4} + 0(kx)^{-5/2}$$

for kx large and >0 ,

(3.12.3)

$$\eta_{S2}(x) = -\frac{\lambda k(1+m)}{\sqrt{2\pi}\, c\gamma(k)\sinh\gamma(k)d(-kx)^{3/2}}e^{-ikx-i\pi/4} + 0(kx)^{-5/2}$$

for kx large and <0 ,

(3.12.4)

The first term in these expansions is $0(kx)^{-3/2}$ since the integrand has a double zero at $\Theta = 0$ and π unlike that for ϕ_2 which only had a single zero in these planes.

We now use Cauchy's Residue theorem to take into account any poles that contribute significantly on capture, giving the following for $\eta(x)$ in the far field.

Above Coincidence ($\mu < k$)

$$\eta(x) \sim \eta_{S2}(x) + \{ \frac{\lambda\varepsilon^3(1+m)}{c^2k(k^4-\mu^4)^3\gamma(k)\sinh\gamma(k)d} + O(\varepsilon^4)\} e^{-ixS_5}$$

$$+ \{ \frac{\lambda\varepsilon}{2ck^2d(1-m)(k^4-\mu^4)} + O(\varepsilon^2)\} e^{-ixS_8}$$

for $x < 0$
kx large

(3.12.5)

which is the $(kx)^{-3/2}$ radiating sonic wave plus two subsonic waves, one of them being small and only marginally subsonic;

$$\eta(x) \sim \eta_{S1}(x) + [\frac{\lambda\varepsilon^3(1-m)}{c^2k(k^4-\mu^4)\gamma(-k(\sinh\gamma(-k)d} + O(\varepsilon^4)]e^{-ixS_6}$$

for $x > 0$
kx large

(3.12.6)

which is the $(kx)^{-3/2}$ radiating sonic wave plus a small marginally subsonic wave.

Below Coincidence ($k < \mu$)

$$\eta(x) \sim \eta_{S2}(x) + [\frac{\lambda\varepsilon}{ck}(1+\frac{m\mu}{k})\{4\mu^3\gamma(\mu)\sinh\gamma(\mu)d\}^{-1} + O(\varepsilon^2)]\ e^{-ixS_1}$$

$$+ [\frac{\lambda\varepsilon}{2ck^2d(1-m)(k^4-\mu^4)} + O(\varepsilon^2)]\ e^{-ixS_8} \quad \text{for } x < 0,\ kx \text{ large}$$

(3.12.7)

which is the radiating sonic wave plus two subsonic waves of comparable amplitude;

$$\eta(x) \sim \eta_{S1}(x) + \left[\frac{\lambda\varepsilon}{ck} \left(1 - \frac{m\mu}{k}\right) \{4\mu^3\gamma(-\mu)\sinh\gamma(-\mu)d\} + O(\varepsilon^2)\right] e^{-ixS_2}$$

$$\text{for } x > 0,\ kx \text{ large} \qquad (3.12.8)$$

which is the radiating sonic wave plus one subsonic wave.

3.13 SUMMARY

In this chapter the interaction between a time periodic disturbance in a flow and the wave-bearing duct containing that flow has been modelled, in two dimensions and in an inviscid régime, by the linearised system of a line source vibrating in an elastic walled duct containing flow. The thin elastic plate has been taken as the model of the wave-bearing surface.

The effects of the flow in the duct, evident throughout the analysis, are of particular interest. In section 3.3. we have the convected forms of the material derivative and Helmholtz's equations, equations (3.3.5) and (3.3.10), as opposed to their non-convected counterparts, equations (3.3.4) and (3.3.11).

In section 3.4 the ratio of the free wave numbers of the plate in vacuo to the wave number of the fluid source is seen to be an important parameter in determining the nature of the solutions present.

After substituting in an asymptotic expansion, in small ε, (the light fluid loading limit) for the zeros of F(s) we were able to find the positions of the poles of the integrand and thus identify the types of solution present. Results could not be obtained analytically, by this method, without the light fluid loading assumption because it is this that permits the identification of the approximate positions of the zeros. Even for large ε, the heavy fluid loading limit, the identification of the approximate zero position is difficult.

We can see in the transform inversion integrals (3.5.14), (3.5.15) and (3.5.16) that the convection term (1+ms/k) is introduced by the flow into all the integrands. In particular in equation (3.6.1) for ϕ_1 the flow has an effect not only in the first but also in the second term which although representing the elasticity in the walls and the presence of the duct also reflects the presence of the flow.

It is the flow that in section 3.12, in equation (3.12.1) for $\eta(x)$, gives different upstream and downstream contributions to $\eta(x)$, ($\eta_{S2}(x)$ and $\eta_{S1}(x)$) and also results in two subsonic surface waves upstream but only one downstream; this applies for $\mu > k$ and $\mu < k$ even though the wave speeds are different in each case.

For the external potential ϕ_2 the asymptotic methods used, casting into Fresnel integrals in the style of Clemmow or "steepest descents", permit the identification of a range of beaming effects, present in the middle field and identified as Leaky waves or Modified Leaky waves, as well as the classic radiating far field. The contributions from those poles that give such Leaky waves do not propagate energy to infinity but decay away to the radiating field at sufficiently large distances.

Notice that here also, as with ϕ_1 and $\eta(x)$, the conditions $\mu < k$ and $\mu > k$ determine whether some waves are or are not present in the solution.

CHAPTER 4: Radiation from a partly elastic infinite duct containing flow

4.1 INTRODUCTION AND SUMMARY

In this chapter we investigate the two-dimensional problem of disturbances caused by a plane wave in a duct (containing uniform flow) that has parallel semi-infinite rigid and semi-infinite elastic walls. The thin elastic plate has again been chosen as a model of a wave-bearing surface. We elect to consider this simple model, losing the algebraic complexity of the Timoshenko-Mindlin model in favour of the clear identification of the major potential contributions.

As the geometries here are semi-infinite the Wiener-Hopf technique will be employed.

The system to be examined is defined in section 4.2. Linearisation of the equation of motion and the boundary conditions gives the model problem (see section 4.3).

In sections 4.4 and 4.5 a Fourier transform analysis and Wiener-Hopf analysis are undertaken. From the solutions obtained from this, an approximation for the wall displacement is found and discussed in section 4.6 and similar results for the interior duct potential are found in section 4.7.

The solution, for the potential inside the duct, is given in the form of a complex integral, and significant contributions arise from the poles of the integrand. In section 4.7 a change in integration variable leads to an identification of the location of these poles. The structure is different according as the frequency is above, or below its 'coincidence' value. In section 4.9 the first estimate, valid for large kr at most angles θ, is found. In sections 4.10, 4.11 and 4.12 more complicated asymptotics for θ near 'Mach Angles' are developed. It is here that sometimes, for light fluid loading, local reinforcement occurs and the existence of Leaky waves is detected. Section 4.13 is a summary of the external velocity potential. Section 4.14 contains observations on the analysis.

4.2 PROBLEM DEFINITION

Inviscid compressible fluid flows adiabatically in a two-dimensional infinite duct. The duct consists of two joined semi-infinite sections. One section has rigid parallel walls, where (x,y) are cartesian coordinates, these are at $y = \pm d$, $x < 0$. The other section has elastic walls about $y = \pm d$ $x < 0$. The basic flow is uniform and constant from the rigid walled section into the elastic walled section. It has a velocity potential ϕ given by $\phi = Ux$, and there is identical still fluid outside the duct.

A plane wave is travelling down the duct from $x = -\infty$ to $x = \infty$. This wave, of maximum amplitude λ, has a velocity potential given by ϕ_0 where

$$\phi_0 = \mathrm{Re}\{\lambda \exp[\frac{ikx}{1+m} - i\omega t]\} , \qquad (4.2.1)$$

The interaction of this wave with the elastic walls of the duct and with the discontinuity at the joins produces disturbances both inside and outside the duct.

As can be seen from Figure 1 the problem is symmetric about the centre line $y = 0$ of the duct.

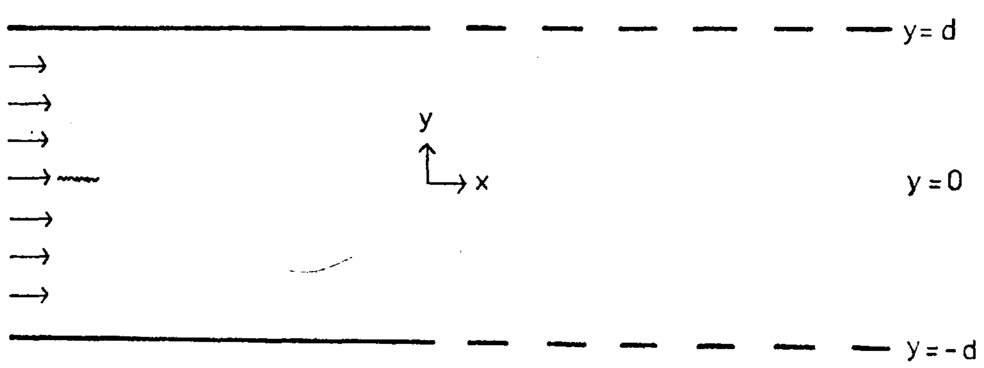

Figure 1: ___ denotes rigid wall
--- denotes elastic wall

The half-space problem to which this is equivalent will be solved; see Figure 2. This consists of a basic uniform flow U down a duct of width d disturbed by a plane wave with potential ϕ_0 given by equation (4.2.1). The plane y = -d consists of an infinite rigid baffle. The half-plane y = 0, x < 0 also consists of a rigid baffle whereas the half-plane y = 0, x > 0 is a semi-infinite elastic plate. There is one other boundary condition, at the joint, where the pressure is taken to have at worst an integrable singularity.

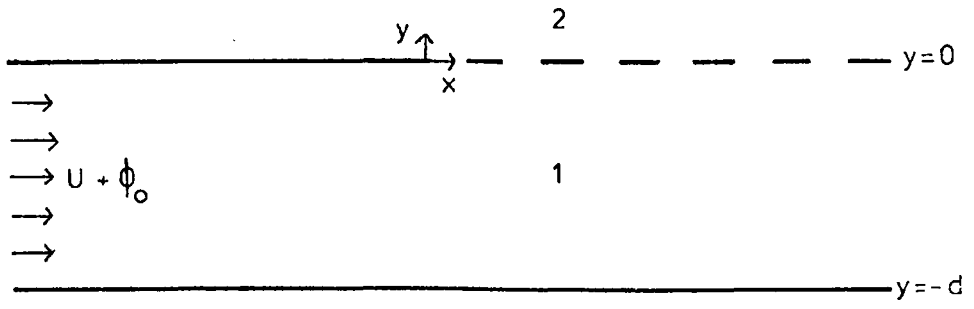

Figure 2

Once again the small amplitude vertical displacement, $\eta(x,t)$, of the elastic plate is governed by the equation (3.2.1)

$$D \frac{\partial^4 \eta}{\partial x^4}(x,t) + 2Mh \frac{\partial^2 \eta}{\partial t^2} = - p]_-^+ \quad \text{on } y = \quad \eta(x,t) \qquad (4.2.2)$$

All quantities here have the same definitions as in the previous chapter.

If Φ is the total velocity potential for the fluid then write

$$\Phi = \begin{cases} Ux + \phi_0 + \hat{\phi}_1 & \text{in Region 1} \\ \hat{\phi}_2 & \text{in Region 2} \end{cases} \qquad (4.2.3)$$

$\hat{\phi}_1$ and $\hat{\phi}_2$ are the velocity potentials of the disturbances and Φ satisfies Lighthill's non-linear acoustic equation (3.3.9)

$$c^2\nabla^2\Phi = \frac{\partial^2\Phi}{\partial t^2} + \frac{2\partial\Phi}{\partial x_i}\frac{\partial^2\Phi}{\partial x_i\partial t} + \frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\frac{\partial^2\Phi}{\partial x_i\,\partial x_j} . \qquad (4.2.4)$$

If we let

$$\hat{\phi}_j = \mathrm{Re}\{\phi_j e^{-i\omega t}\} \quad \text{and} \quad \Phi = \Phi_j \qquad j = 1,2 \qquad (4.2.5)$$

to take into account the periodic response of the fluid to the wave then we have

$$\Phi_1 = Ux + \mathrm{Re}\{\phi_1 e^{-i\omega t} + \lambda \exp[\frac{ikx}{1+m} - i\omega t]\} \qquad (4.2.6)$$

$$\Phi_2 = \mathrm{Re}\{\phi_2 e^{-i\omega t}\} .$$

As before we obtain approximations for ϕ_1, ϕ_2 and η by assuming that they are small, linearising, and replacing the boundary conditions on $y = \eta(x,t)$, $x > 0$ by ones on $y = 0$, $x > 0$.

4.3 FORMULATION OF THE BOUNDARY VALUE PROBLEM

The linearised form of (4.2.2) is

$$\frac{\partial^4 \eta}{\partial x^4}(x,t) + \frac{\mu^4}{\omega^2}\frac{\partial^2 \eta}{\partial t^2}(x,t) = \frac{-\varepsilon}{\rho_0 \omega^2}\, p]_-^+ \text{ on } y=0 \ . \qquad (4.3.1)$$

We are going to consider $\varepsilon \ll 1$ but at this stage we will conduct the analysis for all ε.

Putting

$$\eta(x,t) = \mathrm{Re}\{\eta(x)e^{-i\omega t}\} \ ; \quad \eta(x) = \eta \qquad (4.3.2)$$

to take into account the response of the elastic plate to the disturbance gives for the linearised boundary conditions between the fluid and the baffles at $y = -d$ and at $y=0, x<0$,

$$\frac{\partial \phi_2}{\partial y} = 0 \quad \text{on} \quad y=0 \quad x<0 \ ,$$

$$\frac{\partial \phi_1}{\partial y} = 0 \begin{cases} \text{on} \quad y=0 \quad x<0 \\ \text{on} \quad y=-d \quad \text{for all } x \ . \end{cases} \qquad (4.3.3)$$

For the linearised conditions between the elastic plate and the fluid we have

$$\frac{\partial \phi_1}{\partial y} = -i\omega\eta + U\frac{\partial \eta}{\partial x} \qquad \text{on} \quad y=0, \quad x>0$$

$$\frac{\partial \phi_2}{\partial y} = -i\omega\eta \qquad \text{on} \quad y=0, \quad x>0 \ . \qquad (4.3.4)$$

Additional conditions are that $\hat{\phi}_1$ and $\hat{\phi}_2$ represent outgoing waves at infinity and that $\eta(0)$ and $\eta''(0)$ are zero, that is one section of duct is joined to the other in a locally hinge-like manner.

As before the linearised form of (4.2.4) for ϕ_2 in Region 2 is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right)\phi_2 = 0 \qquad (4.3.5)$$

and for ϕ_1 in Region 1 is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + [k + im\frac{\partial}{\partial x}]^2 \right)\phi_1 = 0 \, . \qquad (4.3.6)$$

For this problem the pressure difference $p]_-^+$ across the plate is given by

$$p]_-^+ = p_2 - p_1 \qquad (4.3.7)$$

where pressure excesses $\hat{p}_1$ and $\hat{p}_2$ are given by

$$\hat{p}_j = \mathrm{Re}\{p_j e^{-i\omega t}\} \qquad j = 1,2, \qquad (4.3.8)$$

and

$$p_2 = i\omega\rho_0\phi_2, \; p_1 = \rho_0\{i\omega\phi_1 - U\frac{\partial\phi_1}{\partial x}\} + i\rho_0 \frac{ck\lambda}{1+m} \exp\{\frac{ikx}{1+m}\} \, , \qquad (4.3.9)$$

which is expression (3.3.14) with $\hat{\phi}$ replaced by $\hat{\phi}_1 + \phi_0$.

This then gives for equation (4.2.2) ,

$$\frac{\partial^4\eta}{\partial x^4} - \mu^4\eta = -\frac{i\varepsilon}{\omega}[\phi_2 - \phi_1 - \frac{im}{k}\frac{\partial\phi_1}{\partial x} - \frac{\lambda}{1+m}\exp\{\frac{ikx}{1+m}\}] \quad . \qquad (4.3.10)$$

It is not instructive to look at the free wave solutions this system can support as we did in the previous chapter because of a basic difference in the two problems. This problem has an inhomogeneous boundary at y = 0 with semi-infinite geometries. Trying to see what solutions this could support brings us back to the original problem (with the semi-infinite boundary conditions). The presence of the semi-infinite geometry implies that a simple Fourier transform analysis will not be sufficient and the Wiener-Hopf technique will be employed as well (see NOBLE (16)). However, it is anticipated that the kernel of the Wiener-Hopf equation is the function F(s) already studied. Our knowledge of the positions of its zeros and poles will be of great use when we come to split it into a product of 'upper' and 'lower' regular functions (which are defined later on). Some of the zeros of F(s) will be poles of the integrand in the solution we obtain, so information about the types of waves each produces will also be of use. Remember that these asymptotic solutions for the zeros rely on m being finite; the solutions are non-uniform and do not hold for m small and, particularly, m equals zero.

4.4. TRANSFORM ANALYSIS

Using the definitions from equations (3.5.1) and (3.5.2) for the Fourier transform and its inverse we now also define the half-range plus transform given by

$$\bar{\phi}_{j+} = \int_0^\infty \phi_j \, e^{isx} \, dx \quad , \quad j = 1,2 \; , \tag{4.4.1}$$

known as a plus function because it is regular for all $s = S_r + iS_i$ (S_r and S_i real) such that $S_i > S_-$ for some $S_- < 0$ provided k and μ have small positive imaginary parts. (This region is known as R_+, see Figure 3.) We also define a corresponding half-range minus transform given by

$$\bar{\phi}_{j-} = \int_{-\infty}^0 \phi_j \, e^{isx} \, dx \quad , \quad j = 1,2 \; , \tag{4.4.2}$$

which is known as a minus function since it is regular for all s such that $S_i < S_+$ for some $S_+ > 0$. (This region is known as R_-, see Figure 3.) Once again k and μ have small positive imaginary parts. We assume $k = k_r + ik_i$ and $\mu = \mu_r + i\mu_i$ to improve the convergence of the transform integrals and then we let $\mu_i, k_i \to 0$ to obtain the solutions for real k and μ .

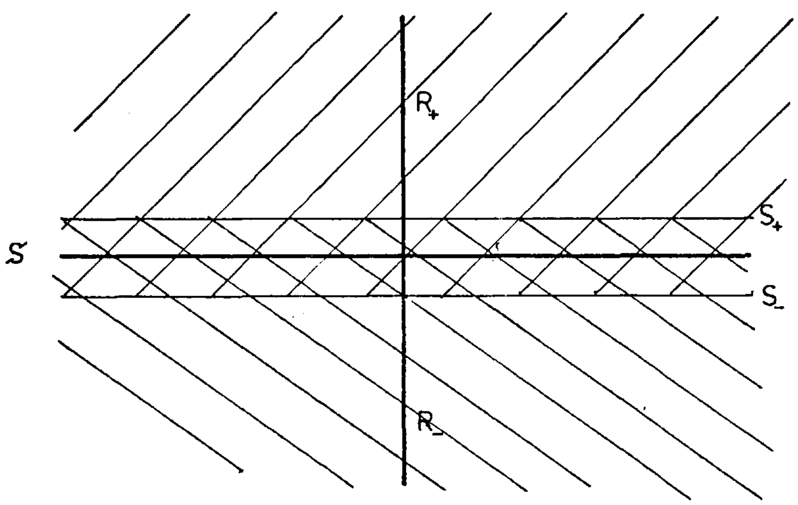

Figure 3: S-plane

Taking half range plus transforms of equation (4.3.10) yields

$$(s^4-\mu^4)\bar{\eta} = -\frac{i\varepsilon}{\omega}\left[\bar{\phi}_{2+} - (1+\frac{ms}{k})\bar{\phi}_{1+} + \frac{i\lambda}{k+s(1+m)} + \frac{im}{k}\phi_1(0)\right] + \frac{\partial^3\eta(0)}{\partial x^3} - s^2\frac{\partial\eta(0)}{\partial x} \quad (4.4.3)$$

where $$\bar{\eta} = \int_0^\infty \eta(x)e^{isx}dx = \int_{-\infty}^{+\infty}\eta(x)e^{isx}dx \quad (4.4.4)$$

since $\eta(x) = 0$ for $x < 0$.

Letting

$$\frac{\partial^3\eta(0)}{\partial x^3} = q \text{ and } \frac{\partial\eta(0)}{\partial x} = -p \quad (4.4.5)$$

where p and q are as yet undetermined we have

$$(s^4-\mu^4)\bar{\eta} = -\frac{i\varepsilon}{\omega}\left[\bar{\phi}_{2+} - (1+\frac{ms}{k})\bar{\phi}_{1+}\right] - \frac{\varepsilon\lambda}{\varepsilon(k+s(1+m))} + \frac{m\varepsilon}{\omega k}\phi_1(0) + ps^2 + q \;. \quad (4.4.6)$$

Taking transforms of equations (4.3.5) and (4.3.6) and using the outgoing wave at infinity condition with equation (4.3.3c) gives

$$\bar{\phi}_1 = B(s)\cosh\gamma(s)(y+d) \quad (4.4.7)$$

$$\bar{\phi}_2 = A(s)\exp\{-\hat{\gamma}(s)y\} \quad (4.4.8)$$

where A and B are functions of s and as yet unknown.

The transform of (4.3.3a) gives

$$\frac{d}{dy}\bar{\phi}_{2-} = 0 \quad \text{on } y = 0 \quad (4.4.9)$$

and that of (4.3.3b) gives

$$\frac{d}{dy}\bar{\phi}_{1-} = 0 \quad \text{on } y = 0 \quad (4.4.10)$$

whereas the transforms of (4.3.4a) and (4.3.4b) yield

$$\frac{d}{dy}\bar{\phi}_{2+} = -i\omega\bar{\eta} \quad \text{on} \quad y = 0 \tag{4.4.11}$$

and

$$\frac{d}{dy}\bar{\phi}_{1+} = -\, i(\omega+Us)\bar{\eta} \quad \text{on} \quad y = 0\ . \tag{4.4.12}$$

Substituting equations (4.4.8) and (4.4.9) into (4.4.11) gives

$$-\hat{\gamma}A = -i\omega\bar{\eta} \tag{4.4.13}$$

and equations (4.4.7) and (4.4.10) into (4.4.12) gives

$$-i(\omega+Us)\bar{\eta} = \gamma(s)B\cosh\gamma(s)d\ . \tag{4.4.14}$$

So on $y = 0$

$$\bar{\phi}_{2+} = \bar{\phi}_2 - \bar{\phi}_{2-} = i\omega\hat{\gamma}^{-1}\bar{\eta} - \bar{\phi}_{2-} \tag{4.4.15}$$

and

$$\begin{aligned}\bar{\phi}_{1+} &= \bar{\phi}_1 - \bar{\phi}_{1-}\\ &= -i\omega(1+\frac{ms}{k})\bar{\eta}\,[\gamma(s)\sinh\gamma(s)d]^{-1}\cosh\gamma(s)d - \bar{\phi}_{1-}\end{aligned} \tag{4.4.16}$$

and on substituting in for $\bar{\phi}_{1+}$ and $\bar{\phi}_{2+}$ and then for $\bar{\eta}$ from equation (4.4.11), equation (4.4.6) becomes

$$\begin{aligned}K(s)\frac{d}{dy}\bar{\phi}_{2+} &= -i\omega\,[ps^2+q+\frac{m\varepsilon}{ck^2}\phi_1(0) - \frac{\varepsilon\lambda}{ck(k+s(1+m))}]\\ &\quad + \frac{i\varepsilon}{ck}[\bar{\phi}_{2-} - (1+\frac{ms}{k})\bar{\phi}_{1-}]\end{aligned} \tag{4.4.17}$$

where

$$\begin{aligned}K(s) &= (s^4-\mu^4) - \varepsilon\{\hat{\gamma}^{-1} + (1+\frac{ms}{k})^2\gamma^{-1}\coth\gamma d\}\\ &= \{\hat{\gamma}\gamma\sinh\gamma d\}^{-1}F(s)\ ,\end{aligned} \tag{4.4.18}$$

K(s) therefore has zeros in precisely the same places as F(s). K(s) also has poles at the zeros of $\hat{\gamma}\gamma\sinh\gamma d$. So as k and μ are slightly complex it follows that there is a strip of regularity, $\mathcal{S}$, for K(s) about the real S axis within which it also has no zeros. iS_- is thus given by the minimum (in magnitude) imaginary part, for complex k and μ, of those S_j (all j) and $\hat{S}_j$ (j=5,6,7,8 and $\pm$n) that have negative imaginary parts. Conversely iS_+ is given by the minimum imaginary part of the same S_j's and $\hat{S}_j$'s that have positive imaginary parts. $\mathcal{S}$ consists of the region where R_+ and R_- overlap. Now as equation (4.4.17) is in an appropriate form we can proceed to apply the Wiener-Hopf technique.

4.5. WIENER-HOPF ANALYSIS

First we define a product factorisation of the kernel

$$K(s) = K_+(s)/K_-(s) \; ; \tag{4.5.1}$$

$K_+(s)$ is regular and non-zero in the upper half-plane, R_+, while $K_-(s)$ is regular and non-zero in R_-. We accept equation (4.5.1) as a formal definition; explicit determination of the factors $K_+(s)$ and $K_-(s)$ is undertaken in Appendix 4 (for $\varepsilon << 1$).

The Wiener-Hopf equation (4.4.17) can thus be rewritten

$$K_+(s)\bar{\phi}_{2+}' = [N(s) + Q_-(s) + \frac{i\varepsilon\lambda}{k+s(1+m)}]K_-(s) \tag{4.5.2}$$

where ' denotes partial differentiation with respect to y; thus $\bar{\phi}_{2+}'$ means $\frac{\partial^2\bar{\phi}_{2+}}{\partial y}$ (s,y) evaluated at y = 0. Also

$$N(s) = -i\omega[ps^2 + q + \frac{\varepsilon mr}{ck^2}], \text{ a regular function }, \tag{4.5.3}$$

$$Q_-(s) = \varepsilon[\bar{\phi}_{2-} - (1+\frac{ms}{k})\bar{\phi}_{1-}], \text{ a minus function }. \tag{4.5.4}$$

The last term in equation (4.5.2) can be split into a sum of plus and minus functions:—

$$\frac{i\varepsilon\lambda K_-(s)}{k+s(1+m)} = i\varepsilon\lambda\,\{\frac{K_-(s)-K_-(-k/1+m)}{k+s(1+m)}\} + \frac{i\varepsilon\lambda K_-(-k/1+m)}{k+s(1+m)} , \tag{4.5.5}$$

and equation (4.5.2) can be subsequently rearranged to give

$$K_+(s)\bar{\phi}_{2+}' - \frac{i\varepsilon\lambda K_-(-k/1+m)}{k+s(1+m)} = N(s)K_-(s) + Q_-(s)K_-(s)$$
$$+ \frac{i\varepsilon\lambda}{k+s(1+m)}\,\{\frac{K_-(s)-K_-(-k/1+m)}{k+s(1+m)}\} . \tag{4.5.6}$$

Now the left hand side of equation (4.5.6) is regular in R_+ and the right hand side is regular in R_-. In the strip of regularity, S, both sides of equation (4.5.6) are regular and we now let them be equal to a regular function R(s). In S

$$R(s) = K_+(s)\bar{\phi}_{2+}{}' - \frac{i\varepsilon\lambda K_-(-k/1+m)}{k+s(1+m)} \tag{4.5.7}$$

and

$$R(s) = N(s)K_-(s) + Q_-(s)K_-(s) + \frac{i\varepsilon\lambda}{k+s(1+m)}\left\{\frac{K_-(s)-K_-(-k/1+m)}{k+s(1+m)}\right\} \tag{4.5.8}$$

Equations (4.5.7) and (4.5.8) thus continue R(s) analytically throughout the whole S-plane. The behaviour of the right hand side of equations (4.5.7) and (4.5.8) as s goes to infinity in the upper and lower half-planes therefore gives the behaviour of the analytic function R(s) at infinity and R(s) can thus be determined. In Appendix 5 the appropriate limits are taken and the function R(s) is found to satisfy

$$R(s) = -ip\omega \, . \tag{4.5.9}$$

In Appendix 5 we find the first two terms in the power series for p. Then from equation (4.5.7)

$$\bar{\phi}_{2+}{}' = -iK_+(s)^{-1}\left[p\omega - \frac{\varepsilon\lambda K_-(-k/1+m)}{k+s(1+m)}\right] \tag{4.5.10}$$

and substituting into equations (4.4.7), (4.4.8), (4.4.11), (4.4.13) and (4.4.15) gives, after applying the inversion formula ,

$$\phi_1 = \frac{-i}{2\pi}\int_{\Gamma_1} \gamma^{-1}\left(1+\frac{ms}{k}\right)\frac{\cosh\gamma(y+d)}{\sinh\gamma d}\left\{p\omega - \frac{\lambda\varepsilon K_-(-k/1+m)}{k+s(1+m)}\right\} K_+(s)^{-1} e^{-isx}\, ds \, , \tag{4.5.11}$$

$$\phi_2 = \frac{i}{2\pi} \int_{\Gamma_1} \hat{\gamma}^{-1} \{p\omega - \frac{\lambda\varepsilon K_-(-k/1+m)}{k+s(1+m)}\} K_+(s)^{-1} \exp\{-\hat{\gamma}y - isx\} ds \, , \qquad (4.5.12)$$

$$\eta = \frac{1}{2\pi} \int_{\Gamma_1} \{p - \frac{\varepsilon\lambda K_-(-k/1+m)}{\omega(k+s(1+m))}\} K_+(s)^{-1} e^{-isx} ds \qquad (4.5.13)$$

where Γ_1 is as before.

4.6 THE WALL DISPLACEMENT $\eta(x)$

Consider now the wall displacement $\eta(x)$ given by equation (4.5.13) as

$$\eta(x) = \frac{\varepsilon\lambda}{2\pi} \int_{\Gamma_1} \{\hat{p} - \frac{(1+m)^2}{\omega(k+\mu(1+m))(k+i\mu(1+m))(k+s(1+m))} + O(\varepsilon)\} K_+(s)^{-1} e^{-isx} ds \quad (4.6.1)$$

To analyse this integral the second of the two options available will be used. Instead of transforming to the Θ-plane as before the integral will be deformed to one around a semi-circular contour is the S-plane, integrating along both sides of a branch cut if necessary. Employing Cauchy's Residue theorem to account for any poles picked up in the process and Jordan's Lemma to confirm that the semi-circular contribution goes to zero as its radius goes to infinity means the integral can be recast into either (i) an integral plus the residue contributions or

(ii) just the residue contribution.

An approximation to this second integral, if it is present, can be found.

First consider $\eta(x)$ for $x < 0$. The path Γ_1 can be deformed onto Γ_5, as shown in Figure 4, because the integrand has no branch cuts in R_+. As the integrand is regular in R_+ there are no residues to be picked up and Jordan's lemma gives the integral along Γ_5 as zero as $L \to \infty$, so $\eta(x) = 0$ for $x < 0$.

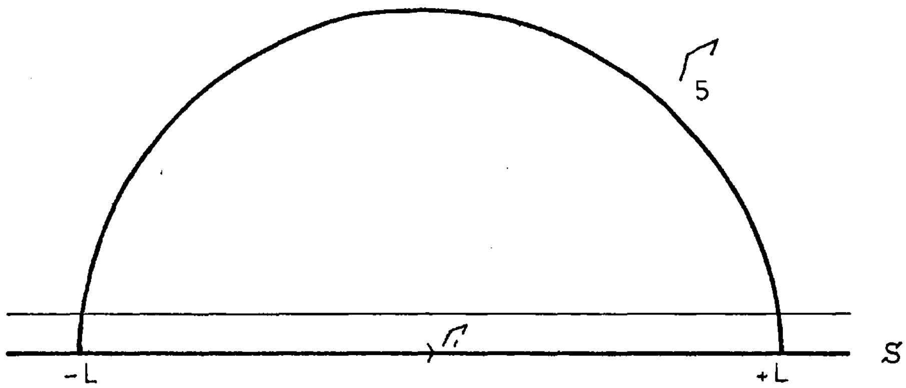

Figure 4: S-plane

For $\eta(x)$, $kx > 0$ and large, the deformation is more complex. The integrand has a branch cut from $-k$ to $-\infty$ along a line of constant argument and the semi-circular path Γ_6 in the lower half-plane R_- has to be indented around this branch cut as shown in Figure 5. The integrand has poles at S_{-n}, S_2, S_4, S_6, S_7 so there are residue contributions to be taken into account.

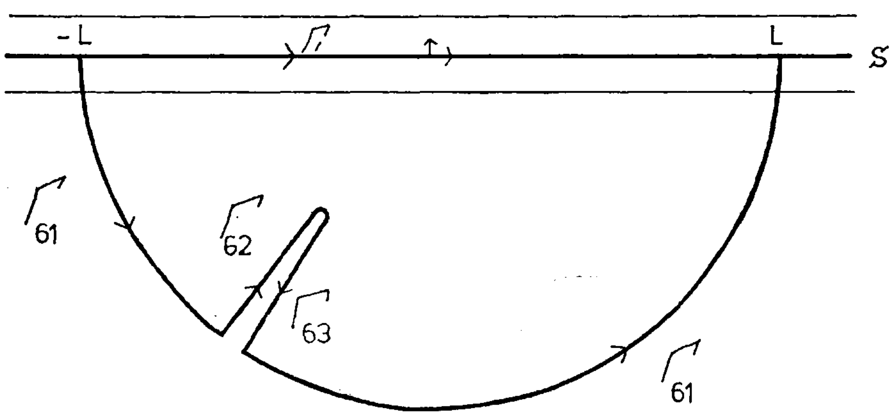

Figure 5: S-plane

From Cauchy's Residue theorem it is known that

$$\int_{\Gamma_1} = \int_{\Gamma_{61}} + \int_{\Gamma_{62}} + \int_{\Gamma_{63}} -2\pi i\{ \sum \text{Residues Captured}\} \tag{4.6.2}$$

From Jordan's lemma it can be shown that

$$\int_{\Gamma_{61}} \equiv 0 \quad . \qquad (4.6.3)$$

This leaves $\int_{\Gamma_{62}}$, $\int_{\Gamma_{63}}$ and the residue contributions to be found.

The integral has poles at S_2, S_4, S_6, S_7, S_{-n}. The contributions from S_6, S_2 (for $\mu > k$) will be travelling waves whereas those from S_2 (for $\mu < k$) S_4, S_7 and S_{-n} will be decaying exponentially at some order in ε. There is no pole at $s = \hat{S}_7 = -k/1+m$ because it is a zero of $(K_+(s))^{-1}$. In Appendix 6 the method for finding the residues of the integrand for $\eta(x)$, partly applicable for ϕ_1 and ϕ_2 is given, and the residue contributions are listed. This then only leaves the contributions from Γ_{62} and Γ_{63} to be found.

Expanding in powers of ε gives

$$\eta_s(x) \sim \{-\varepsilon\lambda(1+m)+0(\varepsilon^2)\}[2\pi\omega\{k+\mu(1+m)\}\{k+i\mu(1)m)\}]^{-1} \times$$

$$\times \int_{\Gamma_{62}+\Gamma_{63}} [\frac{1}{(1+i)\mu} + \frac{1+m}{k+s(1+m)}] K_+(s)^{-1} e^{-isx} ds \qquad (4.6.4)$$

where the integral along $\Gamma_{62}+\Gamma_{63}$ can be written

$$\int_{-\infty}^{-k} [\frac{1}{(1+i)\mu} + \frac{1+m}{k+s(1+m)}] [\frac{1}{K_+(s+)} - \frac{1}{K_+(s-)}] e^{-isx} ds \qquad (4.6.5)$$

and s+ denotes above the branch cut and s- below it.

The method of steepest descents, applied after putting $K_+(s) = K_-(s)K(s)$ then gives

$$\eta_s(x) \sim \frac{i(i+1)k^2\pi\lambda(1+m)(k+\mu)(k+i\mu)}{2\omega[k+\mu(1+m)][k+i\mu(1+m)]} \left[\frac{1-i}{2\mu} - \frac{1+m}{km} + O(\varepsilon)\right] (kx\pi)^{-3/2} e^{ikx} + O(kx)^{-5/2} \text{ for large } kx. \quad (4.6.6)$$

Notice that although this contribution is as small as $O(kx)^{-3/2}$ it is $O(1)$ with respect to ε.

Combining all these separate contributions to $\eta(x)$ then gives

$$\eta(x) \sim \eta_s(x) - 2\pi i\left[\eta_{\ell 2} + \eta_{\ell 4} + \eta_{\ell 6} + \eta_{\ell 7} + \sum_1^\infty \eta_{\ell - n}\right], \quad \ell = 1,2, \text{ for large } kx. \quad (4.6.7)$$

The wall displacement at large x thus consists not only of the $O(kx)^{-3/2}$ part due to η_s but also an $O(\varepsilon^3)$ travelling wave $\eta_{\ell 6}$ with exact wave number ς_6, and, when $\mu > k$, a second travelling wave due to η_{12} or, when $\mu < k$, a contribution η_{22} which is a wave in the exponent at $O(1)$ but decay at $O(\varepsilon)$. $\eta_{\ell 4}$ and $\eta_{\ell - n}$ $(n > N)$ give exponentially decaying terms but $\eta_{\ell 7}$ and $\eta_{\ell - n}$ $(n \le N)$ are waves at $O(1)$ and $O(\varepsilon)$ in the exponent but decay at $O(\varepsilon^2)$.

4.7 THE DUCT POTENTIAL $\phi_1(x,y)$

Equation (4.5.11) gives for the duct potential ϕ_1,

$$\phi_1(x,y) = \frac{i\lambda\varepsilon(1+m)}{2\pi[k+\mu(1+m)][k+i\mu(1+m)]}\int_{\Gamma_1}(1+\frac{ms}{k})\frac{\cosh\gamma(y+d)}{\gamma\sinh\gamma d}[\frac{1}{(1+i)\mu}+\frac{1+m}{k+s(1+m)}]$$

$$\times K_+(s)^{-1}e^{-isx}ds + O(\varepsilon^2) \qquad (4.7.1)$$

where this integrand has poles at S_2, S_4, S_6, S_7, S_{-n}, as did the one for $\eta(x)$. It also has poles at $\hat{S}_7$, $\hat{S}_8$ and $\hat{S}_{+n}$. It does not have any at $\hat{S}_{-n}$ because these are also zeros of $K_+(s)^{-1}$. This integrand, just like the one for $\eta(x)$, has no branch cuts in the upper half-plane R_+.

For $kx < 0$ the same procedure as was used for $\eta(x)$ can be employed, namely deforming the path Γ_1 onto Γ_5 as shown in Figure 4 and collecting the residues from the poles $\hat{S}_8$ and $\hat{S}_{+n}$ in the process. From Jordan's lemma it can again be shown that the contribution from integrating around Γ_5 is zero, giving

$$\phi_1(xy) = 2\pi i \sum \text{Residues} \qquad \text{for } kx<0 \qquad (4.7.2)$$

In Appendix 6 these residues are calculated and give, when substituted into expression (4.7.2)

$$\phi_1(x,y) = -\frac{\lambda\varepsilon(1-m^2)}{2dk[(k+\mu)^2-\mu^2m^2][(k+i\mu)^2+\mu^2m^2]}[\frac{1}{(1+i)\mu}+\frac{1-m^2}{2k}+O(\varepsilon)]\exp\{-\frac{ikx}{1-m}\}$$

$$+\sum_1^{\infty}\frac{-\lambda\varepsilon(1+m)\cos n\pi y/d}{[k+i\mu(1+m)][k+\mu(1+m)]dB_n}[\frac{1}{(1+i)\mu}+\frac{1+m}{k+(1+m)\hat{S}_{+n}}](1+\frac{m}{k}\hat{S}_{+n})\times$$

$$\times\frac{1}{(\mu+\hat{S}_{+n})(i\mu+\hat{S}_{+n})}e^{-i\hat{S}_{+n}x}+O(\varepsilon^2) \qquad (4.7.3)$$

This velocity potential is due solely to the presence of the elasticity in the duct walls because the rigid walled duct potential ϕ_0 was separated away at the outset. It can be seen that the modification to the duct potential, in the rigid walled part of the duct (kx<0), due to the elastic walled part, consists of a main reflected wave, the first term in expression (4.7.3), with a set of subsidiary reflected waves, the sum up to N of the second term, plus a set of disturbances that decay exponentially away from the origin, the sum from N+1 to infinity of the second term. The duct potential is thus known explicitly for kx<0.

For kx large and positive the procedure of deforming the path Γ_1 onto Γ_6 in the lower half plane R_- is again complicated by the necessity to integrate around the branch cut of $K_+(s)^{-1}$ as in Figure 5 in section 6. As in expression (4.6.2) ϕ_1 is given by

$$\phi_1 = \int_{\Gamma_{62}} + \int_{\Gamma_{63}} - 2\pi i \sum \text{ Residues Captured,} \qquad (4.7.4)$$

Jordan's lemma and Cauchy's Residue theorem having both been applied as before. This time however the poles are at S_2, S_4, S_6, S_7, S_{-n} and also $\hat{S}_7$. In Appendix 6 the residues for ϕ_1 at these poles are calculated.

Using the method of steepest descents an asymptotic expression for the integral terms, ϕ_{1S}, can be found:

$$\phi_{1S} = \frac{\lambda(1-m^2)(k+\mu)(k+i\mu)\cosh[\gamma(-k)(y+d)](1+i)k^2\pi}{2[k+\mu(1+m)][k+i\mu(1+m)]\gamma(-k)\sinh\gamma(-k)d} \left[\frac{1}{(1+i)\mu} - \frac{1+m}{km}\right] \times$$

$$\times (kx\pi)^{-3/2} + O(kx)^{-5/2} + O(\varepsilon) \text{ for kx large } . \qquad (4.7.5)$$

The duct velocity potential, ϕ_1, is then given by

$$\phi_1 = \phi_{1S} - 2\pi i \sum_{j=2,4,6,7,-n} R'_{\ell j} + 2\pi i \hat{R}_{\ell 7} \quad \text{for } kx \text{ large .} \tag{4.7.6}$$

The travelling wave $\hat{R}_{\ell 7}$ exactly cancels the imposed wave ϕ_0 and this shows that the imposed travelling wave does not propagate undeformed past the duct discontinuity because it is replaced by $R'_{\ell 7}$ which is a slowly (exponentially) decaying wave with a slightly distorted wave number. It can be intepreted as the travelling wave decaying slowly as it gives up its energy to excite the duct walls and then the fluid beyond. The other travelling wave terms are $R'_{\ell 2}$ (for $\mu > k$) and $R'_{\ell 6}$ which have exact wave numbers S_2 and S_6 respectively; the other terms present decay on going away from the origin, ϕ_{1S} like $(kx)^{-3/2}$ and the others exponentially.

For the downstream potential it is not possible to split the potential into one part due solely to the rigid duct and one part due to the elasticity of the walls as both phenomena are intricately intertwined even for large kx.

4.8 THE EXTERNAL VELOCITY POTENTIAL $\phi_2(x,y)$

Equation (4.5.12) gives for the external velocity potential ϕ_2.

$$\phi_2 = \frac{i\varepsilon\lambda}{2\pi} \int_{\Gamma_1} \{\hat{p}\omega - \frac{K_-(-k/1+m)}{k+s(1+m)}\} \frac{1}{\hat{\gamma}K_+(s)} e^{-\hat{\gamma}y-isx} ds \tag{4.8.1}$$

and as in Chapter 3 a new coordinate system (r,θ) and a new integration variable Θ are introduced where

$$x = r\cos\theta \ ; \quad y = r\sin\theta \ ; \quad s = k\cos\Theta \ ; \quad \hat{\gamma}(s) = -ik\sin\Theta$$

$$0 \le r < \infty \ ; \quad 0 \le \theta \le \pi \ ; \quad \Theta = \alpha + i\beta \ ; \quad 0 \le \alpha \le \pi; \ -\infty < \beta < \infty \tag{4.8.2}$$

giving

$$\phi_2 = \frac{\varepsilon\lambda}{2\pi} \int_{\Gamma_2} \{\hat{p}\omega - \frac{K_-(-k/1+m)}{k+(1+m)k\cos\Theta}\} K_+(k\cos\Theta)^{-1} \exp\{-ikr\cos(\theta+\Theta)\} d\Theta \ . \tag{4.8.3}$$

An approximation to ϕ_2, in the far field, will be found. As in the previous chapter the path Γ_2 can be deformed onto the 'steepest descents' path Γ_3 and any residues captured in the process accounted for. The poles of the integrand of expression (4.8.3) are at Θ_2, Θ_4, Θ_6, Θ_7, Θ_{-n} and its zeros are at $\hat{\Theta}_6$, $\hat{\Theta}_{-n}$. In Figures 6,7, and 8 the positions of the poles relative to the zeros, for the two alternatives $\mu > k$ and $\mu < k$, are shown .

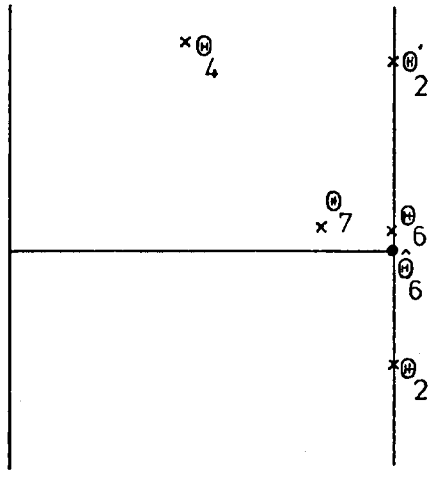

Figure 6: Θ-plane μ > k

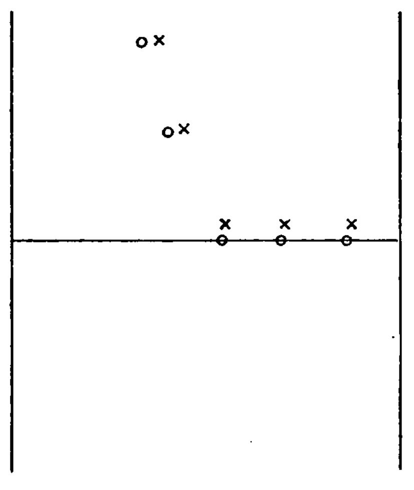

Figure 7: Θ-plane μ < k and μ > k

x denotes a pole at S_{-n}

o denotes a zero at $\hat{S}_{-n}$

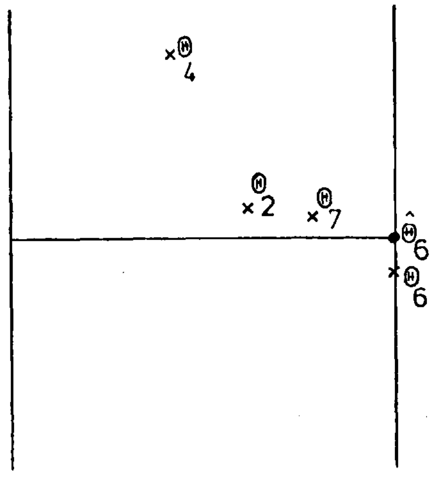

Figure 8: Θ-plane, μ < k .

By using similar deformations to those used in the previous chapter it can be shown that the residue at Θ_4 and those at $\Theta_{-n}(n > N)$, along with their associated zeros at $\hat{\Theta}_{-n}$, never contribute significantly to the result, so can be ignored.

The pole at Θ_2 for (k < μ) contributes a subsonic surface wave and its residue is calculated in Appendix 6 where it is found to be

$$\{ \frac{\varepsilon(1+m)i\lambda}{4\pi\mu^2[k+(1+m)\mu][k-(1+m)\mu]\hat{\gamma}(-\mu)} + O(\varepsilon^2)\} \exp\{-iS_2x-\hat{\gamma}(S_2)y\} . \tag{4.8.4}$$

The steepest descent path Γ_3 passes through the saddle point $\Theta = \pi - \theta$ and there are four possible sets of conditions which will be considered separately.

The first possibility is that the saddle point is not near any pole or zero. The second is that the saddle point is near a pole but not a zero. The third condition is that the saddle point is near a pole and a zero at $\hat{\Theta}_6$. The last possibility is that the saddle point is near a pole and a real zero where the zero is not at $\hat{\Theta}_6$.

4.9 SADDLE POINT NOT NEAR A POLE OR ZERO

This, the first possibility, is the easiest to consider. An approximation to the integral along Γ_3, ϕ_S, can be found, by the 'method of steepest descents', to be ϕ_S' where for $kr >> 1$

$$\phi_S' = \varepsilon\lambda(1+m)\left[\frac{1}{(1+i)\mu} + \frac{1+m}{k[1-(1+m)\cos\theta]}\right]\left[(k\cos\theta-\mu)(k\cos\theta-i\mu)(k+\mu(1+m))\right.$$
$$\left.\times(k+i\mu(1+m))(2\pi kr)^{\frac{1}{2}}\right]^{-1} e^{ikr-i\pi/4} + O(\varepsilon^2) + O(kr)^{-3/2}. \qquad (4.9.1)$$

For below coincidence, $k < \mu$,

$$\phi_2 \sim \phi_S' - 2\pi i\Psi_{12} \qquad \text{for } 0 < \theta \le \frac{\pi}{2}\ ;\quad kr >> 1 \qquad (4.9.2a)$$

and

$$\phi_2 \sim \phi_S' \qquad \text{for } \frac{\pi}{2} < \theta \le \pi\ ;\quad kr >> 1 \qquad (4.9.2b)$$

all other terms being an order of magnitude smaller.

For above coincidence, $\mu < k$,

$$\phi_2 \sim \phi_S' \qquad \text{for } 0 < \theta \le \pi\ ;\quad kr >> 1\ , \qquad (4.9.3)$$

These potentials thus consist of an $O(kr)^{-\frac{1}{2}}$ θ - dependent wave plus, in the case $\mu > k$, a subsonic surface wave on the wall of the duct which decays slowly out into the fluid.

4.10 SADDLE POINT NEAR A POLE BUT NOT A ZERO

The saddle point $\Theta = \pi - \theta$ can be very close to the poles at Θ_2 (within $0(\varepsilon)$ for $\mu < k$) and at Θ_7 (within $0(\varepsilon^2)$) without there being any zero in the vicinity. This permits the type of analysis used in section 9 of the previous chapter to be employed.

Consider first the contribution due to Θ_2 ($\mu < k$). The 'Mach Angle' Θ_{2m} is given by $k\cos\Theta_{2m} = \text{Re}(S_2) + 0(\varepsilon^2)$ and θ_{2m} by $\theta_{2m} = \pi - \Theta_{2m}$. The associated 'Capture Angles' are: $\Theta_2' = \Theta_{2m} + i(k^2-\mu^2)^{-\frac{1}{2}}\,\text{Im}(S_2) + 0(\varepsilon^2)$ and $\theta_2' = \pi - \Theta_2$ just as they were in Chapter 3. Applying the Fresnel analysis as before and denoting the potential contribution from the 'steepest descents' path and the pole by ϕ_{SP} gives:

(i) When $|\theta-\Theta_2'|$ is $0(1)$ and $kr \gg 1$ or
when $|\theta-\theta_2'|$ is $0(\varepsilon)$ and $kr \gg \varepsilon^{-2}$

$$\phi_{SP} \sim -\sqrt{\frac{2\pi}{kr}}\,\hat{F}(\pi-\theta)e^{ikr-i\pi/4} \tag{4.10.1}$$

where $$\hat{F}(\Theta) = \frac{\varepsilon\lambda}{2\pi}\{\hat{p}\omega - \frac{K_-(-k/1+m)}{k[1+(1+m)\cos\Theta]}\}\,K_+(k\cos\Theta)^{-1}. \tag{4.10.2}$$

(ii) When $|\theta-\theta_2'|$ is $0(\varepsilon)$ and $1 \ll kr \ll \varepsilon^{-2}$

$$\phi_{SP} \sim -2\pi i\;F_2(\pi-\theta)\;\exp\{-ikr\cos(\theta+\Theta_2)\} \tag{4.10.3}$$

where

$$F_j(\Theta) = \sin\left(\frac{\Theta-\Theta_j}{2}\right)\hat{F}(\Theta) \tag{4.10.4}$$

An approximation to ϕ_2 in the vicinity of this Mach Angle θ_2, for $\mu > k$, is therefore expressions (4.9.2a) and (4.9.2b) with ϕ_S' replaced by ϕ_{SP} from above.

For the contribution due to Θ_7 the Mach Angles Θ_{7m} and θ_{7m} and the Capture Angles Θ_7' and θ_7' are defined in expressions (3.9.15) to (3.9.18) with j = 7. ϕ_{SP} for the saddle point and this pole is thus given by:

(i) When $|\theta-\theta_7'|$ is $O(1)$ and $kr \gg 1$ or
when $|\theta-\theta_7'|$ is $O(\varepsilon)$ and $kr \gg \varepsilon^{-2}$ or
when $|\theta-\theta_7'|$ is $O(\varepsilon^2)$ and $kr \gg \varepsilon^{-4}$

$$\phi_{SP} \sim \sqrt{\frac{2\pi}{kr}}\ \hat{F}(\pi-\theta)e^{ikr-i\pi/4} \quad . \qquad (4.10.5)$$

(ii) When $|\theta-\theta_7'|$ is $O(\varepsilon^2)$ and $1 \ll kr \ll \varepsilon^{-4}$ or
when $|\theta-\theta_7'|$ is $O(\varepsilon)$ and $1 \ll kr \ll \varepsilon^{-2}$

$$\phi_{SP} \sim -2\pi i\ F_7(\pi-\theta)\ \exp\{-ikr\cos(\theta+\Theta_7)\} \qquad (4.10.6)$$

where $F_7(\Theta)$ is given by expression (4.10.4) with j = 7.

So, in both cases, when θ is near the Mach Angle θ_2 or θ_7 the potential exhibits a Leaky wave or a Modified Leaky wave structure that has a beaming effect of intermediate range. However, as before, for large kr($kr \gg \varepsilon^{-2}$ for θ_2 and $kr \gg \varepsilon^{-4}$ for θ_7) the usual $(kr)^{-\frac{1}{2}}$ decaying potential is identified.

4.11 SADDLE POINT NEAR A POLE AND A ZERO, $\hat{\Theta}_6$

When the saddle point on the path of integration is near both a zero and a pole of the integrand we need special asymptotics as before. The case of $\hat{\Theta}_6$ is different from the rest in that the zero (on the real axis) has the same real part as the pole. This situation will be investigated first.

As the saddle point $\Theta = \pi - \theta$ approaches $\hat{\Theta}_6$ it enters a small region in which the integrand has a pole as well as a zero (for $\mu < k$). This pole's residue is a non-decaying subsonic travelling wave. Note that this does not happen as Θ approaches $\hat{\Theta}_5$ (s=k) because the integrand has neither a pole nor a zero there so the standard steepest descents analysis will hold giving an $O(kr)^{-\frac{1}{2}}$ decaying potential.

In the "below-coincidence" case ($k < \mu$) the pole is not captured and when the saddle point is within $O(\varepsilon)$ of the zero the second term in the steepest descents expansion of the integral for ϕ_S, of order $(kr)^{-3/2}$, combines, as before, with the term already found to give the dominant contribution. When $\Theta = \Theta_6$ the first term is zero so it is the second term which becomes the leading one.

In the "above-coincidence" case ($\mu < k$) the saddle point goes near to a pole which can be captured and also approaches the zero. A similar analysis to that used in the previous chapter will be used. ϕ_S can be rewritten in the form

$$\phi_S = \int_{\Gamma_3} \hat{F}(\Theta)G_6(\Theta)^{-1}G_6(\Theta)\ \exp\{-ikr\cos(\theta+\Theta)\}d\Theta \qquad (4.11.1)$$

where $G_6(\Theta)$ is chosen to be as before (see expression (3.10.4)) for the same reasons. Splitting the integrals as before gives

$$\phi_S = \int_{\Gamma_3} \hat{F}(\Theta) G_6(\Theta)^{-1} \exp\{-ikr\cos(\theta+\Theta)\} d\Theta$$

$$+ \int_{\Gamma_3} \hat{F}(\Theta) \sin \frac{i\varepsilon}{2k(k^4-\mu^4)} \sin\frac{\Theta}{2} \left[\cos\frac{1}{2}\left(\Theta + \frac{i\varepsilon}{k(k^4-\mu^4)}\right)\right]^{-1} G_6(\Theta)^{-1} \times$$

$$\times \exp\{-ikr\cos(\theta+\Theta)\} d\Theta = I_1 + I_2 \quad . \qquad (4.11.2)$$

The integrand of the first integral is regular in the region of both the saddle point and the pole so the standard method of steepest descents, for $kr \gg 1$, may be applied to this integral in this region to give

$$I_1 \sim -\sqrt{\frac{2\pi}{kr}} \frac{\hat{F}(\pi-\theta)}{G_6(\pi-\theta)} e^{ikr-i\pi/4} + O(kr)^{-3/2} \quad \text{for } kr \gg 1 \; . \qquad (4.11.3)$$

Using the usual techniques the second integral can be recast in the form of a Fresnel integral

$$I_2 \sim \mp 4\sqrt{\pi}\, e^{ikr-i\pi/4} \frac{\hat{F}(\pi-\theta)}{G_6(\pi-\theta)} \cos\frac{\theta}{2} \sin\frac{i\varepsilon}{2k(k^4-\mu^4)} F\left(\pm -i\sqrt{2kr}\sin\left(\frac{\theta}{2} - \frac{i\varepsilon}{2k(k^4-\mu^4)}\right)\right)$$

$$(4.11.4)$$

with ± signs in F as before.

Employing the Fresnel expansions already obtained, along with Cauchy's Residue theorem, gives for ϕ_{SP} the combined potential due to the pole and the integral:

(i) When $|\theta-\theta_6'|$ is $O(1)$ and $kr \gg 1$

$$\phi_{SP} \sim -\sqrt{\frac{2\pi}{kr}} \hat{F}(\pi-\theta) e^{ikr-i\pi/4} + O(kr)^{-3/2} \; ; \qquad (4.11.5)$$

(ii) When $|\theta-\theta_6'|$ is $O(\varepsilon)$ and $kr \gg \varepsilon^{-2}$

$$\phi_{SP} \sim -\sqrt{\frac{2\pi}{kr}} \hat{F}(\pi-\theta) e^{ikr-i\pi/4} + O(kr)^{-3/2} - 2\pi i \times \text{Residue}$$

$$(4.11.6)$$

where the residue is found in Appendix 6 to be Ψ_{26} of expression (A6.18b), which is exponentially small unless θ is at or near θ_6 in which case it is, or is almost, a marginally subsonic travelling wave.

(iii) When $|\theta-\theta_6'|$ is $O(\varepsilon)$ and $1 \ll kr \ll \varepsilon^{-2}$

$$\phi_{SP} \sim -\sqrt{\frac{2\pi}{kr}}\,\frac{\hat{F}(\pi-\theta)}{G_6(\pi-\theta)}\,e^{ikr-i\pi/4} + O(kr)^{-3/2} - 2\pi i\Psi_{26}$$
$$- 2\pi i\, G(\pi-\theta)\cos\frac{\theta}{2}\sin\frac{i\varepsilon}{2k(k^4-\mu^4)}\exp\{ikr\cos(\theta - \frac{i\varepsilon}{k(k^4-\mu^4)})\} \tag{4.11.7}$$

which is a Leaky wave plus the pole contribution.

When θ is O(1) away from θ_6' the potential ϕ_{SP} consists of just the $O(kr)^{-\frac{1}{2}}$ radiating far field at $kr \gg 1$. However if θ is within $O(\varepsilon)$ of θ_6' this far field is not detected until $kr \gg \varepsilon^{-2}$ and even then it looks increasingly like $O(kr)^{-3/2}$ plus a subsonic surface wave as θ approaches zero. The Leaky wave, due to the proximity of the pole to the saddle point, is also detectable in the middle ground, $1 \ll kr \ll \varepsilon^{-2}$, when θ is within $O(\varepsilon)$ of θ_6'.

4.12 SADDLE POINT NEAR A POLE AND A ZERO

Consider the pairs of points $\hat{\Theta}_{-n}$ and Θ_{-n} $(n \le N)$; $\hat{\Theta}_{-n}$ is a zero of the integrand, Θ_{-n} is a pole. These pairs are not similar to $\hat{\Theta}_6$ and Θ_6 because for that pair the Mach Angle Θ_{6m} is $\hat{\Theta}_6$ and the distances from the pole and the zero, (and thus the Mach Angle) $\hat{\Theta}_6$ and Θ_6, to the Capture Angle are of the same order, $O(\varepsilon)$. In the case of $\hat{\Theta}_{-n}$ and Θ_{-n} however the Mach Angle Θ_{-nm} is $O(\varepsilon^2)$ from both the Capture Angle Θ_{-n}' and the pole Θ_{-n} whereas all these are $O(\varepsilon)$ from the zero at $\hat{\Theta}_{-n}$ (see Figure 9).

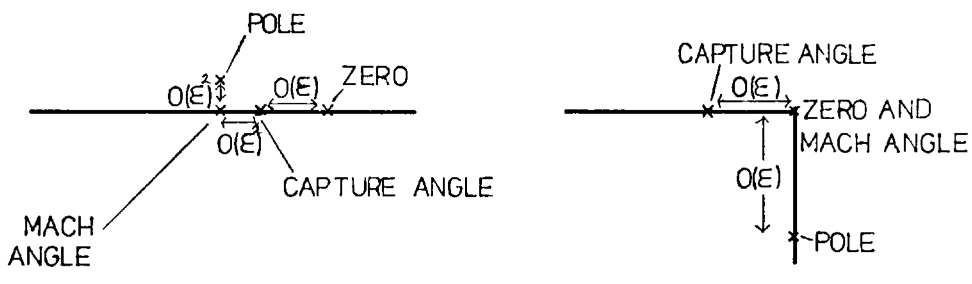

Figure 9: Θ-plane

In the Θ-plane, if we multiply and divide by some function, $G_{-n}(\Theta)$ say, that has a zero at $\hat{\Theta}_{-n}$ and a pole at Θ_{-n}, just as we did in section 6, a somewhat similar analysis can then be undertaken.

For $G_{-n}(\Theta)$ choose

$$G_{-n}(\Theta) = \sin \frac{1}{2} (\Theta-\hat{\Theta}_{-n})[\sin \frac{1}{2} (\Theta-\Theta_{-n})]^{-1} , \tag{4.12.1}$$

because it has its pole and its zero in the right places, and then substitute into ϕ_S so that, as in equation (4.11.1),

$$\phi_S = \int_{\Gamma_3} \hat{F}(\Theta)G_{-n}(\Theta)^{-1} \; G_{-n}(\Theta) \; \exp\{-ikr\cos(\theta+\Theta)\}d\Theta \; . \tag{4.12.2}$$

$G_{-n}(\Theta)$ can be rearranged to give

$$G_{-n}(\Theta) = \cos\tfrac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n}) - [\sin\tfrac{1}{2}(\Theta-\Theta_{-n})]^{-1}\cos\tfrac{1}{2}(\Theta-\Theta_{-n})\,\sin\tfrac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n}) \tag{4.12.3}$$

and then substituted into expression (4.12.2) where $\hat{F}(\Theta)G_{-n}(\Theta)^{-1}$, as it has no poles or zeros near the saddle point and the main contribution comes from there, can be replaced by $\hat{F}(\pi-\theta)\; G_{-n}(\pi-\theta)^{-1}$ to give

$$\begin{aligned}\phi_S \sim\; & \frac{\hat{F}(\pi-\theta)}{G_{-n}(\pi-\theta)} \cos\tfrac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n}) \int_{\Gamma_3} \exp\{-ikr\cos(\theta+\Theta)\}d\Theta \\ & - \frac{\hat{F}(\pi-\theta)}{G_{-n}(\pi-\theta)} \sin\tfrac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n}) \int_{\Gamma_3} \cos\tfrac{1}{2}(\Theta-\Theta_{-n})[\sin\tfrac{1}{2}(\Theta-\Theta_{-n})]^{-1} \times \\ & \times \exp\{-ikr\cos(\theta+\Theta)\}d\Theta \; .\end{aligned} \tag{4.12.4}$$

In the second term we can replace $\cos\frac{1}{2}(\Theta-\Theta_{-n})$ by $\cos\frac{1}{2}(\pi-\theta-\Theta_{-n})$ and rewrite in the form of a Fresnel integral

$$\begin{aligned}\phi_S \sim\; & \frac{\hat{F}(\pi-\theta)}{G_{-n}(\pi-\theta)} \cos\tfrac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n}) \int_{\Gamma} \exp\{-ikr\cos(\theta+\Theta)\}d\Theta \\ & - \frac{\hat{F}(\pi-\theta)}{G_{-n}(\pi-\theta)} \cos\tfrac{1}{2}(\pi-\theta-\Theta_{-n}) \sin\tfrac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n}) [\pm 4\sqrt{\pi}\, e^{ikr-i\pi/4} \\ & \cdot\; F(\pm i\sqrt{2kr}\cos\tfrac{1}{2}(\theta+\Theta_{-n}))] \end{aligned} \tag{4.12.5}$$

$$= I_1 + I_2 \; .$$

Now for $kr \gg 1$ and all θ the method of steepest descents applied to I_1 gives

$$I_1 \sim -\sqrt{\frac{2\pi}{kr}}\,\frac{\hat{F}(\pi-\theta)}{G_{-n}(\pi-\theta)} \cos\tfrac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n}) e^{ikr-i\pi/4} + O(kr)^{-3/2}. \tag{4.12.6}$$

The results for I_2 are not so simple; applying the Fresnel approximations we have:

(i) When $|\theta-\theta_{-n}'|$ is of $O(\varepsilon^2)$ and $kr \gg \varepsilon^{-4}$ or
when $|\theta-\theta_{-n}'|$ is of $O(\varepsilon)$ and $kr \gg \varepsilon^{-2}$ or
when $|\theta-\theta_{-n}'|$ is of $O(1)$ and $kr \gg 1$

$$I_2 \sim \frac{\hat{F}(\pi-\theta)}{G_{-n}(\pi-\theta)} \cos\frac{1}{2}(\pi-\theta-\Theta_{-n}) \sin\frac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n}) \sqrt{\frac{2\pi}{kr}} \times$$
$$\times \left[\sin\frac{1}{2}(\pi-\theta-\Theta_{-n})\right]^{-1} e^{ikr-i\pi/4} + O(kr)^{-3/2} \qquad (4.12.7)$$

which gives for ϕ_{SP}, the potential due to the steepest descents path and the pole's residue.

$$\phi_{SP} \sim -\sqrt{\frac{2\pi}{kr}}\, e^{ikr-i\pi/4}\, \hat{F}(\pi-\theta) + O(kr)^{-3/2} \qquad (4.12.8)$$

Note that when θ is near $\hat{\theta}_{-n}$ the amplitude of the $(kr)^{-\frac{1}{2}}$ term is small, actually equalling zero when $\theta = \hat{\theta}_{-n}$, so this term combines with the rest, $O(kr)^{-3/2}$, to give the dominant contribution.

(ii) When $|\theta-\theta_{-n}'| = O(\varepsilon^2)$ and $1 \ll kr \ll \varepsilon^{-4}$ the argument of the Fresnel function is small giving no change in the leading order contribution from I_1 but giving as the leading order contribution from I_2

$$I_2 \sim \pm 2\pi i\, \frac{\hat{F}(\pi-\theta)}{G_{-n}(\pi-\theta)} \cos\frac{1}{2}(\pi-\theta-\Theta_{-n}) \sin\frac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n}) \exp\{-ikr\cos(\theta+\Theta_n)\} \qquad (4.12.9)$$

with $\pm$ depending, as before, on the capture of the pole. So for $|\theta-\theta_{-n}'| = O(\varepsilon^2)$, $1 \ll kr \ll \varepsilon^{-2}$, the contribution from I_1 dominates that from I_2 to give

$$\phi_{SP} \sim -\sqrt{\frac{2\pi}{kr}}\ \frac{\hat{F}(\pi-\theta)}{G_{-n}(\pi-\theta)}\ \cos\frac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n})e^{ikr-i\pi/4} + O(kr)^{-3/2} \qquad (4.12.10)$$

and for $|\theta-\theta_{-n}'| = O(\varepsilon^2)$, $\varepsilon^{-2} \ll kr \ll \varepsilon^{-4}$ the contribution from I_2 dominates that from I_1 to give

$$\phi_{SP} \sim -i\pi\ \frac{\hat{F}(\pi-\theta)}{G_{-n}(\pi-\theta)}\ \cos\frac{1}{2}(\pi-\theta-\Theta_{-n})\ \sin\frac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n})\ \exp\{-ikr\cos(\theta+\Theta_{-n})\}. \qquad (4.12.11)$$

(iii) Lastly we must consider $|\theta-\theta_{-n}'| = O(\varepsilon)$ and $1 \ll kr \ll \varepsilon^{-2}$. Expression (4.12.6) holds for I_1 and expression (4.12.8) holds for I_2. The contribution from I_1 dominates that from I_2 to give

$$\phi_{SP} \sim -\sqrt{\frac{2\pi}{kr}}\ \frac{\hat{F}(\pi-\theta)}{G_{-n}(\pi-\theta)}\ \cos\frac{1}{2}(\hat{\Theta}_{-n}-\Theta_{-n})e^{ikr-i\pi/4}, \qquad (4.12.12)$$

Now if $|\theta-\theta_{-n}'|$ is $O(\varepsilon)$ and $|\theta-\theta_{-n}'|$ is $O(\varepsilon)$, that is θ is on the opposite side of θ_{-n}' to $\hat{\theta}_{-n}$, then $G_{-n}(\pi-\theta)$ and $\hat{F}(\pi-\theta)$ are both $O(1)$, whereas when $|\theta-\theta_{-n}'|$ is $O(\varepsilon)$ but $|\theta-\hat{\theta}_{-n}|$ can be arbitrarily small, although $\hat{F}(\pi-\theta)/G_{-n}(\pi-\theta)$ is $O(1)$, $\hat{F}(\pi-\theta)$ and $G_{-n}(\pi-\theta)$ are both as small as $|\theta-\hat{\theta}_{-n}|$.

The contribution from the saddle point, near one of these pairs of pole and zero, is different from anything yet described. For sufficiently large kr at any θ the contribution looks like a simple radiating and algebraically decaying wave with a corresponding drop in strength to $O(kr)^{-3/2}$ when the saddle point is near the zero. For the intermediate range $\varepsilon^{-2} \ll kr \ll \varepsilon^{-4}$ and θ within $O(\varepsilon^2)$ of Θ_{-n}' a Modified Leaky wave can be detected. However this does not stretch all the way back to $1 \ll kr \ll \varepsilon^{-2}$ because in this region for $|\theta-\theta_{-n}'|$ of $O(\varepsilon)$ or of $O(\varepsilon^2)$ a second algebraically decaying wave is detected as the dominant one having a different amplitude to both the far field decaying wave and the Modified Leaky wave (which is like the pole's residue). This new decaying wave has an amplitude of O(1) and thus represents another reinforcement of the far field and therefore a locally detectable stronger field.

One of these pairs, under certain circumstances, is of particular interest. Let us consider them in the S and Θ-planes. If B_{-n}, for some n, is such that the real part of $\hat{S}_{-n}$ is positive, that is

$$(1-m^2)^{-1} \left[mk - \left(k^2-(1-m^2)\left(\frac{n^2\pi^2}{d^2}\right)\right)^{\frac{1}{2}} \right] > 0 \quad , \qquad (4.12.13)$$

which is possible, then although the root Θ_{-n} lies above the real axis in the Θ-plane the zero $\hat{\Theta}_{-n}$ lies on the real axis (to the left of $\pi/2$), so the capture angle θ_{-n}' and Mach Angle θ_{-nm} are both greater than $\pi/2$. This means that the flow in the duct has 'pushed' this disturbance around so that it lies upstream of the duct discontinuity not downstream as the others do.

4.13 THE EXTERNAL VELOCITY POTENTIAL : SUMMARY

The main contributions to the estimate of ϕ_2, in the far field,are as follows.

4.13.1 BELOW COINCIDENCE ($k < \mu$)

The potential ϕ_2 consists of the radiating far field of order $(kr)^{-\frac{1}{2}}$ and $(kr)^{-3/2}$ plus: a subsonic surface wave (given by $2\pi i\Psi_{12}$), Modified Leaky waves as described and also these other intermediate range reinforcements, near certain θ's, caused by the proximity of the saddle point and poles of the integral to its zeros.

4.13.2 ABOVE COINCIDENCE ($\mu < k$)

Here the potential ϕ_2 consists basically of the radiating far field plus a weak marginally subsonic surface wave $-2\pi i\Psi_{26}$, Leaky waves and Modified Leaky waves, and these other new contributions, just described,at intermediate ranges near certain θ's.

A sketch of maximum amplitude against angle θ would be similar to Figure 16 of Chapter 3,but this time for $\pi/2 < \theta < \pi$ there would only be radiating field (4) and no Leaky waves (3), Modified Leaky waves (3) or subsonic surfaces waves (2).

4.14 SUMMARY

In this chapter the analysis and techniques of Chapter 3 have been extended from an infinite to a semi-infinite geometry. The interaction between a time periodic disturbance to a uniform flow and the duct, partly wave-bearing, containing that flow has been modelled by the superposition of a plane wave onto the flow of an inviscid fluid down a duct which changes from being rigid walled to being elastic walled.

The flow within the duct again alters the wave speeds slightly and displaces the resonant frequencies for the duct away from those for the duct with no flow. The flow's effect is again most evident in the model problem in the introduction of the convected terms to the material derivative and Helmholtz's equations.

The Wiener-Hopf technique was applied to find an asymptotic solution for the wall displacement and the sound field's velocity potential both inside and outside the duct. Because the first part of the duct is rigid, almost no Leaky wave type disturbances are to be found upstream of the duct discontinuity. This contrasts with the previous analysis where they were present upstream and downstream in almost equal numbers.

The poles of the integrands were a selection of those from the previous analysis and as such their positions and the nature of the solutions they produce was already known. The Wiener-Hopf product factorisation, for small ε, relies on the positions of the zeros and poles of the function $K(s)$ and is, like them, non-uniform in s, ε and m. In the regions where the original factorisation is not valid an alternative, local, method is available for finding an approximation to the $K_+(s)$ or $K_-(s)$ and hence the residue of the integrand.

In the external velocity potential ϕ_2 the modified asymptotics used again identify Leaky waves in the middle field. This time however, due to the presence of zeros in the integrand, they are sometimes accompanied locally by a radiating far field of considerably diminished strength.

APPENDIX 1

THE ZEROS OF F(s)

The problem is to find the zeros of $F(s) = f(s) + \varepsilon g(s)$, for small ε, where

$$f(s) = (s^4-\mu^4)\hat{\gamma}(s)\gamma(s)\sinh\gamma(s)d \tag{A1.1a}$$

$$g(s) = -\gamma(s)\sinh[\gamma(s)d] - (1+\frac{ms}{k})^2\hat{\gamma}(s)\cosh[\gamma(s)d] \tag{A1.1b}$$

Now the zeros $\hat{S}_j$ of f(s) are at

$$\hat{S}_j = \pm\mu, \quad \pm i\mu, \quad \pm k, -k\ (1+m)^{-1},\ k(1-m)^{-1} \qquad j=1,2,\ldots,8 \tag{A1.2a}$$

and

$$\hat{S}_{\pm n} = (1-m^2)^{-1}[\,mk\pm(k^2-(1-m^2)\ \frac{n^2\pi^2}{d^2})^{\frac{1}{2}}\,] \qquad n = 1,2\ldots \tag{A1.2b}$$

Using Rouché's theorem, integrating F(s) around a closed path in the S-plane containing all the zeros of f(s), it can be proved that, provided the zeros $\hat{S}_j$ and $\hat{S}_{\pm n}$ are well separated, then the zeros of F(s), at $S_1, S_2,\ldots,S_8, S_{\pm n}$ can be approximated to by putting

$$S_j = \hat{S}_j + \varepsilon\tau_{1j} + \varepsilon^2\tau_{2j} + O(\varepsilon^3) \tag{A1.3}$$

substituting into $F(S_j) = 0$, expanding for small ε and solving for τ_{1j}, τ_{2j} etc. The restrictions on the separation of the zeros for Rouché's theorem are the same as those needed for the expansion (A1.3) to hold; that is all those conditions needed for the τ_{ij} to be of at most O(1) so that successive terms are an order higher in ε. For the pairs, k, $k(1-m)^{-1}$ and $-k,-k(1+m)^{-1}$ to be well separated m must not be too small. These asymptotics then are non-uniform in m and $\hat{\varepsilon}$ and if we wish to consider both $\hat{\varepsilon}$ and m small the correction τ to say the approximation k would need to satisfy the cubic in $(\tau/k)^{\frac{1}{2}}$;

$$[(\tau/k)^{\frac{1}{2}}]^3 - \left(m + \frac{\hat{\varepsilon}\,\mu^4}{4kd(\mu^4-k^4)}\right)(\tau/k)^{\frac{1}{2}} + \frac{\hat{\varepsilon} m\mu^4}{2\sqrt{2}(k^4-\mu^4)} = 0 \qquad \text{(A1.4)}$$

to first order. The solutions to this cubic are very sensitive to the relative magnitudes of $\hat{\varepsilon}$ and m.

The following approximations are then obtained for the zeros of F(s) :–

$$S_1 = \mu + \frac{\varepsilon}{4\mu^3}\left[\hat{\gamma}(\mu)^{-1} + \left(1+\frac{\mu m}{k}\right)^2 \gamma(\mu)^{-1}\coth\gamma(\mu)d\right] + 0(\varepsilon^2)\,, \qquad \text{(A1.5a)}$$

$$S_2 = S_1 \text{ with } \mu \text{ replaced by } -\mu\,, \qquad \text{(A1.5b)}$$

$$S_3 = S_1 \text{ with } \mu \text{ replaced by } i\mu\,, \qquad \text{(A1.5c)}$$

$$S_4 = S_1 \text{ with } \mu \text{ replaced by } -i\mu\,, \qquad \text{(A1.5d)}$$

$$S_5 = k + \frac{\varepsilon^2}{2k(k^4-\mu^4)^2} + 0(\varepsilon^3) \text{ with } (S_5{}^2-k^2)^{\frac{1}{2}} = \frac{\varepsilon}{k^4-\mu^4} + 0(\varepsilon^2)\,, \qquad \text{(A1.5e)}$$

$$S_6 = -k - \frac{\varepsilon^2}{2k(k^4-\mu^4)^2} + 0(\varepsilon^2) \text{ with } (S_6{}^2-k^2)^{\frac{1}{2}} = \frac{\varepsilon}{k^4-\mu^4} + 0(\varepsilon^2)\,, \qquad \text{(A1.5f)}$$

$$\begin{aligned} S_7 = &-k/1+m - \frac{\varepsilon}{2kd(1+m)^2}\left(\frac{k^4}{(1+m)^4} - \mu^4\right)^{-1} \\ &+ \frac{\varepsilon^2}{2kd}\left(k^4(1+m)^{-4}-\mu^4\right)^{-2}\Big[\frac{1+3m}{4k^2d(1+m)^3} - \frac{i}{k(2m+m^2)^{\frac{1}{2}}(1+m)} \\ &- \frac{d}{3(1+m)^4} + \frac{2k^2}{d(1+m)^7}\left(k^4(1+m)^{-4}-\mu^4\right)^{-1}\Big] + 0(\varepsilon^3)\,, \end{aligned} \qquad \text{(A1.5g)}$$

$$S_8 = \frac{k}{1-m} + \frac{\varepsilon}{2kd(1-m)^2}\left(k^4(1-m)^{-4}-\mu^4\right)^{-1} + 0(\varepsilon^2)\,, \qquad \text{(A1.5h)}$$

$$S_{\pm n} = (1-m^2)^{-1}[mk \pm B_n] \pm \frac{\varepsilon}{d}(\hat{S}_{\pm n}{}^4 - \mu^4)^{-1}(1+\frac{m}{k}\hat{S}_{\pm n})^2 B_n{}^{-1}$$
$$+ \varepsilon^2 \times \text{Real}$$
$$\pm \frac{\varepsilon^2}{d}(1+\frac{m}{k}\hat{S}_{\pm n})^2 B_n{}^{-1}(\hat{S}_{\pm n}{}^4 - \mu^4)^{-2}\hat{\gamma}(\hat{S}_{\pm n})^{-1} + 0(\varepsilon^3) , \quad \text{(A1.5i)}$$

where

$$B_n = (k^2 - (1-m^2)\frac{n^2\pi^2}{d^2})^{\frac{1}{2}} , \quad n^2 \leq \frac{k^2 d^2}{(1-m^2)\pi^2} ,$$
$$= i((1-m^2)\frac{n^2\pi^2}{d^2} - k^2)^{\frac{1}{2}}, \quad n^2 > \frac{k^2 d^2}{(1-m^2)\pi^2} . \quad \text{(A1.6)}$$

The points in the Θ-plane corresponding to these zeros are $\Theta_1,\ldots,\Theta_8$ and $\Theta_{\pm n}$. Primes denote where a second zero in the Θ-plane has come from the S-plane.

In order to discuss the nature of these zeros with respect to their real and imaginary parts we must recognise that the cases $k > \mu$ and $\mu > k$ have to be treated separately as the positions of these zeros in the S-plane are different in each case. We refer to these conditions as above or below coincidence respectively.

ABOVE COINCIDENCE $k > \mu$

S_1 and S_2 have real 0(1) terms but mixed (real and imaginary) $0(\varepsilon)$ terms.

S_3 and S_4 have imaginary 0(1) terms but mixed $0(\varepsilon)$ terms.

S_5 and S_6 are wholly real.

S_7 has real 0(1) and $0(\varepsilon)$ terms but mixed $0(\varepsilon^2)$.

S_8 is wholly real.

$S_{\pm n}$ $n \leq N$ have real 0(1) and $0(\varepsilon)$ but mixed $0(\varepsilon^2)$.

$S_{\pm n}$ $n > N$ are mixed at 0(1).

BELOW COINCIDENCE $k < \mu$

S_1 and S_2 are wholly real.

S_3 and S_4 have imaginary 0(1) but mixed $0(\varepsilon)$ terms.

S_5 and S_6 are wholly real.

S_7 has real 0(1) and $0(\varepsilon)$ but mixed $0(\varepsilon^2)$ terms.

S_8 is wholly real.

$S_{\pm n}$ $n \leq N$ have real 0(1) and $0(\varepsilon)$ terms but mixed $0(\varepsilon^2)$ terms.

$S_{\pm n}$ $n > N$ are mixed at 0(1).

The $0(\varepsilon^2)$ terms for S_7 and $S_{\pm n}$ are listed whereas for the others they are not, although they have been calculated, because it is at this order that the full nature of the zero is revealed.

APPENDIX 2

RESIDUES AND POTENTIALS

The residues of $F(s)^{-1}$, and the corresponding potential contributions, will be calculated in the S-plane as this is easier than in the Θ-plane.

We wish to find the residue of $F(s)^{-1}$ at each of its poles. Now the residue R_j of $F(s)^{-1}$ at $s = S_j$ is given by

$$R_j = \frac{1}{F'(S_j)} \quad . \tag{A2.1}$$

Substituting for F(s) from (3.4.10), differentiating and then substituting for S_j from Appendix 1 and then expanding for small ε gives

$$R_1 = \{4\mu^3\hat{\gamma}(\mu)\gamma(\mu)\sinh\gamma(\mu)d^{\ -1}\} + O(\varepsilon)\ , \tag{A2.2a}$$

$$R_2 = R_1 \text{ with } \mu \text{ replaced by } -\mu, \tag{A2.2b}$$

$$R_3 = R_1 \text{ with } \mu \text{ replaced by } i\mu, \tag{A2.2c}$$

$$R_4 = R_1 \text{ with } \mu \text{ replaced by } -i\mu, \tag{A2.2d}$$

$$R_5 = \varepsilon[\,(k^4-\mu^4)^2\ k\gamma(k)\sinh\gamma(k)d]^{-1} + O(\varepsilon^2), \tag{A2.2e}$$

$$R_6 = R_5 \text{ with } k \text{ replaced by } -k, \tag{A2.2f}$$

$$R_7 = -\,[2kd\hat{\gamma}(-k(1+m)^{-1})(k^4(1+m)^{-4}-\mu^4)]^{-1} + O(\varepsilon), \tag{A2.2g}$$

$$R_8 = [2kd\hat{\gamma}(k(1-m)^{-1})(k^4(1-m)^{-4}-\mu^4)]^{-1} + O(\varepsilon)\ , \tag{A2.2h}$$

$$R_{\pm n} = \pm[\,(-1)^n d(\hat{S}_{\pm n}{}^4-\mu^4)\hat{\gamma}(\hat{S}_{\pm n})B_n]^{-1} + O(\varepsilon)\ , \tag{A2.2i}$$

B_n is given by (A1.6).

For all R_j except R_5 and R_6 the $O(\varepsilon)$ terms can be calculated but (due to their complexity and lack of interesting features) are not listed here.

Now the potential contributions ψ_{ℓ_j}, $\ell = 1$ or 2 are given by

$$\psi_{\ell_j} = \frac{\lambda\varepsilon}{2\pi} P_j \quad \left(\ell = 1 \text{ is } \mu > k; \quad \ell = 2 \text{ is } \mu < k\right) \qquad \text{(A2.3)}$$

where P_j is the residue of the integrand and is given by

$$P_j = R_j \exp\{-isr\cos\theta - \hat{\gamma}(s)r\sin\theta\} \left(1+\frac{ms}{k}\right)\Big|_{s=S_j} \quad . \qquad \text{(A2.4)}$$

This yields

$$\psi_{\ell 1} = \left[\frac{\lambda\varepsilon}{2\pi}\left[4\mu^3\hat{\gamma}(\mu)\gamma(\mu)\sinh\gamma(\mu)d\right]^{-1}\left(1+\frac{m\mu}{k}\right) + O(\varepsilon^2)\right] \times$$

$$\times \exp\left[-ix\left(\mu + \frac{\varepsilon\tau}{4\mu^3}\right) - \bar{y}\hat{\gamma}(\mu)\left(1 + \frac{\varepsilon\tau}{4\mu^2(\mu^2-k^2)}\right) + O(\varepsilon^2)\right] \qquad \text{(A2.5)}$$

where $\tau = \hat{\gamma}(\mu)^{-1} + \left(1+\frac{m\mu}{k}\right)^2\gamma(\mu)^{-1}\coth\gamma(\mu)d$.

$$\psi_{\ell 2} = \psi_{\ell 1} \text{ with } \mu \text{ replaced by } -\mu. \qquad \text{(A2.5b)}$$

$$\psi_{\ell 3} = \psi_{\ell 1} \text{ with } \mu \text{ replaced by } i\mu. \qquad \text{(A2.5c)}$$

$$\psi_{\ell 4} = \psi_{\ell 1} \text{ with } \mu \text{ replaced by } -i\mu. \qquad \text{(A2.5d)}$$

$$\psi_{\ell 5} = \frac{\lambda\varepsilon^2(1+m)}{2\pi k}\left[\left[(k^4-\mu^4)^2\gamma(k)\sinh\gamma(k)d\right]^{-1} + O(\varepsilon)\right] \times$$

$$\times \exp\left[-ix\left(k + \frac{\varepsilon^2}{2k(k^4-\mu^4)^2}\right) - \bar{y}\left(\frac{\varepsilon}{k^4-\mu^4} + O(\varepsilon^2)\right) + O(\varepsilon^3)\right] \qquad \text{(A2.5e)}$$

$$\psi_{\ell 6} = - \frac{\lambda\varepsilon^2(1-m)}{2\pi k}[[(k^4-\mu^4)^2\gamma(-k)\sinh\gamma(-k)d]^{-1} + O(\varepsilon)] \times$$

$$\times \exp[ix(k+\frac{\varepsilon^2}{2k(k^4-\mu^4)^2}) - \bar{y}(\frac{\varepsilon}{k^4-\mu^4} + O(\varepsilon^2)) + O(\varepsilon^3)]$$

(A2.5f)

$$\psi_{\ell 7} = [\frac{-\lambda\varepsilon}{4\pi dk(1+m)}[\hat{\gamma}(\frac{-k}{1+m})(\frac{k^4}{(1+m)^4}-\mu^4)]^{-1} + O(\varepsilon^2)] \times$$

$$\times \exp[-ix(\frac{-k}{1+m}\ \frac{-\varepsilon}{2kd(1+m)^2}[k^4(1+m)^{-4}-\mu^4]^{-1})$$

$$- \bar{y}\hat{\gamma}(\frac{-k}{1+m})[1- \frac{\varepsilon}{2k^2(2+m)md(1+m)}[k^4(1+m)^{-4}-\mu^4]^{-1}]$$

$$+ ix\varepsilon^2[\frac{1}{2dk(1+m)^2}(k^4(1+m)^{-4}-\mu^4)^{-2}\hat{\gamma}(\frac{-k}{1+m})^{-1}]$$

$$+ i\varepsilon_x^2\ \mathrm{Real} + i\varepsilon^2\bar{y}\gamma(\frac{-k}{1+m})\frac{1}{2k^3(2m+m^2)^{3/2}d}[k^4(1+m)^{-4}-\mu^4]^{-2} + O(\varepsilon^3)]$$

(A2.5g)

$$\psi_{\ell 8} = [\frac{\lambda\varepsilon}{4\pi dk(1-m)}[\hat{\gamma}(\frac{k}{1-m})(k^4(1-m)^{-4}-\mu^4)]^{-1} + O(\varepsilon^2)] \times$$

$$\times \exp[-ix(\frac{k}{1-m} + \frac{\varepsilon}{2kd(1-m)^2}[k^4(1-m)^{-4}-\mu^4]^{-1})$$

$$- \bar{y}\hat{\gamma}(\frac{k}{1-m})[1+ \frac{\varepsilon}{2k^2(2m-m^2)d(1-m)}(k^4(1-m)^{-4}-\mu^4)^{-1}] + O(\varepsilon^2)]$$

(A2.5h)

$$\psi_{\ell\pm n} = [\pm\ \frac{\lambda}{2\pi d}\varepsilon(-1)^n(1+\frac{m}{k}\hat{S}_{\pm n})[(\hat{S}_{\pm n}{}^4-\mu^4)\hat{\gamma}(\hat{S}_{\pm n})B_n]^{-1} + O(\varepsilon^2)] \times$$

$$\times \exp[-ix(\hat{S}_{\pm n} \pm \frac{\varepsilon}{d}(\hat{S}_{\pm n}^4-\mu^4)^{-1}(1+\frac{m}{k}\hat{S}_{\mp n})^2B_n{}^{-1} + \varepsilon^2 \times \mathrm{Real})$$

$$- \bar{y}\hat{\gamma}(\hat{S}_{\pm n})[1 \pm \frac{\varepsilon}{d}\hat{S}_{\pm n}(1+\frac{m}{k}\hat{S}_{\pm n})^2(\hat{S}_{\pm n}{}^4-\mu^4)^{-1}(\hat{S}_{\pm n}{}^2-k^2)B_n{}^{-1} + \varepsilon^2 \times \mathrm{Real}]$$

$$\mp \frac{\varepsilon^2}{d}\hat{S}_{\pm n}(1+\frac{m}{k}\hat{S}_{\pm n})^2(\hat{S}_{\pm n}{}^2-k^2)^{-1}(\hat{S}_{\pm n}{}^4-\mu^4)^{-2}B_n{}^{-1} + O(\varepsilon^3)$$

$$\pm \frac{\varepsilon^2 x}{d}(1+\frac{m}{k}\hat{S}_{\pm n})^2(\hat{S}_{\pm n}{}^4-\mu^4)^{-2}(k^2-\hat{S}_{\pm n}{}^2)^{-\frac{1}{2}}B_n{}^{-1}]$$

for $n \le N$. (A2.5i)

$$\psi_{\ell\pm n} = \pm\left[\frac{-i\lambda\varepsilon}{2\pi d}(-1)^n\left(1+\frac{m}{k}\hat{S}_{\pm n}\right)\left[(\hat{S}_{\pm n}^{4}-\mu^4)\hat{\gamma}(\hat{S}_{\pm n})B_n\right]^{-1} + O(\varepsilon^2)\right] \times$$

$$\times \exp\left[-ix\hat{S}_{\pm n} - \bar{y}\hat{\gamma}(\hat{S}_{\pm n}) + O(\varepsilon)\right]$$

for $n > N$. (A2.5j)

As was the case for the R_j's the $O(\varepsilon^2)$ terms have been calculated but are not included here.

APPENDIX 3

THE FRESNEL INTEGRAL $F(a)$

It can be shown that

$$\int_{-\infty}^{+\infty} \frac{b}{\tau^2+ib^2} \exp\{-kr\tau^2\}d\tau = 2\sqrt{\pi}F(+b\sqrt{kr}) \text{ for } -\frac{3\pi}{4} < \arg \ b < \pi/4$$

$$= -2\sqrt{\pi}\ F(-b\sqrt{kr}) \text{ for } \pi/4 < \arg b < \frac{5\pi}{4} \qquad \text{(A3.1)}$$

where $F(a)$ is defined by

$$F(a) = e^{ia^2} \int_a^{\infty} e^{-i\tau^2}\ d\tau \qquad \text{(A3.2)}$$

We also define the complementary function

$$F_0(a) = e^{ia^2} \int_0^{a} e^{-i\tau^2} d\tau \qquad \text{(A3.3)}$$

We use the relations

$$F(a) + F_0(a) = \frac{\sqrt{\pi}}{2}\ e^{-i\pi/4+ia^2} \qquad \text{(A3.4)}$$

$$F(a) + F(-a) = \sqrt{\pi}\ e^{-i\pi/4+ia^2} \qquad \text{(A3.5)}$$

$$F(a) \sim \frac{1}{2ia}(1- \frac{1}{2ia^2} + 0(\frac{1}{a^4})) \text{ for a large } -\frac{3\pi}{4} < \arg\ a < \frac{\pi}{4} \qquad \text{(A3.6)}$$

$$F_0(a) \sim ae^{ia^2}(1+0(a^2)) \text{ for a small } -\frac{3\pi}{4} < \arg\ a < \frac{\Pi}{4} \qquad \text{(A3.7)}$$

to find approximations for equation (3.9.11) under various conditions.

For $-\frac{3\pi}{4} < \arg\ b < \frac{\pi}{4}$ and $b\sqrt{kr}$ large we employ expansion (A3.6) to give us

$$\phi_S \sim - \sqrt{\frac{2\pi}{kr}}\, F_j(\pi-\theta) e^{-i\pi/4+ikr} \cos\left[\frac{\theta+\Theta_j}{2}\right]^{-1} \tag{A3.8}$$

$$= - \sqrt{\frac{2\pi}{kr}}\, \hat{F}(\pi-\theta) e^{-i\pi/4+ikr} \tag{A3.9}$$

whereas for $-3\pi/4 < \arg b < \pi/4$ and $b\sqrt{kr}$ small we employ equations (A3.4) and (A3.7) to give

$$\phi_S \sim - 2\pi i\, F_j(\pi-\theta)\exp\{-ikr\cos(\theta+\Theta_j)\} \tag{A3.10}$$

For $\pi/4 < \arg b < 5\pi/4$ and $b\sqrt{kr}$ large we employ (A3.1) and (A3.6) with a replaced by $-b\sqrt{kr}$ to give

$$\phi_S \sim - \sqrt{\frac{2\pi}{kr}}\, \hat{F}(\pi-\theta) e^{-i\pi/4+ikr} \tag{A3.11}$$

and for $\pi/4<\arg b < 5\pi/4$ with $b\sqrt{kr}$ small expressions (A3.4), (A3.5) and (A3.7) give

$$\phi_S \sim 2\pi i\, F_j(\pi-\theta)\exp\{-ikr\cos(\theta+\Theta_j)\} \quad . \tag{A3.12}$$

APPENDIX 4

QUOTIENT FACTORISATION OF K(s)

We seek a factorisation of K(s) of the kind

$$K(s) = K_+(s)/K_-(s) \ . \tag{A4.1}$$

We are not able to produce, in general, an expansion that is uniformly valid in both s and ε. Papers by KOITER (14) and CARRIER (5) suggest the replacement of the kernel K(s) by a more easily factorised one, f(s), that is K(s) with $\varepsilon = 0$. Also KRANZER & RADLOW (15) showed for fixed s and $\varepsilon \to 0$ the error in doing this is $O(\varepsilon)$. However, we know from our investigations of the zeros of F(s) that the $O(\varepsilon^2)$ terms are often relevant and can play a significant part. We thus choose to follow a method due to NOBLE (25) and employed in a similar situation by CANNELL (1), for small ε, which relies on an iterative factorisation. In the few places where this method fails we use a locally based one.

The first thing that we need to show is that $K_+(s)$ and $K_-(s)$ tend to s^2 and s^{-2} respectively as $s \to \infty$ in the appropriate half-plane. We do this by showing that $\log\hat{K}_+(s)$ and $\log\hat{K}_-(s)$ tend to zero. $\hat{K}_+(s)$ and $\hat{K}_-(s)$ are defined by

$$\begin{aligned} \hat{K}_+(s) &= [(s+\mu)(s+i\mu)]^{-1} \ K_+(s) \\ \hat{K}_-(s) &= [(s-\mu)(s-i\mu)] \quad K_-(s) \ . \end{aligned} \tag{A4.2}$$

So

$$\hat{K}(s) = \hat{K}_+(s)/\hat{K}_-(s) = (s^4-\mu^4)^{-1} \ K_+(s)/K_-(s) = (s^4-\mu^4)^{-1}K(s) \ . \tag{A4.3}$$

We can show that

$\int_{-\infty}^{+\infty} |\log K(z)| dz$ exists by considering an expansion of the integrand in z for $z \to \pm\infty$.

Now if

$$J_+(s) = \log \hat{K}_+(s) + \frac{1}{2\pi i s} \int_{-\infty}^{+\infty} \log K(z) dz \qquad \text{(A4.4)}$$

we know from Cauchy's theorem that

$$\log \hat{K}_+(s) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{1}{z-s} \log K(z) dz \qquad \text{(A4.5)}$$

so

$$0 \le |J_+(s)| \le \frac{1}{|s|} \int_{-\infty}^{+\infty} \left|\frac{z}{z-s}\right| |\log K(z)| dz \le \frac{A}{|s|} \int_{-\infty}^{+\infty} |\log K(z)| dz \le \frac{B}{|s|}$$

(A4.6)

where the constant $A = \max_z \left|\frac{1}{2\pi i} \left(\frac{z}{z-s}\right)\right|$ and B is some other finite constant.

So $\lim_{|s| \to \infty \text{ in } R_+} J_+(s) = 0$

and as

$$\lim_{|s| \to \infty \text{ in } R_+} \frac{1}{2\pi i s} \int_{-\infty}^{+\infty} \log K(z) dz = 0$$

it follows that

$$\lim_{s \to \infty \text{ in } R_+} \log \hat{K}_+(s) = 0; \quad \lim_{s \to \infty \text{ in } R_+} \hat{K}_+(s) = 1 ,$$

Similarly it can be shown that

$$\lim_{s \to \infty \text{ in } R_-} \hat{K}_-(s) = 1 .$$

Now we assume expansions in ε for $K_+(s)$ and $K_-(s)$ of the form

$$K_+(s) = A(\varepsilon) \; [K_+^0(s) + \varepsilon K_+^1(s) + \varepsilon^2 K_+^2(s) + \ldots]$$
$$K_-(s) = A(\varepsilon) \; [K_-^0(s) + \varepsilon K_-^1(s) + \varepsilon^2 K_-^2(s) + \ldots] . \qquad \text{(A4.7)}$$

So $\quad K(s) = K_+(s)/K_-(s)$

$$= [K_+^0(s)+\varepsilon K_+^1(s)+\varepsilon^2 K_+^2(s)+\ldots][K_-^0(s)+\varepsilon K_-^1(s)+\varepsilon^2 K_-^2(s)+\ldots]^{-1}. \tag{A4.8}$$

Expanding this for small ε and comparing with equation (4.4.18) for K(s) gives, for the $0(\varepsilon^0)$ terms

$$K_+^0(s)/K_-^0(s) = s^4-\mu^4 \quad . \tag{A4.9}$$

We choose

$$K_+^0(s) = (s+\mu)(s+i\mu), \; K_-^0(s) = [(s-\mu)(s-i\mu)]^{-1} \tag{A4.10}$$

although both could be multiplied by the same constant.

For the $0(\varepsilon)$ terms we have

$$-\frac{K_+^1(s)}{K_+^0(s)} + \frac{K_-^1(s)}{K_-^0(s)} = \hat{\gamma}^{-1}(s^4-\mu^4)^{-1} + (1+\frac{ms}{k})^2 \gamma^{-1} \coth\gamma d \tag{A4.11}$$

If we denote the right hand side of equation (A4.11) G(s) and assume a sum split of G(s) into plus and minus functions, to be evaluated later, we have

$$G(s) = G_+(s) + G_-(s) \tag{A4.12}$$

and

$$\begin{aligned} K_+^1(s) &= -K_+^0(s)[G_+(s)-L(s)] \\ K_-^1(s) &= \quad K_-^0(s)[G_-(s)+L(s)] \end{aligned} \tag{A4.13}$$

where L(s) is any regular function.

By substituting equations (A4.13) and (A4.10) into equation (A4.7) and using the knowledge that we have about the behaviour of $K_+(s)$ and $K_-(s)$ at infinity we can tell that L(s) is a constant L and that $A(\varepsilon)$ is $(1+\varepsilon L)^{-1} + 0(\varepsilon^2)$, that is

$$K_+(s) = K_+^{\,0}(s)\ (1-\varepsilon G_+(s)) + 0(\varepsilon^2)$$
$$K_-(s) = K_-^{\,0}(s)\ (1+\varepsilon G_-(s)) + 0(\varepsilon^2) \quad . \qquad \text{(A4.14)}$$

Examination of the $0(\varepsilon^2)$ terms of the expansion (A4.8) would allow us to find the next term in the series and so on for all powers of ε.

All that remains for us to do in this section is to outline the method that can be used for the sum split on G(s).

Considering first the non-hyperbolic part of G(s), $G_1(s)$,

$$G_1(s) = \hat{\gamma}^{-1}(s^4-\mu^4)^{-1} \quad . \qquad \text{(A4.15)}$$

Following CANNELL (1) we have

$$G_{1+}(s) = \frac{1}{4\mu^3}\left[\frac{P_+(s)-P_+(\mu)}{s-\mu} + i\,\frac{P_+(s)-P_+(i\mu)}{s-i\mu} - \frac{P_+(s)+P_+(\mu)}{s+\mu} - i\,\frac{P_+(s)+P_+(i\mu)}{s+i\mu}\right] \qquad \text{(A4.16)}$$

$$G_{1-}(s) = \frac{1}{4\mu^3}\left[\frac{P_-(s)+P_+(\mu)}{s-\mu} + i\,\frac{P_-(s)+P_+(i\mu)}{s-i\mu} - \frac{P_-(s)-P_+(\mu)}{s+\mu} - i\,\frac{P_-(s)-P_+(i\mu)}{s+i\mu}\right]$$

where

$$P_+(s) = \frac{2}{\pi}\,(s^2-k^2)^{-\frac{1}{2}}\tan^{-1}\left(\frac{k-s}{k+s}\right)^{\frac{1}{2}};\quad P_-(s) = \frac{2}{\pi}\,(s^2-k^2)^{-\frac{1}{2}}\tan^{-1}\left(\frac{k+s}{k-s}\right)^{\frac{1}{2}}. \qquad \text{(A4.17)}$$

Next we consider $G_2(s)$ the hyperbolic part of G(s),

$$G_2(s) = \left(1+\frac{ms}{k}\right)^2 \gamma(s)^{-1}\coth\gamma(s)d\ (s^4-\mu^4)^{-1} \qquad \text{(A4.18)}$$

$G_2(s)$ is not so readily split into plus and minus functions. However $G_2(s)$ is $0\left(\frac{1}{|s|}\right)^3$ as $s \to \infty$ in the strip of regularity so we can find $G_{2+}(s)$ and $G_{2-}(s)$ explicitly by using Cauchy's theorem,

$$G_2(s) = \frac{1}{2\pi i} \int_{L_1+C_1-L_2+C_2} \frac{G_2(z)}{z-s}\, dz \qquad (A4.19)$$

where L_1, C_1, L_2, and C_2 lie in the strip of regularity S and are shown in Figure 4.1

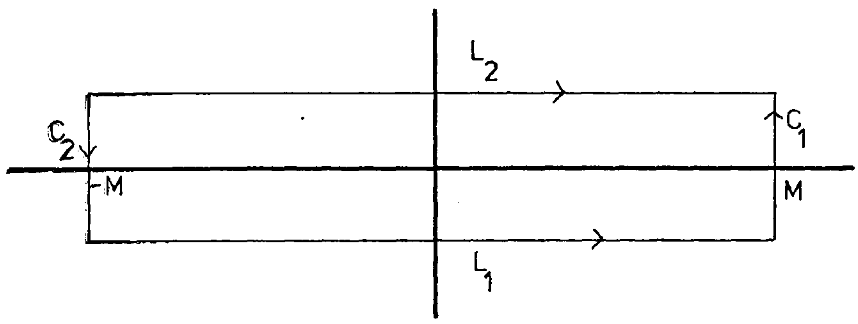

Figure 4.1: z-plane

Now as $M \to \infty$ the contributions from C_1 and C_2 decay to zero to give

$$G_2(s) = \frac{1}{2\pi i} \int_{L_1} \frac{G_2(z)}{z-s}\, dz - \frac{1}{2\pi i} \int_{L_2} \frac{G_2(z)}{z-s} \qquad (A4.20)$$

where L_1 is $z = \xi + i\eta_1$; $-\infty < \xi < \infty$, $0 > \eta_1 > s_-$

L_2 is $z = \xi + i\eta_2$; $-\infty < \xi < \infty$, $s_+ > \eta_2 > 0$.

The first of these two integrals is regular for all s above L_1 so is the plus function $G_{2+}(s)$ whereas the second is regular for all s below L_2 so is the minus function $G_{2-}(s)$.

Note the integrand here contains no branch cuts at all.

To find $G_{2+}(s)$ then we close the contour in the lower half-plane, chosen such that the integrand is bounded, and then use Cauchy's Residue theorem to collect the residue contributions from the poles of the integrand at $-\mu$, $-i\mu$, $-k/(1+m)$, $\hat{S}_{-n}$ to give

$$G_{2+}(s) = -\frac{(1-\frac{m\mu}{k})^2}{4\mu^3(s+\mu)\gamma(-\mu)}\coth\gamma(-\mu)d - i\frac{(1-\frac{im\mu}{k})^2}{4\mu^3(s+i\mu)\gamma(-i\mu)}\coth\gamma(-i\mu)d$$

$$-\left[2kd(1+m)^2(s+\frac{k}{1+m})((\frac{k}{1+m})^4-\mu^4)\right]^{-1} - \sum_1^N \frac{(1+\frac{m}{k}\hat{S}_{-n})^2}{d(s-\hat{S}_n)(\hat{S}_{-n}^4-\mu^4)(k^2-(1-m^2)\frac{n^2\pi^2}{d^2})^{\frac{1}{2}}}$$

$$+\sum_{N+1}^{\infty} i(1+\frac{m}{k}\hat{S}_{-n})^2\left[d(\hat{S}_{-n}^4-\mu^4)((1-m^2)(\frac{n\pi}{d})^2-k^2)^{\frac{1}{2}}(s-\hat{S}_{-n})\right]^{-1},$$

(A4.21)

Similarly $G_{2-}(s)$ can be found by closing the contour in the upper half-plane

$$G_{2-}(s) = -\frac{(1+\frac{m\mu}{k})^2}{4\mu^3\gamma(\mu)(s-\mu)}\coth\gamma(\mu)d - i\frac{(1+\frac{im\mu}{k})^2}{4\mu^3\gamma(i\mu)(s-i\mu)}\coth\gamma(i\mu)d$$

$$-\left[(1-m)^2((\frac{k}{1-m})^4-\mu^4)2kd(s-\frac{k}{1-m})\right]^{-1}$$

$$+\sum_1^N -(1+\frac{m\hat{S}_{+n}}{k})^2\left[(\hat{S}_{+n}^2-\mu^4)d(k^2-(1-m^2)(\frac{n\pi}{d})^2)^{\frac{1}{2}}(s-\hat{S}_{+n})\right]^{-1}$$

$$+\sum_{N+1}^{\infty} i(1+\frac{m\hat{S}_{+n}}{k})^2\left[(\hat{S}_{+n}^4-\mu^4)d((1-m^2)(\frac{n\pi}{d})^2-k^2)^{\frac{1}{2}}(s-\hat{S}_{+n})\right]^{-1}.$$

(A4.22)

We now have

$$G_+(s) = G_{1+}(s) + G_{2+}(s)$$
$$G_-(s) = G_{1-}(s) + G_{2-}(s) \qquad (A4.23)$$

and have found an asymptotic quotient factorisation of K(s) for small ε.

APPENDIX 5

DETERMINATION OF R(s) AND P

Here we investigate the behaviour of R(s) as $s \to \infty$ in R_+ and R_-. We then use this behaviour and the fact that R(s) is analytic, with Liouville's theorem, to identify R(s).

Consider first R(s) in the upper half-plane, R_+, it satisfies equation (4.5.7),

$$R(s) = K_+(s)\Phi_{2+}' - \frac{i\varepsilon\lambda K_-(-k/1+m)}{k+s(1+m)} \tag{A5.1}$$

and we wish to investigate $s \to \infty$.

We know from Appendix 4 that $K_+(s) \sim s^2$ as $s \to \infty$ in R_+.

As the pressure difference is at worst integrable across the plate at x = y = 0 equation (4.2.2) tells us that the following must be asymptotic expansion for $\eta(x)$ for small $x > 0$.

$$\eta(x) \sim -\eta(0) + x\eta_x(0) + \frac{x^2}{2!}\eta_{xx}(0) + \frac{x^3}{3!}\eta_{xxx}(0) + 0(x^3) . \tag{A5.2}$$

We know that

$$\eta(0) = \eta_{xx}(0) = 0 \text{ and } \eta_{xxx}(0) = q \text{ and } \eta_x(0) = -p \tag{A5.3}$$

So, for small x,

$$\eta(x) \sim -px + \frac{qx^3}{3!} + 0(x^3) . \tag{A5.4}$$

Applying an Abelian theorem we get

$$\bar{\eta}_x \sim p/S^2 + q/S^4 + 0(\frac{1}{S^4}) \text{ as } s \to \infty \text{ in } R_+, \tag{A5.5}$$

which then gives

$$\bar{\phi}_{2+}'(s,0) \sim -i\omega\{p/S^2 + O(1/S^4)\} \text{ as } s \to \infty \text{ in } R_+ . \qquad (A5.6)$$

On substitution into equation (A5.1) this gives

$$R(s) \sim -i\omega p, \quad \text{as} \quad s \to \infty \quad \text{in } R_+ . \qquad (A5.7)$$

Next we consider R(s) in the lower half-plane, R_-; it satisfies equation (4.5.8)

$$R(s) = N(s)K_-(s) + Q_-(s)K_-(s) + \frac{i\varepsilon\lambda}{k+s(1+m)}\{\frac{K_-(s)-K_-(-k/(1+m))}{k+s(1+m)}\} . \qquad (A5.8)$$

We know that $N(s) \sim -i\omega ps^2$ and from Appendix 4 we know that $K_-(s) \sim S^{-2}$. The sum of the first and last terms of expression (A5.8) therefore $\sim -i\omega p + O(1/S)$ as $s \to \infty$ in R_-. This leaves us with only the behaviour of $Q_-(s)$, that is $\bar{\phi}_{2-}(s)$ and $\bar{\phi}_{1-}(s)$, to find before we can calculate R(s). Now the integrability of the pressures tells us that the velocity potential ϕ_2 is at worst integrable as $(x,y) \to (0,0)$ so $\bar{\phi}_{2-}$ is at worst a constant and that ϕ_1 is at worst a constant so ϕ_{1-} is at worst order $^{-1}$.

Substitution into $Q_-(s)$ gives

$$R(s) \sim -i\omega p \text{ as } s \to \infty \text{ in } S_- . \qquad (A5.9)$$

So R(s) is bounded by a polynomial (a constant) for all s as $s \to \infty$ and is analytic, by an extension to Liouville's theorem this means

$$R(s) \equiv -i\omega p . \qquad (A5.10)$$

We wish to determine p or the power series in small ε of p. From equation (4.5.7) using the previous result we now know

$$-i\omega p = K_+(s)\Phi_{2+}' - \frac{i\varepsilon\lambda K_-(-k/(1+m))}{k+s(1+m)} \qquad \text{(A5.11)}$$

Letting $s \to \infty$ in (A5.11) is not sufficient to give us a solution for p since it was by this method that we found R(s) and it now tells us no more. Instead we consider simultaneously the limits $s \to \infty$ and $\varepsilon \to 0$.

If we write p as a power series in ε

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots \qquad \text{(A5.12)}$$

and expand $\Phi_{2+}'(s)$, $K_+(s)$ and $K_-(-k/(1+m)$ from equations (A5.6) and (A4.14) in powers of s^{-1} and ε and compare terms of similar order after substitution into equation (A5.11) we get, for the $O(\varepsilon^0)/O(s^{-1})$ expression

$$(1+i)\mu p_0 = 0 \qquad \text{i.e. } p_0 = 0 \quad . \qquad \text{(A5.13)}$$

For the $O(\varepsilon^1)/O(s^{-1})$ expression we get

$$p_1 = -(1+m)\lambda\,[(1+i)\mu\omega\{k+(1+m)\mu\}\{k+(1+m)i\mu\}]^{-1} \qquad \text{(A5.14)}$$

and for the $O(\varepsilon^2)/O(s^{-1})$ term

$$p_2 = -(1+m)\lambda\,[(1+i)\mu\omega\{k+(1+m)\mu\}\{k+i(1+m)\mu\}]^{-1}\,[G_-(\frac{-k}{1-m}) + \frac{1}{(1+i)\mu}\{O(\frac{1}{s}) \text{ of } G_+(s)\}] \qquad \text{(A5.15)}$$

and so on giving the terms of the power series for p.

APPENDIX 6

RESIDUES OF INTEGRANDS INVOLVING $K_+(s)^{-1}$

The problem is to find the residues, $\hat{R}_j$; of $K_+(s)^{-1}$ at its poles, S_2, S_4, S_6, S_7 and S_{-n} which are given by

$$\hat{R}_j = \frac{1}{K_+'(S_j)} \quad . \tag{A6.1}$$

However the expansion already obtained for $K_+'(S_j)$ does not necessarily hold near its zeros or poles so an alternative derivation must be found.

A zero of $K_+(s)$ is also a zero of $K(s)$ and the behaviour of $K(s)$ near one of its zeros S_j is known

$$K(s) \sim K'(S_j)\ (s-S_j) \tag{A6.2}$$

Also the behaviour of $K_-(s)$ near one of the zeros of $K_+(s)$ is known because this is in a region in which $K_-(s)$ is regular and the expansion already found will hold.

Thus

$$K_-(s) \sim K_-(S_j) \tag{A6.3}$$

giving for the behaviour of $K_+(s)$ as $s \to S_j$.

$$K_+(s) \sim L_j(s-S_j) \tag{A6.4}$$

where

$$L_j(s-S_j)\ K_-(S_j)^{-1} \sim K'(S_j)(s-S_j) \tag{A6.5}$$

or

$$L_j = K'(S_j)K_-(S_j) \quad \text{(A6.6)}$$

So the residue of $K_+(s)^{-1}$ at $s = S_j$ is given by

$$\hat{R}_j = L_j^{-1} = [K'(S_j)K_-(S_j)]^{-1} . \quad \text{(A6.7)}$$

Substituting in for $K'(S_j)$ from the differential of equation (4.4.18) and for $K_-(S_j)$ from equation (A4.14) gives for the residues the following

$$\hat{R}_2 = -\frac{(1+i)}{2\mu} + 0(\varepsilon) \quad \text{(A6.8a)}$$

$$\hat{R}_4 = \frac{i(1+i)}{2\mu} + 0(\varepsilon) \quad \text{(A6.8b)}$$

$$\hat{R}_6 = -\varepsilon^2(k+\mu)(k+i\mu)k^{-1}(k^4-\mu^4)^{-3} + 0(\varepsilon) \quad \text{(A6.8c)}$$

$$\hat{R}_7 = \varepsilon(k+\mu(1+m))(k+i\mu(1+m))\,[2kd(1+m)^4]^{-1}[(\frac{k}{1-m})^4-\mu^4]^{-2} \quad \text{(A6.8d)}$$

$$\hat{R}_{-n} = -\varepsilon[dB_n(\hat{S}_{-n}^4-\mu^4)^2]^{-1}(1+\frac{m\hat{S}_{-n}}{k})^2(\hat{S}_{-n}-\mu)(\hat{S}_{-n}-i\mu) \quad \text{(A6.8e)}$$

The residues of the integrand of $\eta(x)$, $\eta_{\ell j}$, are given by

$$\eta_{\ell j} = -\frac{\varepsilon\lambda}{2\pi\omega}(1+m)[(k+\mu(1+m))(k+i\mu(1+m))]^{-1}[\frac{1}{(1+i)\mu}+\frac{1+m}{k+s(1+m)}]\hat{R}_j e^{-isx}\Big|_{s=S_j} \quad \ell=1,2 \quad \text{(A6.9)}$$

and the corresponding contributions to $\eta(x)$ are

$$\eta_{\ell 2} = [\frac{\varepsilon\lambda(1+m)}{4\pi\mu^2\omega}[(k+(1+m)\mu)(k-(1+m)\mu)]^{-1}+0(\varepsilon^2)]e^{-iS_2x} \quad \text{(A6.10a)}$$

$$\eta_{\ell 4} = [-\frac{\varepsilon\lambda(1+m)}{4\pi\mu^2\omega}[(k+i(1+m)\mu)(k-i(1+m)\mu)]^{-1}+0(\varepsilon^2)]e^{-iS_4x} \quad \text{(A6.10b)}$$

$$\eta_{\ell 6} = \varepsilon^3(k+\mu)(k+i\mu)\lambda(1+m)[2\pi\omega k\{k+\mu(1+m)\}\{k+i\mu(1+m)\}(k^4-\mu^4)^{-3}]^{-1}\times$$

$$\times\,[\frac{1}{(1+i)\mu}-\frac{1+m}{km}+0(\varepsilon)]\,e^{-iS_6x} \quad \text{(A6.10c)}$$

$$\eta_{\ell 7} = [\frac{\varepsilon\lambda}{2\pi\omega(1+m)} \{(\frac{k}{1+m})^4 - \mu^4\}^{-1} + 0(\varepsilon^2)] e^{-iS_7 x} \qquad (A6.10d)$$

$$\eta_{\ell -n} = \{\lambda\varepsilon^2(1+m)(\hat{S}_{-n}-\mu)(\hat{S}_{-n}-i\mu) + 0(\varepsilon^2)\}[\frac{1}{(1+i)\mu} + \frac{1+m}{k+\hat{S}_{-n}(1+m)}] \times$$
$$\times (1+\frac{m\hat{S}_{-n}}{k})^2 [2\pi\omega d B_n(\hat{S}_{-n}^{\;4}-\mu^4)\{k+\mu(1+m)\}\{k+i\mu(1+m)\}]^{-1} e^{-iS_{-n}x} \qquad (A6.10e)$$

The residues of the integrand of ϕ_1, $\hat{R}_{\ell j}$, for $x < 0$ are given by

$$\hat{R}_{\ell j} = i\lambda\varepsilon(1+m)[\frac{1}{(1+i)\mu} + \frac{1+m}{k+s(1+m)}] (1+\frac{ms}{k}) \frac{1}{K_+(s)} [2\pi\{k+i\mu(1+m)\}$$
$$\{k+\mu(1+m)\}]^{-1} \times \cosh\gamma(y+d) e^{-isx} [\frac{d}{ds}\gamma\sinh\gamma d]^{-1}\Big|_{s=\hat{S}_j} \qquad (A6.11)$$

which on substitution yields

$$\hat{R}_{\ell 8} = i\lambda\varepsilon(1-m^2)[\frac{1}{(1+i)\mu} + \frac{(1-m^2)}{2k} + 0(\varepsilon)][4\pi d k(k+\mu(1+m))(k+\mu(1-m))$$
$$(k+i\mu(1+m))(k+i\mu(1-m))]^{-1} \exp\{-\frac{ikx}{1-m}\} \qquad (A6.12)$$

$$\hat{R}_{\ell +n} = i\lambda\varepsilon(1+m)(1+\frac{m\hat{S}_{+n}}{k})[\frac{1}{(1+i)\mu} + \frac{1+m}{k+\hat{S}_+(1+m)}] \cos\frac{n\pi y}{d} \times$$
$$\times [2\pi\{k+i\mu(1+m)\}\{k+\mu(1+m)\}(\mu+\hat{S}_{+n})(i\mu+\hat{S}_{+n}) d B_n]^{-1} e^{-i\hat{S}_{-n}x} \qquad (A6.13)$$

The residues of the integrand of ϕ_1, $R'_{\ell j}$ for $x > 0$ at S_2, S_4, S_6, S_7 and S_{-n} are given by

$$R'_{\ell j} = \{i\lambda\varepsilon(1+m) + O(\varepsilon)\}[\frac{1}{(1+i)\mu} + \frac{(1+m)}{k+s(1+m)}](1+\frac{ms}{k})\cosh\gamma(y+d)\hat{R}_j \times$$

$$\times[2\pi\{k+i\mu(1+m)\}\{k+\mu(1+m)\}\gamma\sinh\gamma d]^{-1}e^{-isx}\Big|_{s=\hat{S}_j} \qquad \text{(A6.14)}$$

which after substitution gives

$$R'_{\ell 2} = \{-i\lambda\varepsilon(1+m) + O(\varepsilon^2)\}(1-\frac{m\mu}{k})\cosh\gamma(-\mu)(y+d) \times$$

$$\times[4\pi\mu^2\{k+\mu(1+m)\}\{k-\mu(1+m)\}\gamma(-\mu)\sinh\gamma(-\mu)d]^{-1}e^{-iS_2x} \qquad \text{(A6.15a)}$$

$$R'_{\ell 4} = \{i\lambda\varepsilon(1)m) + O(\varepsilon^2)\}(1-\frac{im\mu}{k})\cosh\gamma(-i\mu)(y+d) \times$$

$$\times[4\pi\mu^2\{k+i\mu(1+m)\}\{k-i\mu(1+m)\}\gamma(-i\mu)\sinh\gamma(-i\mu)d]^{-1}e^{-iS_4x} \qquad \text{(A6.15b)}$$

$$R'_{\ell 6} = \{-i\lambda\varepsilon^3(1-m^2)(k+\mu)(k+i\mu) + O(\varepsilon^4)\}[\frac{1}{(1+i)\mu} - \frac{1+m}{km}] \times$$

$$\times \cosh\gamma(-k)(y+d)[2\pi k\{k+i\mu(1+m)\}\{k+\mu(1+m)\}\gamma(-k)\sinh\gamma(-k)d(k^4-\mu^4)^3]^{-1} \times$$

$$\times e^{-iS_6x} \qquad \text{(A6.15c)}$$

$$R'_{\ell 7} = \{-\frac{i\lambda}{2\pi} + O(\varepsilon)\}e^{-iS_7x} \qquad \text{(A6.15d)}$$

$$R'_{\ell -n} = \{-i\lambda\varepsilon(1+m)(\hat{S}_{-n}-\mu)(\hat{S}_{-n}-i\mu) + O(\varepsilon^2)\}(1+\frac{m\hat{S}_{-n}}{k})[\frac{1}{(1+i)\mu} + \frac{1+m}{k+(1+m)\hat{S}_{-n}}]\times$$

$$\times[2\pi dB_n\{k+i\mu(1+m)\}\{k+\mu(1+m)\}(\hat{S}_{-n}^{\,4}-\mu^4)]^{-1}\cos\frac{n\pi y}{d}e^{-iS_{-n}x}$$

$$\text{(A6.15e)}$$

The residue at $s = \hat{S}_7$ is given by expression (A6.11) evaluated at $s = \hat{S}_7$, noting that both $(1+m)/(k+s(1+m))$ and $[K_+(s)]^{-1}$ have zeros there, and is

$$\hat{R}_{\ell 7} = -\frac{i\lambda}{2\pi} e^{-i\hat{S}_7 x} = 0(\varepsilon) \tag{A6.16}$$

The residues of the integrand of ϕ_2, $\Psi_{\ell j}$, are given by

$$\Psi_{\ell j} = -i\varepsilon\lambda(1+m)\left[\frac{1}{(1+i)\mu} + \frac{1+m}{k+s(1+m)} + 0(\varepsilon)\right]\hat{R}_j \times$$

$$\times\left[2\pi\{k+\mu(1+m)\}\{k+i\mu(1+m)\}\hat{\gamma}(S)\right]^{-1} \exp\{-\hat{\gamma}(s)y-isx\}\Big|_{s=S_j} \quad . \tag{A6.17}$$

After substitution this yields, for j = 2

$$\Psi_{\ell 2} = \left\{\frac{i\varepsilon\lambda(1+m)\hat{\gamma}(S_2)^{-1}}{4\pi\mu^2[k+\mu(1+m)][k-\mu(1+m)]} + 0(\varepsilon^2)\right\}\exp\{-\hat{\gamma}(S_2)y-iS_2x\} \quad . \tag{A6.18a}$$

and for j = 6

$$\Psi_{\ell 6} = i\varepsilon^2\lambda(1+m)(k+\mu)(k+i\mu)\left[\frac{1}{(1+i)\mu} - \frac{1+m}{km} + 0(\varepsilon)\right] \times$$

$$\times\left[2\pi k\{k+\mu(1+m)\}\{k+i\mu(1+m)\}(k^4-\mu^4)^2\right]^{-1} \exp\{-\hat{\gamma}(S_6)y-iS_6x\} \tag{A6.18b}$$

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