JOINT ASYMPTOTIC DISTRIBUTION OF CERTAIN PATH FUNCTIONALS OF THE REFLECTED PROCESS

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ABSTRACT. Let $\tau(x)$ be the first time that the reflected process Y of a Lévy process X crosses $x > 0$. The main aim of this paper is to investigate the joint asymptotic distribution of $Y(t) = X(t) - \inf_{0\leq s\leq t} X(s)$ and the path functionals $Z(x) = Y(\tau(x)) - x$ and $m(t) = \sup_{0\leq s\leq t} Y(s) - y^*(t)$, for a certain non-linear curve $y^*(t)$. We restrict to Lévy processes X satisfying Cramér's condition, a non-lattice condition and the moment conditions that $E[|X(1)|]$ and $E[\exp(\gamma X(1))|X(1)|]$ are finite (where γ denotes the Cramér coefficient). We prove that $Y(t)$ and $Z(x)$ are asymptotically independent as $\min\{t, x\} \to \infty$ and characterise the law of the limit (Y_∞, Z_∞). Moreover, if $y^*(t) = \gamma^{-1}\log(t)$ and $\min\{t, x\} \to \infty$ in such a way that $t\exp\{-\gamma x\} \to 0$, then we show that $Y(t)$, $Z(x)$ and $m(t)$ are asymptotically independent and derive the explicit form of the joint weak limit $(Y_\infty, Z_\infty, m_\infty)$. The proof is based on excursion theory, Theorem 1 in [7] and our characterisation of the law (Y_∞, Z_∞).

1. Introduction and main results

The reflected process Y of a Lévy process X is a strong Markov process on $\mathbb{R}_+$ equal to X reflected at its running infimum. The reflected process is of great importance in many areas of probability, ranging from the fluctuation theory for Lévy processes (e.g. [2, Chapter VI] and the references therein) to mathematical statistics (e.g. [19, 22], CUSUM method of cumulative sum), queueing theory (e.g. [1, 20]), mathematical finance (e.g. [11, 17], drawdown as risk measure), mathematical genetics (e.g. [14] and references therein) and many more. The aim of this paper is to study the weak limiting behaviour of the reflected process $Y = (Y(t))_{t\geq 0}$, $Y(t) = X(t) - \inf_{0\leq s\leq t} X(s)$, and the overshoot $Z(x)$ and the centered running maximum $m(t)$ of Y given by

$$Z(x) \doteq Y(\tau(x)) - x, \qquad m(t) \doteq Y^*(t) - y^*(t), \qquad t, x \in \mathbb{R}_+, \tag{1.1}$$

where y^* is a specific non-linear curve to be specified shortly. Here $\tau(x)$ and $Y^*(t)$ denote the first entry time of Y into the interval (x, ∞), which is finite almost surely, and the supremum up to time t of the reflected process respectively,

$$\tau(x) \doteq \inf\{t \geq 0 : Y(t) > x\} \quad (\inf\emptyset \doteq \infty), \qquad Y^*(t) \doteq \sup_{0\leq s\leq t} Y(s).$$

In this paper we restrict to Lévy processes satisfying the conditions:

Assumption 1. (i) *Cramér's condition*, $E[e^{\gamma X(1)}] = 1$ for $\gamma > 0$, holds, (ii) $E[|X(1)|] + E[e^{\gamma X(1)}|X(1)|] < \infty$ and (iii) either the Lévy measure of X is non-lattice or 0 is regular for $(0, \infty)$.

Under Cramér's condition, it is well-known that X tends to $-\infty$ almost surely which yields by a classical time reversal argument that the reflected process Y has a weak limit Y_∞ equal in distribution to the ultimate supremum $\sup_{t\geq 0} X(t)$ (see e.g. [1, Chapter IX] or Section 2 below). The second functional, the overshoot $Z(x)$, also admits a weak limit Z_∞ (the form of which is given in Proposition 8 below). The following result addresses the question of the weak asymptotics of the vector $(Y(t), Z(x))$.

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Theorem 1. *$Y(t)$ and $Z(x)$ are asymptotically independent, as $\min\{t,x\}\to\infty$, in the sense that*

$$\begin{aligned}\lim_{\min\{x,t\}\to\infty} E[\exp(-uY(t)-vZ(x))] &= E[\exp(-uY_\infty)]E[\exp(-vZ_\infty)] \\ &= \lim_{\min\{x,t\}\to\infty} E[\exp(-uY(t))]E[\exp(-vZ(x))],\end{aligned}$$

for $u,v\in\mathbb{R}_+$. $(Y(t),Z(x))$ converges weakly to the law (Y_∞,Z_∞) determined by the Laplace transform

$$E[\exp(-uY_\infty-vZ_\infty)] = \frac{\gamma}{\gamma+v}\cdot\frac{\phi(v)}{\phi(u)}, \qquad \text{for all} \quad u,v\in\mathbb{R}_+, \tag{1.2}$$

where ϕ is the Laplace exponent of the ascending ladder-height subordinator of X which satisfies $\phi(0)>0$. In particular, the law of the sum $Y_\infty+Z_\infty$ is exponential with mean $1/\gamma$.

We turn next to the weak asymptotics of the triplet $(Y(t),\ Z(x),\ m(t))$. To avoid degeneracies we specify the centering curve to be given by

$$y^*(t)=\gamma^{-1}\log(t), \qquad t\in\mathbb{R}_+\backslash\{0\}. \tag{1.3}$$

This choice is informed by Iglehart [12], where in the analogous random walk setting $x(n)=\gamma^{-1}\log n$ was chosen as centering sequence, and by the main result in Doney & Maller [7], which implies that the running maximum $m(t)$ of Y after centering by the curve $y^*(t)$ given in (1.3) converges weakly to a Gumbel distribution (see [8, Chapter 3] for the form of the Gumbel distribution and Section 2.2 below for a simple derivation of the distribution of m_∞ deploying [7, Theorem 1]). A question of interest is if and when the asymptotic independence of $Y(t)$ and $Z(x)$ extends to that of the triplet $Y(t)$, $Z(x)$ and $m(t)$. A priori, it appears unlikely that $Z(x)$ and $m(t)$ are asymptotically independent in general, for x and t tending to infinity in an arbitrary way. In the next result we give a sufficient condition for such asymptotic independence to hold, namely that $\min\{x,t\}\to\infty$ such that

$$x-y^*(t)\to\infty, \qquad \text{or equivalently } t\exp\{-\gamma x\}\to 0. \tag{1.4}$$

Since, by [7], the process Y^* has weakly convergent random fluctuations around the deterministic curve y^*, the assumption $x-y^*(t)\to\infty$ in effect forces the process Y to reach the level x for the first time after time t. The result is as follows.

Theorem 2. *Let $\min\{t,x\}\to\infty$ such that $t\exp\{-\gamma x\}\to 0$. Then $(Y(t),Z(x),m(t))$ converges weakly and the law of the weak limit $(Y_\infty,Z_\infty,m_\infty)$ is determined by the Fourier-Laplace transform*

$$\begin{aligned}&E\left[\exp(-uY_\infty-vZ_\infty+\mathrm{i}\beta m_\infty)\right] \\ &\quad=\frac{\gamma}{\gamma+v}\cdot\frac{\phi(v)}{\phi(u)}\cdot\Gamma\left(1-\frac{\mathrm{i}\beta}{\gamma}\right)\cdot\exp\left[\mathrm{i}\beta\gamma^{-1}\log\left(\ell C_\gamma\widehat{\phi}(\gamma)\right)\right]\end{aligned} \tag{1.5}$$

for all $u,v\in\mathbb{R}_+$, $\beta\in\mathbb{R}$, where $\widehat{\phi}$ is the Laplace exponent of the decreasing ladder-height process, $\widehat{L}^{-1}$ is the decreasing ladder-time processes with $\ell\doteq 1/E[\widehat{L}^{-1}(1)]$ (see Section 2 for the definitions of $\widehat{\phi}$ and $\widehat{L}^{-1}$), $\Gamma(\cdot)$ denotes the gamma function and the constant C_γ is given by

$$C_\gamma\doteq\frac{\phi(0)}{\gamma\phi'(-\gamma)}, \tag{1.6}$$

where $\phi'(-\gamma)\in\mathbb{R}_+\backslash\{0\}$. In particular, $Y(t)$, $Z(x)$ and $m(t)$ are asymptotically independent: for any $a,b\in\mathbb{R}_+$ and $c\in\mathbb{R}$

$$P(Y(t)\le a,Z(x)\le b,m(t)\le c)=P(Y(t)\le a)P(Z(x)\le b)P(m(t)\le c)+o(1).^1$$

The remainder of the paper is devoted to the proofs of Proposition 8 (in which the form of the law of the asymptotic overshoot Z_∞ is identified) and Theorems 1 and 2. Section 2 is concerned with preliminary results, Proposition 8 is established in Section 3 and the proof of the asymptotic independence is given in Section 4. The proofs of Theorems 1 and 2 draw on these results and are presented in Section 5.

[1] Here we use the definiton $f(t,x)=o(1)$ if $\lim_{\min\{t,x-y*(t)\}\to\infty} f(t,x)=0$.

2. Preliminaries

In this section we briefly define the setting and collect results that are deployed throughout. We refer to [2, Chapter I] and [16] for background on the fluctuation theory of Lévy processes.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, P)$ be a filtered probability space that carries a Lévy process X satisfying Assumption 1. Here $\Omega \doteq D(\mathbb{R}_+, \mathbb{R})$ is the Skorokhod space of real-valued functions that are right-continuous on $\mathbb{R}_+$ and have left-limits on $\mathbb{R}_+\backslash\{0\}$, X is the coordinate process, $\{\mathcal{F}(t)\}_{t\geq 0}$ denotes the completed filtration generated by X, which is right-continuous, and $\mathcal{F}$ is the completed σ-algebra generated by $\{X(t)\}_{t\geq 0}$. For any $x \in \mathbb{R}$ denote by P_x the probability measure on $(\Omega, \mathcal{F})$ corresponding to the Lévy process X shifted by x and let $P_x^{(\gamma)}$ (with $P^{(\gamma)} \doteq P_0^{(\gamma)}$) be the Cramér measure on $(\Omega, \mathcal{F})$, that is, the unique measure such that its restriction to $\mathcal{F}(t)$ is given by

$$P_x^{(\gamma)}(A) \doteq E_x[\mathrm{e}^{\gamma(X(t)-x)}\mathbf{I}_A], \qquad A \in \mathcal{F}(t), \qquad t \in \mathbb{R}_+,$$

where E_x is the expectation under P_x and $\mathbf{I}_A$ is the indicator of A. Under Assumption 1, $P_x^{(\gamma)}$ is a probability measure with $P_x^{(\gamma)}(X(0) = x) = 1$ and the convexity of the Laplace exponent $\theta \mapsto \log E_x[\exp(\theta(X(1) - x))]$ on $[0, \gamma]$ implies

$$E_x[X_1 - x] \in (-\infty, 0), \quad E_x^{(\gamma)}[X(1) - x] \in (0, \infty). \tag{2.1}$$

As $E[X(1)]$ is strictly negative and finite, $X^*(t) \doteq \sup_{0\leq s\leq t} X(s)$ converges almost surely as $t \uparrow \infty$ to $X^*_\infty \doteq \sup_{t\geq 0} X(t)$, which is finite almost surely. Moreover, since $X^*(t)$ and $Y(t)$ have the same distribution for any $t > 0$ (by the duality lemma for Lévy processes—see [2]), it follows that $Y(t)$ converges in distribution to a limit Y_∞ that has the same distribution as X^*_∞. The distribution of the latter can be expressed explicitly in terms of that of the ladder-height process H of X, as we recall below.

Let L be a local time[2] at zero of the reflected process $\widehat{Y} = \{\widehat{Y}(t)\}_{t\geq 0}$ of the dual process $\widehat{X} \doteq -X$, that is, $\widehat{Y}(t) \doteq X^*(t) - X(t)$. The ladder-time process $L^{-1} = \{L^{-1}(t)\}_{t\geq 0}$ is equal to the right-continuous inverse of L. Analogously, let $\widehat{L}$ be a local time of Y at zero, with inverse denoted by $\widehat{L}^{-1}$. Denote by $\kappa(q) \doteq -\log E[\exp\{-qL^{-1}(1)\}\mathbf{I}_{\{L^{-1}(1)<\infty\}}]$ and $\widehat{\kappa}(q) \doteq -\log E[\exp\{-q\widehat{L}^{-1}(1)\}\mathbf{I}_{\{\widehat{L}^{-1}(1)<\infty\}}]$ the Laplace exponents of L^{-1} and $\widehat{L}^{-1}$. For later reference we record that the mean of $\widehat{L}^{-1}(1)$ is finite.

Lemma 3. *We have* $E^{(\gamma)}[L^{-1}(1)] \in \mathbb{R}_+\backslash\{0\}$ *and*

$$1/\ell \doteq E[\widehat{L}^{-1}(1)] = 1/\kappa(0) \in \mathbb{R}_+\backslash\{0\}. \tag{2.2}$$

Proof. From the Wiener-Hopf factorisation of X (see e.g. [16, p. 166]), we have that for some $k \in \mathbb{R}_+\backslash\{0\}$

$$q = k\,\kappa(q)\,\widehat{\kappa}(q), \qquad q \in \mathbb{R}_+\backslash\{0\}. \tag{2.3}$$

Since $X_t \to -\infty$ a.s. under P, $\widehat{Y}$ is transient while Y is recurrent, so that we have $P(L^{-1}(1) < \infty) < 1$ and $P(\widehat{L}^{-1}(1) < \infty) = 1$ and $\kappa(0) > 0 = \widehat{\kappa}(0)$. Differentiating (2.3) at $q \in \mathbb{R}_+\backslash\{0\}$ and letting $q \searrow 0$ yields $1 = \kappa(0)\widehat{\kappa}'(0) \Leftrightarrow E[\widehat{L}_1^{-1}] = 1/\kappa(0)$. By a similar argument it follows that $E^{(\gamma)}[L_1^{-1}] = 1/\widehat{\kappa}^{(\gamma)}(0) < \infty$, where $\widehat{\kappa}^{(\gamma)}$ denotes the Laplace exponent of L^{-1} under $P^{(\gamma)}$. □

By the strong law of large numbers and the fact that $\widehat{L}^{-1}$ is a Lévy process (see e.g. [2, p.92]) we have that

$$\frac{\widehat{L}(t)}{t} \sim \frac{t}{\widehat{L}^{-1}(t)} \sim \ell \quad \text{a.s. as } t \to \infty, \tag{2.4}$$

where we denote $f(x) \sim g(x)$ as $x \uparrow \infty$ if the functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+\backslash\{0\}$ satisfying $\lim_{x\uparrow\infty} f(x)/g(x) = 1$.

The ladder-height process $H = \{H(t)\}_{t\geq 0}$ is given by $H(t) \doteq X(L^{-1}(t))$ for all $t \geq 0$ with $L^{-1}(t)$ finite and by $H(t) \doteq +\infty$ otherwise. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be the Laplace exponent of H, with

$$\phi(\theta) \doteq -\log E[\mathrm{e}^{-\theta H(1)}\mathbf{I}_{\{H(1)<\infty\}}], \quad \theta \in \mathbb{R}_+.$$

[2] In the case 0 is not regular for $[0, \infty)$, only a finite number of maxima of X are attained in any compact time interval. In this case we work with the right-continuous version of local time L.

The Lévy-Khintchine formula for ϕ and an integration-by-parts imply

$$\phi(v) = \phi(0) + \int_{\mathbb{R}_+\backslash\{0\}} (1 - \mathrm{e}^{-vx})\nu_H(\mathrm{d}x) = \phi(0) + v\left(m + \int_0^\infty \mathrm{e}^{-vx}\overline{\nu}_H(x)\,\mathrm{d}x\right), \quad v \in \mathbb{R}_+, \tag{2.5}$$

where $\overline{\nu}_H(x) \doteq \nu_H((x,\infty))$ for $x \in \mathbb{R}_+\backslash\{0\}$ is the tail-function of ν_H, the Lévy measure of H, and

$$m \doteq \lim_{u\to\infty} \phi(u)/u$$

denotes the drift of the ladder-height process H. Note that

$$\phi(0) = \kappa(0) \in \mathbb{R}_+\backslash\{0\}. \tag{2.6}$$

The decreasing ladder-height process $\widehat{H}$ is defined similarly, and its Laplace exponent is denoted by $\widehat{\phi}$. By analytical continuation, continuity and Assumption 1(i), the domains of definition of $\phi(\theta)$, $\widehat{\phi}(\theta)$ and the characteristic exponent $\Psi(\theta) = -\log E[\exp\{\mathbf{i}\theta X(1)\}]$, $\theta \in \mathbb{R}$, can be extended to $\{\theta \in \mathbb{C} : \Re(\theta) \in (-\gamma,\infty)\}$, $\{\theta \in \mathbb{C} : \Re(\theta) \in [0,\infty)\}$ and $\{\theta \in \mathbb{C} : \Im(\theta) \in (-\gamma, 0]\}$, respectively. Denoting this extensions again by ϕ, $\widehat{\phi}$ and Ψ, the Wiener-Hopf factorisation of X [2, p. 166] implies that the following holds for some $k' \in \mathbb{R}_+\backslash\{0\}$:

$$\Psi(-\mathbf{i}\theta) = k'\phi(-\theta)\widehat{\phi}(\theta), \qquad \theta \in \mathbb{C},\ \Re(\theta) = 0. \tag{2.7}$$

By uniqueness of analytical extension, the validity of (2.7) extends to all $\theta \in \mathbb{C}$ with $\Re(\theta) \in (-\gamma, 0]$. From (2.7) we have in particular that the law of X^*_∞ and hence of Y_∞ is characterised (see [2, p. 163]) by its Laplace transform

$$E[\mathrm{e}^{-uY_\infty}] = E[\mathrm{e}^{-uX^*_\infty}] = \frac{\phi(0)}{\phi(u)}, \quad u \in \mathbb{R}_+. \tag{2.8}$$

Furthermore, denoting by $K(x) \doteq X(T(x)) - x$ the overshoot of X at $T(x)$ on the event $\{T(x) < \infty\}$, the second Wiener-Hopf factorisation of X (see e.g. [2, p.183]) implies

$$\int_0^\infty q\mathrm{e}^{-qx} E\left[\mathrm{e}^{-uK(x)}\right] \mathrm{d}x = \frac{q}{\phi(q)} \cdot \frac{\phi(q) - \phi(u)}{q - u}, \qquad q, u > 0, \tag{2.9}$$

As the identity (2.7) holds for any $\theta \in \mathbb{C}$ with $\Re(\theta) \in (-\gamma, 0]$, we may define $\phi(-\gamma) \doteq \lim_{\theta\searrow-\gamma} \Psi(-\mathbf{i}\theta)/k'\widehat{\phi}(\theta)$. The values of $\phi(-\gamma)$ and the right-derivative $\phi'(-\gamma)$ of ϕ at $x = -\gamma$ are given as follows:

Lemma 4. *The function ϕ is right-differentiable at $-\gamma$ and we have*

$$\text{(i) } \phi(-\gamma) = 0, \qquad \text{(ii) } \phi'(-\gamma) = E^{(\gamma)}[H(1)] \in \mathbb{R}_+\backslash\{0\}. \tag{2.10}$$

Furthermore, the Laplace exponent $\phi^{(\gamma)}$ of H under $P^{(\gamma)}$ satisfies $\phi^{(\gamma)}(\gamma + u) = \phi(u)$ for any $u \geq -\gamma$ and

$$\int_0^\infty \mathrm{e}^{\gamma y}\overline{\nu}_H(y)\mathrm{d}y = \frac{\phi(0)}{\gamma} - m. \tag{2.11}$$

Proof. **(i)** As $\widehat{H}$ is a non-zero subordinator we have $\widehat{\phi}(\gamma) > 0$ and hence $\phi(-\gamma) = 0$ from (2.7) with $\theta = \gamma$.
(ii) The concavity of ϕ, part (i) and (2.6) imply that the right-derivative $\phi'(-\gamma)$ is strictly positive, and equal to $\phi'(-\gamma) = E[\mathrm{e}^{\gamma H(1)}H(1)] = E^{(\gamma)}[H(1)]$. We next show that $\phi'(-\gamma)$ is finite. As $\widehat{\phi}(\gamma)$ and $\widehat{\phi}'(\gamma)$ are strictly positive and finite, (left-)differentiation of (2.7) at $\theta = \gamma$ yields, by deploying by part (i) and the fact $E[\mathrm{e}^{\gamma X(1)}] = 1$,

$$-E[X(1)\mathrm{e}^{\gamma X(1)}] = k\phi(-\gamma)\widehat{\phi}'(\gamma) - k\phi'(-\gamma)\widehat{\phi}(\gamma) = -k\phi'(-\gamma)\widehat{\phi}(\gamma). \tag{2.12}$$

As the left-hand side of (2.12) is finite (by Assumption 1(ii)) it follows that $\phi'(-\gamma) < \infty$. By a change-of-measure argument (see [16, Corollary 3.10]) and (2.10)(i) it follows that $\phi^{(\gamma)}(u) = \phi(u+\gamma) - \phi(-\gamma) = \phi(u+\gamma)$ for $u \geq 0$, while part (i) and (2.5) yield (2.11). □

2.1. **Asymptotic first-passage probabilities and overshoot distributions.** We next turn to asymptotics of first-passage probabilities and the distribution of overshoots for large starting values of X. Let $T(x)$ and $\widehat{T}(x)$ denote the first-passage times of X into the intervals (x,∞) and $(-\infty,-x)$ respectively for any $x\in\mathbb{R}_+$,

$$T(x) \doteq \inf\{t\geq 0 : X(t)\in(x,\infty)\}, \qquad \widehat{T}(x) \doteq \inf\{t\geq 0 : X(t)\in(-\infty,-x)\}. \tag{2.13}$$

It is shown in [3] that, under Assumption 1, Cramér's estimate, which was first-established for random walks, remains valid for the Lévy process X (with C_γ defined in (1.6)):

$$P(T(y)<\infty)\sim C_\gamma \mathrm{e}^{-\gamma y} \quad \text{as } y\to\infty. \tag{2.14}$$

From (2.14) the following asymptotic results can be derived:

Proposition 5. **(i)** *(Asymptotic two-sided exit probability) For any $z>0$ we have*

$$P(T(x)<\widehat{T}(z))\sim C_\gamma \mathrm{e}^{-\gamma x}\left(1-E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\right]\right) \qquad \text{as} \quad x\to\infty, \tag{2.15}$$

where C_γ is given in (1.6).
(ii) *(Asymptotic overshoot) Let $u\in\mathbb{R}_+$ and fix $z>0$. Then $E^{(\gamma)}[\mathrm{e}^{-uK(\infty)}]=\phi(u-\gamma)/(u\phi'(-\gamma))$ and we have*

$$E\left[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\widehat{T}(z)\}}\right] \;\sim\; C(u)\mathrm{e}^{-\gamma x}\left(1-E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\right]\right), \quad \text{as } x\to\infty, \text{ with} \tag{2.16}$$
$$C(u) \;\doteq\; \frac{\gamma}{\gamma+u}\cdot\frac{\phi(u)}{\phi(0)}\cdot C_\gamma.$$

Proof. **(i)** By the strong Markov property and spatial homogeneity of X it follows from (2.14) that

$$P(T(x)<\widehat{T}(z)) = P(T(x)<\infty) - \int_{(-\infty,-z]} P_y(T(x)<\infty)P(X(\widehat{T}(z))\in \mathrm{d}y, \widehat{T}(z)<T(x)). \tag{2.17}$$

The translation invariance of X and Cramér's estimate imply the following equality

$$P_y(T(x)<\infty) \;=\; C_\gamma \mathrm{e}^{-\gamma x}\mathrm{e}^{\gamma y}\left(1+r(x-y)\right) \quad \text{for all} \quad x>y, \tag{2.18}$$

where $\lim_{x'\to\infty} r(x')=0$. Equality (2.18) applied to the identity in (2.17) yields

$$\begin{aligned} C_\gamma^{-1}\mathrm{e}^{\gamma x}P(T(x)<\widehat{T}(z)) \;&=\; 1-E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}\right] \\ &+\; r(x)-E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}r(x-X(\widehat{T}(z)))\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}\right]. \end{aligned} \tag{2.19}$$

Since $X(\widehat{T}(z))\leq -z<0$ on the event $\{\widehat{T}(z)<\infty\}$, which satisfies $P(\widehat{T}(z)<\infty)=1$ by Assumption 1, the dominated convergence theorem implies $E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\right]=E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}\right]+o(1)$ as $x\to\infty$. An application of the dominated convergence theorem to the second expectation on the right-hand side of equality (2.19), together with the fact that r vanishes in the limit as $x\to\infty$, proves the first statement in the proposition.
(ii) Since by Assumption 1 and Lemma 4 H is a non-lattice subordinator with $E^{(\gamma)}\left[H(1)\right]\in\mathbb{R}_+\backslash\{0\}$ and the overshoot $K(x)$ is equal to that of H over x, [4, Theorem 1] implies that the weak limit $K(x)\xrightarrow{\mathcal{D}}K_\infty$, as $x\to\infty$, exists under $P^{(\gamma)}$. Hence the continuity theorem [5, p. 16, Theorem 2.1] implies $\lim_{x\uparrow\infty}E^{(\gamma)}[\mathrm{e}^{-uK(x)}]=E^{(\gamma)}[\mathrm{e}^{-uK_\infty}]$ for any fixed $u\geq 0$. Combining this limit, which is bounded, with the second Wiener-Hopf factorisation (2.9) under the measure $P^{(\gamma)}$, which reads as

$$\int_0^\infty q\mathrm{e}^{-qx}E^{(\gamma)}\left[\mathrm{e}^{-uK(x)}\right]\,\mathrm{d}x=\frac{q}{\phi(q-\gamma)}\cdot\frac{\phi(q-\gamma)-\phi(u-\gamma)}{q-u}, \qquad q,u>0,$$

we have in the limit as $q\downarrow 0$ that $E^{(\gamma)}[\mathrm{e}^{-uK_\infty}]=\phi(u-\gamma)/(u\phi'(-\gamma))$. Thus, changing measure from P to $P^{(\gamma)}$ leads to the following for any $u\geq 0$ ($C(u)$ is defined in (2.16)):

$$E[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\infty\}}]=\mathrm{e}^{-\gamma x}\cdot E^{(\gamma)}[\mathrm{e}^{-(\gamma+u)K(x)}]\sim C(u)\mathrm{e}^{-\gamma x} \quad \text{as } x\to\infty. \tag{2.20}$$

Furthermore, since the expectation in (2.20) is bounded as $x \to \infty$, there exists a bounded function $R : \mathbb{R}_+ \to \mathbb{R}$, such that $E[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\infty\}}] = C(u)\mathrm{e}^{-\gamma x}(1 + R(x))$ for $x > 0$, and $\lim_{x\to\infty} R(x) = 0$. The strong Markov property at $\widehat{T}(z)$ and an argument analogous to the one used in the proof of part (i) (cf. (2.19)) yields

$$\begin{aligned} & C(u)^{-1}\mathrm{e}^{\gamma x}E[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\widehat{T}(z)\}}] \\ = \;& 1 - E[\mathrm{e}^{\gamma X(\widehat{T}(z))}\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}] + R(x) - E[\mathrm{e}^{\gamma X(\widehat{T}(z))}R(x - X(\widehat{T}(z)))\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}], \end{aligned}$$

which implies equivalence (2.16). □

2.2. **Asymptotic distribution of $m(t)$.** To establish the existence and forms of the asymptotic distribution of $m(t)$ as t tend to infinity we draw on excursion theory. We refer to [2, Chapters O.5, IV], [16, Chapter 6] and [6, 9] for treatments of excursion theory.

Let $\epsilon = \{\epsilon_t\}_{t\geq 0}$ denote the excursion process of Y away from zero, with $\epsilon_t \in \mathcal{E} = \{\varepsilon \in \Omega : \varepsilon \geq 0\} \cup \{\partial\}$, where ∂ denotes an isolated state and where ϵ_t we recall is given by

$$\epsilon_t \doteq \begin{cases} \left.\begin{cases} Y\left(s + \widehat{L}^{-1}(t-)\right), & s \in [0, \widehat{L}^{-1}(t) - \widehat{L}^{-1}(t-)) \\ 0, & s \geq \widehat{L}^{-1}(t) - \widehat{L}^{-1}(t-) \end{cases}\right\}, & \text{if } \widehat{L}^{-1}(t-) < \widehat{L}^{-1}(t), \\ \partial, & \text{otherwise.} \end{cases} \tag{2.21}$$

Since Y is a recurrent strong Markov process under P, Itô's characterisation [13] implies that ϵ is a Poisson point process under P, with intensity (or excursion) measure n defined on $(\mathcal{E}, \mathcal{G})$, where $\mathcal{G}$ is the Borel sigma-algebra on the Polish space $\mathcal{E}$. Under $P^{(\gamma)}$, Y is transient (as $E^{(\gamma)}[X(1)] > 0$), so that $\widehat{L}_\infty \doteq \lim_{t\in\infty} \widehat{L}(t)$ is finite almost surely and furthermore $\widehat{L}_\infty$ is an exponential random variable independent of the killed Lévy process $\{(\widehat{L}^{-1}(t), \widehat{H}(t))\}_{t\in[0,\widehat{L}_\infty)}$. The process $\epsilon' = \{\epsilon'_t\}_{t\geq 0}$, defined by $\epsilon'_t \doteq \epsilon_t$ for $t < \widehat{L}_\infty$ and by $\epsilon'_t \doteq \partial$ otherwise, is under $P^{(\gamma)}$ a Poisson point process killed at an independent exponential random time with mean $E^{(\gamma)}[\widehat{L}_\infty]$; we denote by $n^{(\gamma)}$ its intensity measure.

For an excursion $\varepsilon \in \mathcal{E}$ and $x \in \mathbb{R}_+\backslash\{0\}$ let $\rho(x, \varepsilon)$ and $\zeta(\varepsilon)$ be the first time that ε enters the interval (x, ∞) and the lifetime of ε respectively:

$$\rho(x, \varepsilon) \doteq \inf\{s \in \mathbb{R}_+ : \varepsilon(s) > x\}, \qquad \zeta(\varepsilon) \doteq \inf\{t \in \mathbb{R}_+\backslash\{0\} : \varepsilon(t) = 0\}. \tag{2.22}$$

For brevity we sometimes write $\rho(x)$ and ζ instead of $\rho(x, \varepsilon)$ and $\zeta(\varepsilon)$. Note that $\zeta(\epsilon(t))$ is given in terms of $\widehat{L}^{-1}$ by $\zeta(\epsilon(t)) = \widehat{L}^{-1}(t) - \widehat{L}^{-1}(t-)$ for any $t \in \mathbb{R}_+$ (with $\widehat{L}^{-1}(0-) \doteq 0$).

The distribution of $\widehat{L}(\tau(x))$ can be expressed in terms of n as follows.

Lemma 6. *For any $x \in \mathbb{R}_+\backslash\{0\}$ the random variable $\widehat{L}(\tau(x))$ is exponentially distributed under P (resp. $P^{(\gamma)}$) with parameter $n(\rho(x) < \zeta)$ (resp. $n^{(\gamma)}(\rho(x) < \zeta)$).*

Proof. The definition of the first-passage time $\rho(x, \varepsilon)$ in (2.22) implies the equality $\widehat{L}(\tau(x)) = T_A \doteq \inf\{t \geq 0 : \epsilon(t) \in A\}$ where $A \doteq \{\varepsilon \in \mathcal{E} : \rho(x, \varepsilon) < \zeta(\varepsilon)\}$. The statement follows since T_A is exponentially distributed with parameter $n(A)$ (e.g. [16, Lemma 6.18(i)]). □

In [7, Theorem 1][3] it is shown that the following version of Cramér's estimate holds under under the excursion measure n (with C_γ defined in (1.6)):

$$n(\rho(x) < \zeta) \sim C_\gamma\, \widehat{\phi}(\gamma)\, \mathrm{e}^{-\gamma x} \qquad \text{as} \quad x \to \infty. \tag{2.23}$$

Using the estimate (2.23) the asymptotic distribution of $m(t)$ may be identified as follows:

Proposition 7. *If $t \to \infty$ then $m(t)$ converges in distribution to m_∞, which follows a Gumbel distribution,*

$$P(m_\infty < z) = \exp\left(-\ell\, C_\gamma\, \widehat{\phi}(\gamma)\, \mathrm{e}^{-\gamma z}\right), \qquad \textit{for all } z \in \mathbb{R}, \tag{2.24}$$

where $\ell = 1/\phi(0)$ and C_γ is given in (1.6).

[3][7, Theorem 1] is established under the same hypotheses as in Assumption 1. In particular, the condition $E[|X(1)|] < \infty$ is used in the proof of [7, Theorem 1].

Proof. We give a short proof of (2.24) based on [7, Theorem 1]. To establish (2.24) we show that the following holds if $\min\{x,t\}\to\infty$ and $t\mathrm{e}^{-\gamma x}\to 1$:

$$P(Y^*(t)-x<z) \;=\; \exp(-t\,\ell\, n(\rho(x+z)<\zeta))+o(1)\quad \text{for any } z\in\mathbb{R}. \tag{2.25}$$

Since (2.23) implies $tn(\rho(x+z)<\zeta)\to C_\gamma\widehat{\phi}(\gamma)\mathrm{e}^{-\gamma z}$ as $\min\{x,t\}\to\infty$ and $t\mathrm{e}^{-\gamma x}\to 1$, the limit in (2.24) follows from (2.25). To complete the proof we now verify the claim in (2.25). Note that as $\tau(x)\to\infty$ P-a.s. as $x\to\infty$, the law of large numbers implies that $\widehat{L}(\tau(x))/\tau(x)\to\ell$ P-a.s. as $t\to\infty$, where $\ell\in\mathbb{R}_+\backslash\{0\}$ (by (2.2), (2.4) and (2.6)). In particular, for any $\delta>0$ and $z\in\mathbb{R}_+$, we have $P(\widehat{L}(\tau(x+z))/\tau(x+z)\in(\ell-\delta,\ell+\delta))=1+o(1)$ as $x\to\infty$. Hence as $\min\{x,t\}\to\infty$ we have

$$\begin{aligned}P(Y^*(t)<x+z) &= P(\tau(x+z)>t,\widehat{L}(\tau(x+z))/\tau(x+z)\geq\ell-\delta)+o(1)\\ &\leq P(\widehat{L}(\tau(x+z))>t(\ell-\delta))+o(1).\end{aligned}$$

Similarly, it follows that as $\min\{x,t\}\to\infty$ we have

$$\begin{aligned}P(Y^*(t)<x+z) &\geq P(\widehat{L}(\tau(x+z))>\widehat{L}(t),\widehat{L}(t)\leq t(\ell+\delta))\\ &\geq P(\widehat{L}(\tau(x+z))>t(\ell+\delta),\widehat{L}(t)\leq t(\ell+\delta))=P(\widehat{L}(\tau(x+z))>t(\ell+\delta))+o(1).\end{aligned}$$

By Lemma 6, $\widehat{L}(\tau(x+z))$ is exponentially distributed with parameter $n(\rho(x+z)<\zeta)$ and hence we find

$$\exp(-(\ell+\delta)t\,n(\rho(x+z)<\zeta))+o(1)\leq P(Y^*(t)<x+z)\leq\exp(-(\ell-\delta)t\,n(\rho(x+z)<\zeta))+o(1).$$

Since this result holds for any $\delta>0$, the equality in (2.25) follows. □

3. Limiting overshoot of the reflected process

In this section we prove the following result, which also plays a role in the proofs of Theorems 1 and 2.

Proposition 8. (i) *The weak limit Z_∞ of $Z(x)$ as $x\to\infty$ has Laplace transform*

$$E[\mathrm{e}^{-vZ_\infty}]=\frac{\gamma}{\gamma+v}\cdot\frac{\phi(v)}{\phi(0)}\qquad \textit{for all}\quad v\in\mathbb{R}_+. \tag{3.1}$$

(ii) *The law of the asymptotic overshoot Z_∞ is given by*

$$P(Z_\infty>x)=\frac{\gamma}{\phi(0)}\mathrm{e}^{-\gamma x}\int_x^\infty \mathrm{e}^{\gamma y}\,\overline{\nu}_H(y)\,\mathrm{d}y,\quad x\in\mathbb{R}_+,\quad \textit{and}\quad P(Z_\infty=0)=\frac{\gamma}{\phi(0)}m. \tag{3.2}$$

Remark. Note that Z_∞ is continuous on $\mathbb{R}_+\backslash\{0\}$ and has a non-zero atom at zero precisely if the drift m of H is strictly positive, in which case the probability of creeping of X is strictly positive if X is not a compound Poisson process (see e.g. [16, Lemma 7.10]). In fact, as shown in [10], $\frac{\gamma}{\phi(0)}m$ is equal to the asymptotic (conditional) probability $\lim_{x\to\infty}P(X(T(x))=x|T(x)<\infty)$ of creeping of X.

The formula in (3.1) of Proposition 8, which characterises the law of the limiting overshoot Z_∞, is implied by the main result in [18]. As this formula constitutes a key step in the proofs of Theorems 1 and 2, we give in this section an independent proof of Proposition 8 based on excursion theory alone. This approach is in the spirit of the present paper and should be contrasted with the result in [18], which crucially relies on the renewal theorem.

The proof relies on the expression of the distribution of $Z(x)$ in terms of the excursion measure n and on representation results for random variables $K_F(x)$ that are defined for $x\in\mathbb{R}_+\backslash\{0\}$ and Borel-measurable and non-negative functions $F:\mathcal{E}\to\mathbb{R}$ by

$$K_F(x)\doteq\sum_g F(\epsilon_{\widehat{L}(g)})\mathbf{I}_{\{g\leq\tau(x)<g+\zeta(\epsilon_{\widehat{L}(g)})\}}, \tag{3.3}$$

where the sum runs over all left-end points g of excursion intervals. We write $n(F)=n(F(\varepsilon))\doteq\int_{\mathcal{E}}F(\varepsilon)\,n(\mathrm{d}\varepsilon)$ for any Borel-measurable non-negative (or integrable) functional $F:\mathcal{E}\to\mathbb{R}$. In this notation we have $n(A)=n(\mathbf{I}_A)$ for any $A\in\mathcal{G}$.

Lemma 9. **(i)** *We have*

$$P(Z(x)>y) \quad = \quad n(\varepsilon(\rho(x,\varepsilon))-x>y|\rho(x)<\zeta) \qquad \text{for any} \quad x,y\in\mathbb{R}_+,$$

where $n(B|A)\doteq n(B\cap A)/n(A)$ *for any* $A,B\in\mathcal{G}$ *with* $n(A)\in\mathbb{R}_+\backslash\{0\}$.
(ii) *Define* $\widehat{\mathcal{V}}(x)\doteq E\left[\widehat{L}(\tau(x))\right]$ *and* $\widehat{\mathcal{V}}^{(\gamma)}(x)\doteq E^{(\gamma)}\left[\widehat{L}(\tau(x))\right]$ *and let* $G(\varepsilon)\doteq F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}$. *Then the following hold:*

$$n(G) \quad = \quad \widehat{\mathcal{V}}(x)^{-1}E\left[K_F(x)\right], \qquad n^{(\gamma)}(G)=\widehat{\mathcal{V}}^{(\gamma)}(x)^{-1}E^{(\gamma)}\left[K_F(x)\right]. \tag{3.4}$$

(iii) *The following identity holds:* $n^{(\gamma)}(F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}})=n(\mathrm{e}^{\gamma\varepsilon(\rho(x,\varepsilon))}F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}})$. *Hence we have*

$$n^{(\gamma)}(\rho(x,\varepsilon)<\zeta(\varepsilon))=n(\mathrm{e}^{\gamma\varepsilon(\rho(x,\varepsilon))}\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}). \tag{3.5}$$

(iv) *For any* $z\in\mathbb{R}_+\backslash\{0\}$ *we have as* $x\to\infty$:

$$n^{(\gamma)}(\rho(x,\varepsilon)<\zeta(\varepsilon))\sim\widehat{\phi}(\gamma) \qquad \text{and} \qquad \mathrm{e}^{\gamma x}n(\varepsilon(\rho(z,\varepsilon))>x,\rho(z,\varepsilon)<\zeta(\varepsilon))=o(1). \tag{3.6}$$

Proof of Lemma 9. **(i)** The assertion is a consequence of the fact that $\epsilon(T_A)$ follows an n-uniform distribution (that is, $P(\epsilon(T_A)\in B)=n(B|A)$ for any $B\in\mathcal{G}$, see e.g. [2, Sec. O.5, Proposition O.2]) and taking B to be equal to $\{\varepsilon\in\mathcal{E}:\rho(x,\varepsilon)<\zeta(\varepsilon),\varepsilon(\rho(x,\varepsilon))-x>y\}$.
(ii) As the proof of the two identities (3.4) is identical, we derive only the left-hand side of (3.4). Since for every $\varepsilon\in\mathcal{E}$ the process $t\to F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}\mathbf{I}_{\{\widehat{L}^{-1}(t-)\leq\tau(x)\}}$ is left-continuous and $\mathcal{F}(\widehat{L}^{-1}(t-))$-adapted, an application of the compensation formula to the Poisson point process ϵ (see e.g. [2, Chapter O.5] or [15]) yields

$$\begin{aligned}E[K_F(x)] &= E\left[\sum_g \mathbf{I}_{\{g\leq\tau(x)\}}\cdot F(\epsilon_{\widehat{L}(g)})\mathbf{I}_{\{\tau(x)-g<\zeta(\epsilon_{\widehat{L}(g)})\}}\right]\\ &= E\left[\sum_{t\geq0}\mathbf{I}_{\{\widehat{L}^{-1}(t-)\leq\tau(x)\}}\cdot\left\{F(\epsilon_t)\mathbf{I}_{\{\tau(x)-\widehat{L}^{-1}(t-)<\zeta(\epsilon_t\}}\right\}\right]=I_1\cdot I_2,\end{aligned} \tag{3.7}$$

where $I_2=n(F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}})$ and

$$\begin{aligned}I_1 &= E\left[\int_0^\infty\mathbf{I}_{\{\widehat{L}^{-1}(t-)\leq\tau(x)\}}\mathrm{d}t\right]=E\left[\int_0^\infty\mathbf{I}_{\{t\leq\widehat{L}(\tau(x))\}}\mathrm{d}t\right]\\ &= E\left[\widehat{L}(\tau(x))\right]=\widehat{\mathcal{V}}(x),\end{aligned} \tag{3.8}$$

where we used that $\{\widehat{L}^{-1}(t-)\leq\tau(x)\}=\{t\leq\widehat{L}(\tau(x))\}$. Inserting I_1 and I_2 in (3.7) and dividing by I_1 yields the left-hand side in (3.4).
(iii) Another application of the compensation formula yields

$$\begin{aligned}E^{(\gamma)}\left[K_G(x)\right] &= E\left[\mathrm{e}^{\gamma X(\tau(x))}\sum_g F(\epsilon_{\widehat{L}(g)})\mathbf{I}_{\{g\leq\tau(x)<g+\zeta(\epsilon_{\widehat{L}(g)})\}})\right]\\ &= E\left[\sum_g \mathrm{e}^{\gamma X(g)}\mathbf{I}_{\{g\leq\tau(x)\}}\cdot\mathrm{e}^{\gamma\epsilon_{\widehat{L}(g)}(\rho(x,\epsilon_{\widehat{L}(g)}))}\mathbf{I}_{\{\rho(x,\epsilon_{\widehat{L}(g)})<\zeta(\epsilon_{\widehat{L}(g)})\}}\right]\\ &= E\left[\sum_{t\geq0}\left\{\mathrm{e}^{\gamma X(\widehat{L}^{-1}(t-))}\mathbf{I}_{\{\widehat{L}^{-1}(t-)\leq\tau(x)\}}\right\}\cdot\mathrm{e}^{\gamma\epsilon_t(\rho(x,\epsilon_t))}\mathbf{I}_{\{\rho(x,\epsilon_t)<\zeta(\epsilon_t)\}}\right]=J_1\cdot J_2,\end{aligned} \tag{3.9}$$

where $J_2=n\left(\mathrm{e}^{\gamma\varepsilon(\rho(x,\varepsilon))}F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}\right)$ and, by an application of Fubini's theorem,

$$\begin{aligned}J_1 &= E\left[\int_0^\infty\mathrm{e}^{\gamma X(\widehat{L}^{-1}(t-))}\mathbf{I}_{\{\widehat{L}^{-1}(t-)\leq\tau(x)\}}\mathrm{d}t\right]=E^{(\gamma)}\left[\int_0^\infty\mathbf{I}_{\{\widehat{L}^{-1}(t-)\leq\tau(x)\}}\mathrm{d}t\right]\\ &= E^{(\gamma)}[\tau(x)]=\widehat{\mathcal{V}}^{(\gamma)}(x).\end{aligned}$$

Combining the right-hand side in (3.4) with (3.9) and the forms of J_1 and J_2 yields the stated identity.

(iv) Since $\widehat{L}(\tau(x))$ under $P^{(\gamma)}$ follows an exponential distribution with mean $1/n^{(\gamma)}(\rho(x) < \zeta))$ (see Lemma 6) we have

$$n^{(\gamma)}(\rho(x) < \zeta) = -\log P^{(\gamma)}(\widehat{L}(\tau(x)) > 1),$$

so that

$$\lim_{x\uparrow\infty} n^{(\gamma)}(\rho(x) < \zeta) = -\log P^{(\gamma)}(\widehat{L}^{-1}(1) < \infty),$$

which is equal to $\widehat{\phi}^{(\gamma)}(0) = \widehat{\phi}(\gamma)$ (as $\widehat{\phi}^{(\gamma)}(u) = \widehat{\phi}(\gamma + u)$, $u \geq 0$). Chebyshev's inequality and part (ii) of the lemma imply

$$\begin{aligned} \mathrm{e}^{\gamma x} n(\varepsilon(\rho(z,\varepsilon)) > x, \rho(z,\varepsilon) < \zeta(\varepsilon)) &\leq n(\mathrm{e}^{\gamma\varepsilon(\rho(z,\varepsilon))}\mathbf{I}_{\{\varepsilon(\rho(z,\varepsilon))>x,\rho(z,\varepsilon)<\zeta(\varepsilon)\}}) \\ &= n^{(\gamma)}(\varepsilon(\rho(z,\varepsilon)) > x, \rho(z,\varepsilon) < \zeta(\varepsilon)). \end{aligned}$$

As the latter tends to zero as $x \uparrow \infty$, the second assertion in (3.6) follows. □

We next apply Lemma 9 to establish the asymptotic behaviour of certain integrals against the excursion measure as $x \to \infty$.

Lemma 10. *Let $u \geq 0$. Then, as $x \to \infty$, we have*

$$n(\mathrm{e}^{-u(\varepsilon(\rho(x))-x)}|\rho(x) < \zeta) \longrightarrow C(u) \cdot C_\gamma^{-1} = \frac{\gamma}{\gamma + u} \cdot \frac{\phi(u)}{\phi(0)}. \tag{3.10}$$

In particular, $Z(x)$ converges weakly to a random variable Z_∞ with Laplace transform $E[\exp(-uZ_\infty)] = C(u) \cdot C_\gamma^{-1}$.

Proof of Lemma 10. Fix $M > 0$ and recall that, under the probability measure $n(\,\cdot\,|\rho(M) < \zeta)$, the coordinate process has the same law as the first excursion of Y away from zero with height larger than M. For any $x > M$, the following identity holds:

$$n(\mathrm{e}^{-u(\varepsilon(\rho(x))-x)}|\rho(x) < \zeta) \quad = \quad n(\mathrm{e}^{-u(\varepsilon(\rho(x))-x)}\mathbf{I}_{\{\rho(x)<\zeta\}}|\rho(M) < \zeta)\frac{n(\rho(M) < \zeta)}{n(\rho(x) < \zeta)}. \tag{3.11}$$

The strong Markov property under the probability measure $n(\,\cdot\,|\rho(M) < \zeta)$, implies that $\varepsilon \circ \theta_{\rho(M)}$ has the same law as the process X with entrance law $n(\varepsilon(\rho(M,\varepsilon)) \in \mathrm{d}z|\rho(M) < \zeta)$ and killed at the epoch of the first passage into the interval $(-\infty, 0]$. We therefore find

$$\begin{aligned} n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}\mathbf{I}_{\{\rho(x)<\zeta\}}|\rho(M) < \zeta) &= n\left(\mathrm{e}^{-u(\varepsilon(\rho(M,\varepsilon))-x)}\mathbf{I}_{\{\varepsilon(\rho(M,\varepsilon))>x\}}|\rho(M) < \zeta\right) \\ + \int_{[M,x]} E_z\left[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\widehat{T}(0)\}}\right] &n(\varepsilon(\rho(M,\varepsilon)) \in \mathrm{d}z|\rho(M) < \zeta). \end{aligned} \tag{3.12}$$

By the second equality in (3.6) of Lemma 9, we have as $x \uparrow \infty$:

$$\mathrm{e}^{\gamma x} n\left(\mathrm{e}^{-u(\varepsilon(\rho(M,\varepsilon))-x)}\mathbf{I}_{\{\varepsilon(\rho(M,\varepsilon))>x\}}\middle|\,\rho(M) < \zeta\right) \leq \mathrm{e}^{\gamma x}\frac{n\left(\varepsilon(\rho(M,\varepsilon)) > x, \rho(M,\varepsilon) < \zeta(\varepsilon)\right)}{n(\rho(M) < \zeta)} = o(1).$$

This estimate, spatial homogeneity of X and equations (3.11) and (3.12) yield as $x \to \infty$:

$$\begin{aligned} &n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x) < \zeta) \\ &\quad= o(1) + \int_{[M,x]} E\left[\mathrm{e}^{-uK(x-z)}\mathbf{I}_{\{T(x-z)<\widehat{T}(z)\}}\right] \frac{n(\varepsilon(\rho(M,\varepsilon)) \in \mathrm{d}z, \rho(M) < \zeta)}{n(\rho(x) < \zeta)}. \end{aligned} \tag{3.13}$$

Formula (2.16) of Proposition 5 implies the following equality:

$$E\left[\mathrm{e}^{-uK(x-z)}\mathbf{I}_{\{T(x-z)<\widehat{T}(z)\}}\right] = C(u)\mathrm{e}^{-\gamma x}\left(1 - G(z) + R(x-z)\right)\mathrm{e}^{\gamma z}, \tag{3.14}$$

where $G, R : \mathbb{R}_+ \to \mathbb{R}$ are bounded functions such that $G(z) = E[\mathrm{e}^{\gamma X(\widehat{T}(z))}]$ and $\lim_{x'\to\infty} R(x') = 0$. Therefore the equality in (3.13), the asymptotic behaviour of $n(\rho(x) < \zeta)$ given in (2.23) and Lemma 9 (ii) imply the following identity as $x \to \infty$:

$$
\begin{aligned}
(3.15)\qquad n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x) < \zeta) &= A_\gamma(u) n^{(\gamma)}(\varepsilon(\rho(M,\varepsilon)) \in [M,x], \rho(M,\varepsilon) < \zeta(\varepsilon)) + o(1) \\
&+ \quad A_\gamma(u) n^{(\gamma)} \left([R(x - \varepsilon(\rho(M,\varepsilon))) - G(\varepsilon(\rho(M,\varepsilon)))] \, I_{\{\varepsilon(\rho(M,\varepsilon))\in[M,x], \rho(M,\varepsilon)<\zeta(\varepsilon)\}} \right),
\end{aligned}
$$

where $A_\gamma(u) \doteq C(u)/(C_\gamma \widehat{\phi}(\gamma))$. By (3.15) the limit $\lim_{x\to\infty} n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x) < \zeta)$ exists and the dominated convergence theorem yields

$$
\lim_{x\to\infty} n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x) < \zeta) = A_\gamma(u) \left(n^{(\gamma)}(\rho(M) < \zeta) - n^{(\gamma)} \left(G(\varepsilon(\rho(M,\varepsilon))) I_{\{\rho(M,\varepsilon)<\zeta(\varepsilon)\}} \right) \right).
$$

Since this equality holds for any $M > 0$ and the left-hand side does not depend on M, if the right-hand side has a limit as $M \to \infty$, then the equality also holds in this limit. Note that (3.6) of Lemma 9 (iii) implies $\lim_{M\to\infty} n^{(\gamma)}(\rho(M) < \zeta) = \widehat{\phi}(\gamma)$. Since $G(\varepsilon(\rho(M,\varepsilon))) \le \mathrm{e}^{-\gamma M}$ on $\{\rho(M,\varepsilon) < \zeta(\varepsilon)\}$, an application of the dominated convergence theorem yields (3.10). By combining with Lemma 9(i) we find the stated form of Laplace transform of Z_∞. □

With the previous results in hand we complete next the proof of Proposition 8.

Proof of Proposition 8. **(i)** Equation (3.1) is established in Lemma 10.
(ii) Straightforward algebra, starting from (3.1), shows that the Laplace transform of $x \mapsto \exp(\gamma x) P(Z_\infty > x)$ is given by

$$
\begin{aligned}
\int_0^\infty \mathrm{e}^{-vx}\mathrm{e}^{\gamma x} P(Z_\infty > x)\mathrm{d}x &= \frac{1}{v-\gamma}\left(1 - \frac{\gamma}{\phi(0)}\frac{\phi(v-\gamma)}{v}\right) \\
&= \frac{\gamma}{\phi(0)}\left[\frac{1}{v}\left(\frac{\phi(0)}{\gamma} - m\right) - \frac{1}{v}\left(\frac{\phi(v-\gamma) - \phi(0) - (v-\gamma)m}{v-\gamma}\right)\right], \quad v > \gamma.
\end{aligned}
$$

A direct Laplace inversion, based on the representation (2.5) of ϕ and (2.11) in Lemma 4, yields the left-hand side of formula (3.2). The atom at zero is obtained by taking the limit in (3.1) of part (i) as $v \to \infty$. □

4. Asymptotic independence

In this section we establish the asymptotic independence of $Y(t)$, $Z(x+y)$ and $M(t,x)$ as $\min\{t,x,y\} \to \infty$, *i.e.* for any $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}$

$$
P(Y(t) \le a, Z(x+y) \le b, M(t,x) \le c) = P(Y(t) \le a) P(Z(x+y) \le b) P(M(t,x) \le c) + o(1),
$$ [4]

where $M(t,x) \doteq Y^*(t) - x$, $t, x \in \mathbb{R}_+$. From this we deduce (see Lemma 14 below) the asymptotic independence of $(Y(t), X(x), m(t))$ as $\min\{t,x\} \to \infty$ and $x - y^*(t) \to \infty$, described in Theorem 2. We start with the following observations concerning the large-time behaviour of the local time $\widehat{L}$:

Lemma 11. *The following statements hold true:*
(i) *As in Theorem 2 denote $\ell = 1/E[\widehat{L}^{-1}(1)]$. For any $\delta \in (0, \ell/2)$ we have*

$$
\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}(\tau(x)) \in t[\ell - \delta, \ell + \delta]) \le \frac{4}{\mathrm{e}\ell}\delta.
$$

(ii) *The following limit holds: $P(\widehat{L}(t) = \widehat{L}(\tau(x))) \longrightarrow 0$ as $\min\{x,t\} \to \infty$;*
(iii) *For any $\delta_1, \delta_2 \in [0, 1/4)$ we have*

$$
(4.1)\qquad \limsup_{\min\{x,t\}\to\infty} P(\widehat{L}(t(1-\delta_1)) \le \widehat{L}(\tau(x)) \le \widehat{L}(t(1+\delta_2))) \le \frac{8}{\mathrm{e}} \max\{\delta_1, \delta_2\}.
$$

For any fixed $s \in \mathbb{R}_+ \backslash \{0\}$ it holds $P(\widehat{L}((t-s) \vee 0) \le \widehat{L}(\tau(x)) < \widehat{L}(t)) \longrightarrow 0$ as $\min\{x,t\} \to \infty$.

[4] Here again $f(t,x,y) = o(1)$ $(\min\{x,y,t\} \to \infty)$ if $\lim_{\min\{t,x,y\}\to\infty} f(t,x,y) = 0$.

Proof of Lemma 11. **(i)** Recall ℓ is finite (Lemma 3). For any $x, t \in \mathbb{R}_+\backslash\{0\}$, Lemma 6 implies $P(\widehat{L}(\tau(x)) > t) = \mathrm{e}^{-t\, n(B(x))}$ for all $t \in \mathbb{R}_+\backslash\{0\}$, where $B(x) \doteq \{\rho(x) < \zeta\}$ with ρ defined in (2.22). Therefore for any $\delta \in (0, \ell/2)$ the following holds:

$$P(\widehat{L}(\tau(x)) \in t[\ell-\delta, \ell+\delta]) = \mathrm{e}^{-t\,\ell\, n(B(x))}\left(\mathrm{e}^{\delta\, t\, n(B(x))} - \mathrm{e}^{-\delta\, t\, n(B(x))}\right).$$

The Mean-Value Theorem implies that there exists $\xi_{t,x} \in (-\delta, \delta)$ such that

$$\begin{aligned} P(\widehat{L}(\tau(x)) \in t[\ell-\delta, \ell+\delta]) &= 2\delta t n(B(x))\mathrm{e}^{(\xi_{t,x}-\ell)tn(B(x))} \\ &\leq 2\delta t n(B(x))\mathrm{e}^{-tn(B(x))\ell/2} \leq \delta 4/(\mathrm{e}\ell), \end{aligned}$$

where the inequality follows from $|\xi_{t,x}| < \ell/2$. Since $t, x \in \mathbb{R}_+\backslash\{0\}$ are arbitrary, this concludes the proof of part (i).

(ii) As the ratio $t/\widehat{L}^{-1}(t)$ tends to ℓ almost surely (see 2.4), we have for any $\delta \in (0, \ell/2)$,

$$P\left(\widehat{L}(t)/t \in [\ell-\delta, \ell+\delta]\right) = 1 + o(1), \qquad \text{as } t \to \infty. \tag{4.2}$$

Equation (4.2) yields the following as $\min\{x, t\} \to \infty$:

$$\begin{aligned} P(\widehat{L}(t) = \widehat{L}(\tau(x))) &= P(\widehat{L}(t) = \widehat{L}(\tau(x)), \widehat{L}(t) \in t[\ell-\delta, \ell+\delta]) + o(1) \\ &\leq P(\widehat{L}(\tau(x)) \in t[\ell-\delta, \ell+\delta]) + o(1). \end{aligned}$$

Hence part (ii) yields $\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}(t) = \widehat{L}(\tau(x))) \leq \delta 4/(\mathrm{e}\ell)$. Since $\delta \in (0, \ell/2)$ was arbitrary and probabilities are non-negative quantities, the limit in part (ii) follows.

(iii) Note that for any $\alpha \geq 0$ the quotient $\widehat{L}(t\alpha)/t$ tends to $\ell\alpha$ P-a.s. as $t \to \infty$. For any $\delta_1, \delta_2 \in [0, 1/4)$ we therefore find that the probability of the event

$$A_{\delta_1,\delta_2}(t, x) = \{\widehat{L}(t(1-\delta_1)) \leq \widehat{L}(\tau(x)) \leq \widehat{L}(t(1+\delta_2))\}$$

satisfies the following as $\min\{x, t\} \to \infty$:

$$\begin{aligned} P(A_{\delta_1,\delta_2}(t, x)) &= P(A_{\delta_1,\delta_2}(t, x), \widehat{L}(t(1-\delta_1)), \widehat{L}(t(1+\delta_2)) \in t[\ell(1-\delta), \ell(1+\delta)]) + o(1) \\ &\leq P(\widehat{L}(\tau(x)) \in t[\ell(1-\delta), \ell(1+\delta)]) + o(1), \end{aligned} \tag{4.3}$$

for any $\delta \in (2\max\{\delta_1, \delta_2\}, 1/2)$. Since $0 < \delta\ell < \ell/2$, part (ii) of the lemma and inequality (4.3) imply that $\limsup_{\min\{x,t\}\to\infty} P(A_{\delta_1,\delta_2}(t, x)) \leq \delta 4/\mathrm{e}$. Therefore the first inequality in part (iv) is satisfied. The second limit in part (iv) follows by noting that, for any $s \in \mathbb{R}_+$ and $\delta_1 \in (0, 1/4)$, the inclusion $\{\widehat{L}((t-s) \vee 0) \leq \widehat{L}(\tau(x)) < \widehat{L}(t)\} \subset A_{\delta_1,0}(t, x)$ holds for all (t, x) with large $\min\{x, t\}$. Hence by (4.1) we have

$$\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}((t-s) \vee 0) \leq \widehat{L}(\tau(x)) < \widehat{L}(t)) \leq \delta_1 8/\mathrm{e}.$$

Since δ_1 can be chosen arbitrarily small, this proves part (iv) and hence the lemma. □

Before moving to the proof of the asymptotic independence of $Y(t)$, $Z(x+y)$ and $M(x, t)$, we establish the asymptotic behaviour of certain convolutions that will arise in the proof.

Lemma 12. *For $a \in [0, \infty)$ and any family of sets $F(t) \in \mathcal{F}$, $t \in \mathbb{R}_+$, we have as* $\min\{y, t\} \to \infty$

$$\int_{[0,t]} P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t-s))\, P(T(a) \in \mathrm{d}s) = P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t))\, P(Y(t) > a) + o(1), \quad . \tag{4.4}$$

Proof of Lemma 12. The proof of this lemma is based on Lemma 11. Since $Y(t)$ and $\sup_{0\leq s\leq t} X(s)$ are equal in law (by time reversal) and $P(T(a) = t) \to 0$ as $t \to \infty$[5] (as $X_t \to -\infty$ by Assumption 1), it follows that $P(T(a) \leq t) = P(Y(t) > a) + o(1)$ as $t \to \infty$. Thus, to prove equality (4.4), it is sufficient to establish

$$\int_{[0,t]} \left(P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t)) - P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t-s))\right) P(T(a) \in \mathrm{d}s) = o(1) \tag{4.5}$$

[5]Note $P(T(a) = t) = 0$ if X is not equal to the sum of a compound Poisson process and a deterministic drift. Indeed, in this case $X(t)$ is continuous (see [21, Theorem 27.4]), so that $P(T(a) = t) \leq P(X(t) = a) = 0$.

as $\min\{y,t\}\to\infty$. Since the local time $\widehat{L}$ is non-decreasing, the integrand in (4.5) satisfies

$$|P(F(t),\widehat{L}(\tau(y))<\widehat{L}(t))-P(F(t),\widehat{L}(\tau(y))<\widehat{L}(t-s))|\leq P(\widehat{L}(t-s)\leq\widehat{L}(\tau(y))<\widehat{L}(t)).$$

Hence Lemma 11(iv) and the dominated convergence theorem imply that (4.5) holds. □

We move next to the asymptotic independence of $Y(t), Z(x+y)$ and $M(t,x)$.

Lemma 13. *For any $t,x\in\mathbb{R}_+\backslash\{0\}$, $a,b\in\mathbb{R}_+$, $c\in\mathbb{R}$, $y\in[0,x]$ and Borel sets $A,B,C\in\mathcal{B}(\mathbb{R})$ with $A=(-\infty,a]$, $B=(-\infty,b]$ and $C=(-\infty,c]$ denote*

$$\pi_1(t,A)=P(Y(t)\in A),\quad \pi_2(x,B)=P(Z(x)\in B),\quad \pi_3(t,y)=P(\widehat{L}(\tau(y))<\widehat{L}(t)).$$

We have as $\min\{t,y,x-y\}\to\infty$

$$P(Y(t)\in A,Z(x)\in B) = \pi_1(t,A)\pi_2(x,B)+o(1), \tag{4.6}$$
$$P(Y(t)\in A,Z(x)\in B,\widehat{L}(\tau(y))<\widehat{L}(t)) = \pi_1(t,A)\pi_2(x,B)\pi_3(t,y)+o(1), \tag{4.7}$$
$$P(Y(t)\in A,Z(x)\in B,M(t,y)\in C) = \pi_1(t,A)\pi_2(x,B)P(M(t,y)\in C)+o(1), \tag{4.8}$$
$$P(Y(t)\in A,Z(x)\in B,m(t)\in C) = \pi_1(t,A)\pi_2(x,B)P(m(t)\in C)+o(1). \tag{4.9}$$

Proof of Lemma 13. Fix $t,x\in\mathbb{R}_+\backslash\{0\}$, $y\in[0,x]$, $a,b\in\mathbb{R}_+$ arbitrary, with $A=(-\infty,a]$, $B=(-\infty,b]$. As a first step we note that by a classical application of excursion theory[6] involving $G(\tau(x))=\sup\{s<\tau(x):Y(s)=0\}=\widehat{L}^{-1}(\widehat{L}(\tau(x))-)$ the random elements $\mathcal{A}:=\{Y(s):0\leq s\leq G(\tau(x))\}$ and $\mathcal{A}':=\epsilon(\widehat{L}(\tau(x)))$ are independent. Hence the sets $\{Z(x)\in B\}$ and $\{\widehat{L}(\tau(y))>\widehat{L}(t),Y(t)\in A\}$, which are measurable with respect to $\sigma(\mathcal{A}')$ and $\sigma(\mathcal{A})$ respectively, are independent, that is,

$$P(\widehat{L}(\tau(y))>\widehat{L}(t),Y(t)\in A,Z(x)\in B)=P(\widehat{L}(\tau(y))>\widehat{L}(t),Y(t)\in A)P(Z(x)\in B). \tag{4.10}$$

Next we establish additional (asymptotic) factorisation. Let $A'\in\mathcal{B}(\mathbb{R})$ arbitrary. Since $s\mapsto\mathbf{I}_{\{\tau(x)<s\leq t,Z(x)\in B\}}$ is left-continuous and adapted an application of the compensation formula of excursion theory (see e.g. [2, Cor. IV.11]) yields

$$\begin{aligned} &P\left(\widehat{L}(\tau(x))<\widehat{L}(t),Y(t)\in A',Z(x)\in B\right)\\ &= E\left[\sum_g \mathbf{I}_{\{\tau(x)<g\leq t,Z(x)\in B\}}\mathbf{I}_{\{\epsilon_{\widehat{L}(g)}(t-g)\in A',t-g<\zeta(\epsilon_{\widehat{L}(g)})\}}\right]\\ &= E\left[\int_{[0,t]}\mathbf{I}_{\{\tau(x)<s\leq t,Z(x)\in B\}}\,n(\varepsilon(t-s)\in A',t-s<\zeta(\varepsilon))\mathrm{d}\widehat{L}(s)\right], \end{aligned} \tag{4.11}$$

where the sum is over all left-end points of excursion intervals. Denote by $e(q)$ an exponential random time with mean $1/q$, defined by extending the probability space to $(\Omega\times\Omega',\mathcal{F}\otimes\mathcal{F}',P\times P')$. Replacing t by the exponential time $e(q)$ in (4.11) and denoting $\mathbb{P}=P\times P'$ we have by the lack of memory property of $e(q)$ (taking $A'=\mathbb{R}$ and $B=\mathbb{R}$, $x=0$, respectively)

$$\begin{aligned} &\mathbb{P}\left(\widehat{L}(\tau(x))<\widehat{L}(e(q)),Z(x)\in B\right)\\ &= \mathbb{E}\left[\int_{[0,e(q)]}\mathbf{I}_{\{\tau(x)<s\leq e(q),Z(x)\in B\}}\mathrm{d}\widehat{L}(s)\right]\mathbb{E}[n(e(q)<\zeta(\varepsilon))], \end{aligned} \tag{4.12}$$

$$\mathbb{P}\left(Y(e(q)\in A'\right)=\mathbb{E}\left[\widehat{L}(e(q)\right]\mathbb{E}[n(\varepsilon(e(q))\in A',e(q)<\zeta(\varepsilon))] \tag{4.13}$$

$$= \frac{\mathbb{E}[n(\varepsilon(e(q))>y,e(q)<\zeta(\varepsilon))]}{\mathbb{E}[n(e(q)<\zeta(\varepsilon))]}, \tag{4.14}$$

[6] This can be seen to follow directly as a consequence of the splitting property [2, Sec O.5, Proposition O.2] of the Poisson point process ϵ at the first entrance time $\mathbb{H}_{B'}=\inf\{s\leq 0:\epsilon(s)\in B'\}$ of ϵ into the set $B'=\{\varepsilon\in\mathcal{E}:\rho(x,\varepsilon)<\zeta(\varepsilon)\}$.

where the last equality follows by taking $A' = \mathbb{R}_+$ in (4.13). Using (4.12) and (4.14) and again replacing t by $e(q)$ in (4.11) we have

$$\begin{aligned}
&\mathbb{P}\left(\widehat{L}(\tau(x)) < \widehat{L}(e(q)), Y(e(q)) \in A', Z(x) \in B\right) \\
&= \mathbb{E}\left[\int_{[0,e(q)]} \mathbf{I}_{\{\tau(x)<s\leq e(q), Z(x)\in B\}} \mathrm{d}\widehat{L}(s)\right] \mathbb{E}[n(\varepsilon(e(q)) \in A', e(q) < \zeta(\varepsilon))] \\
&= \mathbb{E}\left[\int_{[0,e(q)]} \mathbf{I}_{\{\tau(x)<s\leq e(q), Z(x)\in B\}} \mathrm{d}\widehat{L}(s)\right] \mathbb{E}[n(e(q) < \zeta(\varepsilon))] \frac{\mathbb{E}[n(\varepsilon(e(q)) \in A', e(q) < \zeta(\varepsilon))]}{\mathbb{E}[n(e(q) < \zeta(\varepsilon))]} \\
&= \mathbb{P}\left(\widehat{L}(\tau(x)) < \widehat{L}(e(q)), Z(x) \in B\right) \mathbb{P}(Y(e(q)) \in A'). \qquad (4.15)
\end{aligned}$$

Dividing the left-hand and right-hand sides of (4.15) by q with $A' = A^c = (a, \infty)$, inverting the Laplace transform in q, noting $q^{-1}\mathbb{P}(Y(e(q)) \in A^c) = q^{-1}\mathbb{P}(X^*(e(q)) > a) = q^{-1}\mathbb{P}(T(a) \leq e(q)) = \int_0^\infty \mathrm{e}^{-qt} P(T(a) \in \mathrm{d}t)$, and deploying (4.4) in Lemma 12 we have

$$\begin{aligned}
&P(\widehat{L}(\tau(x)) < \widehat{L}(t), Y(t) \in A^c, Z(x) \in B) \qquad (4.16)\\
&\quad = P(\widehat{L}(\tau(x)) < \widehat{L}(t), Z(x) \in B) P(Y(t) \in A^c) + o(1), \qquad \text{as } \min\{x, t\} \to \infty.
\end{aligned}$$

Subtracting $P(\widehat{L}(\tau(x)) < \widehat{L}(t), Z(x) \in B)$ on the left-hand and right-hand sides of (4.16) shows that (4.16) is also valid with A^c replace by A.

Taking note of the following equality for any $y, t \in \mathbb{R}_+\backslash\{0\}$ and set $E \in \mathcal{F}$:

$$\begin{aligned}
&\mathbb{P}(E, \widehat{L}(\tau(y)) > \widehat{L}(t)) + P(E, \widehat{L}(\tau(y)) = \widehat{L}(t)) \qquad (4.17)\\
&\quad = P(E) - P(E, \widehat{L}(\tau(y)) < \widehat{L}(t)),
\end{aligned}$$

and applying (4.10) and (4.16) with $B = \mathbb{R}_+$ yields as $\min\{x, t\} \to \infty$

$$\begin{aligned}
&P(Y(t) \in A, Z(x) \in B) \\
&= \pi_1(t, A) P(\widehat{L}(\tau(x)) < \widehat{L}(t), Z(x) \in B) + P(\widehat{L}(\tau(x)) > \widehat{L}(t), Y(t) \in A) \pi_2(x, B) \\
&\quad + P(\widehat{L}(\tau(x)) = \widehat{L}(t), Y(t) \in A, Z(x) \in B) + o(1) \\
&= \pi_1(t, A) \pi_2(x, B) + R(t, x) + o(1),
\end{aligned}$$

where $R(t, x) = P(\widehat{L}(\tau(x)) = \widehat{L}(t), Y(t) \in A, Z(x) \in B) - P(\widehat{L}(\tau(x)) = \widehat{L}(t), Y(t) \in A)\pi_2(x, B) - P(\widehat{L}(\tau(x)) = \widehat{L}(t), Z(x) \in B)\pi_1(t, A) + \pi_1(t, A)\pi_2(x, B) P(\widehat{L}(\tau(x)) = \widehat{L}(t))$. Observing that $R(t, x) = o(1)$ when $\min\{x, t\} \to \infty$ by Lemma 11(iii) the proof of (4.6) is complete.

Equation (4.7) follows similarly, by combining the equality (4.17) (with $E = \{Y(t) \in A, Z(x) \in B\}$) with Lemma 11(iii) and the identities (4.6), (4.10), and (4.16) (with $B = \mathbb{R}_+$).

Finally, take $C = (-\infty, c]$ for an arbitrary fixed $c \in \mathbb{R}$. In order to prove equality (4.8) note that the following inclusions hold for any $y \in \mathbb{R}_+$:

$$\begin{aligned}
\{M(t, y) \in C\} = \{Y^*(t) \leq y + c\} &\subset \{\widehat{L}(t) \leq \widehat{L}(\tau((y + c)^+))\} \quad \text{and} \\
\{\widehat{L}(t) \leq \widehat{L}(\tau((y + c)^+))\} \cap \{M(t, y) \notin C\} &\subset \{\widehat{L}(\tau((y + c)^+)) = \widehat{L}(t)\}
\end{aligned}$$

(recall that $\tau(x)$ is defined for $x \in \mathbb{R}_+$). These inclusions, together with Lemma 11(iii), imply that the following equality holds for any family of events $E(t, x) \in \mathcal{F}$, $t, x \in \mathbb{R}_+$, as $\min\{t, y, x - y\} \to \infty$:

$$P\left(E(t, x), \widehat{L}(t) \leq \widehat{L}(\tau((y + c)^+))\right) = P(E(t, x), M(t, y) \in C) + o(1). \qquad (4.18)$$

Since $\min\{t, y, x - y\} \to \infty$, for the fixed $c \in \mathbb{R}$ the inequalities $0 \leq y + c \leq x$ hold for all large y and x. In particular (4.7), applied to the complement $\{\widehat{L}(\tau(y + c)) < \widehat{L}(t)\}^c = \{\widehat{L}(\tau(y + c)) \geq \widehat{L}(t)\}$, Lemma 11(iii) and (4.18) yield the following equalities:

$$\begin{aligned} P(Y(t)\in A, Z(x)\in B, M(t,y)\in C) &= P(Y(t)\in A, Z(x)\in B, \widehat{L}(t)\leq \widehat{L}(\tau(y+c))) + o(1) \\ &= P(Y(t)\in A)P(Z(x)\in B)P(\widehat{L}(t)\leq \widehat{L}(\tau(y+c))) + o(1) \\ &= P(Y(t)\in A)P(Z(x)\in B)P(M(t,y)\in C) + o(1) \end{aligned}$$

as $\min\{t,y,x-y\}\to\infty$, which establishes (4.8). Taking $y=y^*(t)$ in (4.8) (recalling $m(t)=M(t,y^*(t))$ and using (1.4)) yields (4.9), and the proof is complete. □

Lemma 14. **(i)** *As* $\min\{x,t\}\to\infty$*,* $Y(t)$ *and* $Z(x)$ *satisfy*

$$E[\exp(-uY(t)-vZ(x))] = E[\exp(-uY(t))]E[\exp(-vZ(x))] + o(1), \quad \text{for any } u,v\in\mathbb{R}_+\backslash\{0\}. \tag{4.19}$$

(ii) *As* $\min\{x,t\}\to\infty$ *such that* $t\exp(-\gamma x)\to 0$*,* $Y(t)$*,* $Z(x)$ *and* $m(t)$ *satisfy*

$$\begin{aligned} E[\exp(-uY(t)-vZ(x)\pm\beta m(t))\mathbf{I}_{\{-m(t)\in\mathbb{R}_\pm\}}] &= E[\exp(-uY(t))]\,E[\exp(-vZ(x))]\times \\ \times\, E[\exp(\pm\beta m(t))\mathbf{I}_{\{-m(t)\in\mathbb{R}_\pm\}}] + o(1), &\qquad \text{for any } u,v,\beta\in\mathbb{R}_+\backslash\{0\}, \end{aligned} \tag{4.20}$$

where $\mathbb{R}_-=\mathbb{R}\backslash\mathbb{R}_+$*. In particular, we have*

$$\begin{aligned} E[\exp(-uY(t)-vZ(x)-\beta|m(t)|-b\,s(m(t)))] &= E[\exp(-uY(t))]\,E[\exp(-vZ(x))]\times \\ \times\, E[\exp(-\beta|m(t)|-b\,s(m(t)))] + o(1), &\qquad \text{for any } u,v,\beta,b\in\mathbb{R}_+\backslash\{0\}, \end{aligned} \tag{4.21}$$

where $s:\mathbb{R}\to(-\infty,\infty]$ *is given by* $s(x)=\pm 1$ *for* $-x\in\mathbb{R}_\mp$*.*

Proof. **(i)** Fix $u,v\in\mathbb{R}_+\backslash\{0\}$ arbitrary. By integrating both sides of the identity in (4.6) in Lemma 13 over $\mathbb{R}^2$ against the measure $\mathbf{I}_{\mathbb{R}_+\times\mathbb{R}_+}(a,b)a\,b\,\exp(-ua-vb)\mathrm{d}a\mathrm{d}b$ we have (4.19) by noting that the integral of the $o(1)$ term in (4.6) tends to zero by the dominated convergence theorem (as it is bounded by one).
(ii) The proof is a modification of the argument in part (i). Note first that (4.9) in Lemma 13 also holds with C replaced by its complement $C^c=\mathbb{R}\backslash C$. For given $a,b,c\in\mathbb{R}_+$ it follows from (4.9) in Lemma 13 (taking $C=(-\infty,c]$ and $C=(-\infty,0]$ and substracting)

$$\begin{aligned} &P(Y(t)\leq a, Z(x)\leq b, m(t)\in(0,c]) \\ &\qquad = P(Y(t)\leq a)P(Z(x)\leq b)P(m(t)\in(0,c]) + o(1), \qquad \min(t,x)\to\infty, \end{aligned} \tag{4.22}$$

and similarly (taking $C=(-\infty,-c]^c=(-c,\infty)$ and $C=(-\infty,0]^c=(0,\infty]$ and substracting)

$$\begin{aligned} &P(Y(t)\leq a, Z(x)\leq b, -m(t)\in[0,c)) \\ &\qquad = P(Y(t)\leq a)P(Z(x)\leq b)P(m(t)\in(-c,0]) + o(1), \qquad \min(t,x)\to\infty. \end{aligned} \tag{4.23}$$

Let next $u,v,w\in\mathbb{R}_+\backslash\{0\}$ be arbitrary. Integrating both sides of the identity in (4.22) over $\mathbb{R}^3$ against the measure

$$\mu(\mathrm{d}a,\mathrm{d}b,\mathrm{d}c)=\mathbf{I}_{\mathbb{R}_+^2\times\mathbb{R}_+}(a,b,c)a\,b\,c\,\exp(-ua-vb-wc)\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$$

and applying the dominated convergence theorem shows that also the integral of the $o(1)$-term tends to zero, which yields the " $-$ "-version of (4.20). The " $+$ "-version of follows similarly by integrating both sides of the identity in (4.23) against μ. As (4.21) follows as direct consequence of (4.20), the proof is complete. □

5. Proofs of Theorems 1 and 2

Proof of Theorem 1. As $Y(t)$ and $Z(x)$ each admit a weak limit Y_∞, Z_∞ as $t,x\to\infty$, given in (2.8) and in Proposition 10, the joint Laplace transform of (Y_∞,Z_∞) follows from (4.19) in Lemma 14(i). Finally, the factorisation of the exponential distribution is obtained by setting $u=v$ in (1.2). □

Proof of Theorem 2. The asymptotic independence of $Y(t)$, $Z(x)$ and $m(t)$ follows from (4.8) in Lemma 13. The joint Fourier-Laplace transform then follows from a direct calculation using (4.21) in Lemma 14(ii) and the laws of Y_∞, Z_∞ and m_∞ given in (2.8) and Propositions 8 and 7, respectively. □

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