Boundary Singularities of Functions in Symplectic and Volume-Preserving Geometry

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Abstract

In this thesis we study the classification problem of boundary singularities of functions in symplectic and volume-preserving geometry. In particular we generalise several well known theorems concerning the classification of isolated singularities of functions and volume forms in the presence of a “boundary”, i.e. a germ of a fixed smooth hypersurface. The results depend in turn on a generalisation of the relative de Rham cohomology and the corresponding Gauss-Manin theory to the case of isolated boundary singularities and in particular, on a relative version of the so called Brieskorn-Deligne-Sebastiani theorem, concerning the finiteness and freeness of certain cohomology modules.
To my family
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1 Introduction

This thesis is devoted to the local classification problem of boundary singularities of functions on the symplectic and Martinet plane (i.e. relative to a nondegenerate and a degenerate 2-form) as well as to some generalisations in higher dimensions, related to the classification of functions and volume forms on manifolds with boundary.

The subject is motivated from several studies related to local analysis problems in the geometric theory of Hamiltonian systems with constraints and in particular to the classification of pairs of hypersurfaces in a symplectic manifold, as well as to the possibility of extending these results in volume-preserving (isochore or else unimodular) geometry.

The local classification problem of pairs of hypersurfaces (and of the functions defining them) in a symplectic space was first studied by R. B. Melrose [70], [71] for the purposes of the diffraction problem in the theory of wave propagation and later by V. I. Arnol’d [7] in the more general setting of variational problems with (1-sided) constraints, such as the problem of bypassing an obstacle, the theory of asymptotic rays in Riemannian geometry, the theory of caustics and Lagrangian singularities and many more (c.f. [9], [10], [11] and references therein). Arnol’d’s works led in turn to many interesting discoveries, such as for example the open swallowtail appearing as a Lagrangian variety of rays in the problem of bypassing an obstacle, the normal form of the bundle of geodesics emanating at points of asymptotic and biasymptotic directions of a surface in 3-dimensional space, as well as to the so called Darboux-Givental theorem, proved by A. B. Givental in [11], i.e. that the symplectic type of a germ of a submanifold in symplectic space is completely determined by the restriction of the symplectic structure on its tangent bundle. The latter statement, along with the theory of singularities of 2-forms studied by J. Martinet [66], R. Roussarie [85] and others (c.f. [39], [83]) led to the classification of generic submanifolds in symplectic space (c.f. [9] for the corresponding list of simple normal forms).

The problem of classification of pairs of curves in a symplectic space, was already considered by Melrose as an important special case [70]. In fact, he gave a symplectic normal form (in the smooth category) of the first occurring singularities, i.e. at points of first order tangency of the pair of curves:

\[ \omega = dx \wedge dy, \quad H = \{ x = 0 \}, \quad F = \{ x + y^2 = 0 \}, \]

whereas he addressed the fact that the corresponding symplectic classification of even one of the functions defining the curves has moduli. He didn’t continue more though to the description of these moduli \(^1\). One of our main objectives (Chapter 3) is to continue this study and also to extend

\(^1\) neither to a further classification of the pair of curves at more degenerate singularities.
it in the volume-preserving case (Chapter 5).

In fact, the degenerate singularities of functions on the plane (or more generally in a space of any dimension) relative to a fixed smooth curve (a smooth hypersurface in higher dimensions) were studied only later by V. I. Arnol’d [6], under the name “boundary singularities”. The motivation was coming again from certain constrained variational problems, such as for example the problem of evolutes of a smooth plane curve at an inflection point: the corresponding Lagrangian variety was shown to be diffeomorphic to the bifurcation diagram of the $B_3$ boundary singularity $f(x, y) = x^4 + y^2, H = \{ x = 0 \}$. In that manner, Arnol’d extended the ordinary $A, D, E$ correspondence between the classification of Weyl groups with the simple singularities of functions and the corresponding singularities of Lagrangian maps (already proved by him in [5], see also [3], [4], [8]), in order to include in the scheme of singularity theory the Weyl groups $B, C, F$ (with roots of different length), corresponding now to the simple singularities of functions on a manifold with boundary i.e. with a fixed smooth hypersurface. The relation with the singularities of Lagrangian maps, was proved only later (c.f. [22], [96] and also [10] for general references), where it was shown that to a boundary singularity, a pair of Lagrangian submanifolds with certain properties is naturally associated. Several other interesting properties of boundary singularities were also discovered: for example V. I. Matov extended Arnol’d’s list to more degenerate singularities [68] and also proved (among other things [69]) that the group of diffeomorphisms preserving the boundary is a “nice geometric group” (in the sense of J. Damon [19]) so that Mather’s unfolding and determinacy theorems [67] do indeed hold in the presence of a boundary.

Despite these many studies the isochore-symplectic classification of the boundary singularities has not yet been treated in the literature and one of our objectives is to fill in this gap. On the contrary, the isochore-symplectic classification of the ordinary singularities of functions (i.e. without boundary), has been studied extensively, starting already from G. D. Birkhoff’s work [12] on the symplectic normal forms of Hamiltonian functions at nondegenerate critical points (Morse singularities) and then, much later on, by J. Vey [109] who extended the results in the volume-preserving case. It is important to notice here an important difference between the symplectic and volume-preserving categories: generically, the symplectic diffeomorphisms (normalising transformations) bringing a Hamiltonian function to its normal form diverge (this was shown by C. L. Siegel [98]) and in fact, convergence of the corresponding diffeomorphisms poses very strict conditions on the Hamiltonian system, such as for example Liouville integrability (as a theorem of H. Rüssmann implies [86], see also [2] and the many references therein). On the other hand, the volume-preserving diffeomorphisms bringing a Morse function to its normal form do indeed converge, as is implied by Vey’s isochore Morse lemma referred to above. Of course. the 2-dimensional case is special, since both these theories trivially coincide: a volume (area) form on the plane is just a symplectic form and the function $f$ defines an integrable Hamiltonian system, and in fact, a “singular Lagrangian fibration” (c.f. [18], [35], [36], [37], [38], [93], [99] for general results in Lagrangian deformation theory).

Apart from Vey’s original proof of the isochore Morse lemma, there exist two more modern proofs

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\footnotetext[2]{who later extended these results for other geometric subgroups}  
\footnotetext[3]{among many others, such as for example the geometry of submanifolds mentioned before.
in the literature: one is due to J.-P. Françoise [29], [30] (see also [32]) who gave a generalisation for all isolated singularities and the other, much later on, due to M. D. Garay [33] who proved an isochore version of Mather’s versal unfolding theorem [67]. One of our objectives in this thesis is a generalisation of Vey-Françoise-Garay theorems in the presence of a boundary, i.e. for the case of isolated boundary singularities discussed before.

In terms of methodology, it is important to notice here that Vey’s isochore Morse lemma and its generalisations, rely upon (in fact they are more or less direct applications of) a general theory known as Gauss-Manin theory. This is cohomological theory after A. Groethendieck, P. Deligne, P. A. Griffiths, Y. Manin, E. Brieskorn, B. Malgrange, F. Pham, A. N. Varchenko, J. S. Steenbrink and many others, devised for the description of the variations in the cohomology on the fibers of maps between analytic and/or algebraic manifolds, such as the monodromy properties, the variations of Hodge structures e.t.c. (c.f. [15], [20], [21], [43], [46], [64], [81], [89], [90], [91], [106], [107] and also the books [4], [8], [49], [56], [79], [87] for more modern general references). The relation with the classification problem of functions and volume forms comes from the fact that the unique (functional) invariants can be expressed in terms of integrals of a primitive of the volume form (the so called period integrals) along a set of generators of the middle homology group of the smooth level sets of the function, the so called vanishing cycles (after S. Lefschetz [57]). Thus stated, the local classification problem becomes a problem of local analytic geometry and in fact a topological problem.

So, in order to extend the isochore deformation theory to the case of boundary singularities, we come across to a rather non-trivial problem, with its own important implications, that is, to extend the relative de Rham cohomology and the corresponding Gauss-Manin theory to the relative case, i.e. in the presence of a fixed smooth hypersurface (a boundary, or a smooth divisor in algebraic geometry terms).

Let us describe now in more detail the contents of the thesis:

Chapter 2 is introductory and contains no new results. We consider the simplest problem, i.e. the classification of pairs \((\omega, f)\) on the plane \(\mathbb{C}^2\), where \(\omega\) is a germ of a holomorphic symplectic form and \(f\) is holomorphic function germ with an isolated singularity at the origin. We follow first J.-P. Françoise’s approach [30] who studied the corresponding classification problem of pairs \((\omega, f)\) in the volume-preserving category. We start with the first occuring singularities, i.e. the well known Morse-Darboux lemma (c.f. [14]), or equivalently, the 2-dimensional isochore Morse lemma, concerning the symplectic normal form of the germ \(f\) at a nondegenerate critical point. We show, by simple computations, how to interpret the unique functional invariant associated to the pair \((\omega, f)\), in terms of the integral of a primitive of the form \(\omega\) along the vanishing cycle, i.e. the distinguished cycle in the smooth level sets of \(f\) generating the first homology group. This motivates a short review of the basic results of Gauss-Manin theory for isolated singularities, mostly due to E. Brieskorn [15] and B. Malgrange [64], concerning the general properties of the integrals of holomorphic forms along the vanishing cycles of a singularity and the relations with the (topological) Picard-Lefschetz monodromy. We then show how we may interpret Françoise’s generalisation cohomologically, in terms of Gauss-Manin theory. Finally we review the proof of Mather-Garay’s isochore deformation
theorem, and we deduce from it both Vey’s isochrome Morse lemma, as well as a conjecture of Y. Colin de Verdière [18], i.e. that a symplectic infinitesimally versal deformation of a curve germ $X_0 = \{ f = 0 \}$, is in fact symplectically versal.

In Chapter 3 we extend the construction of Chapter 2 in the presence of a boundary. The results here are new and they are part (i.e. the 2-dimensional case) of [53]. Here we consider the classification of triples $(\omega, f, H)$ on the plane, where $\omega$ is again a germ of a holomorphic symplectic form, $H$ is a smooth plane curve germ (the “boundary”) and $f$ is such that, it either has an isolated critical point at the origin, or it is smooth, but its restriction $f|_H$ on the boundary has an isolated critical point at the origin. According to the standard terminology (c.f. [3], [4], [6], [8]), we say that the pair $(f, H)$ defines an isolated boundary singularity. We start again with the first occurring singularity, i.e. when the germ $f$ is smooth but its restriction on the boundary $f|_H$ has a nondegenerate critical point at the origin (relative Morse-Darboux lemma). The functional invariant associated to the triple can be computed, as in the ordinary case, by considering the integrals of a primitive of the symplectic form $\omega$ along the vanishing half-cycle (according to the terminology of V. I. Arnol’d) which is the generator of the relative homology group of the level sets of $f$ modulo the level sets (points) of its restriction $f|_H$ on the boundary. We then, following Françoise, generalise this construction to the more degenerate case of isolated boundary singularities and we show how the classification problem reduces to a problem of relative de Rham cohomology, and in fact to a relative version of Gauss-Manin theory, in the presence of a boundary (which will be studied extensively in Chapter 5). Finally we present the main ingredients which constitute the proof of the relative analog of Mather-Garay’s versal unfolding theorem, which lead in turn to another proof of the relative isochrome Morse lemma, as well as to Y. Colin de Verdière’s relative analog on the symplectic versal unfoldings of curve germs.

Let us now describe the contents of Chapter 4. The results in this chapter are again new and they are contained partially in [51] and wholly in [52]. They concern again the classification (in the analytic category) of pairs $(\omega, f)$ on the plane, but now $\omega$ is a germ of a Martinet 2-form (in honour of J. Martinet [60]), i.e. a degenerate 2-form (not symplectic) which vanishes along the points of a smooth line $H(\omega)$. The singularities of the pair $(\omega, f)$ are to be determined initially by the relative positions of the germ $f$ with respect to the Martinet curve $H(\omega)$. In particular, as long as the singularities of $f$ are isolated, then we can view the pair $(f, H(\omega))$ as defining a germ of an isolated boundary singularity, as in the previous chapter. Thus the classification problem reduces to the study of the possible normal forms and the invariants of degenerate 2-forms $\omega$ whose zero set is exactly equal to a fixed smooth curve $H = H(\omega)$ (we call these, Martinet 2-forms), with respect to the symmetries of the boundary singularity $(f, H)$. We show that, at least as long as the boundary singularity is quasihomogeneous, both Vey’s isochrome Morse lemma and Françoise’s generalisation, extend with minor modifications to the Martinet case. Finally we give an application to a specific problem arising in several instances in mathematical physics, that is the classification of constrained Hamiltonian systems on 2-manifolds:

$$X_{f, \omega} = df,$$
and the associated singular Lagrangian functions (i.e. of first order in the velocities):

\[ L = \alpha(x) \cdot \dot{x} - f(x), \]

where \( \omega = d\alpha \) is a Martinet 2-form. These systems arise naturally, as their name suggests, when considering general Hamiltonian systems in phase spaces (of higher dimension) forced to evolve on some submanifold representing the constraints (c.f. [26], [28], [50], [62], [82], [84]). In turn, these systems form a particular class of the more general constraint systems, whose singularities (called impasse singularities in the literature) have been analysed extensively by M. Zhitomirskii in [112], [113] and also [100] in higher dimensions. The higher dimensional analog of the constrained Hamiltonian systems though, has not yet been considered and it still remains an open problem (even for orbital equivalence).

Finally, in Chapter 5 we study extensively the Gauss-Manin connections on the relative cohomology of an isolated boundary singularity and we give some applications in isochore deformation theory. The results here are new and they are contained in [53]. Apparently, a detailed description of the Gauss-Manin connections for boundary singularities has not yet been treated, except the closely related studies [22], [23], [24] (and also [81] and references therein) on the Gauss-Manin systems with boundary and regular analytic interactions of pairs of Lagrangian manifolds. Here we give a generalisation, for the boundary case, of some fundamental results obtained mainly by E. Brieskorn [15], M. Sebastiani [92] and B. Malgrange [64]. More specifically we prove a relative analog of the Brieskorn-Deligne-Sebastiani theorem, concerning the finiteness and freeness of the de Rham cohomology modules and of the corresponding Brieskorn lattices associated to the boundary singularity. We also give a relative analog of the regularity theorem according to which, the restriction of the natural Gauss-Manin connection on the localisation of the Brieskorn modules at the critical value, has regular singularities. According to the work of Brieskorn [15] for the ordinary, without boundary case, the regularity of the Gauss-Manin connection, along with the algebraicity theorem and the positive solution of Hilbert’s VII’th problem, give also a direct analytic proof of a relative version of the monodromy theorem, i.e. that the eigenvalues of the Picard-Lefschetz monodromy operator in the relative vanishing (co)homology (i.e. the one generated by the vanishing cycles and half-cycles), are indeed roots of unity. Following Malgrange [64], we show that the relative monodromy theorem along with the regularity theorem, give also the asymptotic expansion of the integrals of holomorphic forms along the vanishing cycles and half-cycles of the boundary singularity, when the values of the function tend to the critical one.

These results in turn can be viewed as the first steps for the establishment of several important (Hodge theoretic) invariants for boundary singularities, extending those for the ordinary singularities, such as the spectrum, the spectral pairs and eventually, the mixed Hodge structure in the relative vanishing cohomology (c.f. [106], [107] and also [4], [56] and references therein). Here we don’t take this step but instead we give a direct application in the classification problem of triples (volume form, function, hypersurface). In particular we prove a relative analog of a J. Vey’s isochore Morse lemma [109], J. -P. Françoise’s generalisation on the local normal forms of volume forms with respect to the boundary singularity preserving diffeomorphisms [29], [30], as well as
Mather-Garay’s isochore versal unfolding theorem [33] for boundary singularities. These results are the higher dimensional analogs of the corresponding theorems proved in Chapter 3 for the 2-dimensional case.

It is important to notice finally that there are two natural ways to study a boundary singularity. The first one is due to Arnol’d [6] according to which a boundary singularity can be viewed as an ordinary $\mathbb{Z}_2$-symmetric singularity after passing to the double covering space branched along the boundary (see also [110] and [40] for generalisations for other symmetric singularities). There is also another approach due to A. Szpirglas [103], [104], according to which a boundary singularity can be viewed, at least in a (co)homological level, as an extension of two ordinary singularities, namely the ambient singularity and its restriction on the boundary. Our approach is in accordance with the second one, i.e. we show that the relative cohomology, the relative Gauss-Manin connection and the corresponding Brieskorn lattices associated to a boundary singularity, are indeed extensions of the corresponding ordinary objects associated to the pair of isolated singularities.

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2 Singularities of Functions on the Symplectic Plane

2.1 Introduction

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function at the origin, $f(0) = 0$. Denote by $\mathcal{R}$ the group of germs of holomorphic diffeomorphisms (coordinate transformations) of $(\mathbb{C}^n, 0)$, fixing the origin. The group $\mathcal{R}$ acts naturally in the space of holomorphic functions vanishing at the origin $m \subset \mathcal{O}$ (where $\mathcal{O}$ denotes the space of germs of holomorphic functions at the origin and $m$ its maximal ideal). By classification of functions (or of the singularities of functions) we mean the corresponding classification under this action, i.e. up to right-equivalence as is usually called.

The classification of isolated singularities of functions, i.e. those having an isolated critical point at the origin, has been carried out by V. I. Arnol’d [5] and his coworkers (c.f. [3], [8] as general references). In Table 2.1 we present for convenience the list of normal forms for the simple singularities $A, D, E$, i.e. those which do not contain moduli and which are the first in hierarchy in the $\mathcal{R}$-classification.

<table>
<thead>
<tr>
<th>$A_{\mu}$</th>
<th>$D_{\mu}$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^{\mu+1} + y^2$</td>
<td>$x^2y + y^{\mu-1}$</td>
<td>$x^3 + y^4$</td>
<td>$x^3 + xy^3$</td>
<td>$x^3 + y^5$</td>
</tr>
<tr>
<td>$\mu \geq 1$</td>
<td>$\mu \geq 4$</td>
<td>$\mu = 6$</td>
<td>$\mu = 7$</td>
<td>$\mu = 8$</td>
</tr>
</tbody>
</table>

The number $\mu$ in the list is called the *multiplicity*, or the *Milnor number* of the corresponding singularity and it is an important invariant, intimately related to the analytical and topological properties of the singularity.

Suppose now that along with a germ $f$ at the origin of the plane $\mathbb{C}^2$, a germ of a symplectic form is given. The question is as to whether it is possible to reduce, not only $f$, but the pair $(\omega, f)$ simultaneously to normal form under right-equivalences. We start with the simplest example, i.e. the Morse singularity $A_1$.

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*rather loosely, those that do not contain continuous parameters in their normal forms*
2.2 The Morse-Darboux Lemma

Let \( \omega \) be a germ of a holomorphic symplectic form at the origin of the plane \( \mathbb{C}^2 \) and let \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) be a germ of a holomorphic function, \( f(0) = 0 \) with a nondegenerate critical point at the origin, i.e. such that \( df(0) = 0 \) and the Hessian matrix \( d^2 f(0) \) (i.e. the matrix of second partial derivatives) of \( f \) is nondegenerate at the origin: \( \det d^2 f(0) \neq 0 \). The well known Morse lemma (c.f. [3], [72]) implies that there exists a change of coordinates, i.e. a germ of a diffeomorphism at the origin \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \), which reduces \( f \) to its quadratic part:

\[
\Phi^* f = x^2 + y^2.
\]

On the other hand, for any germ of a symplectic structure \( \omega \), there exists a diffeomorphism \( \Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \) which reduces it to the so called Darboux normal form (c.f. [1], [10]):

\[
\Psi^* \omega = dx \wedge dy.
\]

The question as to whether it is possible to reduce the pair \((\omega, f)\) simultaneously to its normal form \((dx \wedge dy, x^2 + y^2)\) has a negative answer. Despite this fact, there exists a normal form for the pair involving exactly one functional invariant \( \xi \) as was proved by J. Vey [109]:

**Theorem 2.2.1.** Fix the symplectic form \( \omega = dx \wedge dy \). Then, there exists a symplectic diffeomorphism germ \( \Phi \) preserving the germ \( \omega \) and an analytic function \( \psi \in \mathbb{C}\{t\} \), \( \psi(0) = 0 \), \( \psi'(0) = 1 \) such that \( f \) is reduced to the normal form:

\[
\Phi^* f = \psi(x^2 + y^2).
\]

Moreover, the function \( \psi \) is unique and it is characteristic (functional invariant) for the pair \((\omega, f)\).

**Remark 2.2.1.** Before we proceed to the proof of the theorem it is important to notice here that the diffeomorphism \( \Phi \) bringing the pair to its normal form is far from being unique. To see this, denote by \( X_f \) the Hamiltonian vector field of \( f \) obtained by symplectic duality:

\[
X_f \omega = df.
\]

Then any other vector field \( v \) preserving \( f \) will be of the form \( v = gX_f \), where \( g \) is some arbitrary function germ (obviously \( v(f) = gX_f(f) = 0 \)). The point is that we can choose the function \( g \) so that \( v \) preserves also \( \omega \). Indeed,

\[
L_v \omega = d(v \omega) = d(gdf) = dg \wedge df = \psi'(h)dg \wedge dh,
\]

where \( h(x, y) = x^2 + y^2 \) is the quadratic part of \( f \). Thus, if \( g = g(h) \) is a function of \( h \), then \( L_v \omega = 0 \). Hence, the flow \( \tilde{\Phi}_t \) of \( v \) preseves both \( f \) and \( \omega \) and for different choices of \( g \) gives an abundance of such diffeomorphisms.

\footnote{or a functional modulus as is usually called}

\footnote{all the diffeomorphisms considered here are tangent to the identity, i.e. such that \( \Phi_*(0) = 1 \).}
Now for the proof, we will use the following:

**Main Lemma 2.2.2.** Let $f$ be a function germ with a nondegenerate critical point at the origin of $\mathbb{C}^2$. Then, for any germ of a symplectic form $\omega$ there exists a function germ $h$ and a uniquely defined analytic function $c \in \mathbb{C}\{t\}$, $c(0) = 1$, such that:

$$\omega = c(f)dx \wedge dy + df \wedge dh. \quad (2.1)$$

Let us see how we may deduce the Morse-Darboux lemma using the Main Lemma above, whose proof we postpone until Section 2.4.

**Proof of Theorem 2.2.1.** The proof brakes up into two steps. The first relies in an application of Moser’s homotopy method: choose coordinates such that $f$ is already in Morse normal form, i.e. $f(x, y) = x^2 + y^2$. Consider the family of symplectic forms

$$\omega_t = c(f)dx \wedge dy + tdf \wedge dh.$$

We will find first a 1-parameter family of diffeomorphisms $\Phi_t$, $\Phi_0 = Id$, $\Phi_t(0) = 0$ (where $t \in [0, 1]$), preserving the Morse germ $f$, $\Phi_t^* f = f$ and such that:

$$\Phi_t^* \omega_t = c(f)dx \wedge dy.$$

To do this, it suffices to solve the infinitesimal equation:

$$L_{v_t} \omega_t = -\frac{\partial \omega_t}{\partial t},$$

for the family of vector fields $v_t$ (where $L_{v_t}$ stands as usual for the Lie derivative along $v_t$) such that:

$$\frac{d\Phi_t}{dt} = v_t(\Phi_t), \quad v_t(0) = 0$$

and such that it preserves $f$, i.e. $L_{v_t} f = 0$. By Cartan’s identity the infinitesimal equation reduces to:

$$d(v_t \omega_t) = d(-hdf) \Rightarrow v_t \omega_t = -hdf + d\xi.$$

Now, if we choose $v_t$ as a solution of the equation:

$$v_t \omega_t = -hdf,$$

then this immediately responds to our problem since:

$$L_{v_t}(f) \omega_t = df \wedge (v_t \omega_t) = df \wedge (-hdf) = 0,$$

i.e. $L_{v_t} f = 0$ and since $v_t$ vanishes at the origin for all $t$, the time 1-map $\Phi_1$ provides the required
diffeomorphism. Thus, we have reduced the pair $(\omega, f)$ to the form:
\[
\Phi_1^* \omega = c(f) dx \wedge dy, \quad \Phi_1^* f = f.
\]
The second step of the proof is the following: consider the diffeomorphism $\Psi(x, y) = (x v(f), y v(f))$ where $v \in \mathbb{C}\{t\}$ is some function with $v(0) = 1$ (so $\Psi$ is indeed a diffeomorphism tangent to the identity). With any such function $v$ we have $\Psi_* f = \psi(f)$ for the function $\psi(f) = f v^2(f)$, with $\psi(0) = 0$ and $\psi'(0) = 1$. Now it suffices to choose $v$ so that the map $\Psi$ satisfies $\det \Psi_* = c(t)$, i.e. such that the following initial value problem is satisfied for the function $w = v^2$:
\[
t u' + w(t) = c(t), \quad w(0) = 1.
\]
As is easily verified this admits an analytic solution given by the formula:
\[
w(t) = t^{-1} \int_0^t c(s) ds.
\]
From this it follows that the diffeomorphism $\Psi \circ \Phi_1$ reduces the pair $(\omega, f)$ to the required normal form $(dx \wedge dy, \psi(f))$ and the theorem is proved.

The proof presented above is due to J. P. Françoise [30]. Its advantage is that its first part can be extended to the case of more complicated singularities (see Françoise's Theorem 2.4.1) and that it also gives a precise description of the functional invariants $c(t)$ and $\psi(t)$. In fact, as we shall indicate below, Main Lemma 2.2.2 itself, is a statement concerning the so called relative de Rham cohomology of $f$ and in particular the finiteness and freeness of the Brieskorn module; the functional invariants can be expressed in turn in terms of integrals along certain generators of the homology groups of the smooth level sets of $f$, the so called vanishing cycles (introduced by S. Lefschetz in [57]). These results form a part of what is now known as Gauss-Manin theory, the simplest case of which is the Morse case, presented below.

### 2.2.1 Geometric Description of the Invariants: Integrals Along Vanishing Cycles

Consider for instance the real case, i.e. $\omega$ is a germ of a symplectic form on the real plane and $f(x, y) = x^2 + y^2$ is real. Then, the function
\[
V(t) = \int_{f \leq t} \omega,
\]
measures, for $t \geq 0$, the area enclosed by the level sets of $f$, which are closed discs centered at the origin of radius $t^{1/2}$. Obviously, the function $V(t)$ is an invariant of the pair $(\omega, f)$ (i.e. independent of the coordinate systems). In the case where $\omega = c(f) dx \wedge dy + df \wedge dh$ as in Main Lemma 2.2.2 then the invariant $c(t)$ (and thus also $\psi(t)$) above can be expressed concretely in terms of the integral
$V(t)$: indeed, in polar coordinates $(r, \theta)$ we have:

$$V(t) = \int_0^{1/2} \int_0^{2\pi} r c(r^2) dr d\theta = 2\pi \int_0^{1/2} r c(r^2) dr$$

and taking the derivative we obtain:

$$c(t) = \frac{V'(t)}{\pi},$$

i.e. $c(t)$ measures the rate of change of the area enclosed by the real level sets (discs) $f(x, y) \leq t$. In fact, this real picture is not deceiving at all even when we consider the complex analytic case.

We recall the geometry of the problem (see figure 2.2.1). The level sets $X_t = \{(x, y) \in \mathbb{C}^2 / f(x, y) = t\}$ of the function $f$ are Riemann surfaces which, for $t \neq 0$ are all diffeomorphic to the infinite cylinder: indeed, the Riemann surface of the function $y = \sqrt{t - x^2}$ is obtained by two copies of the complex plane branched along the points $x = \pm \sqrt{t}$. Going once around one of the two points we arrive at a point lying in the other sheet from the initial one, while going once around both points we arrive at the same sheet. The image of the branch cut is exactly the cycle which corresponds to the circumference of the cylinder. Denote it by $\gamma(t)$. It is obvious that for $t \to 0$, the two branch points collide and the corresponding cycle $\gamma(t)$ shrinks to a point, the vertex point of the cone $X_0 = \{y^2 = -x^2\}$ (the singular level set of $f$). For this reason, Lefschetz called the cycle $\gamma(t)$ a vanishing cycle (c.f. [4] and references therein). Notice also that for all $t \neq 0$ the (compact) first homology group $H_1(X_t; \mathbb{Z})$ is 1-dimensional (i.e. it has rank equal to 1) and is generated exactly by the vanishing cycle $\gamma(t)$. Moreover, for all $t$ real and positive the cylinder $X_t$ contracts onto its real part, i.e. onto the vanishing cycle $\gamma(t)$ (see lower part of figure 2.2.1).

Given now an arbitrary holomorphic 1-form $\alpha$ in $\mathbb{C}^2$, its restriction $\alpha|_{X_t}$ on each one of the smooth level sets $X_t$, $t \neq 0$, gives for every $t$ a holomorphic 1-form which is closed (because it is of maximal degree on $X_t$) and thus we may consider its integral along the vanishing cycle $\gamma(t)$ as a function of $t$:

$$V(t) = \int_{\gamma(t)} \alpha.$$

It is important to notice first that if $\alpha$ is of the form $g df + dh$ for some functions $g$, $h$, then the above integral vanishes identically and for this reason we can consider 1-forms modulo elements of the form $\Omega^0 df + d\Omega^0$ (where $\Omega^0$ denotes the space of holomorphic functions). Moreover, the integral $V(t)$ is a holomorphic function of $t \neq 0$ and its derivative is given by:

$$V'(t) = \int_{\gamma(t)} \frac{d\alpha}{df},$$

where the integrand is defined as follows: since $X_t = \{f = t\}$ is smooth it means that $df \neq 0$ at points of $X_t$ and thus there exists a 1-form $\eta$ such that $d\alpha = df \wedge \eta$ (this can be easily verified by taking local coordinates). Of course $\eta$ can be chosen modulo terms of the form $h df$. Then

$$\frac{d\alpha}{df} = \eta.$$

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Figure 2.1: Degeneration of the family of Riemann surfaces $x^2 + y^2 = t$. The bolded lines correspond to the vanishing cycle, whereas in the lower part of the figure the real part is depicted.
Notice also that the 1-form $\eta$ is not uniquely defined, but its restriction $\eta|_{X_t}$ on the level set $X_t$ is. It is called the Gelfand-Leray form of $da$. These claims, i.e. holomorphicity of the integral, the formula for its derivative and the existence of the Gelfand-Leray form, are rather non immediate and they will be verified later in this chapter in terms of Gauss-Manin theory.

Suppose now that $\alpha$ is a primitive of $\omega = c(f)dx \wedge dy + df \wedge dh$, i.e. $da = \omega$. From Main Lemma 2.2.2 it follows that the 1-form $\alpha$ can be chosen in the form:

$$\alpha = w(f)\alpha_0 - hdf + dg,$$

(2.3)

where $\alpha_0 = (xdy - ydx)/2$ is a primitive of the standard symplectic form $\omega_0 = dx \wedge dy$ and the function $w \in \mathbb{C}\{t\}$ satisfies an equation of the form $\frac{2.2}{5}$:

$$tw'(t) + w(t) = c(t), \quad w(0) = 1.$$

Indeed, this follows from the relation:

$$df \wedge \alpha_0 = f\omega_0.$$

Then, we immediately obtain:

$$V(t) = w(t)V_0(t),$$

where $V_0(t) = \int_{\gamma(t)} \alpha_0$. Of course, the function $V_0(t)$ is again a holomorphic function of $t$ and moreover, it satisfies the differential equation $\frac{5}{5}$

$$tV'_0(t) = V_0(t).$$

The latter claim can be verified as follows: write $\alpha_0$ for the integrand 1-form. Then $d\alpha_0 = \omega_0$ is the standard symplectic form and the Gelfand-Leray 1-form associated to it is:

$$\frac{\omega_0}{df} = \frac{\alpha_0}{f}.$$

Thus

$$V'_0(t) = \int_{\gamma(t)} \frac{\alpha_0}{f} = \frac{1}{t} \int_{\gamma(t)} \alpha_0,$$

which proves the claim (of course $V_0(t) = \pi t$ measures the area enclosed by the vanishing cycle $\gamma(t)$). Taking now the derivative of the function $V(t)$ we have from the above:

$$V'(t) = w'(t)V_0(t) + w(t)V'_0(t) = (w'(t) + \frac{w(t)}{t})V_0(t),$$

i.e.

$$tV'(t) = c(t)V_0(t),$$

\[\text{because both the integrand and the cycle of integration depend on the parameter } t \text{ in the integral } V(t)\]

\[\text{this is the simplest example of what is called Picard-Fuchs equation of the singularity.}\]
which gives the desired expression of the invariant \( c(t) \) in terms of the integral \( V(t) \):

\[
c(t) = \frac{tV'(t)}{V_0(t)} = \frac{V'(t)}{\pi}.
\]

From the result above and from equation (2.2), we immediately obtain also the expression of the invariant \( \psi(t) \) in the normal form \((dx \wedge dy, \psi(x^2 + y^2))\) of the Morse-Darboux lemma:

\[
\psi(t) = \frac{V(t)}{\pi}.
\]

2.3 A Review of Gauss-Manin Theory for Isolated Singularities on the Plane

The results of the previous sections admit a generalisation in the case where the function \( f \) has more complicated (degenerate) singularities. The description of the invariants associated to the pair \((\omega, f)\) in terms of integrals along the vanishing cycles of the singularity, is a part of what is known as Gauss-Manin theory, the main results of which we describe below. We follow closely the works of Brieskorn \[15\] and Malgrange \[64\]. Most of the results are valid in higher dimensions and in the case of isolated complete intersections as well, studied by G. M. Greuel \[43\]. In Chapter 5 (and partially in Chapter 3) we will present another generalisation for the case of isolated boundary singularities.

2.3.1 (Co)homological Milnor Bundles, Topological Gauss-Manin Connection and Picard-Lefschetz Monodromy

To a singularity \( f = f(x, y) \) at the origin of the plane \( \mathbb{C}^2 \) we associate its local algebra (c.f. \[6\]):

\[
\mathcal{Q}_f = \frac{\mathcal{O}}{(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})},
\]

where the ideal \( J_f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \) in the denominator is the tangent space to the \( \mathcal{R} \)-orbit of \( f \), i.e. under right-equivalences (as usual \( \mathcal{O} \) is the algebra of germs of analytic functions at the origin of \( \mathbb{C}^2 \)). The \( \mathbb{C} \)-dimension \( \mu \) of this vector space is called the multiplicity, or the Milnor number of the singularity \( f \) and it is an important invariant. Recall that a singularity \( f \) is called isolated (in the sense that its critical points are isolated) whenever its Milnor number \( \mu \) is finite\(^6\). Topologically, the Milnor number can be interpreted as the rank of the first homology group \( H_1(X_t; \mathbb{Z}) \) of the smooth Milnor fiber of \( f \), i.e. of the level set \( f^{-1}(t) \) intersected with a small ball centered at the origin (see figure 2.3.1). Let us be somewhat more precise on that:

In order to study the topology of the level sets of the germ \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) it is convenient, following Milnor \[73\], to intersect them with a sufficiently small ball \( B_r \) centered at the origin. We

\(^6\)this follows for example by the analytic Nullstellensatz c.f. \[59\] (see also \[3\] for a standard proof)
do this as follows (c.f. [4]): choose a holomorphic representative $g : B_r \to T$, $T = g(B_r)$ of the germ $f$ such that:

(a) the level set $g^{-1}(0)$ is transverse to the boundary circle $\partial B_\epsilon$ for all $\epsilon \leq r$ and it has an isolated singularity at the origin (i.e. it is not smooth at the origin),

(b) the level sets $g^{-1}(t)$ are transverse to some boundary circle $\partial B_\epsilon$ above the closure $t \in \bar{S}$ of an open disc $S \subset T$ centered at the origin.

A standard or Milnor representative $f : X \to S$ of the germ $f$ is then the restriction of $g$ on the set $X = g^{-1}(S) \cap \mathring{B}_\epsilon$. Write $X_0$ for the singular fiber. Then, by Ehresmann’s fibration theorem\footnote{this is a classical result in differential topology which states that a proper surjective submersion between two $C^\infty$-manifolds is a locally trivial fibration.} the restriction $f : X \setminus X_0 \to S \setminus 0$ is a locally trivial $C^\infty$-fibration over the complement of the origin in $S$. We denote this map by $f : X^* \to S^*$. Now, each of the smooth fibers $X_t$, $t \in S^*$ is an open Riemann surface and thus it has the homotopy type of a bouquet of circles. The so called Milnor-Palamodov theorem gives the precise number of these circles:

\textbf{Theorem 2.3.1 (}\cite{[13], [18]}\textbf{).} The manifold $X_t$ has the homotopy type of a bouquet of $\mu$ circles, where $\mu = \dim_{\mathbb{C}}Q_f$.

It follows that the first (co)homology group $H_1(X_t; \mathbb{Z})$ (resp. $H^1(X_t; \mathbb{Z})$) is free of rank $\mu$:

$$H_1(X_t; \mathbb{Z}) \cong \mathbb{Z}^\mu.$$
and a basis can be obtained by the classes of the $\mu$ cycles defining the homotopy type of $X_t$. Notice that for $t \to 0$ these cycles shrink at the origin and for this reason they are usually called vanishing cycles (and the first (co)homology is called vanishing (co)homology as well, c.f. [4] for a detailed description).

In order to study the variations in (co)homology of the fibers $X_t$ as we travel along the base, it is convenient to consider the cohomology with complex coefficients:

$$H^1(X_t; \mathbb{C}) \cong H^1(X_t; \mathbb{Z}) \otimes \mathbb{C},$$

and accordingly for the homology. Indeed, since the map $f : X^* \to S^*$ is a $C^\infty$-fiber bundle, the cohomology vector spaces $H^1(X_t; \mathbb{C})$ glue together to form the fibers of a vector bundle, with locally constant transition functions, i.e. a local system in algebraic terminology (c.f. [20]):

$$R^1 f_* \mathbb{C}_{X^*} := \bigcup_{t \in S^*} H^1(X_t; \mathbb{C}) \to S^*.$$

This is known as the cohomology (Milnor) bundle. Dually, there is also defined a homology (Milnor) bundle:

$$\left(R^1 f_* \mathbb{C}_{X^*}\right)^* := \bigcup_{t \in S^*} H_1(X_t; \mathbb{C}) \to S^*,$$

with fibers the first homology vector spaces $H_1(X_t; \mathbb{C})$ of the Milnor fibers $X_t$. These bundles are holomorphic, flat vector bundles, endowed with connections $\nabla$ (resp. $\nabla^*$), the so called (topological) Gauss-Manin connections, defined by the condition that their horizontal sections are spanned by the local systems $R^1 f_* \mathbb{C}_{X^*}$ (resp. $(R^1 f_* \mathbb{C}_{X^*})^*$). More precisely, let

$$\mathcal{H}^1(X^*/S^*) = R^1 f_* \mathbb{C}_{X^*} \otimes_{\mathcal{O}_{S^*}} \mathcal{O}_{S^*}$$

be the sheaf of sections of the cohomology bundle (this is a coherent, locally free sheaf of $\mathcal{O}_{S^*}$-modules as the sheaf of sections of a vector bundle). Then, the covariant derivative associated to the connection $\nabla$:

$$D : \mathcal{H}^1(X^*/S^*) \to \mathcal{H}^1(X^*/S^*)$$

is determined by the condition:

$$\ker D \cong R^1 f_* \mathbb{C}_{X^*}.$$

In particular, if $f = t$ is a local coordinate in a neighborhood of $t \in S^*$, then the covariant derivative $D$ is just differentiation along the vector field $\frac{d}{dt}$. Of course, one obtains dually in homology:

$$D^* : \mathcal{H}_1(X^*/S^*) \to \mathcal{H}_1(X^*/S^*)$$

---

*the locally constant sheaf $R^1 f_* \mathbb{C}_{X^*}$ denotes the 1st direct image sheaf of the constant sheaf $\mathbb{C}_{X^*}$ and it is defined by the homology presheaf:

$$S^* \ni U \mapsto H^1(f^{-1}(U), \mathbb{C}_{X^*}).$$
and
\[ \ker D^* \cong (R^1f_*C_{X^*})^*. \]

Now, it is a well known fact (c.f. [20]) that any local system is determined by its monodromy and conversely, the monodromy determines the connection. In the case of the (co)homological Milnor bundle with the Gauss-Manin connection, going around the origin \( 0 \in S \) on the base, gives a diffeomorphism of the initial fiber \( X_t \) to itself, and thus, an isomorphism in the level of (co)homology:

\[ T : H^1(X_t; \mathbb{C}) \to H^1(X_t; \mathbb{C}) \]

(resp. \( T^* : H_1(X_t; \mathbb{C}) \to H_1(X_t; \mathbb{C}) \)).

This is called the Picard-Lefschetz monodromy transformation and it is exactly the monodromy of the corresponding Gauss-Manin connection \( D \) (resp. \( D^* \)). A famous result here is the so called Monodromy Theorem, concerning the Jordan structure of the Picard-Lefschetz monodromy:

**Theorem 2.3.2.** The eigenvalues of the Picard-Lefschetz monodromy operator \( T \) are roots of unity and the maximal size of Jordan blocks does not exceed 2.

The original proofs of this theorem concern the global case, i.e. for a proper morphism between algebraic manifolds \( f : X \to S \), and they are topological in nature (due to A. Groethendieck, A. Landman, C. H. Clemens, c.f. [4], [56] and references therein). In [15], E. Brieskorn gave an analytic proof of the part of the monodromy theorem concerning the eigenvalues of the monodromy operator using holomorphic differential forms. According to P. Deligne [20] Brieskorn’s proof can be transferred to the global case as well. Below we review some of the main parts of Brieskorn’s construction.

### 2.3.2 Relative de Rham Cohomology, Brieskorn Modules and the Analytic Gauss-Manin Connection

Brieskorn’s construction relies on the fact that since the fiber \( X_t \) is Stein, its cohomology can be computed using holomorphic differential forms.

**Coherence and Freeness of the Relative de Rham Cohomology Sheaves**

To the standard representative \( f : X \to S \) (which is a Stein morphism) we associate the relative de Rham complex (c.f. [15], [45], [46]):

\[ \Omega^*_X/S : 0 \to \Omega^0_X \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to 0, \]

where

\[ \Omega^p_{X/S} := \frac{\Omega^p_X}{df \wedge \Omega^{p-1}_X}, \quad p = 1, 2, \]
and the (relative) differential $d_{X/S}$ is induced from the (absolute) differential $d_X$ of the complex $\Omega^\bullet_X$ of holomorphic forms on $X$ (we will write simply $d$ when there is no confusion). The relative de Rham complex is a complex of coherent sheaves of $O_X$-modules with a $f^{-1}O_S$-linear differential:

$$d_{X/S}(f\alpha) = fd_{X/S}\alpha, \quad \alpha \in \Omega^p_{X/S}.$$  

Notice that the stalk at the origin of the relative de Rham complex $\Omega^\bullet_{X/S}$, is exactly the complex:

$$\Omega^\bullet_{X/S,0} := \Omega^\bullet_f : 0 \to \Omega^0 \to \Omega^1_f \to \Omega^2_f \to 0,$$

where

$$\Omega^p_f := \frac{\Omega^p}{df \wedge \Omega^{p-1}}, \quad p = 1, 2,$$

and $\Omega^\bullet_{X,0} = \Omega^\bullet$ is the (absolute) complex of germs of holomorphic forms at the origin. The relative de Rham complex $\Omega^\bullet_f$ is in turn independent of the standard representative of $f$, but it depends only on the germ of the singularity at the origin. Moreover, as it is easy to see, multiplication with the 2-form $dx \wedge dy$ gives a (non-canonical) isomorphism of the local algebra of the singularity $Q_f$ with the last term $\Omega^2_f$, i.e. there is an isomorphism of $\mu$-dimensional $\mathbb{C}$-vector spaces:

$$\Omega^2_f \cong Q_f.$$

Let us describe now the cohomologies of these complexes. The *relative de Rham cohomology* sheaves $H^p_{dR}(X/S), p = 0, 1, 2$, are sheaves of $O_S$-modules and they are defined by the hyperdirect image sheaves $\mathcal{R}^p f_* \Omega^\bullet_{X/S}$, the latter defined in turn by the hypercohomology presheaves:

$$S \supset U \mapsto \mathbb{H}^p(f^{-1}(U), \Omega^\bullet_{X/S}) \cong H^p(\Omega^\bullet_{X/S}(f^{-1}(U))) \cong H^p(\Omega^\bullet_{X/S}|_{f^{-1}(U)}),$$

where the isomorphisms follow from Cartan Theorems A and B. If $U$ does not contain the origin $0 \in S$, i.e. for $t \in S^*$, there is a relative de Rham theorem (see below) according to which the de Rham cohomology sheaves are isomorphic to the sheaf of sections of the cohomology Milnor bundle:

$$\mathcal{H}^p_{dR}(X^*/S^*) \cong H^p(X^*/S^*). \tag{2.4}$$

Indeed, at the smooth points of $f : X^* \to S^*$ there is a relative Poincaré lemma which implies that the complex $\Omega^\bullet_{X^*/S^*}$ is a resolution of the sheaf $f^{-1}O_{S^*}$. This establishes an isomorphism of $O_{S^*}$-modules:

$$\mathcal{R}^p f_* \Omega^\bullet_{X^*/S^*} \cong R^p f_* f^{-1}O_{S^*},$$

and the required isomorphism (2.4) is obtained by showing (c.f. [61]) that the natural homomorphism:

$$R^p f_* C_{X^*} \otimes_{C_{S^*}} O_{S^*} \to R^p f_* f^{-1}O_{S^*}$$

is an isomorphism of $O_{S^*}$-modules. If $U$ contains the origin though, the Milnor cohomology sheaves are not defined (because of the singularity of the fiber $X_0$) but the de Rham cohomology sheaves are well defined instead. The following theorem, known as the Brieskorn-Deligne theorem, concerns
the properties of the extension of the de Rham cohomology sheaves at the origin:

**Theorem 2.3.3** ([15]). The sheaves $\mathbb{R}^p f_* \Omega^\bullet_{X/S}$ are coherent sheaves of locally free $\mathcal{O}_S$-modules. Moreover there is a canonical isomorphism of $\mathcal{O}_{S,0}$-modules:

$$\left(\mathbb{R}^p f_* \Omega^\bullet_{X/S}\right)_{0} \cong H^p(\Omega^\bullet_f).$$  \hfill (2.5)

**Sketch of the Proof:** In [15] Brieskorn proved the coherence of the de Rham cohomology sheaves by embedding the morphism $f : X \to S$ in a projective morphism $f : Y \to S$ and then he used Grauert’s coherence theorem [43] and a long exact sequence in cohomology argument, induced by the embedding $X \to Y$. In more modern terms, the coherence theorem, as well as the isomorphism (2.5), is a direct corollary of a relative version of Kielh-Verdier type theorem, which relates the coherence of the direct image sheaves with the relative constructibility of a complex (c.f. [34] and references therein). To prove that the de Rham cohomology sheaves are locally free, it suffices to prove that their stalks at the origin $H^p(\Omega^\bullet_f)$ are free $\mathbb{C}\{f\}(=\mathcal{O}_{S,0})$-modules. Indeed, as we shall see below, away from the origin the sheaves $\mathcal{H}^p_{dR}(X^*/S^*)$ carry natural Gauss-Manin connections which makes them locally free\[9\]. The freeness of the modules $H^p(\Omega^\bullet_f)$, $p = 0, 1, 2$ and in particular of $H^1(\Omega^\bullet_f)$ (which is the only interesting case) was also proved by Brieskorn in [15], by showing that freeness of a certain extension of this module (the so called Brieskorn module c.f. below) is equivalent to the absence of torsion in the de Rham cohomology of the singular curve germ $X_0 = \{ f = 0 \}$, a fact already known to D. Mumford c.f. [15]. Later we will give also a different proof of this result in terms of “Lagrangian deformation theory” (see Theorem 2.4.2). \hfill \Box

From the above it immediately follows that the de Rham cohomology sheaves $\mathcal{H}^p_{dR}(X/S)$ are the natural coherent extensions at the origin $0 \in S$ of the sheaves of sections $\mathcal{H}^p(X^*/S^*)$ of the corresponding cohomology bundles. Moreover, these extensions $H^p(\Omega^\bullet_f)$ are free $\mathbb{C}\{f\}$-modules of finite type and they depend only on the germ $f$ (and not on the choice of the standard representative). In particular, from Milnor’s theorem 2.3.1 it follows:

**Corollary 2.3.4** ([15] (see also [64] for a different proof)). Let $f$ be an isolated singularity at the origin of the plane of Milnor number $\mu$. Then:

$$H^p(\Omega^\bullet_f) \cong \begin{cases} \mathbb{C}\{f\}, & p = 0, \\ \mathbb{C}\{f\}^\mu, & p = 1, \\ 0, & p > 1. \end{cases}$$

**Remark 2.3.1.** The cohomologies $H^p(\Omega^\bullet_f)$ for $p = 0$ and $p = 2$ are in fact easy to compute by hand. Indeed, for the zeroth cohomology module we have:

$$H^0(\Omega^\bullet_f) = \ker(d : \Omega^0 \to \Omega^1_f) \cong \mathbb{C}\{f\},$$

because if $g$ is such that $dg = hdf$ for some function germ $h$, then $g$ is a function of $f$ (because $g$ is

---

9because a module of finite type endowed with a connection is free, c.f. [64].
constant on the fibers of \( f \). For the second cohomology module we have:

\[
H^2(\Omega^*_f) = \frac{\Omega^2_f}{d^\Omega^*_f} = 0,
\]

because any 2-form of the form

\[
\omega = \omega' + df \wedge \beta,
\]

can be represented in the form:

\[
\omega = d\alpha + df \wedge \beta,
\]

where \( \omega' = d\alpha \) (by Poincaré lemma).

The Analytic Gauss-Manin Connection and Brieskorn modules

Below we will define, following Brieskorn [15], the \textit{(analytic) Gauss-Manin connection} on the de Rham cohomology sheaf \( H^1_{dR}(X^*/S^*) \), and we will show its coincidence with the topological Gauss-Manin connection on the sheaf \( H^1(X^*/S^*) \) of sections of the cohomology bundle. Then we will study the corresponding Brieskorn modules (lattices) which are certain extensions of the first cohomology module \( H^1(\Omega^*_f) \). The extension of the analytic Gauss-Manin connection at the origin “lives” naturally on the localisation of these modules at the origin, and it is meromorphic, with a regular-singular point at the origin. To start, let us analyse in more detail first the de Rham isomorphism \( (2.4) \):

Let \( \gamma(t) \) be a family of cycles representing a locally constant (horizontal) section of the Milnor homology bundle \( H^1_{dR}(X^*/S^*) \cong (H^1(X^*/S^*))^* \). Let also \( \alpha \in \Omega^1_{X/S} \) be a relative form representing a cohomology class in \( H^1_{dR}(X/S) \). Then, for each \( t \in S^* \) the integral

\[
I(t) := \langle \alpha, \gamma(t) \rangle := \int_{\gamma(t)} \alpha
\]

is well defined, it is nondegenerate (it takes zero values only if \( \alpha \) is relatively exact or \( \gamma(t) \) is a relative boundary) and it defines a holomorphic (multivalued) function of \( t \). Indeed, the latter claim can be proved using the Leray residue theorem (c.f. [58] and also [4], [56], [82]), according to which:

\[
I(t) = \int_{\gamma(t)} \alpha = \frac{1}{2\pi i} \int_{\sigma(t)} \frac{df \wedge \alpha}{f-t},
\]

where

\[
\sigma : H_1(X_t; \mathbb{C}) \to H_2(X \setminus X_t; \mathbb{C})
\]

is the Leray boundary operator. This is defined briefly as follows: choose a tubular neighborhhod \( N \) of the fiber \( X_t \) and let \( \nu : \partial N \to X_t \) be the fibration by circles, obtained by the restriction of the natural projection \( N \to X_t \) on the boundary \( \partial N \). Then the image of a cycle \( \gamma(t) \) under the Leray
boundary operator \( \sigma \) is just the cycle defined by the preimage of \( \gamma(t) \) under the map \( \nu \):

\[
\sigma \gamma(t) = \nu^{-1}(\gamma(t)).
\]

Using the Leray residue formula above one shows immediately that the integral \( I(t) \) is holomorphic in \( t \neq 0 \), since the integrand in the right hand-side is holomorphic in \( t \) and the cycle \( \sigma \gamma(t) \) is independent of \( t \). Thus, the integration map:

\[
I : \mathcal{H}^1_{dR}(X^*/S^*) \times \mathcal{H}_1(X^*/S^*) \rightarrow \mathcal{O}_{S^*},
\]

defines a duality isomorphism [2,4 for \( p = 1 \]):

\[
\mathcal{H}^1_{dR}(X^*/S^*) \cong (\mathcal{H}_1(X^*/S^*))^* \cong \mathcal{H}^1(X^*/S^*),
\]

obtained by integration of holomorphic 1-forms along the (locally constant) cycles of the singularity.

To define now the analytic Gauss-Manin connection on the de Rham cohomology sheaf:

\[
D : \mathcal{H}^1_{dR}(X^*/S^*) \rightarrow \mathcal{H}^1_{dR}(X^*/S^*),
\]

it suffices to calculate the derivative of the integral \( I(t) \), since:

\[
I'(t) = \frac{d}{dt} \langle \alpha, \gamma(t) \rangle = \langle D\alpha, \gamma(t) \rangle.
\]

But

\[
I'(t) = \frac{d}{dt} \int_{\gamma(t)} \alpha = \frac{1}{2\pi i} \int_{\sigma\gamma(t)} \frac{df \wedge \alpha}{(f-t)^2} = \frac{1}{2\pi i} \int_{\sigma\gamma(t)} \frac{d\alpha}{f-t} = \int_{\gamma(t)} \eta,
\]

where the 1-form \( \eta \in \Omega^1_{X^*/S^*} \) is the Gelfand-Leray form of \( d\alpha \):

\[
\eta := \frac{d\alpha}{df} := D\alpha,
\]

defined by the condition \( d\alpha = df \wedge \eta \) (because \( \alpha \) is relatively closed). Notice now that away from the origin, the 1-form \( \eta \) is also relatively closed, i.e. there exists a 1-form \( \beta \in \Omega^1_{X^*/S^*} \), such that \( d\eta = df \wedge \beta \) (this can be verified for example by taking local coordinates). Thus, we have defined a map:

\[
D : \mathcal{H}^1_{dR}(X^*/S^*) \rightarrow \mathcal{H}^1_{dR}(X^*/S^*),
\]

\[
D\alpha = \frac{d\alpha}{df} = \eta,
\]

which, as is easily verified, it is \( \mathbb{C} \)-linear and satisfies the Leibniz rule over \( \mathcal{O}_{S^*} \), i.e. it defines a connection on \( \mathcal{H}^1_{dR}(X^*/S^*) \). This is the (analytic) Gauss-Manin connection which, by definition (the formula of the derivative \( I'(t) \) above), coincides with the topological Gauss-Manin connection.
on the cohomology sheaf $\mathcal{H}^1(X^*/S^*)$.

The de Rham cohomology sheaf $\mathcal{H}^1_{dR}(X^*/S^*)$ extends naturally at the origin to the module $H^1(\Omega^*_f)$, but the Gauss-Manin connection does not extend to a connection on $H^1(\Omega^*_f)$ (because the germ of the form $\eta = D\alpha$ might not be relatively closed at the origin). To study the extension of the Gauss-Manin connection at the origin, Brieskorn [15] defined two extensions of the cohomology module $H^1(\Omega^*_f)$ as follows: denote by $E := H^1(\Omega^*_f)$ and consider the natural inclusion of this module in the cokernel of the differential $d : \Omega^0 \rightarrow \Omega^1_f$:

$$E \subset F := \frac{\Omega^1_f}{d\Omega^0} \cong \frac{\Omega^1}{\Omega^0 df + d\Omega^0}.$$

Consider now multiplication by $df \wedge$ in $F$. It defines an isomorphism:

$$F \xrightarrow{\text{df} \wedge} \frac{df \wedge \Omega^1}{df \wedge d\Omega^0}$$

and we thus obtain another natural inclusion:

$$F \subset G := \frac{\Omega^2}{df \wedge d\Omega^0}.$$

We have thus a sequence of inclusions of $\mathbb{C}\{f\}$-modules:

$$E \subset F \subset G,$$

whose cokernels are both isomorphic to the same $\mu$-dimensional $\mathbb{C}$-vector space:

$$\frac{F}{E} \cong \Omega^2_f, \quad \frac{G}{F} \cong \Omega^2_f.$$

Hence, we may view these modules as defining lattices in the same $\mu$-dimensional $\mathbb{C}$-vector space over the field of quotients $\mathbb{C}(f)$ of $\mathbb{C}\{f\}$:

$$\mathcal{M} = E \otimes_{\mathbb{C}(f)} \mathbb{C}(f) = F \otimes_{\mathbb{C}(f)} \mathbb{C}(f) = G \otimes_{\mathbb{C}(f)} \mathbb{C}(f).$$

The modules $F$ and $G$ are known as Brieskorn modules (or lattices) of the singularity $f$ and they are important analytic invariants. For them we have the well known equivalent of Brieskorn-Deligne’s theorem (proved by Sebastiani for the higher dimensional case and also known as the Brieskorn-Sebastiani theorem):

**Theorem 2.3.5** ([15], [92] and also [64]). *If $f$ is an isolated singularity of Milnor number $\mu$, then the Brieskorn module $G$ (and thus also $F$ and $E$) is a free module of rank $\mu$ over $\mathbb{C}\{f\}$:***

$$G \cong \mathbb{C}\{f\}^\mu.$$

**Proof.** See the references above and also Theorem 2.4.2 for an alternative proof for the 2-dimensional case.
It follows from the theorem above that a basis of the Brieskorn module $G$ is obtained, by Nakayama’s lemma, by lifting a basis of the $\mu$-dimensional $\mathbb{C}$-vector space:

$$\frac{G}{fG} \cong \frac{\Omega^2}{df \wedge d\Omega^0 + f\Omega^2}.$$ 

Let us see now how, starting from a basis of the Brieskorn module $G$, we may obtain a basis of each cohomology fiber $H^1(X_t; \mathbb{C})$, i.e. a trivialisation of the cohomology bundle $\cup_{t \in S^*} H^1(X_t; \mathbb{C})$. Notice first that from the Brieskorn-Deligne theorem, a basis $\{\alpha_1, \ldots, \alpha_\mu\}$ of the cohomology module $E = H^1(\Omega^*_X)$ extends to a basis of the locally free sheaf $\mathcal{H}^1_{dR}(X/S)$ in a neighborhood of the origin, and each fiber $\mathcal{H}^1_{dR}(X/S)|_t$ is isomorphic to the cohomology $H^1(X_t; \mathbb{C}) \otimes_{\mathbb{C}_{S^*}} \mathcal{O}_{S^*, t}$ for $t \neq 0$. Thus, the map $t \in S^* \mapsto \{\alpha_1|_{X_t}, \ldots, \alpha_\mu|_{X_t}\} \in H^1(X_t; \mathbb{C})$ gives a trivialisation of the cohomology bundle. Consider now the sheafification of the first Brieskorn module $F$:

$$\mathcal{F} := \frac{f_*\Omega^1_{X/S}}{d(f_*\Omega^0_X)},$$

and the natural short exact sequence:

$$0 \to \mathcal{H}^1_{dR}(X/S) \to \mathcal{F} \xrightarrow{d} f_*\Omega^2_{X/S} \to 0.$$ 

Since the sheaf on the right is concentrated at the origin $0 \in S$, there is an isomorphism away from the origin:

$$\mathcal{H}^1_{dR}(X^*/S^*) \cong \mathcal{F}|_{S^*},$$

and so, we may define a trivialisation of the cohomology bundle by starting from a basis of $F$ instead, and in fact of $F \otimes_{\mathbb{C}(f)} \mathbb{C}(f)$. Such a basis can be found in turn as follows: let $\{\omega_1, \ldots, \omega_\mu\}$ be a basis of the second Brieskorn module $G$. Then division by $df$ gives a basis $\{\frac{\omega_1}{df}, \ldots, \frac{\omega_\mu}{df}\}$ of $F \otimes_{\mathbb{C}(f)} \mathbb{C}(f)$. If we consider now the sheafification of $G$:

$$\mathcal{G} := \frac{f_*\Omega^2_X}{df \wedge d(f_*\Omega^0_X)},$$

and the natural short exact sequence:

$$0 \to \mathcal{F} \to \mathcal{G} \to f_*\Omega^2_{X/S} \to 0,$$

then, by the same argument as before, there is an isomorphism:

$$\mathcal{F}|_{S^*} \cong \mathcal{G}|_{S^*}.$$ 

By coherence and freeness of the Brieskorn module $G$ the basis $\{\omega_1, \ldots, \omega_\mu\}$ extends to a basis of $\mathcal{G}$ in a neighborhood of the origin, so that $\{\frac{\omega_1}{df}, \ldots, \frac{\omega_\mu}{df}\}$ extends to a basis of $\mathcal{F}$ as well. It follows that the map $t \in S^* \mapsto \{\frac{\omega_1}{df}|_{X_t}, \ldots, \frac{\omega_\mu}{df}|_{X_t}\} \in H^1(X_t; \mathbb{C})$ defines a trivialisation of the cohomology bundle.

**Remark 2.3.2.** In fact, for any $\omega \in G$, the holomorphic form $\frac{\omega}{df}|_{X_t}$ is nothing but the Poincaré
residue at $X_t$ of the form $\frac{\omega}{f-t}$:

$$\text{Res}_{X_t}(\frac{\omega}{f-t}) = \frac{\omega}{df}|_{X_t}.$$  

The map $t \in S^* \mapsto s[\omega](t) = \frac{\omega}{df}|_{X_t} \in H^1(X_t; \mathbb{C})$ is what A. N. Varchenko called the geometric section associated to $\omega$ (c.f. [107] and also [4], [56] and references therein).

Now, using the Brieskorn modules we may extend the map $D$ to two maps (which we denote by the same symbol):

$$D : E \to F, \quad D\alpha = \frac{d\alpha}{df} = \eta,$$

$$D : F \to G, \quad D\eta = D(df \wedge \eta) = d\eta,$$

which, as is easily verified, are $\mathbb{C}$-linear and satisfy the Leibniz rule of $\mathbb{C}\{f\}$ (they define “connections” on the corresponding pairs of modules in the sense of Malgrange [64]). These maps induce isomorphisms of the underlying $\mathbb{C}$-vector spaces, i.e. there exists a commutative diagram (c.f. [15], [56]):

$$\begin{array}{ccc}
F & \xrightarrow{D} & G \\
\sim & & \sim \\
D\uparrow \wr & & D\uparrow \wr \\
E & \xrightarrow{D} & F \\
\sim & & \sim \\
& & \Omega_f^2
\end{array}$$

and in fact, they induce the same meromorphic connection:

$$D : \mathcal{M} \to \mathcal{M},$$

on the localisation $\mathcal{M}$ of the Brieskorn modules at the origin. This is defined as follows: let $\omega \in \Omega^2$ be a representative of a class in $G$. Since the singularity $f$ is isolated there exists a natural number $k < \infty$ such that $f^k \omega = df \wedge \eta$, where $\eta \in \Omega^1$. Then $D(f^k \omega) = D(df \wedge \eta) = d\eta$ and by the Leibniz rule we obtain in $\mathcal{M}$:

$$D\omega = \frac{d\eta}{f^k} - k \frac{\omega}{f}.$$  

It is easy now to verify that the map thus defined is $\mathbb{C}$-linear and satisfies the Leibniz rule over $\mathbb{C}\{f\}$, i.e. it indeed defines a connection on $\mathcal{M}$. It is obviously meromorphic with a pole of degree at most $k$ at the origin.

In the next Section 2.3.3 we will discuss the important example of quasihomogeneous singularities, for which the order of the pole of the Gauss-Manin connection at the origin is equal to 1. For general isolated singularities though the order of the pole is greater than 1. One of the main properties of the meromorphic Gauss-Manin connection is the so called regularity, i.e. that the linear differential operator $D$ has a regular-singular point at the origin (c.f. [17], [20]), which implies in particular that the connection is meromorphically equivalent to a connection with a pole of order exactly equal to 1. Let us describe this in more detail:
The Regularity Theorem

Let \( \omega_1, \ldots, \omega_\mu \) be 1-forms whose classes give a basis of \( F \). Then the connection matrix \( \Gamma(t) = (\Gamma_{ij}(t)) \) is the \( \mu \times \mu \) meromorphic matrix obtained by the expression:

\[
D\omega_i = \Gamma_{ij}(f)\omega_j.
\]

Let now \( \gamma(t) \in H_1(X_t; \mathbb{C}) \) be a locally constant section of the homology bundle and consider the (multivalued) holomorphic function of \( t \neq 0 \):

\[
I_i(t) = \int_{\gamma(t)} \omega_i.
\]

Then differentiation gives a system of ordinary differential equations:

\[
I_i'(t) = \int_{\gamma(t)} D\omega_i = \int_{\gamma(t)} \Gamma_{ij}(f)\omega_j = \Gamma_{ij}(t) \int_{\gamma(t)} \omega_j = \Gamma_{ij}(t)I_j(t),
\]

which, in matrix form is written:

\[
I'(t) = \Gamma(t)I(t), \quad (2.6)
\]

where \( I(t) = (I_1(t), \ldots, I_\mu(t))^T \). The system of equations above is known as the Picard-Fuchs equations (of the Gauss-Manin connection) of the singularity \( f \). It can be viewed as expressing the condition of horizontality of the section \( \gamma(t) \) with respect to the dual Gauss-Manin connection, in a basis of \( H_1(X^*/S^*) \) dual to the one in \( F \). In fact, if \( \{\gamma_1(t), \ldots, \gamma_\mu(t)\} \in H_1(X_t; \mathbb{C}) \) is such a dual basis, then the map:

\[
(I_1(t), \ldots, I_\mu(t)) \mapsto \sum_{i=1}^\mu I_i(t)\gamma_i(t),
\]

establishes an isomorphism between the space of solutions of the Picard-Fuchs equation \((2.6)\) with the locally constant (horizontal) sections of the homology bundle. Under this isomorphism, the monodromy \( M \) of the Picard-Fuchs equation (or equivalently of the Gauss-Manin connection), obtained by analytic continuation of solutions around the origin \( t = 0 \), can be identified with the classical Picard-Lefschetz monodromy \( T^* \in \text{Aut}(H_1(X_t; \mathbb{C})) \) in homology. But the monodromy of the Picard-Fuchs equations can be easily expressed in terms of its fundamental matrix of solutions; indeed, the fundamental solution of system \((2.6)\) is given by the matrix:

\[
Y(t) = Q(t)t^R,
\]

where \( Q(t) \) is a single-valued holomorphic matrix in a punctured neighborhood of the origin \( 0 \in S \) and \( R \) is a constant matrix. The monodromy \( M \) of the Picard-Fuchs equations (and thus the Picard-Lefschetz monodromy) is then given by the matrix:

\[
M = e^{2\pi i R}.
\]

Remark 2.3.3. If \( \{\gamma_1(t), \ldots, \gamma_\mu(t)\} \in H_1(X_t; \mathbb{C}) \) is a locally constant section of the homology bundle,
then the fundamental matrix $Y(t)$ is exactly the $\mu \times \mu$-matrix whose entries are given by the integrals:

$$Y_{ij}(t) = \int_{\gamma_{ij}(t)} \omega_i,$$

where the classes of the 1-forms $\{\omega_1, ..., \omega_\mu\}$ form a basis of $F$. It is called the period matrix of the singularity $f$ (c.f. [4] and references therein).

Recall now from the classical theory of differential equations (c.f. [17]) that the point $t = 0$ is called singular if the matrix $Q(t)$ in the expression of the fundamental solution of system (2.6) cannot be continued holomorphically at the origin. The singularity is called regular (or the origin is a regular-singular point), if the matrix $Q(t)$ is meromorphic at the origin. Equivalently, the solutions $I_j(t)$ of the differential system (2.6) are of moderate growth, i.e. for $t \to 0$ and in each fixed sector $a \leq \arg t \leq b$, there exist natural numbers $K$ and $N$ such that for all $j = 1, ..., \mu$:

$$|I_j(t)| \leq K|t|^{-N}. \quad (2.7)$$

Recall also that according to a classical theorem of Sauvage (c.f. [54]), the regularity condition is equivalent to the existence of a meromorphic change of basis, such that in this new basis, the connection matrix $\Gamma(t)$ has a pole of at most first order at the origin, i.e. it is of classical Eulerian type.

The following theorem, known as the regularity theorem, was first proved by Brieskorn and it is the local analog of the regularity theorem obtained by P. A. Griffiths [44] and N. Nilson [77] for the Gauss-Manin connection on the cohomology of families of algebraic manifolds:

**Theorem 2.3.6** ([15], see also [20], [64]). The Gauss-Manin connection $D : \mathcal{M} \to \mathcal{M}$ is regular.

Brieskorn’s proof [15] relies on the embedding of the singularity $f : X \to S$ in a proper family of algebraic manifolds $f : Y \to \mathbb{P}^1$ which satisfies Griffiths’ assumptions. In [64], Malgrange gave two different proofs of the regularity theorem; one uses the properties of the analytical index of the Gauss-Manin connection (see also [63]) and the Bloom-Brieskorn theorem [15], whose proof in turn relies on resolution of singularities. The other does not use resolution of singularities and it is closer to Griffiths-Nilson original proofs. It uses in fact a better estimate of the asymptotics of integrals along the vanishing cycles:

**Proposition 2.3.7** ([64]). For any 1-form $\omega \in \Omega^1_{X/S}$ and any locally constant section $\gamma(t) \in H_1(X_t; \mathbb{C})$ in a sector containing the zero ray:

$$\lim_{t \to 0, \arg t = 0} \int_{\gamma(t)} \omega = 0.$$

**Proof.** Let us review the main parts in Malgrange’s proof [10] let $\omega \in \Omega^1_X$ represent the class of $\omega$. Fix a real $t_0 > 0$ and let $Y = f^{-1}([0, t_0]) \subset X$. Let $\gamma(t_0)$ be a 1-dimensional cycle in $X_{t_0}$ and let $\Gamma$ be a representative. By the fact that the fiber $X_{t_0}$ is contractible, it follows that $Y$ is contractible as well.

\textsuperscript{10}see also Proposition 5.2.6 in Chapter 5 for a generalisation in the relative case.
Since \( Y \) is semianalytic we may find semianalytic triangulation such that \( X_{t_0} \) is a subcomplex of \( Y \) (c.f. [60]). Thus, there exists a 2-dimensional chain \( \Delta \) such that \( \Gamma = \partial \Delta \). By the Stokes-Herrera theorem [48], we have that the integrals:

\[
I(t_0) = \int_{\gamma(t_0)} \omega = \int_{\Gamma} \omega = \int_{\Delta} d\omega
\]

are well defined. Consider now a 2-dimensional chain \( \Delta_t = f^{-1}([0, t]) \cap \Delta, \ t \in (0, t_0] \). Then \( \Delta = \Delta_t + \Delta' \) where \( \Delta' \) is a 2-dimensional chain on \( f^{-1}([t, t_0]) \) and \( \partial \Delta' = \Gamma - \Gamma_t \). It follows that \( \Gamma_t \) is a cycle representing \( \gamma(t) \) and

\[
I(t_0) = \int_{\Delta} d\omega = \int_{\Delta_t} d\omega + \int_{\Delta'} d\omega = \int_{\Delta_t} d\omega + \int_{\Gamma} \omega - \int_{\Gamma_t} \omega = \int_{\Delta_t} d\omega + \int_{\Delta_t} d\omega + I(t_0) - I(t),
\]

i.e.

\[
I(t) = \int_{\Gamma_t} \omega = \int_{\Delta_t} d\omega.
\]

But

\[
\lim_{t \to 0} \int_{\Delta_t} d\omega = \int_{\Delta_0} d\omega
\]

where \( \Delta_0 = X_0 \cap \Delta \) is a relative 1-dimensional chain on \( X_0 \). By the fact that the restriction of \( d\omega \) on the smooth part of \( X_0 \) is zero, it follows that \( \lim_{t \to 0} I(t) = 0 \) as was asserted.

**Proof of the Regularity Theorem 2.3.6.** It suffices to show the estimate (2.7). This follows from the above Proposition 2.3.7 and the Phragmén-Lindelöf theorem for the strip \( a \leq \arg t \leq b \) as in [64].

2.3.3 Monodromy and Asymptotics of Integrals Along Vanishing Cycles

From the results of the previous section it follows that the monodromy of the analytic Gauss-Manin connection \( D \) can be naturally identified with the classical Picard-Lefschetz monodromy \( T \) of the singularity \( f \). Using this identification, Brieskorn gave an analytic proof of the first part of the Monodromy Theorem 2.3.2, which can be stated as follows:

**Theorem 2.3.8 ([15]).** The eigenvalues of the Picard-Lefschetz monodromy operator are roots of unity.

**Sketch of the Proof:** The Gauss-Manin connection \( D \) is regular and thus there exists a meromorphic change of basis such that it has a pole of degree at most 1 at the origin. The residue \( \text{Res}_0 D \) of the connection is then the constant matrix \( \Gamma \) in the representation:

\[
I' = (\frac{\Gamma}{t} + \tilde{\Gamma}(t))I,
\]

of the differential system of horizontal sections in this basis, where \( \tilde{\Gamma}(t) \) is a holomorphic matrix. Since the characteristic polynomial of the Picard-Lefschetz monodromy is integral, it is constant.
under variations of $t$ and thus its roots $\lambda_j$ coincide with the numbers $e^{-2\pi i\alpha_j}$, where $\alpha_j$ are the eigenvalues of $\text{Res}_0 D$. Moreover, Brieskorn shows in [15] that the connection $D$ is algebraically defined, i.e. that for any automorphism $\phi : \mathbb{C} \to \mathbb{C}$ of the field of complex numbers, the following relation holds:

$$D_{\phi,f,H} = \phi \circ D.$$ 

It follows then from the solution of Hilbert’s VII problem that the eigenvalues $\alpha_j$ of $\text{Res}_0 D$ are rational numbers and thus, the eigenvalues of the relative monodromy operator are indeed roots of unity.

As another application of the regularity theorem, along with the monodromy theorem, we obtain an exact calculation of the asymptotics of integrals along vanishing cycles, due to Malgrange [64] and others (see [4], [56] and references therein):

**Theorem 2.3.9.** Let $\omega \in \Omega^2$ and let $\gamma(t) \in H_1(X_t; \mathbb{C})$ be a locally constant section of the homology Milnor bundle. Then, for $|t|$ sufficiently small and in each sector of $\arg t$, there is a convergent expansion of the integral:

$$I_{\omega,\gamma}(t) := \int_{\gamma(t)} \frac{\omega}{df} = \sum_{a,k} a_{a,k} t^a \frac{(\ln t)^k}{k!},$$

where:

(i.) $a_{a,k}$ are vectors in $\mathbb{C}^\mu$,

(ii.) the numbers $\alpha$ are rational numbers $> -1$ which belong in a set of arithmetic progressions with the property that $\lambda = e^{-2\pi i\alpha}$ is an eigenvalue of the Picard-Lefschetz monodromy operator in homology $H_1(X_t; \mathbb{C})$,

(iii.) the numbers $k$ are integers $0 \leq k \leq 1$, equal to the maximal size of Jordan blocks corresponding to the eigenvalue $\lambda = e^{-2\pi i\alpha}$.

**Proof.** Let $\eta \in \mathcal{F}$ be a local section of the Brieskorn module such that $D\eta = d\eta = \omega \in \mathcal{G}$. Then

$$I_{\omega,\gamma}(t) = \int_{\gamma} \frac{d\eta}{df} = \frac{d}{dt} \int_{\gamma(t)} \eta = V'_{\eta,\gamma}(t),$$

where $V_{\eta,\gamma}(t) = \int_{\gamma(t)} \eta$. Since the map $D : \mathcal{F} \to \mathcal{G}$ is an isomorphism we may study first the expansion of the integral $V_{\eta,\gamma}(t)$ into asymptotic series. Let $\Lambda = \{\lambda_1, ..., \lambda_\mu\}$ be the eigenvalues of the monodromy operator $T$ in cohomology $H^1(X_t; \mathbb{C})$. Then $\{\lambda_1, ..., -\lambda_\mu\}$ are the eigenvalues of the monodromy operator $T^*$ in homology $H_1(X_t; \mathbb{C})$. Let

$$\alpha_j = -\frac{1}{2\pi i} \ln \lambda_j$$

be the eigenvalues of the matrix $R$, where:

$$T^* = e^{2\pi i R}.$$
By the monodromy Theorem 2.3.2, the eigenvalues $\lambda_j = e^{-2\pi i \alpha_j}$ are roots of unity and so $\alpha_j$ are rational numbers defined modulo $\mathbb{Z}$. Denote by

$$L(\lambda_j) = \{\alpha_j^0, \alpha_j^0 + 1, \alpha_j^0 + 2, \ldots\}$$

the arithmetic progression with one suitable value of $\alpha_j$. Let now $\{\omega_1, \ldots, \omega_\mu\}$ be a local basis of the sheaf $\mathcal{F}$. Then the vector:

$$V(t) = (\int_{\gamma(t)} \omega_1, \ldots, \int_{\gamma(t)} \omega_\mu)^T$$

is a solution of the Picard-Fuchs equation:

$$V'(t) = \Gamma(t)V(t),$$

where $\Gamma(t)$ is the connection matrix of the Gauss-Manin connection $D$ with respect to the basis $\{\omega_1, \ldots, \omega_\mu\}$. A fundamental solution of this equation is given by the period matrix:

$$Y(t) = (\int_{\gamma_j(t)} \omega_i)_{1,j=1,\ldots,\mu},$$

where $\{\gamma_1(t), \ldots, \gamma_\mu(t)\}$ is a locally constant (horizontal) basis of the homology bundle $\cup H_1(X_t; \mathbb{C})$. By well known theorems of differential equations discussed above, the period matrix can be represented in the form:

$$Y(t) = Q(t)t^R,$$

where $Q(t)$ is a single-valued holomorphic matrix on $\mathcal{S}$. In particular, there is a constant matrix $C$ such that:

$$V(t) = Q(t)t^RC.$$

By the regularity Theorem 2.3.6, the matrix $Q(t)$ is meromorphic at the origin. After a choice of a Jordan basis of the monodromy operator and the corresponding structure of the matrix $t^R$, we obtain an expansion:

$$V(t) = \sum_{\lambda \in \Lambda} \sum_{\alpha \in L(\lambda)} \frac{1}{k!} a_{\alpha,k} t^\alpha (\ln t)^k.$$

But by Proposition 2.3.7 we have $\lim_{t \to 0} V(t) = 0$ and thus all $\alpha \geq 0$. Moreover, if $\alpha = 0$ then $a_{0,1} = 0$. Thus we have obtained the required expansion for the function $V(t) = V_{\eta,\gamma}(t)$. Then, by differentiating and using equation (2.8) we obtain the required expansion for $I_{\omega,\gamma}(t)$. Thus, it suffices to prove only (ii.) But for $\alpha = 0$ we have only constants in the expansion of $V(t)$ and thus all $\alpha > -1$ in the expansion of $I_{\omega,\gamma}(t)$. This finishes the proof. \hfill \Box

**Example 2.3.1** (Quasihomogeneous Singularities on the Plane.). Recall that a function germ $f = f(x, y)$ is called quasihomogeneous with weights $(w_1, w_2) \in \mathbb{Q}_+$ if there exists a natural number $\nu$ (which can be taken equal to 1, c.f. [3]) such that:

$$f(t^{w_1}x, t^{w_2}y) = t^{\nu}f(x, y).$$
This implies the existence of an Euler vector field $E_f$ for $f$, i.e. such that:

$$E_f(f) = \nu f.$$ 

In local coordinates $(x,y)$ the Euler vector field can be written as:

$$E_f = w_1 x \frac{\partial}{\partial x} + w_2 y \frac{\partial}{\partial y},$$

which shows in particular that if $f$ is quasihomogeneous then $f$ belongs to its Jacobian ideal $f \in J_f$. According to a famous theorem of K. Saito [88] a converse to this statement also holds, i.e. if $f$ belongs to its Jacobian ideal then it is (right-equivalent) to a quasihomogeneous germ.

Below we suppose w.l.o.g. that $\nu = 1$. In terms of differential forms, the condition $f \in J_f$ means that for any 2-form $\omega$ there exists a 1-form $\eta$ (in fact $\eta = E_f \omega$) such that $f \omega = df \wedge \eta$, i.e.:

$$fG \subseteq F.$$ 

The main thing here is that for a quasihomogeneous germ $f$ there exists an inverse inclusion as well, i.e.:

$$F \subseteq fG.$$ 

Indeed, it suffices to show that for a given 1-form $\eta$ there exists a 2-form $\theta$ and a function germ $h$ such that:

$$df \wedge \eta = f\theta + df \wedge dh. \quad (2.9)$$

This identity (called “quasihomogeneous division in the Brieskorn module” in [111]) is a restatement of the so called Poincaré lemma property for the singular curve germ $X_0 = \{ f = 0 \}$ (c.f. [38], [108]) and it can be proved easily as follows: since $\nu f \theta = df \wedge E_f \theta$, we obtain after division of (2.9) with $df$ the identity:

$$\eta - dh = E_f \theta,$$

and after taking exterior derivatives we seek a $\theta$ such that:

$$d\eta = L_{E_f} \theta.$$ 

But the mapping

$$L_{E_f} : \Omega^2 \rightarrow \Omega^2,$$

$$L_{E_f} = w_1 x \frac{\partial}{\partial x} + w_2 y \frac{\partial}{\partial y} + (w_1 + w_2)$$

is an isomorphism, viewed as a mapping in formal series, and thus, for a given $\eta$ there always exists a formal solution $\theta$. This solution can be extended then to an analytic solution in a fundamental system of neighborhoods of the origin.

Thus we have proved that for a quasihomogeneous germ $f$ there is an isomorphism:

$$fG \cong F.$$
A basis of the Brieskorn module $G$ is obtained then, by Nakayama’s lemma, by lifting a basis of the $\mu$-dimensional vector space:

$$\frac{G}{fG} \cong \frac{G}{F} \cong \Omega_f^2 \cong Q_f,$$

i.e. of the local algebra of the singularity.

Let us calculate now the Gauss-Manin connection $D = \frac{d}{df}$. Since $f G = df \wedge F$, it follows that:

$$f \frac{G}{df} = f \frac{d}{df} F = f DF = F,$$

which implies in particular that the operator $D$ has a pole of first order at the origin as was asserted.

Let now $\omega_m = e_m dx \wedge dy$ be a basis of $G$, where $e_m = x^{m_1}y^{m_2}$, $m = (m_1, m_2) \in A$, $|A| = \mu$, is a basis of monomials of the vector space $Q_f$. Since $f dx \wedge dy = df \wedge (E_f dx \wedge dy)$ it follows that $f \omega_m = df \wedge \omega_m = df \wedge (e_m E_f dx \wedge dy)$, and thus:

$$f D \omega_m = \frac{1}{f}(-\omega_m + d(e_m E_f dx \wedge dy)) = \frac{1}{f}(-\omega_m + \sum_{i=1}^{2} w_i (m_1 + 1) \omega_m),$$

i.e. the 2-forms $\omega_m$ are exactly the eigenvectors of the operator $f D$:

$$f D \omega_m = (\alpha(m) - 1) \omega_m,$$

where:

$$\alpha(m) = \sum_{i=1}^{2} w_i (m_i + 1).$$

Thus, the residue $\text{Res}_0 D$ is a semisimple operator and in particular, the Picard-Lefschetz monodromy operator:

$$T = e^{-2\pi i \text{Res}_0 D}$$

is semisimple, with eigenvalues:

$$\lambda_m = e^{-2\pi i \alpha(m)}.$$

Moreover, for any 2-form $\omega$ and any locally constant cycle $\gamma(t) \in H_1(X_t; \mathbb{C})$ there exists an asymptotic expansion for $t \to 0$:

$$I(t) = \int_{\gamma(t)} \omega = \sum_{\lambda \in \Lambda} \sum_{\alpha \in L(\lambda)} a_\alpha t^{\alpha-1},$$

where for each $\lambda_m$, $\alpha \in L(\lambda_m) = \{\alpha(m), \alpha(m) + 1, \alpha(m) + 2, \ldots\}$ and $a_\alpha \in \mathbb{C}^\mu$.

Let us calculate the numbers $\alpha(m)$ for the simple singularities on the plane, i.e. the $A, D, E$ singularities in Arnol’d’s list \[6\]. These are all quasihomogeneous polynomials with positive rational weights and quasihomogeneous degree equal to 1.

$A_\mu$: $f = x^{\mu+1} + y^2$, $\mu \geq 1$, $(w_1, w_2) = (\frac{1}{\mu+1}, \frac{1}{2})$.

$$Q_f = \text{span}_\mathbb{C}\{1, x, x^2, \ldots, x^{\mu-1}\},$$

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\[ G = \text{span}_{\mathbb{C}(f)} \{ dx \wedge dy, xdx \wedge dy, \ldots, x^{\mu-1}dx \wedge dy \}, \]
\[ \alpha(m) = \{ \frac{\mu + 3}{2\mu + 2}, \frac{\mu + 5}{2\mu + 2}, \ldots, \frac{3\mu + 1}{2\mu + 2} \}. \]

\( D_\mu: f = x^2y + y^{\mu-1}, \mu \geq 4, (w_1, w_2) = (\frac{\mu - 2}{2\mu - 2}, \frac{1}{\mu - 1}). \)

\[ Q_f = \text{span}_{\mathbb{C}} \{ 1, x, y, \ldots, y^{\mu-2} \}, \]
\[ G = \text{span}_{\mathbb{C}(f)} \{ dx \wedge dy, xdx \wedge dy, ydx \wedge dy, \ldots, y^{\mu-2}dx \wedge dy \}, \]
\[ \alpha(m) = \{ \frac{\mu}{2\mu - 2}, \frac{2\mu - 2}{2\mu - 2} = 1, \frac{\mu + 2}{2\mu - 2}, \frac{\mu + 4}{2\mu - 2}, \ldots, \frac{3\mu - 2}{2\mu - 2} \}. \]

\( E_6: f = x^3 + y^4, \mu = 6, (w_1, w_2) = (\frac{1}{3}, \frac{1}{4}). \)

\[ Q_f = \text{span}_{\mathbb{C}} \{ 1, x, y, y^2, xy, xy^2 \}, \]
\[ G = \text{span}_{\mathbb{C}(f)} \{ dx \wedge dy, xdx \wedge dy, ydx \wedge dy, y^2dx \wedge dy, xydx \wedge dy, xy^2dx \wedge dy \}, \]
\[ \alpha(m) = \{ \frac{7}{12}, \frac{11}{12}, \frac{10}{12}, \frac{14}{12}, \frac{17}{12} \}. \]

\( E_7: f = x^3 + xy^3, \mu = 7, (w_1, w_2) = (\frac{1}{3}, \frac{3}{5}). \)

\[ Q_f = \text{span}_{\mathbb{C}} \{ 1, x, y, y^2, xy, x^2y \}, \]
\[ G = \text{span}_{\mathbb{C}(f)} \{ dx \wedge dy, xdx \wedge dy, ydx \wedge dy, y^2dx \wedge dy, x^2dy \wedge dy, xydx \wedge dy, x^2ydx \wedge dy \}, \]
\[ \alpha(m) = \{ \frac{5}{9}, \frac{8}{9}, \frac{7}{9}, \frac{11}{9}, \frac{9}{9} = 1, \frac{10}{9}, \frac{13}{9} \}. \]

\( E_8: f = x^3 + y^5, \mu = 8, (w_1, w_2) = (\frac{1}{3}, \frac{1}{5}). \)

\[ Q_f = \text{span}_{\mathbb{C}} \{ 1, x, y, y^2, xy, x^2y, xy^3 \}, \]
\[ G = \text{span}_{\mathbb{C}(f)} \{ dx \wedge dy, xdx \wedge dy, ydx \wedge dy, y^2dx \wedge dy, xydx \wedge dy, y^3dx \wedge dy, xy^2dx \wedge dy, xy^3dx \wedge dy \}, \]
\[ \alpha(m) = \{ \frac{8}{15}, \frac{13}{15}, \frac{11}{15}, \frac{14}{15}, \frac{16}{15}, \frac{17}{15}, \frac{19}{15}, \frac{22}{15} \}. \]

### 2.4 Applications in Symplectic Deformation Theory

Here we give an application of the results obtained so far in the problem of classification of pairs \((\omega, f)\) where \(\omega\) is a germ of a symplectic form and \(f\) is an isolated singularity at the origin of the plane.
2.4.1 Equivalence of Symplectic Structures: Normal Forms and Functional Invariants

We start first with a direct corollary of the finiteness and freeness of the Brieskorn module $G$ concerning the classification of symplectic forms on the plane relative to diffeomorphisms tangent to the identity and preserving the singularity $f$. Write $\mathcal{R}_f$ for the group of germs of these diffeomorphisms, i.e. such that:

$$\Phi^* f = f, \quad \Phi(0) = 0, \quad \Phi_*(0) = Id.$$  

Two germs of symplectic forms at the origin will be called $\mathcal{R}_f$-equivalent (or equivalent for brevity) if they belong in the same orbit under the action of $\mathcal{R}_f$ in the space of germs of symplectic forms $\Omega^{n+1}_f$. The following theorem is due to J. -P. Françoise and concerns the local normal forms of symplectic forms and their functional invariants:

**Theorem 2.4.1** ([29], [30]) (see also [32]). Two germs of symplectic forms are equivalent if and only if they define the same class in the Brieskorn module $G$. In particular any germ of a symplectic form is equivalent to the form

$$\omega = \sum_{i=1}^{\mu} c_i(f) \omega_i,$$  

(2.10)

where $c_i \in \mathbb{C}\{t\}$ and the classes of the 2-forms $\omega_i$ form a basis of $G$. The $\mu$ holomorphic functions $c_i(t)$ are unique and they are characteristic (functional invariants) for the pair $(\omega, f)$.

**Proof.** The one direction is rather straightforward (see also [30] for an alternative proof): if two germs of symplectic forms are equivalent then their Poincaré residues define the same cohomology class in each fiber $H^1(X_t; \mathbb{C})$ of the cohomological Milnor fibration of $f$ in a sufficiently small neighborhood of the origin. Indeed, since the diffeomorphism realising the equivalence is tangent to the identity, it induces the identity in the cohomology of each fiber $X_t$ with constant coefficients. It follows by the coherence and freeness of the Brieskorn module $G$ that the diffeomorphism $\Phi$ induces the identity morphisms in both $F$ and $G$. The other direction is an application of Moser's homotopy method, whose proof goes exactly as in the proof of the Morse-Darboux lemma 2.2.1): consider a family of symplectic forms $\omega_s = \omega_0 + sdf \wedge dg, s \in [0, 1]$. Then the vector field $v_s$ defined by:

$$v_s \cdot \omega_s = g \wedge df$$

is a solution of the homological equation:

$$L_{v_s} \omega_s = -df \wedge dg$$

and thus, its time-1 map $\Phi_1$ is the desired diffeomorphism between $\omega_1$ and $\omega_0$. Choosing now a basis of $G$ and $\omega_0$ as the representative of $\omega_1$ in this basis, then we obtain the normal form (2.10), as well as the uniqueness of the coefficients $c_i(t)$. To finish the proof it suffices to show the functions $c_i(t)$ are independent of the coordinates systems, i.e. they are characteristic for the pair $(\omega, f)$. This is a consequence of the description of the functions $c_i(t)$ in terms of integrals along vanishing cycles: let
\{ \gamma_1(t), \ldots, \gamma_\mu(t) \} \in H_1(X_t; \mathbb{C}) \) be a locally constant (horizontal) basis of relative cycles and consider the (multivalued) vector-valued map: \( I(t) = (I_1(t), \ldots, I_\mu(t))^T \):

\[
I_j(t) = \int_{\gamma_j(t)} \frac{\omega}{df}.
\]

Then the holomorphic vector-valued map \( c(t) = (c_1(t), \ldots, c_\mu(t)) \) may be obtained as a solution of the linear system:

\[
Y(t)c(t) = I(t),
\]

where

\[
Y(t) = (\int_{\gamma_j(t)} \frac{\omega_i}{df})
\]

is the period matrix of the singularity \( f \). It follows by Cramer’s rule that:

\[
c_i(t) = \frac{\det \bar{Y}_i(t)}{\det Y(t)},
\]

where the matrix \( \bar{Y}_i(t) \) is obtained by the period matrix \( Y(t) \) after replacing its \( i \)’th column with \( I(t) \). Thus the vector \( c(t) = (c_1(t), \ldots, c_\mu(t)) \) is indeed characteristic for the triple \( (\omega, f) \).

**Remark 2.4.1.** Since the singularity \( f \) is isolated, we may always choose local coordinates \( (x, y) \) such that in the theorem above \( f(x, y) \) is a polynomial of sufficiently high degree (by Tougeron’s determinacy theorem [105]).

**Remark 2.4.2.** For the case \( \mu = 1 \), i.e. for a Morse germ \( f \), the statement of the theorem above is the first part of the proof of the Morse-Darboux Lemma 2.2.1. Indeed, the class of the 2-form \( dx \wedge dy \) generates the Brieskorn module \( G \) (c.f. Example (2.3.1)), which is equivalent to the statement of Main Lemma 2.2.2.

### 2.4.2 Symplectic Deformations of Isolated Singularities

Françoise’s theorem presented above concerns the deformation theory of symplectic structures (volume forms in higher dimensions) with respect to diffeomorphisms preserving a fixed isolated singularity \( f \). Here we will discuss instead the Lagrangian deformation theory of isolated singularities \( f \), i.e. with respect to a fixed symplectic form \( \omega = dx \wedge dy \). The results presented below are due to M. D. Garay [33] who proved them for the isochore case. The main theorem is a symplectic version of Mather’s unfolding theorem [67], i.e. that a symplectic infinitesimally versal deformation of an isolated singularity, is in fact symplectically versal, at least as long as we allow changes of coordinates in the target space as well (i.e. for a symplectic version of the so called right-left, or \( A \)-equivalence). Indeed, it is already obvious from the Morse-Darboux lemma 2.2.1 that such a change of coordinates should be allowed (notice that the function \( \psi(t) \) in the Morse-Darboux normal form is a diffeomorphism of \( (\mathbb{C}, 0) \) tangent to the identity). The question of the symplectic deformation theorem was raised by Y. Colin de Verdière in [18] for the case of Lagrangian singularities, i.e. for the symplectic equivalence of plane curve germs \( X_0 = \{ f = 0 \} \) (the symplectic analog of the so
called contact, or else K-equivalence). Garay’s theorem for the “wider” group \( \mathcal{A} \) gives in an obvious way a positive answer to Verdier’s question.

To start let \( F : (\mathbb{C}^2 \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0), F(,0) = f \) be a deformation of a singularity \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \). To the deformation \( F \) we associate its unfolding, i.e. the map:

\[
\tilde{F} : (\mathbb{C}^2 \times \mathbb{C}^k, 0) \to (\mathbb{C} \times \mathbb{C}^k, 0), \quad \tilde{F}(,\lambda) = (F(,\lambda), \lambda).
\]

Fix now a germ of a symplectic form \( \omega = dx \wedge dy \) at the origin. All the notions of right-left (or \( \mathcal{A} \)-equivalence) between deformations, versality, infinitesimal versality e.t.c. (c.f. [3]) carry over to the subgroup \( \mathcal{A}_\omega \) of right-left equivalences, where the right diffeomorphism has to preserve the symplectic form \( \omega \). In particular, a deformation \( F \) (or the unfolding \( \tilde{F} \)) of a boundary singularity \( f \) will be called symplectically versal if any other deformation \( F' \) (or unfolding \( \tilde{F}' \) respectively) is \( \mathcal{A}_\omega \)-equivalent to a deformation induced from \( F \), i.e. there exists a relative symplectic diffeomorphism \( \phi : (\mathbb{C}^2 \times \mathbb{C}^k, 0) \to (\mathbb{C}^2, 0), \phi(,0) = , \), preserving \( \omega \), a relative diffeomorphism \( \psi : (\mathbb{C} \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0), \psi(,0) = . \) and a map germ \( g : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0) \) such that:

\[
\psi(F(\phi(x,y;\lambda);g(\lambda))) = F'(x,y;\lambda').
\]

Let us consider now the corresponding infinitesimal symplectic deformations of \( f \). The space of non-trivial symplectic deformations is, as is easily seen, the space:

\[
\tilde{I}_f^1 = \frac{\mathcal{O}}{\{Lv f + k(f)/Lv \omega = 0\}} \cong \frac{\mathcal{O}}{\{f, \mathcal{O}\} + \mathbb{C}\{f\}},
\]

where \( \{,\} \) is the Poisson bracket induced by the symplectic structure \( \omega \). This is a \( \mathbb{C}\{f\} \)-module which can be viewed as the quotient of the “symplectic Jacobian module” of the singularity \( f \):

\[
I_f^1 = \frac{\mathcal{O}}{\{Lv f/Lv \omega = 0\}} \cong \frac{\mathcal{O}}{\{f, \mathcal{O}\}},
\]

by the submodule generated by the class of the constant function 1. The latter module is in turn isomorphic to the Brieskorn module \( G \) of \( f \), the isomorphism given by multiplication with the symplectic form \( \omega \):

\[
I_f^1 \cong G.
\]

Consequently:

**Theorem 2.4.2.** The symplectic Jacobian module \( I_f^1 \) of a singularity \( f \) of Milnor number \( \mu \), is a free module of rank \( \mu \):

\[
I_f^1 \cong \mathbb{C}\{f\}^\mu.
\]

**Proof.** Let us prove here the freeness part, as an alternative proof (for the 2-dimensional case only) of Theorem 2.3.5 on the freeness of the Brieskorn module \( G \). It is distilled from [36], where the topology of Lagrangian Milnor fibers is studied. To show freeness it suffices to show that the module \( I_f^1 \) is torsion free. To do this we interpret this module cohomologically as the first cohomology \( H^1(C_f^*) \)
of the relative complex of Lagrangian deformations of $f$ (as in [36] and also [33], [93], [99] in higher dimensions):

$$C_f^\bullet : 0 \rightarrow \mathcal{O} \xrightarrow{(f,)} \mathcal{O} \rightarrow 0.$$ 

Indeed

$$H^1(C_f^\bullet) = \text{Coker}\{f,\} = I^1_f.$$ 

Consider now multiplication by $f$ in $C_f^\bullet$. Since it is a complex of free modules multiplication by $f$ gives a short exact sequence:

$$0 \rightarrow C_f^\bullet \xrightarrow{f} C_f^\bullet \rightarrow C_f^\bullet \xrightarrow{f} 0,$$

which induces a long exact sequence in cohomology:

$$0 \rightarrow H^0(C_f^\bullet) \xrightarrow{f} H^0(C_f^\bullet) \rightarrow H^0(C_f^\bullet) \xrightarrow{f} H^1(C_f^\bullet) \rightarrow H^1(C_f^\bullet) \rightarrow H^1(C_f^\bullet) \rightarrow 0.$$ 

The main thing here is that this sequence splits at $H^0(C_f^\bullet)$: indeed, this follows from the fact that any function commuting with $f$ (with respect to the Poisson bracket of course) is a function of $f$. Thus the long exact sequence above reduces to the short exact sequence:

$$0 \rightarrow H^1(C_f^\bullet) \xrightarrow{f} H^1(C_f^\bullet) \rightarrow H^1(C_f^\bullet) \xrightarrow{f} 0,$$

which is exactly what we wanted to prove (i.e. multiplication by $f$ is injective).

From the above it follows that a necessary condition for a deformation $F$ of $f$ to be symplectically versal is that the classes of the velocities $\partial_\lambda F := \frac{\partial F}{\partial \lambda}|_{\lambda=0}$ along with the class of 1, span the symplectic Jacobian module $I^1_f$ over $\mathbb{C}\{f\}$. The following theorem is a symplectic version of Mather’s unfolding theorem and says that this condition is also sufficient:

**Theorem 2.4.3** ([33], [36]). A deformation $F : (\mathbb{C}^2 \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ of a singularity $f$ is symplectically versal if it is infinitesimally symplectically versal, i.e.

$$I^1_f = \text{span}_{\mathbb{C}\{f\}}\{1, \partial_{\lambda_1} F, ..., \partial_{\lambda_k} F\} \leftrightarrow G = \text{span}_{\mathbb{C}\{f\}}\{\omega, \partial_{\lambda_1} F \omega, ..., \partial_{\lambda_k} F \omega\}$$  \hspace{1cm} (2.11)

Following [33] we may prove this theorem as follows: first we show that any 1-parameter deformation of an infinitesimal versal deformation $F$ is symplectically trivial (we call $F$ symplectically rigid). Then we conclude by using J. Martinet’s trick, according to which any $k$-parameter deformation can be considered as a “sum” of 1-parameter deformations. The symplectic rigidity in turn can be interpreted cohomologically in terms of a parametric version of the Brieskorn module which we present below.

---

\footnote{which is not the case in higher dimensions.}
The Parametric Brieskorn Module and Symplectic Rigidity

Let $\Omega_{2+k}^*$ denote the complex of germs of holomorphic forms at the origin of $\mathbb{C}^2 \times \mathbb{C}^k$. In analogy with the case of the germ $f$, there is naturally defined a relative de Rham cohomology for the map $F$ as well as the corresponding Brieskorn modules and the Gauss-Manin connection, due to G. M. Greuel [43], who studied the Gauss-Manin theory of isolated complete intersections (see also [61]). Here we will only need to consider the parametric version of the Brieskorn module, i.e the $\mathbb{C}F$-module:

$$G_F := \frac{\Omega_{2+k}^*}{d\lambda_1 \wedge \ldots \wedge d\lambda_k \wedge dF \wedge d\Omega_{2+k}^*},$$

which plays a crucial role in the proof of the symplectic deformation Theorem 2.4.3. The following finiteness result due to Greuel, is a parametric version of the Sebastiani theorem:

**Proposition 2.4.4** ([43]). The parametric Brieskorn module $G_F$ of a deformation $F$ of a singularity $f$ is finitely generated over $\mathbb{C}\{F, \lambda\}$ and it is free\(^{12}\) of rank $\mu$. Moreover, its restriction on $\mathbb{C}^2 = \{\lambda_1 = 0, ..., \lambda_k = 0\}$ is isomorphic to the Brieskorn module $G_f$ of $f$:

$$G_F|_{\lambda=0} \cong G_f.$$

**Proof.** Since the map $F$ defines an isolated complete intersection singularity, the proof of the finiteness of the parametric Brieskorn module $G_F$ is again a straightforward corollary of the relative analog of the Kiehl-Verdier theorem (c.f. [34] and references). The freeness of $G_F$ is proved also in [43] along the lines of Malgrange [64] for the ordinary (non-parametric) case. \(\square\)

Consider now a 1-parameter deformation $F'_t$ of $F$:

$$F'_t := F' : (\mathbb{C}^2 \times \mathbb{C}^k \times \mathbb{C}, 0) \to (\mathbb{C}, 0), \quad (x, y; \lambda, t) \mapsto F'(x, y; \lambda, t),$$

$$F'(x, y; \lambda, 0) = F(x, y; \lambda).$$

Then, as is easily seen, $F'_t$ is symplectically trivial provided that there exists a decomposition:

$$\partial_t F' = k(F', \lambda, t) + \sum_{i=1}^{k} c_i(F', \lambda, t) \partial_{\lambda_i} F' + L_v F',$$

where $v$ is a relative vector field preserving $\omega$. Multiplying with $\tilde{\omega} = \omega \wedge d\lambda^k \wedge dt$ (where we denote $d\lambda^k = d\lambda_1 \wedge \ldots \wedge d\lambda_k$), the condition of symplectic triviality above can be viewed as the condition that the class of the form $\partial_t F' \tilde{\omega}$ in the Brieskorn module $G_{F'}$ of $F'$ (of the unfolding $F'$) belongs to the $\mathbb{C}\{F', \lambda, t\}$-module spanned by the classes of form $\tilde{\omega}$ and of the initial velocities $\partial_{\lambda_i} F' \tilde{\omega}$:

$$\partial_t F' \tilde{\omega} \in M = \text{span}_{\mathbb{C}\{F', \lambda, t\}}\{\tilde{\omega}, \partial_{\lambda_1} F' \tilde{\omega}, ..., \partial_{\lambda_k} F' \tilde{\omega}\}.$$

We will show that if $F$ is infinitesimally symplectically versal, then in fact $M = G_{F'}$, which implies

\(^{12}\)in fact, the freeness of the parametric Brieskorn module is irrelevant for the symplectic deformation problem.
in turn the existence of a solution of the homological equation (2.12). To prove the assertion, notice that since the Brieskorn module $G_{F'}$ is finitely generated, by the above Proposition 2.4.4 it suffices to show, by Nakayama’s lemma, that the image of $M$ by the natural projection:

$$\pi : G_{F'} \to \frac{G_{F'}}{mG_{F'}},$$

coincides with the whole $\mu$-dimensional $\mathbb{C}$-vector space:

$$\pi(M) = \frac{G_{F'}}{mG_{F'}}. \tag{2.13}$$

Here $m$ is the maximal ideal at the origin of $\mathbb{C} \times \mathbb{C}^k \times \mathbb{C}$. But according to Proposition 2.4.4 again, there is an isomorphism of $\mu$-dimensional vector spaces:

$$\frac{G_{F'}}{mG_{F'}} \cong \frac{G_f}{fG_f}.$$

Thus the condition (2.13) above reduces to the condition:

$$\pi(M) = \text{span}_{\mathbb{C}(f)} \{\omega, \partial_{\lambda_1} F\omega, ..., \partial_{\lambda_k} F\omega\} = \frac{G_f}{fG_f}, \tag{2.14}$$

which is in turn equivalent, by Nakayama’s lemma, to the assumption (2.11) of infinitesimal symplectic versality of $F$. Thus we have proved:

**Proposition 2.4.5.** An infinitesimal symplectic versal deformation of an isolated singularity is symplectically rigid.

**Proof of the Symplectic Versal Deformation Theorem and Corollaries**

**Proof of Theorem 2.4.3.** The proof relies in a standard trick of J. Martinet: let $F$ be a deformation of $f$, $f = F(.,0)$ and $F'$ another deformation of $f$. Define the sum $F \oplus F'$ by:

$$F \oplus F'(x,y;\lambda,\lambda') = F(x,y;\lambda) + F'(x,y;\lambda') - f(x,y).$$

The restriction of $F \oplus F'$ on $\lambda = 0$ is equal to $F'$ and thus, in order to show that $F'$ is symplectically equivalent to a deformation induced by $F$, it suffices to show that the deformation $F \oplus F'$ is a symplectically trivial deformation of $F$. This can be shown inductively as follows: denote by $F_j$ the restriction of $F \oplus F'$ to $\{\lambda_j = \ldots = \lambda_k = 0\}$. Then $F_1 = F$ and $F_k = F \oplus F'$. It follows from Proposition 3.4.11 that for each $j$ the deformation $F_{j-1}$ is isochore rigid and thus $F_j$ is an isochore trivial deformation of $F_{j-1}$. We conclude by induction that $F_k$ is a symplectically trivial deformation of $F_1$. \qed

As an immediate corollary we obtain another proof of the Morse-Darboux lemma 2.2.1 which can be now stated as follows:
Corollary 2.4.6. Any Morse germ $f$ on the symplectic plane with a fixed symplectic form $\omega = dx \wedge dy$, is symplectically (right-left, or $\mathcal{A}_\omega$-)equivalent to its quadratic part:

$$f_0 = x^2 + y^2.$$ 

Moreover, the left diffeomorphism is unique.

Proof. Consider $f_t = f_0 + th, t \in [0,1]$, a 1-parameter deformation of $f_0$, $f_1 = f$, such that $f_t$ has a nondegenerate critical point at the origin for all $t$. Then for any point $t_0 \in [0,1]$ the germ at $t_0$ of the deformation $f_t$ is a symplectically trivial deformation of $f_{t_0}$. Indeed, the parametric Brieskorn module $G_{f_t}$ is generated by the class of the form $dx \wedge dy \wedge dt$ and the claim follows from the symplectic deformation Theorem 2.4.3. Thus, for any $\epsilon$ sufficiently small, the germ $f_{t_0 + \epsilon}$ is symplectically equivalent to $f_{t_0}$, and thus $f_0$ is symplectically equivalent to $f_1$ as well. The uniqueness of the left diffeomorphism follows from the fact that a symplectically versal deformation of a Morse germ, is in fact universal. 

As another immediate corollary we obtain also a relative version of a theorem of Y. Colin de Verdière, according to which:

Corollary 2.4.7 ([IS]). A versal deformation of a quasihomogeneous singularity $f$ is symplectically (right-left, or $\mathcal{A}_\omega$-)versal.

Proof. Indeed, in this case there is a canonical isomorphism (c.f. Example 2.3.1):

$$\frac{G_f}{J_f} \cong Q_f$$

given by the symplectic form $\omega = dx \wedge dy$ and thus, the classes of 1 with the initial velocities of the deformation generate the symplectic Jacobian module $I^1_f$ of $f$. 

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3 Boundary Singularities of Functions on the Symplectic Plane

3.1 Introduction

Here we generalise the results obtained in the previous chapter for the classification of pairs \((\omega, f)\), in the presence of a “boundary”, i.e. of a fixed smooth plane curve germ at the origin \(H \subset \mathbb{C}^2\). We will show that the corresponding results on the number of moduli, on normal forms e.t.c. extend, with minor modifications, to the classification of triples \((\omega, f, H)\), where \(\omega\) is again a germ of a holomorphic symplectic form and \(f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)\) is holomorphic function germ such that:

- either it has an isolated critical point at the origin,
- or it is smooth, but its restriction \(f|_H : (H, 0) \to (\mathbb{C}, 0)\) on the boundary has an isolated critical point at the origin.

In this situation we call the pair \((f, H)\) (or abusively \(f\)) an isolated boundary singularity, as is usual in the literature (see references below).

The study of isolated boundary singularities was initiated by V. I. Arnol’d in \([6]\) (see also \([3], [4], [8]\) for general references) where he extended the \(A, D, E\) classification of simple singularities to include also the \(B, C, F\) series of Weyl groups in the scheme of singularity theory. The list of simple normal forms is given for convenience in Table 3.1.

<table>
<thead>
<tr>
<th>(A_\mu)</th>
<th>(B_\mu)</th>
<th>(C_\mu)</th>
<th>(F_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x + y^{\mu+1})</td>
<td>(x^\mu + y^2)</td>
<td>(xy + y^\mu)</td>
<td>(x^2 + y^3)</td>
</tr>
<tr>
<td>(\mu \geq 1)</td>
<td>(\mu \geq 2)</td>
<td>(\mu \geq 2)</td>
<td>(\mu = 4)</td>
</tr>
</tbody>
</table>

The equivalence of functions \(f\) on manifolds with boundary \(H\), is given naturally by the action of the subgroup \(\mathcal{R}_H \subset \mathcal{R}\) in the space of functions \(\mathcal{O}\), i.e. by the action of the group of germs of diffeomorphisms (right-equivalences) preserving the boundary \(H\). The number \(\mu\) in the table is called the multiplicity or the Milnor number of the boundary singularity and it is the invariant replacing the ordinary Milnor number in all considerations in the presence of a boundary. In fact, it is related to the ordinary Milnor number \(\mu_1\) of \(f\) (viewed as an ordinary singularity in the ambient
space) and the Milnor number $\mu_0$ of its restriction $f|_H$ on the boundary, by the formula (c.f. \[6\], \[103\], \[110\] and also Chapter 5):

$$\mu = \mu_1 + \mu_0.$$  

As it turns out, the Milnor number $\mu$ of a boundary singularity $(f, H)$ is the most important (arithmetic) invariant in the classification of triples $(\omega, f, H)$ and it again replaces the ordinary Milnor number in all considerations: in particular we will show that the number of moduli in the classification problem is exactly equal to $\mu$, that these moduli are analytic functions of 1-variable and that they enter in a normal form analogous to Françoise’s normal form presented in the previous chapter. Finally, we prove a relative version of Garay’s symplectic versal unfolding theorem, i.e. that a symplectic infinitesimally versal deformation of a boundary singularity on the plane, is in fact symplectically versal.

The results depend again on the relative de Rham cohomology naturally associated to the boundary singularity $(f, H)$, and in particular, on the first cohomology of the complex:

$$\Omega^1_{f, H} := \frac{\Omega_H^1}{df \wedge \Omega_H^{-1}},$$

where now $\Omega_H^1 \subset \Omega^1$ is the subcomplex of germs of holomorphic differential forms “vanishing on the boundary $H$”. In fact, the main result here is a relative version of the Brieskorn-Deligne(-Sebastiani) theorem, i.e. that the relative Brieskorn module:

$$G = \frac{\Omega^2}{df \wedge d\Omega_H^1},$$

is a free module of rank $\mu$ over $\mathbb{C}\{f\}$, where $\mu$ is the Milnor number of the boundary singularity.

In Chapter 5 we will study extensively the cohomological properties and the corresponding Gauss-Manin theory for the boundary singularities. Here we will restrict only to the description of those analytical results which are intimately related with the classification of triples $(\omega, f, H)$.

Let us start with the first simplest example, that is the analog of the Morse-Darboux lemma in the presence of a boundary.

### 3.2 The Relative Morse-Darboux Lemma

The following theorem describes the symplectic normal form of the first occurring boundary singularity $f = x + y^2$, $H = \{x = 0\}$ (i.e. the $A_1$ case in Table 3.1):

**Theorem 3.2.1.** Fix a symplectic form $\omega = dx \wedge dy$. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a function germ such that the origin is a regular point for $f$ but nondegenerate (Morse) critical point for its restriction $f|_H$ on the boundary $H = \{x = 0\}$. Then there exists a symplectic diffeomorphism $\Phi$, preserving both $\omega$ and the boundary $H$, and an analytic function $\psi \in \mathbb{C}\{t\}$, $\psi(0) = 0$, $\psi'(0) = 1$
such that $f$ is reduced to the normal form:

$$
\Phi^*f = \psi(x + y^2).
$$

(3.1)

Moreover, the function $\psi(t)$ is unique and it is characteristic (functional invariant) for the triple $(\omega, f, H)$.

The proof is similar to the ordinary Morse-Darboux lemma and it again relies on the following:

**Main Lemma 3.2.2.** Let $(f, H)$ be a nondegenerate boundary singularity $A_1$ and $\omega$ a germ of a symplectic form. Then there exists a uniquely defined analytic function $c \in \mathbb{C}\{t\}$, $c(0) = 1$ and a function germ $h \in \Omega^0_H$ vanishing on the boundary, such that:

$$
\omega = c(f)dx \wedge dy + df \wedge dh.
$$

Notice that a function $h$ “vanishes on the boundary $H$”, when it is divisible by the equation defining the curve $H$ (so if $H = \{x = 0\}$ then there exists a function $h_1$ such that $h = xh_1$). The proof of the Main Lemma above will be postponed again until later. It relies on the finiteness and freeness of a relative version of the Brieskorn module for the boundary singularity and the quasihomogeneity of the germ $f$.

**Proof of Theorem 3.2.1.** As in the proof of the ordinary Morse-Darboux lemma we work in two steps. The first is an application of Moser’s homotopy method: choose coordinates such that the pair $(f, H)$ is already in relative Morse normal form, i.e. $f(x, y) = x + y^2$ and $H = \{x = 0\}$. Consider the family of symplectic forms

$$
\omega_t = c(f)dx \wedge dy + tdf \wedge dh,
$$

where $h \in \Omega^0_H$. We will find a 1-parameter family of diffeomorphisms $\Phi_t$, $\Phi_0 = I_d$, $\Phi_t(0) = 0$ (where $t \in [0, 1]$), preserving the pair $(f, H)$, $\Phi_t^*f = f$, $\Phi(H) = H$ and such that:

$$
\Phi_t^*\omega_t = c(f)dx \wedge dy.
$$

To do this, it suffices to solve the infinitesimal equation:

$$
L_{v_t}\omega_t = -\frac{\partial \omega_t}{\partial t},
$$

for the family of vector fields $v_t$ such that:

$$
\frac{d\Phi_t}{dt} = v_t(\Phi_t), \quad v_t(0) = 0
$$

and such that it preserves $(f, H)$, i.e. $L_{v_t}f = 0$ and $v_t(x)|_{x=0} = 0$ (which means that $v_t$ is tangent
to $H = \{x = 0\}$). By Cartan’s identity the infinitesimal equation reduces to:

$$d(v_t \omega_t) = d(-hdf) \Leftrightarrow v_t \omega_t = -hdf + d\xi.$$ 

Now, if we choose $v_t$ as a solution of the equation:

$$v_t \omega_t = -hdf,$$

then this immediately responds to our problem. Indeed, since:

$$L_{v_t}(f)\omega_t = df \wedge (v_t \omega_t) = df \wedge (-hdf) = 0,$$

it follows that $L_{v_t}f = 0$, and since $h$ vanishes on $H$, $v_t$ also vanishes on $H$ (and then of course, it is trivially tangent to $H$). Thus, the time 1-map $\Phi_1$ provides the required diffeomorphism. This reduces the pair $(\omega, f, H)$ to the normal form:

$$\Phi_1^* \omega = c(f)dx \wedge dy, \quad \Phi_1^* f = f, \quad \Phi_1(H) = H.$$ 

For the second step of the proof we consider the change of coordinates $(x, y) \mapsto (xv(f), y\sqrt{v(f)})$, where $v \in \mathbb{C}\{t\}$ is some function with $v(0) = 1$ (so $\Psi$ is indeed a boundary-preserving diffeomorphism). With any such function $v$ we have $\Psi^*f = \psi(f)$ for the function $\psi(f) = fv(f)$, with $\psi(0) = 0$ and $\psi'(0) = 1$. Now it suffices to choose $v$ so that the map $\Psi$ satisfies $\text{det} \Psi = c(f)$, i.e. such that the following initial value problem is satisfied for the function $w = v^{3/2}$:

$$2 \frac{2}{3}tw'(t) + w(t) = c(t), \quad w(0) = 1. \quad (3.2)$$

As is easily verified this admits an analytic solution given by the formula

$$w(t) = t^{-\frac{2}{3}} \int_0^t \frac{3}{2} s^{\frac{1}{2}} c(s) ds.$$ 

Thus, the diffeomorphism $\Phi = \Psi \circ \Phi_1$ is the required diffeomorphism. 

\[3.2.1\] Geometric Description of the Invariants: Integrals along Vanishing Half-Cycles

As in the ordinary case, the geometric description of the invariant $c(t)$ (and thus $\psi(t)$) can be achieved in terms of integrals of a primitive of the symplectic form along the vanishing half-cycle of the boundary singularity (according to the terminology of Arnol’d [4]).

For this, let us consider a small ball at the origin of $\mathbb{C}^2$ such that the fibers of $f(x, y) = t$ are transversal to the boundary of this ball over the points of a sufficiently small disc in $\mathbb{C}$, centered at the origin (the critical value of $f|_H$). Modifying the neighborhoods under consideration sufficiently, we may suppose that the fibers of the restriction $f|_H$ on the boundary (which consists of two points away from the origin) are also transversal to the restriction of the boundary of the initial ball (a
circle) on the boundary $H$ (i.e. they do not meet). The intersection of each of the fibers of $f$ with the interior of the chosen ball, is an open Riemann surface $X_t$ with a set of distinguished points $X'_t = X_t \cap H$. Let $\gamma(t)$ be a 1-parameter family of relative cycles on the pair of fibers representing a relative homology class in $H_1(X_t, X'_t; \mathbb{C})$, so that $\gamma(t)$ is obtained by continuous deformation preserving $H$, of some relative cycle $\gamma(t_0)$ over a smooth pair $(X_{t_0}, X'_{t_0})$. As is easily seen, for $t$ real and positive, the pair of fibers $(X_t, X'_t)$ is contractible to its real part and as $t \to 0$ the fiber $X'_t$ shrinks to a point (see Figure 3.2.1). Arnol’d called the relative cycle $\gamma(t)$ arising this way, vanishing half-cycle [6]:

$$\gamma(t) = \{(x, y) \in \mathbb{R}^2 / x \geq 0, \ x + y^2 = t, \ t < \epsilon\}.$$  

![Figure 3.1: Local model for a vanishing half-cycle of a boundary singularity. The area of the shaded region is $\text{const.} t^{3/2}$](image)

Let now $\alpha$ be a primitive of the symplectic form $\omega = c(f)dx \wedge dy + df \wedge dh$, i.e. such that $\omega = da$. From Main Lemma 3.2.2 it follows that the 1-form $\alpha$ can be chosen in the form:

$$\alpha = w(f)\alpha_0 - h df + dg,$$

where both $h$ and $g$ vanish on $H$ and $\alpha_0 = xdy - (y/2)dx$ obviously vanishes on $H$ (i.e. it has vanishing pull back under the natural embedding $y \mapsto (0, y)$ of $H$ in $\mathbb{C}^2$) and is such that:

$$df \wedge \alpha_0 = f \omega_0, \quad d\alpha_0 = \frac{3}{2} \omega_0,$$

where $\omega_0 = dx \wedge dy$ is the standard symplectic form. From this it follows that the function $w \in \mathbb{C}\{t\}$ satisfies an equation of the form (3.2):

$$\frac{2}{3} tw'(t) + w(t) = c(t), \quad w(0) = 1.$$

Thus we obtain

$$V(t) = w(t)V_0(t),$$

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where \( V_0(t) = \int_{\gamma(t)} \alpha_0 \). Of course, the function \( V_0(t) \) is again a holomorphic function of \( t \) and moreover, it satisfies the differential equation

\[
\frac{2}{3} t V_0'(t) = V_0(t).
\]

Indeed, the latter claim is verified by the relation

\[
\frac{\omega_0}{df} = \frac{\alpha_0}{f}.
\]

Thus

\[
V_0'(t) = \int_{\gamma(t)} \frac{d\alpha_0}{df} = \frac{3}{2} \int_{\gamma(t)} \frac{\omega_0}{df} = \frac{3}{2} \int_{\gamma(t)} \frac{\alpha_0}{f} = \frac{3}{2t} \int_{\gamma(t)} \alpha_0,
\]

which proves the claim (of course \( V_0(t) = 2t^{3/2} \) measures the area enclosed by the vanishing half-cycle \( \gamma(t) \)). Taking now the derivative of the function \( V(t) \) we have from the above:

\[
V'(t) = w'(t)V_0(t) + w(t)V_0'(t) = (w'(t) + \frac{3w(t)}{2t})V_0(t),
\]

i.e.

\[
\frac{2}{3} t V'(t) = c(t)V_0(t),
\]

which gives the desired expression of the invariant \( c(t) \) in terms of the integral \( V(t) \):

\[
c(t) = \frac{2t V'(t)}{3V_0(t)} = \frac{t^{-1/2} V'(t)}{3}.
\]

From the result above and from equation \([3.2]\), we immediately obtain also the expression of the invariant \( \psi(t) \) in the normal form \((dx \wedge dy, \psi(x + y^2), x = 0)\) of the relative Morse-Darboux lemma:

\[
\psi(t) = \left( \frac{V(t)}{2} \right)^{2/3}.
\]

### 3.3 A Generalisation for Arbitrary Boundary Singularities of Finite Codimension

As in the ordinary case for the classification of pairs \((\omega, f)\), it is possible to give a generalisation for the classification of triples \((\omega, f, H)\) where the boundary singularity \((f, H)\) is more degenerate than the relative Morse case, but still isolated and thus of finite codimension.

---

\footnote{as we shall see in Chapter 5, this is the simplest \textit{Picard-Fuchs equation} of a boundary singularity.}
3.3.1 Milnor Numbers, Relative de Rham Cohomology and Relative Brieskorn Modules

We fix a coordinate system \((x, y)\) at the origin of \(\mathbb{C}^2\) such that the boundary is given by the equation \(H = \{ x = 0 \}\). To a boundary singularity \((f, H)\) we associate its local algebra (c.f. [6])

\[
Q_{f,H} = \mathcal{O} \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right),
\]

where the ideal \(J_{f,H} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})\) in the denominator is the tangent space to the \(\mathcal{R}_H\)-orbit of \(f\) (as usual \(\mathcal{O}\) is the algebra of germs of analytic functions at the origin). We call it, in analogy with the ordinary case the Jacobian ideal of the boundary singularity \((f, H)\). Its codimension, i.e. the \(\mathbb{C}\)-dimension \(\mu\) of the vector space \(Q_{f,H}\) is called the multiplicity or Milnor number of the boundary singularity \((f, H)\) and it is an important invariant: it is related to the ordinary Milnor number \(\mu_1\) of \(f\):

\[
\mu_1 = \dim_C \mathcal{O} \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right),
\]

and the ordinary Milnor number \(\mu_0\) of its restriction \(f|_H\) on the boundary:

\[
\mu_0 = \dim_C \mathcal{O} \left|_H \left( \frac{\partial f}{\partial y} \right) \right.,
\]

by the formula:

\[
\mu = \mu_1 + \mu_0.
\]

To interpret this topologically we may, as in the ordinary case, intersect the level sets of \(f\) with a small ball centered at the origin (c.f. [6] and also Chapter 5 for more details). The level set (Milnor fiber) \(X_t, t \neq 0\), is an open Riemann surface of Betti number \(\mu_1\) and its intersection \(X'_t = X_t \cap H\) with the boundary \(H\) consists of a finite number of points (by transversality) whose number is equal to \(\mu_0 + 1\). For \(t \to 0\), the \(\mu_1\) cycles of \(X_t\) and the \(\mu_0\) segments in \(H\) joining the distinguished points \(X'_t\), both shrink at the origin: the first are the ordinary vanishing cycles of \(f\) and the latter are the ordinary vanishing cycles of its restriction \(f|_H\) on the boundary and they form respectively a basis of \(H_1(X_t; \mathbb{Z})\) and of \(\widetilde{H}_0(X'_t; \mathbb{Z})\) (where \(\widetilde{H}\) means reduced homology). In fact, as is easily seen, the ordinary vanishing cycles and the relative cycles in the complement \(X_t \setminus X'_t\) covering the vanishing segments on \(H\), the so called vanishing half-cycles [6], form a basis of the relative homology group \(H_1(X_t, X'_t; \mathbb{Z})\), whose rank is a priori equal to

\[
\mu = \mu_1 + \mu_0,
\]

as is easily deduced from the long exact sequence in homology:

\[
0 \to H^1(X_t; \mathbb{Z}) \to H_1(X_t, X'_t; \mathbb{Z}) \to \widetilde{H}^0(X'_t; \mathbb{Z}) \to 0,
\]

induced by the embedding \(X'_t \hookrightarrow X_t\). Moreover:
Theorem 3.3.1 ([6],[110]). The space $X_t/X'_t$ has the homotopy type of a bouquet of $\mu$ circles.

Let us consider now a relative analog of the relative de Rham cohomology associated to the germ $f$ in the presence of a boundary $H$. Write $\Omega^*$ for the complex of germs of holomorphic forms at the origin ($\Omega^0 = \mathcal{O}$) and let $\Omega^*_H$ be the subcomplex of forms that vanish on the boundary $H$, in the sense that their pull back by the natural embedding $i(y) = (0, y)$ of $H$ in $\mathbb{C}^2$, is identically zero. So for example, $\Omega^1_H = x\mathcal{O}$ is the ideal generated by the equation of the boundary, $\Omega^1_H = x\Omega^1 + \Omega^0 dx$ and $\Omega^2_H = \Omega^2$ (every 2-form vanishes on the boundary). The differential $df$ defines an ideal in the algebra $\Omega^*$ and thus in the subalgebra $\Omega^*_H$ as well. Hence, in analogy with the ordinary case, to the boundary singularity $(f, H)$ we may associate the following relative de Rham complex:

$$
\Omega^*_f,H : 0 \to \Omega^0_H \to \Omega^1_{f,H} \to \Omega^2_{f,H} \to 0,
$$

where each term is defined by

$$
\Omega^p_{f,H} = \frac{\Omega^p_H}{df \wedge \Omega^p_H}, \quad p = 1, 2,
$$

and the differential $d$ is induced by the relative differential of the complex $\Omega^*_H$ of forms vanishing on the boundary. Notice also that multiplication with the 2-form $dx \wedge dy$ defines a (non-canonical) isomorphism of the last term of this complex with the local algebra of the boundary singularity,

$$
\Omega^2_{f,H} \cong \mathcal{Q}_{f,H},
$$

and it is thus a $\mu$-dimensional vector space.

There is yet another equivalent interpretation of the local algebra $\mathcal{Q}_{f,H}$ of the boundary singularity, as the highest order cohomology of the following complex (which we will call the relative Koszul complex of $(f, H)$):

$$
K^*_{f,H} : 0 \to \Omega^0_H \xrightarrow{df \wedge} \Omega^1_H \xrightarrow{df \wedge} \Omega^2 \to 0.
$$

The cohomologies of this complex are again defined by

$$
H^p(K^*_{f,H}) = \frac{\text{Ker}(df \wedge : \Omega^p_H \to \Omega^{p+1}_H)}{df \wedge \Omega^{p-1}_H}, \quad p = 0, 1, 2
$$

and obviously we have:

$$
H^2(K^*_{f,H}) = \frac{\Omega^2}{df \wedge \Omega^1_H} = \Omega^2_{f,H} \cong \mathcal{Q}_{f,H},
$$

as was asserted.

Again, the zeroth cohomology is easy to compute by the same reasoning as in the ordinary case:

$$
H^0(K^*_{f,H}) = \text{Ker}(df \wedge : \Omega^0_H \to \Omega^1_H) = 0.
$$

The description of first cohomology is the content of the following relative analog of the de Rham complex:

\footnote{throughout the text the word ‘relative’ has two meanings: relative to $H$ and/or relative to $f$, the difference should be understood contextually.}
division lemma:

**Lemma 3.3.2** (see also Lemma 5.2.4). Let \( \alpha \in \Omega^1_H \) be a 1-form vanishing on the boundary and such that \( df \wedge \alpha = 0 \). Then there exists a function \( g \in \Omega^0_H \) vanishing on the boundary and such that \( \alpha = g df \), i.e. \( H^1(K_{f,H}^\bullet) = 0 \) (the converse is obvious).

**Proof.** Notice first that if \( \alpha \in \Omega^1_H \) is such that \( df \wedge \alpha = 0 \), then \( \alpha \in x \Omega^1 \subset \Omega^1_H \), i.e. \( \alpha \) vanishes on \( H \), in the sense that it vanishes when evaluated at any point of \( H \) (i.e. its coefficients are divisible by \( x \)). Indeed this follows from the fact that \( (df \wedge \alpha)|_{x=0} = 0 \) holds and that \( df|_{x=0} \) does not vanish identically. Thus \( \alpha = x \alpha_1 \) for some 1-form \( \alpha \in \Omega^1 \) and then \( df \wedge (x \alpha_1) = x(df \wedge \alpha_1) = 0 \) implies that \( df \wedge \alpha_1 = 0 \) (because \( x \) is a nonzero divisor). Thus, by the ordinary de Rham division lemma [25], there exists a function germ \( g_1 \in \Omega^0 \) such that \( \alpha_1 = g_1 df \) and thus \( \alpha = x g_1 df \). The lemma is proved.

Let us now describe the relative de Rham cohomology \( H^p(\Omega^\bullet_{f,H}) \) of the boundary singularity, i.e. the cohomology of the complex \( \Omega^\bullet_{f,H} \). Notice first that the differential \( d \) is \( \mathbb{C} \{ f \} \)-linear and so the cohomology spaces \( H^p(\Omega^\bullet_{f,H}) \) have a natural \( \mathbb{C} \{ f \} \)-module structure with multiplication induced by \( f \). Again as in the ordinary case, a \( p \)-form \( \alpha \in \Omega^p_H \) vanishing on the boundary is relatively closed (resp. relatively exact) if \( d \alpha = df \wedge h \) (resp. \( \alpha = d \eta + df \wedge g \)) for some \( h \in \Omega^p_H \) (resp. \( (\eta, g) \in \Omega^p_H \)). The cohomology modules \( H^p(\Omega^\bullet_{f,H}) \) thus defined, measure relatively closed forms vanishing on the boundary \( H \), modulo relatively exact forms vanishing on the boundary.

It is easy to calculate the cohomologies for \( p = 0, 2 \). The zeroth cohomology module is equal to zero:

\[ H^0(\Omega^\bullet_{f,H}) = 0. \]

This follows for example from the fact that \( H^0(\Omega^1_H) = 0 \). Equivalently, \( H^0(\Omega^\bullet_{f,H}) \) consists of analytic function germs \( h \in \Omega^0_H \) such that \( dh = g df \) for some function germ \( g \in \Omega^0_H \). Then \( h \) is constant on the fibers of \( f \) and vanishes on \( x = 0 \) and thus it vanishes everywhere.

On the other hand, by the fact that every 2-form can be viewed as the differential of a 1-form vanishing on \( H \) (relative Poincaré lemma for \( \Omega^1_H \)) we immediately obtain:

\[ H^2(\Omega^\bullet_{f,H}) = 0. \]

It remains to describe the first cohomology module. The following theorem is an immediate consequence of the analog of the Brieskorn-Deligne(-Sebastiani) theorem (see Theorem 5.2.2) and it is the relative analog of Corollary [23.4] in the presence of a boundary \( H \).

**Theorem 3.3.3.** If \( (f,H) \) is an isolated boundary singularity at the origin of the plane of multiplicity \( \mu = \mu_1 + \mu_0 \), then \( H^1(\Omega^\bullet_{f,H}) \) is a free module of rank \( \mu \) over \( \mathbb{C} \{ f \} \):

\[ H^1(\Omega^\bullet_{f,H}) \cong \mathbb{C} \{ f \}^\mu. \]
Proof. The proof of this theorem will be given in Chapter 5 for the general case of isolated boundary singularities and in any dimension as well. It can be indeed obtained by the ordinary Brieskorn-Deligne-Sebastiani theorem applied to the pair \((f, f|_H)\) of isolated singularities and a natural long exact sequence in relative cohomology. An alternative proof of the freeness part, for the 2-dimensional case only, will be given below (Theorem 3.4.8) in terms of “Lagrangian deformations” as in the ordinary case.

We define now extensions of the first cohomology module, analogous to those defined by Brieskorn in [15] for the ordinary case, presented in the previous chapter. Write \(E = H^1(\Omega^\bullet_{f,H})\) and let

\[
F = \frac{\Omega^1_H}{\Omega^0_H df + d\Omega^0_H}
\]

denote the cokernel of the differential \(d : \Omega^0_H \to \Omega^1_{f,H}\). After multiplication with \(df \wedge\) we may view \(F\) naturally as a submodule of the \(\mathbb{C}\{f\}\)-module:

\[
G = \frac{\Omega^2}{df \wedge d\Omega^0_H}.
\]

It is easy to see that the cokernels of the natural inclusions of \(\mathbb{C}\{f\}\)-modules thus obtained:

\[
E \subset F \subset G,
\]

are both isomorphic to the same \(\mu\)-dimensional vector space \(\Omega^2_{f,H} \cong \mathbb{C}^\mu\). It follows that, as in the ordinary case, the \(\mathbb{C}\{f\}\)-modules \(F\) and \(G\) can be viewed as lattices spanning the same \(\mu\)-dimensional vector space

\[
\mathcal{M} = E \otimes \mathbb{C}\{f\} = F \otimes \mathbb{C}\{f\} = G \otimes \mathbb{C}\{f\},
\]

over the field of quotients \(\mathbb{C}\{f\}\) of \(\mathbb{C}\{f\}\). In analogy with the ordinary case, we call the \(\mathbb{C}\{f\}\)-module \(G\) (and/or \(F\)) the relative Brieskorn module of the boundary singularity \((f, H)\). The following theorem is the analog of Sebastiani’s Theorem 2.3.5 (and it is also equivalent to Theorem 3.3.3 above):

**Theorem 3.3.4.** Let \((f, H)\) be an isolated boundary singularity of Milnor number \(\mu = \mu_1 + \mu_0\). Then the relative Brieskorn module \(G\) (and thus also \(F\) and \(E\)) is a free module of rank \(\mu\) over \(\mathbb{C}\{f\}\):

\[
G \cong \mathbb{C}\{f\}^\mu.
\]

**Proof.** See Theorem 3.4.8 for the 2-dimensional case and Theorem 5.2.7 in higher dimensions.

Thus, to obtain a basis of the relative Brieskorn module \(G\), it suffices to obtain a basis of the \(\mu\)-dimensional \(\mathbb{C}\)-vector space:

\[
\frac{G}{fG} \cong \frac{\Omega^2}{df \wedge d\Omega^0_H + f\Omega^2}.
\]
and lift it, by Nakayama’s lemma, to a basis of $G$. Below we will give an important example which generalises the case of quasihomogeneous singularities in the presence of a boundary.

**Remark 3.3.1.** In Chapter 5 we will show how, starting from a basis of the relative Brieskorn module $G$, we may obtain a basis of the relative cohomology $H^1(X_t, X'_t; \mathbb{C})$, using the corresponding geometric sections, as in the ordinary, without boundary case. Moreover, we will show that on the localisation $\mathcal{M}$ of the relative Brieskorn modules defined above, there is naturally defined a meromorphic Gauss-Manin connection (which we will call relative), which has a regular-singular point at the origin, and it is exactly the extension at the origin of the topological Gauss-Manin connection on the relative cohomology bundle $\cup_{t \neq 0} H^1(X_t, X'_t; \mathbb{C})$ over the base of the Milnor fibration.

**Construction of a Basis of the Relative Brieskorn Module for Quasihomogeneous Boundary Singularities**

Recall that a function germ $f = f(x, y)$ is quasihomogeneous with weights $(w_1, w_2) \in \mathbb{Q}_+$ if there exists a natural number $\nu$ such that:

$$f(t^{w_1}x, t^{w_2}y) = t^\nu f(x, y).$$

This implies the existence of an *Euler vector field* $E_f$ for $f$, i.e. such that:

$$E_f(f) = \nu f.$$

For a boundary singularity $(f, H)$ to be quasihomogeneous, it means that both $f$ and its restriction on the boundary $f|_H$ are quasihomogeneous (the latter always holds in dimension 2). Then, in local coordinates $(x, y)$ for which $H = \{x = 0\}$, the Euler vector field of $f$ can be written as:

$$E_f = w_1 x \frac{\partial}{\partial x} + w_2 y \frac{\partial}{\partial x},$$

(i.e. it is tangent to the boundary $H$). It follows in particular that if the boundary singularity $(f, H)$ is quasihomogeneous, then $f$ belongs to its Jacobian ideal $f \in J_{f,H}$. A relative version of K. Saito’s theorem, implies a converse to this statement, i.e. that if $f$ belongs to the Jacobian ideal $J_{f,H}$ then it is $R_H$-equivalent to a quasihomogeneous germ.

In terms of differential forms, the condition $f \in J_{f,H}$ means that for any 2-form $\omega$ there exists a 1-form $\eta \in \Omega^1_H$ such that $f \omega = df \wedge \eta$, i.e.:

$$fG \subseteq F.$$

Indeed, $\eta = E_f \omega$ is such a 1-form (it vanishes on $H$ because $E_f$ is tangent to $H$). As in the ordinary case, the main thing is that for a quasihomogeneous boundary singularity $(f, H)$ there exists an inverse inclusion as well, i.e.:

$$F \subseteq fG.$$

---

3 which is easy to prove.
Indeed, it suffices to show that for a given 1-form $\eta \in \Omega^1_H$ there exists a 2-form $\theta$ and a function germ $h \in \Omega^0_H$ such that:

$$df \wedge \eta = f\theta + df \wedge dh.$$  \hfill (3.3)

This identity (the relative analog of the “quasihomogeneous division in the Brieskorn module”\(^{(2.9)}\)) can be proved exactly as in the ordinary case (using the relative Poincaré lemma for $\Omega^*_H$). Thus for a quasihomogeneous boundary singularity $(f, H)$ there is an equality:

$$fG = F.$$

A basis of the relative Brieskorn module $G$ is obtained then, by Nakayama’s lemma, by lifting a basis of the $\mu$-dimensional vector space:

$$\frac{G}{fG} \cong \frac{G}{F} \cong \Omega^2_{f,H} \cong Q_{f,H},$$

i.e. of the local algebra of the boundary singularity.

**Example 3.3.1** (Basis of the Relative Brieskorn Module for Simple Boundary Singularities). As an example we compute here the basis of the Brieskorn module $G$ for the simple singularities $(f, H)$, i.e. those belonging in Arnol’d’s list \(^{(3.1)}\) presented in the introduction. These singularities are all quasihomogeneous and thus it suffices to find a basis of the local algebra $Q_{f,H}$. In particular:

- **$A_\mu$:** $f = x + y^{\mu+1}$, $\mu \geq 1$.
  
  $$Q_f = \text{span}_\mathbb{C}\{1, y, y^2, \ldots, y^{\mu-1}\},$$
  
  $$G = \text{span}_\mathbb{C}\{dx \wedge dy, ydx \wedge dy, \ldots, y^{\mu-1}dx \wedge dy\}.$$

- **$B_\mu$:** $f = x^{\mu} + y^2$, $\mu \geq 2$.
  
  $$Q_f = \text{span}_\mathbb{C}\{1, x, x^2, \ldots, x^{\mu-1}\},$$
  
  $$G = \text{span}_\mathbb{C}\{dx \wedge dy, xdx \wedge dy, \ldots, x^{\mu-1}dx \wedge dy\}.$$

- **$C_\mu$:** $f = xy + y^\mu$, $\mu \geq 2$.
  
  $$Q_f = \text{span}_\mathbb{C}\{1, y, y^2, \ldots, y^{\mu-1}\},$$
  
  $$G = \text{span}_\mathbb{C}\{dx \wedge dy, ydx \wedge dy, x^2dx \wedge dy, \ldots, x^{\mu-1}dx \wedge dy\}.$$

- **$F_4$:** $f = x^2 + y^3$, $\mu = 4$.
  
  $$Q_f = \text{span}_\mathbb{C}\{1, x, y, xy\},$$
  
  $$G = \text{span}_\mathbb{C}\{dx \wedge dy, xdx \wedge dy, ydx \wedge dy, xydx \wedge dy\}.$$
3.4 Applications in Symplectic Deformation Theory

Below we will describe the relationship of the cohomology modules defined above with the classification of triples \((\omega, f, H)\).

3.4.1 Equivalence of Symplectic Structures: Normal Forms and Functional Invariants

We present first the relative analog of Françoise’s Theorem 2.4.1 in the presence of a boundary. As in the ordinary case, it is a consequence of the finiteness and freeness of the relative Brieskorn module \(G\) associated to the boundary singularity \((f, H)\). We follow closely Françoise’s approach in [30]. A more geometrical approach (as in the previous Chapter for the ordinary case) using the Gauss-Manin connection on the relative Brieskorn modules, will be given in Chapter 5.

Fix the boundary singularity \((f, H)\). According to a theorem of V. I. Matov [69] (which generalises the classical Tougeron theorem for finitely determined isolated singularities), one may choose coordinates such that the germ \(H\) is given by \(H = \{x = 0\}\) and \(f = f(x, y)\) is represented by a polynomial. Write \(R_{f,H}\) for the subgroup of diffeomorphisms preserving the germ \((f, H)\) and tangent to the identity. Let also \(r_{f,H}\) the tangent space of \(R_{f,H}\) at the identity:

\[
r_{f,H} = \{v \in T/L(f) = 0, \quad v|_H \in TH\},
\]

where \(T\) is the algebra of vector fields at the identity. Let \(\omega\) be a germ of a symplectic form at the origin. Denote by \(R_{f,H}(\omega)\) its \(R_{f,H}\)-orbit and \(r_{f,H}(\omega)\) for its tangent space at \(\omega\). Then, the quotient space

\[
D_{f,H}(\omega) = \Omega^2 / r_{f,H}(\omega)
\]

is the space of non-trivial infinitesimal deformations of \(\omega\) relative to the boundary singularity-preserving subgroup \(R_{f,H}\). We call \(D_{f,H}(\omega)\) the (relative to the boundary singularity \((f, H)\)) deformation space of \(\omega\). We have:

**Proposition 3.4.1.** The deformation space \(D_{f,H}(\omega)\) of a symplectic form \(\omega\) is exactly equal to the relative Brieskorn module \(G\) of the boundary singularity:

\[
D_{f,H}(\omega) = G
\]

Proof. It suffices to show that \(r_{f,H}(\omega) = df \wedge d\Omega^0_H\). An infinitesimal deformation of \(\omega\) is by definition an element of the form \(L_v\omega\) for some \(v \in r_{f,H}\). We have that \(df \wedge (v, \omega) = L_v(f)\omega = 0\) and thus there exists (by the relative de Rham division lemma [3.3.2]) a function germ \(g \in \Omega^0_H\) such that \(v, \omega = gdf\).

It follows that \(d(v, \omega) = L_v\omega = df \wedge d(-g)\). Conversely, let \(g \in \Omega^0_H\) be such that there exists a vector field \(v\) with \(L_v\omega = df \wedge dg\). In fact define \(v\) as the dual of the 1-form \(gdf\) through symplectic
duality: \( v \omega = -gdf \). Obviously \( L_v(f) = 0 \) and \( v \) is tangent to \( H \) (in particular it vanishes on \( H \)) since \( g \) vanishes on \( H \). Thus \( v \in \Gamma_{f,H} \).

From this and Theorem 3.3.4 it follows:

**Corollary 3.4.2.** The deformation space \( \mathcal{D}_{f,H}(\omega) \) of \( \omega \) is a free module of rank \( \mu = \mu_1 + \mu_0 \) over \( \mathbb{C}\{f\} \):

\[
\mathcal{D}_{f,H}(\omega) \cong \mathbb{C}\{f\}^\mu.
\]

Thus, if \( \omega \) is any germ of symplectic form, there exist \( \mu \) analytic functions \( c_i \in \mathbb{C}\{t\} \) and a function germ vanishing on \( H \), \( h \in \Omega^0_H \) such that

\[
\omega = \sum_{i=1}^\mu c_i(f)\omega_i + df \wedge dh, \tag{3.4}
\]

where the classes of the \( \mu \) 2-forms \( \omega_i \) form a basis of the relative Brieskorn module \( G \) of \( (f,H) \). For example, for \( \mu = \mu_0 = 1 \), i.e. for a relative Morse germ \( f \), we have the decomposition of Main Lemma 3.2.2

\[
\omega = c(f)dx \wedge dy + df \wedge dh.
\]

Indeed, as is easily seen from Example 3.3.1 the class of the form \( dx \wedge dy \) generates the relative Brieskorn module \( G \).

We will show now that the coefficients \( \{c_i\}_{i=1}^\mu \) in the expansion (3.4) above, are uniquely determined by the triple \( (\omega, f,H) \) (in particular by the choice of the basis \( \omega_i \)), and in particular, that they are the unique invariants of the \( \mathcal{R}_{f,H} \)-orbit of \( \omega \). Thus, if

\[
\Phi^* \omega = \sum_{i=1}^\mu \tilde{c}_i(f)\omega_i + df \wedge dh
\]

is the decomposition of \( \Phi^* \omega \) in the relative Brieskorn module, it suffices to show that \( c_i = \tilde{c}_i \) for all \( i = 1, \ldots, \mu \). To prove this we take the difference \( \omega - \Phi^* \omega = \sum_{i=1}^\mu (c_i(f) - \tilde{c}_i(f))\omega_i + df \wedge dh' \) of the decomposition of \( \omega \) in the two coordinate systems and use the following:

**Lemma 3.4.3.** For any \( \omega \in \Omega^2 \) and any \( \Phi \in \mathcal{R}_{f,H} \), there exists an \( h \in \Omega^0_H \) such that:

\[
\omega - \Phi^* \omega = df \wedge dh.
\]

Following [30] we will prove the lemma in the formal category first and then, using an interpolation lemma for the elements of the group \( \mathcal{R}_{f,H} \), we pass to the analytic category.

Denote by

\[
\hat{G} = \frac{\hat{\Omega}^2}{df \wedge d\Omega^0_H}
\]

the \( \mathbb{C}[[f]] \)-module induced by the relative Brieskorn module \( G \) under formal completion. Then:
Lemma 3.4.4. A basis of the $\mathbb{C}[f]$-module $G$ goes to a basis of the $\mathbb{C}[[f]]$-module $\hat{G}$ under formal completion:

$$\hat{G} \cong G \otimes_{\mathbb{C}[f]} \mathbb{C}[[f]].$$

Proof. As in the ordinary case, the lemma follows either from a relative version of the Bloom-Brieskorn theorem \[15\], or from the regularity of the relative Gauss-Manin connection \[64\], proved in Chapter 5.

We will also need the following interpolation lemma, which is a simple variant of a lemma obtained in \[30\] for the group $R_f$:

Lemma 3.4.5. Any diffeomorphism $\Phi \in \mathcal{R}_{f,H}$ can be interpolated by a 1-parameter family of formal diffeomorphisms $\Phi_t \in \mathcal{R}_{f,H}$, i.e. such that:

$$\Phi_0 = Id, \quad \Phi_1 = \Phi,$$
$$\Phi_t^* f = f, \quad \Phi_t(H) = H \ \forall t.$$

Proof. For convenience in notation let $(x_1, x_2)$ be a coordinate system at the origin of $\mathbb{C}^2$ for which $H = \{x_1 = 0\}$. Let also $\beta = (\beta_1, \beta_2) \in \mathbb{N}_+^2$, $|\beta| = \beta_1 + \beta_2$ and $x^\beta = x_1^{\beta_1} x_2^{\beta_2}$. Let $\Phi_i$, $i = 1, 2$ be the components of the diffeomorphism $\Phi$:

$$\Phi_i(x) = x_i + \sum_{j \geq 1} \sum_{|\beta| = j} \phi^j_{\beta} x^\beta$$

and notice that since $\Phi$ preserves $H$ the function $\Phi_1$ should be divisible by $x_1$. In terms of the above expression this means that $\phi^1_{0, \beta_2} = 0$ for all $\beta_2 \geq 1$. We will seek now the interpolation diffeomorphism $\Phi_t$ in the form:

$$\Phi_{t,i}(x) = x_i + \sum_{j \geq 1} \sum_{|\beta| = j} \phi^j_{\beta}(t) x^\beta, \quad i = 1, 2,$$

where again $\phi^1_{0, \beta_2}(t) = 0$ for all $\beta_2 \geq 1$, as a solution of the differential equation (c.f. \[101\]):

$$\Phi_t' = \Phi_0' \circ \Phi_t,$$

with the boundary conditions $\Phi_0 = Id$ (i.e. $\phi^j_{\beta}(0) = 0$), $\Phi_1 = \Phi$ (i.e. $\phi^j_{\beta}(1) = \phi^j_{\beta}$). By induction on $|\beta| = j$ we may assume that the functions $\phi^j_{\beta}(t)$ are known for $j \leq k - 1$. For $j = k$, the differential equation above imposes the conditions:

$$\phi^j_{\beta}(t)' = \phi^j_{\beta}(0)' + \psi^j_{\beta}(t),$$

where the functions $\psi^j_{\beta}(t)$ are known by the induction process and they vanish at the origin. Then
we can determine the coefficients \( \phi^i_\beta(t) \) by integration:

\[
\phi^i_\beta(t) = \phi^i_\beta(0)' t + \int_0^t \psi^i_\beta(\tau) d\tau.
\]

Obviously the initial condition \( \phi^i_\beta(0) = 0 \) is satisfied and it suffices to choose \( \phi^i_\beta(0)' \) such that the condition \( \phi^i_\beta(1) = \phi^i_\beta \) is satisfied as well. Now, the coefficients \( \phi^i_\beta(t) \) are polynomials in \( t \) and since \( \Phi_t \) is an interpolation of \( \Phi \) it follows that, for any fixed \( k \), the homogeneous term of order \( k \) in the Taylor expansion of \( \Phi^*_tf - f \) vanishes for all integer values of \( t \) and it thus vanishes everywhere. By construction, the diffeomorphism \( \Phi_t \) preserves also \( H \) and this finishes the proof.

**Proof of Lemma 3.4.3.** Interpolate \( \Phi \) by a 1-parameter formal subgroup \( \Phi_t \in \hat{\mathcal{R}}_{f,H} \), i.e. \( \Phi_0 = Id, \Phi_1 = \Phi \) and

\[
\Phi^*_tf = f, \quad \Phi_t(H) = H
\]
as in the lemma above. Then

\[
\omega - \Phi^*\omega = \int_0^1 \frac{d}{dt} \Phi^*_t \omega dt = \int_0^1 L_{\hat{X}} \omega dt,
\]
where \( \hat{X} \) is the formal vector field generated by the 1-parameter subgroup \( \Phi_t \). Since \( \Phi_t \) preserves \( (f,H) \) for all \( t \in [0,1] \) we have that

\[
L_{\hat{X}} \omega = d(\hat{X} \omega) = df \wedge d\hat{g},
\]
for some formal function \( \hat{g} \in \hat{\Omega}^0_H \) and in particular:

\[
\omega - \Phi^*\omega = df \wedge d \int_0^1 \Phi^*_t g dt = df \wedge d\hat{h}.
\]

Now if we consider the decomposition of the 2-form \( \omega - \Phi^*\omega \) in the relative Brieskorn module \( G \):

\[
\omega - \Phi^*\omega = \sum_{i=1}^\mu \psi_i(f) \omega_i + df \wedge dh,
\]
then by Lemma 3.4.4 it can be read as a decomposition in \( \hat{G} \). Comparing these two decompositions we immediately obtain \( \psi_i(f) = 0 \) for all \( i = 1, ..., \mu \). □

From this we obtain:

**Corollary 3.4.6.** The \( \mu \) functions \( c_i \in \mathbb{C}\{t\} \) in the decomposition of a symplectic form \( \omega \) in the relative Brieskorn module \( G \), are invariants of its \( \mathcal{R}_{f,H} \)-orbit.

The following theorem is a statement on normal forms of symplectic structures under the action of the group of diffeomorphisms \( \mathcal{R}_{f,H} \) preserving the boundary singularity \( (f,H) \). It is the relative analog of Françoise’s Theorem 2.4.1 and it implies that the \( \mu \) invariants \( c_i(t) \) are the unique functional invariants for the triple \( (\omega, f, H) \).
Theorem 3.4.7. For any germ $\omega \in \Omega^2$ of a symplectic form at the origin of the plane there exists a diffeomorphism $\Phi \in \mathcal{R}_{f,H}$ and $\mu$ analytic functions $\{c_i\}_{i=1}^\mu \in \mathbb{C}\{t\}$ such that:

$$\Phi^*\omega = \sum_{i=1}^\mu c_i(f)\omega_i,$$

(3.5)

where the $\mu$ classes of the 2-forms $\omega_i$ form a basis of the relative Brieskorn module $G$ of the boundary singularity $(f,H)$. Moreover, the $\mu$ functions $c_i(t)$ are characteristic (functional invariants) for the triple $(\omega, f, H)$.

Proof. The fact that the functions $c_i(t)$ are characteristic for the triple $(\omega, f, H)$ follows immediately from Lemma 3.4.3. The existence of the normal form follows from the homotopy method, along the same lines of Theorem 2.4.1. We present it for completeness: Let

$$\omega = \omega_0 + df \wedge dh,$$

be the decomposition of $\omega$ in the relative Brieskorn module $G$ of $(f,H)$, where $\omega_0 = \sum_{i=1}^\mu c_i(f)\omega_i$ and $h \in \Omega^0_{H,F}$. Consider the family of symplectic forms,

$$\omega_t = \omega_0 + tdf \wedge dh, \quad t \in [0,1].$$

We will find a 1-parameter family of diffeomorphisms $\Phi_t \in \mathcal{R}_{f,H}$ such that:

$$\Phi_t^*\omega_t = \omega_0.$$

To do this, it suffices to solve the infinitesimal equation

$$L_{v_t}\omega_t = -\frac{\partial\omega_t}{\partial t} \iff d(v_t.\omega_t) = df \wedge d(-h),$$

for the vector field $v_t$ in the Lie algebra $\mathfrak{r}_{f,H}$, i.e. tangent to both $H$ and $f$ and defined by

$$\frac{d\Phi_t}{dt} = v_t(\Phi_t), \quad \Phi_0 = Id, \quad \Phi_t(0) = 0.$$

Choose $v_t$ as the solution of the equation $v_t.\omega_t = hdf$. Then obviously $v_t$ preserves $\omega_t$, it is tangent to $H$ (because of $h$ vanishing on $H$) and it also preserves $f$ since

$$(L_{v_t}f)\omega_t = df \wedge (v_t.\omega_t) = 0.$$

Thus, the time one map $\Phi_1$ of the flow $\Phi_t$ of $v_t$ is the desired diffeomorphism. 

Remark 3.4.1. In Chapter 5, Theorem 5.3.1 we will show that the relative versions of Françoise’s results presented above, may be summarised in the following statement (compare with Theorem 2.4.1): two germs of symplectic forms are equivalent if and only if their Poincaré residues define the same relative cohomology class in each fiber $H^1(X_t, X^*_t; \mathbb{C})$ of the relative cohomology bundle.
3.4.2 Symplectic Deformations of Boundary Singularities

In [69], V. I. Matov showed that Mather’s determinacy and unfolding (or else deformation) theorems [67] do indeed hold for the subgroup \( \mathcal{R}_H \subset \mathcal{R} \) of boundary-preserving diffeomorphisms. In more modern terms it can be stated that the group \( \mathcal{R}_H \) is a “nice geometric subgroup” of \( \mathcal{R} \) in the sense of J. Damon [19]. Here we will discuss the symplectic version of these theorems for the “wider” group \( \mathcal{A}_H \) of right-left equivalences, where the right diffeomorphism has to preserve both the boundary \( H \) and a fixed symplectic structure \( \omega \) as well. The results can be viewed as the relative analogs of Garay’s results [33], presented in the previous chapter. The corresponding generalisation to the isochore case is rather straightforward and it will be presented for convenience in Chapter 5.

To start recall (c.f. [3], [6], [103]) that a deformation \( F : (\mathbb{C}^2 \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0) \) of a boundary singularity \((f, H)\) is just a deformation of \( f \), \( F(\cdot; 0) = f \), such that its restriction \( F|_H : (H \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0) \) on the boundary \( H = \mathbb{C} \subset \mathbb{C}^2 \), is a deformation of \( f|_H, F|_H(\cdot; 0) = f|_H \). To the deformation \( F \) of the boundary singularity we associate its unfolding, i.e. the map:

\[
\tilde{F} : (\mathbb{C}^2 \times \mathbb{C}^k, 0) \to (\mathbb{C} \times \mathbb{C}^k, 0), \quad \tilde{F}(\cdot; \lambda) = (F(\cdot; \lambda), \lambda)
\]

and accordingly we define also \( \tilde{F}|_H \). Fix now the equation of the boundary \( H = \{ x = 0 \} \) and fix also a germ of a symplectic form \( \omega = dx \wedge dy \) at the origin of \( \mathbb{C}^2 \). All the notions of right-left equivalence between deformations, versality, infinitesimal versality e.t.c. (c.f. [3]) carry over to the subgroup \( \mathcal{A}_{\omega, H} \) of right-left equivalences, where the right diffeomorphism has to preserve both the boundary \( H \) and the volume form \( \omega \). In particular, a deformation \( F \) (or the unfolding \( \tilde{F} \)) of a boundary singularity \((f, H)\) will be called symplectically versal if any other deformation \( F' \) (or unfolding \( \tilde{F}' \) respectively) is \( \mathcal{A}_{\omega, H} \)-equivalent to a deformation induced from \( F \), i.e. there exists a relative diffeomorphism \( \phi : (\mathbb{C}^2 \times \mathbb{C}^k, 0) \to (\mathbb{C}^2, 0), \phi(\cdot; 0) = . , \) preserving both \( H \) and \( \omega \), a relative diffeomorphism \( \psi : (\mathbb{C} \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0), \psi(\cdot; 0) = . \) and a map germ \( g : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0) \) such that:

\[
\psi(F(\phi(x, y; \lambda); g(\lambda')) = F'(x, y; \lambda').
\]

Let us consider first the corresponding infinitesimal symplectic deformations. The space of non-trivial symplectic deformations of the germ \((f, H)\) is, as is easily seen, the space:

\[
\mathcal{I}_{f, H}^1 = \frac{\mathcal{O}}{\{ L_v f + k(f)/L_v \omega = 0, \: v|_H \in TH \}} = \frac{\mathcal{O}}{\{ f, x \mathcal{O} \} + \mathcal{C}\{ f \}}.
\]

where \( \{ . , \} \) is the Poisson bracket induced by \( \omega \). This is a \( \mathcal{C}\{ f \} \)-module which can be viewed as the quotient of the *symplectic Jacobian module* of the boundary singularity \((f, H)\):

\[
\mathcal{I}_{f, H}^1 = \frac{\mathcal{O}}{\{ L_v f/L_v \omega = 0, \: v|_H \in TH \}} = \frac{\mathcal{O}}{\{ f, x \mathcal{O} \}}.
\]

by the submodule generated by the class of the constant function 1. The latter module is in turn isomorphic to the relative Brieskorn module \( G \) of the boundary singularity, the isomorphism given

---

*In analogy with the symplectic Jacobian module of an ordinary singularity in the previous chapter.*
by multiplication with the symplectic form $\omega$.

**Theorem 3.4.8.** The symplectic Jacobian module $I_{f,H}^1$ of a boundary singularity $(f,H)$ of Milnor number $\mu = \mu_1 + \mu_0$, is a free module of rank $\mu$:

$$I_{f,H}^1 \cong \mathbb{C}\{f\}^\mu.$$ 

**Proof.** To show freeness it suffices to show that the module $I_{f,H}^1 \cong G$ is torsion free. As in the ordinary case, we interpret this module as the first cohomology $H^1(C_f^\bullet)$ of the relative complex of Lagrangian deformations of the boundary singularity $(f,H)$:

$$C_{f,H}^\bullet : 0 \to xO \xrightarrow{f} \mathcal{O} \to 0.$$ 

Indeed

$$H^1(C_{f,H}^\bullet) = \text{Coker}\{,\} = I_{f,H}^1.$$ 

Consider now multiplication by $f$ in $C_{f,H}^\bullet$. Since it is a complex of free modules, multiplication by $f$ gives a short exact sequence:

$$0 \to C_{f,H}^\bullet \xrightarrow{f} C_{f,H}^\bullet \to \frac{C_{f,H}^\bullet}{fC_{f,H}^\bullet} \to 0,$$

which induces a long exact sequence in cohomology:

$$0 \to H^0(C_{f,H}^\bullet) \xrightarrow{f} H^0(C_{f,H}^\bullet) \to H^0\left(\frac{C_{f,H}^\bullet}{fC_{f,H}^\bullet}\right) \to H^1(C_{f,H}^\bullet) \xrightarrow{f} H^1(C_{f,H}^\bullet) \to H^1\left(\frac{C_{f,H}^\bullet}{fC_{f,H}^\bullet}\right) \to 0.$$ 

But since any function commuting with $f$ and vanishing on $x = 0$ is identically zero, it follows that $H^0(C_{f,H}^\bullet) = 0$ and thus, the long exact sequence above reduces to the short exact sequence:

$$0 \to H^1(C_{f,H}^\bullet) \xrightarrow{f} H^1(C_{f,H}^\bullet) \to H^1\left(\frac{C_{f,H}^\bullet}{fC_{f,H}^\bullet}\right) \to 0,$$

which is exactly what we wanted to prove. \hfill $\square$

Thus, as in the ordinary case, a necessary condition for a deformation $F$ of $(f,H)$ to be symplectically versal is that the classes of the velocities $\partial_\lambda F := \frac{\partial F}{\partial \lambda}|_{\lambda=0}$ along with the class of 1, span the isochore Jacobian module $I_{f,H}^1$ over $\mathbb{C}\{f\}$. The following theorem is the relative analog of the Garay-Mather theorem \[33\] and says that this condition is also sufficient:

**Theorem 3.4.9.** A deformation $F : (\mathbb{C}^2 \times \mathbb{C}^k,0) \to (\mathbb{C},0)$ of a boundary singularity $(f,H)$ is symplectically versal if it is infinitesimally symplectically versal, i.e.

$$I_{f,H}^1 = \text{span}_{\mathbb{C}\{f\}}\{1, \partial_{\lambda_1} F, ..., \partial_{\lambda_k} F\} \Leftrightarrow G = \text{span}_{\mathbb{C}\{f\}}\{\omega, \partial_{\lambda_1} F \omega, ..., \partial_{\lambda_k} F \omega\} \quad (3.6)$$

Following \[33\] we may prove this theorem as in the ordinary case: first we show that any 1-parameter
deformation $F'$ of an infinitesimally versal deformation $F$ is symplectically trivial (i.e. $F$ is symplectically rigid in analogy with the ordinary case). Then we conclude by using J. Martinet’s trick again, according to which any $k$-parameter deformation can be considered as a “sum” of 1-parameter deformations. The symplectic rigidity in turn can be interpreted cohomologically in terms of a parametric version of the relative Brieskorn module:

$$G_F = \frac{\Omega^2_{2+k}}{d\lambda^k \wedge dF \wedge d\Omega^0_{2+k,H}},$$

where $d\lambda^k = d\lambda_1 \wedge ... \wedge d\lambda_k$, $\Omega^p_{2+k}$ is the module of germs of $p$-forms at the origin of $\mathbb{C}^2 \times \mathbb{C}^k$ and $\Omega^0_{2+k,H}$ is the submodule of forms vanishing on $H$. The following finiteness result is a relative analog of a result of G. M. Greuel [43] (see Proposition 2.4.4) and it is a parametric version of the relative Sebastiani theorem:

**Proposition 3.4.10.** The parametric Brieskorn module $G_F$ of a deformation $F$ of a boundary singularity $(f,H)$ is finitely generated over $\mathbb{C}\{F,\lambda\}$ and it is of rank $\mu = \mu_1 + \mu_0$. Moreover, its restriction on $\mathbb{C}^2 = \{\lambda_1 = 0, ..., \lambda_k = 0\}$ is isomorphic to the relative Brieskorn module $G_f$ of $f$:

$$G_F|_{\lambda=0} \cong G_f.$$

**Proof.** Since the map $\tilde{F}$ defines an isolated complete intersection singularity, the proof of the finiteness is again a straightforward corollary of the relative analog of the Kiehl-Verdier theorem (c.f. Proposition 5.3.4 in Chapter 5 for more details). \hfill $\Box$

**Remark 3.4.2.** It can be shown that the module $G_F$ is also free, but this is irrelevant to the symplectic deformation theorem.

Consider now a 1-parameter deformation $F'_t$ of $F$:

$$F'_t := F' : (\mathbb{C}^2 \times \mathbb{C}^k \times \mathbb{C}, 0) \to (\mathbb{C}, 0), \quad (x, y; \lambda, t) \mapsto F'(x, y; \lambda, t),$$

$$F'(x, y; \lambda, 0) = F(x, y; \lambda).$$

Then, as is easily seen, $F'_t$ is symplectically trivial provided that there exists a decomposition:

$$\partial_t F' = k(F', \lambda, t) + \sum_{i=1}^k c_i(F', \lambda, t)\partial_{\lambda_i} F' + L_v F',$$

where $v$ is a relative vector field preserving $\omega$ and tangent to $H$. Multiplying with $\tilde{\omega} = \omega \wedge d\lambda^k \wedge dt$ we see that the condition of symplectic triviality above can be viewed as the condition that the class of the form $\partial_t F' \tilde{\omega}$ in the relative Brieskorn module $G_{F'}$ of $F'$ (of the unfolding $\tilde{F}'$) belongs to the $\mathbb{C}\{F', \lambda, t\}$-module spanned by the classes of form $\tilde{\omega}$ and of the initial velocities $\partial_{\lambda_i} F' \tilde{\omega}$:

$$\partial_t F' \tilde{\omega} \in M = \text{span}_{\mathbb{C}\{F', \lambda, t\}} \{\tilde{\omega}, \partial_{\lambda_1} F' \tilde{\omega}, ..., \partial_{\lambda_k} F' \tilde{\omega}\}.$$

As in the ordinary case, we will show that if $F$ is infinitesimally symplectically versal, then in fact
$M = G_{F'}$, which implies in turn the existence of a solution of the homological equation \((3.7)\). To prove the assertion, notice that since the Brieskorn module $G_{F'}$ is finitely generated, by the above Proposition 3.4.10, it suffices to show, by Nakayama’s lemma, that the image of $M$ by the natural projection:

$$
\pi : G_{F'} \to \frac{G_{F'}}{mG_{F'}}
$$

coincides with the whole $\mu$-dimensional $\mathbb{C}$-vector space:

$$
\pi(M) = \frac{G_{F'}}{mG_{F'}}. \tag{3.8}
$$

Here $m$ is the maximal ideal at the origin of $\mathbb{C} \times \mathbb{C}^k \times \mathbb{C}$. But according to Proposition 3.4.10 again, there is an isomorphism of $\mu$-dimensional vector spaces:

$$
\frac{G_{F'}}{mG_{F'}} \cong \frac{G_f}{fG_f}.
$$

Thus the condition \((3.8)\) above reduces to the condition:

$$
\pi(M) = \text{span}_{\mathbb{C}(f)} \{ \omega, \partial_{\lambda_1} F \omega, ..., \partial_{\lambda_k} F \omega \} = \frac{G_f}{fG_f}, \tag{3.9}
$$

which is in turn equivalent, by Nakayama’s lemma, to the assumption \((3.6)\) of infinitesimal symplectic versality of $F$. Thus we have proved:

**Proposition 3.4.11.** An infinitesimal symplectic versal deformation of a boundary singularity is symplectically rigid.

**Proof of the Symplectic Versal Deformation Theorem and Corollaries**

_Proof of Theorem 2.4.3._ We use again Martinet’s trick: let $F$ be a deformation of $f$, $f = F(., 0)$ and $F'$ another deformation of $f$. Define the sum $F \oplus F'$ by:

$$
F \oplus F'(x, y; \lambda, \lambda') = F(x, y, \lambda) + F'(x, y; \lambda') - f(x, y).
$$

The restriction of $F \oplus F'$ on $\lambda = 0$ is equal to $F'$ and thus, in order to show that $F'$ is symplectically equivalent to a deformation induced by $F$, it suffices to show that the deformation $F \oplus F'$ is a symplectically trivial deformation of $F$. This can be shown inductively as follows: denote by $F_j$ the restriction of $F \oplus F'$ to $\{ \lambda_j = ... = \lambda_k = 0 \}$. Then $F_1 = F$ and $F_k = F \oplus F'$. It follows from Proposition 3.4.11 that for each $j$ the deformation $F_{j-1}$ is isochore rigid and thus $F_j$ is an isochore trivial deformation of $F_{j-1}$. We conclude by induction that $F_k$ is an isochore trivial deformation of $F_1$. 

As an immediate corollary we obtain another proof of the relative Morse-Darboux lemma 3.2.1, which can now be stated as follows:
Corollary 3.4.12. Any relative Morse germ \( f \) on the symplectic plane with a fixed symplectic form \( \omega = dx \wedge dy \) and a fixed boundary \( H = \{x = 0\} \), is symplectically (right-left, or \( \mathcal{A}_{\omega,H} \)-)equivalent to the germ:

\[
f_0 = x + y^2.
\]

Moreover, the left diffeomorphism is unique.

Proof. Consider \( f_t = f_0 + th, t \in [0,1] \), a 1-parameter deformation of \( f_0, f_1 = f \), such that \( f_t|_H \) has a nondegenerate critical point at the origin for all \( t \). Then for any point \( t_0 \in [0,1] \) the germ at \( t_0 \) of the deformation \( f_t \) is an isochore trivial deformation of \( f_{t_0} \). Indeed, the relative Brieskorn module \( G_{f_t} \) is generated by the class of the form \( dx \wedge dy \wedge dt \) and the claim follows from the symplectic deformation theorem. Thus, for any \( \epsilon \) sufficiently small, the germ \( f_{t_0+\epsilon} \) is isochore equivalent to \( f_{t_0} \), and thus \( f_0 \) is isochore equivalent to \( f_1 \) as well. Finally, the uniqueness of the left diffeomorphism follows from the fact that a symplectically versal deformation of a boundary singularity, is in fact universal.

As another immediate corollary we obtain also a relative version of the theorem of Y. Colin de Verdière [18]:

Corollary 3.4.13. A versal deformation of a quasihomogeneous boundary singularity is symplectically versal.

Proof. Indeed, in this case there is an isomorphism:

\[
\frac{G}{fG} \cong Q_{f,H}
\]

and thus the classes of 1 with the initial velocities of the deformation generate the symplectic Jacobian module \( J^1_{f,H} \).
4 Singularities of Functions on the Martinet Plane, Constrained Hamiltonian Systems and Singular Lagrangians

4.1 Introduction

In several local analysis problems arising in mathematical physics, control theory, dynamical systems e.t.c. one is led to consider the classification problem for pairs \((\omega, f)\), where \(\omega\) is a germ of a closed 2-form on a manifold \(M\) and \(f\) is a function germ, with or without singularities. The most studied case is when the 2-form is nondegenerate, i.e. it defines a symplectic structure on \(M\). Then \(f\) can be viewed as a Hamiltonian function and the classification problem reduces to the well known problem of symplectic classification of singularities of functions presented in Chapter 2. Here we will consider a generalisation of the classification problem for pairs \((\omega, f)\) on a 2-manifold \(M\), where now the 2-form \(\omega\) is allowed to have singularities and thus it does not define a symplectic structure everywhere on \(M\). This situation is typical when we consider Hamiltonian systems with constraints (c.f. [26], [28], [50], [62], [74], [75], [76], [82], [84]).

In analogy with the unconstrained case, we may define a Constrained Hamiltonian System (CHS) on a 2-manifold \(M\), simply as a pair \((\omega, f)\) consisting of a function \(f\) and a singular 2-form \(\omega\) on \(M\) as above. Let \(X_f\) be the “Hamiltonian vector field” associated to the pair \((\omega, f)\) through the equation:

\[ X_f \cdot \omega = df. \]

This vector field is in general not defined and smooth everywhere on \(M\); the obstruction to the existence and/or uniqueness of \(X_f\) is obviously the set of zeros \(H(\omega)\) of the 2-form \(\omega\). In the theory of singularities of Constraints Systems (c.f. [100], [112]) it is called the Impasse Hypersurface, while in the theory of differential systems is usually called the Martinet hypersurface (c.f. [73]), in honor of J. Martinet who was the first who studied systematically singularities of differential forms [66]. The problem is thus to classify CHS at impasse points.

It is easy to see that the germ of a generic singular 2-form \(\omega\) on the plane can be reduced to Martinet normal form [66]:

\[ \omega = xdx \wedge dy. \]

The geometric invariants of the 2-form \(\omega\) on the plane are just its Martinet curve of zeros \(H(\omega) = \{x = 0\}\), along with an orientation (in the real analytic-smooth case) induced by the two symplectic
structures in its complement. The orientation of the Martinet curve plays no role in the initial
definition of singularity classes for the pair \((f, \omega)\) and thus, all the singularities are to be defined
by the relative positions of the germ \(f\) with the Martinet curve \(H(\omega)\). In particular, as long as
the function \(f\) is non-singular, or its singularities are isolated, the pair \((f, H(\omega))\) can be viewed as
defining a germ of a “boundary singularity” at the origin of the plane, i.e. such that:

- either \(f\) has an isolated critical point at the origin,
- or \(f\) is non-singular but its restriction \(f|_{H(\omega)}\) on the Martinet curve has an isolated critical
  point at the origin.

Thus, in order to study the singularities of pairs \((\omega, f)\), one may fix an arbitrary boundary singularity
\((f, H)\) and study possible normal forms of degenerate 2-forms \(\omega\), whose zero set \(H(\omega)\) is exactly
the curve \(H\). We call these 2-forms Martinet 2-forms. The corresponding group of diffeomorphisms
acting on the space of Martinet 2-forms is then the group \(R_{f,H}\) of diffeomorphisms preserving the
boundary singularity. Thus stated, the problem is very much alike with the problem presented in
the previous chapter for the classification of symplectic forms relative to the group \(R_{f,H}\).

As is expected, the Milnor number \(\mu\) of the boundary singularity \((f, H)\) plays again a significant
role in the classification problem; indeed, it will be shown that, at least as long as the germ \(f\) is
quasihomogeneous, the corresponding deformation space of a germ of a Martinet 2-form \(\omega\) on the
plane (relative to diffeomorphisms preserving the pair \((f, H)\) of course) is again a free module of rank
\(\mu\) over \(\mathbb{C}\{f\}\). From this we derive immediately, merely by the same arguments as in the previous
two chapters, a normal form for the pair \((\omega, f)\) involving exactly \(\mu\) functional moduli, which are
again analytic functions of 1-variable (Theorem 4.4.1).

Indeed, as it will become apparent in the text, the classification of pairs \((\omega, f)\) where \(\omega\) is a Martinet
2-form, relies on the first cohomology of the quotient complex \(d\Omega_H^2 / df\), where \(H = \{ x = 0 \}\) and
\(x\Omega_H^2 \subset \Omega_H^2\) is the subcomplex of forms vanishing on \(H\) (in the sense of pullback) and whose
coefficients vanish on \(H\) as well. In particular, the corresponding “Brieskorn module” can be
identified with the deformation module of \(\omega\), i.e. the \(\mathbb{C}\{f\}\)-module:

\[ D_{f,H}(\omega) = \frac{x\Omega^2}{df \wedge d(x\Omega_H^2)}. \]

The main result here is, as in the ordinary and boundary case, the finiteness and freeness of the
deformation module:

\[ D_{f,H}(\omega) \cong \mathbb{C}\{f\}^\mu. \]

We will give a proof of this fact only for the quasihomogeneous case (Theorem 4.3.1), using a variant
of a proof obtained by J. P. Françoise in [31], for the finiteness and freeness of the Brieskorn module
of a quasihomogeneous isolated singularity. Despite the fact that Françoise’s proof does not cover
all the isolated singularities, it has the advantage that it is algorithmic in nature and moreover, one
may obtain exact bounds (in appropriate norms) for the decomposition of a form in the deformation
module. In any case, the general isolated case can be easily deduced from the results of the previous
and the next chapter for the boundary singularities on the symplectic plane.
One final remark: the classification of functions $f$ and Martinet 2-forms $\omega$ in higher dimensions is a much more complicated problem and the results obtained here do not generalise in this case. Probably, analogous formal normal forms do exist but we don’t discuss this here. Instead, we give an application of the 2-dimensional results in a problem arising in the geometric theory of Hamiltonian systems with constraints, that is, the problem of classification of generic singular Lagrangians (of first order in the velocities) on the plane, under variational (gauge) equivalence (Theorem 4.5.1). Such Lagrangians, which appear in high energy physics (c.f. [28], [50], [84]), in hydrodynamics and general vortex theory (c.f. [55] and references therein) and also in control theory and sub-riemannian geometry [74], [75], [76], give rise to Euler-Lagrange equations which define a constrained Hamiltonian system $(\omega, f)$ and thus, they are subjectable to our analysis.

4.2 Deformations of Singular Symplectic Structures and Boundary Singularities

Denote by $\mathcal{O}^p$ the space of germs of analytic differential $p$-forms at the origin and by $x \mathcal{O}^p$ those forms that “vanish on $H = \{x = 0\}$”, in the sense that their coefficients belong in the ideal $x \mathcal{O}^0(= (x) \subset \mathcal{O}^0)$ generated by the equation of $H$. Notice that any form vanishing on $H$ vanishes automatically when evaluated at tangent vectors of $H$ (i.e. it has zero pull back by the embedding $H \hookrightarrow (\mathbb{C}^2, 0)$), but the converse does not hold (take for example the 1-form $dx$). We distinguish by writing $\mathcal{O}^p_H$ for the space of $p$-forms with zero pull-back on $H$. Notice that with this notation $0_H = x \mathcal{O}^0$, $\mathcal{O}^2_H = \mathcal{O}^2$ (identically) and $x \mathcal{O}^1_H = x \mathcal{O}^0 dx + x^2 \mathcal{O}^0 dy$. The space $x \mathcal{O}^2$ may be identified with the space of 2-forms whose zero set contains the curve $H = \{x = 0\}$. We write $x \mathcal{O}^2_H$ for the space of Martinet 2-forms (with zero set exactly equal to $H$). We will need the following local version of a type of relative Poincaré lemma for the complex $x \mathcal{O}^\bullet_H$ (c.f. [27], [38]):

**Lemma 4.2.1.** For any closed $i$-form $\alpha \in x \mathcal{O}^i_H$ there exists an $(i-1)$-form $\beta \in x \mathcal{O}^{i-1}_H$ such that $\alpha = d\beta$.

**Proof.** By the classical Poincaré lemma the 1-parameter family of maps $F_t(x, y) = (tx, ty)$, $t \in [0, 1]$, is a contraction at the origin, it preserves $H$, $F_t(H) \subset H$ and is such that: $F'_1 \alpha = \alpha$, $F'_0 \alpha = 0$ and $F'_1 \alpha = d\beta$, where $\beta$ is defined by

$$
\beta = \int_0^1 F'_t(V_t \omega \alpha) dt
$$

and the vector field $V_t = dF_t/\mathcal{O}_H$ is defined as the generator of $F_t$. Notice that by definition $V_t$ is tangent to $H$ for all $t$. Now, since $\alpha$ vanishes on $H$ to second order, the $(i-1)$-form $V_t \omega \alpha$ vanishes also on $H$ and since $V_t$ is tangent to $H$ it follows that $V_t \omega \alpha \in x \mathcal{O}^{i-1}_H$. By the fact that $F_t(H) \subset H$, it follows that $\beta \in x \mathcal{O}^{i-1}_H$. $\square$

Fix now a pair $(f, H)$, where $f$ has an isolated singular point at the origin of finite multiplicity $\mu = \mu_1 + \mu_0$. The differential $df$ defines an ideal in the algebra $\mathcal{O}^\bullet$ of germs of differential forms at the origin, which induces an ideal in the subalgebra $x \mathcal{O}^\bullet_H$. The lemma below gives necessary and
sufficient conditions for the ideal membership problem. It is an analog of the relative de Rham’s division lemma presented in the previous chapter.

Lemma 4.2.2.

(a) Let $\omega \in x\Omega^2$. Then $\omega = df \wedge \eta$ holds for some $\eta \in x\Omega^1_H$ if and only if $\omega \in J_{f,H}x\Omega^2$.

(b) For any 1-form $\alpha \in x\Omega^1_H$ such that $\alpha \wedge df = 0$, there exists a function germ $g \in x\Omega^0_H$ such that $\alpha = gd\!f$.

Proof.

(a) the proof is an obvious calculation

(b) It follows by the fact that the relative de Rham’s division lemma is true for the complex $\Omega^*_H$ (see Lemma 3.3.2): indeed, if we write $\alpha = x\alpha_1$ for some $\alpha_1 \in \Omega^1_H$, then $df \wedge (x\alpha_1) = xdf \wedge \alpha_1 = 0$ and since $x$ is a non-zero divisor it follows from the relative de Rham division lemma in $\Omega^*_H$ that there exists a function germ $g_1 \in \Omega^0_H$ such that $\alpha_1 = g_1df$. But then $\alpha = xg_1df$ and the germ $g = xg_1 \in x\Omega^0_H$ as we wanted.

By the extension of Tougeron’s theorem on the finite determinacy of boundary singularities $(f,H)$ proved by V. I. Matov [69] we may suppose that $H = \{x = 0\}$ and $f$ is polynomial of sufficiently high degree $(\geq \mu + 1)$. Write $\mathcal{R}_{f,H}$ for the pseudogroup of symmetries of the pair $(f,H)$. In the following lemma we identify the set of (infinitesimal) trivial deformations of Martinet 2-forms relative to $\mathcal{R}_{f,H}$-action:

Lemma 4.2.3. Let $\omega$ be a germ of a Martinet 2-form. The tangent space to the orbit of $\omega$ under the $\mathcal{R}_{f,H}$-action is:

$$\mathfrak{r}_{f,H}(\omega) = df \wedge d(x\Omega^0_H).$$

Proof. Let $v$ be an element of the Lie algebra $\mathfrak{r}_{f,H}$. The infinitesimal deformation of $\omega$ associated to $v$ is by definition an element of the form $L_v\omega$ (where $L$ is the Lie derivative). We have that $df \wedge (v,\omega) = L_v(f)\omega = 0$ and thus, by de Rham’s division with $df$, there exists a function germ $g$ such that $v,\omega = gd\!f$. Since the 1-form $v,\omega$ vanishes on $H$ to second order (because both $v$ and $\omega$ vanish on $H$) we conclude by Lemma 4.2.2 above that $g \in x\Omega^0_H$. It follows that $d(v,\omega) = L_v\omega = df \wedge d(-g)$, $g \in x\Omega^0_H$. Conversely, let $g \in x\Omega^0_H$ be such that there exists a vector field $v$ with $L_v\omega = df \wedge dg$. In fact define $v$ as the dual of the 1-form $gd\!f$ through $\omega$: $v,\omega = -gd\!f$ (this is possible because $g$ vanishes on $H$). Obviously $L_v(f) = 0$ and $v$ is vanishes on $H$ since $g$ vanishes on $H$ to second order. Thus $v \in \mathfrak{r}_{f,H}$ and the lemma is proved.

It follows that the quotient space

$$\mathcal{D}_{f,H}(\omega) = \frac{x\Omega^2}{df \wedge d(x\Omega^0_H)}$$
consists of the nontrivial infinitesimal deformations of the Martinet 2-form $\omega$ relative to the symmetries of the boundary singularity. Along with the $\mathbb{C}$-linear space structure, the deformation space $\mathcal{D}_{f,H}(\omega)$ (which we will denote simply by $\mathcal{D}(\omega)$) has a natural $\mathbb{C}\{f\}$-module structure with multiplication by $f$. We call it the deformation module of the Martinet germ $\omega$. In the next section we will show that this module is a free module of rank $\mu = \mu_1 + \mu_0$ over $\mathbb{C}\{f\}$. This statement is analogous to the relative Brieskorn-Deligne-Sebastiani Theorem 3.3.4 in the previous chapter. This finiteness result along with the following proposition, are cornerstones in the classification problem.

**Proposition 4.2.4.** Fix a boundary singularity $(f, H)$. Let $\omega$ and $\omega'$ be two germs of Martinet 2-forms at the origin, such that $\omega - \omega' \in df \wedge d(x\Omega^0_H)$. Then there exists a diffeomorphism $\Phi \in \mathcal{R}_{f,H}$ such that $\Phi^* \omega' = \omega$.

**Proof.** The proof of the existence of the diffeomorphism is again by the homotopy method: consider a 1-parameter family of Martinet 2-forms connecting $\omega$ and $\omega'$:

$$\omega_t = \omega + tdf \wedge dh, \quad h \in x\Omega^0_H,$$

so that $\omega_0 = \omega$ and $\omega_1 = \omega'$. We seek a 1-parameter family of vector fields $v_t \in \mathfrak{r}_{f,H}$ such that

$$L_{v_t} \omega_t = 0 \iff d(v_t \omega_t) = df \wedge d(-h),$$

for all $t \in [0,1]$. Choose $v_t$ by $v_t \omega_t = hdf$. It preserves $\omega_t$ and it is also in $\mathfrak{r}_{f,H}$ by the same reasoning as in the previous lemma. It follows that the time 1-map of the flow $\Phi_t$ of $v_t$ sends $\omega_0$ to $\omega_1$.

### 4.3 Finiteness and Freeness of the Deformation Module

We prove here the finiteness and freeness of the deformation module $\mathcal{D}_{f,H}(\omega)$ for the class of all quasi-homogeneous boundary singularities, using Françoise’s algorithm. In particular we will prove:

**Theorem 4.3.1.** Let $(f, H)$ be a quasihomogeneous boundary singularity at the origin of Milnor number $\mu = \mu_1 + \mu_0$ and let $\omega$ be a germ of a Martinet 2-form whose zero set is exactly the curve $H = H(\omega)$. Then the deformation module of $\omega$ is a free module of rank $\mu$ over $\mathbb{C}\{f\}$:

$$\mathcal{D}_{f,H}(\omega) \cong \mathbb{C}\{f\}^\mu.$$

Suppose that $f$ is a quasihomogeneous polynomial of type $(m_1, m_2; 1)$, $m_i \in \mathbb{Q}_+$, i.e. such that $f(t^{m_1}x, t^{m_2}y) = tf(x, y)$. Denote by

$$E_f = m_1 x \frac{\partial}{\partial x} + m_2 y \frac{\partial}{\partial y}$$

the Euler vector field of $f$, i.e. such that $E_f(f) = f$. Write also $M = m_1 + m_2$. Then the following division lemma holds:
Lemma 4.3.2. If \((f, H)\) is a quasihomogeneous boundary singularity at the origin of the plane, then the following identity holds:

\[
df \wedge x \Omega^1_H = fx \Omega^2 + df \wedge d(x \Omega^1_H). \tag{4.1}
\]

Proof. It suffices to find, for a given 1-form vanishing on the boundary \(\eta \in x \Omega^1_H\), a 2-form \(\theta \in x \Omega^2\) and a function \(h \in x \Omega^0_H\) vanishing on the boundary to second order, such that

\[
df \wedge \eta = f \theta + df \wedge dh.
\]

But \(f \theta = df \wedge (E_f \theta)\) and so the equality above reduces to

\[
df \wedge (\eta - E_f \theta - dh) = 0,
\]

i.e. to

\[
E_f \theta = \eta - dh.
\]

Taking exterior differential, it suffices to find \(\theta\) such that:

\[
L_{E_f} \theta = d\eta.
\]

We view \(L_{E_f}\) as an operator in formal series:

\[
L_{E_f} : \hat{x} \Omega^2 \to \hat{x} \Omega^2, \quad L_{E_f} = m_1 x \frac{\partial}{\partial x} + m_2 y \frac{\partial}{\partial y} + M.
\]

This is obviously an invertible operator since for any monomial \(x^i y^j\), \(b = (i, j), i > 0\) we have:

\[
L_{E_f} x^i y^j = (<m,b> + M)x^i y^j,
\]

where \(<m,b> + M\) never vanishes. Thus we can find a formal solution to this equation. This solution can easily be extended to an analytic solution \(\eta \in x \Omega^1_H\) in a fundamental system of neighborhoods of the origin (see Lemma 4.3.3 below). The lemma is proved. \(\Box\)

4.3.1 Construction of a Formal Basis of the Deformation Module

First we construct a formal basis of the \(\mathbb{C}[[f]]\)-module \(\hat{D}_{f,H}(\omega)\) (i.e. of the formal deformation module):

Choose a monomial basis \(\{e_i(x, y)\}_{i=1}^\mu\) of the local algebra \(Q_{f,H}\) of the boundary singularity and lift it to a basis of monomial 2-forms \(\omega_i = \{e_i(x, y)dx \wedge dy\}_{i=1}^\mu\) in

\[
\Omega^2_{f,H} = \frac{\Omega^2}{df \wedge \Omega^1_H}.
\]

Any 2-form \(\omega \in x \Omega^2\) can be written as \(\omega = x\tilde{\omega}\) for some 2-form \(\tilde{\omega}\). Decompose now \(\tilde{\omega} \in \hat{\Omega}^2\) in
\[ \tilde{\Omega}^2_{f,H} : \]
\[ \tilde{\omega} = \sum_{i=1}^{\mu} c_i \omega_i + df \wedge \tilde{\eta}, \]
where \( \tilde{\eta} \in \tilde{\Omega}_H \) is a 1-form vanishing on \( H \) (and defined uniquely by \( \omega \) modulo terms of the form \( gdf \)) and \( c_i \in \mathbb{C} \) for \( i = 1, \ldots, \mu \). This decomposition induces also a decomposition, after multiplication with the function \( x \), in the space \( x\tilde{\Omega}^2 \), in the sense that:
\[ \omega = x \sum_{i=1}^{\mu} c_i \omega_i + df \wedge x\tilde{\eta}. \quad (4.2) \]
Write now \( \eta = x\tilde{\eta} \in x\tilde{\Omega}^1_H \) and decompose the 2-form \( df \wedge \eta \) according to the division Lemma 4.3.2 and plug it to equation (4.2) above:
\[ \omega = x \sum_{i=1}^{\mu} c_i \omega_i + f\theta + df \wedge dh, \quad (4.3) \]
where \( \theta = x\tilde{\theta} \in x\tilde{\Omega}^2 \) and \( h \in x\tilde{\Omega}^0_H \). Continuing that way, decompose \( \tilde{\theta} \) in \( \tilde{\Omega}^2_{f,H} \):
\[ \tilde{\theta} = \sum_{i=1}^{\mu} c_i^1 \omega_i + df \wedge \tilde{\eta}_1, \]
where again \( \tilde{\eta}_1 \in \tilde{\Omega}_H \) (is defined by \( \tilde{\theta} \) modulo terms \( gdf \)) and the \( c_i^1 \in \mathbb{C} \) are constants. Multiplying by \( x \) and plugging this back to equation (4.3) we obtain the new decomposition:
\[ \omega = x \sum_{i=1}^{\mu} (c_i + c_i^1 f) \omega_i + df \wedge \eta_1 + df \wedge dh, \quad (4.4) \]
where \( \eta_1 = x\tilde{\eta}_1 \in x\tilde{\Omega}^1_H \). Now use again the division Lemma 4.3.2 for the 2-form \( df \wedge \eta_1 \) to obtain new \( \theta_1 = x\tilde{\theta}_1 \in x\tilde{\Omega}^2 \), \( h_1 \in x\tilde{\Omega}^0_H \) such that:
\[ df \wedge \eta_1 = f\theta_1 + df \wedge dh_1 \]
and plug it back to (4.4) above to get:
\[ \omega = x \sum_{i=1}^{\mu} (c_i + c_i^1 f) \omega_i + f^2\theta_1 + df \wedge d(fh_1 + h). \]
Continuing that way with the 2-form \( \theta_1 \), e.t.c. we obtain at the \( p \)-th iterate a decomposition of the form:
\[ \omega = x \sum_{i=1}^{\mu} (c_i + c_i^1 f + \ldots + c_i^p f^p) \omega_i + df \wedge d(f^p h_p + \ldots + f h_1 + h) + o(f^{p+1}). \]
The term \( o(f^{p+1}) \) belongs, for \( p \to \infty \), to the intersection of all maximal ideals \( \cap_p \mathfrak{m}_p \), i.e. it goes to zero in the \( \mathbb{C}[[f]] \)-module \( D_{f,H}(\omega) \). Thus, the algorithm converges in the Krull topology.
4.3.2 Proof of Convergence in the Analytic Category

For convenience in notation, we change coordinates from \((x, y)\) to \(x = (x_1, x_2)\) (so that \(H = \{x_1 = 0\}\)). We write in this notation \(x^b = x_1^{b_1} x_2^{b_2}\) for a monomial \(x^i y^j\) and a vector \(b = (b_1, b_2)\) in \(\mathbb{N}^2\). Let \(r = (r_1, r_2) \in \mathbb{R}^2_+\) and let

\[
D(r) = \{(x_1, x_2) \in \mathbb{C}^2 | |x_1| \leq r_1, |x_2| \leq r_2\}
\]

be a polycylinder in \(\mathbb{C}^2\). We consider the pseudo-norm \(|.|_r\) in \(\mathcal{O}\) defined by:

\[
|\phi|_r = \sum_b |\phi_b|r^b,
\]

where \(\phi = \sum_b \phi_b x^b\) is an analytic function. We denote by \(\mathcal{O}_r\) the subset of \(\mathcal{O}\) for which the pseudo-norm \(|.|_r\) is finite and thus defines a norm. For \(\phi = (\phi_1, ..., \phi_k) \in \mathcal{O}_r^k\) we have accordingly:

\[
|\phi|_r = \sum_{i=1}^k |\phi_i|_r.
\]

Identify now \(\Omega^2_\mathcal{O}\) with \(\mathcal{O}_r\) and \(\Omega^1_\mathcal{O}\) with \(\mathcal{O}_r \times \mathcal{O}_r\). Obviously \((\Omega^1_H)_r\) can be identified with the subspace \(\mathcal{O}_r \times (x\mathcal{O})_r\) and \((x\Omega^1_H)_r\) with \((x\mathcal{O})_r \times (x^2\mathcal{O})_r\). The map \(u = df \wedge : x\Omega^1_H \to x\Omega^2\) is a \(\mathcal{O}\)-linear map and induces a map:

\[
u : (x\Omega^1_H)_r \to (x\Omega^2)_r.
\]

A section of \(u\) is a \(\mathbb{C}\)-linear map:

\[
\lambda : (x\Omega^2)_r \to (x\Omega^1_H)_r,
\]

such that \(u = u\lambda u\). We can see from the definition that \(\lambda = \frac{1}{n}df\) is division with \(df\).

Recall that a section \(\lambda\) is adapted to the polydisc \(D(r)\) (or to \(r\)) if \(\lambda\) is a continuous mapping between Banach spaces, i.e. there exists a constant \(C_r\) such that:

\[
|\lambda(\theta)|_r \leq C_r|\theta|_r,
\]

for all \(\theta \in (x\Omega^2)_r\). By Malgrange’s priviledged neighborhoods theorem [65] we have that given \(u\), there is a section \(\lambda\) such that the set of polydiscs \(D(r)\) onto which is adapted, forms a fundamental system of neighborhoods of the origin.

Now will also need the following two lemmata concerning the bounds obtained by division with \(df\) (Lemma 4.3.2) and the corresponding bounds obtained by the relative Poincaré lemma (Lemma 4.2.1).

**Lemma 4.3.3.** If \(\theta \in (x\Omega^2)_r\) is such that \(L_{E_f}\theta = d\eta\), then:

\[
|\theta|_r \leq \frac{1}{m_0 r_0} |\eta|_r,
\]
where \( m_0 = \min(m_1, m_2) \) and \( r_0 = \min(r_1, r_2) \).

**Proof.** It is exactly the same as in [31], Lemma 3.1.2. For completeness, write \( \eta = \eta_2 dx_1 + \eta_1 dx_2 \) with \( \eta_i = \sum b \eta_i^b x^b \), \( i = 1, 2 \), where, since \( \eta \) vanishes on \( H = \{ x_1 = 0 \} \) to second order, the vector \( b \) in \( \eta_1 \) is of the form \( b = (b_1, b_2) \), \( b_1 \geq 2 \) and in \( \eta_2 \), \( b_1 \geq 1 \). Then by direct computation:

\[
\theta = \sum_{i=1}^{2} \sum_{b} \frac{b_i}{m_i} \eta_i^b x^b - I_i,
\]

where \( I_1 = (1, 0) \), \( I_2 = (0, 1) \) are unit vectors in \( \mathbb{N}^2 \). From this the lemma follows. \(\square\)

**Lemma 4.3.4.** For any closed 1-form \( \pi \in (x\Omega^1_H)_r \), there exists a function \( \zeta \in (x\Omega^0_H)_r \) such that:

\[
\pi = d\zeta, \quad |\zeta|_r \leq R|\pi|_r,
\]

where \( R = r_1 + r_2 \).

**Proof.** It follows by the proof of the relative Poincaré Lemma 4.2.1. \(\square\)

Now, the finiteness of the deformation module in the analytic category may be stated in the following form:

**Proposition 4.3.5.** Let \( D(r) \) be a polydode onto which the section \( \lambda \) is adapted. Then there exists a smaller polydode \( D(r') \subset D(r) \) such that for any \( \omega \in (x\Omega^2)_r \) there exist an analytic function \( \xi \in (x\Omega^0_H)_r \) and \( \mu \) analytic functions \( c_i(f) \in \Omega^0_r \) such that:

\[
\omega = x \sum_{i=1}^{\mu} c_i(f) \omega_i + df \wedge d\xi,
\]

with the following explicit bounds:

\[
|\xi|_{r'} \leq \frac{RC_r(1 + \frac{MR}{m_0 r_0})}{1 - |f|_r} \frac{|\omega|_r}{C_r m_0 r_0},
\]

\[
|c_i(f)|_{r'} \leq \frac{|\omega|_r}{1 - |f|_r} \frac{C_r}{m_0 r_0}.
\]

**Proof.** The proof is again the same as in the ordinary case. We present it for completeness. Start with the first decomposition

\[
\omega = \sum_{i=1}^{\mu} c_i \omega_i + df \wedge \eta,
\]

with the bound \( |\eta|_r \leq C_r |\omega|_r \). Then we solve the equation \( d\eta = L_{E_\omega} \theta \) and we find a \( \theta \) such that, according to Lemma 4.3.3:

\[
|\theta|_r \leq \frac{1}{m_0 r_0} |\eta|_r \leq \frac{C_r}{m_0 r_0} |\omega|_r.
\]
Then by the relation $dh = E_f \theta - \eta$ and Lemma 4.3.4 we obtain the bounds:

$$|h|_r \leq R(1 + \frac{MR}{m_0r_0})Cr_\omega_r.$$

Decomposing this way we obtain at the $p$-th iterate:

$$|\theta_p|_r \leq \left(\frac{Cr}{m_0r_0}\right)^p |\omega|_r,$$

$$|h_p|_r \leq CRC_r(1 + \frac{MR}{m_0r_0})^{p-1} |\omega|_r.$$

Choose now $r'$ such that $|f|_{r'}(C_r/m_0r_0) < 1$. Then, for the term $df \wedge d(\sum_{i=0}^p f^i h_i)$ (where $h_0 = h$) we have the bounds:

$$\left|\sum_{i=0}^p f^i h_i_{r'}\right| \leq \sum_{i=0}^p \left|f\right|_{r'} |h|_{r'} \leq \sum_{i=0}^p \left|f\right|_{r'} |h|_{r}$$

and for the term $\sum_{i=1}^p \sum_{j=0}^p c_i^j f^j \omega_i$ (where $c_0^i = c_i$) we have the corresponding bounds:

$$\left|\sum_{j=0}^p c_i^j f^j\right|_{r'} \leq \sum_{j=0}^p \left|c_i^j\right|_{r'} |f|_{r'}.$$

From this the theorem follows.

\[
\square
\]

### 4.3.3 Proof of Freeness of the Deformation Module

Again it suffices to show that the $\mathbb{C}\{f\}$-module $\mathcal{D}_{f,H}(\omega)$ is torsion free. Suppose then that there exists a function germ $h \in x\Omega^0_H$ such that $f \omega = df \wedge dh$ for some 2-form $\omega \in x\Omega^2$. We will need to show that there exists an analytic function $\xi \in x\Omega^1_H$ such that $\omega = df \wedge d\xi$. But by assumption, we have that $df \wedge (E_f \omega - dh) = 0$ and thus, by the relative de Rham division Lemma 4.2.2, there exists a germ $g \in x\Omega^0_H$ such that $E_f \omega - dh = gdf$. Taking the exterior differential, this relation reads:

$$L_{E_f} \omega = df \wedge d(-g).$$

Take now quasihomogeneous decomposition of the form $\omega$, $\omega = \sum_k \omega_k$. For any $k$ we have

$$\omega_k = df \wedge \frac{-(dg)_{k-1+M}}{k + M},$$

and since the Lie derivative commutes with the differential, we obtain the existence of an analytic function:

$$\xi = \sum_k \frac{-g_{k-1+M}}{k + M},$$

such that $\omega = df \wedge d\xi$. Obviously $\xi \in x\Omega^1_H$ and freeness is proved.
4.3.4 Choice of a Basis

To construct a basis of $\mathcal{D}_{f,H}(\omega)$ we consider the $\mathbb{C}[f]$-module

$$F = \frac{df \wedge x\Omega^1_H}{df \wedge d(x\Omega^1_H)}.$$

We have a natural inclusion of $\mathbb{C}[f]$-modules

$$F \subset \mathcal{D}_{f,H}(\omega).$$

Multiplication by $f$ in $\mathcal{D}_{f,H}(\omega)$ gives obviously elements inside $F$ and so $f\mathcal{D}_{f,H}(\omega) \subset F$. On the other hand, by the quasihomogeneous division with $df$ (Lemma 4.3.2) we obtain that the class of any 2-form of the form $df \wedge \eta$, $\eta \in x\Omega^1_H$, can be represented by the class of a 2-form $f\theta$, $\theta \in x\Omega^2$. From this it follows that:

$$f\mathcal{D}_{f,H}(\omega) = F.$$

Thus we obtain a sequence of isomorphisms of $\mathbb{C}$-vector spaces:

$$\frac{\mathcal{D}_{f,H}(\omega)}{f\mathcal{D}_{f,H}(\omega)} \cong \frac{\mathcal{D}_{f,H}(\omega)}{F} \cong \frac{x\Omega^2}{xdf \wedge \Omega^1} \cong \frac{x\mathcal{O}}{xJ_{f,H}},$$

which is again a $\mu$-dimensional vector space (a shift by $x$ of the local algebra $\mathcal{Q}_{f,H}$ of the boundary singularity). Thus, by Nakayama lemma a basis of monomials $e_i(x,y)$, $i = 1,\ldots,\mu$ of the local algebra $\mathcal{Q}_{f,H}$ lifts to a basis $xe_i(x,y)dx \wedge dy$ in the deformation module $\mathcal{D}_{f,H}(\omega)$.

4.4 Normal Forms and Functional Invariants

The following theorem concerns the local normal forms of Martinet 2-forms under the action of the boundary-singularity preserving diffeomorphism group $\mathcal{R}_{f,H}$.

**Theorem 4.4.1.** Let $(f,H)$ be a quasihomogeneous boundary singularity of finite multiplicity $\mu$. Then, for any germ of a Martinet 2-form $\omega$ at the origin, there exist $\mu$ analytic functions $c_i(x,y)$ and a diffeomorphism $\Phi \in \mathcal{R}_{f,H}$, such that $\omega$ is reduced to the normal form:

$$\Phi^*\omega = x \sum_{i=1}^{\mu} c_i(f)e_i(x,y)dx \wedge dy,$$

where the classes of the monomials $e_i(x,y)$ form a basis of the local algebra $\mathcal{Q}_{f,H}$. Moreover, the $\mu$ functions $c_i$ are characteristic for the pair $(\omega,f)$.

**Proof.** The existence of the normal form is obtained immediately by the homotopy method of Proposition 4.2.4 and the finiteness Theorem 4.3.1. It remains to prove only that the coefficients $c_i \in \mathbb{C}\{t\}$ in the normal form are indeed characteristic for the pair $(\omega,f)$. For this we will need the following lemma:
Lemma 4.4.2. For any $\omega \in x\Omega^2$ and any $\Phi \in \mathcal{R}_{f,H}$, there exists an $h \in x\Omega^0_H$ such that:

$$\omega - \Phi^* \omega = df \wedge dh.$$ 

Proof of the Lemma. Interpolate $\Phi$ by a 1-parameter formal subgroup $\Phi_t \in \hat{\mathcal{R}}_{f,H}$, i.e. $\Phi_0 = Id$, $\Phi_1 = \Phi$ and

$$\Phi_t f = f, \quad \Phi_t(H) = H.$$ 

Then

$$\omega - \Phi^* \omega = \int_0^1 \frac{d}{dt} \Phi_t^* \omega dt = \int_0^1 \Phi_t^* (L_{\hat{X}} \omega) dt,$$

where $\hat{X}$ is the formal vector field generated by the 1-parameter subgroup $\Phi_t$. Since $\Phi_t$ preserves $(f,H)$ for all $t \in [0,1]$ we have that

$$L_{\hat{X}} \omega = d(\hat{X} \omega) = df \wedge d\hat{g},$$

for some formal function $\hat{g} \in x\hat{\Omega}^0_H$ (because $\omega$ vanishes also on $H$) and in particular:

$$\omega - \Phi^* \omega = df \wedge d \int_0^1 \Phi_t^* g dt = df \wedge d \hat{h},$$

where again $\hat{h} \in x\hat{\Omega}^0_H$. Now if we consider the decomposition of the 2-form $\omega - \Phi^* \omega$ in the deformation module $\mathcal{D}_{f,H}(\omega)$:

$$\omega - \Phi^* \omega = \sum_{i=1}^\mu \psi_i(f) \omega_i + df \wedge dh,$$

then this can be read as a decomposition in the formal module $\hat{\mathcal{D}}(\omega)$ and after comparing the two decompositions thus obtain, we immediately deduce: $\psi_i(f) = 0$ for all $i = 1, \ldots, \mu$. 

(Continuation of the proof of Theorem 4.4.1). Decompose first $\omega$ in the deformation module $\mathcal{D}_{f,H}(\omega)$:

$$\omega = \sum_{i=1}^\mu \tilde{c}_i(f) \omega_i + df \wedge dh,$$

and take the difference of $\omega$ with $\Phi^* \omega$:

$$\omega - \Phi^* \omega = \sum_{i=1}^\mu (\tilde{c}_i(f) - c_i(f)) \omega_i + df \wedge dh.$$ 

Then from Lemma 4.4.2 above, it immediately follows that $\tilde{c}_i(t) = c_i(t)$. 

4.4.1 Geometric Description of the Moduli for the Nondegenerate case

Fix the pair:

$$\omega = xc(f)dx \wedge dy, \quad f(x,y) = x + y^2, \quad \text{(4.5)}$$
where $c(0) \neq 0$.

Let us consider first a small ball at the origin of $\mathbb{C}^2$ such that the fibers of $f(x, y) = t$ are transversal to the boundary of this ball over the points $t$ of a sufficiently small disc in $\mathbb{C}$, centered at the origin (the critical value of the restriction $f|_H$). Modifying the neighborhoods under consideration sufficiently, we may suppose that the fibers of the restriction $f|_H$ on the Martinet curve (which consists of two points away from the origin, for $t \neq 0$) are also transversal to the restriction of the boundary of the initial ball on the Martinet curve $H$. The latter consists of two points and thus transversality with the fibers of $f|_H$ means simply that they do not meet on $H$, i.e., the fibers $f^{-1}(t) \cap H$ are bounded within a sufficiently small segment of the Martinet curve. The intersection of each of the fibers of $f$ with the interior of the chosen ball, is an open Riemann surface $X_t$ with a set of distinguished points $X_t \setminus H$. Let $(t)$ be a 1-parameter family of relative cycles on the pair of fibers representing a relative homology class in $H_1(X_t, X_t \setminus H; \mathbb{C})$, so that $(t)$ is obtained by continuous deformation of some relative cycle $(t_0)$ over a smooth pair $(X_{t_0}, X_{t_0} \setminus H)$. As is easily seen, for $t$ real and positive, the pair of fibers is contractible to its real part and as $t \to 0$ the fiber $X_t \setminus H$ shrinks to a point (see figure 4.4.1). Arnol’d called the relative cycle $(t)$ arising this way, vanishing half cycle $[6]:$

$$\gamma(t) = \{(x, y) \in \mathbb{R}^2/x \geq 0, \ x + y^2 = t, \ t < \epsilon\}.$$

Obviously, if $\omega$ is a germ of a Martinet 2-form and $\alpha$ is a primitive of $\omega$, then the integral

$$V(t) = \int_{(t)} \alpha,$$

is an invariant of the pair $(\omega, f)$. In a realisation of the 2-form as a magnetic (curvature) 2-form, the integral $V(t)$ is nothing more that the magnetic flux on the 2-cell enclosed by the vanishing half-cycle. It is easy to see that this integral is a holomorphic function of $t$ and thus we may consider its derivative $V'(t)$:

$$V'(t) = \int_{(t)} \frac{d\alpha}{df} = \int_{(t)} \frac{\omega}{df}.$$

The integrand of this integral is the Gelfand-Leray form of $\omega$ and is defined as follows: let $\omega_0 = xdx \wedge dy$ be the standard Martinet 2-form. Then if we denote by $E_f$ the Euler vector field of $f$ the following relation holds:

$$f\omega_0 = df \wedge \alpha_0,$$

where $\alpha_0 = E_f \omega_0 = x^2dy - (xy/2)dx$ and of course $d\alpha_0 = (5/2)\omega_0$. Now, since $\omega = c(f)\omega_0$ we have that

$$\frac{\omega}{df} = \frac{c(f)}{f}\alpha_0$$

and thus:

$$V'(t) = \frac{c(t)}{t} \int_{(t)} \alpha_0.$$

Now the latter integral $V_0(t)$ can be evaluated immediately, $V_0(t) = (4/3)t^{5/2}$ and it can be inter-
Figure 4.1: Local model for a vanishing half-cycle of a boundary singularity. The area of the shaded region is $\text{const} \cdot t^{3/2}$, while the magnetic flux is $\text{const} \cdot t^{5/2}$.

preted as the standard magnetic flux enclosed by the vanishing half-cycle $\gamma(t)$. Thus we have:

$$tV'(t) = c(t)V_0(t). \quad (4.6)$$

From this equation we obtain the expression for the invariant:

$$c(t) = (3/4)t^{-3/2}V'(t).$$

Its geometric explanation is direct: it measures the rate of change of the magnetic flux on the family of fibers, enclosed by the vanishing half-cycle $\gamma(t)$ for $t$ varying close to zero.

### 4.4.2 Martinet Normal Form for Nondegenerate Boundary Singularities

Fix now the Martinet germ $\omega = xdx \wedge dy$, $H(\omega) = \{x = 0\}$. The following theorem describes the $R_\omega$-orbit of the $A_1$-boundary singularity $f = x + y^2$. It is the analog of the relative Morse-Darboux lemma (Theorem 3.2.1) in the Martinet case:

**Corollary 4.4.3.** Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a function germ such that the origin is a regular point for $f$ but nondegenerate (Morse) critical point for the restriction $f|_{H(\omega)}$ on the Martinet curve. Then there exists a diffeomorphism $\Phi \in R_\omega$ and a unique analytic function $\psi \in \mathbb{C}\{t\}$, $\psi(0) = 0$, $\psi'(0) = 1$ such that

$$\Phi^*f = \psi(x + y^2). \quad (4.7)$$

**Proof.** By Theorem 4.4.1 above we may choose a coordinate system $(x, y)$ such that $(x = 0, f = x + y^2)$ and $\omega = xc(f)dx \wedge dy$, where $c \in \mathbb{C}\{t\}$ is a function, nonvanishing at the origin. We may suppose that $c(0) = 1$. We will show that there exists a change of coordinates $\Phi$ such that the pair $(x = 0, f = x + y^2)$ goes to $(x = 0, \psi(f))$ for some function $\psi$ and $\omega$ is reduced to Martinet normal form. To do this, we set $\Phi(x, y) = (xv(f), y\sqrt{v(f)})$, where $v \in \mathbb{C}\{t\}$ is some function with $v(0) = 1$ (so $\Phi$ is indeed a boundary-preserving diffeomorphism). With any such function $v$ we have $\Phi^*f = \psi(f)$ for the function $\psi(f) = fv(f)$, with $\psi(0) = 0$ and $\psi'(0) = 1$. Now it suffices to choose
so that the map $\Phi$ satisfies $v(f) \det \Phi_s = c(f)$, i.e. such that the following initial value problem is satisfied for the function $w = v^{5/2}$:

$$\frac{2}{5}tw'(t) + w(t) = c(t), \quad w(0) = 1. \quad (4.8)$$

As is easily verified this admits an analytic solution given by the formula

$$w(t) = t^{-\frac{5}{2}} \int_0^t \frac{5}{2} s^{\frac{3}{2}} c(s) ds.$$

**Remark 4.4.1.** From the geometric description of the modulus $c(t)$ and equation (4.8) above, we immediately obtain also the relation of the modulus $\psi$ with the magnetic flux of the vanishing half-cycle $\gamma(t)$ of $f$:

$$\psi(t) = \left( \frac{15V(t)}{8} \right)^{2/5}.$$

### 4.5 Application: Motions of Generalised Particles in the Quantisation Limit and Local Normal Forms of Generic Singular Lagrangians

We consider an example from Lagrangian mechanics which concerns the motion of a charged particle on a Riemann surface in the strong coupling (quantisation) limit with an electromagnetic field, or more generally with an Abelian gauge field ([28], [50], [84]). There are several approaches and reformulations of the problem, the most appropriate for our case being that of a generalised particle (c.f. [2], [12]), a variant of which we present below.

We fix a 2-dimensional riemannian manifold $M$. We consider the family of mechanical systems described by a regular Lagrangian function $L : TM \to \mathbb{R}$ “quadratic in the velocities”, i.e. such that in any local trivialisation of the tangent bundle with coordinates $(x, \dot{x})$ it can be expressed as:

$$L = mL_2 + eL_1 + \nu L_0, \quad (4.9)$$

where the functions $L_i = L_i(x, \dot{x})$ are homogeneous in the velocities $\dot{x}$ of degree $i$ and:

- $L_2(x, \cdot) = \sum g_{ij}(x)dx_idx_j$ is a nondegenerate quadratic form $g$ on $M$ (the riemannian metric) representing the kinetic energy of the system
- $L_1(x, \cdot) = \sum \alpha_i(x)dx_i$ corresponds to a 1-form (vector potential) $\alpha$ of gyroscopic forces (such as magnetic forces e.t.c.) represented by the 2-form $\omega = da$,
- $L_0(x, \cdot) = -f(x)$ is independent in the velocities and represents the scalar potential of other external forces acting upon the system (such as electric e.t.c.),
- $m \in \mathbb{R}$, $e \in \mathbb{R}$ and $\nu \in \mathbb{R}$ are the coupling constants ($m$ is the mass, $e$ is the charge e.t.c.),
which may be viewed as formal parameters.

It is known that motions of the generalised particle emanating from a point \( x_0 \in M \) are smooth curves \( t \mapsto x(t) \) on \( M \), \( x(0) = x_0 \), which satisfy the generalised Euler-Lagrange equations:

\[
m \frac{D\dot{x}}{dt} = e \dot{x} \omega - \nu df,
\]

where \( D/dt \) is the covariant derivative associated to the riemannian metric \( g \) and \( \omega \) is the interior multiplication of a vector field with the 2-form \( \omega = da \). The right hand-side of equation (4.10) is known in the theory of electromagnetism as the Lorentz force. In particular, the motions \( x : [t_0, t_1] \to M \) of the particle are exactly the critical points of the action functional:

\[
\mathcal{A} = \int_{t_0}^{t_1} L(x(t), \dot{x}(t)) dt.
\]

For the case \( \nu = 0 \), i.e. in the absence of external forces, the geometry of this variational problem with Lagrangian \( L = mL_2 + eL_1 \), has been studied extensively in terms of Subriemannian geometry and control theory (c.f. [74], [75], [76]). There, the eventual singularities \( H(da) \) of the 2-form \( da \) play a special role: they are the abnormal geodesics of the corresponding Pontryagin maximum principle and they can be obtained formally by the Euler-Lagrange equations (4.10) for \( m = 0 \).

For the case \( \nu \neq 0 \), the geometry of this variational problem for the values \( m \neq 0 \) can be studied in terms of Jacobi metrics and for most of the cases where the 1-form \( L_1 = \alpha \) of gyroscopic forces is nonsingular, in the sense that the 2-form \( \omega = da \) is nondegenerate (symplectic) on \( M \). It is important to notice also that for any \( m \neq 0 \) the Legendre transform \( \mathcal{L} : TM \to T^*M \) of \( L \) is a diffeomorphism and thus the phase space of the Lagrangian system can be identified with cotangent bundle \( T^*M \) with Hamiltonian \( F = (\mathcal{L}^{-1})^* L \) and symplectic form \( \Omega = dp \wedge dx \) the natural symplectic form of the cotangent bundle. The generalised Euler-Lagrange equations (4.10) transform in that way to the canonical Hamilton’s equations:

\[
X_F \Omega = dF.
\]

We will be interested here in the quantisation limit equations, i.e. for \( m \to 0 \) (or \( e \to \infty, \nu \to \infty \)). Notice that for \( m = 0 \) the Lagrangian \( L = eL_1 + \nu L_0 \) is linear in the velocities and thus its Legendre transform \( \mathcal{L} : TM \to T^*M \) is not a diffeomorphism. For this reason, Lagrangians linear in the velocities are called singular (or constrained) and the dynamics that they define through the Euler-Lagrange equations:

\[
e \dot{x} \omega = \nu df,
\]

is also called singular (or constrained) Lagrangian dynamics. The equation (4.12) above can be viewed as a Constrained Hamiltonian System \( (f, \omega) \) with 2-form \( \omega = da \) the form of gyroscopic forces and “Hamiltonian” \( f \) defined by the potential energy \( L_0 \) of the initial system. To describe this geometrically notice that in the limit \( m = 0 \), the image of the Legendre transform \( \mathcal{L}(TM) \subset T^*M \) defines naturally a constraint submanifold in the phase space which is exactly equal to the image
of the 1-form $L_1 = \alpha$, viewed as a local section $\alpha : M \to T^*M$ of the cotangent bundle:

$$ \text{Im} \mathcal{L}_L = \text{Im} \alpha \subset T^*M $$

and it is thus diffeomorphic to the configuration space $M$. One has thus defined a diagram of maps:

$$ M \overset{\alpha}{\to} T^*M \overset{F}{\to} \mathbb{R}, $$

whose left arrow is the embedding of $M$ in $T^*M$ through $\alpha$ and the right arrow is the Hamiltonian $F$. This shows that the motions of the generalised particle can be viewed inside the general scheme of the theory of Hamiltonian systems with constraints. As is easily verified, the Euler-Lagrange equations (4.10) for $m = 0$, correspond to the restriction of the Hamiltonian system $(F, \Omega)$ on the constrained submanifold, i.e. on the image of the 1-form $\alpha$. Indeed, one has

$$ \alpha^* \Omega = da, \quad \alpha^* F = f, $$

and thus the restriction $\alpha^* (F, \Omega)$ defines the Constrained Hamiltonian System:

$$ X_f da = df, $$

which is exactly the system of Euler-Lagrange equations (4.12) (were we have put $e = \nu = 1$).

Now let us consider the problem to determine the motions of the particle in the quantisation limit. Fix a point $x_0 \in M$ and identify the germs of singular Lagrangians $L = L_1 + L_0$ at $x_0$ with the germs of the corresponding pairs of potentials $L := (\alpha, f)$. A singular Lagrangian $L$ will be called “generic” if the corresponding pair of potentials $(\alpha, f)$ is in general position (relative to diagram (4.13)) or equivalently, the codimension of the singularities of the pair $(da, f)$ is less or equal to $2 = \text{dim} M$. The fact that this definition is correct is verified by the Darboux-Givental theorem [10].

Replace now the 1-form $\alpha$ by a 1-form $\alpha + d\xi$, where $\xi$ is some arbitrary function, and the potential $f$ by $f + c$, $c$ some constant. Then the form of the Euler-Lagrange equations (4.12) does not change and so there is naturally defined an equivalence relation between singular Lagrangians:

**Definition 4.5.1.** Two germs $L = (\alpha, f)$ and $L' = (\alpha', f')$ of singular Lagrangians at $x_0 \in M$ will be called variationally (or gauge) equivalent if their Euler-Lagrange equations (4.12) are equivalent, i.e. there exists a diffeomorphism germ $\Phi$, $\Phi(x_0) = x_0$, an arbitrary function germ $\xi$ and a constant $c$ such that:

$$ \Phi^* \alpha' = \alpha + d\xi, \quad \Phi^* f' = f + c. $$

From the results of the previous sections on the classification of the pair $(da, f)$ we immediately obtain:

**Theorem 4.5.1.** The germ of a generic analytic Lagrangian $L$ at a point $x_0$ on a manifold $M$, is
variationally equivalent to one of the following four invariant normal forms:

\begin{align*}
L(x, \dot{x}) &= x_1 \dot{x}_2 - x_1, \quad (4.14) \\
L(x, \dot{x}) &= x_1 \dot{x}_2 - \phi(\pm x_1^2 \pm x_2^2), \quad (4.15) \\
L(x, \dot{x}) &= \frac{x_1^2}{2} \dot{x}_2 \mp x_2, \quad (4.16) \\
L(x, \dot{x}) &= \frac{x_1^2}{2} \dot{x}_2 - \psi(x_1 \pm x_2^2), \quad (4.17)
\end{align*}

where the functions germs $\psi$ and $\phi$ are analytic functions of one variable with a simple zero at the origin and they are the unique functional invariants of the (variational orbits of the) corresponding Lagrangians.

**Proof.** Normal forms (4.14), (4.15) correspond to the well known regular and Morse cases respectively at a point $x_0 \in M \setminus H(\omega)$ in the symplectic plane. Local normal forms (4.16) and (4.17) correspond to regular, transversal points $x_0 \in H(\omega) = \{x_1 = 0\}$ of $f$ on the Martinet curve (open and dense) and to (isolated) points of first order tangency of $f$ with the Martinet curve respectively. Normal form (4.17) is obtained immediately from Corollary 4.4.3 of Theorem 4.4.1. The normal form (4.16) is a simple exercise; the $\mp$ sign comes from the fact that in the real case the Martinet curve $H(\omega) = \{x_1 = 0\}$ has an invariant orientation induced by the two symplectic structures in its complement in $M$. In fact, there is no real analytic diffeomorphism preserving $x_1 dx_1 \wedge dx_2$ and sending $x_2$ to $-x_2$. \qed
5 Singularities of Functions on Manifolds with Boundary, Relative Cohomology, Gauss-Manin Connections and Volume Forms

5.1 Introduction

We study here the Gauss-Manin connections on the relative cohomology of an isolated boundary singularity, i.e. of an isolated singularity of a function in the presence of a fixed hyperplane section, called “the boundary” as is usual in the literature (c.f. [3], [4], [8], [68], [69], [103], [104] for several classification results and topological properties). In particular, we give a generalisation, for the boundary case, of some fundamental results obtained mainly by E. Brieskorn [15], M. Sebastiani [92] and B. Malgrange [64], such as the Brieskorn-Deligne-Sebastiani theorem, concerning the finiteness and freeness of the de Rham cohomology modules and of the corresponding Brieskorn lattices associated to the boundary singularity (Theorems 5.2.2, 5.2.7). We also give a relative analog of the regularity theorem (Theorem 5.2.8) according to which, the restriction of the natural Gauss-Manin connection on the localisation of the Brieskorn modules at the critical value, has regular singularities. As in Brieskorn’s work for the ordinary case [15], the regularity of the Gauss-Manin connection, along with the algebraicity theorem and the positive solution of Hilbert’s VII’th problem, give also a direct analytic proof of a relative version of the monodromy theorem (Theorem 5.2.1), i.e. that the eigenvalues of the Picard-Lefschetz monodromy operator in the relative vanishing cohomology, are indeed roots of unity. Following Malgrange [64], we show that the relative monodromy theorem along with the regularity theorem, give also the asymptotic expansion of the integrals of holomorphic forms along the vanishing cycles and half-cycles of the boundary singularity, when the values of the function tend to the critical one (Theorem 5.2.9). We then give a direct application in the classification problem of triples (volume form, function, hypersurface). In particular we prove a relative analog of a J. Vey’s isochore Morse lemma (Theorem 5.3.2), J. -P. Francoise’s generalisation on the local normal forms of volume forms with respect to the boundary singularity preserving diffeomorphisms (Theorem 5.3.1), as well as Mather-Garay’s isochore versal unfolding theorem for boundary singularities (Theorem 5.3.3). These results are the higher dimensional analogs of the corresponding theorems proved in Chapter 3 for the 2-dimensional case.

It is important to notice finally that there are two natural ways to study a boundary singularity. The first one is due to Arnol’d [6] according to which a boundary singularity can be viewed as an
ordinary $\mathbb{Z}_2$-symmetric singularity after passing to the double covering space branched along the boundary (see also [110] and [10] for generalisations for other symmetric singularities). There is also another approach due to A. Szpirglas [103], [104], according to which a boundary singularity can be viewed, at least in a (co)homological level, as an extension of two ordinary singularities, namely the ambient singularity and its restriction on the boundary. Our approach is in accordance with the second one, i.e. we show that the relative cohomology, the relative Gauss-Manin connection and the corresponding Brieskorn lattices associated to a boundary singularity, are indeed extensions of the corresponding ordinary objects associated to the pair of isolated singularities.

5.2 Relative Cohomology, Brieskorn Modules and Gauss-Manin Connections for Boundary Singularities

We review first some basic facts concerning the topology of isolated boundary singularities.

5.2.1 Milnor Numbers, (Co)homological Milnor Bundles and Topological Gauss-Manin Connections

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ and let $H = \mathbb{C}^n \subset \mathbb{C}^{n+1}$ be a germ of a hyperplane section at the origin, which we call “the boundary”, such that $f$ and its restriction $f|_H$ on the boundary has an isolated critical point at the origin. Fix a coordinate system $(x, y_1, ..., y_n)$ such that the equation of the boundary is given by $H = \{x = 0\}$. The multiplicity $\mu$ of the critical point, or else, the Milnor number of the boundary singularity, is the dimension of the local algebra:

$$Q_{f,H} = \frac{\mathcal{O}_{n+1}}{(x \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y_1}, ..., \frac{\partial f}{\partial y_n})}, \quad \mu = \dim \mathbb{C} Q_{f,H}.$$ 

The Milnor number of the boundary singularity is related to the ordinary Milnor number $\mu_1$ of $f$:

$$Q_f = \frac{\mathcal{O}_{n+1}}{(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y_1}, ..., \frac{\partial f}{\partial y_n})}, \quad \mu_1 = \dim \mathbb{C} Q_f,$$

and the Milnor number $\mu_0$ of its restriction on the boundary:

$$Q_{f|_H} = \frac{\mathcal{O}_n}{(\frac{\partial f}{\partial y_1}|_{x=0}, ..., \frac{\partial f}{\partial y_n}|_{x=0})}, \quad \mu_0 = \dim \mathbb{C} Q_{f,H},$$

by the formula (c.f. [6], [103], [110]):

$$\mu = \mu_1 + \mu_0.$$ 

The Milnor number of a boundary singularity is an important topological invariant; let $B_{r+1}^n$ be a sufficiently small ball at the origin of $\mathbb{C}^{n+1}$ and choose a holomorphic representative $g : B_{r+1}^n \to T = g(B_{r+1}^n)$ such that its restriction $g' : B_r^n \to T$ on the boundary ball $B_r^n = B_{r+1}^n \cap H$ is a
holomorphic representative of the germ $f|_H$. By choosing the radius of the ball appropriately, as well as the representatives $(g, g')$, we may succeed that:

- the pair of fibers $(g^{-1}(0), g'^{-1}(0))$ is transversal to the pair of boundary spheres $(\partial B^n \cap g^{-1}(S), \partial B^n \cap g'^{-1}(S))$ for all $\epsilon < r$, and it has an isolated singularity at the origin (the fiber $g^{-1}(0)$ might be smooth but not transversal to the hyperplane $H$),
- the pair of fibers $(g^{-1}(t), g'^{-1}(t))$ is smooth and transversal to the boundary spheres $(\partial B^n \cap g^{-1}(S), \partial B^n \cap g'^{-1}(S))$ for some $\epsilon$ over all points $t \in S$ of the closure of a sufficiently small open disc $S \subset T$ centered at the origin.

The standard representative $f : X \to S$ is obtained by restricting $g$ to $X = \hat{B}^{n+1}_\epsilon \cap g^{-1}(S)$ and is such that its restriction $f' : X' = X \cap H \to S$ is a standard representative of $f|_H$ in the sense that it is obtained by the restriction of $g'$ on $X' = \hat{B}^{n}_\epsilon \cap g'^{-1}(S)$. Thus one obtains a diagram of standard representatives:

\[
\begin{array}{c}
X \\ f \downarrow \\
\hat{X}' \\ f' \downarrow \\
S
\end{array}
\]

which we denote by $(f, f') : (X, X') \to S$. We will call it the standard (or Milnor) representative of the boundary singularity $(f, H)$.

Denote now by $(X_0 = f^{-1}(0), X'_0 = f'^{-1}(0))$ the pair of singular fibers and let $(X^* = X \setminus X_0, X'^* = X' \setminus X'_0)$ be their corresponding complements. Then for $S^* = S \setminus 0$, the restriction of $(f, f')$ on $(X^*, X'^*)$ induces a $C^\infty$-fiber bundle pair (by Ehresmann’s fibration theorem), i.e. a diagram of $C^\infty$-fiber bundles:

\[
\begin{array}{c}
X^* \\ f \downarrow \\
\hat{X}'^* \\ f' \downarrow \\
S^*
\end{array}
\]

which we denote again by $(f, f') : (X^*, X'^*) \to S^*$. Let $(X_t = f^{-1}(t), X'_t = f'^{-1}(t))$ be a pair of regular fibers. In particular the fiber $X_t$ is smooth and transversal to the boundary $X'_t$, so that its intersection $X'_t$ with the boundary is a smooth submanifold of both $X'$ and $X_t$. According to a theorem of Arnol’d [6] which generalises the Milnor-Palamodov theorem [73, 78] for the boundary case, the manifold $X_t/X'_t$ has the homotopy type of a bouquet of $\mu$ $n$-dimensional spheres, where $\mu = \dim_{\mathbb{C}} Q_{f,H}$ is the Milnor number of the boundary singularity $(f, H)$. In particular, $\mu$ is exactly equal to the rank of the relative homology group $H_n(X_t, X'_t)$ (it can be considered with integer coefficients). The equality $\mu = \mu_1 + \mu_0$ follows then from the long exact sequence in homology induced by the embedding $i_t : X'_t \hookrightarrow X_t$ and the Milnor-Palamodov theorem for the pair $(f, f')$ respectively, according to which:

\[
H_n(X_t) \cong \mathbb{Z}^{\mu_1}, \quad H_{n-1}(X'_t) \cong \mathbb{Z}^{\mu_0}
\]

(all other homologies of $X_t$ and $X'_t$ are zero, except in zero degree). Indeed, the long exact homology
sequence reduces to the short exact sequence:

\[ 0 \to H_n(X_t) \to H_n(X_t, X'_t) \xrightarrow{\partial} H_{n-1}(X'_t) \to 0 \]

and thus

\[ H_n(X_t, X'_t) \cong \mathbb{Z}^{\mu_1 + \mu_0}. \]

A basis of the relative homology group \( H_n(X_t, X'_t) \) is obtained by the \( \mu_1 \) ordinary vanishing cycles of \( f \) and the \( \mu_0 \) vanishing half-cycles, i.e. those relative cycles of \( X_t \) which cover the \( \mu_0 \) ordinary vanishing cycles of \( f|_H \) inside \( X \setminus X'_t \) (c.f. [6], [103]).

By obvious duality, to the short exact homology sequence above there corresponds a short exact sequence in cohomology:

\[ 0 \to H^{n-1}(X'_t) \xrightarrow{\delta} H^n(X_t, X'_t) \to H^n(X_t) \to 0, \tag{5.1} \]

with the standard formal adjoint formula for the boundary and coboundary operators \( (\partial, \delta) \):

\[ < \delta \alpha, \gamma > = < \alpha, \partial \gamma >, \]

where \( < , , > \) is the natural duality morphism between relative homology and cohomology:

\[ < , , > : H^n(X_t, X'_t) \times H_n(X_t, X'_t) \to \mathbb{Z}. \]

In order to study the variations in cohomology of the Milnor fibers as \( t \) varies in \( S^* \) it is convenient to consider the cohomologies above as with complex coefficients, and endowed with their canonical integral lattices. Since the pair \((f, f') : (X^*, X'^*) \to S^*\) is a \( C^\infty\)-fiber bundle pair over the 1-dimensional manifold \( S^* \), the vector spaces \( H^p(X_t; \mathbb{C}) \), \( H^p(X'_t; \mathbb{C}) \) and \( H^p(X_t, X'_t; \mathbb{C}) \), glue together to form the fibers of the corresponding cohomological (or Milnor) vector bundles:

\[ R^p f_* \mathbb{C}_{X^*} := \bigcup_{t \in S^*} H^p(X_t; \mathbb{C}) \to S^*, \]

\[ R^p f_* \mathbb{C}_{X'^*} := \bigcup_{t \in S^*} H^p(X'_t; \mathbb{C}) \to S^*, \]

\[ R^p f_* \mathbb{C}_{X^* \setminus X'^*} := \bigcup_{t \in S^*} H^p(X_t, X'_t; \mathbb{C}) \to S^*, \]

where the sheaves \( \mathbb{C}_{X'^*} \), \( \mathbb{C}_{X^* \setminus X'^*} \) are the extensions by zero of the restrictions of the constant sheaf \( \mathbb{C}_{X^*} \) on the closed subspace \( X'^* \) and its open complement \( X^* \setminus X'^* \) respectively (see below). The vector bundles above are holomorphic flat vector bundles, each endowed with its own topological Gauss-Manin connection, defined by the condition that the horizontal sections are generated by the corresponding local systems \( R^p f_* \mathbb{C}_{X^*} \), \( R^p f_* \mathbb{C}_{X'^*} \), and \( R^p f_* \mathbb{C}_{X^* \setminus X'^*}. \) In particular, consider the sheaves of sections of each of the cohomological fibrations:

\[ \mathcal{H}^p(X^*/S^*) = R^p f_* \mathbb{C}_{X^*} \otimes_{\mathcal{O}_S} \mathcal{O}_S. \]

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\[ \mathcal{H}^p(X^*/S^*) = R^p f_* \mathbb{C}_{X^*} \otimes_{\mathcal{O}_S} \mathcal{O}_{S^*} \]

and
\[ \mathcal{H}^p(X^*, X'^*/S^*) = R^p f_* \mathbb{C}_{X^* \setminus X'^*} \otimes_{\mathcal{O}_{S^*}} \mathcal{O}_{S^*}. \]

Then, the (topological) Gauss-Manin connections are defined by:

\[ R^p f_* \mathbb{C}_{X^*} = \ker D_1, \quad R^p f_* \mathbb{C}_{X^*} = \ker D_0, \]
\[ R^p f_* \mathbb{C}_{X^* \setminus X'^*} = \ker D, \]

where
\[ D_1 : \mathcal{H}^p(X^*/S^*) \to \mathcal{H}^p(X^*/S^*), \quad D_0 : \mathcal{H}^p(X'^*/S^*) \to \mathcal{H}^p(X'^*/S^*), \]

and
\[ D : \mathcal{H}^p(X^*, X'^*/S^*) \to \mathcal{H}^p(X^*, X'^*/S^*), \]

are the covariant derivatives of the corresponding connections on the sheaves of sections of the cohomological bundles. Each one of these connections is determined in turn by differentiating locally constant sections of the corresponding cohomology bundle along the vector field \( \frac{d}{dt} \) on the base \( S^* \) (where \( f = t \) is a local coordinate) by the rule:

\[ D(c \otimes g) = c \otimes \frac{dg}{dt}, \]

where \( c \) is a section of the corresponding local system and \( g \) is a holomorphic function of \( t \). We will call the two Gauss-Manin connections \( D_1 \) and \( D_0 \) ordinary, and the Gauss-Manin connection \( D \) relative.

The cohomological Milnor bundles and the Gauss-Manin connections above are not independent with each other but they are connected through long exact sequences; first there is a long exact sequence of local systems:

\[ \cdots \to R^{p-1} f_* \mathbb{C}_{X^*} \to R^p f_* \mathbb{C}_{X^* \setminus X'^*} \to R^p f_* \mathbb{C}_{X^*} \to R^p f_* \mathbb{C}_{X'^*} \to \cdots, \]

obtained by applying the direct image functor \( Rf_* \) to the short exact sequence of constant sheaves\[1\]

\[ 0 \to \mathbb{C}_{X^* \setminus X'^*} \to \mathbb{C}_{X^*} \to \mathbb{C}_{X'^*} \to 0. \]

There is also a long exact sequence of sheaves of sections of the cohomology bundles:

\[ \cdots \to \mathcal{H}^{p-1}(X'^*/S^*) \to \mathcal{H}^p(X^*, X'^*/S^*) \to \mathcal{H}^p(X^*/S^*) \to \mathcal{H}^p(X'^*/S^*) \to \cdots \quad (5.2) \]

---

\[1\]this is just the well known adjunction triangle for any sheaf \( \mathcal{F} \):

\[ 0 \to j_! j^{-1} \mathcal{F} \to \mathcal{F} \to i_* i^{-1} \mathcal{F} \to 0, \]

where \( i : X' \to X \) and \( j : X \setminus X' \to X \) is the embedding of the closed subspace \( X' \) and of its open complement \( X \setminus X' \) respectively in \( X \).
obtained by the long exact sequence of local systems above after tensoring with $\mathcal{O}_{S^*}$. In particular, the long exact sequence of the cohomology sheaves is a long exact sequence of locally free sheaves of coherent $\mathcal{O}_{S^*}$-modules which, according to Milnor’s (or Arnol’d’s) theorem reduces to the short exact sequence:

$$0 \to \mathcal{H}^{n-1}(X^*/S^*) \to \mathcal{H}^n(X^*, X^*/S^*) \to \mathcal{H}^n(X^*/S^*) \to 0. \quad (5.3)$$

It follows that the relative cohomology sheaf $\mathcal{H}^n(X^*, X^*/S^*)$ is an extension of the sheaf $\mathcal{H}^{n-1}(X^*/S^*)$ by $\mathcal{H}^n(X^*/S^*)$ and the relative Gauss-Manin connection $D$ on it is an extension of the two ordinary Gauss-Manin connections $D_0$, $D_1$. In particular the restriction of the relative Gauss-Manin connection $D$ on the sheaf $\mathcal{H}^{n-1}(X^*/S^*)$ can be identified with the ordinary Gauss-Manin connection $D_0$ while the quotient connection induced on $\mathcal{H}^n(X^*/S^*)$ can be identified with the ordinary Gauss-Manin connection $D_1$.

On the other hand, it is well known (c.f. [20]) that any local system on $S^*$ with a flat connection is determined by the monodromy, i.e. the representation of the fundamental group $\pi_1(S^*, t)$ on its fibers, and conversely, the monodromy determines the connection. Here we may choose the standard representatives $(f, f')$ in such a way so that the geometric monodromy on the fibers $X_t$ induced by travelling once around the origin in the positive direction, leaves the subfiber $X'_t$ invariant. Thus we obtain representations of the fundamental group $\pi_1(S^*, t) = \mathbb{Z}$ in the group of automorphisms of the fibers of the corresponding cohomological bundles. Let $T_0 \in \text{Aut} H^{n-1}(X'_t; \mathbb{C})$, $T_1 \in \text{Aut} H^n(X_t; \mathbb{C})$ be the ordinary linear transformations in cohomology, i.e. the well known Picard-Lefschetz monodromy transformations, and denote by $T \in \text{Aut} H^n(X_t, X'_t; \mathbb{C})$ the linear transformation induced in relative cohomology. We will call this transformation the relative Picard-Lefschetz monodromy (as in [103]). By the above, it is an extension of the two ordinary Picard-Lefschetz monodromies, i.e. there is a commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^{n-1}(X'_t; \mathbb{C}) & \delta & H^n(X_t, X'_t; \mathbb{C}) & \longrightarrow & H^n(X_t; \mathbb{C}) & \longrightarrow & 0 \\
& & T_0 \downarrow & & T \downarrow & & T_1 \downarrow & & \\
0 & \longrightarrow & H^{n-1}(X'_t; \mathbb{C}) & \delta & H^n(X_t, X'_t; \mathbb{C}) & \longrightarrow & H^n(X_t; \mathbb{C}) & \longrightarrow & 0 
\end{array}
$$

(5.4)

By the fact that both $T_0$ and $T_1$ are isomorphisms it follows that $T$ is also an isomorphism. Concerning its eigenvalues we have the following relative analog of the monodromy theorem:

**Theorem 5.2.1.** *The eigenvalues of the relative monodromy operator $T$ are roots of unity.*

The proof follows immediately by the fact that the characteristic polynomial of $T$ is the product of the characteristic polynomials of $T_0$ and $T_1$, whose roots are, by the ordinary monodromy theorem (c.f. Brieskorn [15]), roots of unity. Another straightforward analytic proof of the relative monodromy theorem may be derived, following Brieskorn, by the results of the next sections (see Remark 5.2.3).

**Remark 5.2.1.** The statement of the theorem above is, as is usually called, the first part of the monodromy theorem. The second part, concerning the bound on the maximal size of the Jordan
blocks, is more complicated and it will not be discussed here. Possibly, a sharper bound than the obvious one \( n - 1 + n = 2n - 1 \), may be obtained either using resolution of singularities and a Clemens construction as in [10], or using the eventual mixed Hodge structure on the vanishing relative cohomology \( H^n(X_t, X'_t; \mathbb{C}) \) (as for example in [101], [107]).

5.2.2 Relative de Rham Cohomology, Analytic Gauss-Manin Connections and Brieskorn Modules

Since the pair of Milnor fibers \((X_t, X'_t)\) is Stein, its cohomologies can be computed using holomorphic differential forms and the corresponding relative de Rham cohomologies.

The Brieskorn-Deligne-Sebastiani Theorem for Boundary Singularities

Recall (for example from Chapter 2) that for a single morphism \( f : X \to S \) the complex of holomorphic relative differential forms \( \Omega^\bullet_{X/S} \) is defined as the quotient complex (c.f. [15]):

\[
\Omega^\bullet_{X/S} = \frac{\Omega^\bullet_X}{df \wedge \Omega^\bullet_{X,T}},
\]

where \( \Omega^\bullet_X \) is the complex of holomorphic forms on \( X \) and \( f^* \Omega^1(S) = df \wedge \) is the ideal sheaf generated by the differential of \( f \). The differential \( d \) (called the relative differential and denoted also by \( d_{X/S} \)) of the relative de Rham complex \( \Omega^\bullet_{X/S} \) is the one induced by the absolute differential \( d_X \) of the complex \( \Omega^\bullet_X \) and it is \( f^{-1}O_S \)-linear. For a pair of standard representatives \((f, f') : (X, X') \to S\), one may define several other relative de Rham complexes, with the most obvious one being the relative de Rham complex \( \Omega^\bullet_{X'/S} \) of the map \( f' : X' \to S \), viewed independently of the embedding \( i : X' \hookrightarrow X \). Indeed, we have as above:

\[
\Omega^\bullet_{X'/S} := \frac{\Omega^\bullet_{X'}}{df' \wedge \Omega^\bullet_{X',T}},
\]

where the relative differential \( d_{X'/S} \) is induced by the differential \( d_{X'} \) and it is also \( f'^{-1}O_S \)-linear.

Consider now its extension by zero \( i_* \Omega^\bullet_{X'/S} \) in \( X \). Since \( X' \) is closed and smooth we have an epimorphism of analytic modules, which is the restriction morphism induced by the pullback map:

\[
i^* : \Omega^\bullet_{X/S} \to i_* \Omega^\bullet_{X'/S}.
\]

The kernel of this morphism is the subcomplex \( \Omega^\bullet_{X,X'/S} \subset \Omega^\bullet_{X/S} \) consisting of relative differential forms whose support lies in the complement \( X \setminus X' \) and in particular they vanish when restricted to the hypersurface \( X' \). More specifically, let \( \Omega^\bullet_{X,X'} \subset \Omega^\bullet_X \) be the subcomplex of holomorphic forms on \( X \) which vanish when restricted on \( X' \). This fits in a short exact sequence of complexes:

\[
0 \to \Omega^\bullet_{X,X'} \to \Omega^\bullet_X \to \Omega^\bullet(X') \to 0,
\]
where we denote by $\Omega^\bullet(X')$ the quotient complex $\Omega^\bullet_{X'}/\Omega^\bullet_{X,X'}$. Obviously there is an isomorphism $i_*\Omega^\bullet_{X'} \cong \Omega^\bullet(X')$ obtained by the fact that the complex $i^*\Omega^\bullet(X')$ is supported on the closed smooth subspace $X'$ and its restriction on $X'$ is identified with $\Omega^\bullet_{X'}$. Consider now multiplication with $df \wedge$ in the short exact sequence above. It gives a commutative diagram:

$$
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & df \wedge \Omega^{\bullet-1}_{X,X'} & df \wedge \Omega^{\bullet-1}_{X} & df \wedge \Omega^{\bullet-1}(X') & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Omega^{\bullet}_{X,X'} & \Omega^{\bullet}_{X} & \Omega^{\bullet}(X') & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Omega^{\bullet}_{X,X'/S} & \Omega^{\bullet}_{X/S} & \Omega^{\bullet}(X'/S) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

(5.5)

where the last row consists of the relative de Rham complexes:

$$
\Omega^{\bullet}_{X,X'/S} := \frac{\Omega^{\bullet}_{X,X'}}{df \wedge \Omega^{\bullet-1}_{X,X'}}, \quad \Omega^{\bullet}_{X/S} := \frac{\Omega^{\bullet}_{X}}{df \wedge \Omega^{\bullet-1}_{X}},
$$

$$
\Omega^{\bullet}(X'/S) := \frac{\Omega^{\bullet}(X')}{df \wedge \Omega^{\bullet-1}(X')}.
$$

By the fact that all the columns and the first two rows in the above diagram are exact, it follows from the 9-lemma that the lower sequence of relative de Rham complexes is also exact and thus there is an isomorphism:

$$
\Omega^{\bullet}(X'/S) \cong \frac{\Omega^{\bullet}_{X/S}}{\Omega^{\bullet}_{X,X'/S}}.
$$

Moreover, in analogy with the absolute case, there is also a natural isomorphism:

$$
i_*\Omega^{\bullet}_{X'/S} \cong \Omega^{\bullet}(X'/S),
$$

from which it follows that the complex $\Omega^{\bullet}_{X,X'/S}$ can indeed be identified with the kernel of the restriction morphism $i^* : \Omega^{\bullet}_{X'/S} \to i_*\Omega^{\bullet}_{X'/S}$.

Recall now that if $\mathcal{F}^\bullet$ is a complex of analytic sheaves with an $f^{-1}\mathcal{O}_S$-linear differential, then its cohomology sheaves are defined by the hyperdirect image sheaves $\mathbb{R}^p f_* \mathcal{F}^\bullet$, which are defined in turn by the hypercohomology presheaves $S \supset U \mapsto H^p(f^{-1}(U), \mathcal{F}^\bullet)$. Moreover, for a Stein morphism, it follows from Cartan theorems that these do indeed compute the cohomology $H^p(\mathcal{F}^\bullet)|_{f^{-1}(U)}$. If $\mathcal{F}^\bullet$ is one of the above complexes of relative forms then we write the relative de Rham cohomology sheaves as:

$$
\mathcal{H}^p_{dR}(X, X'/S) = \mathbb{R}^p f_* \Omega^\bullet_{X,X'/S}, \quad \mathcal{H}^p_{dR}(X/S) = \mathbb{R}^p f_* \Omega^\bullet_{X,X'/S},
$$

$$
\mathcal{H}^p_{dR}(X'/S) = \mathbb{R}^p f_* \Omega^\bullet(\mathcal{X}'/\mathcal{S}).
$$
respectively. The short exact sequence:

$$0 \to \Omega^*_{X,X'/S} \to \Omega^*_{X/S} \to \Omega^*(X'/S) \to 0$$

(5.6)

gives, after application of the hyperdirect image functor \( \mathbb{R}f_* \), a long exact sequence in cohomology:

$$\cdots \to \mathcal{H}^p_{dR}(X'/S) \to \mathcal{H}^p_{dR}(X,X'/S) \to \mathcal{H}^p_{dR}(X/S) \to \mathcal{H}^p_{dR}(X'/S) \to \cdots$$

(5.7)

which possesses the following important properties summarised in the following relative analog of the Brieskorn-Deligne-Sebastiani theorem:

**Theorem 5.2.2.**

(i.) The long exact sequence (5.7) is a long exact sequence of coherent sheaves of \( \mathcal{O}_S \)-modules.

(ii.) It is isomorphic over \( S^* \) with the long exact sequence (5.2) of sheaves of sections of the corresponding cohomological Milnor bundles.

(iii.) The stalk at the origin of the long exact sequence (5.7) is isomorphic to the long exact sequence of finitely generated \( \mathcal{O}_{S,0} \)-modules:

$$\to H^{p-1}(\Omega^*(X'/S,0)) \xrightarrow{\delta} H^p(\Omega^*_{X,X'/S,0}) \to H^p(\Omega^*_{X/S,0}) \to H^p(\Omega^*(X'/S,0)) \to \cdots$$

(5.8)

which is the long exact cohomology sequence induced from the stalk at the origin of the short exact sequence (5.6).

(iv.) The sheaves in the long exact sequence (5.7) are locally free sheaves of \( \mathcal{O}_S \)-modules. In particular, the \( \mathcal{O}_{S,0} \)-modules in the long exact sequence (5.8) are free.

**Proof.** (i.) (iii.) Since the singularities are isolated the proof follows immediately from Kiehl-Verdier type theorems related to the relative constructibility of these sheaves (c.f. [34]). Alternatively, we know from the ordinary Brieskorn-Deligne theorem that the sheaves \( \mathbb{R}f_* \Omega^*_{X/S} \) are already coherent. Thus, it suffices to show that one of the remaining cohomologies in the long exact sequence is coherent. Notice now that because the direct image functor \( i_* \) of the closed embedding \( i : X' \hookrightarrow X \) is exact we have the following isomorphism of sheaves of \( \mathcal{O}_S \)-modules:

$$\mathbb{R}^p f_* \Omega^*(X'/S) \cong \mathbb{R}^p f_* (i_* \Omega^*_{X'/S}) \cong \mathbb{R}^p f'_* \Omega^*_{X'/S},$$

where the second isomorphism follows from the Grothendieck spectral sequence for the composition \( f' = (f \circ i)_* \). Thus the relative de Rham cohomology of the restriction \( f' \) is independent of the embedding in \( X \). It follows by the ordinary Brieskorn-Deligne theorem again that the sheaves \( \mathbb{R}^p f'_* \Omega^*_{X'/S} \) are coherent and thus, the remaining sheaves \( \mathbb{R}^p f_* \Omega^*_{X,X'/S} \) are also coherent. The property (iii) also holds for \( \mathbb{R}^p f_* \Omega^*_{X,X'/S} \) because it holds for the other two sheaves; indeed if \( X_0 = \)

\[ ^2 \text{in the ordinary case, the freeness of the cohomology modules was proved by Sebastiani [92].} \]
$f^{-1}(0)$ is the singular fiber, one has a commutative diagram of canonical restriction morphisms:

$$
\begin{array}{ccc}
0 & \longrightarrow & \Gamma(X_0, \Omega^\bullet_{X/X'/S}) \\
& & \downarrow \\
0 & \longrightarrow & \Omega^\bullet_{X,X'/S,0}
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \Gamma(X_0, \Omega^\bullet_{X'/S}) \\
& & \downarrow \\
\longrightarrow & \longrightarrow & \Omega^\bullet_{X'/S,0}
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & 0
\end{array}
$$

where the middle and right morphisms are quasi-isomorphisms. It follows by the 5-lemma that the left morphism is a quasi-isomorphism as well.

(ii.) This property is also classical and it guarantees that the de Rham cohomology sheaves are indeed coherent extensions of the sheaves of sections of the corresponding cohomological bundles at the origin. Briefly, one uses the relative Poincaré lemma according to which over the smooth points $S^*$, the short exact sequence (adjunction triangle):

$$
0 \rightarrow f^{-1}O_{S^*}|_{X^* \setminus X^{**}} \rightarrow f^{-1}O_{S^*} \rightarrow f^{-1}O_{S^*}|_{X^{**}} \rightarrow 0,
$$

where the left and right terms are the extension by zero of the restriction of the sheaf $f^{-1}O_{S^*}$ on $X^* \setminus X^{**}$ and $X^{**}$ respectively, is a resolution of the short exact sequence (5.6), i.e. there is a commutative diagram:

$$
\begin{array}{ccc}
0 & 0 & 0 \\
& & \downarrow \\
0 & \longrightarrow & f^{-1}O_{S^*}|_{X^* \setminus X^{**}} \\
& & \downarrow \\
0 & \longrightarrow & \Omega^\bullet_{X^* \setminus X^{**}}
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & f^{-1}O_{S^*} \\
& & \downarrow \\
\longrightarrow & \longrightarrow & \Omega^\bullet_{X^*/S^*}
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \Omega^\bullet(X^{**}/S^*) \\
& & \downarrow \\
& & \longrightarrow
\end{array}
$$

From this one obtains the required isomorphisms (c.f. [15], [59]):

$$
\mathbb{R}^p f_* \Omega^\bullet_{X^*/S^*} \cong \mathbb{R}^p f_* f^{-1}O_{S^*} \cong \mathbb{R}^p f_* \mathcal{C}_{X^*} \otimes_{\mathcal{C}_{S^*}} O_{S^*},
$$

$$
\mathbb{R}^p f_* \Omega^\bullet_{X^* \setminus X^{**}/S^*} \cong \mathbb{R}^p f_* (f^{-1}O_{S^*}|_{X^* \setminus X^{**}}) \cong \mathbb{R}^p f_* \mathcal{C}_{X^* \setminus X^{**}} \otimes_{\mathcal{C}_{S^*}} O_{S^*},
$$

and finally:

$$
\mathbb{R}^p f_* \Omega^\bullet_{X^* \setminus X^{**}/S^*} \cong \mathbb{R}^p f_* (f^{-1}O_{S^*}|_{X^* \setminus X^{**}}) \cong \mathbb{R}^p f_* \mathcal{C}_{X^* \setminus X^{**}} \otimes_{\mathcal{C}_{S^*}} O_{S^*}.
$$

(iv.) For $p < n$ all the sheaves in (5.7) are endowed with Gauss-Manin connections which makes them locally free. Indeed, for the sheaves $\mathbb{R}^p f_* \Omega^\bullet_{X/S}$ and $\mathbb{R}^{p-1} f_* \Omega^\bullet_{X'/S}$ this was proved by Brieskorn, whereas for $\mathbb{R}^p f_* \Omega^\bullet_{X,X'/S}$ it will be shown in the next section. For $p = n$ it follows from Milnor’s (or Arnol’d’s) theorem that there is a short exact sequence of coherent sheaves:

$$
0 \rightarrow \mathbb{R}^{n-1} f_* \Omega^\bullet_{X/S} \rightarrow \mathbb{R}^n f_* \Omega^\bullet_{X,X'/S} \rightarrow \mathbb{R}^n f_* \Omega^\bullet_{X'/S} \rightarrow 0
$$

By the Sebastiani theorem [92] the sheaves on the left and on the right are locally free and it follows that the middle one is also locally free.
In the theorem above, property (iii) is of great significance in the sense that the long exact sequence (5.8) is an invariant of the boundary singularity germ \((f, H)\), i.e. it does not depend on all other choices (e.g. the standard representatives). For convenience in the following let us change notation for the relative de Rham complexes associated to the germ \((f, H)\):

\[
\Omega^\bullet_{X/S,0} := \Omega_f^\bullet = \frac{\Omega^\bullet}{df \wedge \Omega^{\bullet-1}}, \quad \Omega^\bullet_{X',X/S,0} := \Omega^\bullet_{f,H} = \frac{\Omega^\bullet_H}{df \wedge \Omega^{\bullet-1}_H},
\]

\[
\Omega^\bullet(X'/S, 0) := \Omega^\bullet_{f,H} = \frac{\Omega^\bullet(H)}{df \wedge \Omega^{\bullet-1}(H)},
\]

where \(\Omega^\bullet\) is the complex of germs of holomorphic forms at the origin of \(\mathbb{C}^{n+1}\), \(\Omega^\bullet_H = x\Omega^\bullet + dx \wedge \Omega^{\bullet-1} \subset \Omega^\bullet\) is the subcomplex of forms vanishing on \(H\) and \(\Omega^\bullet(H) = \Omega^\bullet / \Omega^\bullet_H\) is the quotient complex, which can be viewed as the extension by zero of the complex of sheaves of germs of holomorphic forms defined on \(H = \mathbb{C}^n \subset \mathbb{C}^{n+1}\). The stalk at the origin of the short exact sequence (5.6) is written now:

\[
0 \to \Omega^\bullet_{f,H} \to \Omega^\bullet_f \to \Omega^\bullet_{f,H} \to 0,
\]

whereas the induced long exact cohomology sequence (5.8) is written:

\[
\ldots \to H^{p-1}(\Omega^\bullet_{f,H}) \overset{\delta}{\to} H^p(\Omega^\bullet_{f,H}) \to H^p(\Omega^\bullet_f) \to H^p(\Omega^\bullet_{f,H}) \to \ldots
\]

and it is a long exact sequence of free \(\mathbb{C}\{f\}\)-modules of finite type. In particular, the long exact sequence (5.9) above reduces to the short exact sequence:

\[
0 \to H^{n-1}(\Omega^\bullet_{f,H}) \overset{\delta}{\to} H^n(\Omega^\bullet_{f,H}) \to H^n(\Omega^\bullet_f) \to 0.
\]

The connecting morphism \(\delta\) is defined as follows: let \(\bar{\alpha} \in \Omega^{n-1}_f\) represent a class \(\alpha \in E_0 \cong H^n(\Omega^{n-1}_{f,H})\). Then \(\delta \bar{\alpha} \in \Omega^n_{f,H}\) is closed and defines a class \(\delta \bar{\alpha} \in E\). By definition \(\delta \alpha = \delta \bar{\alpha}\). Obviously this map is \(\mathbb{C}\{f\}\)-linear and it is independent of the representatives, but depends only on the class \(\alpha\).

As a corollary we obtain:

**Corollary 5.2.3.**

\[
H^p(\Omega^\bullet_{f,H}) \cong \begin{cases} 
\mathbb{C}\{f\}, & p = 0 \\
0, & 0 < p < n - 1, \\
\mathbb{C}\{f\}^{\mu_0}, & p = n - 1, \\
\mathbb{C}\{f\}^{\mu_1}, & p = n,
\end{cases} \quad H^p(\Omega^\bullet_f) \cong \begin{cases} 
\mathbb{C}\{f\}, & p = 0 \\
0, & 0 < p < n, \\
\mathbb{C}\{f\}^{\mu_1}, & p = n,
\end{cases}
\]

\[
H^p(\Omega^\bullet_{f,H}) \cong \begin{cases} 
0, & 0 \leq p < n, \\
\mathbb{C}\{f\}^{\mu}, & p = n,
\end{cases}
\]

where \(\mu = \mu_0 + \mu_1\) is the Milnor number of the boundary singularity \((f, H)\).
The Relative Gauss-Manin Connection and Relative Brieskorn Modules

Here we will define first the analytic relative Gauss-Manin connection $D$ on the de Rham cohomology sheaves $\mathcal{H}^p_{dR}(X, X'/S)$ and we will show that it coincides with the topological one defined on the cohomology sheaves $\mathcal{H}^p(X^*, X'^*/S^*)$. This will imply also that the de Rham cohomology sheaves are indeed locally free and will finish the proof of Theorem 5.2.2 (iv). To start let us make explicit the isomorphism:

$$
\mathcal{H}^p_{dR}(X^*, X'^*/S^*) \cong \mathcal{H}^p(X^*, X'^*/S^*),
$$

(5.11)

which is a simple variant of the relative de Rham theorem, for holomorphic forms vanishing on the boundary. Let $\gamma(t) \in \cup_{t \in S} H_p(X_t, X'_t; \mathbb{C})$ be a locally constant (horizontal) section of the relative homology bundle, i.e. a section of the local system $(R^p f_* \mathbb{C}_{X^* \setminus X'^*})^*$, dual to the local system $R^p f_* \mathbb{C}_{X^* \setminus X'^*} = \ker D$. Let $\omega \in \mathcal{H}^p_{dR}(X^*, X'^*/S^*)$ be a relative cohomology class represented by a holomorphic form $\omega \in \Omega^p_{X^*, X'^*/S^*}$. Then, the integral:

$$
I(t) = \int_{\gamma(t)} \omega
$$

is well defined (because $\omega$ vanishes on the boundary $X'^*$), nondegenerate (it takes zero values on relatively exact forms and relative boundaries) and it is also a holomorphic (multivalued) function of $t \in S^*$. The verification of the holomorphicity comes from a relative version of the Leray residue formula:

$$
\int_{\gamma(t)} \omega = \frac{1}{2\pi i} \int_{\sigma_{\gamma(t)}} \frac{df \wedge \omega}{f - t},
$$

(5.12)

where the relative Leray boundary operator

$$
\sigma : H_p(X_t, X'_t; \mathbb{C}) \to H_{p+1}(X \setminus X_t, X' \setminus X'_t; \mathbb{C})
$$

is defined as follows: choose a tubular neighborhood $N$ of the fiber $X_t$ whose intersection with the boundary $X'$ gives a tubular neighborhood $N'$ of the subfiber $X'_t$ (such a choice is always possible by the transversality of $X_t$ with $X'$). The image of a relative cycle $\gamma(t)$ under $\sigma$ is then the relative cycle obtained by the preimage of $\gamma(t)$ under the natural projection (fibration by circles $S^1$) of the boundary of the tubular neighborhood $\partial N$ over $X_t$. In particular, the relative Leray boundary operator is such that it makes the following diagram of long exact homology sequences
where the upper and lower arrows are the ordinary Leray boundary operators. The proof of the formula (5.12) is then the same as in the ordinary case. From this it follows that indeed the function $I(t)$ is holomorphic in $t$, from which we immediately obtain the isomorphism (5.11):

$$\mathcal{H}^p_{dR}(X^*, X'^*/S) \cong (\mathcal{H}^p(X^*, X'^*/S))^* \cong \mathcal{H}^p(X^*, X'^*/S^*)$$

The analytic Gauss-Manin connection on the relative de Rham cohomology sheaves $\mathcal{H}^p_{dR}(X^*, X'^*/S^*)$ can now be defined as follows: calculate first the formula of the derivative of $I(t)$:

$$I'(t) = \frac{d}{dt} \int_{\gamma(t)} \omega = \frac{1}{2\pi i} \int_{\gamma(t)} \frac{df \wedge \omega}{(f-t)^2} = \frac{1}{2\pi i} \int_{\gamma(t)} \frac{d\omega}{f-t} = \frac{1}{2\pi i} \int_{\gamma(t)} \frac{df \wedge \eta}{f-t} = \int_{\gamma(t)} \eta,$$

where $\eta \in \Omega^p_{X^*, X'^*/S^*}$ is the Gelfand-Leray form of $d\omega$:

$$\eta = \frac{d\omega}{df},$$

defined by the condition $d\omega = df \wedge \eta$ (because $\omega$ is relatively closed). Notice now that the condition $0 = d(d\omega) = df \wedge d\eta$ implies the existence of a $p$-form vanishing on the boundary $\alpha \in \Omega^p_{X^*, X'^*}$, such that $d\eta = df \wedge \alpha$ (this can be verified for example by taking local coordinates). Thus, we may define a map:

$$D : \mathcal{H}^p_{dR}(X^*, X'^*/S^*) \to \mathcal{H}^p_{dR}(X^*, X'^*/S^*),$$

by the rule:

$$D\omega = \frac{d\omega}{df} = \eta,$$

which, as is easily verified, it is $\mathbb{C}$-linear and satisfies the Leibniz rule over $\mathcal{O}_{S^*}$, i.e. it defines a connection on $\mathcal{H}^p_{dR}(X^*, X'^*/S^*)$. Moreover, by the formula of the derivative $I'(t)$ above, the connection $D$ coincides with the topological Gauss-Manin connection on $\mathcal{H}^p(X^*, X'^*/S^*)$. We will call it the relative (analytic) Gauss-Manin connection.
Now we will show that for all $p < n$, the relative Gauss-Manin connection $D$ can be extended at the origin $0 \in S$, i.e. to a map:

$$D : H^p(\Omega^*_{f,H}) \to H^p(\Omega^*_{f,H})$$

defined by the same rule:

$$D\omega = \frac{d\omega}{df} = \eta.$$ 

To do this, it suffices to verify that the germ of the $p$-form $\eta \in \Omega^p_{f,H}$ is indeed relatively closed. This follows from the lemma below, which is a relative analog of the de Rham division lemma [25]:

**Lemma 5.2.4.** For all $p \leq n$ and any relative form $\omega \in \Omega^p_H$ such that $df \wedge \omega = 0$, there exists a $(p-1)$-form $\alpha \in \Omega^{p-1}_H$ such that $\omega = df \wedge \alpha$.

**Proof.** It follows from the fact that the de Rham division lemma holds for both $f$ and $f|_H$ because their singularities are isolated. Briefly, consider the Koszul complexes $K^*_f = (\Omega^*, df \wedge)$, $K^*_{f,H} = (\Omega^*_H, df \wedge)$ and $K^*_{f|_H} = (\Omega^*(H), df \wedge)$ and the corresponding short exact sequence:

$$0 \to K^*_{f,H} \to K^*_f \to K^*_{f|_H} \to 0.$$ 

The statement of the lemma is then equivalent to the cohomologies $H^p(K^*_{f,H})$ being all zero for $p \leq n$. This follows in turn by the long exact cohomology sequence and the fact that $H^p(K^*_f)$ and $H^{p-1}(K^*_{f|_H})$ are both zero for all $p \leq n$.

**Remark 5.2.2.** It follows from the argument above that the nonzero cohomologies of the Koszul complexes are in degree $n + 1$:

$$H^{n+1}(K^*_f) = \Omega^{n+1}_f, \quad H^n(K^*_{f|_H}) = \Omega^n_{f|_H},$$

$$H^{n+1}(K^*_{f,H}) = \Omega^{n+1}_{f,H}$$

and thus, there is a short exact sequence:

$$0 \to \Omega^n_{f|_H} \xrightarrow{df \wedge} \Omega^{n+1}_{f,H} \to \Omega^{n+1}_f \to 0.$$ (5.13)

But after a choice of coordinates $(x, y_1, \ldots y_n)$ for which $H = \{ x = 0 \}$ and division with the form $\omega = dx \wedge dy_1 \wedge \ldots \wedge dy_n$, the short exact sequence above reduces to a short exact sequence of the corresponding local algebras (c.f. [103]):

$$0 \to Q_{f|_H} \to Q_{f,H} \to Q_f \to 0.$$ 

This gives also another proof of the formula for the Milnor number of a boundary singularity:

$$\mu = \mu_1 + \mu_0.$$ 

Thus, the map $D$ can indeed be extended at the origin and consequently it defines a connection in the usual sense for all $p < n$ as expected. Attempting now to extend the relative Gauss-Manin
connection at the origin for \( p = n \) we come to the obstruction that the form \( d\eta = d\left(\frac{df}{df}\right) \) may not be relatively closed, being of maximal degree \( n + 1 \). To study the Gauss-Manin connection in this case we may, following Brieskorn [15], define two extensions of the cohomology module \( H^n(\Omega^*_f,H) \) (the relative Brieskorn modules) as follows: denote by \( E := H^n(\Omega^*_f,H) \) and consider the natural inclusion of this module in the cokernel of the differential \( d : \Omega^{n-1}_f,H \to \Omega^n_f,H \):

\[
E \subset F := \frac{\Omega^n_f,H}{d\Omega^{n-1}_f,H} \cong \frac{\Omega^n_H}{df \land \Omega^{n-1}_H + d\Omega^{n-1}_H}.
\]

Consider now multiplication by \( df \land \) on \( F \). It defines an isomorphism:

\[
F \xrightarrow{df \land} \frac{df \land \Omega^n_H}{df \land d\Omega^{n-1}_H}
\]

and we thus obtain another natural inclusion:

\[
F \xrightarrow{df \land} G := \frac{\Omega^{n+1}_f,H}{df \land d\Omega^{n-1}_H}.
\]

We have thus a sequence of inclusions of \( \mathbb{C}\{f\}\)-modules:

\[
E \subset F \subset G,
\]

whose cokernels are both isomorphic to the same \( \mu \)-dimensional \( \mathbb{C} \)-vector space:

\[
\frac{E}{d \Omega^{n+1}_f,H} \xrightarrow{E} \frac{G}{d \Omega^{n+1}_f,H} \cong \Omega^{n+1}_f,H.
\]

Hence, we may view these modules as defining lattices in the same \( \mu \)-dimensional vector space over the field of quotients \( \mathbb{C}(f) \) of \( \mathbb{C}\{f\} \):

\[
\mathcal{M} = E \otimes_{\mathbb{C}(f)} \mathbb{C}(f) = F \otimes_{\mathbb{C}(f)} \mathbb{C}(f) = G \otimes_{\mathbb{C}(f)} \mathbb{C}(f)
\]

In analogy with the ordinary case we call the modules \( F \) and \( G \) the relative Brieskorn modules (or lattices) of the boundary singularity \((f,H)\).

Now, using the relative Brieskorn modules we may extend the map \( D \) to two maps (which we denote by the same symbol):

\[
D : E \to F, \quad D\alpha = \frac{d\alpha}{df} = \eta,
\]

\[
D : F \to G, \quad D\eta = D(df \land \eta) = d\eta,
\]

which, as is easily verified, are \( \mathbb{C} \)-linear and satisfy the Leibniz rule over \( \mathbb{C}\{f\} \) (they define “connections” on the corresponding pairs of modules in the sense of Malgrange [64]). For these maps we have first the following important proposition:

**Proposition 5.2.5.** The maps \( D \) defined above induce isomorphisms of the underlying \( \mathbb{C} \)-vector
spaces, i.e. there exists a commutative diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{D'} & G \\
\sim & & \sim \\
\downarrow D & & \downarrow D \\
E & \xrightarrow{D} & F \\
\end{array}
\]

Proof. We will show that the map \( D : F \to G \) is indeed an isomorphism (for the other map see Proposition 5.2.9). It is obviously surjective since for any \( \omega \in \Omega_{H}^{n+1} \) representing a class in \( G \) there exists a form \( \eta \in \Omega_{H}^{n} \) such that \( \omega = d\eta \) (by the Poincaré lemma for \( \Omega_{H}^{n} \)). To show that it is injective, let \( D\eta = d\eta = 0 \). This means that for a representative \( d\eta \in \Omega_{H}^{n+1} \) of the class \( \eta \) there exists a form \( h \in \Omega_{H}^{n-1} \) such that \( d\eta = df \wedge dh \). Thus \( \eta = df \wedge h + dg \) for some \( g \in \Omega_{H}^{n-1} \), i.e. the class of \( \eta \) is indeed zero in \( F \).

Despite the fact that these maps do not define connections in the ordinary sense, it follows that they induce the same meromorphic connection \( D \) on the localisation \( \mathcal{M} \) of the relative Brieskorn modules:

\[ D : \mathcal{M} \to \mathcal{M} \]

defined as follows: let \( \omega \in \Omega_{H}^{n+1} \) be a representative of a class in \( G \). Since the boundary singularity \((f, H)\) is isolated there exists a natural number \( k < \infty \) such that \( f^{k}\omega = df \wedge \eta \), where \( \eta \in \Omega_{H}^{n} \). Then \( D(f^{k}\omega) = D(df \wedge \eta) = d\eta \) and by the Leibniz rule we obtain in \( \mathcal{M} \):

\[ D\omega = \frac{d\eta}{f^{k}} - k \frac{\omega}{f}. \]

It is easy now to verify that the map thus defined is \( \mathbb{C} \)-linear and satisfies the Leibniz rule over \( \mathbb{C}(f) \), i.e. it indeed defines a connection on \( \mathcal{M} \), with a pole of degree at most \( k \) at the origin.

Remark 5.2.3. In the next section we will show that the relative Gauss-Manin connection thus defined is regular, i.e. there exists a (meromorphic) change of coordinates such that \( D \) has a pole of degree at most 1 at the origin. The residue \( \text{Res}_{0}D \) of the connection is then the constant matrix \( \Gamma \) in the representation:

\[ y' = (\frac{\Gamma}{t} + \Gamma(t))y, \]

of the differential system of horizontal sections in this basis, where \( \Gamma(t) \) is a holomorphic matrix. Since the characteristic polynomial of the relative Picard-Lefschetz monodromy \( T \) is integral, it is constant under variations of \( t \) and thus its roots \( \lambda_{j} \) coincide with the numbers \( e^{-2\pi i \alpha_{j}} \), where \( \alpha_{j} \) are the eigenvalues of \( \text{Res}_{0}D \). Moreover, one may show\(^3\) that the connection \( D \) is algebraically defined, i.e. that for any automorphism \( \phi : \mathbb{C} \to \mathbb{C} \) the following relation holds:

\[ D_{\phi f, H} = \phi \circ D. \]

It follows then from the solution of Hilbert’s VII problem that the eigenvalues \( \alpha_{j} \) of \( \text{Res}_{0}D \) are

\(^{3}\)following for example the same construction as in [13].
rational numbers and thus, the eigenvalues of the relative monodromy operator $T$ are indeed roots of unity.

Asymptotics of Integrals along Vanishing Cycles: the Relative Sebastiani Theorem and Regularity of the Relative Gauss-Manin Connection

We give here a direct extension of some results obtained by Malgrange in [64] (see also Chapter 2), concerning the asymptotics of integrals of holomorphic forms along relative vanishing cycles. First we will need the following estimate which we will use to prove the relative Sebastiani theorem as well as the regularity theorem for the relative Gauss-Manin connection:

**Proposition 5.2.6.** For any relative $n$-form $\omega \in \Omega^n_{X,X'/S}$ and any section $\gamma(t) \in H_n(X_t, X'_t; \mathbb{C})$ in a sector containing the zero ray:

$$\lim_{t \to 0, \text{arg} t = 0} \int_{\gamma(t)} \omega = 0.$$  

**Proof.** The proof is the same as in [64] with simple modifications: let $\omega \in \Omega^n_{X,X'}$ represent the class of $\omega$. Fix a real $t_0 > 0$ and let $Y = f^{-1}([0,t_0]) \subset X$, $Y' = f^{-1}([0,t]) \cap X' = f'^{-1}([0,t_0]) \subset X'$. Let $\gamma(t_0)$ be a relative $n$-cycle on $X_{t_0}$ and let $\Gamma$ be a representative. By the fact that the pair $(X_{t_0}, X'_{t_0})$ is contractible, it follows that the pair $(Y, Y')$ is contractible as well. Since $Y$ is semianalytic and $Y'$ is a semianalytic subset, we may find semianalytic triangulation of $Y$ such that both $Y'$ and $X_{t_0}$ are subcomplexes of $Y$ and such that $X'_{t_0} = X_{t_0} \cap Y'$ is a subcomplex of both $Y'$ and $X_{t_0}$ (c.f. [60]). Thus, there exists a relative $(n+1)$-chain $\Delta$ such that $\Gamma = \partial \Delta$ (here the boundary operator $\partial$ is the one induced on the relative chains). By an immediate extension of Stokes-Herrera theorem [48] for the relative case, we have that the integrals

$$I(t_0) = \int_{\gamma(t_0)} \omega = \int_{\Gamma} \omega - \int_{\Delta} d\omega$$

are well defined. Consider now a relative $(n+1)$-chain $\Delta_t = f^{-1}([0,t]) \cap \Delta$, $t \in (0,t_0]$. Then $\Delta = \Delta_t + \Delta'$ where $\Delta'$ is a relative $(n+1)$-chain on $f^{-1}([t,t_0])$ and $\partial \Delta' = \Gamma - \Gamma_t$. It follows that $\Gamma_t$ is a relative cycle representing $\gamma(t)$ and

$$I(t_0) = \int_{\Delta} d\omega = \int_{\Delta_t} d\omega + \int_{\Delta'} d\omega = \int_{\Delta_t} d\omega + \int_{\Gamma_t} \omega - \int_{\Gamma_t} \omega = \int_{\Delta_t} d\omega + I(t_0) - I(t),$$

i.e.

$$I(t) = \int_{\Gamma_t} \omega = \int_{\Delta_t} d\omega.$$  

But

$$\lim_{t \to 0} \int_{\Delta_t} d\omega = \int_{\Delta_0} d\omega$$

where $\Delta_0 = X_0 \cap \Delta$ is a relative $n$-chain on $X_0$. By the fact that the restriction of $d\omega$ on the smooth part of $X_0$ is zero, it follows that $\lim_{t \to 0} I(t) = 0$ as was asserted. 

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As an immediate corollary of this proposition we obtain the following relative analog of the Sebastiani theorem [92]:

**Theorem 5.2.7.** The relative Brieskorn module $G$ (and thus $F$ and $E$) is a free module of rank $\mu$.

**Proof.** The proof is again the same as in [64]. Briefly, let $H^T$ and $H'^T$ be the torsion submodules of the corresponding Brieskorn modules with $H'^T \neq 0$. We have $DH^T \subset H'^T$ and necessarily $H^T \neq H'^T$ because then the restriction of $D$ will give a connection on $H^T = H'^T$ and thus $H'^T = 0$. Since $D : F \to G$ is an isomorphism (Proposition 5.2.5) it follows that there exists nonzero $\omega \in F$ such that $\omega \neq H^T$ and $D\omega \in H'^T$. After tensoring with $\mathbb{C}(f)$ we find a form $\omega \in \Omega^n_{D,H}$ such that its class $\omega \in F \otimes_{\mathbb{C}(f)} \mathbb{C}(f)$ satisfies $\omega \neq 0$ and $D\omega = 0$. But then, for any section $\gamma(t) \in H_n(X_t, X'_t; \mathbb{C})$ we have:

$$I'(t) = \frac{d}{dt} \int_{\gamma(t)} \omega = \int_{\gamma(t)} D\omega = 0,$$

i.e. $I(t)$ is constant. From Proposition 5.2.6 we have that $I(t) = 0$ and thus $\omega = 0$ in $F \otimes_{\mathbb{C}(f)} \mathbb{C}(f)$ which is a contradiction. Thus $H'^T = 0$ which proves the theorem.

Now we will prove the following relative analog of the regularity theorem:

**Theorem 5.2.8.** The relative Gauss-Manin connection $D : \mathcal{M} \to \mathcal{M}$ is regular.

**Proof.** The proof is again the same as in [64]. Recall (c.f [20]) that the condition of regularity of a connection is equivalent to the fact that each of the components $I_j(t)$ of the (multivalued) solutions $I(t) = (I_1(t), ..., I_\mu(t))^T$ of the differential system:

$$\frac{dI}{dt} = \Gamma(t)I(t), \quad (5.14)$$

where $\Gamma(t)$ is the connection matrix, is of moderate growth, i.e. for $t \to 0$ and in a fixed sector $a \leq \arg t \leq b$, $(a, b) \in \mathbb{R}^2$, there exist natural numbers $K$ and $N$ such that:

$$|I_j(t)| \leq K|t|^{-N}.$$

Fixing a basis $\{\omega_1, ..., \omega_\mu\} \subset F \otimes_{\mathbb{C}(f)} \mathbb{C}(f)$ we may consider for a locally constant section $\gamma(t) \in H_n(X_t, X'_t; \mathbb{C})$ the multivalued functions

$$I_j(t) = \int_{\gamma(t)} \omega_j$$

and the corresponding vector-valued map $I(t) = (I_1(t), ..., I_\mu)$ as a solution of the equation (5.14) above (the Picard-Fuchs equation, expressing the condition of horizontality of the section $\gamma(t)$ with respect to the dual Gauss-Manin connection in a basis dual to $\omega_j$). Indeed,

$$I'_j(t) = \int_{\gamma(t)} D\omega_j = \int_{\gamma(t)} \sum_{i=1}^\mu \Gamma_{ij}(f)\omega_i = \sum_{i=1}^\mu \Gamma_{ij}(t)I_i(t).$$
Thus, to prove regularity it suffices to prove that these integrals are indeed of moderate growth. This follows immediately from Proposition 5.2.6 applied to \( I(t) \) and an application of the Phragmén-Lindelöf theorem for the strip \( a \leq \arg t \leq b \) as in [63].

Combining the regularity of the relative Gauss-Manin connection with the relative monodromy theorem we may obtain a more exact calculation of the asymptotics of integrals of holomorphic forms along the relative vanishing cycles of the boundary singularity. Let us define now the theorem we may obtain a more exact calculation of the asymptotics of integrals of holomorphic forms along the relative vanishing cycles of the boundary singularity.

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Poincaré residue at $X_t$ of the form $\frac{\omega}{f-t}$:

$$\text{Res}_{X_t}(\frac{\omega}{f-t}) = \frac{\omega}{df}|_{X_t}.$$  

The map $t \in S^* \mapsto s[\omega](t) = \frac{\omega}{df}|_{X_t} \in H^n(X_t, X'_t; \mathbb{C})$ is what A. N. Varchenko called “a geometric section” (c.f. [107] and also [4], [56] and references therein). Thus, in order to obtain a trivialisation of the relative cohomology bundle, it suffices to find a basis of $G$ and by Nakayama’s lemma, a basis of the $\mu$-dimensional $\mathbb{C}$-vector space $\frac{G}{fG}$ (c.f. Example 5.2.1 below for the quasihomogeneous case).

Fix now a form $\omega \in G$ and denote by:

$$I_{\omega, \gamma}(t) = \int_{\gamma(t)} \frac{\omega}{df},$$

where $\gamma(t) \in \cup H_n(X_t, X'_t; \mathbb{C})$ is a locally constant section of the relative homology bundle. The theorem below is a relative analog of the classical theorem on the asymptotics of integrals obtained by Malgrange [64] and others (see again [4], [56] and references therein):

**Theorem 5.2.9.** For $|t|$ sufficiently small there is a convergent expansion in each sector of $\arg t$:

$$I_{\omega, \gamma}(t) = \sum_{\alpha, k} a_{\alpha, k} t^{\alpha} \frac{(\ln t)^k}{k!},$$

where:

(i.) $a_{\alpha, k}$ are vectors in $\mathbb{C}^\mu$,

(ii.) the numbers $\alpha$ are rational numbers $>-1$ which belong in a set of arithmetic progressions with the property that $\lambda = e^{-2\pi i \alpha}$ is an eigenvalue of the relative Picard-Lefschetz monodromy operator in relative homology $H_n(X_t, X'_t; \mathbb{C})$,

(iii.) the numbers $k$ are integers $0 \leq k \leq N$ where $N$ is the maximal size of Jordan blocks of the relative monodromy operator. In particular, if the size of the Jordan blocks corresponding to the eigenvalue $\lambda = e^{-2\pi i \alpha}$ is $\leq r$ then $0 \leq k \leq r$.

**Proof.** Let $\eta \in \mathcal{H}'_{X,X'/S}$ be a local section of the Brieskorn module such that $D\eta = d\eta = \omega \in \mathcal{H}'_{X,X'/S}$. Then

$$I_{\omega, \gamma}(t) = \int_{\gamma(t)} \frac{d\eta}{df} = \frac{d}{dt} \int_{\gamma(t)} \eta = V_{\eta, \gamma}(t),$$

where $V_{\eta, \gamma}(t) = \int_{\gamma(t)} \eta$. Since the map $D : \mathcal{H}'_{X,X'/S} \to \mathcal{H}'_{X,X'/S}$ is an isomorphism we may study first the expansion of the integral $V_{\eta, \gamma}(t)$ into asymptotic series. Let $\Lambda = \{\lambda_1, ..., \lambda_\mu\}$ be the eigenvalues of the relative monodromy operator $T$ in cohomology $H^n(X_t, X'_t; \mathbb{C})$. Then $\{-\lambda_1, ..., -\lambda_\mu\}$ are the eigenvalues of the relative monodromy operator $T'$ in homology $H_n(X_t, X'_t; \mathbb{C})$. Let

$$\alpha_j = -\frac{1}{2\pi i} \ln \lambda_j$$

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be the eigenvalues of the matrix $R$, where:

$$T^i = e^{2\pi i R}.$$ 

By the relative monodromy Theorem 5.2.1, the eigenvalues $\lambda_j = e^{-2\pi i \alpha_j}$ are roots of unity and so $\alpha_j$ are rational numbers defined modulo $\mathbb{Z}$. Denote by

$$L(\lambda_j) = \{\alpha_j^0, \alpha_j^0 + 1, \alpha_j^0 + 2, \ldots\}$$

the arithmetic progression with one suitable value of $\alpha_j$. Let now $\{\omega_1, \ldots, \omega_\mu\}$ be a local basis of the sheaf $\mathcal{H}_{X,X'/S}$. Then the vector:

$$V(t) = (\int_{\gamma(t)}^{\mu} \omega_1, \ldots, \int_{\gamma(t)}^{\mu} \omega_\mu)^T$$

is a solution of the Picard-Fuchs equation:

$$y'(t) = \Gamma(t)y(t),$$

where $\Gamma(t)$ is the connection matrix of the Gauss-Manin connection $D$ with respect to the basis $\{\omega_1, \ldots, \omega_\mu\}$. A fundamental solution of this equation is given by the period matrix:

$$Y(t) = (\int_{\gamma(t)}^{\mu} \omega_i)_{i,j=1,\ldots,\mu},$$

where $\{\gamma_1(t), \ldots, \gamma_\mu(t)\}$ is a locally constant (horizontal) basis of the homology bundle $\cup H_n(X_t, X'_t; \mathbb{C})$. By well known theorems of differential equations (c.f. [20]), the period matrix can be represented in the form:

$$Y(t) = Q(t)t^R,$$

where $Q(t)$ is a single-valued holomorphic matrix on $S^*$. In particular, there is a constant matrix $C$ such that:

$$V(t) = Q(t)t^RC.$$ 

By the regularity Theorem 5.2.8, the matrix $Q(t)$ is meromorphic at the origin. After a choice of a Jordan basis of the relative monodromy operator and the corresponding structure of the matrix $t^R$, we obtain an expansion:

$$V(t) = \sum_{\lambda \in \Lambda} \sum_{\alpha \in L(\lambda)} \sum_{k=0}^N a_{\alpha,k}t^\alpha \frac{(\ln t)^k}{k!}.$$ 

But by Proposition 5.2.6 we have $\lim_{t \to 0} V(t) = 0$ and thus all $\alpha \geq 0$. Moreover, if $\alpha = 0$ then $a_{\alpha,k} = 0$ for all $k \geq 1$. Thus we have obtained the required expansion for the function $V(t) = V_{\mu,\gamma}(t)$. Then, by differentiating and using equation (5.15) we obtain the required expansion for $I_{\omega,\gamma}(t)$. Thus, it suffices to prove only (ii.) But for $\alpha = 0$ we have only constants in the expansion of $V(t)$ and thus all $\alpha > -1$ in the expansion of $I_{\omega,\gamma}(t)$. This finishes the proof. \qed
Example 5.2.1 (Quasihomogeneous Boundary Singularities.) By a quasihomogeneous boundary singularity \((f, H)\) we mean a quasihomogeneous germ \(f\) at the origin of \(\mathbb{C}^{n+1}\) such as its restriction \(f|_H\) on the boundary \(H = \{ x = 0 \}\) is also quasihomogeneous. For example, all the simple boundary singularities in Arnol’d’s list [6] are quasihomogeneous. It is easy to see (analogously with [88]) that this is equivalent to

\[
f^2 J_f, H
\]

where \(J_f, H = (x \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n})\) is the Jacobian ideal of the boundary singularity. Equivalently this implies that \(f G = df \wedge F\), i.e.

\[
Df = F,
\]

that is, the operator \(D = \frac{d}{df}\) has a pole of first order at the origin. The residue of the connection is then the linear operator between the \(\mu\)-dimensional \(\mathbb{C}\)-vector spaces:

\[
Res_0 D : \frac{G}{f G} \rightarrow \frac{G}{f G},
\]

where:

\[
G \approx \frac{G}{df \wedge F} \approx \Omega_{f, H}^{n+1} \approx Q_{f, H}.
\]

In particular, by Nakayama’s lemma, a monomial basis \(e_m = x^{m_1}y_1^{m_2} \ldots y_n^{m_{n+1}}, m = (m_1, \ldots, m_{n+1}) \in A, |A| = \mu\) of the vector space \(Q_{f, H}\), lifts to a basis \(\omega_m = e_m dx \wedge dy^n\) of the relative Brieskorn module \(G\). An easy calculation shows that the forms \(\omega_m\) are exactly the eigenvectors of the operator \(fD\):

\[
fD \omega_m = (\alpha(m) - 1) \omega_m,
\]

where:

\[
\alpha(m) = \sum_{i=1}^{n+1} w_i (m_i + 1),
\]

and \((w_1, \ldots w_{n+1})\) are the quasihomogeneous weights of \(f\). Thus, the residue \(Res_0 D\) is a semisimple operator and in particular, the relative Picard-Lefschetz monodromy operator:

\[
T = e^{-2\pi i Res_0 D}
\]

is semisimple, with eigenvalues:

\[
\lambda_m = e^{-2\pi i \alpha(m)}.
\]

Moreover, for any \((n+1)\)-form \(\omega\) and any locally constant relative cycle \(\gamma(t) \in H_n(X_t, X'_t; \mathbb{C})\) there exists an asymptotic expansion for \(t \rightarrow 0\):

\[
I(t) = \int_{\gamma(t)} \omega \frac{df}{df} = \sum_{\lambda \in \Lambda} \sum_{\alpha \in L(\lambda)} a_\alpha t^{\alpha-1},
\]

where for each \(\lambda_m \in L(\lambda_m)\) \(\{\alpha(m), \alpha(m) + 1, \alpha(m) + 2, \ldots\}\) and \(a_\alpha \in \mathbb{C}\).

Let us calculate the numbers \(\alpha(m)\) for the \(A_k, B_k, C_k\) and \(F_4\) singularities on the plane \(\mathbb{C}^2\) with boundary \(H = \{ x = 0 \}\), i.e. the simple boundary singularities in Arnol’d’s list [6]:

\(A_k\): The normal form is: \(f = x + y^{k+1}, k = \mu \geq 1\). It is quasihomogeneous with weights

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\((w_1 = 1, w_2 = \frac{1}{k+1})\). The monomials \(1, y, \ldots, y^{k-1}\) form a basis of \(\mathcal{Q}_{f,H}\) and thus:

\[
G = \text{span}_{\mathbb{C}\{f\}}\{dx \wedge dy, ydx \wedge dy, \ldots, y^{k-1}dx \wedge dy\}.
\]

In particular:

\[
\alpha(m) = \{\frac{k+2}{k+1}, \ldots, \frac{2k+1}{k+1}\}.
\]

**B_k:** The normal form is: \(f = x^k + y^2, k = \mu \geq 2\). It is quasihomogeneous with weights \((w_1 = \frac{1}{k}, w_2 = \frac{1}{2})\). The monomials \(1, x, \ldots, x^{k-1}\) form a basis of \(\mathcal{Q}_{f,H}\) and thus:

\[
G = \text{span}_{\mathbb{C}\{f\}}\{dx \wedge dy, xdx \wedge dy, \ldots, x^{k-1}dx \wedge dy\}.
\]

In particular:

\[
\alpha(m) = \{1, \frac{k}{2}, \frac{k+1}{2}, \ldots, \frac{2k-1}{2}\}.
\]

**C_k:** The normal form is: \(f = xy + y^k, k = \mu \geq 2\). It is quasihomogeneous with weights \((w_1 = \frac{k-1}{k}, w_2 = \frac{1}{k})\). The monomials \(1, y, \ldots, y^{k-1}\) form a basis of \(\mathcal{Q}_{f,H}\) and thus:

\[
G = \text{span}_{\mathbb{C}\{f\}}\{dx \wedge dy, ydx \wedge dy, \ldots, y^{k-1}dx \wedge dy\}.
\]

In particular:

\[
\alpha(m) = \{1, \frac{k}{k}, \frac{k+1}{k}, \frac{2k-1}{k}\}.
\]

**F_4:** The normal form is: \(f = x^2 + y^3, \mu = 4\). It is quasihomogeneous with weights \((w_1 = \frac{1}{2}, w_2 = \frac{1}{3})\). The monomials \(1, x, y, xy\) form a basis of \(\mathcal{Q}_{f,H}\) and thus:

\[
G = \text{span}_{\mathbb{C}\{f\}}\{dx \wedge dy, xdx \wedge dy, ydx \wedge dy, xydx \wedge dy\}.
\]

In particular:

\[
\alpha(m) = \{\frac{5}{6}, \frac{4}{3}, \frac{7}{6}, \frac{5}{3}\}.
\]

**5.2.3 Relations between the Relative and Ordinary Brieskorn Modules**

In the previous section we showed the regularity of the relative Gauss-Manin connection \(D\) and the freeness of the Brieskorn module \(G\) independently of the regularity of the ordinary Gauss-Manin connections \(D_1\) and \(D_0\), and the freeness of the ordinary Brieskorn modules \(G_1\) and \(G_0\) respectively. On the other hand we know from the short exact cohomology sequence (5.10) that the relative cohomology module \(E := H^n(\Omega^*_{f,H})\) is an extension of the two ordinary cohomology modules \(E_1 := H^n(\Omega^*_f)\) and \(E_0 := H^{n-1}(\Omega^*_{f,H})\), i.e. there is a short exact sequence of free \(\mathbb{C}\{f\}\)-modules of finite type:

\[
0 \to E_0 \xrightarrow{\delta} E \to E_1 \to 0.
\]  

(5.16)
Here we will show that the relative Brieskorn modules $F$ and $G$ are also extensions of the two ordinary Brieskorn modules:

$$F_0 := \frac{\Omega_{\mathcal{J}|\mathcal{U}}^{n-1}}{d\Omega_{\mathcal{J}|\mathcal{U}}^{n-2}} \cong \frac{\Omega_{\mathcal{Y}}^{n-1}(H)}{df \wedge \Omega_{\mathcal{Y}}^{n-2}(H) + d\Omega_{\mathcal{Y}}^{n-2}(H)} \subset G_0 := \frac{\Omega_{\mathcal{J}}^n(H)}{df \wedge d\Omega_{\mathcal{Y}}^{n-2}(H)},$$

$$F_1 := \frac{\Omega_{\mathcal{J}}^n}{d\Omega_{\mathcal{J}}^{n-1}} \cong \frac{\Omega_{\mathcal{Y}}^n}{df \wedge \Omega_{\mathcal{Y}}^{n-1} + d\Omega_{\mathcal{Y}}^{n-1}} \subset G_1 := \frac{\Omega_{\mathcal{J}}^{n+1}}{df \wedge \Omega_{\mathcal{Y}}^{n-1}}.$$ 

The statement for $F$ is proved in the proposition below and for $G$ immediately after that:

**Proposition 5.2.10.** There exist $C\{f\}$-linear map $\delta'$ that makes the following diagram commutative:

$$0 \longrightarrow E_0 \xrightarrow{\delta} E \xrightarrow{p} E_1 \longrightarrow 0$$

Moreover, there exists a $C$-linear map $\delta''$ which extends the above diagram to a commutative diagram:

$$0 \longrightarrow F_0 \xrightarrow{\delta'} F \xrightarrow{\nu'} F_1 \longrightarrow 0$$

**Proof.** Let us prove first the claim for the diagram (5.17). It depends on the algebraic definition of the Gauss-Manin connections involved, i.e. as connecting homomorphisms in certain long exact cohomology sequences (c.f. [54] for the ordinary case). More specifically, consider the stalk at the origin of the diagram of short exact sequences (5.5):

$$0 \longrightarrow df \wedge \Omega_{\mathcal{Y}}^{*1} \longrightarrow df \wedge \Omega^{*1} \longrightarrow df \wedge \Omega^{*1}(H) \longrightarrow 0$$

Taking the corresponding long exact cohomology sequences we obtain a commutative diagram whose
part containing the corresponding connecting homomorphisms is depicted below:

$\begin{array}{ccccccccc}
 & & & & & & & & & \\
\downarrow & & & & & & & & & \\
H^p-1(\Omega^\bullet_{f|H}) & \stackrel{\delta}{\longrightarrow} & H^p(\Omega^\bullet_{f,H}) & \longrightarrow & H^p(\Omega^\bullet_f) & \longrightarrow & \\
\partial_0 & & & & & & & & & \\
\downarrow & & & & & & & & & \\
H^p(df \wedge \Omega^\bullet(H)) & \stackrel{\delta'}{\longrightarrow} & H^p+1(df \wedge \Omega^\bullet_H) & \longrightarrow & H^p+1(df \wedge \Omega^\bullet) & \longrightarrow & \\
\partial & & & & & & & & & \\
\downarrow & & & & & & & & & \\
H^p(\Omega^\bullet(H)) & \longrightarrow & H^p+1(\Omega^\bullet_H) & \longrightarrow & H^p+1(\Omega^\bullet) & \longrightarrow & \\
\downarrow & & & & & & & & & \\
H^p(\Omega^\bullet_{f|H}) & \stackrel{\delta}{\longrightarrow} & H^p+1(\Omega^\bullet_{f,H}) & \longrightarrow & H^p+1(\Omega^\bullet_f) & \longrightarrow & \\
\end{array}$

(5.20)

Consider now multiplication by $df\wedge$ in each of the complexes $\Omega^\bullet$, $\Omega^\bullet_H$ and $\Omega^\bullet(H)$. By the relative de Rham division lemma 5.2.4 it induces, for all $p \leq n$ a commutative diagram:

$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^p_{f,H} & \longrightarrow & \Omega^p_f & \longrightarrow & \Omega^p_{f|H} & \longrightarrow & 0 \\
& & \downarrow df\wedge & & \downarrow df\wedge & & \downarrow df\wedge & & \\
0 & \longrightarrow & df \wedge \Omega^p_H & \longrightarrow & df \wedge \Omega^p & \longrightarrow & df \wedge \Omega^p(H) & \longrightarrow & 0 \\
\end{array}$

(5.21)

where the vertical arrows are isomorphisms. Since $df\wedge$ commutes with each of the differentials in the relative complexes we obtain isomorphisms in cohomologies for all $p$:

$H^p(\Omega^\bullet_f) \cong H^p+1(df \wedge \Omega^\bullet), \quad H^p(\Omega^\bullet_{f,H}) \cong H^p+1(df \wedge \Omega^\bullet),$

$H^{p-1}(\Omega^\bullet_{f|H}) \cong H^p(df \wedge \Omega^\bullet(H)),$

where $\Omega^\bullet_f$, $\Omega^\bullet_{f,H}$ and $\Omega^\bullet_{f|H}$ are the complexes $\Omega^\bullet_f$, $\Omega^\bullet_{f,H}$ and $\Omega^\bullet_{f|H}$ with their last terms replaced by zero. Putting these back in the diagram (5.20) we obtain:

$\begin{array}{ccccccccc}
 & & & & & & & & & \\
\downarrow & & & & & & & & & \\
H^p-1(\Omega^\bullet_{f|H}) & \stackrel{\delta}{\longrightarrow} & H^p(\Omega^\bullet_{f,H}) & \longrightarrow & H^p(\Omega^\bullet_f) & \longrightarrow & \\
D_0 & & & & & & & & & \\
\downarrow & & & & & & & & & \\
H^p-1(\Omega^\bullet_{f|H}) & \stackrel{\delta'}{\longrightarrow} & H^p(\Omega^\bullet_{f,H}) & \longrightarrow & H^p(\Omega^\bullet_f) & \longrightarrow & \\
D & & & & & & & & & \\
\downarrow & & & & & & & & & \\
H^p(\Omega^\bullet(H)) & \longrightarrow & H^p+1(\Omega^\bullet_H) & \longrightarrow & H^p+1(\Omega^\bullet) & \longrightarrow & \\
\downarrow & & & & & & & & & \\
H^p(\Omega^\bullet_{f|H}) & \stackrel{\delta}{\longrightarrow} & H^p+1(\Omega^\bullet_{f,H}) & \longrightarrow & H^p+1(\Omega^\bullet_f) & \longrightarrow & \\
\end{array}$

(5.22)
where the map $\delta'$ is the connecting homomorphism in the long exact cohomology sequence induced by the short exact sequence:

$$0 \to \Omega^\bullet_{f,H} \to \Omega^\bullet_f \to \Omega^\bullet_{f/H} \to 0,$$

and it is thus $\mathbb{C}\{f\}$-linear. An easy calculation shows also that it is defined by the same rule with $\delta$. The first series of vertical maps in (5.22) are the corresponding Gauss-Manin connections which are obtained as the composition of the maps in (5.20) $\partial_0, \partial$ and $\partial_1$ respectively, with the following isomorphisms:

$$H^p(\Omega^\bullet_f) \cong H^p(\Omega^\bullet_f), \quad H^p(\Omega^\bullet_{f,H}) \cong H^p(\Omega^\bullet_f),$$

$$H^{p-1}(\Omega^\bullet_{f/H}) \cong H^{p-1}(\Omega^\bullet_f),$$

for all $p < n$, whereas for $p = n$:

$$H^n(\Omega^\bullet_f) \cong F_1, \quad H^n(\Omega^\bullet_{f,H}) \cong F,$$

$$H^{n-1}(\Omega^\bullet_{f/H}) \cong F_0.$$

But for all $p < n$ all the cohomologies (except the zero ones) in the diagram (5.22) above are zero, while for $p = n$ we obtain the commutative diagram (5.17). Finally, to obtain the commutative diagram (5.18) it suffices to set

$$\delta'' = D\delta' D_0^{-1}, \quad \rho'' = D_1 p' D^{-1}.$$

The map $\delta''$ takes a class $\omega \in G_0$ to the class of the differential $d\bar{\omega} \in G$, where $\bar{\omega} \in \Omega^n$ is a lift of a representative of $\omega$. It is obvious that this map is $\mathbb{C}$-linear. This finishes the proof.

In the proposition above the map $\delta''$ is not $\mathbb{C}\{f\}$-linear and so the short exact sequence in the bottom row of diagram (5.18) is only short exact for the underlying $\mathbb{C}$-vector spaces. To show that the relative Brieskorn module $G$ is an extension of the two ordinary Brieskorn modules $G_0, G_1$, we identify first $G_0$ with $D_0 F_0 = dF_0$, which is a free $\mathbb{C}\{f\}$-module of rank $\mu_0$. The inclusion $df \wedge d\Omega^{n-1}_H \subset df \wedge d\Omega^{n-1}$ induces a natural projection $\pi : G \to G_1$ whose kernel is exactly the module $df \wedge dF_0$. By the fact that $G_1$ is free, we obtain a split short exact sequence of $\mathbb{C}\{f\}$-modules:

$$0 \to dF_0 \overset{df \wedge}{\to} G \overset{\pi}{\to} G_1 \to 0,$$

which is what we wanted to prove. This gives also another direct proof of the relative Sebastiani Theorem 5.2.7

$$G \cong \mathbb{C}\{f\}^{\mu}.$$

As another immediate corollary of the above proposition we obtain a second proof of the regularity Theorem 4.4.1 for the relative Gauss-Manin connection: indeed, both of the commutative diagrams (5.17), (5.18) give, after localisation, the following commutative diagram of finite dimensional $\mathbb{C}(f)$-
vector spaces:

\[ 0 \rightarrow M_0 \rightarrow M \rightarrow M_1 \rightarrow 0 \]

\[ D_0 \downarrow \quad D \downarrow \quad D_1 \downarrow \]

\[ 0 \rightarrow M_0 \rightarrow M \rightarrow M_1 \rightarrow 0 \]  \hspace{1cm} (5.23)

The claim follows then from a well known proposition \[20\] according to which the connection \( D \) is regular if and only if both \( D_0 \) and \( D_1 \) are.

\[ \] 5.3 Boundary Singularities in Isochore Geometry

We give here some more applications of the results obtained so far in isochore deformation theory, i.e. the deformation theory of boundary singularities with respect to a volume form.

5.3.1 Local Classification of Volume Forms and Functional Invariants

We start first with a direct corollary of the finiteness and freeness of the relative Brieskorn module \( G \) concerning the classification of volume forms relative to diffeomorphisms tangent to the identity and preserving the boundary singularity \( (f, H) \). Write \( \mathcal{R}_{f,H} \) for the group of germs of these diffeomorphisms, i.e. such that:

\[ \Phi^* f = f, \quad \Phi(H) = H, \]

\[ \Phi(0) = 0, \quad \Phi_*(0) = Id. \]

Two germs of volume forms at the origin will be called \( \mathcal{R}_{f,H} \)-equivalent (or equivalent for brevity) if they belong in the same orbit under the action of \( \mathcal{R}_{f,H} \) in the space of germs of volume forms \( \Omega_{n+1} \).

The following theorem is a relative analog of a theorem obtained by J. -P. Françoise \[29\], \[30\] (see also \[32\]) for the ordinary singularities, concerning the local normal forms of volume forms and their functional invariants. It is the higher dimensional analog of the 2-dimensional results presented in Chapter 3:

\textbf{Theorem 5.3.1.} Two germs of volume forms are equivalent if and only if they define the same class in the relative Brieskorn module \( G \). In particular any germ of a volume form is equivalent to the form

\[ \omega = \sum_{i=1}^{\mu} c_i(f)\omega_i, \]  \hspace{1cm} (5.24)

where \( c_i \in \mathbb{C}\{t\} \) and the classes of the forms \( \omega_i \) form a basis of \( G \). The \( \mu \) holomorphic functions \( c_i(t) \) are unique and they are the characteristic (functional invariants) for the triple \( (\omega, f, H) \).

\textbf{Proof.} The one direction is rather straightforward: if two germs of volume forms are equivalent then their Poincaré residues define the same cohomology class in each fiber \( H^n(X_t, X'_t; \mathbb{C}) \) of the cohomological Milnor fibration in a sufficiently small neighborhood of the origin. Indeed, since the diffeomorphism realising the equivalence is tangent to the identity, it induces the identity in the
cohomology of each pair of fibers \((X_t, X'_t)\) with constant coefficients. It follows by the coherence and freeness of the Brieskorn module \(G\) that the diffeomorphism \(\Phi\) induces the identity morphisms in both \(F\) and \(G\). The other direction is an application of Moser’s homotopy method, whose proof goes briefly as follows: consider a family of volume forms \(\omega_s = \omega_0 + sdf \wedge dg, s \in [0, 1]\). Then the vector field \(v_s\) defined by:

\[
v_s, \omega_s = g \wedge df
\]

is a solution of the homological equation:

\[
L_{v_s} \omega_s = -df \wedge dg
\]

and thus, its time-1 map \(\Phi_1\) is the desired diffeomorphism between \(\omega_1\) and \(\omega_0\). Choosing now a basis of \(G\) and \(\omega_0\) as the representative of \(\omega_1\) in this basis, then we obtain the normal form (5.24), as well as the uniqueness of the coefficients \(c_i(t)\). To finish the proof it suffices to show the functions \(c_i(t)\) are independent of the coordinates systems (characteristic). This is a consequence of the description of the functions \(c_i(t)\) in terms of integrals along vanishing cycles: let \(\{\gamma_1(t), ..., \gamma_\mu(t)\} \in \cup H_n(X_t, X'_t; \mathbb{C})\) be a locally constant (horizontal) basis of relative cycles and consider the (multivalued) vector-valued map: \(I(t) = (I_1(t), ..., I_\mu(t))^T:\)

\[
I_j(t) = \int_{\gamma_j(t)} \frac{\omega_i}{df}.
\]

Then the holomorphic vector-valued map \(c(t) = (c_1(t), ..., c_\mu(t))\) may be obtained as a solution of the linear system:

\[
Y(t)c(t) = I(t),
\]

where

\[
Y(t) = (\int_{\gamma_j(t)} \frac{\omega_i}{df})
\]

is the period matrix of the boundary singularity. It follows by Cramer’s rule that:

\[
c_i(t) = \frac{\det \tilde{Y}_i(t)}{\det Y(t)},
\]

where the matrix \(\tilde{Y}_i(t)\) is obtained by the period matrix \(Y(t)\) after replacing its \(i\)'th column with \(I(t)\). Thus the vector \(c(t) = (c_1(t), ..., c_\mu(t))\) is indeed characteristic for the triple \((\omega, f, H)\).

**Remark 5.3.1.** Since the boundary singularity \((f, H)\) is isolated, we may always choose local coordinates \((x, y_1, ..., y_n)\) such that in the theorem above \(H = \{x = 0\}\) and \(f(x, y_1, ..., y_n)\) is a polynomial of sufficiently high degree (by a relative analog of the determinacy theorem c.f. [69]).

The case \(\mu = \mu_0 = 1\) i.e. the first occurring boundary singularity \((A_1\text{ in Arnol’d’s list [6]}\)), with normal form \(f(x, y) = x + y_1^2 + ... + y_n^2, H = \{x = 0\}\), is of special interest. The following theorem is a direct corollary of the above theorem and it may be interpreted as the relative analog of J. Vey’s isochore Morse lemma [109]. For its proof we follow [29] (for another proof see next section).

**Theorem 5.3.2.** Let \((f, H)\) be a boundary singularity such that the origin is a regular point for \(f\) but nondegenerate critical point for the restriction \(f|_H\) on the boundary. Then there exists a
diffeomorphism \( \Psi \), preserving both the boundary \( H = \{ x = 0 \} \) and the standard volume form \( \omega = dx \wedge dy_1 \wedge \ldots \wedge dy_n \), as well as a unique function \( \psi \in \mathbb{C}\{t\} \), \( \psi(0) = 0, \psi'(0) = 1 \) such that

\[
\Psi^* f = \psi(x + y_1^2 + \ldots + y_n^2),
\]

(5.25)

Proof. The proof goes exactly as the relative Morse-Darboux Lemma [3.2.1] for the 2-dimensional case. Briefly, by Theorem 5.3.1 above, we may choose a coordinate system \((x, y_1, \ldots, y_n)\) such that \( H = \{ x = 0 \} \), \( f(x, y) = x + y_1^2 + \ldots + y_n^2 \) and \( \omega = c(f) dx \wedge dy_1 \wedge \ldots \wedge dy_n \), where \( c \in \mathbb{C}\{t\} \) is a function, nonvanishing at the origin, \( c(0) = 1 \). We will show that there exists a change of coordinates \( \Psi(x, y_1, \ldots, y_n) = (x', y'_1, \ldots, y'_n) \) such that the pair \((f, H)\) goes to \((\psi(f), H)\) for some function \( \psi \) and \( \omega \) is reduced to normal form \( dx \wedge dy_1 \wedge \ldots \wedge dy_n \). To do this, we set \( x' = x\psi(f) \), \( y'_i = y_i \sqrt{v(f)} \), where \( v \in \mathbb{C}\{t\} \) is some function with \( v(0) = 1 \) (so \( \Psi \) is indeed a boundary-preserving diffeomorphism tangent to the identity). With any such function \( v \) we have \( \Phi^* f = \psi(f) \), for some function \( \psi(t) = tv(t) \) with \( \psi(0) = 0 \) and \( \psi'(0) = 1 \). Now it suffices to choose \( v \) so that \( \Phi_x \) has determinant equal to \( c(f) \), i.e. such that the following initial value problem is satisfied for the function \( w = v^{\frac{n+2}{2}} \):

\[
\frac{2}{n+2} tw'(t) + w(t) = c(t), \quad w(0) = 1.
\]

(5.26)

As is easily verified this admits an analytic solution given by the formula:

\[
w(t) = t^{-\frac{n+2}{2}} \int_0^t \frac{n+2}{2} s^\frac{n}{2} c(s) ds.
\]

This also shows the uniqueness of the function \( \psi(t) \), which can be written as:

\[
\psi(t) = \left( \int_0^t \frac{n+2}{2} s^\frac{n}{2} c(s) ds \right)^{\frac{2}{n+2}}.
\]

\( \square \)

5.3.2 Isochore Versal Deformations of Boundary Singularities

In [33], M. D. Garay gave a different proof of Vey’s isochore Morse lemma which, according to his results, is a simple consequence of an isochore version of Mather’s versal unfolding theorem proved by him (as a positive answer to a question asked by Y. Colin de Verdière in [18]). Here we will present the main parts of the proof of a relative version of the isochore unfolding theorem, i.e. for the isochore unfoldings of boundary singularities, by considering only the main modifications needed in order to adapt the same proof as in [33].

To start recall that a deformation \( F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0) \) of a boundary singularity \((f, H)\) is just a deformation of \( f \), \( F(\cdot; 0) = f \), such that its restriction \( F|_H : (H \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0) \) on the boundary \( H = \mathbb{C}^n \subset \mathbb{C}^{n+1} \), is a deformation of \( f|_H, F|_H(\cdot; 0) = f|_H \). To the deformation \( F \) of the boundary singularity we associate its unfolding, i.e. the map:

\[
\tilde{F} : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \to (\mathbb{C} \times \mathbb{C}^k, 0), \quad \tilde{F}(\cdot; \lambda) = (F(\cdot; \lambda), \lambda)
\]

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and accordingly we define also $\tilde{F}|_H$. Fix now the equation of the boundary $H = \{x = 0\}$ and fix also a germ of a volume form $\omega = dx \wedge dy^n$ (where $dy^n = dy_1 \wedge ... \wedge dy_n$) at the origin of $\mathbb{C}^{n+1}$. All the notions of Right-Left (or $A$-)equivalence between deformations, versality, infinitesimal versality e.t.c. (c.f. [3]) carry over to the subgroup $A_{\omega, H}$ of right-left equivalences, where the right diffeomorphism has to preserve both the boundary $H$ and the volume form $\omega$. In particular, a deformation $F$ (or the unfolding $\tilde{F}$) of a boundary singularity $(f, H)$ will be called isochore versal if any other deformation $F'$ (or unfolding $\tilde{F}'$ respectively) is $A_{\omega, H}$-equivalent to a deformation induced from $F$, i.e. there exists a relative diffeomorphism $\phi : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$, $\phi(\cdot, 0) = .$, preserving both $H$ and $\omega$, a relative diffeomorphism $\psi : (\mathbb{C} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$, $\psi(\cdot, 0) = .$ and a map germ $g : (\mathbb{C}^k, 0) \rightarrow (\mathbb{C}^k, 0)$ such that:

$$
\psi(F(\phi(x, y; \lambda'); g(\lambda')) = F'(x, y; \lambda').
$$

Let us consider now the corresponding infinitesimal isochore deformations. The space of non-trivial isochore deformations of the germ $(f, H)$ is, as is easily seen, the space:

$$
I_{f, H} = \{L_v f + k(f)/L_v \omega = 0, \ v|_H \in TH\}.
$$

This is a $\mathbb{C}\{f\}$-module which can be viewed as the quotient of the “isochore Jacobian module” of the boundary singularity $(f, H)$ by the submodule generated by the class of the constant function 1. The latter module is in turn isomorphic to the relative Brieskorn module $G$ of the boundary singularity, the isomorphism given by multiplication with the volume form $\omega$, and consequently it is free of rank $\mu$. Thus, a necessary condition for a deformation $F$ of $(f, H)$ to be isochore versal is that the classes of the velocities $\partial_{\lambda_i} F := \frac{\partial F}{\partial \lambda_i}|_{\lambda=0}$ along with the class of 1, span the isochore Jacobian module $I_{f, H}$ over $\mathbb{C}\{f\}$. The following theorem is an analog of the Garay-Mather theorem [34] and says that this condition is also sufficient:

**Theorem 5.3.3.** A deformation $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ of a boundary singularity $(f, H)$ is isochore versal if and only if it is infinitesimally isochore versal, i.e.

$$
I_{f, H} = \text{span}_{\mathbb{C}\{f\}}\{1, \partial_{\lambda_1} F, ..., \partial_{\lambda_k} F\} \leftrightarrow G = \text{span}_{\mathbb{C}\{f\}}\{\omega, \partial_{\lambda_1} F \omega, ..., \partial_{\lambda_k} F \omega\} \quad (5.27)
$$

Following [33] we may prove this theorem as follows: first we show that any 1-parameter deformation $G$ of an infinitesimally versal deformation $F$ is isochore trivial (we call $F$ isochore rigid in analogy with the ordinary case). Then we conclude by using J. Martinet’s trick, according to which any $k$-parameter deformation can be considered as a “sum” of 1-parameter deformations. The isochore

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4. In analogy with the isochore Jacobian module of an ordinary singularity [33], it is the space of non-trivial infinitesimal deformations with respect to (right) $R_{\omega, H}$-equivalence.
rigidity in turn can be interpreted cohomologically in terms of a parametric version of the relative Brieskorn module which we present below.

The Parametric Relative Brieskorn Module and Isochore Rigidity

Let $\Omega^\bullet_{n+1+k}$ denote the complex of germs of holomorphic forms at the origin of $\mathbb{C}^{n+1} \times \mathbb{C}^k$ and let $\Omega^\bullet_{n+1+k,H}$ denote the subcomplex of forms vanishing on $H$. In a coordinate system $(x, y_1, ..., y_n; \lambda_1, ..., \lambda_k)$ for which $H = \{x = 0\}$ we have explicitly $\Omega^\bullet_{n+1+k,H} = x\Omega^\bullet_{n+1+k} + dx \wedge \Omega^\bullet_{n+1+k}$. In analogy with the case of the germ $(f, H)$ we may define a relative de Rham cohomology for the map $\tilde{F}$ (and for the map $\tilde{F}|_H$) as well as the corresponding Brieskorn modules. Here we will only need to consider the parametric version of the relative Brieskorn module $G_f$, i.e the $\mathbb{C}[F, \lambda]$-module:

$$G_F := \frac{\Omega^{n+1+k}_{n+1+k}}{d\lambda_1 \wedge ... \wedge d\lambda_k \wedge dF \wedge d\Omega_{n+1+k,H}^{-1}}$$

which plays a crucial role in the proof of the isochore unfolding Theorem 5.3.2. In the ordinary case [33], the finiteness (and freeness) of the parametric Brieskorn module follows from the results of G. M. Greuel [43] on the isolated complete intersection singularities. For the boundary case we will only need the following relative part:

**Proposition 5.3.4.** The parametric Brieskorn module $G_F$ of a deformation $F$ of a boundary singularity $(f,H)$ is finitely generated over $\mathbb{C}[F,\lambda]$ and it is of rank $\mu$. Moreover, its restriction on $\mathbb{C}^{n+1} = \{\lambda_1 = 0, ..., \lambda_k = 0\}$ is isomorphic to the Brieskorn module $G$ of $(f,H)$:

$$G_F|_{\lambda=0} \cong G.$$

**Proof.** Since the singularities of $\tilde{F}$ are isolated, the proof of the finiteness of the Brieskorn module $G$ is again a straightforward corollary of the relative analog of the Kiehl-Verdier theorem (c.f. [34] and references therein). The rank of this module is then equal to the dimension of its fiber for any $(t, \lambda)$ sufficiently close to the origin and in the complement of the discriminant of $\tilde{F}$. By the same reasoning as in Section 2 (a parametric version of the de Rham theorem), this is exactly equal to the dimension of the relative cohomology $H^n(X_t, X^t_0; \mathbb{C})$, i.e. equal to $\mu$. The fact that the restriction of $G_F$ to $\{\lambda_1 = 0, ..., \lambda_k = 0\}$ is isomorphic to $G$ is obvious from the definition. \qed

**Remark 5.3.2.** Another proof of the finiteness (and freeness) of the parametric Brieskorn module of a boundary singularity, could be derived, as in the non-parametric case, by showing that this module is just an extension of the two ordinary parametric Brieskorn modules, i.e. those associated to $F$ and to its restriction $F|_H$ on the boundary.

Consider now a 1-parameter deformation $F'_t$ of $F$:

$$F'_t := F' : (\mathbb{C}^{n+1} \times \mathbb{C}^k \times \mathbb{C}, 0) \to (\mathbb{C}, 0), \quad (x, y; \lambda, t) \mapsto F'(x, y; \lambda, t),$$

$$F'(x, y; \lambda, 0) = F(x, y; \lambda).$$
Then, as is easily seen, $F'_0$ is isochore trivial provided that there exists a decomposition:

$$\partial_t F' = k(F', \lambda, t) + \sum_{i=1}^{k} c_i(F', \lambda, t) \partial_{\lambda_i} F' + L_v F',$$  

(5.28)

where $v$ is a relative vector field tangent to $H$ and preserving $\omega$. Multiplying with $\bar{\omega} = \omega \wedge d\lambda^k \wedge dt$ (where we denote $d\lambda^k = d\lambda_1 \wedge ... \wedge d\lambda_k$), the condition of isochore triviality above can be viewed as the condition that the class of the form $\partial_t F' \bar{\omega}$ in the Brieskorn module $G_{F'}$ of $F'$ (of the unfolding $\bar{F}'$) belongs to the $\mathbb{C}\{F', \lambda, t\}$-module spanned by the classes of form $\bar{\omega}$ and of the initial velocities $\partial_{\lambda_i} F' \bar{\omega}$:

$$\partial_t F' \bar{\omega} \in M = \text{span}_{\mathbb{C}\{F', \lambda, t\}} \{ \bar{\omega}, \partial_{\lambda_1} F' \bar{\omega}, ..., \partial_{\lambda_k} F' \bar{\omega} \}.$$  

We will show that if $F$ is infinitesimally isochore versal, then in fact $M = G_{F'}$, which implies in turn the existence of a solution of the homological equation (5.28). To prove the assertion, notice that since the Brieskorn module $G_{F'}$ is finitely generated, by the above Proposition 5.3.4, it suffices to show, by Nakayama’s lemma, that the image of $M$ by the natural projection:

$$\pi : G_{F'} \rightarrow \frac{G_{F'}}{m G_{F'}},$$

coincides with the whole $\mu$-dimensional $\mathbb{C}$-vector space:

$$\pi(M) = \frac{G_{F'}}{m G_{F'}}.$$  

(5.29)

Here $m$ is the maximal ideal at the origin of $\mathbb{C} \times \mathbb{C}^k \times \mathbb{C}$. But according to Proposition 5.3.4 again, there is an isomorphism of $\mu$-dimensional vector spaces:

$$\frac{G_{F'}}{m G_{F'}} \cong \frac{G}{f G}.$$  

Thus the condition (5.29) above reduces to the condition:

$$\pi(M) = \frac{\text{span}_{\mathbb{C}\{f\}} \{ \omega, \partial_{\lambda_1} F \omega, ..., \partial_{\lambda_k} F \omega \}}{f G} = \frac{G}{f G},$$  

(5.30)

which is in turn equivalent, by Nakayama’s lemma, to the assumption (5.27) of infinitesimal isochore versality of $F$. Thus we have proved:

**Proposition 5.3.5.** An infinitesimally isochore versal deformation of a boundary singularity is isochore rigid.
Proof of the Isochore Versal Deformation Theorem and Corollaries

Proof of Theorem 5.3.3. It goes exactly as in [33]: let $F$ be a deformation of $(f,H)$, $f = F(.,0)$ and $F'$ another deformation of $(f,H)$. Define the sum $F \oplus F'$ by:

$$F \oplus F'(x,y;\lambda,\lambda') = F(x,y;\lambda) + F'(x,y;\lambda') - f(x,y).$$

The restriction of $F \oplus F'$ on $\lambda = 0$ is equal to $F'$ and thus, in order to show that $F'$ is isochore equivalent to a deformation induced by $F$, it suffices to show that the deformation $F \oplus F'$ is an isochore trivial deformation of $F$. This can be shown inductively as follows: denote by $F_j$ the restriction of $F \oplus F'$ to $\{\lambda_j = \ldots = \lambda_k = 0\}$. Then $F_1 = F$ and $F_k = F \oplus F'$. It follows from Proposition 5.3.5 that for each $j$ the deformation $F_j$ is isochore rigid and thus $F_j$ is an isochore trivial deformation of $F_{j-1}$. We conclude by induction that $F_k$ is an isochore trivial deformation of $F_1$.

Corollary 5.3.6. Any relative Morse germ $f$ at the origin of $\mathbb{C}^{n+1}$ with a fixed volume form $\omega = dx \wedge dy^n$ and a fixed boundary $H = \{x = 0\}$, is isochore (right-left, or $A_{\omega,H}$-)equivalent to the germ:

$$f_0 = x + y_1^2 + \ldots + y_n^2.$$  
Moreover, the left diffeomorphism is unique.

Proof. The proof is the same as in the 2-dimensional case presented in Chapter 3. Briefly, Consider $f_t = f_0 + th$, $t \in [0,1]$, a 1-parameter deformation of $f_0$, $f_1 = f$, such that $f_t|_H$ has a nondegenerate critical point at the origin for all $t$. Then for any point $t_0 \in [0,1]$ the germ at $t_0$ of the deformation $f_t$ is an isochore trivial deformation of $f_{t_0}$. Indeed, the relative Brieskorn module $G_{f_t}$ is generated by the class of the form $dx \wedge dy^n \wedge dt$ and the claim follows from the isochore deformation theorem. Thus, for any $\epsilon$ sufficiently small, the germ $f_{t_0+\epsilon}$ is isochore equivalent to $f_{t_0}$, and thus $f_0$ is isochore equivalent to $f_1$ as well. Finally, the uniqueness of the left diffeomorphism follows from the fact that an isochore versal deformation of a boundary singularity, is in fact universal.

As another immediate corollary we obtain also a relative version of a theorem of Y. Colin de Verdière [18]:

Corollary 5.3.7. A versal deformation of a quasihomogeneous boundary singularity is isochore versal.

Proof. Indeed, in this case there is an isomorphism:

$$\frac{G}{fG} \cong Q_{f,H}$$

and thus the classes of 1 with the initial velocities of the deformation generate the isochore Jacobian module $I^1_{f,H}$. 

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