

A FROBENIUS THEOREM FOR CONTINUOUS DISTRIBUTIONS IN DIMENSION THREE

STEFANO LUZZATTO, SINA TURELI, AND KHADIM WAR

ABSTRACT. We formulate a notion of *(uniform) asymptotic involutivity* and show that it implies (unique) integrability of two-dimensional continuous distributions in dimension three. This generalizes a classical theorem of Frobenius Theorem which says that an involutive C^1 distribution is uniquely integrable.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let M be a smooth Riemannian manifold. A k -dimensional distribution Δ on M is a choice of k -dimensional linear subspaces $\Delta_p \subset T_p M$ at each point $p \in M$. A (local) integral manifold N of Δ is a submanifold $N \subset M$ such that $T_p N = \Delta_p$ at each point $p \in N$. The distribution Δ is *integrable* if there exists an integral manifold through every point, and *uniquely integrable* if this integral manifold is unique. The integrability and unique integrability of a given distribution are classical questions which generalize the problem of existence and uniqueness of solutions for ODE's. Unlike in the case of ODE's, there are counterexamples to show that for higher-dimensional distributions smoothness plays a role but is, in general, not sufficient to guarantee integrability. Indeed, in the higher-dimensional setting, some *involutivity* conditions are required over and above some regularity conditions (which are, moreover, sometimes necessary even for the formulation of the involutivity conditions).

One of the main points of this paper is to formulate some involutivity conditions for *continuous* distributions. We will formulate these conditions in two stages: the first one more natural and the second one more general and more technical and more useful for applications. Since integrability is a local property we will work in some fixed local chart of the manifold; in particular the distribution in this local chart can always be written as the kernel of some 1-form η . If η is sufficiently regular (e.g. C^1), it admits an exterior derivative which we will denote by $d\eta$. All norms to be used below will be the norms induced by the Riemannian volume. Unless specified otherwise all norms and converging sequences refer to the C^0 topology.

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1.1. Asymptotic involutivity.

Definition 1.1. A continuous distribution $\Delta = \ker(\eta)$ is *asymptotically involutive* if there exists a sequence of C^1 differential 1-forms η_k with $\eta_k \rightarrow \eta$ such that

$$\|\eta_k \wedge d\eta_k\| e^{\|d\eta_k\|} \rightarrow 0$$

as $k \rightarrow \infty$. Δ is *uniformly asymptotically involutive* if moreover we have

$$\|\eta_k - \eta\| e^{\|d\eta_k\|} \rightarrow 0.$$

Theorem 1.2. *Let Δ be a 2-dimensional distribution on a 3-dimensional manifold. If Δ is asymptotically involutive then it is integrable. If Δ is uniformly asymptotically involutive then it is uniquely integrable.*

We recall that the classical Frobenius Theorem [11] yields unique integrability for C^1 distributions (in arbitrary dimension) under the assumption that Δ is *involutive*:

$$(1.1) \quad \eta \wedge d\eta = 0.$$

This can be seen as a special case of Theorem 1.2 by choosing $\eta_k \equiv \eta$. Other generalizations and extensions of Frobenius' Theorem exist in the literature, related both to the regularity of the distribution and to the setting of the problem, see [5, 6, 7, 8, 16, 17, 19, 20, 14, 15, 22, 24, 25, 26, 27, 29, 33, 36, 37, 38], including generalizations to Lipschitz distributions, for which condition (1.1) can be formulated almost everywhere, and the interesting, though apparently not very well known, generalization of Hartman [16, 17] to *weakly differentiable* distributions, i.e. distributions defined by a 1-form η which may not be differentiable or even Lipschitz but still admits continuous exterior derivative¹ $d\eta$ and for which, therefore, condition (1.1) can also still be formulated.

Our definition of asymptotic involutivity allows for a significant relaxation of the assumptions on the regularity of Δ and in particular *does not require* that η admit a continuous exterior derivative. Indeed, Hartman [16] showed that the existence of $d\eta$ is equivalent to the existence of a sequence of C^1 differential 1-forms η_k such that $\eta_k \rightarrow \eta$ and $d\eta_k \rightarrow d\eta$. Replacing, as we do here, the conditions $d\eta_k \rightarrow d\eta$ and $\eta \wedge d\eta = 0$ with $\|\eta_k \wedge d\eta_k\| e^{\|d\eta_k\|} \rightarrow 0$ and $\|\eta_k - \eta\| e^{\|d\eta_k\|} \rightarrow 0$ relaxes the assumption on the existence of $d\eta$.

1.2. Asymptotic involutivity on average. A key feature of the asymptotic involutivity condition is that it allows $\|d\eta_k\|$ to blow up, as should be expected to happen in the case of the approximation of distributions which are not Lipschitz, albeit at some controlled rate. Our argument however yields some rather technical, but significantly more general, conditions which relax to some extent the requirement on the rate at which $\|d\eta_k\|$ is allowed to blow up, and instead only require some control on the rate at which the “average” value of $\|d\eta_k\|$ blows up. It seems that these weaker

¹More precisely we say that η is “weakly differentiable” if there exists a differential 2-form $d\eta$ that satisfies Stokes' Formula: $\int_J \eta = \int_S d\eta$ for every piece of C^1 surface S bounded by a C^1 piecewise Jordan curve J . Note that this condition holds for example under the assumption that η is Lipschitz (and therefore differentiable almost everywhere) and is therefore strictly weaker than assuming that η is C^1 .

conditions are significantly easier to verify in applications and therefore we give here a precise formulation and statement of results in terms of these conditions.

We fix some arbitrary point $x_0 \in M$ and a local coordinate system $(x^1, x^2, x^3, \mathcal{U})$ around x_0 . We suppose we are given a continuous form η defined in \mathcal{U} and the corresponding distribution $\Delta = \ker(\eta)$, and assume without loss of generality that Δ is everywhere transversal to the coordinate axis $\partial/\partial x^3$. For any sequence of C^1 forms η_k defined in \mathcal{U} we write the corresponding exterior derivative $d\eta_k$ in coordinates as

$$d\eta_k = d\eta_{k,1}dx^1 \wedge dx^3 + d\eta_{k,2}dx^2 \wedge dx^3 + d\eta_{k,3}dx^1 \wedge dx^2$$

where $d\eta_{k,1}, d\eta_{k,2}, d\eta_{k,3}$ are C^1 functions defined in \mathcal{U} . If $\eta_k \rightarrow \eta$ then, for all k sufficiently large, the corresponding distributions $\Delta_k = \ker(\eta_k)$ are also transversal to the coordinate axis $\partial/\partial x^3$ and therefore there exist C^1 frames $\{X_k, Y_k\}$ for Δ_k in \mathcal{U} where X_k, Y_k are C^1 vector fields of the form

$$(1.2) \quad X_k = \frac{\partial}{\partial x^1} + a_k \frac{\partial}{\partial x^3}, \quad Y_k = \frac{\partial}{\partial x^2} + b_k \frac{\partial}{\partial x^3}$$

for some C^1 functions a_k, b_k . We let $e^{\tau X_k}, e^{\tau Y_k}$ denote the flows induced by the vector fields X_k, Y_k respectively. Fixing some smaller neighbourhood $\mathcal{U}' \subset \mathcal{U}$ we can choose $t_0 > 0$ such that the flow is well defined and $e^{\tau X_k}(x), e^{\tau Y_k}(x) \in \mathcal{U}$ for all $x \in \mathcal{U}'$ and $|\tau| \leq t_0$. Then, for every $x \in \mathcal{U}', |t| \leq t_0$ we define

$$\widetilde{d\eta}_{k,1}(x, t) := \int_0^t d\eta_{k,1} \circ e^{\tau X^{(k)}}(x) d\tau,$$

$$\widetilde{d\eta}_{k,2}(x, t) := \int_0^t d\eta_{k,2} \circ e^{\tau Y^{(k)}}(x) d\tau$$

and

$$\widetilde{d\eta}_k(x, t) := \max\{\widetilde{d\eta}_{k,1}(x, t), \widetilde{d\eta}_{k,2}(x, t)\}.$$

Definition 1.3. A continuous distribution $\Delta = \ker(\eta)$ is *asymptotically involutive on average* if, for every $x_0 \in M$, there exist local coordinates around x_0 and a sequence of C^1 differential 1-forms η_k with $\eta_k \rightarrow \eta$ and corresponding C^1 distributions $\Delta_k = \ker(\eta_k)$ and C^1 local frames $\{X_k, Y_k\}$, and a neighbourhood $\mathcal{U}' \subset \mathcal{U}$ such that for every $x \in \mathcal{U}'$ and every $|t| \leq t_0$

$$\|\eta_k \wedge d\eta_k\|_x e^{\widetilde{d\eta}_k(x,t)} \rightarrow 0$$

as $k \rightarrow \infty$. Δ is *uniformly asymptotically involutive on average* if, moreover, for every $x \in \mathcal{U}'$ and every $|t| \leq t_0$,

$$\|\eta_k - \eta\|_x e^{\widetilde{d\eta}_k(x,t)} \rightarrow 0$$

as $k \rightarrow \infty$.

Theorem 1.4. *Let Δ be a 2-dimensional distribution on a 3-dimensional manifold. If Δ is asymptotically involutive on average then it is integrable. If Δ is uniformly asymptotically involutive on average then it is uniquely integrable.*

Notice that $\widetilde{d\eta}_k(x, t) \leq \|d\eta_k\|$ and therefore Theorem 1.2 follows immediately from Theorem 1.4.

We conclude this section with a question motivated by the observation that in the C^1 setting the involutivity condition (1.1) is both necessary and sufficient for unique integrability. It seems natural to ask whether the same is true for uniform asymptotic involutivity on average.

Question 1. *Let Δ be a 2-dimensional continuous uniquely integrable distribution on a 3-dimensional manifold. Is Δ uniformly asymptotically involutive on average?*

1.3. Applications. We discuss here three applications of our results: to the problem of the uniqueness of solutions of ODE's, of existence and uniqueness of solutions of PDE's, and to the problem of integrability of invariant bundles in Dynamical Systems. While none of these applications perhaps has the status of a major result in itself, we believe they are good "examples" and indicate the potential applicability of our main integrability results to a wide range of problems in different areas of mathematics.

1.3.1. Uniqueness of solutions for ODE's. We consider a vector field

$$(1.3) \quad X = f(x)$$

defined in some local chart \mathcal{U} of a two-dimensional Riemannian manifold M by a non-vanishing continuous function f . By a classical result of Peano, X admits locally defined integral curves at every point in \mathcal{U} but uniqueness is not guaranteed as there exist simple counterexamples even if f is Hölder continuous. A natural question concerns the "weakest" form of continuity which guarantees uniqueness. We recall that the *modulus of continuity* of a continuous function f defined on \mathcal{U} is a continuous function $w : [0, \infty) \rightarrow [0, \infty)$ such that $w(t) \rightarrow 0$ as $t \rightarrow 0$ and, for all $x, y \in \mathcal{U}$,

$$|f(x) - f(y)| \leq w(|x - y|).$$

As a Corollary of our arguments we obtain the following result which we will prove in Section 9.1.

Theorem 1.5. *Suppose the modulus of continuity of f satisfies*

$$(1.4) \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^\epsilon \omega(t) dt \cdot \exp\left(\frac{1}{\epsilon^2} \int_0^\epsilon \omega(t) dt\right) = 0.$$

Then X has a unique local integral curve through every point in \mathcal{U} .

It would be interesting to compare our condition (1.4) with other conditions such as the classical and the well-known *Osgood* condition

$$(1.5) \quad \int_0^\epsilon \frac{1}{w(t)} dt = \infty$$

which also implies unique integrability [32]. For the moment however we have not been able to establish a relationship between the two conditions.

Question. Does either of (1.4) or (1.5) imply the other? Are there examples of functions which satisfy one and not the other?

We remark that any function which admits a modulus of continuity also admits an *increasing* modulus of continuity $\hat{w}(t) \geq \omega(t)$. Then if (1.4) holds for the modulus of continuity $\omega(t)$ it clearly holds also for the increasing modulus of continuity $\hat{\omega}(t)$ (but notice that the converse is not true). If the modulus of continuity is increasing, then (1.4) is equivalent to the following more natural-looking condition

$$(1.6) \quad \lim_{t \rightarrow 0} \omega(t) e^{\frac{\omega(t)}{t}} = 0.$$

It is easy to check that Lipschitz functions satisfy (1.6) (and therefore also (1.4)) as well as (1.5), as do some standard non-Lipschitz functions such as $w(t) = t \ln t$, $w(t) = t \ln \dots \ln t$ and more “exotic” examples such as $w(t) = t \ln^{1+t} t$, whereas functions such as $w(t) = t^\alpha$, $\alpha \in (0, 1)$ and $w(t) = t \ln^\alpha t$, $\alpha > 1$ satisfy neither our condition (1.4) nor (1.5). It seems likely that if there exists any example of a modulus of continuity which satisfies (1.4) but not (1.5) it would have to have some significant amount of oscillation which could be controlled by the integrals in (1.4) but not by the simpler condition (1.6).

1.3.2. *Pfaff Equations.* Besides the intrinsic interest of the question of integrability from a purely geometric point of view, the issue of integrability classically arises in the context of the problem of existence and uniqueness of solutions of PDE’s. Indeed, this seems to have been the main motivation of Frobenius [11], who applied previous results of Clebsch [9] and Deahna [10], see discussion in [23], to Pfaff equations, i.e. equations of the form

$$(P) \quad \begin{cases} \frac{\partial f}{\partial x}(x, y) = a(x, y, f(x, y)) \\ \frac{\partial f}{\partial y}(x, y) = b(x, y, f(x, y)) \end{cases}$$

where $a(x, y, z), b(x, y, z)$ are scalar functions defined on $\mathcal{U} = \mathcal{V} \times I \subset \mathbb{R}^3$. When $f = f(x, y)$ exists (and is unique), with the initial condition $f(x_0, y_0) = z_0$, the system (P) is said to be (*uniquely*) *integrable* at (x_0, y_0, z_0) .

The existence and uniqueness of the Pfaff system of equations clearly depends on the properties of the functions a and b . The classical Theorem of Frobenius gives some *involutivity* conditions which imply integrability if the functions a, b are C^1 . As a relatively straightforward application of our more general result, we can consider the situation where a, b are just continuous but have a particular, though not very restrictive, form. More specifically suppose that a, b have the form

$$(\tilde{P}) \quad a(x, y, z) := A(x, y)F(z) \quad \text{and} \quad b(x, y, z) := B(x, y)F(z)$$

for continuous functions $A(x, y), B(x, y), F(z)$ satisfying:

- (\tilde{P}_1) F is Lipschitz continuous
- (\tilde{P}_2) There exist sequences $A^{(k)}, B^{(k)}$ of C^1 functions such that
 - i) $A^{(k)} \rightarrow A$ and $B^{(k)} \rightarrow B$,
 - ii) $A_y^{(k)} - B_x^{(k)} \rightarrow 0$.

Note that $A_y^{(k)}, B_x^{(k)}$ denote the partial derivatives of $A^{(k)}, B^{(k)}$ with respect to x and y respectively, and that the convergence in *i*) and *ii*) of $(\tilde{\mathcal{P}}_2)$ are intended in the C^0 topology.

Theorem 1.6. *The Pfaff system (\mathcal{P}) defined by functions of the form $(\tilde{\mathcal{P}})$ satisfying $(\tilde{\mathcal{P}}_1), (\tilde{\mathcal{P}}_2)$ is uniquely integrable.*

We will prove Theorem 1.6 in Section 9.2.

Remark 1.7. We remark that condition $(\tilde{\mathcal{P}}_2)$ seems relatively abstract but it is quite easy to construct examples of continuous functions A, B which satisfy it. Suppose for example that $\tilde{A}(x), \tilde{B}(y)$ are continuous functions and that $\varphi(x, y)$ is a C^2 function, and let $A(x, y) = \tilde{A}(x) + \varphi_x(x, y)$ and $B(x, y) = \tilde{B}(y) + \varphi_y(x, y)$. Let $\tilde{A}^{(k)}, \tilde{B}^{(k)}$ be sequences of C^1 functions with $\tilde{A}^{(k)} \rightarrow \tilde{A}, \tilde{B}^{(k)} \rightarrow \tilde{B}$. Then $A^{(k)}(x, y) = \tilde{A}^{(k)}(x) + \varphi_x(x, y)$ and $B^{(k)}(x, y) = \tilde{B}^{(k)}(y) + \varphi_y(x, y)$ are C^1 functions and it follows that $A^{(k)} \rightarrow A, B^{(k)} \rightarrow B$. Moreover the partial derivatives are $A_y^{(k)} = \varphi_{xy}, B_x^{(k)} = \varphi_{yx}$ and therefore $A_y^{(k)} - B_x^{(k)} = 0$.

1.3.3. Dominated decompositions with linear growth. Continuous distributions arise naturally in Dynamical Systems as $D\varphi$ -invariant distributions for some diffeomorphism $\varphi : M \rightarrow M$. The integrability (or not) of such distributions can have significant implications for ergodic and topological properties of the dynamics generated by φ . The classical Frobenius Theorem and its various extensions have generally not been suitable for studying the integrability of such “dynamically defined” distributions which are usually given implicitly by asymptotic properties of the dynamics and therefore have low regularity. The conditions we give here, on the other hand, are naturally suited to treat these kind of distributions because they allow distributions with low regularity and also because they formulate the notion of involutivity in an asymptotic way which lends itself to be verified by sequences of dynamically defined approximations to the invariant distributions.

A first non-trivial application of the results of this paper is given in [39] for a class of C^2 diffeomorphisms $\varphi : M \rightarrow M$ of a 3-dimensional manifold which admit a *dominated splitting*: there exists a continuous $D\varphi$ -invariant tangent bundle decomposition $TM = E \oplus F$ and a Riemannian metric for which derivative restricted to the 1-dimensional distribution F is *uniformly expanding*, i.e. $\|D\varphi_x\| > 1$ for all $x \in M$, and the derivative restricted to the 2-dimensional distribution E may have a mixture of contracting, neutral, or expanding behaviour but is in any case *dominated* by the derivative restricted to F , i.e. $\|D\varphi_x(v)\| < \|D\varphi_x(w)\|$ for all $x \in M$ and all unit vectors $v \in E_x, w \in F_x$.

Dominated splittings, even on 3-dimensional manifolds, are not generally uniquely integrable [31] but the main result of [39] is the unique integrability of dominated splittings on 3-dimensional manifolds under the additional assumption that the derivative restricted to E admits *at most linear growth*, i.e. there exists a constant $C > 0$ such that $\|D\varphi_x^k v\| \leq Ck$ for all $x \in M$, all unit vectors $v \in E_x$, and all $k \in \mathbb{N}$. This result is obtained by a non-trivial argument which leads to the verification that the distribution E is

uniformly asymptotically involutive on average and therefore Theorem 1.4 can be applied, giving unique integrability.

Previous related results include unique integrability for splittings on the torus \mathbb{T}^3 which admit a strong form of domination [4] and other results which assume various, rather restrictive, geometric and topological conditions [3, 13, 28, 30]. The assumption on linear growth is a natural extension of the most classical of all integrability results in the dynamical systems setting, that of *Anosov diffeomorphisms*, where E is uniformly contracting, i.e. $\|D\varphi_x(v)\| < 1 < \|D\varphi_x(w)\|$ for all $x \in M$ and for every unit vector $v \in E_x, w \in F_x$. For Anosov diffeomorphisms the integrability of the invariant distributions can be obtained by a very powerful set of techniques which yield so-called “Stable Manifold” Theorems, which go back to Hadamard and Perron, see [21]. These techniques however generally break down in settings where the domination is weaker. The application of our Theorem 1.4, as implemented in [39], includes the setting of Anosov diffeomorphisms on 3-dimensional manifolds and thus represents a perhaps more flexible, and maybe even more powerful in some respects, alternative to the standard/classical techniques.

2. STRATEGY AND MAIN TECHNICAL STEPS

Our approach is quite geometrical and implements the simple idea that if Δ is a distribution which is “almost involutive”, it should be possible to apply a small perturbation to obtain a new distribution $\tilde{\Delta}$ which is involutive. It turns out that finding *some* perturbation to make the distribution involutive is easy, but making sure this perturbation is small is non-trivial and essentially constitutes the key estimate in our argument. We will obtain an estimate on the size of the perturbation which allows us to conclude that the asymptotic involutivity on average condition implies that Δ can be approximated by involutive distributions. This can then be shown to imply (not necessarily unique) integrability of Δ . Finally, we will use an additional argument to show that we have unique integrability with the additional assumption of *uniform* asymptotic involutivity on average.

In this section we reduce the proof of Theorem 1.4 to the proof of some more technical statements, albeit also of independent interest. Before we proceed, however, we remark that we can assume without loss of generality that the forms η_k approximating η in the definition of asymptotic involutivity are actually C^2 . Indeed, by a standard “mollification” procedure we can replace the original sequence with smoother ones which still satisfy the (uniform) asymptotic involutivity on average conditions. Thus, from now on and for the rest of the paper we assume that the approximating forms η_k are C^2 .

We will prove the following general perturbation result which does not require any involutivity or asymptotic involutivity assumptions. For two distributions Δ, Δ' defined in some local chart \mathcal{U} , we will use the notation $\angle(\Delta, \Delta')$ to denote the maximum angle between subspaces of Δ and Δ' at all points of \mathcal{U} .

Theorem 2.1. *Let Δ be a continuous 2-dimensional distribution on a 3-dimensional manifold M . Then, for every $x_0 \in M$, there exist neighborhoods*

$\mathcal{U}' \subset \mathcal{U}$ of x_0 and $\epsilon > 0$ such that if Δ_ϵ is a C^2 distribution with $\angle(\Delta_\epsilon, \Delta) \leq \epsilon$ then there exists a local frame $\{X, Y\}$ of Δ_ϵ and a C^1 vector field W such that the distribution

$$\tilde{\Delta}_\epsilon = \text{span}\{X + W, Y\}$$

is involutive. Moreover, X, Y and W can be chosen so that for every C^2 form η with $\Delta_\epsilon = \ker(\eta)$ and $\|\eta\|_x \geq 1$ for every $x \in \mathcal{U}$, we have

$$\|W\| \leq \sup_{x \in \mathcal{U}, |t| \leq t_0} \{\|\eta\|_x \|\eta \wedge d\eta\|_x e^{\tilde{d}\eta(x,t)}\}.$$

Theorem 2.1 will be proved in Sections 3-6. We remark that the condition $\|\eta\|_x \geq 1$ is not a restriction since the condition $\Delta_\epsilon = \ker(\eta)$ is preserved under multiplication of η by a scalar and therefore we can always assume without loss of generality that this lower bound holds. Its purpose is just to simplify the form of the upper bound on $\|W\|$ (where, as mentioned in the introduction, the norm $\|\cdot\|$ refers to the C^0 topology). Notice that this bound is perfectly adapted to work with the asymptotic involutivity assumption of our main theorem. Indeed, by this assumption, for sufficiently large k we have that $\Delta^{(k)}$ is close to Δ and we can apply Theorem 2.1 to get a corresponding involutive distribution $\tilde{\Delta}^{(k)}$ after a perturbation whose norm is bounded by $\|\eta\| \|\eta \wedge d\eta\| e^{\tilde{d}\eta(x,t)}$. Since $\eta_k \rightarrow \eta$ we have that $\|\eta_k\|$ is uniformly bounded, hence $\|\eta\| \|\eta \wedge d\eta\| e^{\tilde{d}\eta(x,t)} \rightarrow 0$ and therefore the sequence of perturbed involutive distributions $\tilde{\Delta}^{(k)}$ approximates the original distribution Δ . In Section 7 we will show that this implies that Δ is (weakly) integrable in the sense that it admits (not necessarily unique) local integral surfaces through every point. We formalize this statement in the following

Proposition 2.2. *Suppose there exists a sequence of involutive distributions $\tilde{\Delta}^{(k)}$ which converges to a continuous distribution Δ uniformly on some open set \mathcal{U} . Then there exists an open subset $\mathcal{V} \subset \mathcal{U}$ such that Δ is (not necessarily uniquely) integrable at every $x \in \mathcal{V}$.*

We note that the (local) convergence of a family of hyperplanes and line bundles is to be understood here in terms of the maximal angle going to zero uniformly in a given neighbourhood. To get uniqueness of these integral manifolds we will prove the following statement.

Proposition 2.3. *Let Δ be a continuous 2-dimensional distribution on a 3-dimensional manifold M . Suppose that Δ is uniformly asymptotically involutive on average. Then Δ is locally spanned by two uniquely integrable vector fields X, Y .*

The unique integrability of Δ follows from Proposition 2.3 by a simple contradiction argument: if there are two integral manifolds of Δ through a point then at least one of the vector fields X and Y does not satisfy uniqueness of solutions, thus contradicting the statement of Proposition 2.3.

We have thus reduced the proof of Theorem 1.4 to the proofs of Theorem 2.1 which will be given in Sections 3-6, Proposition 2.2 which will be given in Section 7, and Proposition 2.3, which will be given in Section 8.

3. THE PERTURBATION

We now fix once and for all an arbitrary point $x_0 \in M$. Our aim in this section is to define a neighbourhood \mathcal{U} of x_0 and a perturbation of a C^2 distribution Δ_ϵ sufficiently close to our original distribution Δ which yields a new C^1 distribution $\tilde{\Delta}_\epsilon$. In the following sections we will show that $\tilde{\Delta}_\epsilon$ satisfies the required properties for the conclusions of Theorem 1.4, in particular that it is involutive and that it is a *small* perturbation of Δ_ϵ .

First of all we fix a local chart $(x^1, x^2, x^3, \mathcal{U}_0)$ centered at x_0 . Notice that we can (and do) assume without loss of generality that Δ is everywhere transversal to the coordinate axes in \mathcal{U}_0 and that therefore this transversality also holds for Δ_ϵ if ϵ is sufficiently small. In particular this implies that we can define a local frame $\{X, Y\}$ for Δ_ϵ in \mathcal{U}_0 where X, Y are C^2 vector fields of the form

$$(3.1) \quad X = \frac{\partial}{\partial x^1} + a \frac{\partial}{\partial x^3} \quad \text{and} \quad Y = \frac{\partial}{\partial x^2} + b \frac{\partial}{\partial x^3}.$$

for suitable C^2 functions $a(x), b(x)$. Notice that the transversality condition implies that the C^0 norms of a and b are uniformly bounded below for all Δ_ϵ with ϵ sufficiently small.

Remark 3.1. The vector fields X, Y are C^2 and thus define local flows, which we will denote by e^{tX}, e^{tY} respectively, and admit unique integral curves through every point $x \in \mathcal{U}_0$, which we denote by $\mathcal{X}_x, \mathcal{Y}_x$ respectively. These integral curves will play an important role in the following construction and it will be sometimes convenient to mix the notation a little bit. For example we will refer to the “natural” parametrization of an integral curve \mathcal{Y}_x to intend parametrization by the flow so that once we specify some point $y = \mathcal{Y}_x(0)$ (which may be different from the point x which we use to specify the curve) we then have $\mathcal{Y}_x(t) := e^{tY}(y)$.

We are now ready to fix the neighbourhood \mathcal{U} in which we define the perturbation. At this stage we make certain choices motivated by the fact that the involutive distribution we are constructing is given by the span of two vector fields of the form $\{X + W, Y\}$. We could similarly obtain a pair of vector fields of the form $\{X, Y + Z\}$ since the situation is completely symmetric. We let \mathcal{S} denote the integral manifold through x_0 of the coordinate planes given by $\langle \partial/\partial x^1, \partial/\partial x^3 \rangle$ in the local chart \mathcal{U}_0 . Then the vector field Y and its unique integral curves are everywhere transversal to \mathcal{S} and indeed, by the uniform bounds on $|b(x)|$, this transversality is uniform in Δ_ϵ as long as ϵ is sufficiently small. In particular this means that we can choose a smaller neighbourhood $\mathcal{U} \subset \mathcal{U}_0$ which is “saturated” by the integral curves of Y in the sense that every point $x \in \mathcal{U}$ lies on an integral curve of Y through some point of $\mathcal{S} \cap \mathcal{U}$. Moreover, this saturation condition can be guaranteed for a fixed neighbourhood \mathcal{U} for any Δ_ϵ sufficiently close to Δ . For every $x \in \mathcal{U}$ we let \mathcal{Y}_x denote the integral curve through x of the vector field Y . We consider the natural parametrization of each integral curve \mathcal{Y}_x by fixing the initial condition $\mathcal{Y}_x(0) \in \mathcal{S}$ and then let t_x be the time such that $\mathcal{Y}_x(t_x) = x$. Notice that by choosing our neighbourhood \mathcal{U} sufficiently small, we can also assume that the integration time t_x is bounded by any a priori given arbitrarily small constant. To simplify the final expression it

will be convenient to have

$$(3.2) \quad |t_x| \leq \frac{1}{\|X\| \cdot \|Y\|} = \frac{1}{\sqrt{(1+a^2)(1+b^2)}} \leq 1.$$

This upper bound is uniform for all distributions Δ_ϵ is ϵ if sufficiently small.

We are now ready to define our perturbation. Notice first that the explicit forms of the vector fields X and Y implies that the Lie bracket $[X, Y]$ always lies in the $\partial/\partial x^3$ direction. Indeed, we can compute explicitly the Lie bracket and use it to define a function $h : \mathcal{U} \rightarrow \mathbb{R}$ by

$$(3.3) \quad [X, Y] = \left(\frac{\partial b}{\partial x^1} - \frac{\partial a}{\partial x^2} + a \frac{\partial b}{\partial x^3} - b \frac{\partial a}{\partial x^3} \right) \frac{\partial}{\partial x^3} =: h \frac{\partial}{\partial x^3}.$$

Thus h is the (signed) magnitude of the Lie bracket $[X, Y]$ (which happens to be always in the $\partial/\partial x^3$ direction). We define the function $\alpha : \mathcal{U} \rightarrow \mathbb{R}$ by

$$(3.4) \quad \alpha(x) := \int_0^{t_x} h(\mathcal{Y}_x(\tau)) \exp \left(\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s)) ds \right) d\tau.$$

At the moment the function α is just defined “out of the blue” with no immediately obvious motivation, but we will show below that it is exactly the right form for the perturbation we seek. Using this function we define the perturbed distribution by

$$(3.5) \quad \tilde{\Delta}_\epsilon := \text{span} \left\{ X + \alpha \frac{\partial}{\partial x^3}, Y \right\}$$

In Section 4 we will show that $\tilde{\Delta}_\epsilon$ is C^1 , in Section 5 that it is involutive, and in Section 6 we will show that the perturbation α satisfies the required upper bounds.

4. DIFFERENTIABILITY

In this section we prove the following

Proposition 4.1. *The function α is C^1 .*

Since X and Y are C^2 , it follows immediately from Proposition 4.1 and (3.5) that $\tilde{\Delta}_\epsilon$ is C^1 as required.

To prove Proposition 4.1, notice first that from the definition of α in (3.4) it follows immediately that α is C^1 in the direction of Y . It is therefore sufficient to prove that α is also C^1 along two other vector fields that together with Y form a coordinate system in \mathcal{U} . The existence of such a coordinate system in \mathcal{U} is guaranteed by classical results on the representation of vector fields near a regular point, see e.g. [24], however we will need here a particular choice of coordinate system, in particular one defined by Y and two additional vector fields Z, V which span the tangent space of \mathcal{S} . Therefore we give a self contained proof.

Proposition 4.2. *There are two C^1 vector fields Z and V that span the tangent space of \mathcal{S} , such that the system $\{Y, V, Z\}$ is a trivialization of the tangent bundle TM by commuting vector fields in the neighborhood \mathcal{U} , which is to say that*

$$[Y, Z] = [Y, V] = [Z, V] = 0.$$

Proof. We first recall that the neighborhood \mathcal{U} is parametrized in such a way that any point can be joined to a point of \mathcal{S} by an integral curve of Y , and so we can choose $\epsilon > 0$, and modify \mathcal{U} slightly, such that the map

$$\phi(t_1, t_2, t_3) = e^{t_1 Y} \circ e^{t_2 \frac{\partial}{\partial x^1}} \circ e^{t_3 \frac{\partial}{\partial x^3}}(x_0)$$

is a diffeomorphism from $(-\epsilon, \epsilon)^3$ to \mathcal{U} . We define

$$Z := \phi_* \frac{\partial}{\partial t_2} \quad \text{and} \quad V := \phi_* \frac{\partial}{\partial t_3}$$

where the subscript $*$, here and below, denotes the standard push-forward of vector fields. Observe that by the chain rule, for every $\bar{t} = (t_1, t_2, t_3) \in (-\epsilon, \epsilon)^3$ we have

$$\phi_* \frac{\partial}{\partial t_2}(\phi(\bar{t})) = \frac{\partial \phi}{\partial t_2}(\bar{t}) = e_*^{t_1 Y} \frac{\partial}{\partial x^1}(\phi(\bar{t}))$$

and

$$\phi_* \frac{\partial}{\partial t_3}(\phi(\bar{t})) = \frac{\partial \phi}{\partial t_3}(\bar{t}) = e_*^{t_1 Y} \frac{\partial}{\partial x^3}(\phi(\bar{t})).$$

Since the vector field Y is C^2 it follows that e_*^{tY} is C^1 which implies that the vector fields Z and V are C^1 . By the naturality of the Lie bracket and observing that $Y = \phi_* \frac{\partial}{\partial t_1}$ we have

$$[V, Y] = \left[\phi_* \frac{\partial}{\partial t_3}, \phi_* \frac{\partial}{\partial t_1} \right] = \phi_* \left[\frac{\partial}{\partial t_3}, \frac{\partial}{\partial t_1} \right] = 0$$

and similarly $[Z, Y] = [Z, V] = 0$. This shows that the vector fields commute. Now we are only left to prove that \mathcal{S} is spanned by V and Z . Since \mathcal{S} is by definition the integral surface of the local coordinates $\partial/\partial x^1$ and $\partial/\partial x^3$, it is sufficient to show that V and Z span this plane. We will show this by computing explicit formulas for the vector fields. By standard calculus for vector fields on manifolds [1, 12], for any $x \in \mathcal{U}$, we have that

$$\frac{d}{dt} \left(e_*^{tY} \frac{\partial}{\partial x^3} \Big|_x \right) = e_*^{tY} \left[\frac{\partial}{\partial x^3}, Y \right] \Big|_x = \frac{\partial b}{\partial x^3} \circ e^{-tY}(x) \cdot e_*^{tY} \frac{\partial}{\partial x^3} \Big|_x.$$

Integrating both side gives

$$e_*^{tY} \frac{\partial}{\partial x^3} \Big|_x = \exp \left(\int_0^t \frac{\partial b}{\partial x^3} \circ e^{-\tau Y}(x) d\tau \right) \frac{\partial}{\partial x^3} \Big|_x$$

which shows that V always lies in the direction $\partial/\partial x^3$. By the same calculations we also get

$$e_*^{tY} \frac{\partial}{\partial x^1} \Big|_x = \frac{\partial}{\partial x^1} \Big|_x + \int_0^t \frac{\partial b}{\partial x^1} \circ e^{-sY}(x) \exp \left(\int_s^t \frac{\partial b}{\partial x^3} \circ e^{-\tau Y}(x) d\tau \right) ds \frac{\partial}{\partial x^3} \Big|_x$$

which shows that Z always lies in the span of $\partial/\partial x^1$ and $\partial/\partial x^3$. Therefore we have that V and Z span the tangent space of \mathcal{S} . \square

To complete the proof of Proposition 4.1 it is sufficient to show that α is C^1 along the vector fields Z and V defined above. We will need the following simple fact which constitutes the main motivation for our specific choice of the coordinate system.

Lemma 4.3. *If x and y belong to the same integral curve of V or to the same integral curve of Z , then we have*

$$t_x = t_y.$$

Proof. Let x and y be in the same integral curve of V . Then, since $[Y, V] = 0$, it follows that $e^{t_x Y}(x)$ and $e^{t_x Y}(y)$ are in the same integral curve of V . Since $e^{t_x Y}(x) \in \mathcal{S}$ and $V \in T\mathcal{S}$ then $e^{t_x Y}(y) \in \mathcal{S}$ and it follows that $t_x = t_y$. The proof for Z is exactly the same. \square

Proof of Proposition 4.1. To show that α is differentiable along the vector fields Z and V we will show directly from first principle that for every $x \in \mathcal{U}$, the limits

$$\lim_{\delta \rightarrow 0} \frac{\alpha(x) - \alpha(e^{\delta V}(x))}{\delta} \quad \text{and} \quad \lim_{\delta \rightarrow 0} \frac{\alpha(x) - \alpha(e^{\delta Z}(x))}{\delta}$$

exist. We will prove the statement for the first limit, the second follows by exactly the same arguments. We fix some $x \in \mathcal{U}$ and for $\delta \neq 0$, by Lemma 4.3 we have $t_x = t_{e^{\delta V}(x)} =: t$. To simplify the notation in the calculations below, we shall write

$$\mathcal{B}(\delta, \tau) := \exp \left(\int_{\tau}^t \frac{\partial b}{\partial x^3}(\mathcal{Y}_{e^{\delta V}(x)}(s)) ds \right).$$

Notice that for $\delta = 0$ we have $e^{\delta V}(x) = x$ and so we have

$$\alpha(x) = \int_0^t h(\mathcal{Y}_x(\tau)) \mathcal{B}(0, \tau) d\tau.$$

Then

$$\alpha(x) - \alpha(e^{\delta V}(x)) = \int_0^t h(\mathcal{Y}_x(\tau)) \mathcal{B}(0, \tau) d\tau - \int_0^t h(\mathcal{Y}_{e^{\delta V}(x)}(\tau)) \mathcal{B}(\delta, \tau) d\tau.$$

By adding and subtracting the term

$$\int_0^t h(\mathcal{Y}_x(\tau)) \mathcal{B}(\delta, \tau) d\tau$$

to the right hand side we get

$$\begin{aligned} \alpha(x) - \alpha(e^{\delta V}(x)) &= \int_0^t h(\mathcal{Y}_x(\tau)) \mathcal{B}(0, \tau) d\tau - \int_0^t h(\mathcal{Y}_x(\tau)) \mathcal{B}(\delta, \tau) d\tau \\ &\quad + \int_0^t h(\mathcal{Y}_x(\tau)) \mathcal{B}(\delta, \tau) d\tau - \int_0^t h(\mathcal{Y}_{e^{\delta V}(x)}(\tau)) \mathcal{B}(\delta, \tau) d\tau \\ &= \int_0^t h(\mathcal{Y}_x(\tau)) [\mathcal{B}(0, \tau) - \mathcal{B}(\delta, \tau)] d\tau \\ &\quad + \int_0^t [h(\mathcal{Y}_x(\tau)) - h(\mathcal{Y}_{e^{\delta V}(x)}(\tau))] \mathcal{B}(\delta, \tau) d\tau. \end{aligned}$$

Dividing both sides by δ it is therefore sufficient to show that the limit exists for each integral on the last two lines above. For the first integral notice that $h(\mathcal{Y}_x(\tau))$ does not depend on δ and therefore it is sufficient to show that

$$\lim_{\delta \rightarrow 0} \frac{\mathcal{B}(0, \tau) - \mathcal{B}(\delta, \tau)}{\delta}$$

exists. To see this, notice first that it is equal to

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left[\exp \left(\int_{\tau}^t \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s)) ds \right) - \exp \left(\int_{\tau}^t \frac{\partial b}{\partial x^3}(\mathcal{Y}_{e^{\delta V}(x)}(s)) ds \right) \right],$$

This is by definition the directional derivative of the function

$$(4.1) \quad \exp \int_{\tau}^t \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s)) ds$$

in the direction of V . Since $\partial b / \partial x^3$ is C^1 it follows that (4.1) is also C^1 and therefore this directional derivative exists. Similarly, for the second integral above, the limit of $\mathcal{B}(\delta, \tau)$ as $\delta \rightarrow 0$ is just $\mathcal{B}(0, \tau)$ (and thus exists), and so it is sufficient to show that the limit

$$\lim_{\delta \rightarrow 0} \frac{h(\mathcal{Y}_x(\tau)) - h(\mathcal{Y}_{e^{\delta V}(x)}(\tau))}{\delta}$$

exists. Again, this is exactly the directional derivative of h in the direction of V . Since h is C^1 this derivative exists. This proves that α is C^1 . \square

5. INVOLUTIVITY

In this Section we prove

$$\left[X + \alpha \frac{\partial}{\partial x^3}, Y \right] = 0.$$

This implies involutivity as required, since the vanishing of the Lie bracket for a C^1 local frame of a C^1 distribution is well known to be equivalent to the involutivity condition $\eta \wedge d\eta = 0$ given above; this follows for example from Cartan's formula given in (6.4) below, or see any standard reference such as [24]. By the linearity of the Lie bracket we have

$$\left[X + \alpha \frac{\partial}{\partial x^3}, Y \right] = [X, Y] + \left[\alpha \frac{\partial}{\partial x^3}, Y \right]$$

and, applying the general formula $[\varphi X, \psi Y] = \varphi X(\psi)Y - \psi Y(\varphi)X + \varphi\psi[X, Y]$ for C^1 functions φ, ψ , where $X(\psi), Y(\varphi)$ denote the "directional derivatives" of the functions ψ, φ in the directions of the vector fields X, Y respectively, we get

$$\left[\alpha \frac{\partial}{\partial x^3}, Y \right] = -Y(\alpha) \frac{\partial}{\partial x^3} + \alpha \frac{\partial b}{\partial x^3} \frac{\partial}{\partial x^3}$$

Notice that this bracket lies in the x^3 direction. Substituting above and using the fact that $[X, Y] = h\partial/\partial x^3$ also lies in the x^3 direction we get

$$(5.1) \quad \left[X + \alpha \frac{\partial}{\partial x^3}, Y \right] = \left(h + \alpha \frac{\partial b}{\partial x^3} - Y(\alpha) \right) \frac{\partial}{\partial x^3}.$$

Thus the involutivity of Δ_ϵ is equivalent to the condition that the bracket on the left hand side of (5.1) is equal to 0, or equivalently that $Y(\alpha) = h + \alpha \partial b / \partial x^3$, i.e. that α is a solution to the partial differential equation

$$(5.2) \quad Y(\alpha) = h + \alpha \frac{\partial b}{\partial x^3}.$$

To see that α is a solution of (5.2) note that by the definition of α in (3.4), for any integral curve \mathcal{Y} of Y in \mathcal{U} parametrized so that $\mathcal{Y}(0) \in \mathcal{S}$ and for any $|t| \leq t_0$ we have

$$\alpha(\mathcal{Y}(t)) = \int_0^t h(\mathcal{Y}(\tau)) \exp\left(\int_\tau^t \frac{\partial b}{\partial x^3}(\mathcal{Y}(s)) ds\right) d\tau.$$

Differentiating α along \mathcal{Y} we get

$$\begin{aligned} Y(\alpha)(\mathcal{Y}(t)) &= \frac{d}{dt} \alpha(\mathcal{Y}(t)) \\ &= h(\mathcal{Y}(t)) \exp\left(\int_t^t \frac{\partial b}{\partial x^3}(\mathcal{Y}(s)) ds\right) \\ &\quad + \frac{\partial b}{\partial x^3}(\mathcal{Y}(t)) \int_0^t h(\mathcal{Y}(\tau)) \exp\left(\int_\tau^t \frac{\partial b}{\partial x^3}(\mathcal{Y}(s)) ds\right) d\tau \\ &= h(\mathcal{Y}(t)) + \frac{\partial b}{\partial x^3}(\mathcal{Y}(t)) \alpha(\mathcal{Y}(t)) \end{aligned}$$

which proves that α is the required solution of (5.2).

6. PERTURBATION BOUNDS

In this Section we prove the upper bound on the norm of α which gives the upper bound required in the statement of Theorem 2.1. Notice first of all that by the definition of the function h in (3.3) we have

$$\begin{aligned} (6.1) \quad \alpha(x) &= \int_0^{t_x} h(\mathcal{Y}_x(\tau)) \exp\left(\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s)) ds\right) d\tau \\ &\leq \int_0^{t_x} \|[X, Y](\mathcal{Y}_x(\tau))\| \exp\left(\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s)) ds\right) d\tau \end{aligned}$$

We will estimate the two terms in two Lemmas.

Lemma 6.1. $\|[X, Y](\mathcal{Y}_x(\tau))\| \leq \{\|X\|_{\mathcal{Y}_x(\tau)} \|Y\|_{\mathcal{Y}_x(\tau)} \|\eta \wedge d\eta\|_{\mathcal{Y}_x(\tau)}\}$

Lemma 6.2. $\exp\left(\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s)) ds\right) \leq \|\eta\|_{\mathcal{Y}_x(\tau)} \exp(\widetilde{d\eta}(\mathcal{Y}_x(\tau), t_x - \tau))$

Combining these two estimates, substituting into (6.1), and using the bound on t given by (3.2), we obtain

$$\begin{aligned} |\alpha(x)| &\leq t \sup_{x \in \mathcal{U}} \{\|\eta\|_x \exp(\widetilde{d\eta}(t, x)) \|X\|_x \|Y\|_x \|\eta \wedge d\eta\|_x\} \\ &\leq \sup_{x \in \mathcal{U}} \{\|\eta\|_x \|\eta \wedge d\eta\|_x \exp(\widetilde{d\eta}(t, x))\} \end{aligned}$$

which is the required bound and thus completes the proof of Theorem 2.1 modulo the proof of the two Lemmas. For the proof of both Lemmas, notice first that, since the vector fields X, Y defined in (3.1) lie in $\ker(\eta)$, any C^1 form η such that $\Delta = \ker(\eta)$ is of the form

$$(6.2) \quad \eta = c(dx^3 - adx^1 - bdx^2)$$

for some non-vanishing C^1 function $c(x)$ defined in \mathcal{U} . Notice that $\eta(\partial/\partial x^3) = c$ and therefore $c \leq \|\eta\|$ everywhere and we can even assume, up to multiplying η by a (possibly negative) scalar if necessary, that

$$(6.3) \quad 1 \leq c \leq \|\eta\|$$

We can now prove the two Lemmas.

Proof of Lemma 6.1. All the estimates below are made for a given fixed point in \mathcal{U} and so for simplicity we omit this from the notation. By the definition of h in (3.3) we have $\eta([X, Y]) = h\eta(\partial/\partial x^3) = ch$ and therefore $h = \eta([X, Y])/c$ and in particular

$$\|[X, Y]\| = \frac{|\eta([X, Y])|}{c}$$

Since $X, Y \in \ker(\eta)$, we have $\eta(X) = \eta(Y) = 0$ and the ‘‘Cartan formula’’ gives

$$(6.4) \quad d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) = -\eta([X, Y]).$$

On the other hand, we have

$$\eta \wedge d\eta \left(\frac{\partial}{\partial x^3}, X, Y \right) = \eta \left(\frac{\partial}{\partial x^3} \right) d\eta(X, Y) = cd\eta(X, Y).$$

Substituting into the equations above we then get

$$\|[X, Y]\| = \frac{|\eta([X, Y])|}{c} = \frac{|d\eta(X, Y)|}{c} = \frac{1}{c^2} \left| \eta \wedge d\eta \left(\frac{\partial}{\partial x^3}, X, Y \right) \right|$$

Using that $c > 1$ and the multilinearity of $\eta \wedge d\eta$ this gives the bound $\|[X, Y]\| \leq \|X\| \|Y\| \|\eta \wedge d\eta\|$ as required. \square

Proof of Lemma 6.2. By direct calculation we have

$$\begin{aligned} d\eta &= \left(\frac{\partial c}{\partial x^1} + a \frac{\partial c}{\partial x^3} + c \frac{\partial a}{\partial x^3} \right) dx^1 \wedge dx^3 + \left(\frac{\partial c}{\partial x^2} + b \frac{\partial c}{\partial x^3} + c \frac{\partial b}{\partial x^3} \right) dx^2 \wedge dx^3 \\ &\quad + \left(a \frac{\partial c}{\partial x^2} - b \frac{\partial c}{\partial x^1} + c \frac{\partial a}{\partial x^2} - c \frac{\partial b}{\partial x^1} \right) dx^1 \wedge dx^2 \\ &= (X(c) + c \frac{\partial a}{\partial x^3}) dx^1 \wedge dx^3 + (Y(c) + c \frac{\partial b}{\partial x^3}) dx^2 \wedge dx^3 \\ &\quad + \left(a \frac{\partial c}{\partial x^2} - b \frac{\partial c}{\partial x^1} + c \frac{\partial a}{\partial x^2} - c \frac{\partial b}{\partial x^1} \right) dx^1 \wedge dx^2 \end{aligned}$$

On the other hand we can write

$$d\eta = d\eta_1 dx^1 \wedge dx^3 + d\eta_2 dx^2 \wedge dx^3 + d\eta_3 dx^1 \wedge dx^2$$

and so, by comparing the terms of the two formulae for $d\eta$, we have

$$Y(c) = -c \frac{\partial b}{\partial x^3} + d\eta_2.$$

Dividing both sides by c gives

$$\frac{Y(c)}{c} = -\frac{\partial b}{\partial x^3} + \frac{d\eta_2}{c}.$$

Since $Y(c)$ is exactly the derivative of c along integral curves of Y , integrating along these integral curves we get

$$\log \left| \frac{c(\mathcal{Y}_x(t_x))}{c(\mathcal{Y}_x(\tau))} \right| = - \int_{\tau}^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s)) ds + \int_{\tau}^{t_x} \frac{d\eta_2(\mathcal{Y}_x(s))}{c(\mathcal{Y}_x(s))} ds$$

which implies

$$\int_{\tau}^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s)) ds = \int_{\tau}^{t_x} \frac{d\eta_2(\mathcal{Y}_x(s))}{c(\mathcal{Y}_x(s))} ds - \log \left| \frac{c(\mathcal{Y}_x(t_x))}{c(\mathcal{Y}_x(\tau))} \right|$$

hence we have

$$\exp\left(\int_{\tau}^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))d\tau\right) = \left|\frac{c(\mathcal{Y}_x(\tau))}{c(\mathcal{Y}_x(t_x))}\right| \exp\left(\int_{\tau}^{t_x} \frac{d\eta_2(\mathcal{Y}_x(s))}{c(\mathcal{Y}_x(s))}ds\right).$$

Using that $1 \leq c \leq \|\eta\|$ by (6.3) we then get

$$\exp\left(\int_{\tau}^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))d\tau\right) \leq \|\eta\|_{\mathcal{Y}_x(\tau)} \exp\left(\int_{\tau}^{t_x} d\eta_2(\mathcal{Y}_x(s))ds\right).$$

Notice that the integral on the right hand side is not exactly in the form used in the definition of $\widetilde{d\eta_2}$ in Section 1.2, where the limits in the integral go from 0 to t_x . Recalling that $\mathcal{Y}_x(s) = e^{(s-t_x)Y}(x)$, we have

$$\int_{\tau}^{t_x} d\eta_2(\mathcal{Y}_x(s))ds = \int_{\tau}^{t_x} d\eta_2 \circ e^{(s-t_x)Y}(x)ds = \int_0^{t_x-\tau} d\eta_2 \circ e^{(s-t_x)Y}(e^{\tau Y}(x))ds$$

This last integral is by definition equal to $\widetilde{d\eta_2}(\mathcal{Y}_x(\tau), t_x - \tau)$ and so substituting into the expression above this completes the proof. \square

7. CONVERGENCE

In this section we prove Proposition 2.2. We suppose throughout that we have a sequence $\tilde{\Delta}^{(k)}$ of C^1 involutive distributions converging uniformly to a continuous distribution Δ in some open set \mathcal{U} . The involutivity of the distributions $\tilde{\Delta}^{(k)}$ implies that they are uniquely integrable by the classical Frobenius Theorem, but to prove the required convergence we will need to construct these integral manifolds rather explicitly.

We assume without loss of generality that the open set \mathcal{U} is contained inside some local chart and that Δ is everywhere transversal to the coordinate axes in this local chart. By the convergence of the sequence of distributions $\tilde{\Delta}^{(k)}$ to Δ , the same transversality property holds for $\tilde{\Delta}^{(k)}$ for all sufficiently large k . This implies that each $\tilde{\Delta}^{(k)}$ admits a local frame $\{X_k, Y_k\}$ formed by C^1 vector fields of the form

$$(7.1) \quad X_k = \frac{\partial}{\partial x^1} + a_k \frac{\partial}{\partial x^3} \quad \text{and} \quad Y_k = \frac{\partial}{\partial x^2} + b_k \frac{\partial}{\partial x^3}.$$

for C^1 functions a_k, b_k . By the convergence of the sequence of distributions $\tilde{\Delta}^{(k)}$ to Δ it follows that the sequences of vector fields X_k, Y_k converge to continuous vector fields X, Y which form a continuous local frame of Δ and have the form

$$(7.2) \quad X = \frac{\partial}{\partial x^1} + a \frac{\partial}{\partial x^3} \quad \text{and} \quad Y = \frac{\partial}{\partial x^2} + b \frac{\partial}{\partial x^3}.$$

for continuous functions a, b (cf. (3.1)). Since the approximating vector fields X_k, Y_k are C^1 , their Lie bracket is well defined and their specific form implies it lies in the $\partial/\partial x^3$ direction, see (3.3), and in particular is transversal to $\tilde{\Delta}^{(k)}$. Therefore by the involutivity of $\tilde{\Delta}^{(k)}$ it follows that the vector fields commute, i.e.

$$(7.3) \quad [X_k, Y_k] = 0.$$

Notice that of course we cannot draw the same conclusion for the vector fields X, Y since they are only continuous and the Lie bracket is not defined.

We now fix an open subset $\mathcal{V} \subset \mathcal{U}$ and some sufficiently small $\epsilon > 0$ such that for any $x \in \mathcal{V}$, any $|s_1|, |s_2| \leq \epsilon$ and all sufficiently large k , we have

$$W_x^{(k)}(s_1, s_2) := e^{s_1 X_k} \circ e^{s_2 Y_k}(x) \in \mathcal{U}.$$

Notice that by (7.3) we have that

$$(7.4) \quad \begin{aligned} \frac{\partial W_x^{(k)}}{\partial s_1}(s_1, s_2) &= X_k(W_x^{(k)}(s_1, s_2)) \\ \frac{\partial W_x^{(k)}}{\partial s_2}(s_1, s_2) &= Y_k(W_x^{(k)}(s_1, s_2)). \end{aligned}$$

In particular the Jacobian of $W_x^{(k)}$ is non-zero everywhere on $[-\epsilon, \epsilon]^2$ and therefore we have a sequence of embeddings

$$W_x^{(k)} : [-\epsilon, \epsilon]^2 \rightarrow \mathcal{U}.$$

By (7.4), the tangent spaces of $W_x^{(k)}$ are exactly the hyperplanes of the distributions $\tilde{\Delta}^{(k)}$ and therefore the images of the maps $W_x^{(k)}$ are exactly the local integral manifolds of the distributions through the point x .

Lemma 7.1. *For every $x \in \mathcal{V}$, sequence $\{W_x^{(k)}\}$ is equicontinuous and equibounded*

Proof. We have that $W_x^{(k)}(0) = x$ so $W_x^{(k)}(s_1, s_2) \in U$ and therefore equiboundedness is easy. For equicontinuity note that $DW_x^{(k)}(s_1, s_2)$ is a matrix whose columns are $X_k(W_x^{(k)}(s_1, s_2))$ and $Y_k(W_x^{(k)}(s_1, s_2))$. Therefore the differential is equibounded and so $W_x^{(k)}(s_1, s_2)$ is equicontinuous. \square

By the Arzela-Ascoli Theorem there exists a continuous function

$$W_x : [-\epsilon, \epsilon]^2 \rightarrow \mathcal{U}.$$

which is the uniform limit of some subsequence of $\{W_x^{(k)}\}_{k=1}^\infty$ (which we assume, without loss of generality, to be the full sequence from now on). To complete the proof of Proposition 2.2 it is therefore sufficient to show that $W_x : [-\epsilon, \epsilon]^2 \rightarrow \mathcal{U}$ is an integral manifold of the limiting distribution Δ , i.e. that W_x is actually differentiable and that its tangent spaces coincide with the hyperplanes of Δ . Thus, letting $DW_x = DW_x(s_1, s_2)$ denote the matrix whose columns are $X(W_x(s_1, s_2))$ and $Y(W_x(s_1, s_2))$ it is sufficient to prove the following

Lemma 7.2. *W_x is a differentiable function whose derivative is DW*

Proof. Notice first of all that from (7.4) and the fact that $X_k \rightarrow X$, $Y_k \rightarrow Y$ and $W_x^{(k)}(s_1, s_2) \rightarrow W_x(s_1, s_2)$ it follows that the partial derivatives $\partial W_x^{(k)} / \partial s_i(s_1, s_2)$ converge to $X(W_x(s_1, s_2))$ and $Y(W_x(s_1, s_2))$ respectively, and therefore, the derivative $DW_x^{(k)}$ converges uniformly to DW_x . Now, for any two points $p, q \in W_x : [-\epsilon, \epsilon]^2 \rightarrow \mathcal{U}$ and a smooth curve $\gamma = \gamma(t)$ connecting p to q with $\gamma(0) = p$, $\gamma(\tau) = q$, the fundamental theorem of calculus implies that

$$W_x^{(k)}(q) = W_x^{(k)}(p) + \int_0^\tau DW_x^{(k)}(\gamma(t)) \circ \frac{d\gamma(t)}{dt} dt$$

Thus, taking limits and exchanging the limit and the integral (which can be done due to uniform convergence) we get

$$W_x(q) = W_x(p) + \int_0^\tau DW_x(\gamma(t)) \circ \frac{d\gamma(t)}{dt} dt$$

This implies that DW_x is the derivative of W_x and thus proves in particular that W_x is differentiable as required. \square

8. UNIQUENESS

In this section we will prove Proposition 2.3 and thus complete the proof of our main result, Theorem 1.4. We will first state a general uniqueness result for continuous vector fields on surfaces, Proposition 8.1 below, and then show that there exist vector fields X, Y which span Δ and satisfy the assumptions of Proposition 8.1 and are therefore uniquely integrable. The conditions for uniqueness are given here in the same spirit as the uniqueness conditions of Theorem 1.4 but for completeness and clarity we give the full details in this simpler setting.

We consider a Riemannian surface \mathcal{S} and a vector field X on \mathcal{S} which we can suppose to be given as $X = \ker(w)$ for some continuous 1-form w defined on \mathcal{S} . We can restrict our attention to a local chart \mathcal{U} with local coordinates (z^1, z^2, \mathcal{U}) and suppose without loss of generality that X is everywhere transversal to both coordinate axes $\partial/\partial z^1, \partial/\partial z^2$ in \mathcal{U} and that therefore in particular it can be written in the form

$$X = \frac{\partial}{\partial z^1} + a \frac{\partial}{\partial z^2}$$

for some non-zero continuous function $a(z)$. If w_k is a sequence of C^1 1-forms on \mathcal{U} with $w_k \rightarrow w$ then, for all sufficiently large k , the corresponding vector fields X_k will also be transversal to both axes and the corresponding vector fields $X_k = \ker(w_k)$ can also be written in the form

$$X_k = \frac{\partial}{\partial z^1} + a_k \frac{\partial}{\partial z^2}$$

for C^1 functions a_k . Choosing some smaller domain $\mathcal{U}' \subset \mathcal{U}$ there exists some $t_0 > 0$ such that the flow $e^{\tau X_k}$ is well defined for all $x \in \mathcal{U}'$ and $|\tau| \leq t_0$ and $e^{\tau X_k}(x) \in \mathcal{U}$. Notice that in this case the external derivatives dw_k of the 1-forms w_k have only one component and so, by some slight abuse of notation we can simply write $dw_k = dw_k dz^1 \wedge dz^2$. Then, for every $z \in \mathcal{U}'$ and every $|t| \leq t_0$ define

$$\widetilde{dw}_k(z, t) := \int_0^t dw_k \circ e^{\tau X_k}(z) d\tau.$$

With this notation we then have the analogue of Theorem 1.4 as follows.

Proposition 8.1. *Let $X = \ker(w)$ be a continuous vector field defined on a surface \mathcal{S} . Suppose that for every point there is a local chart \mathcal{U} and a sequence of C^1 differential 1-forms w_k on \mathcal{U} such that $w_k \rightarrow w$, the corresponding C^1 vector fields X_k , and a neighbourhood $\mathcal{U}' \subset \mathcal{U}$ such that for every $z \in \mathcal{U}'$ and every $|t| \leq t_0$*

$$(8.1) \quad \|w_k - w\|_z e^{\widetilde{dw}_k(z, t)} \rightarrow 0$$

as $k \rightarrow \infty$. Then X is uniquely integrable.

It might be interesting to know if this result also admits an analogue for vector fields on \mathbb{R} giving conditions for uniqueness in this most simple setting, and thus allowing comparison with existing results such as [2]. The argument for the proof that we give below does not admit an immediate “restriction” to the one-dimensional setting. We first show how it implies Proposition 2.3.

Proof of Proposition 2.3 assuming Proposition 8.1. Let $\Delta = \ker(\eta)$ be a uniformly asymptotically involutive on average continuous distributions as per the hypotheses of Proposition 2.3. Then, by definition, for every $x_0 \in M$ there exists local coordinates (x^1, x^2, x^3) in some local chart \mathcal{U} in which we have vector fields X_k, Y_k, X, Y as in (7.1) and (7.2). We just need to show that X and Y are uniquely integrable. We will prove unique integrability for X using Proposition 8.1, the argument for Y is completely analogous.

Notice first of all that X is contained in the surface tangent to the coordinate axes $\langle \partial/\partial x^1, \partial/\partial x^3 \rangle$. Moreover, as in (6.2) above, the explicit form of the vector fields X_k and X mean that the forms η_k and η can be written as

$$\eta_k = c_k(dx^3 - a_k dx^1 - b_k dx^2) \quad \text{and} \quad \eta = c(dx^3 - a dx^1 - b dx^2)$$

for some non-vanishing C^1 function $c_k(x)$ and continuous function $c(x)$ respectively, and that their restriction to the surface \mathcal{S} locally tangent to $\langle \partial/\partial x^1, \partial/\partial x^3 \rangle$ yields the forms

$$w_k = c_k(dx^3 - a_k dx^1) \quad \text{and} \quad w = c(dx^3 - a dx^1)$$

with the property that $X_k = \ker(w_k)$ and $X = \ker(w)$. We therefore just need to show that the forms w_k, w satisfy the assumptions of Proposition 8.1 to get unique integrability of the vector field X . The convergence is immediate since the assumption that $\eta_k \rightarrow \eta$ implies that $a_k \rightarrow a, b_k \rightarrow b$ and $c_k \rightarrow c$ and therefore in particular that $w_k \rightarrow w$ as $k \rightarrow \infty$. To show that $\|w_k - w\|_x e^{\tilde{d}w_k(x,t)} \rightarrow 0$ we have, by direct calculation,

$$\begin{aligned} d\eta_k &= \left(X_k(c_k) + c_k \frac{\partial a_k}{\partial x^3} \right) dx^1 \wedge dx^3 + \left(Y_k(c_k) + c_k \frac{\partial b_k}{\partial x^3} \right) dx^2 \wedge dx^3 \\ &\quad + \left(a \frac{\partial c_k}{\partial x^2} - b_k \frac{\partial c_k}{\partial x^1} + c_k \frac{\partial a_k}{\partial x^2} - c \frac{\partial b_k}{\partial x^1} \right) dx^1 \wedge dx^2 \end{aligned}$$

and

$$dw_k = \left(X_k(c_k) + c_k \frac{\partial a_k}{\partial x^3} \right) dx^1 \wedge dx^3.$$

Therefore $dw_k = d\eta_{k,1}$ and thus (8.1) follows immediately from the assumption that η is uniformly asymptotically involutive on average. \square

The rest of this section is devoted to the proof Proposition 8.1 which is a sort of one-dimensional version of Theorem 2.1. Indeed, in Theorem 2.1 we showed that each distribution could be perturbed to yield an involutive distribution and that the size of this perturbation could be controlled. Here we show that any 1-form η defining a vector field on a surface can be “rescaled”

to a closed 1-form defining the same vector field and that this rescaling has controlled norm.

Lemma 8.2. *Let w be C^2 differential 1-form on a surface \mathcal{S} and (z^1, z^2, \mathcal{U}) be a coordinate systems whose axes are transverse to $\ker(w)$ and let \mathcal{L} be an integral curve of $\partial/\partial z^2$. Then there is a C^1 differential 1-form \hat{w} with $\ker(\hat{w}) = \ker(w)$ and $\hat{w}(\partial/\partial z^2) = 1$ along \mathcal{L} such that for every $z \in \mathcal{U}$ we have*

$$d\hat{w} = 0 \quad \text{and} \quad \|\hat{w}\|_z \leq \sup_{|t| \leq t_0} \{e^{\tilde{d}w(z,t)}\} \|w\|_z.$$

Proof. The proof also proceeds along quite similar lines to the proof of Theorem 2.1, though the situation here is considerably simpler. By the transversality of $\ker(w)$ to the axes we have that $w = c(dz^2 - bdz^1)$ for some C^2 functions b, c and, without loss of generality, we assume that $1 \leq |c| \leq \|\eta\|$. We assume that the neighborhood \mathcal{U} is parametrized such that every point $z \in \mathcal{U}$ corresponds to a time t_z such that the integral curve \mathcal{X} of $X := \partial/\partial x^1 + b\partial/\partial x^2$ is so that $e^{-t_z X}(z) \in \mathcal{L}$. For every $z \in \mathcal{U}$ let

$$\beta(z) := \exp\left(-\int_0^{t_z} \frac{\partial b}{\partial x^2}(e^{(\tau-t_z)X}(z))d\tau\right).$$

We can now define the form

$$\hat{w} := \beta(dz^2 - bdz^1).$$

Then, by definition of \hat{w} we have $\ker(\hat{w}) = \ker(w)$ and for $z \in \mathcal{L}$ we have $t_z = 0$ which implies that the integral above vanishes and so $\beta(z) = 1$ and so $\hat{w}(\partial/\partial z^2) = 1$. Therefore we just need to show that \hat{w} is C^1 and satisfies the required bounds. From the form of w and \hat{w} we have

$$(8.2) \quad \|\hat{w}\|_z \leq |\beta(z)| \|dz^2 - bdz^1\|_z = |\beta(z)| \frac{\|w\|_z}{c(z)} = \frac{\beta(z)}{c(z)} \|w\|_z.$$

It is enough therefore to bound $\beta(z)/c(z)$. Notice first that, as can be verified in a straightforward way, the function β is the unique solution of the partial differential equation

$$X(u) = -u \frac{\partial b}{\partial z^2}$$

with boundary conditions $u = 1$ on \mathcal{L} . By the same arguments as in the proof of Proposition 4.1, we have that β is C^1 and, by direct calculation,

$$d\hat{w} = \left(X(\beta) + \beta \frac{\partial b}{\partial z^2}\right) dz^1 \wedge dz^2 = 0.$$

Thus \hat{w} is closed. Finally, to estimate the norm of \hat{w} , again by direct calculation, we have

$$dw = \left(X(c) + c \frac{\partial b}{\partial z^2}\right) dz^1 \wedge dz^2.$$

By a slight abuse of notation again we write $dw = X(c) + c\partial b/\partial z^2$. Then, dividing through by c , we have

$$\frac{X(c)}{c} + \frac{\partial b}{\partial z^2} = \frac{dw}{c}.$$

Hence for every $z \in \mathcal{U}$ and $\tau \in [0, t_z]$ we have

$$\frac{X(c)}{c} \circ e^{(\tau-t_z)X}(z) + \frac{\partial b}{\partial z^2} \circ e^{(\tau-t_z)X}(z) = \frac{dw}{c} \circ e^{(\tau-t_z)X}(z).$$

Integrating this equality along an integral curve of X we get

$$\log \left| \frac{c(z)}{c \circ e^{-t_z X}(z)} \right| - \int_0^{t_z} \frac{dw}{c} \circ e^{(\tau-t_z)X}(z) d\tau = - \int_0^{t_z} \frac{\partial b}{\partial x^2} \circ e^{(\tau-t_z)X}(z) d\tau$$

and hence, taking exponentials,

$$\begin{aligned} \exp \left(- \int_0^{t_z} \frac{\partial b}{\partial x^2} \circ e^{(\tau-t_z)X}(z) d\tau \right) &= \left| \frac{c(z)}{c \circ e^{-t_z X}(z)} \right| \exp \left(- \int_0^{t_z} \frac{dw}{c} \circ e^{(\tau-t_z)X}(z) d\tau \right) \\ &= \left| \frac{c(z)}{c \circ e^{-t_z X}(z)} \right| \exp \left(\int_0^{-t_z} \frac{dw}{c} \circ e^{\tau X}(z) d\tau \right) \end{aligned}$$

Finally, using that $1 \leq |c| \leq \|\eta\|$, this gives

$$\beta(z) \leq c(z) \exp \left(- \int_0^{t_z} \frac{dw}{c} \circ e^{(\tau-t_z)X}(z) d\tau \right) \leq c(z) \exp(\widetilde{dw}(z, -t_z)).$$

Substituting into (8.2) gives the result. \square

Proof of Proposition 8.1. We suppose by contradiction that the vector field X admits two integral curves \mathcal{X}^1 and \mathcal{X}^2 through a given point $z_0 \in \mathcal{S}$ parametrized so that $\mathcal{X}^1(0) = \mathcal{X}^2(0) = z_0$. Now let (z^1, z^2, \mathcal{U}) be the local coordinate system around z_0 given by the assumptions of the Theorem. In particular X is transverse to both coordinate axes, and in particular to $\partial/\partial z^2$ and therefore there exists an integral curve \mathcal{L} of $\partial/\partial x^2$ which joins two points $z_1 = \mathcal{X}^1(s_1)$ and $z_2 = \mathcal{X}^2(s_2)$ for some s_1, s_2 . We can suppose that \mathcal{L} is parametrized such that $\mathcal{L}(0) = z_1$ and $\mathcal{L}(t_2) = z_2$. Let Γ be the closed curve given by union of \mathcal{L} and the two integral curves of X through z_0 , let D be the region bounded by Γ , and let \hat{w}_k be the sequence of 1-forms given by Lemma 8.2. By Stokes' formula we have

$$(8.3) \quad \int_{\Gamma} \hat{w}_k = \int_D d\hat{w}_k = 0$$

and therefore

$$\int_{\Gamma} \hat{w}_k = \int_{\mathcal{X}^1} \hat{w}_k + \int_{\mathcal{X}^2} \hat{w}_k + \int_{\mathcal{L}} \hat{w}_k = 0$$

and so

$$(8.4) \quad \left| \int_{\mathcal{X}^1} \hat{w}_k + \int_{\mathcal{X}^2} \hat{w}_k \right| = \left| \int_{\mathcal{L}} \hat{w}_k \right|.$$

By Lemma 8.2 we have $\hat{w}_k(\partial/\partial z^2) = 1$ along \mathcal{L} and therefore the right hand side of (8.4) is equal to $|t_2|$ where t_2 is the "distance" between z_1 and z_2 along \mathcal{L} . By assumption $t_2 \neq 0$ (and is independent of k) but we will show that the left hand side of (8.4) tends to 0 as $k \rightarrow \infty$, thus giving rise to a contradiction as required.

To estimate the left hand side of (8.4), notice that since the curves \mathcal{X}^i are tangent to X we have

$$\int_{\mathcal{X}^i} \hat{w}_k = \int_0^{s_i} \hat{w}_k(X)(\mathcal{X}^i(t)) dt$$

Also since $\hat{w}_k(X_k) = 0$ (since $X_k = \ker(\hat{w}_k)$) we can write

$$\int_{\mathcal{X}^i} \hat{w}_k = \int_0^{s_i} \hat{w}_k(X - X_k)(\mathcal{X}^i(t)) dt$$

Now let $|t_i| \leq s_i$ be such that

$$\int_0^{s_i} \hat{w}_k(X - X_k)(\mathcal{X}^i(t)) dt = s_i \hat{w}_k(X - X_k)(\mathcal{X}^i(t_i)).$$

Then, letting $y = \mathcal{X}^i(t_i)$ we have

$$(8.5) \quad \begin{aligned} \int_{\mathcal{X}^i} \hat{w}_k &= \int_0^{s_i} \hat{w}_k(X - X_k)(\mathcal{X}^i(t)) dt \\ &\leq s_i \|\hat{w}_k\|_y \|X - X_k\|_y \leq s_i \|\hat{w}_k\|_y \|w - w_k\|_y \end{aligned}$$

By Lemma 8.2 we have

$$\|\hat{w}_k\|_y \leq \sup_{|t| \leq t_0} \{e^{\widetilde{dw}(y,t)}\} \|w\|_y$$

Substituting this into (8.5) and applying the assumptions of the Theorem we get that $\int_{\mathcal{X}^i} \hat{w}_k \rightarrow 0$ for $i = 1, 2$ and, as explained above, this leads to a contradiction and thus completes the proof. \square

9. APPLICATIONS

In this section we prove Theorem 1.5 on the uniqueness of solutions for ODE's, and Theorem 1.6 on the existence and uniqueness of solutions for a class of Pfaff PDE's.

9.1. Uniqueness of solutions for continuous ODE's. To prove Theorem 1.5 we fix once and for all some reference point $x_0 \in \mathcal{U}$ and choose a sufficiently small ball $B(x_0, r) \subset \mathcal{U}$. We will show that the vector field X is uniquely integrable at each point $x \in B(x_0, r)$. Notice first of all that, since $f(x) = (f_1(x), f_2(x))$ is non-vanishing in \mathcal{U} , we can assume, without loss of generality and passing to a smaller radius r if necessary, that there exist local coordinates where $f_1(x) > 0$ for all $x \in B(x_0, r)$. Then we can rescale the vector field X by dividing through by f_1 and define

$$Y := \frac{\partial}{\partial x_1} + g(x) \frac{\partial}{\partial x_2} \quad \text{where} \quad g(x) = \frac{f_2}{f_1}(x)$$

Note that g has the same modulus of continuity $\omega(t)$ as f and unique integrability of Y is equivalent to that of X since the two vector fields are just rescalings one of the other (and thus define the same one-dimensional distributions). It is therefore sufficient to show that Y is uniquely integrable. To do this we first show that we can approximate the function g by a family of smooth functions g^ϵ satisfying certain approximation bounds, and thus approximate the continuous vector field Y by corresponding smooth vector fields Y^ϵ . We then apply Proposition 9.1 to get unique integrability of Y .

For any small $\epsilon > 0$ let $V_{x_0, \epsilon}$ denote the Riemannian volume of the ball $B(x_0, \epsilon)$. Then, letting $w(t)$ denote the modulus of continuity of the function g above, we have the following result.

Proposition 9.1. *There exists a family of smooth functions $\{g^\epsilon\}_{\epsilon>0}$ defined on $B(x_0, r)$ and a constant $K > 0$ such that for every $\epsilon > 0$ we have*

$$|g^\epsilon - g|_\infty \leq \frac{K}{\epsilon} \int_0^\epsilon \omega(t) dt \quad \text{and} \quad \left| \frac{\partial g^\epsilon}{\partial x} \right| \leq \frac{K}{\epsilon^2} \int_0^\epsilon \omega(t) dt$$

Proof of Theorem 1.5 assuming Proposition 9.1. Notice first of all that $Y = \ker(w)$, where w is the continuous 1-form $w := dx - g(x)dt$. Then there exists a family of approximating vector fields Y^ϵ given by the smooth 1-forms $w^\epsilon := dx - g^\epsilon(x)dt$ where the smooth functions g^ϵ are given by Proposition 9.1. Notice that since w^ϵ are smooth they admit exterior derivatives dw^ϵ and $|dw^\epsilon|_\infty = |\partial g/\partial x|_\infty$. Then from Proposition 9.1 we get

$$|w^\epsilon - w|_\infty e^{\epsilon|dw^\epsilon|_\infty} \leq \frac{K}{\epsilon} \int_0^\epsilon \omega(t) dt \exp\left(\frac{K}{\epsilon} \int_0^\epsilon \omega(t) dt\right)$$

and therefore, from the assumptions of Theorem 1.5 we have that

$$|w^\epsilon - w|_\infty e^{\epsilon|dw^\epsilon|_\infty} \rightarrow 0$$

as $\epsilon \rightarrow 0$. Thus the assumptions of Proposition 8.1 are satisfied (note that in Proposition 8.1, for generality the bound is stated in terms of $\widetilde{dw}_k(z, t)$, but since $|\widetilde{dw}_k(z, t)| \leq t|dw_k|_\infty$ the above condition is sufficient) and we get the unique integrability of Y . \square

Proof of Proposition 9.1. We will use some fairly standard approximations by mollifiers. In particular the calculations below follow closely the ones in [34] but are formulated in terms of modulus of continuity rather than the Hölder norm. Let $r \in (0, 1)$ and ϕ be at the standard mollifier supported on $B(x_0, r/2)$ and, for every $\epsilon \in (0, 1)$, let

$$\phi_\epsilon := \frac{1}{V_{x_0, \epsilon r/2}} \phi\left(\frac{x}{\epsilon}\right). \quad \text{and} \quad g^\epsilon(x) := \int_{B(x_0, \epsilon)} \phi_\epsilon(y) g(x-y) dy.$$

By the well-known properties of mollifiers we have that ϕ_ϵ is supported on $B(x_0, \epsilon)$, $\int \phi_\epsilon = 1$, $|\phi_\epsilon|_\infty \leq 1/V_{x_0, \epsilon r/2}$,

$$\int_{B(x_0, \epsilon)} \frac{\partial \phi_\epsilon}{\partial x} = 0, \quad \text{and} \quad \frac{\partial \phi_\epsilon}{\partial x}(x) = \frac{1}{\epsilon V_{x_0, \epsilon r/2}} \frac{\partial \phi}{\partial x}\left(\frac{x}{\epsilon}\right).$$

Therefore, using these properties

$$|g^\epsilon(x) - g(x)| \leq \int_{B(0, \epsilon)} |\phi_\epsilon(y)| |g(x-y) - g(x)| dy \leq |\phi_\epsilon|_\infty \int_{B(x_0, \epsilon)} \omega(|y|) dy.$$

Passing to polar coordinates (r, θ) and noting that $|y| = r$ and that the volume form in polar coordinates has the form $dV = r dr d\theta$, we get

$$\begin{aligned} |g^\epsilon(x) - g(x)| &\leq |\phi_\epsilon|_\infty \int_{B(x_0, \epsilon)} \omega(|y|) dy \leq \frac{2\pi}{V_{x_0, \epsilon r/2}} \int_0^\epsilon t \omega(t) dt \\ &\lesssim \frac{1}{\epsilon^2} \int_0^\epsilon t \omega(t) dt \leq \frac{1}{\epsilon} \int_0^\epsilon \omega(t) dt \end{aligned}$$

which proves the first claim. Similarly

$$\begin{aligned}
\left| \frac{\partial g^\epsilon}{\partial x}(x) \right| &= \left| \int_{B(x_0, \epsilon)} \frac{\partial \phi_\epsilon}{\partial x}(y) g(x-y) dy \right| \\
&= \left| \int_{B(x_0, \epsilon)} \frac{\partial \phi_\epsilon}{\partial x}(y) (g(x-y) - g(x)) dy \right| \\
&\leq \int_{B(x_0, \epsilon)} \left| \frac{\partial \phi_\epsilon}{\partial x}(y) \right| \omega(|y|) dy \\
&= |d\phi_\epsilon|_\infty \int_{B(x_0, \epsilon)} \omega(|y|) dy \\
&= |d\phi|_\infty \frac{2\pi}{\epsilon V_{x_0, \epsilon r/2}} \int_0^\epsilon t \omega(t) dt \\
&\lesssim \frac{1}{\epsilon^3} \int_0^\epsilon t \omega(t) dt \leq \frac{1}{\epsilon^2} \int_0^\epsilon \omega(t) dt
\end{aligned}$$

where last line is again achieved by passing to polar coordinates. This completes the proof of the proposition. \square

9.2. Pfaff's Problem. The proof of Theorem 1.6 is based on the observation that notion of integrability for the PDE (\mathcal{P}) is closely related to the geometric integrability of distributions in the following way. Let Δ be the two-dimensional distribution in \mathcal{U} spanned by the local frame $\{X, Y\}$ with

$$(9.1) \quad X := \frac{\partial}{\partial x} + a \frac{\partial}{\partial z} \quad \text{and} \quad Y := \frac{\partial}{\partial y} + b \frac{\partial}{\partial z}$$

where $a(x, y, z), b(x, y, z)$ are the functions in (\mathcal{P}) .

Proposition 9.2. (\mathcal{P}) is (uniquely) integrable iff Δ is (uniquely) integrable.

Proof. Suppose first that (\mathcal{P}) is integrable. Let $(x_0, y_0) \in \mathcal{V}$ and let f be a solution of (\mathcal{P}) with initial condition $f(x_0, y_0) = z_0$. Then the graph

$$\Gamma(f) = \{(x, y, f(x, y)), (x, y) \in \mathcal{V}\}$$

is an embedded surface in \mathbb{R}^3 and the tangent space at a point $(x, y, f(x, y))$ is spanned by

$$(9.2) \quad \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial y} + \frac{\partial f}{\partial y} \frac{\partial}{\partial z}.$$

Since f is a solution of (\mathcal{P}) , then (9.2) is exactly of the form (9.1) and so the graph $\Gamma(f)$ is a (unique) integral manifold of Δ . Conversely, suppose that the distribution Δ with local frame (9.1) is (uniquely) integrable. Then the integral manifold of Δ through a point $(x_0, y_0, z_0) \in \mathcal{U}$ can be realized as a graph of a (unique) function $f = f(x, y)$. The tangent spaces of this graph are by definition given by the span of (9.1) but also of (9.2) and thus f is a solution of (\mathcal{P}) which is therefore (uniquely) integrable. \square

Proof of Theorem 1.6. We translate the problem into the geometric problem of the integrability of the corresponding distribution Δ spanned by vector fields of the form (9.1). Then Δ can be written as the kernel $\Delta = \ker(\eta)$ of the 1-form

$$\eta = dz - adx - bdy.$$

We will construct a sequence η_k of C^1 with the property that $\eta_k \rightarrow \eta$, $\|d\eta_k\|$ uniformly bounded in k , and $\|\eta_k \wedge d\eta_k\| \rightarrow 0$. This implies that Δ is uniformly asymptotically involute and is thus uniquely integrable by Theorem 1.2. Let $F^{(k)}(z)$ be a sequence of C^1 functions such that $F^{(k)} \rightarrow F$ in the C^0 topology and such that the derivative $F_z^{(k)}$ is uniformly bounded in k (this is possible because F is Lipschitz continuous). Now let

$$\eta_k := dz - A^{(k)}F^{(k)}dx - B^{(k)}F^{(k)}dy.$$

Then η is a C^1 form and clearly $\eta_k \rightarrow \eta$. Moreover, by direct calculation, we have

$$\begin{aligned} d\eta_k &= -d(A^{(k)}F^{(k)}) \wedge dx - d(B^{(k)}F^{(k)}) \wedge dy \\ &= -A_y^{(k)}F^{(k)}dy \wedge dx - A^{(k)}F_z^{(k)}dz \wedge dx - B_x^{(k)}F^{(k)}dx \wedge dy - B^{(k)}F_z^{(k)}dz \wedge dy \\ &= (A_y^{(k)} - B_x^{(k)})F^{(k)}dx \wedge dy + A^{(k)}F_z^{(k)}dx \wedge dz + B^{(k)}F_z^{(k)}dy \wedge dz. \end{aligned}$$

The functions $A^{(k)}, B^{(k)}, F^{(k)}$ are converging to the corresponding functions A, B, F and are therefore uniformly bounded, the functions $F_z^{(k)}$ are uniformly bounded by construction, and $A_y^{(k)} - B_x^{(k)} \rightarrow 0$ by assumption. It follows that $\|d\eta_k\|$ is uniformly bounded in k . Finally, again by direct calculation we have

$$\begin{aligned} \eta_k \wedge d\eta_k &= [(A_y^{(k)} - B_x^{(k)})F^{(k)} - A^{(k)}F^{(k)}B^{(k)}F_z^{(k)} + B^{(k)}F^{(k)}A^{(k)}F_z^{(k)}]dx \wedge dy \wedge dz \\ &= (A_y^{(k)} - B_x^{(k)})F^{(k)}dx \wedge dy \wedge dz. \end{aligned}$$

Therefore $\|\eta_k \wedge d\eta_k\| \rightarrow 0$ and this completes the proof. \square

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Stefano Luzzatto

ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS (ICTP),
STRADA COSTIERA 11, TRIESTE, ITALY

Email address: `luzzatto@ictp.it`

Sina Tureli

ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS (ICTP),
STRADA COSTIERA 11, TRIESTE, ITALY and INTERNATIONAL SCHOOL FOR
ADVANCED STUDIES (SISSA), VIA BONOMEA 265, TRIESTE

Email address: `sinatureli@gmail.com`

Khadim War

ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS (ICTP),
STRADA COSTIERA 11, TRIESTE, ITALY and INTERNATIONAL SCHOOL FOR
ADVANCED STUDIES (SISSA), VIA BONOMEA 265, TRIESTE

Email address: `kwar@ictp.it`