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A Limit Proofs for derivatives of LMTD and RecLMTD

In this section, we evaluate the limits of the log mean temperature difference, LMTD, its reciprocal, RecLMTD, and the respective functions raised to the $\beta$-th power for derivatives of order 0, 1 and 2. We show these results for both functions to support Proposition 4. The limits for the derivatives of order 0 and 1 for LMTD have previously been proven by Zavala-Ríó et al. (2005), our method of proof differs in that we make a coordinate transformation opposed to analysing a series expansion.

Let $S^* = \{(x, y)^T \in \mathbb{R}_+^2 \mid x \neq y\}$. LMTD, RecLMTD : $S^* \rightarrow \mathbb{R}$ and their derivatives of orders 1 and 2 are well defined over the domain $S^*$. Equating the parameters (letting $x = y$) results in, by direct evaluation, the function and all elements of the derivatives of order 1 and 2 evaluating to 0. We show that these limits exist by making a transformation to the polar system; the Cartesian limit $[(x, y) \rightarrow (c, c)]$ becomes $[(r, \theta) \rightarrow (c\sqrt{2}, \frac{\pi}{4})]$. We switch from the Cartesian to the polar system using transformations $x \rightarrow r \cos(\theta)$, $y \rightarrow r \sin(\theta)$ and $w = \frac{x}{y} \rightarrow \cot(\theta)$. These proofs allow us to give well defined functions, given by Definition 1, for LMTD and RecLMTD over $\mathbb{R}_+^2$, and therefore show that LMTD, RecLMTD $\in C^2$.

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Over $S^*$, the functions and their derivatives are given by:

\[
\begin{align*}
LMTD(x, y) &= \frac{x - y}{\ln(x/y)} \\
\nabla LMTD(x, y) &= \ln(w)^{-2} \left( \frac{w^{-1} + \ln(w) - 1}{w - \ln(w) - 1} \right) \\
\n\nabla^2 LMTD(x, y) &= y^{-1} \ln(w)^{-2} k(w) \left( -\frac{w^{-1}}{1} \frac{1}{w} \right) \\
\text{RecLMTD}(x, y) &= \frac{\ln(x/y)}{x - y} \\
\nabla \text{RecLMTD}(x, y) &= \frac{1}{(x - y)^2} \left( 1 - \frac{\ln(w) - w^{-1}}{1 + \ln(w) - w} \right) \\
\n\nabla^2 \text{RecLMTD}(x, y) &= \frac{1}{(x - y)^3} \left( \frac{2 \ln(w) + 4w^{-1} - w^{-2} - 3}{w - w^{-1} - 2 \ln(w)} \frac{w - w^{-1} - 2 \ln(w)}{2 \ln(w) - 4w + w^2 + 3} \right)
\end{align*}
\]

where

\[
k(w) = 1 + w^{-1} + 2w^{-1} \ln(w)^{-1} - 2 \ln(w)^{-1}
\]

**Proposition 1.** Let $c > 0$,

\[
\begin{align*}
\lim_{(x, y) \to (c, c)} LMTD(x, y) &= c \\
\lim_{(x, y) \to (c, c)} \nabla LMTD(x, y) &= \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\lim_{(x, y) \to (c, c)} \nabla^2 LMTD(x, y) &= \frac{1}{6c} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \\
\lim_{(x, y) \to (c, c)} \text{RecLMTD}(x, y) &= \frac{1}{c} \\
\lim_{(x, y) \to (c, c)} \nabla \text{RecLMTD}(x, y) &= -\frac{1}{2c^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\lim_{(x, y) \to (c, c)} \nabla^2 \text{RecLMTD}(x, y) &= \frac{1}{3c^3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\end{align*}
\]

**Proof.** Here the proof is only shown for LMTD and its derivatives, the proofs for RecLMTD can be derived from an application of the chain rule or using a similar process to that given here.

Making a transformation to the polar system, we get

\[
\begin{align*}
\text{LMTD}_p(r, \theta) &= LMTD(r \cos(\theta), r \sin(\theta)) \\
&= r \frac{\cos(\theta) - \sin(\theta)}{\ln(\cot(\theta))} \\
&= f_0(r) \cdot g_0(\theta)
\end{align*}
\]

where $f_0$ is continuous and $g_0$ is defined for $\theta \in \left(0, \frac{\pi}{2}\right), \theta \neq \pi/4$. At $\theta = \frac{\pi}{4}$, corresponds to $x = y$, $g_0 \left( \frac{\pi}{4} \right) = 0$. Applying l'Hôpital’s rule gives (the numerator and denominator are differentiated once)

\[
\lim_{\theta \to \frac{\pi}{4}} g_0(\theta) = \frac{1}{\sqrt{2}}
\]
\[
\lim_{(x,y) \to (c,c)} \text{LMTD}(x,y) = \lim_{(r,\theta) \to (c\sqrt{2},\pi/4)} \text{LMTD}_p(r,\theta) = \left[ \lim_{r \to c\sqrt{2}} f_0(r) \right] \left[ \lim_{\theta \to \pi/4} g_0(\theta) \right] = c \sqrt{2} \cdot \frac{1}{\sqrt{2}} = c
\]

For \( \nabla \text{LMTD} \), making the polar coordinate transformation gives functions that are univariate in \( \theta \)

\[
\nabla \text{LMTD}_p(r,\theta) = \nabla \text{LMTD}(r \cos(\theta), r \sin(\theta)) = \left( g_1^{[x]}(\theta) \right) \left( g_1^{[y]}(\theta) \right)
\]

where

\[
g_1^{[x]}(\theta) = \frac{\tan(\theta) + \ln(\cot(\theta)) - 1}{\ln(\cot(\theta))^2}
\]

\[
g_1^{[y]}(\theta) = \frac{\cot(\theta) - \ln(\cot(\theta)) - 1}{\ln(\cot(\theta))^2}
\]

Applying l’Hôpital’s rule gives (the numerator and denominator of the two functions need to be differentiated twice)

\[
\lim_{\theta \to \pi/4} g_1^{[x]}(\theta) = \lim_{\theta \to \pi/4} g_1^{[y]}(\theta) = \frac{1}{2}
\]

hence

\[
\lim_{(x,y) \to (c,c)} \nabla \text{LMTD}(x,y) = \lim_{(r,\theta) \to (c\sqrt{2},\pi/4)} \nabla \text{LMTD}_p(r,\theta) = \lim_{\theta \to \pi/4} \left( \frac{g_1^{[x]}(\theta)}{g_1^{[y]}(\theta)} \right) = \left( \frac{1}{2} \right)
\]

For \( \nabla^2 \text{LMTD} \), making the polar coordinate transformation gives

\[
\nabla^2 \text{LMTD}_p(r,\theta) = \nabla^2 \text{LMTD}(r \cos(\theta), r \sin(\theta)) = f_2(r) \left( g_2^{[xx]}(\theta) g_2^{[xy]}(\theta) \right) \left( g_2^{[yx]}(\theta) g_2^{[yy]}(\theta) \right)
\]

where

\[
f_2(r) = \frac{1}{r}
\]

\[
g_2^{[xx]}(\theta) = (-1) \cdot \frac{\sec(\theta) \ln(\cot(\theta)) + \sec(\theta) \tan(\theta) \ln(\cot(\theta)) + 2 \sec(\theta) \tan(\theta) - 2 \sec(\theta)}{\ln(\cot(\theta))^3}
\]

\[
g_2^{[xy]}(\theta) = \frac{\csc(\theta) \ln(\cot(\theta)) + \sec(\theta) \ln(\cot(\theta)) + 2 \sec(\theta) - 2 \csc(\theta)}{\ln(\cot(\theta))^3}
\]

\[
g_2^{[yx]}(\theta) = (-1) \cdot \frac{\csc(\theta) \ln(\cot(\theta)) + \csc(\theta) \cot(\theta) \ln(\cot(\theta)) - 2 \csc(\theta) \tan(\theta) + 2 \csc(\theta)}{\ln(\cot(\theta))^3}
\]

Applying l’Hôpital’s rule gives (the numerator and denominator of the three functions parametrised by \( \theta \) need to be differentiated three times)

\[
\lim_{\theta \to \pi/4} g_2^{[xx]}(\theta) = -\frac{\sqrt{2}}{6}, \quad \lim_{\theta \to \pi/4} g_2^{[xy]}(\theta) = \frac{\sqrt{2}}{6}, \quad \lim_{\theta \to \pi/4} g_2^{[yx]}(\theta) = -\frac{\sqrt{2}}{6}
\]
hence

\[
\lim_{(x,y) \to (c,c)} \nabla^2 \text{LMTD}(x, y) = \lim_{(r,\theta) \to (c\sqrt{2}, \frac{\pi}{4})} \nabla^2 \text{LMTD}_p(r, \theta)
\]

\[
= \left[ \lim_{r \to c\sqrt{2}} f_2(r) \right] \left[ \lim_{\theta \to \frac{\pi}{4}} \begin{pmatrix} g_2^{[x]}(\theta) & g_2^{[y]}(\theta) \\ g_2^{[x]}(\theta) & g_2^{[y]}(\theta) \end{pmatrix} \right]
\]

\[
= \frac{1}{c\sqrt{2}} \cdot \sqrt{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}
\]

\[
= \frac{1}{6c} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}
\]

\[
\square
\]

Definition 1. The log mean temperature difference, \( \text{LMTD} : \mathbb{R}_+^2 \rightarrow \mathbb{R} \), and its reciprocal, \( \text{RecLMTD} : \mathbb{R}_+^2 \rightarrow \mathbb{R} \), and their respective gradient and Hessian are defined as

\[
\text{LMTD}(x, y) = \begin{cases} 
  x, & x = y \\
  \frac{x - y}{\ln(x/y)}, & x \neq y
\end{cases}
\]

\[
\nabla \text{LMTD}(x, y) = \begin{cases} 
  \left(\frac{1}{2}\right), & x = y \\
  \ln(w)^{-2} \begin{pmatrix} w^{-1} + \ln(w) - 1 \\ w - \ln(w) - 1 \end{pmatrix}, & x \neq y
\end{cases}
\]

\[
\nabla^2 \text{LMTD}(x, y) = \begin{cases} 
  \frac{1}{6x} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, & x = y \\
  y^{-1} \ln(w)^{-2} k(w) \begin{pmatrix} -w^{-1} & 1 \\ 1 & w \end{pmatrix}, & x \neq y
\end{cases}
\]

\[
\text{RecLMTD}(x, y) = \begin{cases} 
  \frac{1}{x}, & x = y \\
  \frac{\ln(x/y)}{x - y}, & x \neq y
\end{cases}
\]

\[
\nabla \text{RecLMTD}(x, y) = \begin{cases} 
  \begin{pmatrix} -\frac{1}{2w^2} \\ -\frac{1}{2w^2} \end{pmatrix}, & x = y \\
  \frac{1}{(x - y)^2} \begin{pmatrix} 1 - \ln(w) - w^{-1} \\ 1 + \ln(w) - w \end{pmatrix}, & x \neq y
\end{cases}
\]

\[
\nabla^2 \text{RecLMTD}(x, y) = \begin{cases} 
  \frac{1}{3x^3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, & x = y \\
  \frac{1}{(x - y)^3} \begin{pmatrix} 2 \ln(w) + 4w^{-1} - w^{-2} - 3 \\ w - w^{-1} - 2 \ln(w) \end{pmatrix}, & x \neq y
\end{cases}
\]
where

\[ w = \frac{x}{y} \]

\[ k(w) = 1 + w^{-1} + 2w^{-1} \ln(w)^{-1} - 2 \ln(w)^{-1} \]

Use of the chain rule allows us to extend the results of Proposition 1. Here we define \( \text{LMTD}^\beta \) and \( \text{RecLMTD}^\beta \) to be \( \text{LMTD} \) and \( \text{RecLMTD} \) raised to the \( \beta \)-th power respectively.

Corollary 1. Let \( c > 0, \beta \in \mathbb{R} \)

\[ \lim_{(x,y) \to (c,c)} \text{LMTD}^\beta(x,y) = c^\beta \]

\[ \lim_{(x,y) \to (c,c)} \nabla \text{LMTD}^\beta(x,y) = \frac{\beta}{2} c^{\beta-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ \lim_{(x,y) \to (c,c)} \nabla^2 \text{LMTD}^\beta(x,y) = \frac{\beta}{12} c^{\beta-2} \begin{pmatrix} 3\beta - 5 & 3\beta - 1 \\ 3\beta - 1 & 3\beta - 5 \end{pmatrix} \]

\[ \lim_{(x,y) \to (c,c)} \text{RecLMTD}^\beta(x,y) = \frac{1}{c^\beta} \]

\[ \lim_{(x,y) \to (c,c)} \nabla \text{RecLMTD}^\beta(x,y) = -\frac{\beta}{2c^{\beta+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ \lim_{(x,y) \to (c,c)} \nabla^2 \text{RecLMTD}^\beta(x,y) = \frac{\beta}{12c^{\beta+2}} \begin{pmatrix} 3\beta + 5 & 3\beta + 1 \\ 3\beta + 1 & 3\beta + 5 \end{pmatrix} \]

B Convexity Result

In this section, we give a proof of convexity results for \( \text{LMTD} \), the proof is given using the derivatives of \( \text{RecLMTD} \). Throughout this section we assume that the function definitions are as in Proposition 2. We require the following results (Propositions 2 and 3 and Corollary 2) before showing the results relating to convexity (Proposition 4). Note that for Proposition 4, Floudas and Ciric (1989) showed that \( \text{RecLMTD}^\beta, \beta \in (0,1] \) is convex for the half planes \( 0 < y < x \) and \( 0 < x < y \), our proof, after including the limits, holds for \( x,y > 0 \) and applies to \( \text{RecLMTD}^\beta, \beta > -1 \). Also, a similar result to that of Proposition 3 was previously shown by Zavala-Río et al. (2005), our method of proof differs by using variable substitution.

Proposition 2. The log mean temperature difference, \( \text{LMTD} : \mathbb{R}_+^2 \to \mathbb{R} \),

\[ \text{LMTD}(x,y) = \frac{x - y}{\ln(x/y)} \]

is positive

Proof.

When \( x > y \), the numerator and denominator in the function definition are positive hence the result is positive.

When \( x < y \), the numerator and denominator in the function definition are negative hence the result is positive.

When \( x = y \), by Proposition 1 the result is \( x \) which is positive by the domain.

Corollary 2. The reciprocal of the log mean temperature difference, \( \text{RecLMTD} : \mathbb{R}_+^2 \to \mathbb{R} \),

\[ \text{RecLMTD}(x,y) = \frac{\ln(x/y)}{x - y} \]

is positive
Proposition 3. The partial derivatives of $\text{RecLMTD} : \mathbb{R}_+^2 \to \mathbb{R}$
\[
\text{RecLMTD} = \frac{\ln(x/y)}{x-y}
\]
are strictly decreasing.

Proof. Let $w = x/y$, note that $w > 0$. The gradient of $\text{RecLMTD}$ when $x \neq y$ is
\[
\nabla \text{RecLMTD}(x,y) = \frac{1}{(x-y)^2} \left( 1 - \ln(w) - w^{-1} \right).
\]
Since $x \neq y$, \(\frac{1}{(x-y)^2} > 0\), therefore we analyse the functions given by
\[
p_1(w) = 1 - \ln(w) - w^{-1}
q_1(w) = 1 + \ln(w) - w
\]
Since $w > 0$, we can make the substitution $w = e^n$ where $n \in \mathbb{R} \setminus \{0\}$ ($x \neq y$), this gives:
\[
p_2(n) = p_1(e^n) = 1 - n - e^{-n}
q_2(n) = q_1(e^n) = 1 + n - e^n
\]
These functions are strictly concave (second derivatives are negative) and have a maximiser at $n = 0$ ($p_2'(0) = q_2'(0) = 0$) therefore when $n \neq 0$ they are negative. Since $n = 0$ if and only if $x = y$, the partial derivatives are negative when $x \neq y$, i.e. they are strictly decreasing.

When $x = y$, by Proposition 1, the gradient of $\text{RecLMTD}$ is
\[
\nabla \text{RecLMTD}(x,x) = -\frac{1}{2x^2} \left( 1 \right).
\]
Since $x > 0$, the partial derivatives are negative, i.e. they are strictly decreasing. \qed

Proposition 4. Let $\beta \geq -1$ be constant. $\text{RecLMTD}^\beta : \mathbb{R}_+^2 \to \mathbb{R}$, the reciprocal of the log mean temperature difference raised to the $\beta$-th power:
\[
\text{RecLMTD}^\beta(x,y) = \begin{cases} 
  \left( \frac{\ln(x/y)}{x-y} \right)^\beta & x \neq y, \\
  1/x^\beta & x = y,
\end{cases}
\]
is concave if $\beta = -1$, strictly concave if $-1 < \beta < 0$, linear if $\beta = 0$, and strictly convex if $\beta > 0$.

Proof.
For the case of $\beta = 0$:
Since $\text{RecLMTD}$ is well-defined and positive over $\mathbb{R}_+^2$ (i.e. non-zero), $\text{RecLMTD}^0 = 1$ which is linear.

For the case of $\beta > -1$, $\beta \neq 0$:
We shall analyse the Hessian of $\text{RecLMTD}^\beta$, showing that $\nabla^2 \text{RecLMTD}^\beta$ is positive definite for $\beta > 0$ and negative definite for $-1 < \beta < 0$. This is done by analysing it over the sets $S' = \{(x,x)^T \mid x > 0\}$ and $S^* = \mathbb{R}_+^2 \setminus S'$ separately.

In the sequel, we aid readability by using $m$ to represent $\text{RecLMTD}$ (no exponent); the function parameters are also dropped. The Hessian of $\text{RecLMTD}^\beta$ is given by:

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Expanding the factors in $p$ over $\mathbb{R}$ we have that, for fixed $w > 0$ since

$$\frac{\beta}{12x^{2+\beta}} \left[ \begin{array}{cc} 4 & 0 \\ 0 & 4 \end{array} \right] + (3\beta + 1) \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$$

(1)

The derivation of the Hessian over $S'$ in Equation (1) can be found by referring to Appendix A.

$$\nabla^2 \text{RecLMTD}^0(x, x) = \frac{\beta}{12x^{2+\beta}} \left[ \begin{array}{cc} 4 & 0 \\ 0 & 4 \end{array} \right] + (3\beta + 1) \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$$

Analysing the leading principle minors (the upper left element, $D_1$, and the determinant, $D_2$) of the matrix inside the square brackets in Equation (1) gives:

$$D_1 = 5 + 3\beta,$$
$$D_2 = (5 + 3\beta)^2 - (1 + 3\beta)^2 = 24(1 + \beta).$$

For $\beta > -1$, both $D_1$ and $D_2$ are positive therefore the matrix inside the square brackets of Equation (1) is positive definite. Hence, since $x > 0$, when $-1 < \beta < 0$, $\nabla^2 \text{RecLMTD}^0(x, x)$ is negative definite and when $\beta > 0$, $\nabla^2 \text{RecLMTD}^0(x, x)$ is positive definite.

For the remainder of the proof, we use the substitution $w = x/y$. Once again, we analyse the leading principle minors of the Hessian. For positive definiteness, both leading principle minors need to be positive. For negative definiteness, the leading principle minor of order 1 must be negative and the leading principle minor of order 2 needs to be positive.

The leading principle minor of order 1 ($((\text{RecLMTD}^0)''_{xx})$) is:

$$(\text{RecLMTD}^0)''_{xx} = \beta m^{\beta - 1} p^{[\beta]}$$

where

$$p^{[\beta]} = m''_{xx} + (\beta - 1)m^{-1}(m'_x)^2$$

Since $\beta m^{\beta - 1} > 0$ when $\beta > 0$ and $\beta m^{\beta - 1} < 0$ when $\beta < 0$ ($m$ is positive over $\mathbb{R}^+_2$), we require that $p^{[\beta]} > 0$ over $\mathbb{R}^+_2$ when $\beta > -1$, $\beta \neq 0$. Positivity of $m^{-1}(m'_x)^2$ over $S^*$ follows from Propositions 2 and 3 therefore we have that, for fixed $x$ and $y$, $p$ is linear and strictly increasing in $\beta$. Therefore we show that $p^{[\beta = -1]} > 0$. Expanding the factors in $p^{[\beta]}$ gives

$$p^{[\beta]}(x, y) = \left( \frac{\ln(w)^{-1}}{y^2(w-1)^3} \right) q^{[\beta]}(w)$$

(2)

where

$$q^{[\beta]}(w) = (\beta + 1) \ln(w)^2 + 2(\beta + 1)w^{-1} \ln(w) - w^{-2} \ln(w) - (2\beta + 1) \ln(w)$$
$$+ (\beta - 1)w^{-2} - 2(\beta - 1)w^{-1} + \beta - 1$$

In Equation (2), $\left( \frac{\ln(w)^{-1}}{y^2(w-1)^3} \right)$ is positive over $S^*$ therefore we show that $q^{[\beta = -1]}(w) > 0$ over $S^*$.

$$q^{[\beta = -1]}(w) = -w^{-2} \ln(w) + \ln(w) - 2w^{-2} + 4w^{-1} - 2$$

Since $w > 0$ ($w \neq 1$), we can substitute $e^n$ for it ($n \in \mathbb{R} \setminus \{0\}$). Let

$$r(n) = q^{[\beta = -1]}(e^n) = -ne^{-2n} + n - 2e^{-2n} + 4e^{-n} - 2$$

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Differentiating $r(n)$ twice gives

$$r'(n) = 2ne^{-2n} - e^{-2n} + 1 + 4e^{-2n} - 4e^{-n}$$
$$= 2ne^{-2n} + 3e^{-2n} - 4e^{-n} + 1$$
$$r''(n) = -4ne^{-2n} + 2e^{-2n} - 6e^{-2n} + 4e^{-n}$$
$$= 4e^{-2n}(e^n - 1 - n)$$

When $n \neq 0$, $e^n - 1 - n > 0$ hence $r''(n) > 0$. Also, $r''(0) = 0$. This gives that $r$ is strictly convex. We also have that $r'(0) = 0$ and $r(0) = 0$ therefore $r$ is minimised only at $n = 0$ hence $r(n) > 0$, for $n \neq 0$ proving the positivity of $p^{\beta=-1}(x, y)$.

The leading principle minor of order 2 is the determinant of $\nabla^2 \text{RecLMTD}^\beta$:

$$\det \left( \nabla^2 \text{RecLMTD}^\beta \right) = (\beta m^{\beta-1})^2 \left( m_{xx}''m_{yy}'' - (m_{xy}'')^2 \right) + (\beta - 1)m^{-1} \left( m_{xx}''m_y'm_y' + m_{yy}''m_x'm_x' - 2m_{xy}''m_x'm_y' \right).$$

For positive and negative definiteness, we require that the determinant is positive when $\beta > -1$, $\beta \neq 0$. In the above, $(\beta m^{\beta-1})^2 > 0$ therefore we assess the second part of the product in the determinant. Let

$$f = m_{xx}''m_{yy}'' - (m_{xy}'')^2 + (\beta - 1)m^{-1} \left( m_{xx}''m_y'm_y' + m_{yy}''m_x'm_x' - 2m_{xy}''m_x'm_y' \right)$$

expansion of the factors gives:

$$f(x, y) = \left( \frac{2}{(x - y)^6} \right) g(w) + (\beta - 1) \left( \frac{1}{(x - y)^6} \right) g(w)$$
$$= \left( \frac{1}{(x - y)^6} \right) (1 + \beta)g(w),$$

where:

$$g(w) = \ln(w) \left( w^2 - w^{-2} \right) + 2 \ln(w) \left( w^{-1} - w \right) - 2 \left( w^2 + w^{-2} \right) + 8 \left( w + w^{-1} \right) - 12.$$

In Equation (3), $\left( \frac{1}{(x - y)^6} \right)$ $(1 + \beta)$ are positive when $\beta > -1$ therefore only positivity of $g(w)$ has to be shown. Substituting $w = e^n$ gives:

$$h(n) = g(e^n) = n \left( e^{2n} - e^{-2n} \right) + 2n \left( e^{-n} - e^n \right) - 2 \left( e^{2n} + e^{-2n} \right) + 8 \left( e^n + e^{-n} \right) - 12$$
$$= 2n \sinh(2n) - 4n \sinh(n) - 4 \cosh(2n) + 16 \cosh(n) - 12.$$

Taking derivatives, we get:

$$h'(n) = 4n \cosh(2n) - 4n \cosh(n) - 6 \sinh(2n) + 12 \sinh(n),$$
$$h''(n) = 8n \sinh(2n) - 4n \sinh(n) - 8 \cosh(2n) + 8 \cosh(n),$$
$$h^{(3)}(n) = 16n \cosh(2n) - 4n \cosh(n) - 8 \sinh(2n) + 4 \sinh(n),$$
$$h^{(4)}(n) = 32n \sinh(2n) - 4n \sinh(n).$$

Using the identity: $\sinh(2n) = 2 \sinh(n) \cosh(n)$, we get:

$$h^{(4)}(n) = 4n \sinh(n)(16 \cosh(n) - 1).$$
\( h^{(4)}(n) > 0 \) when \( n \neq 0 \). Evaluating at 0, the derivatives of \( h \) of orders 0 to 3 all equal to 0. Using reasoning similar to that of the function \( r \), we get \( h(n) > 0 \) when \( n \neq 0 \). This shows positivity of \( g(w) \).

For the case of \( \beta = -1 \):
Over \( S' \), substituting \( \beta = -1 \) gives (note \( x > 0 \)):

\[
\text{RecLMTD}^{-1}(x, x) = -\frac{1}{6x} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \preceq 0.
\]

Over \( S^* \), substituting \( \beta = -1 \) and taking out common factors gives:

\[
\nabla^2 \text{RecLMTD}^{-1}(x, y) = -(m(x, y))^{-2} \left( \frac{\ln(w)^{-1}}{(x-y)^3} \right) k(w) \begin{pmatrix} w^{-1} & -1 \\ -1 & w \end{pmatrix},
\]

where:

\[
k(w) = w \ln(w) - w^{-1} \ln(w) - 2w - 2w^{-1} + 4.
\]

Equation (4) is negative semi-definite provided that \( k(w) \) is positive for \( w > 0, w \neq 1 \). To see that \( k(w) \in \mathbb{R}^+ \), substitute \( w = e^n \) and reason similarly to the arguments for \( g(w) \).

We have shown that the Hessian of \( \text{RecLMTD} \) is: positive definite for \( \beta > 0 \), negative definite for \( -1 < \beta < 0 \), negative semi-definite for \( \beta = -1 \). Therefore \( \text{RecLMTD} \) is: strictly convex for \( \beta > 0 \), strictly concave for \( -1 < \beta < 0 \), concave for \( \beta = -1 \). This along with the linearity of the function when \( \beta = 0 \) completes our proof.

\section*{C Analysis of Bounds on Errors}

In this section, we will derive bounds on estimating: the concave function \( x^\beta \) with a line segment at the end points assuming a constant error of \( \xi \) on the parameter and the bilinear function \( xy \) with the McCormick hull assuming a constant error \( \xi \) on the parameter \( y \).

\textbf{Proposition 5.} Let \( 0 < \beta < 1, 0 \leq l < u, f : [l, u] \to \mathbb{R} \geq 0, f(x) = x^\beta \). The maximal error attained by approximating \( f \) with the line segment bounded by \( l^\beta \) and \( u^\beta \) is

\[
\frac{(1 - \beta) (u^\beta - l^\beta)}{\beta (u - l)} \cdot \frac{\beta^{-1}}{\sqrt{\beta(u - l)}} + \frac{lu^\beta - l^\beta u}{u - l}
\]

and occurs at the point

\[
x^* = \beta^{-1} \sqrt{\frac{u^\beta - l^\beta}{\beta(u - l)}}.
\]

\textbf{Proof.} The error between \( f \) and the line segment is given by:

\[
E(x) = x^\beta - \frac{(u^\beta - l^\beta) x}{u - l} + \frac{lu^\beta - l^\beta u}{u - l}.
\]

The derivative of \( E \) is:

\[
E'(x) = \beta x^{\beta - 1} - \frac{u^\beta - l^\beta}{u - l}.
\]

At the point of maximal error, \( x^* \), \( E(x^*) = 0 \), giving:

\[
x^* = \beta^{-1} \sqrt{\frac{u^\beta - l^\beta}{\beta(u - l)}}.
\]
This can be shown in a similar process to that of Proposition 5. The error function is:

\[ E(x^*) = \left( \frac{\beta - 1}{\beta(u - l)} \right)^{\beta - 1} \frac{u^\beta - l^\beta}{u - l} - \frac{u^\beta - l^\beta}{u - l} \]

Differentiating with respect to \( \xi \) has a fixed error of

\[ \text{Here we derive bounds on the error attained by approximating } x^\beta, 0 < \beta < 1 \text{ assuming the input parameter has a fixed error of } \xi \text{ subtracted from it and the error attained by making McCormick estimation for } xy \text{ assuming that } y \text{ has a fixed error of } \xi \text{ subtracted from it.}

We begin with the case of approximating \( x^\beta \). Assuming that \( 0 < \beta < 1, 0 \leq l < u, f : [l, u] \rightarrow \mathbb{R}_{\geq 0} \) and \( f(x) = x^\beta \). If we let the input of \( f \) be \( \hat{x} = x + \xi \) where

\[ 0 \leq \xi \leq \frac{\beta - 1}{\beta(u - l)} \]

is fixed. We can show that the maximal error attained by approximating \( g : [l, u - \xi] \rightarrow \mathbb{R}_{\geq 0}, g(x) = f(x + \xi) \) with the line segment with endpoints \((l, l^\beta)\) and \((u, u^\beta)\) is

\[ \frac{(1 - \beta)(u^\beta - l^\beta)}{\beta(u - l)} \frac{\beta - 1}{\beta(u - l)} \sqrt{\frac{u^\beta - l^\beta}{u - l}} + \frac{\xi(u^\beta - l^\beta)}{u - l} + \frac{lu^\beta - l^3 u}{u - l} \]

and occurs at the point

\[ x^* = \sqrt{\frac{u^\beta - l^\beta}{\beta(u - l)}} - \xi. \]

This can be shown in a similar process to that of Proposition 5. The error function is:

\[ E(x) = (x + \xi)^\beta - \frac{(u^\beta - l^\beta)x}{u - l} + \frac{lu^\beta - l^3 u}{u - l} \]

Differentiating with respect to \( x \) and equating to zero results in \( E \) maximising at the point:

\[ x^* = \sqrt{\frac{u^\beta - l^\beta}{\beta(u - l)}} - \xi. \]

with value:

\[ E(x^*) = \frac{(1 - \beta)(u^\beta - l^\beta)}{\beta(u - l)} \frac{\beta - 1}{\beta(u - l)} \sqrt{\frac{u^\beta - l^\beta}{u - l}} + \frac{\xi(u^\beta - l^\beta)}{u - l} + \frac{lu^\beta - l^3 u}{u - l}. \]

For the case of approximating \( xy \), we adapt the reasoning given by Androulakis et al. (1995). Androulakis et al. (1995) showed that the maximal error of underestimating the bilinear term \( xy \) over the domain \([x^L, x^U] \times [y^L, y^U]\) with its convex envelope, \( \max(x^L y + xy^L - x^L y^U, x^U y + xy^U - x^U y^U) \), is:

\[ \frac{(x^U - x^L)(y^U - y^L)}{4} \]
and occurs at the mid point of its domain:
\[
x^m = \frac{x^L + x^U}{2}, \quad y^m = \frac{y^L + y^U}{2}.
\]

We adapt this proof by looking at the system where one of the parameters has been underestimated and has a fixed error of \(\xi\), i.e. we derive the maximal error between \(x(y + \xi)\) and the convex envelope \(\max\{x^L y + x y^L, x^U y + x y^U - x^L y^U\}\). Assume that \(x \in [x^L, x^U]\) and \(y \in [y^L, y^U - \xi]\) with
\[
0 \leq \xi < y^U - y^L,
\]
the maximum error attained by approximating \(x(y + \xi)\) with the convex envelope \(\max\{x^L y + x y^L - x^L y^L, x^U y + x y^U - x^U y^U\}\) occurs at the point
\[
x^* = \frac{x^U + x^L}{2} + \frac{\xi (x^U - x^L)}{2(y^U - y^L)}, \quad y^* = \frac{y^U + y^L - \xi}{2}
\]
and is equal to:
\[
\frac{(x^U - x^L)(y^U - y^L)}{4} + \frac{\xi (x^U + x^L)}{2} + \frac{\xi^2 (x^U - x^L)}{4(y^U - y^L)}.
\]

Following the proof given by Androulakis et al. (1995), the problem can be formulated as:
\[
\begin{align*}
&\min_{x,y} - x(y + \xi) + z \\
&\text{subject to } z \geq x^L y + y^L x - x^L y^L \\
&\quad z \geq x^U y + y^U x + x^U y^U \\
&\quad x^L \leq x \leq x^U \\
&\quad y^L \leq y \leq y^U - \xi
\end{align*}
\]

The KKT conditions give:
\[
\begin{align*}
\mu_1 + \mu_2 - 1 &= 0 \quad (5) \\
\mu_1 x^L + \mu_2 x^U - x &= 0 \quad (6) \\
\mu_1 y^L + \mu_2 y^U - y - \xi &= 0 \quad (7) \\
(-z + x^L y + y^L x - x^L y^L) \mu_1 &= 0 \quad (8) \\
(-z + x^U y + y^U x - x^U y^U) \mu_2 &= 0 \quad (9) \\
\mu_1, \mu_2 &\geq 0. \quad (10)
\end{align*}
\]

We still require that at least one of \(\mu_1, \mu_2\) be nonzero, giving the cases:

(i) \(\mu_1 = 1, \mu_2 = 0\)

(ii) \(\mu_1 = 0, \mu_2 = 1\)

(iii) \(\mu_1 > 0, \mu_2 > 0\).
We cannot have (i) since it means \( y = y^L - \xi \) which is outside the domain of \( y \). If we have (ii) then \( y = y^U - \xi \) which yields a local minimum. Hence we analyse (iii), which gives the system:

\[
\begin{align*}
\mu_1 + \mu_2 - 1 &= 0 \\
\mu_1 x^L + \mu_2 x^U - x &= 0 \\
\mu_1 y^L + \mu_2 y^U - y - \xi &= 0 \\
-z + x^L y + y^L x - x^L y^L &= 0 \\
-z + x^U y + y^U x - x^U y^U &= 0
\end{align*}
\]

Solving this system gives:

\[
\begin{align*}
x &= \frac{x^U + x^L}{2} + \frac{\xi (x^U - x^L)}{2 (y^U - y^L)}, \\
y &= \frac{y^U + y^L - \xi}{2}, \\
z &= \frac{x^L y^U + x^U y^L}{2} + \frac{\xi (x^U y^L - x^L y^U)}{2 (y^U - y^L)}, \\
\mu_1 &= \frac{y^U - y - \xi}{y^U - y^L}, \\
\mu_2 &= \frac{1 + \xi}{2 (y^U - y^L)}
\end{align*}
\]

and a maximal error of:

\[
x(y + \xi) - z = \frac{(x^U - x^L) (y^U - y^L)}{4} + \frac{\xi (x^U + x^L)}{2} + \frac{\xi^2 (x^U - x^L)}{4 (y^U - y^L)}.
\]

**References**

