# UN-REDUCTION IN FIELD THEORY, WITH APPLICATIONS

A. ARNAUDON, M. CASTRILLÓN LÓPEZ, AND D.D. HOLM

ABSTRACT. The un-reduction procedure introduced previously in the context of Mechanics is extended to covariant Field Theory. The new covariant un-reduction procedure is applied to the problem of shape matching of images which depend on more than one independent variable (for instance, time and an additional labelling parameter). Other possibilities are also explored: non-linear $\sigma$-models and the hyperbolic flows of curves.

## 1. Introduction

Symmetry (i.e., invariance under a Lie group of transformations) greatly facilitates the study of variational problems, both for the construction of explicit solutions of the variational equations and for their qualitative analysis. A rich variety of information arises from Lie symmetry of variational problems, especially when they are formulated geometrically. For example, a vast, interesting literature exists on the topic of reduction by symmetry. In reduction by symmetry, the idea is to take advantage of the group of symmetry transformations to reduce the dimension of the configuration and phase spaces of the variational problem, thereby making the problem easier to handle. When performing such a reduction, one must also provide a method of reconstructing the solutions of the original, unreduced, variational problem from solutions of the reduced problem, which sometimes requires additional compatibility conditions.

Surprisingly, there are nice instances where this procedure can be used backwards. For example, suppose a variational problem looks complicated, but it may be recognised as the reduction by a certain group of symmetries of a variational problem formulated in a bigger space. Although the dimension of the corresponding un-reduced configuration space may be larger, the equations or the space itself may be simpler. Furthermore, the existence of the groups of symmetries may shed light on the nature of the initial equations. In this situation, one should notice that reduction by symmetries changes the structure of the equations. For example, in the Lagrange-Poincaré reduction procedure (when the configuration space is a manifold $Q$ on which a Lie symmetry group $G$ acts properly, see [5], [7], [14]), the reduced variational equations split into two different types. The first type is an Euler-Lagrange operator coupled with a gyroscopic term (the curvature of a chosen connection $\mathcal{A}$ in the bundle $Q \to Q/G$). The second type is a conservation law. In order to have a free variational problem in the reduced space, one needs to introduce forces into the un-reduced principle so that the equations will decouple. The choice of this force can be made by splitting the Lagrangian into horizontal and vertical parts with respect to the connection $\mathcal{A}$. This is the un-reduction construction given in [4] for variational problems of a particle (Mechanics) and generalized in this article to a covariant field theoretical

*Key words and phrases.* Field Theory, image matching, Lagrangian, reduction, symmetries.

setting. In particular, we also explore the topological situations which arise when the parameter manifold is not longer simply connected.

The main motivation of [4] was shape matching: given two plane shapes $S_1, S_2 \in \mathrm{Sh}(\mathbb{R}^2)$, understood as closed curves in $\mathbb{R}^2$, one seeks the optimal path of shapes joining $S_1$ and $S_2$. This problem is analysed in [9],[24] and references therein. The space $\mathrm{Sh}(\mathbb{R}^2)$ is a complicated infinite dimensional manifold. However, we have $\mathrm{Sh}(\mathbb{R}^2) = Q/G$, where $G = \mathrm{Diff}^+(S^1)$, and $Q$ is the space $\mathrm{Emb}^+(S^1, \mathbb{R}^2)$ of positively embedded parametrizations of the circle in the plane, which is a much easier functional space than the unparameterised planar curves in $\mathrm{Sh}(\mathbb{R}^2)$. By means of conveniently chosen forces, one may use un-reduction to lift the problem of shape matching to $\mathrm{Emb}^+(S^1, \mathbb{R}^2)$. In this article, this situation becomes richer. In particular, we can study matching of shapes depending on, say, two independent variables. A primary case is where the shapes depend on time (time evolution) and another parameter (space evolution) labelling a set of subjects in a research study. This so-called spatiotemporal analysis of shapes is a recent and active field of research. For details, the reader may consult [11], [15], [26]. In spatiotemporal shape analysis, there are two main approaches. These are the time-specific and subject-specific approaches, indicating the variable which parameterises the evolution in shape comparisons; either for a certain subject at a sequence of times, or for a sequence of subjects at a certain time. This spatio-temporal construction is illustrated in Fig. 1. Note that the $x$ and $t$ variables have interchangeable meanings. A more complex construction is found in [11] where the authors build a subject-specific approach together with a time-reparametrization, with interesting applications to the compared evolution of *Homo Sapiens Neanderthalensis* and *Homo Sapiens Sapiens*, or bonobos and apes. The methodology is meant to couple with statistical analysis. The configuration space of this approach is $\mathrm{Diff}(\mathbb{R}^2)$ together with the time reparametrization in $\mathrm{Diff}(\mathbb{R})$.

The un-reduction procedure in $\mathrm{Emb}^+(S^1, \mathbb{R}^2)$ that we propose here provides *simultaneous* evolution of both types in a single system of partial differential equations. We expect this combined evolution to provide more accurate and versatile information for the problem of spatiotemporal curve matching. Furthermore, we introduce a certain convenient Riemannian metric in the space of embeddings depending on derivatives of the curve (a Sobolev metric) which seems to be appropriate for the evolution in $\mathrm{Sh}(\mathbb{R}^2)$ and $\mathrm{Emb}^+(S^1, \mathbb{R}^2)$ (see [2], [3]), spaces which possess some other natural but pathological metrics. In [1], the authors further investigated this approach with a simple numerical test in the classical mechanical setting, but more work is needed to obtain a reliable scheme.

Because the theory is quite general, the range of potential applications is wide. Apart from the motivation of curve matching, we point out two other completely different areas of mathematical physics where covariant un-reduction is hidden. For example, $\sigma$-models in homogeneous spaces $G/H$ may be written as an un-reduction problem in $G$. Interestingly, we may sometimes combine un-reduction with Euler-Poincaré reduction to the Lie algebra $\mathfrak{g}$ to get a new set of equations. These equations are already in the literature, but we incorporate full geometric meaning to them with this concatenation of un-reduction and reduction, a situation intimately related with dual pairs (see [16]). Finally, covariant un-reduction is also applied to hyperbolic curve evolution, a baby geometric construction of other more sophisticated geometric flow equations.

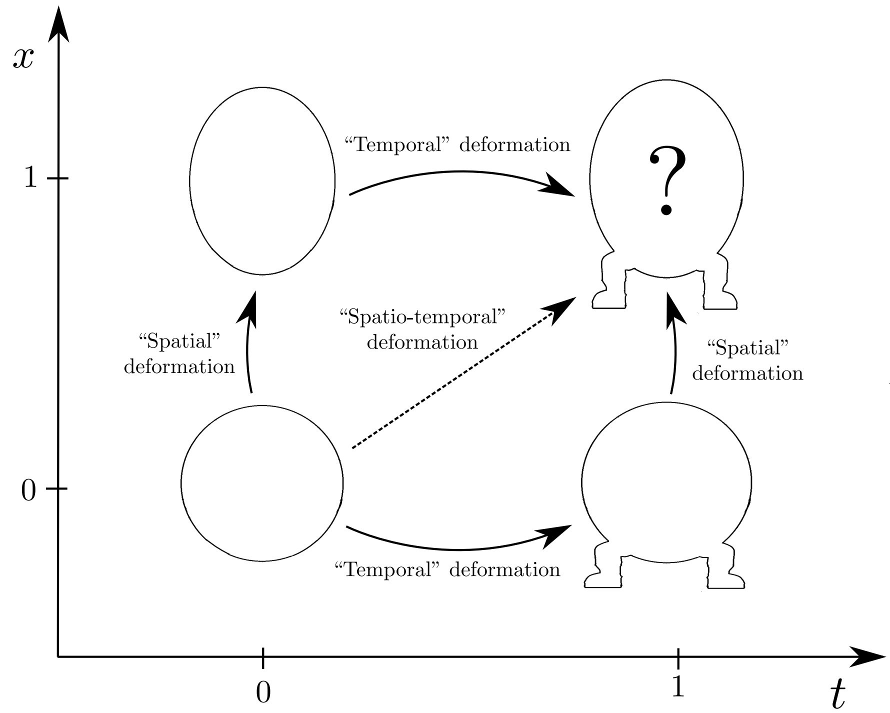


FIGURE 1. This diagram illustrates the spatio-temporal deformation of curves in $\mathrm{Sh}(S^1, \mathbb{R}^2)$ that is considered in this work. The combination of spatial and temporal deformations, where the precise meaning of space and time has to be defined depending on the context, allows for a simultaneous deformation of a curve along two parameters. The solution is then a function of $(x, t)$ which minimises an given energy functional. In the simplest case of quadratic energy functional, the solution is known as being a harmonic map.

**Plan and Main Contents of the Paper.** Section 2 reviews the basic concepts of covariant Lagrange-Poincaré reduction, before formulating the main result of the paper, which is the Un-reduction Theorem 3.1, in Section 3. Section 4 provides examples of explicit applications of the Un-reduction Theorem for (i) curve matching in the plane; (ii) nonlinear sigma model and (iii) hyperbolic curve evolution. Each of these examples demonstrates the method of un-reduction and illustrates different ways to take advantage of the geometry of the reduced space.

## 2. Covariant Lagrange-Poincaré reduction

The main result of the paper will be formulated as Theorem 3.1 in the next section. This section first reviews the basic concepts of covariant Lagrange-Poincaré reduction. The version of this reduction in Mechanics takes place when a Lie group of symmetries $G$ acts properly on the configuration manifold $Q$ of the variational problem under study (for example see [7]). In the field theoretical setting, the group of symmetries acts on a fibre bundle $\pi : E \to N$ by vertical diffeomorphisms, that is, actions such that $\pi(y \cdot g) = \pi(y), \forall y \in E, g \in G$. We refer the reader to [5] and [14] for the exposition of the theoretical framework of this procedure. For our purposes, in this article we have adapted these results as follows. On one hand we just consider trivial bundles $Q \times N \to N$, so that the dynamical objects of interest are mappings from $N$ to $Q$ and the problem is defined by a first order Lagrangian defined in the first jet space $J^1(N,Q)$. This simplification is mainly done for convenience in the applications, though the theoretical core of this work can be done in full generalities. On the other hand, we need to incorporate forces to our scheme, which will induce new terms in the equations in a straightforward manner.

2.1. **Background material.** For the standard notions on bundles and connections, the reader can go, for example, to [19]; and to [18] for the basic definitions on geometric variational calculus in bundles and field theories.

Let $\pi : Q \to Q/G = \Sigma$ be a $G$-principal bundle where the action $R_g : Q \to Q$, $g \in G$, is assumed to be on the right. Recall that a principal connection $\mathcal{A}$ is a $\mathfrak{g}$-valued 1-form in $Q$ such that the equivariance property $R_g^*\mathcal{A} = \mathrm{Ad}_{g^{-1}} \circ \mathcal{A}$ holds, and $\mathcal{A}(\xi_Q) = \xi$, for any $\xi \in \mathfrak{g}$, where $\xi_Q$ is the infinitesimal generator of the action, i.e., $\xi_Q(q) := d/d\varepsilon|_{\varepsilon=0} R_{\exp(\varepsilon\xi)}(q)$. This definition is equivalent to a choice of $G$-invariant splitting of the tangent bundle $TQ$ into horizontal and vertical parts

$$T_qQ = H_qQ \oplus V_qQ,$$

for $q \in Q$, where $V_qQ = \{(\xi_Q)_q : \xi \in \mathfrak{g}\}$ and $H_qQ = \ker \mathcal{A}$. We denote by $p^h : TQ \to HQ$ and $p^v : TQ \to VQ$ the induced projections. The curvature of $\mathcal{A}$ is defined to be the $\mathfrak{g}$-valued two form $\mathcal{B} = d\mathcal{A} + [\mathcal{A}, \mathcal{A}]$ and satisfies the equivariance property $(R_g)^*\mathcal{B} = \mathrm{Ad}_{g^{-1}} \circ \mathcal{B}$. One can also define a 2-form in $\Sigma$, but taking values in the adjoint bundle $\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$ as

$$\bar{\mathcal{B}}(u_\rho, w_\rho) = [q, \mathcal{B}(u_q^h, u_q^h)]_G, \qquad u_\rho, w_\rho \in T_\rho\Sigma,$$

where $u_q^h$ stands for the unique tangent vector (the horizontal lift of $u_q$ with respect to $\mathcal{A}$) in $H_qQ$ such that $T\pi(u_q^h) = u_\rho$. The definition does not depend on $q \in \pi^{-1}(\rho)$ because of the equivariant behaviour of the curvature.

Let $N$ be an oriented manifold endowed with a volume form $\mathbf{v}$ and consider a Lagrangian function $L : J^1(N,Q) \to \mathbb{R}$ defined in the 1-jet space of mappings $s : N \to Q$. As the jet space $J^1(N,Q)$ can be naturally identified with $T^*N \otimes TQ$, we will use both representations of this space in the following. We assume that $L$ is invariant with respect to the lifted action of $G$ in $J^1(N,Q)$, defined as

$$R_g^{(1)}(j_x^1 s) := j_x^1(R_g \circ s)$$

for $g \in G$ and any (local) mapping $s$. We can thus drop $L$ to the quotient to obtain a reduced Lagrangian function

$$\ell : J^1(N,Q)/G \simeq T^*N \otimes (TQ)/G \longrightarrow \mathbb{R}.$$

If we fix a principal connection $\mathcal{A}$ of the bundle $Q \to Q/G$, we have a diffeomorphism

$$\begin{aligned}(TQ)/G &\longrightarrow T\Sigma \oplus \tilde{\mathfrak{g}}\\ ([v_q]_G) &\mapsto (T\pi(v_q), [q, \mathcal{A}(v_q)]_G),\end{aligned}$$

so that the reduced phase space decomposes as

$$(J^1(N,Q))/G = T^*N \otimes (TQ)/G \cong T^*N \otimes (T\Sigma \oplus \widetilde{\mathfrak{g}}) \cong J^1(N,\Sigma) \oplus (T^*N \otimes \widetilde{\mathfrak{g}}),$$

so that the reduced Lagrangian can then be written as

$$\ell : J^1(N,\Sigma) \oplus (T^*N \otimes \tilde{\mathfrak{g}}) \to \mathbb{R}.$$

In the following sections, we will work with variational principles including a force term, that is, a map $F : J^1(N,Q) \to T^*Q$. The connection $\mathcal{A}$ splits the cotangent bundle $T^*Q = V^*Q \oplus H^*Q$ and we can consider the decomposition $F = F^h + F^v$ where $F^h = p^h \circ F$ and $F^v = p^v \circ F$ with $p^v$ and $p^h$ denoting the projections of $V^*Q$ and $H^*Q$ respectively. We use the same notation as for the projection of the tangent bundle as no confusion can occur. If in addition $F$ is $G$-equivariant with respect to the action of $G$ in both the source and target spaces, we can drop $F^h$ and $F^v$ to $J^1(N,Q)/G$ as

$$f^h : J^1(N,\Sigma) \oplus (T^*N \otimes \tilde{\mathfrak{g}}) \to T^*\Sigma \quad \text{and} \quad f^v : J^1(N,\Sigma) \oplus (T^*N \otimes \tilde{\mathfrak{g}}) \to \tilde{\mathfrak{g}}^*.$$

Note that for $f^h$ we use $H^*Q/G \simeq T^*\Sigma$, and for $f^v$ we have the isomorphism $VQ/G \simeq \tilde{\mathfrak{g}}$ given by $[(\xi_Q)_q]_G \mapsto [q,\xi]_G$.

Finally, we recall the definition of the canonical momentum map for the natural lift action of $G$ on $T^*Q$

$$\begin{array}{rcl} \mathbf{J} : T^*Q & \to & \mathfrak{g}^* \\ \langle \mathbf{J}(\alpha_q), \xi \rangle_{\mathfrak{g}\times\mathfrak{g}^*} & = & \langle \alpha_q, \xi_Q \rangle_{TQ\times TQ^*} \end{array}$$

where $\alpha_q \in T^*Q$, $\xi \in \mathfrak{g}$, and $\xi_Q \in TQ$. We can extend $\mathbf{J}$ to a map

$$\mathbf{J} : TN \otimes T^*Q \to TN \otimes \mathfrak{g}^*, \tag{2.1}$$

trivially in the factor $TN$. We note that, if we identify $TN \simeq \wedge^{n-1}T^*N$, $n = \dim N$, by means of a fixed volume form $\mathbf{v}$, the map $\mathbf{J} : TN \otimes T^*Q \to TN \otimes \mathfrak{g}^*$ is the covariant momentum map in field theories (cf. [18][Proposition 4.4]).

2.2. **Lagrange-Poincaré reduction.** In the sequel, we assume that $N$ is compact. If $N$ is not compact, the domain of variations of the maps $s : N \to Q$ will be assumed to be compactly supported. We project the variational principle defined for $L$ from $J^1(N,Q)$ to its quotient $J^1(N,Q)/G$. for $\ell : J^1(N,\Sigma) \times (T^*N \otimes \tilde{\mathfrak{g}}) \to \mathbb{R}$. Critical solutions are maps $\sigma : N \to T^*N \otimes \tilde{\mathfrak{g}}$ which, moreover, project to maps $\rho : N \to \Sigma = Q/G$ as $\rho = \pi_{\tilde{\mathfrak{g}}} \circ \sigma$ according to the diagram

$$\begin{array}{ccc} & & T^*N \otimes \tilde{\mathfrak{g}} \\ & \overset{\sigma}{\nearrow} & \downarrow^{\pi_{\tilde{\mathfrak{g}}}} \\ N & \overset{\rho}{\longrightarrow} & \Sigma \end{array} \tag{2.2}$$

where $\pi_{\tilde{\mathfrak{g}}} : T^*N \otimes \tilde{\mathfrak{g}} \to \Sigma$ is the projection of the adjoint bundle forgetting the $T^*N$ factor. The free variations of the initial problem provide a family of constrained variations that define a new type of variational equations. They are called Lagrange-Poincaré equations (see [5], [14]). The next theorem gives the Lagrange-Poincaré reduction with forces $F$ which is obviously the one in the literature when $F = 0$.

**Theorem 2.1** (Covariant Lagrange-Poincaré reduction with forces)**.** *Let* $\pi : Q \to Q/G = \Sigma$ *be a principal $G$-bundle, $\mathcal{A}$ be a principal connection on it and $N$ be a compact manifold oriented by a volume form $\mathbf{v}$. Given a map $s : N \to Q$, let $\sigma : N \to T^*N \otimes \tilde{\mathfrak{g}}$ be defined as*

$$\sigma(x)(\omega) = [s(x), \mathcal{A}(Ts \cdot (\omega))]_G,$$

*with* $\omega \in T_xN, x \in N$*; and let* $\rho : N \to \Sigma$, $\rho(x) = [s(x)]_G = \pi_{\tilde{\mathfrak{g}}} \circ \sigma$. *We consider a* $G$*-invariant Lagrangian* $L : J^1(N,Q) \to \mathbb{R}$ *and a* $G$*-equivariant force* $F : J^1(N,Q) \to T^*Q$*. Then the following points are equivalent:*

(1) $s$ *is a critical mapping of the variational principle*

$$\delta \int_N L(s, j^1 s)\mathbf{v} + \int_N \langle F(s, j^1 s), \delta s\rangle \mathbf{v} = 0 \tag{2.3}$$

*with free variations* $\delta s$.

(2) *The Euler-Lagrange form of* $L$ *satisfies the relation*

$$\mathcal{EL}(L\mathbf{v})(j^2 s) = F.$$

(3) $\sigma : N \to T^*N \otimes \tilde{\mathfrak{g}}$ *is a critical mapping of the variational principle*

$$\delta \int_N \ell(j^1\rho, \sigma)\mathbf{v} + \int_N \langle f^h(j^1\rho,\sigma), \delta\rho\rangle \mathbf{v} + \int_N \langle f^v(j^1\rho,\sigma), \eta\rangle \mathbf{v} = 0,$$

*for variations of the form* $\delta\sigma = \nabla^{\mathcal{A}}\eta - [\sigma, \eta] + \bar{\mathcal{B}}(\delta\rho, T\rho) \in \tilde{\mathfrak{g}}$*, where* $\delta\rho \in T_\rho\Sigma$ *is a free variation of* $\rho$ *and* $\eta$ *is a free section of* $\tilde{\mathfrak{g}} \to \Sigma$.

(4) $\sigma$ *satisfies the Lagrange-Poincaré equations*

$$\left.\begin{aligned} \mathcal{EL}_\rho(\ell\mathbf{v}) &= f^h - \left\langle \frac{\delta \ell}{\delta\sigma}, i_{T\rho}\bar{\mathcal{B}} \right\rangle, \\ \operatorname{div}^{\mathcal{A}} \frac{\delta\ell}{\delta\sigma} + \operatorname{ad}^*_\sigma \frac{\delta\ell}{\delta\sigma} &= f^v, \end{aligned}\right\} \tag{2.4}$$

*where* $\mathcal{EL}_\rho(\ell\mathbf{v}) : J^2(N,\Sigma) \to T^*\Sigma$ *is the Euler-Lagrange form of* $\ell$ *with respect to the variable* $\rho$ *only and* $\operatorname{div}^{\mathcal{A}}$ *stands for the covariant divergence operator defined by the connection* $\mathcal{A}$.

**Remark 2.2.** *Given a solution of the Lagrange-Poincaré equations* (2.4)*, the reconstruction of a solution of the initial variational problem requires a compatibility condition. Given the map* $\sigma : N \to T^*N \otimes \tilde{\mathfrak{g}}$ *and the induced map* $\rho : N \to \Sigma$*, we consider the pull-back principal bundle* $\rho^*Q \to N$ *and the pull-back of the connection* $\rho^*\mathcal{A}$*. Recall that the space of connections is an affine space modelled over the vector space of* $\tilde{\mathfrak{g}}$*-valued* 1*-forms in the base manifold. We can thus consider the new connection* $\mathcal{A}^\sigma = \rho^*\mathcal{A} + \sigma$*. Then, the compatibility condition is*

$$\operatorname{Curv}(\mathcal{A}^\sigma) = 0. \tag{2.5}$$

*Indeed, if this condition is satisfied, and the manifold* $N$ *is simply connected (see §3.4 below for some topological issues), then the solutions* $s : N \to Q$ *are the integral leaves or sections of that connection. See* [5, 14] *for more details.*

## 3. The covariant un-reduction scheme

We are now almost ready to describe the un-reduction scheme for Field Theories. As in the case of Mechanics (see [4]), this construction requires that the Lagrangian is decomposed into horizontal and vertical parts with respect to the chosen connection $\mathcal{A}$.

3.1. **Vertical and horizontal Lagrangians.** We first give an expanded expression of the Euler-Lagrange form $\mathcal{EL}(L) : J^2(N,Q) \to T^*Q$ for an arbitrary Lagrangian $L : J^1(N,Q) \to \mathbb{R}$ once a linear connection $\overline{\nabla}$ in $Q$ has been fixed. For that, we consider the horizontal lift $v \mapsto \hat{v}$ from $TQ$ to $T(T^*N \otimes TQ)$ with respect to $\overline{\nabla}$ (the lift is done in the $TQ$ part only and is trivial in the $T^*N$ factor). Then we define $\frac{\overline{\nabla} L}{ds} : J^1(N,Q) \to T^*Q$ as

$$\left\langle \frac{\overline{\nabla} L}{ds}(j^1_x s), \delta s \right\rangle_{TQ\times T^*Q} := \mathbf{d}L(j^1_x s)\cdot \widehat{\delta s},$$

for any $\delta s \in T_qQ$, $q = s(x)$. On the other hand, we define the vertical derivative $\frac{\partial L}{\partial j^1 s}$ : $J^1(N,Q) \to TN \otimes T^*Q$ as

$$\left\langle \frac{\partial L}{\partial j^1 s}(j^1_x s), \omega \right\rangle := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(j^1_x s + \epsilon\omega),$$

for any $\omega \in T^*_xN \otimes T_qQ$, $q = s(x)$. The Euler-Lagrange form is thus

$$\mathcal{EL}(L)(j^2 s) = \frac{\overline{\nabla} L}{ds}(j^1 s) - \operatorname{div}^{\overline{\nabla},\mathbf{v}} \frac{\partial L}{\partial j^1 s}(j^1 s), \tag{3.1}$$

where $\operatorname{div}^{\overline{\nabla},\mathbf{v}}$ stands for the divergence operator defined by the volume form $\mathbf{v}$ and the affine connection $\overline{\nabla}$. It acts on $T^*Q$-valued vector fields in $N$ (note that along the map $j^1 s$, $\partial L/\partial j^1 s$ is precisely a section of $TN \otimes s^*T^*Q \to N$) and it is defined as the only operator such that

$$\operatorname{div}^{\mathbf{v}} \langle \mathcal{X}, X\rangle = \left\langle \operatorname{div}^{\overline{\nabla},\mathbf{v}} \mathcal{X}, X \right\rangle + \langle \mathcal{X}, \overline{\nabla} X\rangle$$

for any vector field $\mathcal{X} \in TN \otimes T^*Q$ and any section vector field $X$ in $TQ$.

We now assume that the Lagrangian $L : J^1(N,Q) = T^*N \otimes TQ \to \mathbb{R}$ can be decomposed as $L = L^h + L^v$ with

$$L^h(\omega\otimes v) = L^h(\omega \otimes p^h(v)) \qquad \text{and} \qquad L^v(\omega\otimes v) = L^v(\omega\otimes p^v(v))$$

for any $\omega \otimes v \in T^*N \otimes TQ$, with respect to the connection $\mathcal{A}$. Furthermore, as $TQ = HQ \oplus VQ$, we have

$$L^h : T^*N \otimes HQ \to \mathbb{R} \qquad \text{and} \qquad L^v : T^*N \otimes VQ \to \mathbb{R}.$$

Obviously, the $G$ invariance of $L$ and $\mathcal{A}$ extends to the $G$-invariance of $L^v$ and $L^h$ so that they drop to the quotient as

$$\ell^h : J^1(N,\Sigma) = T^*N \otimes T\Sigma \to \mathbb{R} \qquad \text{and} \qquad \ell^v : T^*N \otimes \tilde{\mathfrak{g}} \to \mathbb{R},$$

to form the reduce Lagrangian $\ell(j^1\rho, \sigma) = \ell^h(j^1\rho) + \ell^v(\rho,\sigma)$. It is easy to see that

$$\frac{\delta \ell}{\delta j^1\rho} = \frac{\delta \ell^h}{\delta j^1 \rho} \qquad \text{and} \qquad \frac{\delta\ell}{\delta\sigma} = \frac{\delta \ell^v}{\delta \sigma}.$$

We then consider that the linear connection $\overline{\nabla}$ in $Q$ is invariant under the action of $G$ so that it projects to a linear connection $\nabla$ in $\Sigma = Q/G$ by the condition $\nabla_X Y = \pi_*(\overline{\nabla}_{X^h} Y^h)$. In addition, the connection $\mathcal{A}$ induces a connection in the associated bundle $\tilde{\mathfrak{g}} \to \Sigma$. With respect to these connections we can compute

$$\frac{\nabla \ell}{d\rho} = \frac{\nabla \ell^h}{d\rho} + \frac{\nabla \ell^v}{d\rho},$$

and the Lagrange-Poincaré equations (2.4) thus read

$$\left.\begin{array}{c} \mathrm{div}^{\nabla,\mathbf{v}}\left(\dfrac{\delta \ell^h}{\delta j^1\rho}\right) - \dfrac{\nabla \ell^h}{\delta\rho} = f^h + \dfrac{\nabla \ell^v}{\delta \rho} - \left\langle \dfrac{\delta \ell^v}{\delta\sigma}, i_{T\rho}\bar{\mathcal{B}}\right\rangle, \\ \mathrm{div}^{\mathcal{A}}\dfrac{\delta\ell^v}{\delta\sigma} + \mathrm{ad}^*_\sigma \dfrac{\delta \ell^v}{\delta\sigma} = f^v. \end{array}\right\} \tag{3.2}$$

The Lagrangian splitting is crucial in this methods and allows the appearance of the standard Euler-Lagrange equations for $\ell^h$ in the left hand side of the first equation. The second important ingredient is the force term $f^h$ which will allow us to exactly obtain the Euler-Lagrange equations by cancelling the right hand side of the same equation.

3.2. **The un-reduction theorem.** We are now ready to state the central theorem of the un-reduction method using the field theoretical context described above.

**Theorem 3.1.** *Let $N$ be a smooth manifold oriented by a volume form $\mathbf{v}$ and $\pi : Q \to \Sigma$ be a $G$-principal bundle equipped with a principal connection $\mathcal{A}$. Let $l : J^1(N,\Sigma) = T^*N \otimes T\Sigma \to \mathbb{R}$ be a first order Lagrangian. We consider a $G$-invariant Lagrangian $L : J^1(N,Q) = T^*N \otimes TQ \to \mathbb{R}$ such that $L = L^h + L^v$ where $L^h \circ p^h = L^h$ is uniquely determined by $l$, $L^v \circ p^v = L^v$ is arbitrary, and $p^h, p^v$ are the projectors of the splitting $TQ = HQ \oplus VQ$ induced by $\mathcal{A}$. We also consider a $G$-equivariant force $F : J^1(N,Q) \to T^*Q$ such that $F^v = p^v \circ F$ is arbitrary and $F^h = p^h \circ F$ is given by the condition*

$$f^h = -\frac{\nabla \ell^v}{\delta\rho} + \left\langle \frac{\delta\ell^v}{\delta\sigma}, i_{T\rho}\bar{\mathcal{B}}\right\rangle, \tag{3.3}$$

*for its projection $f^h : J^1(N,\Sigma) \times (T^*N \otimes \tilde{\mathfrak{g}}) \to T^*\Sigma$. Then, the variational equations of the problem defined by $L$ and $F$ read*

$$\left.\begin{array}{c} \mathcal{EL}(L^h)(j^2 s) = 0 \\ \mathcal{A}^*\mathrm{div}^{\mathbf{v}}\left(\mathbf{J}\left(\dfrac{\delta L^v}{\delta j^1 s}\right)\right) = F^v(j^1 s), \end{array}\right\} \tag{3.4}$$

*where $\mathcal{A}^* : \mathfrak{g}^* \to V^*Q$ is the dual of the connection form. Finally, critical solutions $s : N \to Q$ of (3.4) project to critical solutions $\rho = [s]_G$ of the Euler-Lagrange equations $\mathcal{EL}(l)(j^2\rho) = 0$.*

*Proof.* We follow the notations of the preceding sections. The variational principle of $L$ and $F$ is

$$\begin{aligned} 0 &= \delta\int_N L^h\mathbf{v} + \delta\int_N L^v\mathbf{v} + \int_N \langle F^h, \delta s\rangle \mathbf{v} + \int_N \langle F^v, \delta s\rangle\mathbf{v} \\ &= \delta\int_N L^h\mathbf{v} + \int_N \left\langle \frac{\delta\ell^v}{\delta\sigma}, \delta\sigma\right\rangle\mathbf{v} + \int_N\left\langle\frac{\nabla\ell^v}{\delta\rho},\delta\rho\right\rangle\mathbf{v} \\ &\quad + \int_N \left\langle f^h, \delta\rho\right\rangle\mathbf{v} + \int_N \langle F^v, \mathcal{A}(\delta s)\rangle\mathbf{v} \\ &= \delta\int_N L^h(j^1 s)\mathbf{v} + \int_N\left\langle\frac{\delta\ell^v}{\delta\sigma},\delta\sigma\right\rangle\mathbf{v} + \int_N\left\langle\frac{\delta\ell^v}{\delta\sigma}, \bar{\mathcal{B}}(T\rho,\delta\rho)\right\rangle\mathbf{v} \\ &\quad + \int_N\langle F^v, \mathcal{A}(\delta s)\rangle\mathbf{v}. \end{aligned}$$

From the expression of $\delta\sigma$ in Theorem 2.1 with $\eta(x) = (s(x), \mathcal{A}(\delta s))_G$ we have that

$$\int_N \left\langle \frac{\delta \ell^v}{\delta\sigma}, \delta\sigma + \bar{\mathcal{B}}(T\rho, \delta\rho) \right\rangle \mathbf{v} = \int_N \left\langle \frac{\delta \ell^v}{\delta\sigma}, \nabla^{\mathcal{A}}\eta - [\sigma, \eta] \right\rangle \mathbf{v}.$$

For any $f : N \to \mathfrak{g}$, we recall that the covariant derivative is $\nabla^{\mathcal{A}}(s(x), f(x))_G = (s(x), df(x) + [\mathcal{A}(j^1 s), f])_G = (s(x), df(x))_G + [\sigma, (s(x), f(x))_G]$. Now, for $f = \mathcal{A}(\delta s)$, we have

$$\int_N \left\langle \frac{\delta \ell^v}{\delta\sigma}, \delta\sigma + \bar{\mathcal{B}}(T\rho, \delta\rho) \right\rangle \mathbf{v} = \int_N \left\langle \frac{\delta \ell^v}{\delta\sigma}, (s, d\mathcal{A}(\delta s))_G \right\rangle \mathbf{v}$$
$$= \int_N \left\langle \mathbf{J}\left(\frac{\delta L^v}{\delta j^1 s}\right), d\mathcal{A}(\delta s) \right\rangle \mathbf{v} = -\int_N \left\langle \operatorname{div}^{\mathbf{v}}\left(\mathbf{J}\left(\frac{\delta L^v}{\delta j^1 s}\right)\right), \mathcal{A}(\delta s) \right\rangle \mathbf{v}.$$

Finally note that, as $L^h(j^1 s) = l(j^1\rho)$, the variation of the action defined by $L^h$ with respect to vertical variations of $L^h$ automatically vanishes. The variational principle naturally splits into vertical and horizontal part as equations (3.4).

Solutions of the variational problem defined by $M$ project to solutions of the problem defined by $l = \ell^h$ by Theorem 2.1. □

**Remark 3.2.** *If we have $N = \mathbb{R}$, $\mathbf{v} = dt$, (that is, the case of classical Mechanics) we have* $\operatorname{div}^{\mathbf{v}} = d/dt$ *and we recover the results and equations of* [4].

The expression of the horizontal force $F^h$ defined by condition (3.3) is

$$F^h = -\frac{\overline{\nabla} L^v}{ds} + \left\langle \mathbf{J}\left(\frac{\partial L^v}{\partial j^1 s}\right), i_{Ts}\mathcal{B} \right\rangle.$$

The variational principle on the un-reduced space of equation (2.3) is then defined using this particular force such that the reduced Lagrange-Poincaré equations decouples.

The first equation in (3.4) is the usual Euler-Lagrange equation for the horizontal Lagrangian. With respect to the second, we first note that we cannot exchange the position of $\mathcal{A}$ and $\operatorname{div}^{\mathbf{v}}$ as the authors do in [4]. In fact, the divergence of $\mathcal{A}^*\mathbf{J}(\delta L^v/\delta j^1 s)$ would require an additional (linear) connection in $Q$. Moreover, as we mentioned in the definition (2.1) of $\mathbf{J}$ above, we have that $\mathbf{J}(\delta L^v/\delta j^1 s)$ is a covariant momentum map, so that $\operatorname{div}^{\mathbf{v}}\mathbf{J}(\delta L^v/\delta j^1 s)$ is the expression of a conservation law with respect to the group of symmetries. The second equation in (3.4) equals this to the vertical part of the force. If one set $F^v = 0$, the conservation law is complete, but sometimes it is interesting to keep this vertical force as it might be used to externally control the dynamic along the vertical space.

3.3. **Reconstruction and the surjectivity of the un-reduction scheme.** The theorem 3.1 above says that solutions of the un-reduced problem project to solutions of the Euler-Lagrange equations defined by the Lagrangian $l$. One may ask if this projection is exhaustive, that is, if every solution of the variational equations of $l$ is a projection of a solution of $L$. This question involves some topological constraints concerning $N$ (see §3.4), but we first give an answer assuming that $N$ is simply connected (or we just consider the question from a local point of view). From the Lagrange-Poincaré reduction theorem, the variational equations defined by $L$ are equivalent to

$$\mathcal{EL}(\ell^h)(j^2\rho) = 0, \qquad \operatorname{div}^{\mathcal{A}}\frac{\delta \ell^v}{\delta\sigma} + \operatorname{ad}^*_\sigma \frac{\delta \ell^v}{\delta\sigma} = f^v,$$

that is, they contain the Euler-Lagrange equations of $l = \ell^h$ together with an additional set of equations which might restrict the solution of the first set. They are written in terms of the map $\sigma : N \to T^*N \otimes \tilde{\mathfrak{g}}$ and $\rho : N \to \Sigma$. Recall that $\sigma$ determines $\rho$ as $\rho = \pi_{\tilde{\mathfrak{g}}} \circ \sigma$ (see diagram (2.2)). The key point is that the first reduced equation only involves $\rho$ and its first jet $j^1\rho$. Once we have a solution $\rho$ and $j^1\rho$, we may consider both the second reduced equation and the compatibility condition. They are now equations for maps $\sigma$ seen as sections of the bundle $T^*N \otimes \rho^*\tilde{\mathfrak{g}} \to N$, which means that we "restrict" the vertical part of our construction to the fibers which sits only on the solution $\rho$ on the base manifold. With the solution of these last equations, we can perform reconstruction to get a map $s : N \to Q$ such that $\rho = [s]_G$. Roughly speaking, the reduced equations are uncoupled, so that $\rho$ and $\sigma$ can be treated separately and the surjectivity of the un-reduction technique is guaranteed. The reason of this is the force term which exactly decouples these equations although it is not explicit in the un-reduced equations.

3.4. **Topological constraints and un-reduction.** The topology of the manifold $N$ may create interesting situations in the reconstruction and un-reduction frameworks. If $N$ is not simply connected, the flatness of a connection, that is the compatibility condition (2.5), does not ensure the existence of global integral sections and the surjectivity of the projection $s \mapsto \rho$ of solutions involves some other global considerations.

An example of this situation is the following. Consider $Q = S^3$ and $G = S^1$ so that $Q \to \Sigma = S^2$ is the Hopf fibration. Choose the mechanical connection $\mathcal{A}$ in this bundle, that is, the connection such that $H_qS^3 \perp V_qS^3$ with respect to the standard Riemannian metric in $S^3$. For the sake of simplicity we consider $N = S^1$, that is, a problem of Mechanics with cyclic solutions where, in addition, the compatibility condition (2.5) is satisfied automatically. We denote $\theta$ the coordinate of $S^1$ and we consider the $G$-invariant Lagrangian $L : J^1(N, S^3) \to \mathbb{R}$,

$$L(j^1_\theta s) = \frac{1}{2} \|\dot{s}(\theta)\|^2 ,$$

where $\dot{s} = ds/d\theta \in T_{s(\theta)}S^3$, as well as its decomposition $L = L^h + L^v$ induced by the orthogonal splitting $\dot{s}(\theta) = \dot{s}^h(\theta) + \dot{s}^v(\theta)$ defined by the $\mathcal{A}$. The adjoint bundle $\tilde{\mathfrak{g}} \to S^2$ is a trivial line bundle and the reduced phase space $J^1(N, \Sigma) \times (T^*N \otimes \tilde{\mathfrak{g}})$ becomes $TS^2 \times T^*S^1$. We can write the reduced Lagrangian as $\ell = \ell^h + \ell^v$ with

$$\ell^h(j^1\rho) = \frac{1}{2} \|\dot{\rho}\|^2 , \qquad \ell^v(\sigma) = \frac{1}{2}\varsigma^2,$$

where $\rho : S^1 \to \Sigma = S^2$, $\dot{\rho} = d\rho/d\theta$, and $\sigma = \varsigma d\theta$ with $\varsigma$ a map $S^1 \to \tilde{\mathfrak{g}} \cong \mathbb{R}$. The reduced equations are

$$\nabla\dot{\rho} = 0, \qquad \dot{\varsigma} = f^v.$$

Solutions of the first equation are closed geodesics $\rho$ in $S^2$. Given one of these, the curves $s(\theta)$ of the un-reduced problem will be in the restriction $\rho^*S^3$ of the Hopf fibration along $\rho$. This restriction is a torus and according to the reconstruction process seen in §2.2, the curve $s(\theta)$ must be horizontal with respect to the connection $\mathcal{A} + \varsigma d\theta$. Under these circumstances, the curve $s(\theta)$ need not be closed and in fact, the phase $\varphi \in S^1$ such that $s(2\pi) - s(0) = \varphi$ is precisely the holonomy of the connection $\mathcal{A} + \varsigma d\theta$ along the curve $\rho$. The holonomy of $\mathcal{A}$ alone is $\pi$ (indeed, the connection $\mathcal{A}$ is not flat and the holonomy is related with the Chern number of the Hopf bundle, see [19, Chapter XII]). Hence, besides conditions $\dot{\varsigma} = f^v$ and $\varsigma(2\pi) = \varsigma(0)$, for the closeness of $c(\theta)$ we need $\varsigma(\theta)$ to satisfy

$$\int_0^{2\pi} \varsigma(\theta) d\theta = -\pi,$$

so that we cancel the holonomy of $\mathcal{A}$. Only very specific functions $f^v$ may accomplish these conditions. For example, $f^v(\theta) = \cos(\theta)$ gives $\varsigma(\theta) = \sin(\theta) - 1/2$ as a possible solution. Other functions $f^v$ does not provide closed curves $c(\theta)$. Furthermore, it is important to note that the constant value of the holonomy of the fixed connection $\mathcal{A}$ along geodesics $\rho$ is unusual and other choices of fixed connections $\mathcal{A}$ will define a holonomy depending on $\rho$. In that case, the choice of $f^v$ will depend on the global curve $\rho$ and will not be a local object.

In other words, there are circumstances where one cannot recover all solution of the reduced problem from those of the un-reduced problem. It seems that the freedom in the choice of $L^v$ and, especially, $F^v$ might solve this issue but their specific expression will depend on the solution $\rho$ itself. We refer the reader to [23] and [25] to some related approaches to the problem or, for example, [28] for a similar situation to the example above in the context of isoholonomic problems and quantum computation. The situation for manifolds $N$ of dimension greater than 1 is, of course, much more complicated.

## 4. Applications

4.1. **Planar curve matching.** We begin the application section with curve matching, he main motivation of this work, initiated by [4] and extended here to field theories.

4.1.1. *Geometric setting.* Let $Q = \mathrm{Emb}^+(S^1, \mathbb{R}^2)$ be the manifold of positive oriented embeddings from $S^1$ to $\mathbb{R}^2$. Elements in $Q$ are maps $c(\theta) \in \mathbb{R}^2$ for $\theta \in S^1$ and elements in the tangent space $T_cQ$ are pairs $(c, v)$ with $c \in \mathrm{Emb}^+(S^1, \mathbb{R}^2)$ and $u \in C^\infty(S^1, \mathbb{R}^2)$ a parametrized vector field along the curve $c$. Then

$$TQ = Q \times C^\infty(S^1, \mathbb{R}^2)$$

and we can take a trivial linear connection $\overline{\nabla}$ in $Q$. We consider an open domain $N \subset \mathbb{R} \times \mathbb{R}$ with the Euclidean metric, coordinates $(t, x)$ and volume form $\mathbf{v} = dt \wedge dx$. Elements of the jet bundle $J^1(N, Q) \simeq T^*N \otimes TQ$ are written as

$$j^1_{(x,t)}c = c_t(\theta)(t, x)dt + c_x(\theta)(t, x)dx, \tag{4.1}$$

that is, $c_t$ and $c_x$ are the derivatives of a map $c : N \to Q$ along $t$ and $x$ respectively.

We now consider the group $G = \mathrm{Diff}^+(S^1)$ of orientation preserving diffeomorphisms of $S^1$ and its Lie algebra $\mathfrak{g} = \mathfrak{X}(S^1)$ which consists of vector fields on $S^1$. The group $G$ acts on the right in $\mathrm{Emb}^+(S^1, \mathbb{R}^2)$ as reparametrization of curves $c$ and the reduced space is the space of shapes in $\mathbb{R}^2$

$$\Sigma := \frac{Q}{G} = \frac{\mathrm{Emb}^+(S^1, \mathbb{R}^2)}{\mathrm{Diff}^+(S^1)}. \tag{4.2}$$

The principal bundle $Q \to \Sigma$ is endowed with a canonical principal connection $\mathcal{A}$ as follows. Given $u \in T_cQ$, we consider its tangent and normal decomposition

$$u(\theta) = v(\theta)\mathbf{t}(\theta) + h(\theta)\mathbf{n}(\theta),$$

where $(\mathbf{t}, \mathbf{n})$ is the orthonormal Frenet frame along $c$ and $v(\theta), h(\theta)$ scalar functions along the curve. We clearly have that $v(\theta)\mathbf{t}(\theta)$ is a vector tangent to the orbits of $G = \mathrm{Diff}^+(S^1)$ so that $v(\theta)\mathbf{t}(\theta) \in V_cQ$. We can thus define the horizontal part of $u$ as the part $h(\theta)\mathbf{n}(\theta)$ and we have a decomposition $TQ = HQ \oplus VQ$.

The definition of a convenient Riemannian metric in $Q = \mathrm{Emb}^+(S^1, \mathbb{R}^2)$ invariant with respect to the action of $G = \mathrm{Diff}^+(S^1)$ is an interesting topic which has attired the attention of many research works (see, for example, [2] and [3] and the references therein). The natural $L^2$ metric

$$g(u, v) = \int_{S^1} \langle u(\theta), v(\theta)\rangle dl, \tag{4.3}$$

with $u, v \in T_cQ$, and $dl = |c_\theta| d\theta$ being the arc-length, is not very useful as it defines a zero geodesic distance in both $Q$ and $Q/G$. The problem can be overcome in the shape space $Q/G$ by the metrics

$$g(u, v) = \int_{S^1} (1 + A\kappa(\theta)^2)\langle u(\theta), v(\theta)\rangle dl, \tag{4.4}$$

with $A > 0$ and $\kappa$ the Frenet curvature of $c$. But this metric defines again a zero geodesic distance in $Q$ along the fibers of the fibration $Q \to Q/G$. A metric with a well defined Riemannian distance in both $Q$ and $Q/G$ is obtained by adding higher order derivatives of $u$ and $v$ in a Sobolev-type expression as

$$g(u, v) = \int_{S^1} \left(\langle u(\theta), v(\theta)\rangle + A^2\langle D_\theta u(\theta), D_\theta v(\theta)\rangle\right) dl, \tag{4.5}$$

where $D_\theta = \frac{1}{|c_\theta|}\partial_\theta$ is the arc-length derivative. We can collect these three cases (as well as many others, see [2]) as

$$g_{\mathcal{P}}(u, v) = \int_{S^1} \langle u(\theta), \mathcal{P}v(\theta)\rangle, \tag{4.6}$$

for a convenient choice of a $G$-invariant self-adjoint pseudo-differential operator $\mathcal{P}$ which can depend on the curve and its derivatives. In particular, the operator for (4.4) if $\mathcal{P} = 1 + A\kappa^2$, and for (4.5) we have $\mathcal{P} = 1 - A^2 D_\theta^2$. One additional advantage of the operator associated to (4.5) is that it does not depend on the curve, whereas the operator for (4.4) depends on the curvature of the curve where it is evaluated. This represents a great simplification in the expression of the un-reduced equations.

**Remark 4.1.** *Even if the mechanical connection $\mathcal{A}$ in this context of space of embeddings is easy to visualise and compute, its structure is far from being trivial. The calculation of its holonomy and curvature is a whole subject on its own, which should be addressed in forthcoming works.*

4.1.2. *Reduction and un-reduction.* Elements of the shape space of plane curves $\Sigma = \mathrm{Emb}^+(S^1, \mathbb{R}^2)/\mathrm{Diff}^+(S^1)$ are denoted by $\rho$ and the elements of the jet space $J^1(N, \Sigma) = T^*N \otimes T\Sigma$ are expressed as

$$j^1_{(t,x)}\rho = \rho_t(t, x)dt + \rho_x(t, x)dx.$$

Furthermore, elements of $T^*N \otimes \tilde{\mathfrak{g}}$ are

$$\sigma(t, x) = \sigma_t(t, x)dt + \sigma_x(t, x)dx$$

where $\sigma_t(t,x), \sigma_x(t,x)$ belong to the adjoint bundle $\tilde{\mathfrak{g}} \to \Sigma$ and can be understood as vector fields along a shape $\rho \in \Sigma$ and tangent to it. We consider the $\mathrm{Diff}^+(S^1)$-invariant Lagrangian $L : J^1(N,Q) \simeq T^*N \otimes TQ \to \mathbb{R}$

$$L(j^1_{(x,t)}c) = \frac{1}{2}\int_{S^1} \left(\langle c_t, \mathcal{P}c_t\rangle + \langle c_x, \mathcal{P}c_x\rangle\right) dl \tag{4.7}$$

which can be decomposed as $L = L^h + L^v$ with respect to the connection $\mathcal{A}$ as

$$L^h(j^1_{(x,t)}c) = \frac{1}{2}\int_{S^1} \left(\langle h_t, \mathcal{P}h_t\rangle + \langle h_x, \mathcal{P}h_x\rangle\right) dl,$$
$$L^v(j^1_{(x,t)}c) = \frac{1}{2}\int_{S^1} \left(\langle v_t, \mathcal{P}v_t\rangle + \langle v_x, \mathcal{P}v_x\rangle\right) dl,$$

where

$$c_t = v_t\mathbf{t} + h_t\mathbf{n} \quad \text{and} \quad c_x = v_x\mathbf{t} + h_x\mathbf{n}.$$

The un-reduction equations (3.4) are then computed in the proposition 4.2 below in the case when $\mathcal{P}$ is independent of the curve.

**Proposition 4.2.** *The un-reduced equations* (3.4) *for the bi-dimensional problem of planar simple curves defined by the Lagrangian* (4.7) *and the metric* (4.6) *with* $\mathcal{P}$ *independent of the curve are*

$$\begin{aligned} \partial_x \mathcal{P}h_x + \partial_t \mathcal{P}h_t &= D_\theta(h_x\mathcal{P}v_x + h_t\mathcal{P}v_t) - \kappa H \\ \partial_x \mathcal{P}v_x + \partial_t\mathcal{P}v_t &= F^v \end{aligned} \tag{4.8}$$

*with the decomposition*

$$c_x = v_x\mathbf{t} + h_x\mathbf{n}, \qquad c_t = v_t\mathbf{t} + h_t\mathbf{n}$$

*and for any choice of vertical force* $F^v$*, where*

$$H = \frac{1}{2}(h_x\mathcal{P}h_x + h_t\mathcal{P}h_t). \tag{4.9}$$

*Proof.* The Euler-Lagrange equation contains two terms, the first is readily

$$\mathrm{div}\frac{\delta L^h}{\delta j^1 c} = \partial_t(\mathcal{P}h_t) + \partial_x(\mathcal{P}h_x).$$

Before computing the second term of the EL equation, we rewrite only the temporal part of the Lagrangian in order to simplify the calculation as

$$L^h(c, j^1c)|_t = \frac{1}{2}\int_{S^1}\langle (c_t\cdot\mathbf{n})\mathbf{n}, \mathcal{P}(c_t\cdot\mathbf{n})\mathbf{n}\rangle dl = \frac{1}{2}\int_{S^1}\left(c_t\cdot J\frac{c_\theta}{|c_\theta|}\right)\mathcal{P}\left(c_t\cdot J\frac{c_\theta}{|c_\theta|}\right)|c_\theta| d\theta.$$

This Lagrangian being horizontal, we just need to consider variations of $c$ that are horizontal with respect to $\mathcal{A}$, that is, variations of the form $\delta c = \mathbf{n}\xi$, $\xi \in C^\infty(S^1)$. With the identities $D_\theta\mathbf{n} = -\kappa\mathbf{t}$, $J\mathbf{n} = -\mathbf{t}$ and $J\mathbf{t} = \mathbf{n}$ we compute

$$\begin{aligned} \frac{\partial L^h|_t}{\partial c}\cdot(\mathbf{n}\xi) &= \int_{S^1}\left(c_t\cdot J(\mathbf{n}\xi)_\theta\right)\mathcal{P}\left(c_t\cdot\mathbf{n}\right) d\theta \\ &= \int_{S^1}\xi_\theta\left(c_t\cdot J\mathbf{n}\right)\mathcal{P}\left(c_t\cdot\mathbf{n}\right) d\theta + \int_{S^1}\xi\left(c_t\cdot JD_\theta\mathbf{n}\right)\mathcal{P}\left(c_t\cdot\mathbf{n}\right) dl \\ &= \int_{S^1}\xi D_\theta\left[\left(c_t\cdot\mathbf{t}\right)\mathcal{P}\left(c_t\cdot\mathbf{n}\right)\right] dl - \int_{S^1}\xi\kappa\left(c_t\cdot\mathbf{n}\right)\mathcal{P}\left(c_t\cdot\mathbf{n}\right) dl. \end{aligned}$$

Therefore the derivative of the Lagrangian is

$$\frac{\partial L^h|_t}{\partial c} = D_\theta(h_t \mathcal{P} v_t) - \kappa h_t \mathcal{P} h_t.$$

From the symmetry $t \Leftrightarrow x$, the Euler-Lagrange equation follows. □

**Remark 4.3.** *The term $\kappa H$ can be interpreted as a penalty term for the deformation of most curved regions of the curve. The sign of this term would depend on the concavity or convexity of the curve at this point, and thus this force would try to prevent the curve to be deformed too fast in these regions. The equation (4.8) also shows that the dynamics in $(x,t)$ is governed by the coupling between $h_t$ and $v_t$ required for the shape deformation to be independent of the reparametrisation.*

**Remark 4.4.** *The un-reduced equations with curvature weighted metric (4.4) can be computed directly from the variational principle, as in [4]. The equation will have the same symmetry $x \leftrightarrow t$ but with more complicated terms. Because this metric is not very useful, the covariant un-reduced equations will not be displayed here.*

4.2. **Horizontal Lagrangians and $\sigma$-models.** The freedom in the choice of forces and Lagrangians in Theorem 3.1 permits the trivial choice of $L^v = 0$ and $F^v = 0$. From (3.3), the horizontal part $F^h$ of the force automatically vanishes. This simple situation appears when the un-reduced Lagrangian $L$ is just the pull-back of the Lagrangian $\ell^h = l : J^1\Sigma \to \mathbb{R}$ with respect to the projection $J^1(N,Q) \to J^1(N,\Sigma)$, $j^1 s \mapsto j^1[s]_G = j^1\rho$. A solution of the problem defined by $L$ is any map $s : N \to Q$ such that $\rho = [s]_G$ is a solution for $l$. This means that there is a gauge degeneracy in the sense that, given a solution $s$ and any map $g : N \to G$, the map $\bar{s} = s \cdot g$ is also a solution.

Even though these trivial choices for $F$ and $L^v$ are not always convenient, there are some instances where they appear naturally. This is the case of $\sigma$-models in homogeneous spaces (see for example [12], [13], [17], [10]). Let $Q = G$ be a Lie group and $H$ be a closed subgroup such that the Lie algebra decomposes as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ for certain vector space $\mathfrak{m}$ such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ (that is, we have a reductive decomposition). We can right translate the decomposition $\mathfrak{m} \oplus \mathfrak{h} = \mathfrak{g} = T_eG$ to every $T_gG, g \in G$, thus obtaining a connection $\mathcal{A}$ for the principal bundle $G \to M$ over the homogeneous space $\Sigma = M = G/H$. We consider the harmonic, or $\sigma$-model, problem on maps $\rho : N \to M$ defined by the Lagrangian

$$\begin{aligned} l : J^1(N,M) &\to \mathbb{R} \\ j^1\rho &\mapsto \frac{1}{2}\|d\rho\|^2, \end{aligned}$$

where the norm is taken with respect to a pseudo-Riemannian metric in $N$ and a Riemannian metric in $M$. The lift $L$ of $l$ to $J^1(N,G)$ is

$$\begin{aligned} L : J^1(N,G) &\to \mathbb{R} \\ j^1 g &\mapsto \frac{1}{2}\left\|p^h(dg)\right\|^2, \end{aligned}$$

where $p^h : TG \to HG$ is the horizontal projection defined by $\mathcal{A}$ and the norm is taken with respect to the metric of $N$ and the lift of the metric in $M$ to horizontal vectors in $M$. Theorem 3.1 can apply and solutions of the force-free problem defined by $L$ project to the desired harmonic maps in $M$.

In the majority of the homogeneous spaces where relevant $\sigma$-models are defined, the group $G$ is endowed with a bi-invariant metric. In this case, the reductive decomposition is assumed to be $\mathfrak{m} = \mathfrak{h}^\perp$ and we have a metric in $M$ by imposing the projection $\pi : G \to M$ to be an isometric submersion, that is, the metric in $T_x M$ is the same as the metric in $H_g G$ for any $g$ with $\pi(g) = x$. The group $G$ left-acts on the coset space $M$ by isometries. Hence, the Lagrangians $l$ and $L$ are both $G$ invariant. This group of symmetries is too big for $M$ to do reduction (in fact the orbit space is a single point), but we can perform covariant Euler-Poincaré reduction for $L$. We then get a new reduced Lagrangian

$$\begin{aligned} \bar{l} : J^1(N, G)/G = T^*N \otimes \mathfrak{g} &\to \mathbb{R} \\ \varsigma &\mapsto \frac{1}{2}\left\|\varsigma_{\mathfrak{m}}\right\|^2 \end{aligned}$$

where $\varsigma = \varsigma_{\mathfrak{h}} + \varsigma_{\mathfrak{m}}$ is the splitting defined by the reductive decomposition. It is easy to see that the Euler-Poincaré equations are

$$\operatorname{div}^{\mathbf{v}} \varsigma_{\mathfrak{m}} + [\varsigma_{\mathfrak{h}}, \varsigma_{\mathfrak{m}}] = 0$$

which, together with the suitable compatibility condition, can be used to get solutions of $L$ that, afterwards, can be projected to $\Sigma$. This approach is found, for example, in [13], [10], [21]. The advantage of this un-reduction and reduction procedure relies on the fact that $\mathfrak{g}$ is a simpler space (is a vector space) than either $G$ and $M$.

The situation can be even put in a more general framework as follows. Let $L$ be a first order Lagrangian on a Lie group $G$ as configuration space, which is right invariant under the action of a subgroup $H$ and left invariant under the group $G$ itself. Suppose that we are interested in the induced variational problem in the homogeneous space $G/H$. The un-reduction and reduction procedure will give first a variational problem in $G$ to finally induce a problem in the Lie algebra $\mathfrak{g}$ which, in general, is simpler. See [27] for a description of a similar situation in Mechanics (that is, $N = \mathbb{R}$).

4.3. **Hyperbolic curvature flow.** The hyperbolic curvature flow of plane curves (see for example [22] or [29]) is the variational equation defined by the Lagrangian

$$L : T\mathrm{Emb}(S^1, \mathbb{R}^2) \to \mathbb{R},$$
$$L(c, c_t) = \int_{S^1} \left(\tfrac{1}{2}\left\|c_t\right\|^2 - 1\right) dl.$$

Note that this is not a geodesic variational principle of the $L^2$ metric (which provides null geodesic distances in both the curve and shape spaces) but a Lagrangian involving a kinetic and a potential term. Moreover, the Lagrangian $L$ can be easily split into horizontal and vertical with respect to the connection $\mathcal{A}(c_t)= (c_t \cdot \mathbf{t})\mathbf{t}$ as

$$L^h = \int_{S^1} \left(\tfrac{1}{2}h^2 - 1\right) dl, \qquad L^v = \int_{S^1} \tfrac{1}{2}v^2 dl,$$

where

$$c_t = h\mathbf{n} + v\mathbf{t}.$$

The Lagrangian $L$ (and $L^h$, $L^v$) is $\mathrm{Diff}(S^1)$-invariant as its definition is geometric and does not depend on the parametrization of $c$ but only on its shape.

One of the main features and applications of the hyperbolic flow (as well as of other geometric flows of curves) is the study of the evolution of the shapes of the curves under it. We can now suppose that we just want to study this evolution in the shape space $\mathrm{Emb}(S^1, \mathbb{R}^2)/\mathrm{Diff}^+(S^1)$. The natural Lagrangian in this situation becomes $l = \ell^h$, the projection of $L^h$ to this quotient space. In this context, the un-reduction technique applies and we have the last result of this paper.

**Proposition 4.5.** *The un-reduced equations for the system described above read*

$$\begin{aligned} \partial_t h &= D_\theta(vh) - \kappa(\tfrac{1}{2}h^2 - 1), \\ \partial_t v &= F^v, \\ c_t &= h\mathbf{n} + v\mathbf{t}. \end{aligned}$$

*In particular, if we choose $F^v = 0$ and the initial tangent velocity to vanish ($v(0) = 0$), then $v(t) = 0$ for all times and the velocity of $h$ is proportional to the curvature $\kappa$.*

*Proof.* Following the derivation of the un-reduced equation for curve matching, but in the classical case, one can prove this proposition as well. □

**Remark 4.6.** *The equations of the Proposition above for $F^v = 0$ are the hyperbolic mean flow equations (see for example [22] for a good account of this flow). The usual approach in the literature works in $\mathrm{Emb}(S^1, \mathbb{R}^2)$ and then restrict oneself to the normal part of the flow. The approach here works with shapes in $\mathrm{Emb}(S^1\mathbb{R}^2)/\mathrm{Diff}(S^1)$ so that the trivial choice of $F^v = 0$ gives directly the geometric equations.*

## Acknowledgements

We are grateful to M. Bauer, S. Durrleman, R. Montgomery and T. Ratiu for valuable discussions during the course of this work. We also want to thank H.Dumpty for helpful suggestions during the realisation of Fig. 1. AA acknowledges partial support from an Imperial College London Roth Award, AA and DH from the European Research Council Advanced Grant 267382 FCCA. MCL has been partially funded by MINECO (Spain) under projects MTM2011-22528 and MTM2010-19111. MCL wants to thank Imperial College for its hospitality during the visit in which the main ideas of this work were developed.

## References

[1] A. Arnaudon, M. Castrillón López, D.D. Holm, *Covariant un-reduction for curve matching*, Conference paper for MFCA2015, arXiv:1508.05325 (2015)

[2] M. Bauer, M. Bruveris, S. Marsland, P.W. Michor, *Constructing reparameterization invariant metrics on spaces of plane curves*, Differential Geom. Appl. **34** (2014), 139-165.

[3] M. Bauer, M. Bruveris, P.W. Michor, *Overview of the Geometries of Shape Spaces and Diffeomorphism Groups*, J. Math. Imaging and Vision **50** (2014), 67–90.

[4] M. Bruveris, D.C.P. Ellis, D.D. Holm, F. Gay-Balmaz, *Un-reduction*, Journal of Geometric Mechanics **3** (2011), 363–387.

[5] M. Castrillón López, T.S. Ratiu, *Reduction in Principal Bundles: Covariant Lagrange-Poincaré Equations*, Comm. Math. Phys. **236** (2003), 223-250.

[6] M. Castrillón López, P.L. García, T.S. Ratiu, *Euler–Poincaré reduction on principal bundles*, Lett. Math. Phys. **58** (2001), 167–180.

[7] Cendra, Hernán and Marsden, Jerrold E and Ratiu, Tudor S *Lagrangian reduction by stages*, American Mathematical Soc. **722**, (2001)

[8] C.J. Cotter, A. Clark, J. Peiró, *A reparameterisation based approach to geodesic constrained solvers for curve matching*,Int. J. Comput. Vis. **99**, (2012)

[9] C.J. Cotter and D.D. Holm, *Geodesic boundary value problems with symmetry*, J. Geom. Mech. **2**, no. 1 (2010), 417–444.

[10] Y.J. Dai, M. Shoji, H. Urakawa, *Harmonic maps into Lie groups and homogeneous spaces*, Diff. Geom. Appl. **7** (1997), 143–160.

[11] S. Durrleman, X. Pennec, A. Trouvé, J. Braga, G. Gerig, N. Ayache, *Toward a comprehensive framework for the spatiotemporal statistical analysis of longitudinal shape data*, Int J Comput Vis. **103**, no. 1 (2013), 22–59.

[12] H. Eichenherr, M. Forger *More About Non-Linear Sigma Models on Symmetric Spaces*, Nuclear Physics B **164** (1980), 528–535.

[13] H. Eichenherr, M. Forger *Higher Local Conservation Laws for Nonlinear Sigma Models on Symmetric Spaces*, Comm. Math. Phys **82** (1981), 227–255.

[14] D.C. Ellis, F. Gay-Balmaz, D.D. Holm, T.S. Ratiu, *Lagrange-Poincaré field equations*, Journal of Geometry and Physics **61**, no. 11, (2011), 2120–2146.

[15] G. Derig, B. Davis, P. Lorenzen, S. Xu, M. Jomier, J. Piven, S. Joshi, *Computational anatomy to asses longitudinal trajectory of brain growth*, In Third International Symposium on D Data Processing, Visualization and Transmission, 1041–1047.

[16] D.D. Holm, C. Vizman, *Dual pairs in resonances*, J. Geom. Mech. **4**, no. 3 (2012), 297-311.

[17] M.A. Guest, *Harmonic maps, loop groups, and integrable systems*, London Mathematical Society Student Texts, 38. Cambridge University Press, Cambridge, 1997.

[18] M.J. Gotay, J. Isenberg, J.E. Marsden, R. Montgomery, *Momentum Maps and Classical Relativistic Fields*, Unpublised notes, arXiv:physics/9801019.

[19] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Wiley Classics Library Vol. I, 1963.

[20] P.W. Michor, D. Mumford, *An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach*, Appl. Comput. Harmon. Anal. **23** (1) (2007), 74-113.

[21] M. Higaki, *Actions of Loop Groups on the Space of Harmonic Maps into Reductive Homogeneous Spaces*, J. Math. Sci. Univ. Tokyo **5** (1998), 401-421.

[22] P.G. LeFloch, K. Smoczyk, *The hyperbolic mean curvature flow*, J. Math. Pur App. **90** (2008), 591–614.

[23] J. Marsden, R. Montgomery, T. Ratiu, *Reduction, symmetry, and phases in mechanics*, Mem. Amer. Math. Soc. **88** (1990), no. 436.

[24] P.W. Michor, D. Mumford, *Riemannian geometries on spaces of plane curves*, J. Eur. Math. Soc. **8** (2006), 1–48.

[25] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, 91. American Mathematical Society, Providence, RI, 2002.

[26] J.M. Peyrat, H. Delingette, M. Samersat, X. Pennec, C. Xu, N. Ayachie, *Registration of 4D Time-Series of cardiac images with Multichannel Diffeomorphic Demons*, In Proc. Medical Image computing and computer assisted Intervention, Springer, LNCS **5242**, 972–979.

[27] F. Tığlay, C. Vizman, *Generalized Euler–Poincaré equations on Lie groups and homogeneous spaces, orbit invariants and applications*, L. Math. Phys. **97** (2011), no. 1

[28] S. Tanimura, M. Nakahara, D. Hayashi, *Exact solutions of the isoholonomic problem and the optimal control problem in holonomic quantum computation*, J. Math. Phys. **46** (2005), no. 2, 022101.

[29] Z. Wang, *Hyperbolic mean curvature flow with a forcing term: evolution of plane curves*, Nonlinear Anal. **87** (2014), 65–82.

AA, DDH: Department of Mathematics, South Kensington Campus, Imperial College London, London SW7 2AZ, United Kingdom

*E-mail address*: alexis.arnaudon@imperial.ac.uk , d.holm@imperial.ac.uk

MCL: ICMAT(CSIC-UAM-UC3M-UCM), Dept. Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040, Madrid, Spain

*E-mail address*: mcastri@mat.ucm.es