A geometric characterisation of persistently exciting signals generated by autonomous systems

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Abstract: The persistence of excitation of signals generated by time-invariant, continuous-time, autonomous linear and nonlinear systems is studied. The notion of persistence of excitation is characterised as a rank condition which is reminiscent of a geometric condition used to study the controllability properties of a control system. The notions and tools introduced are illustrated by means of simple examples and of an application in system identification.

Keywords: Persistence of excitation; nonlinear system identification; structural properties.

1. INTRODUCTION

Over the past decades the persistence of excitation condition has found a number of applications in mathematical control theory, see, *e.g.*, Söderström and Stoica [1989], Ljung [1999], Van Overschee and De Moor [1996], Katayama [2006], Huang and Kadali [2008], Pintelon and Schoukens [2012], Sastry and Bodson [1989], Narendra and Annaswamy [1989], Åström and Wittenmark [1995], Ioannou and Sun [1995], Khalil [1996], Tao [2003]. From a historical point of view, the notion of persistence of excitation of a signal has been introduced in Åström and Bohlin [1965] to ensure asymptotic consistency of the solution of a maximum likelihood identification problem. Thereafter, persistently exciting signals have attracted considerable interest in the control engineering area prompted by the possibility of proving stability results for adaptive control algorithms (see, *e.g.*, Morgan and Narendra [1977], Anderson and Johnson [1982], Boyd and Sastry [1983], Bitmead [1984], Goodwin and Teoh [1985], Boyd and Sastry [1986], Narendra and Annaswamy [1987]). This has generated a vast and diverse body of contributions, *e.g.*, Ljung [1971], Yuan and Wonham [1977], Stoica [1981], Moore [1983], Mareels [1984], Mareels and Gevers [1988], Ljung and Glad [1990, 1994], Willems et al. [2005], Gevers et al. [2009], Ortega and Fradkov [1993], Zhang et al. [1996], Panteley et al. [2001], Loría et al. [2005], which have provided extensions and applications of the persistence of excitation condition.

The notion of persistence of excitation is particularly useful in experiment design both for system identification and adaptive control Söderström and Stoica [1989], Ljung [1999], Van Overschee and De Moor [1996], Katayama [2006], Huang and Kadali [2008], Pintelon and Schoukens [2012], Sastry and Bodson [1989], Narendra and Annaswamy [1989], Åström and Wittenmark [1995], Ioannou and Sun [1995], Khalil [1996], Tao [2003]. Roughly speaking, a signal is persistently exciting if its spectrum contains a sufficiently large number of harmonics Ljung [1971]. The use of persistently exciting signals is often necessary to ensure that a system identification experiment produces informative data. For example, these signals are especially important when all the modes of a linear system need to be excited Söderström and Stoica [1989], Ljung [1999], Van Overschee and De Moor [1996], Katayama [2006], Huang and Kadali [2008], Pintelon and Schoukens [2012]. Persistence of excitation conditions are also crucial in the stability analysis of certain classes of nonlinear systems Panteley et al. [2001], Loría et al. [2005], which, in turn, is instrumental for the investigation of convergence properties of adaptive control algorithms Sastry and Bodson [1989], Narendra and Annaswamy [1989], Åström and Wittenmark [1995], Ioannou and Sun [1995], Khalil [1996], Tao [2003].

The first objective of this work is to characterise geometrically the persistence of excitation condition for signals generated by the class of time-invariant, continuous-time, autonomous systems, *i.e.* dynamical systems described by time-invariant ordinary differential equations without forcing term. In particular, our main goal is to show that a rank condition is equivalent to the property that the state of an autonomous system is persistently exciting. The proposed rank condition turns out to have a simple system-theoretic interpretation in terms of controllability properties of a particular system.

The second objective is to use the proposed rank condition to address special issues in control problems dealing with input-output data. To demonstrate the significance of our result we address questions in subspace identification for linear autonomous systems Verhaegen and Verdult [2007] and nonlinear autonomous systems Padoan and Astolfi [2015]. We also envisage the use of the result to deal with issues arising in the study of the steady-state response of nonlinear systems Astolfi [2010] and in data-driven model reduction by moment matching for linear and nonlinear, possibly time-delay, systems Scarciotti and Astolfi [2015].

Note that this is only a subjective selection of issues based on the authors' background and the possibility of finding other applications in the future is not excluded.

The rest of the paper is organized as follows. Section 2 contains basic definitions. Section 3 provides a system-theoretic interpretation of the notions introduced and the main result of the paper, which asserts that the state of a time-invariant, continuous-time, autonomous system is persistently exciting if and only if a rank condition is satisfied. Section 4 illustrates ideas and tools presented in Sections 2 and 3 by means of simple examples. Section 5 presents an application in system identification. Finally, conclusions and future research directions are outlined in Section 6.

Notation: Standard notation is used. $\mathbb{N}$ denotes the set of natural numbers (including 0). $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{p\times m}$ denote the set of real numbers, of n-dimensional vectors with real entries and of $p \times m$-dimensional matrices with real entries, respectively. $\mathbb{R}^+$ and $\mathbb{C}^-$ denote the set of non-negative real numbers and the set of complex numbers with negative real part, respectively. $\|\omega\|$ denotes the standard Euclidean norm of the vector $\omega \in \mathbb{R}^\nu$. M' and $\operatorname{Im} M$ denote the transpose and the image space of the matrix $M \in \mathbb{R}^{p\times m}$, respectively. $\lambda(S)$ denotes the spectrum of the matrix $S \in \mathbb{R}^{\nu\times\nu}$. The function $\varphi(\cdot)$ is said to be of class C^k, with $k > 0$ integer, if it possesses continuous partial derivatives up to the order k. $L^2_{\mathrm{loc}}(\mathbb{R}^+,\mathbb{R}^\nu)$ denotes the Hilbert space of locally square-integrable functions $\omega : \mathbb{R}^+ \to \mathbb{R}^\nu$. The flow of a vector field $s(\cdot)$ passing through $\omega \in \mathbb{R}^\nu$ at time $t = 0$ is denoted by $\phi^s_t(\omega)$, *i.e.* $t \mapsto \phi^s_t(\omega)$ is the unique maximal integral curve of $s(\cdot)$ such that $\phi^s_0(\omega) = \omega$. The positive orbit of a forward-complete vector field $s(\cdot)$ passing through $\omega \in \mathbb{R}^\nu$ at time $t = 0$ is denoted by $\gamma_+(\omega)$, *i.e.* $\gamma_+(\omega) = \{\phi^s_t(\omega) : t \in \mathbb{R}^+\}$. Finally, $\omega_1(t) \equiv \omega_2(t)$ means that the functions $\omega_1(\cdot)$ and $\omega_2(\cdot)$ coincide in their common domain of existence.

2. PRELIMINARIES

Consider a continuous-time, autonomous, nonlinear system described by equations of the form

$$\dot\omega = s(\omega), \tag{1}$$

in which $\omega(t) \in W \subset \mathbb{R}^\nu$ denotes the state of the system at time $t \in \mathbb{R}^+$. Without loss of generality, suppose that the state space W is a compact invariant set containing the origin, that the vector field $s : W \to W$ is analytic, *i.e.* expandable in a convergent power series in its arguments about each point of W, and that the origin is an equilibrium point for the system, *i.e.* $s(0) = 0$.

Definition 1. Consider the system (1). The *excitation space* $\mathcal{E}$ of system (1) is defined as the smallest linear space over $\mathbb{R}$ of vector fields on W containing the identity map and closed under Lie differentiation along the vector field $s(\cdot)$.

An equivalent description of the object defined above can be established as follows. Define recursively the vector fields

$$\theta_{k+1} : W \to W : \omega \mapsto \frac{\partial\theta_k}{\partial\omega}(\omega)s(\omega), \quad k \in \mathbb{N}, \tag{2}$$

with $\theta_0(\cdot)$ the identity map. If $\omega(\cdot)$ is a solution of class C^k of the ordinary differential equation (1), then

$$\omega^{(k)}(t) \equiv \theta_k(\omega(t)),$$

in which $\omega^{(k)}(t)$, with k a positive integer, denotes the k-th order time derivative of the function $\omega(\cdot)$ at time $t \in \mathbb{R}^+$, provided it exists. In other words, along the solutions of the system (1), the vector fields defined above coincide with the state of the system and its successive time derivatives. Hence, we provide an equivalent definition of excitation space.

Definition 2. Consider the system (1). The excitation space of system (1) is given as

$$\mathcal{E} = \operatorname{span}\{\theta_k(\cdot),\ k \in \mathbb{N}\}.$$

Definition 3. Consider the system (1). The *excitation distribution* of system (1) is defined as

$$E(\omega) = \operatorname{span}\{\theta_k(\omega),\ k \in \mathbb{N}\}, \quad \omega \in W.$$

Definition 4. Consider the system (1) and the corresponding excitation distribution $E(\cdot)$. System (1) is said to satisfy the *excitation rank condition* at $\omega_0 \in W$ if $\dim E(\omega_0) = \nu$, *i.e.*

$$\dim \operatorname{Im}[\ \theta_0(\omega_0)\ \theta_1(\omega_0)\ \theta_2(\omega_0)\ \cdots\ ] = \nu.$$

3. A SYSTEM-THEORETIC INTERPRETATION

In this section we show that the definitions introduced above can be used to give a geometric characterisation of the classical notion of persistence of excitation (see, *e.g.*, Sastry and Bodson [1989], Ioannou and Sun [1995]).

Definition 5. A signal $\omega(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}^+,\mathbb{R}^\nu)$ is *persistently exciting* if there exists a real constant $T > 0$ such that the matrix

$$\mathcal{W}(t, t+T) = \int_t^{t+T} \omega(\tau)\omega(\tau)'d\tau \tag{3}$$

is positive definite for all $t \in \mathbb{R}^+$.

To illustrate our results we first consider the case of linear systems and subsequently discuss the extension to nonlinear systems.

3.1 Linear systems

The definitions introduced in the previous section can be given a simple interpretation. Every solution of an autonomous system of the form (1) with non-zero initial condition can always be associated with the impulse response of a specific controlled system with zero initial condition, the controllability properties of which are well-defined. More precisely, consider a continuous-time, autonomous, *linear* system described by equations of the form

$$\dot\omega = S\omega, \tag{4}$$

in which $\omega(t) \in \mathbb{R}^\nu$ and $S \in \mathbb{R}^{\nu\times\nu}$ is a constant matrix. If $\omega(0) = \omega_0$ is the initial condition of the state, then the dynamics of the system (4) coincide with that of the system

$$\dot\eta = S\eta + \omega_0 u, \tag{5}$$

with $\eta(t) \in \mathbb{R}^\nu$, when the initial condition is zero and the input $u(t) \in \mathbb{R}$ is the Dirac δ-function. Therefore, it is always possible to interpret the dynamics of a linear autonomous system of the form (4) with non-zero initial condition as the dynamics of a controlled linear system of the form (5) with zero initial condition and an impulsive input.

The definitions introduced in Section 2 can be now given a direct interpretation as follows. Consider the systems (4) and (5). The special structure of system (4) allows to write

$$\theta_k(\omega) = S^k\omega, \quad \omega \in \mathbb{R}^\nu.$$

Thus, according to Definition 3 and 4, the excitation distribution reads as

$$E(\omega) = \operatorname{Im}\Xi(S,\omega), \quad \omega \in \mathbb{R}^\nu,$$

in which $\Xi(S,\omega)$ denotes the controllability-like matrix with infinitely many columns associated with the pair (S,ω), defined as

$$\Xi(S,\omega) = \begin{bmatrix} \omega & S\omega & S^2\omega & \cdots \end{bmatrix}. \quad (6)$$

As a result, the following result can be established.

Proposition 1. System (4) satisfies the excitation rank condition at $\omega_0 \in \mathbb{R}^\nu$ if and only if the system (5) is controllable, *i.e.* $\dim \operatorname{Im}\Xi(S,\omega_0) = \nu$.

The results developed so far allow to give a geometric characterisation of the persistence of excitation condition.

Theorem 1. Consider the system (4) with initial condition $\omega(0) = \omega_0$. The following statements are equivalent.

(L1) The state of the system is a persistently exciting signal.

(L2) For every $\omega^\circ \in \gamma_+(\omega_0)$, there exists a real constant $T > 0$ such that the controllability Gramian

$$\mathcal{W}(0,T) = \int_0^T e^{S\zeta}\omega^\circ\omega^{\circ\prime}e^{S'\zeta}d\zeta \quad (7)$$

is positive definite.

(L3) The system (4) satisfies the excitation rank condition at every $\omega^\circ \in \gamma_+(\omega_0)$.

Proof 1. (L1) $\Leftrightarrow$ (L2). We only show (L1) $\Rightarrow$ (L2), since the inverse implication it trivial.

Taking into account that the solution of the linear differential equation (4) is $t \to \omega(t) = e^{St}\omega_0$, a shift of the integration variable gives

$$\mathcal{W}(t,t+T) = \int_0^T e^{S\zeta}\omega^\circ\omega^{\circ\prime}e^{S'\zeta}d\zeta,$$

with $\omega^\circ = e^{St}\omega_0$ and $t \in \mathbb{R}^+$. Recalling that the matrix $\mathcal{W}(t,t+T)$ is positive definite by hypothesis yields the claim.

(L2) $\Leftrightarrow$ (L3). This equivalence is a direct consequence of a standard result of linear systems theory Antoulas [2005, Corollary 4.11].

The significance of Theorem 1 is that the state of the system (4) is persistently exciting if and only if the pair (S,ω°) is exciting for every point ω° which belongs to the positive orbit $\gamma_+(\omega_0)$. Note that an objection can be raised: one needs to check the validity of (L2) or (L3) for every $\omega^\circ \in \gamma_+(\omega_0)$ to conclude that the state of system (4) is persistently exciting, thus making the verification practically infeasible. To resolve this issue, the following result, stated without proof for reasons of space, is introduced by strengthening the assumptions of Theorem 1.

Theorem 2. Consider the system (4) with initial condition $\omega(0) = \omega_0$. Assume that the matrix S is skew-symmetrizable [1]. The following statements are equivalent.

[1] A matrix S is skew-symmetrizable if there exist a positive definite diagonal matrix D such that $SD + DS' = 0$. In particular, all the eigenvalues of S lie on the imaginary axis.

(L1)* The state of the system is a persistently exciting signal.

(L2)* There exists a real constant $T > 0$ such that the controllability Gramian

$$\mathcal{W}(0,T) = \int_0^T e^{S\zeta}\omega_0\omega_0'e^{S'\zeta}d\zeta$$

is positive definite.

(L3)* The system (4) satisfies the excitation rank condition at ω_0.

3.2 Nonlinear systems

The arguments used in the linear case can be extended to the case of nonlinear systems.

Theorem 3. Consider the system (1) with initial condition $\omega(0) = \omega_0$. The following statements are equivalent.

(NL1) The state of the system is a persistently exciting signal.

(NL2) For every $\omega^\circ \in \gamma_+(\omega_0)$, there exists a real constant $T > 0$ such that the matrix

$$\mathcal{W}(0,T) = \int_0^T \phi_\zeta^s(\omega^\circ)\phi_\zeta^s(\omega^\circ)'d\zeta \quad (8)$$

is positive definite.

(NL3) The system (1) satisfies the excitation rank condition at every $\omega^\circ \in \gamma_+(\omega_0)$.

The following lemma is crucial in proving Theorem 3.

Lemma 1. Consider the system (1) with initial condition $\omega(0) = \omega_0$. Select $\omega^\circ \in \gamma_+(\omega_0)$ and let $\mathcal{W}(0,T)$ be as in (8). Then $\operatorname{Im}\mathcal{W}(0,T) = E(\omega^\circ)$ for all $T > 0$.

Proof 2. The proof is adapted from the proof of Antoulas [2005, Proposition 4.10]. Fix a real constant $T > 0$ and a constant vector $z \in \mathbb{R}^\nu$ such that $z \neq 0$. To prove the claim it suffices to show that $z'\mathcal{W}(0,T) = 0$ if and only if $z'E(\omega_0) = 0$. Equivalently, in view of the positive semi-definiteness of $\mathcal{W}(0,T)$, it is enough to show that $z'\mathcal{W}(0,T)z = 0$ if and only if $z'E(\omega^\circ) = 0$. Moreover,

$$0 = z'\mathcal{W}(0,T)z = \int_0^T (z'\phi_\zeta^s(\omega^\circ))^2 d\zeta$$

amounts to saying that $z'\phi_t^s(\omega^\circ) = 0$ for every $t \in \mathbb{R}^+$. The latter condition, by analyticity, is equivalent to $t \mapsto z'\phi_t^s(\omega^\circ)$ and all its derivatives being zero for every $t \in \mathbb{R}^+$, *i.e.* $z'\theta_k(\omega^\circ) = 0$ for every $k \in \mathbb{N}$. This, in view of Definition 3, is equivalent to $z'E(\omega^\circ) = 0$, as desired.

Proof 3. (Proof of Theorem 3). (NL1) $\Leftrightarrow$ (NL2). The claim follows directly from Definition 5 using the change of integration variable $\zeta = \tau - t$.

(NL2) $\Leftrightarrow$ (NL3). This equivalence is a direct consequence of Lemma 1.

To state a nonlinear counterpart of Theorem 2 we introduce the following definitions.

Definition 6. Isidori [1995] Consider the system (1). A point $\omega^\circ \in W$ is said to be *Poisson stable* if the following conditions hold.

(PS1) The flow $\phi_t^s(\omega^\circ)$ of the vector field $s(\cdot)$ is defined for every $t \in \mathbb{R}$.

(PS2) For every neighbourhood $W^\circ \subset W$ of ω° and for every real number $T > 0$ there exist real constants

$t_- < -T$ and $t_+ > T$ such that $\phi^s_{t_-}(\omega^\circ) \in W^\circ$ and $\phi^s_{t_+}(\omega^\circ) \in W^\circ$.

Definition 7. Isidori [1995] The system (1) is said to be *neutrally stable* if the following conditions hold.

(NS1) The equilibrium point at $\omega = 0$ is stable in the sense of Lyapunov.
(NS2) There exists an open neighbourhood of the origin in which every point is Poisson stable.

Remark 1. A continuous-time, autonomous, *linear* system described by equations of the form (4) is neutrally stable if and only if the matrix S is skew-symmetrizable.

We are now in the position to establish the following result.

Theorem 4. Consider the system (4) with initial condition $\omega(0) = \omega_0$. Assume that system (4) is neutrally stable. The following statements are equivalent.

(NL1)* The state of the system is a persistently exciting signal.
(NL2)* There exists a real constant $T > 0$ such that the matrix

$$\mathcal{W}(0,T) = \int_0^T \phi^s_\zeta(\omega_0)\phi^s_\zeta(\omega_0)' d\zeta$$

is positive definite.
(NL3)* The system (1) satisfies the excitation rank condition at ω_0.

4. EXAMPLES

This section illustrates the notions and tools introduced in Sections 2 and 3 by means of two simple examples.

Example 1. Consider a continuous-time, autonomous, linear system described by equations of the form (4), with $\omega(t) \in \mathbb{R}^3$, $S \in \mathbb{R}^{3\times 3}$ a constant skew-symmetrizable matrix defined as

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega \\ 0 & -\Omega & 0 \end{bmatrix}, \quad \Omega > 0,$$

and $\omega(0) = [\,\omega_{10}\ \omega_{20}\ \omega_{30}\,]'$ the initial condition of the state. The matrix with infinitely many columns $\Xi(S, \omega(0))$, defined as in (6), reads as

$$\Xi(S,\omega(0)) = \begin{bmatrix} \omega_{10} & 0 & 0 & \cdots \\ \omega_{20} & \Omega\omega_{30} & -\Omega^2\omega_{20} & \cdots \\ \omega_{30} & -\Omega\omega_{20} & -\Omega^2\omega_{30} & \cdots \end{bmatrix}.$$

A direct computation shows that the determinant of the matrix composed by the first three columns of $\Xi(S, \omega(0))$ is zero if and only if

$$\omega_{10} = 0 \quad \text{or} \quad \omega_{20}^2 + \omega_{30}^2 = 0. \tag{9}$$

By Proposition 1 and by Cayley-Hamilton theorem Horn and Johnson [1985], this implies that the excitation rank condition is satisfied if and only if neither of the conditions (9) holds. Hence, in view of Theorem 2, the state of the system is a persistently exciting signal if and only if neither of the conditions (9) holds. Since the system under consideration *de facto* describes the time evolution of two non-interacting autonomous linear systems, conditions (9) have a simple interpretation: to ensure that all the components of the state of the system are persistently exciting both sub-systems need to start from a non-zero initial condition.

Example 2. (Euler's equations). The angular velocity of a rigid body with respect to a fixed reference frame with axes fixed to the body and parallel to the body's principal axes of inertia can be described by the classical Euler's equations Bloch [2003]

$$\begin{aligned} I_1\dot\omega_1 &= (I_2 - I_3)\omega_2\omega_3, \\ I_2\dot\omega_2 &= (I_3 - I_1)\omega_1\omega_3, \\ I_3\dot\omega_3 &= (I_1 - I_2)\omega_1\omega_2, \end{aligned} \tag{10}$$

in which $\omega = [\,\omega_1\ \omega_2\ \omega_3\,]'$ and $I_1 > 0$, $I_2 > 0$, and $I_3 > 0$ denote the vector of angular velocities and the principal moments of inertia with respect to a body fixed coordinates frame with origin located at the center of mass, respectively. Due to their numerous applications in diverse fields, Euler's equations have received considerable attention by the control community over the past decades, *e.g.*, in relation to controllability, observability, and stabilization problems Baillieul [1981], Crouch [1984], Aeyels [1985], Aeyels and Szafranski [1988], Sontag and Sussmann [1989], Outbib and Sallet [1992], Bloch et al. [1992], Astolfi and Rapaport [1998], Astolfi [1999], Mazenc and Astolfi [2000]. Note that system (10) is neutrally stable.

Assuming, without loss of generality, that $I_1 < I_2 < I_3$, system (10) can be re-written as

$$\dot\omega_1 = A_1\omega_2\omega_3, \quad \dot\omega_2 = A_2\omega_1\omega_3, \quad \dot\omega_3 = A_3\omega_1\omega_2, \tag{11}$$

with $A_1 = (I_2 - I_3)/I_3 < 0$, $A_2 = (I_3 - I_1)/I_2 > 0$ and $A_3 = (I_1 - I_2)/I_3 < 0$. Combining (2) and (11) gives

$$\theta_0(\omega) = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \quad \theta_1(\omega) = \begin{bmatrix} A_1\omega_2\omega_3 \\ A_2\omega_1\omega_3 \\ A_3\omega_1\omega_2 \end{bmatrix},$$

and

$$\theta_2(\omega) = \begin{bmatrix} A_1A_2\omega_1\omega_3^2 + A_1A_3\omega_1\omega_2^2 \\ A_1A_2\omega_2\omega_3^2 + A_2A_3\omega_1^2\omega_2 \\ A_1A_3\omega_2^2\omega_3 + A_2A_3\omega_1^2\omega_3 \end{bmatrix}.$$

A direct computation shows that the determinant of the matrix the columns of which are the vectors $\theta_0(\omega)$, $\theta_1(\omega)$ and $\theta_2(\omega)$ is zero if [2]

$$\omega_1 = \omega_2 = 0 \text{ or } \omega_2 = \omega_3 = 0 \text{ or } A_1\omega_3^2 - A_3\omega_1^2 = 0, \tag{12}$$

which are (well-defined) invariant manifolds for the system. By Definition 4 this implies that the excitation rank condition at ω is satisfied only if neither of the conditions (12) holds. Thus, in view of Theorem 4, the state of the system is a persistently exciting signal only if neither of the conditions (12) holds at the initial condition $\omega(0)$. However, these are only sufficient conditions for the system to satisfy the controllability rank condition at a given point $\omega \in \mathbb{R}^\nu$. In other words, the system may satisfy the controllability rank condition at ω even if some of the conditions (12) is satisfied.

Simulations have been run using an explicit Runge-Kutta $(4,5)$-order integration method, with a variable integration step, an absolute tolerance $\varepsilon_{\text{abs}} = 10^{-12}$ and a relative tolerance $\varepsilon_{\text{rel}} = 10^{-12}$. The parameters $I_1 = 1$, $I_2 = 2$ and $I_3 = 3$ have been selected. Fig. 1 displays the state of the system (top) and the smallest singular value $\sigma_{\min}(\mathcal{W}(t-T,t))$ of the matrix $\mathcal{W}(t-T,t)$ (bottom), with $T = 2$ and initial condition selected as

[2] Recalling that $A_1 < 0$, $A_2 > 0$ and $A_3 < 0$ the conditions $A_2\omega_3^2 - A_3\omega_2^2 = 0$ and $A_1\omega_2^2 - A_2\omega_1^2 = 0$ boil down to the first two conditions in (12).

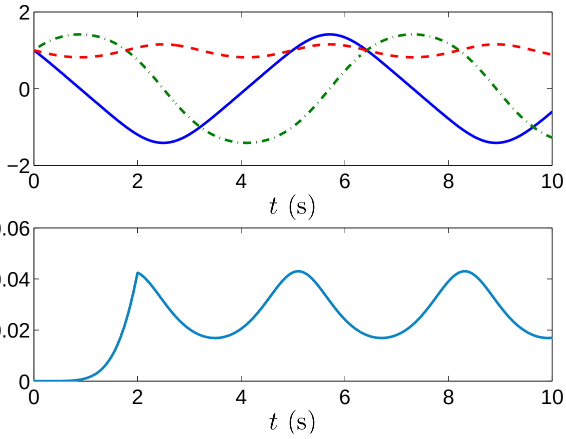

Fig. 1. Time histories of the state of the system $\omega(t)$ (top) and of the least singular value $\sigma_{\min}(\mathcal{W}(t-T,t))$ of the matrix $\mathcal{W}(t-T,t)$ (bottom) with $\omega(0) = [1\ 1\ 1]'$.

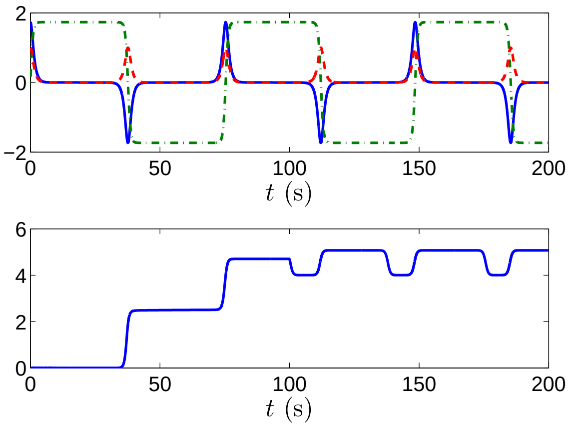

Fig. 2. Time histories of the state of the system $\omega(t)$ (top) and of the least singular value $\sigma_{\min}(\mathcal{W}(t-T,t))$ of the matrix $\mathcal{W}(t-T,t)$ (bottom) with $\omega(0) = [\sqrt{A_1/A_3}\ 0\ 1]'$.

$\omega(0) = [1\ 1\ 1]'$. The matrix $\mathcal{W}(t-T,t)$ has been computed using the fact that $\mathcal{W}(t-T,t) = \mathcal{W}(0,t) - \mathcal{W}(0,t-T)$ and setting $\mathcal{W}(0,t-T) = 0$ for $t \leq T$. Note that this causes a transient behaviour of $\sigma_{\min}(\mathcal{W}(t-T,t))$ for $t < T$. We can see that the system satisfies the excitation rank condition at $\omega(0)$ and that, after an initial period, the quantity $\sigma_{\min}(\mathcal{W}(t-T,t))$ is positive and periodic. Therefore the state of the system is a persistently exciting signal, as expected. Fig. 2 displays the state of the system $\omega(t)$ (top) and the smallest singular value $\sigma_{\min}(\mathcal{W}(t-T,t))$ of the matrix $\mathcal{W}(t-T,t)$ (bottom), with $T = 100$ and initial condition selected as $\omega(0) = [\sqrt{A_1/A_3}\ 0\ 1]'$. Although the system satisfies the third condition in (12), after an initial period, the quantity $\sigma_{\min}(\mathcal{W}(t-T,t))$ is positive and periodic. Therefore the state of the system is a persistently exciting signal and, by Theorem 4, the system satisfies the excitation rank condition at $\omega(0)$. This also proves that the system may satisfy the excitation rank condition even if one of the conditions (12) holds.

5. APPLICATION TO THE PROBLEM OF SUBSPACE IDENTIFICATION

This section contains an application of the notions introduced in Section 2. Consider a continuous-time, autonomous, nonlinear system described by equations of the form

$$\dot{x} = f(x), \quad y = h(x), \tag{13}$$

in which $x(t) \in X \subset \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ denote the unknown state and the measured output of the system at time $t \in \mathbb{R}^+$, respectively. Without loss of generality, suppose that the state space X is a compact invariant set containing the origin and that the mappings $f : X \to X$ and $h : X \to \mathbb{R}^p$ are analytic and such that $f(0) = 0$ and $h(0) = 0$. In addition, assume that the system is observable [3] and that $x(0) = x_0$.

The identification problem considered in this section can be stated as follows.

Problem 1. Determine the dimension n of system (13) and the mappings $f(\cdot)$ and $h(\cdot)$ from a given sequence of observed measurements of the form $\{y^{(k)}(t)\}_{k=0}^{M}$, with $t \in \mathbb{R}^+$ and $M \in \mathbb{N}$.

To make sure that the problem is well-posed, we assume that $t \in \mathbb{R}^+$ is fixed, that $M \in \mathbb{N}$ is sufficiently large, and that a pair of integers (i, j) satisfying the condition $n \leq \bar{n} < i \leq j$, with $\bar{n}$ an upper bound on n that we assume to be *a priori* known, is given.

We now show that a solution can be determined using a subspace approach when the system satisfies the excitation rank condition. The material presented below is partly borrowed and adapted from Verhaegen and Verdult [2007, p. 297] and Padoan and Astolfi [2015].

5.1 Linear systems

Suppose system (13) is *linear* and described by equations of the form

$$\dot{x} = Fx, \qquad y = Hx,$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $F \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{p\times n}$ constant matrices. Define the matrix

$$Y_{i,j} = \begin{bmatrix} y(t) & y^{(1)}(t)\ldots & y^{(j-1)}(t) \\ y^{(1)}(t) & y^{(2)}(t)\ldots & y^{(j)}(t) \\ \vdots & \vdots \quad \ddots & \vdots \\ y^{(i-1)}(t) & y^{(i)}(t)\ldots & y^{(i+j-2)}(t) \end{bmatrix},$$

and note that, using the linearity of the system, it admits the decomposition

$$Y_{i,j} = \Gamma_i X_j, \tag{14}$$

in which $\Gamma_i = [\ H' \ \ F'H' \ \ \cdots \ \ (F')^{i-1}H'\ ]'$ and $X_j = [\ x(t) \ \ Fx(t) \ \ \cdots \ \ F^{j-1}x(t)\ ]$.

To see that the excitation rank condition is sufficient for Problem 1 to be solved, recall that, by observability of the system, the matrix Γ_i is full rank. In view of Proposition 1 and of the decomposition (14), it follows that if system (13) satisfies the excitation rank condition

[3] For nonlinear systems it is possible to define several, non-equivalent notions of observability. Herein, a system is observable if it satisfies the observability rank condition at any point of the state space Nijmeijer and Van der Schaft [1990, p. 96].

at x_0, then $\operatorname{rank} Y_{i,j} = n$ and $\operatorname{Im}\Gamma_i = \operatorname{Im} Y_{i,j}$. This, in turn, allows to solve Problem 1 by performing a singular value decomposition of the matrix $Y_{i,j}$ and solving a linear system (see Padoan and Astolfi [2015] for more detail).

5.2 Nonlinear systems

The treatment of the nonlinear case proceeds along the same lines as the linear one based on the following results.

Lemma 2. Padoan and Astolfi [2015] Consider the system (13). Assume that $\mu(\cdot)$ and $\theta(\cdot)$ are an analytic function and an analytic vector field both defined on X, respectively. For every integer $k > 0$, define recursively [4]

$$\rho_k^\theta(x) = \sum_{q=0}^{k-1} L_f^{k-1-q} L_{[f,\theta]} L_f^q h(x), \text{ for all } x \in X,$$

with $\rho_1^\theta(\cdot) = L_{[f,\theta]}h(\cdot)$. If $\mu(\cdot) = L_\theta h(\cdot)$, then

$$L_f^k \mu(\cdot) = L_\theta L_f^k h(\cdot) + \rho_k^\theta(\cdot)$$

for every integer $k \geq 1$.

Theorem 5. Padoan and Astolfi [2015] Consider the system (13). If there exist vector fields $\theta_0(\cdot), \ldots, \theta_{j-1}(\cdot)$ that satisfy the conditions

$$L_f^k h(\cdot) = L_{\theta_k} h(\cdot), \quad k \in [0,\, j-1], \tag{15}$$

then, for every $x \in X$, the decomposition [5]

$$\mathcal{H}_{i,j}(x) = d\mathcal{H}_i(x)\Theta_j(x) + R_{i,j}(x), \tag{16}$$

in which

$$\mathcal{H}_{i,j}(x) = \begin{bmatrix} h(x) & L_f h(x) \ldots & L_f^{j-1}h(x) \\ L_f h(x) & L_f^2 h(x) \ldots & L_f^j h(x) \\ \vdots & \vdots \quad \ddots & \vdots \\ L_f^{i-1}h(x) & L_f^i h(x) \ldots & L_f^{i+j-2}h(x) \end{bmatrix},$$

$$\mathcal{H}_i(x) = [\; h(x)' \;\; L_f h(x)' \ldots \;\; L_f^{i-1}h(x)' \;]',$$

$$\Theta_j(x) = [\; \theta_0(x) \;\; \theta_1(x) \;\; \ldots \;\; \theta_{j-1}(x) \;],$$

$$R_{i,j}(x) = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \rho_1^{\theta_0}(x) & \rho_1^{\theta_1}(x) & \ldots & \rho_1^{\theta_{j-1}}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{i-1}^{\theta_0}(x) & \rho_{i-1}^{\theta_1}(x) & \ldots & \rho_{i-1}^{\theta_{j-1}}(x) \end{bmatrix},$$

holds.

Theorem 6. Padoan and Astolfi [2015] Consider the system (13) and equation (16). Suppose that the assumptions of Theorem 5 hold and that the intersection of the subspaces spanned by the rows of the matrices $\Theta_j(x)$ and $R_{i,j}(x)$ contains only the zero vector for almost every [6] $x \in X$. If the matrix $\Theta_j(x)$ is full rank, then [7]

$$\operatorname{rank} \mathcal{H}_{i,j}/R_{i,j}^\perp(x) = n, \tag{17}$$

for almost every $x \in X$.

The decomposition (16) along with the rank condition (17) allows to solve the identification problem posed. As in the linear case, since $\operatorname{rank} \mathcal{H}_{i,j}/R_{i,j}^\perp(x) = n$ and $\operatorname{Im} \mathcal{H}_{i,j}/R_{i,j}^\perp(x) = \operatorname{Im} d\mathcal{H}_i(x)$ for almost every $x \in X$, if $R_{i,j}(x)$ can be estimated from the available data, Problem 1 can be solved through a singular value decomposition and the solution of a linear system Padoan and Astolfi [2015]. Note, however, that one needs to find a full rank matrix $\Theta_j(x)$, the columns of which satisfy the conditions (15). The following theorem formalises the role of the proposed excitation notion in this context: the excitation rank condition guarantees that the matrix $\Theta_j(x)$ is full rank for some $j \in \mathbb{N}$.

Theorem 7. Consider the system (13). Define recursively the vector fields $\theta_0(\cdot), \ldots, \theta_{j-1}(\cdot)$ as in (2) and the matrix-valued function $\Theta_j(\cdot)$ as in Theorem 5. If system (13) satisfies the excitation rank condition at $x \in \gamma_+(x_0)$, then there exists $j \in \mathbb{N}$ such that the columns of the matrix $\Theta_j(x)$ are linearly independent and satisfy the conditions (15).

Proof 4. By hypothesis, the system (13) satisfies the excitation rank condition at $x \in \gamma_+(x_0)$. Hence, there exist $j \in \mathbb{N}$ and $t \in \mathbb{R}^+$ such that $\dim \operatorname{Im} X_j = n$, with $X_j(t) = [\, x(t)\, x^{(1)}(t) \ldots x^{(j-1)}(t)\,]$. The claim is now a direct consequence of Padoan and Astolfi [2015, Lemma 3].

6. CONCLUSION AND FUTURE RESEARCH DIRECTIONS

The property of persistence of excitation for signals generated by time-invariant, continuous-time, autonomous linear and nonlinear systems has been studied. After having discussed the system-theoretic interpretation of the proposed notion in terms of a geometric condition, the result has been used to address issues arising in system identification. In order to illustrate notions and tools required to develop our approach, simple examples have also been provided.

We envisage that the notions of excitation space and excitation distribution have a fundamental role in solving specific issues arising in many control problems dealing with input-output data. This is, for instance, immediately evident in the problem of data-driven model reduction by moment matching Scarciotti and Astolfi [2015]. More precisely, consider a linear, single-input, single-output, continuous-time, system described by the equations

$$\dot{x} = Fx + Gu, \qquad y = Hx, \tag{18}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times 1}$ and $H \in \mathbb{R}^{1\times n}$. Let $W(s) = H(sI-F)^{-1}G$ be the associated transfer function and assume that (18) is minimal, *i.e.* controllable and observable.

Definition 8. Let $s_i \in \mathbb{C}$, with $s_i \notin \lambda(F)$. The 0*-moment of system (18) at* s_i is the complex number $\eta_0(s_i) = W(s_i)$. The k*-moment of system (18) at* s_i is the complex number $\eta_k(s_i) = \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}W(s)\right]_{s=s_i}$, with $k \geq 1$ integer.

In Astolfi [2010] it has been noted that the moments of system (18) are in one-to-one relation with the well-defined steady-state response of the output of a particular interconnected system. This interpretation of the notion

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[4] As in Isidori [1995, Chapter 1], $L_f h(\cdot)$ denotes the Lie derivative of the function $h(\cdot)$ along the vector field $f(\cdot)$. $L_f^k h(\cdot)$, with $k \in \mathbb{N}$, is defined recursively as $L_f^{k+1}h(\cdot) = L_f(L_f^k h)(\cdot)$, with $L_f^0 h(\cdot) = h(\cdot)$. $[f,\, g](\cdot)$ denotes the Lie bracket of the vector fields $f(\cdot)$ and $g(\cdot)$.

[5] $d\mathcal{H}(\cdot)$ denotes the differential of the mapping $\mathcal{H}(\cdot)$, as defined in Isidori [1995].

[6] A property holds almost everywhere if the set where the property does not hold has a Lebesgue measure equal to zero.

[7] Following Padoan and Astolfi [2015], $A/B^\perp$ denotes the projection of the row space of the matrix $A \in \mathbb{R}^{p\times j}$ onto the orthogonal complement of the row space of the matrix $B \in \mathbb{R}^{q\times j}$, *i.e.* $A/B^\perp = A(I - B'(BB')^\dagger B)$, with $(BB')^\dagger$ the Moore-Penrose inverse of the matrix BB'.

of moment relies upon the center manifold theory and it has the advantage that it can be extended to nonlinear, possibly time-delay, systems Scarciotti and Astolfi [2016].

Theorem 8. Astolfi [2010] Consider system (18) and suppose $s_i \notin \lambda(F) \subset \mathbb{C}^-$, for $i = 1, \dots, \eta$. Let $S \in \mathbb{R}^{\nu\times\nu}$ be any non-derogatory matrix with characteristic polynomial $p(s) = \prod_{i=1}^{\eta}(s - s_i)^{k_i}$, where $\nu = \sum_{i=1}^{\eta} k_i$. Consider the interconnection of system (18) with the system

$$\dot{\omega} = S\omega, \qquad u = L\omega, \tag{19}$$

with L such that the pair (L, S) is observable. Then there exists a one-to-one relation between the moments $\eta_0(s_1)$, $\dots$, $\eta_{k_1-1}(s_1)$, $\dots$, $\eta_0(s_\eta)$, $\dots$, $\eta_{k_\eta-1}(s_\eta)$ and the steady-state response of the output y of such interconnected system, namely $H\Pi\omega$, with $\Pi \in \mathbb{R}^{n\times\nu}$ the unique solution of

$$F\Pi + GL = \Pi S.$$

Since the moments are in one-to-one relation with the output of a system, it is natural to investigate the possibility of determining the moments of the system, *i.e.* $H\Pi$, from input-output data without any knowledge of F, G, H and $x(0)$. This problem has been solved in Scarciotti and Astolfi [2015]. The proposed solution relies on the following assumption.

Assumption 1. Scarciotti and Astolfi [2015] Let $T_k^r = \{t_{k-r+1}, \dots, t_{k-1}, t_k\}$ with $0 \le t_0 < t_1 < \dots < t_{k-r} < \dots < t_k < \dots < t_q$, with $r > 0$ and $q \ge r$. The elements of T_k^ν are such that $\dim \operatorname{Im} [\, \omega(t_{k-\nu+1}) \; \dots \; \omega(t_k) \,] = \nu$ for all k.

Exploiting the notion of excitation distribution it follows that Assumption 1 can be always satisfied when the excitation rank condition holds for the initial state of system (19).

Proposition 2. Assume $\lambda(F) \cap \lambda(S) = \emptyset$ and that $\omega(0) = \omega_0 \in P = \{\omega \in W : \dim(E(\omega)) = \nu\}$. Then it is always possible to select the elements of T_k^ν such that Assumption 1 holds and, as a consequence, Π is the unique solution of the system

$$F\bar{\Pi}(\omega(t_{k-\nu+1})) - \bar{\Pi}(\omega(t_{k-\nu+1}))S + GL\omega(t_{k-\nu+1}) = 0,$$
$$\vdots$$
$$F\bar{\Pi}(\omega(t_k)) - \bar{\Pi}(\omega(t_k))S + GL\omega(t_k) = 0,$$

where $\bar{\Pi}(\omega) = \Pi\omega$.

In light of these results, we expect that the notions of excitation space and excitation distribution play an important role, not only in solving the problem of data-driven model reduction for linear and nonlinear, possibly time-delay, systems, but also in many other data-driven problems based on the notion of steady-state, *e.g.*, in the output regulation problem. These applications are, among other topics, the subject of current research.

The analysis of the interplay between the stability of specific classes of nonlinear systems and the proposed notions of persistence of excitation is another important direction for future research. As well known, different notions of persistence of excitation may be used to establish global uniform asymptotic stability results for general classes of time-varying nonlinear systems (see, *e.g.*, in Morgan and Narendra [1977], Panteley et al. [2001], Loría et al. [2005]). Consequently, with the geometric characterisation of persistence of excitation provided in the present paper, one may extend these stability results using geometric conditions which may be checked *a priori*, thus bypassing the numerical integration of (possibly nonlinear) ordinary differential equations.

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