A LOGARITHMIC INTERPRETATION OF EDIXHOVEN'S JUMPS FOR JACOBIANS

DENNIS ERIKSSON, LARS HALVARD HALLE, AND JOHANNES NICAISE

ABSTRACT. Let A be an abelian variety over a discretely valued field. Edixhoven has defined a filtration on the special fiber of the Néron model of A that measures the behaviour of the Néron model under tame base change. We interpret the jumps in this filtration in terms of lattices of logarithmic differential forms in the case where A is the Jacobian of a curve C, and we give a compact explicit formula for the jumps in terms of the combinatorial reduction data of C.

1. INTRODUCTION

Let R be a henselian discrete valuation ring, with quotient field K and algebraically closed residue field k. We denote by p the characteristic exponent of k. Let A be an abelian variety over K, and denote by $\mathscr{A}$ its Néron model; loosely speaking, this is the minimal smooth model of A over R. If K' is a finite separable extension of K with valuation ring R', then from the universal property that defines the Néron model we get a canonical morphism of R'-group schemes $h : \mathscr{A} \times_R R' \to \mathscr{A}'$ where $\mathscr{A}'$ denotes the Néron model of $A' = A \times_K K'$. The morphism h is not an isomorphism unless $K = K'$ or A has good reduction (i.e., $\mathscr{A}$ is an abelian R-scheme). One would like to understand the properties of the morphism h, and thus how the Néron model of A behaves under base change.

In [Ed92], Edixhoven constructed a filtration on the special fiber $\mathscr{A}_k$ of $\mathscr{A}$ by closed subgroups that measures the behaviour of the Néron model under *tame* extensions of the base field K, that is, finite extensions of degree prime to the characteristic exponent p of k. This filtration is indexed by rational numbers in the semi-open interval $[0, 1[$. The numbers in $[0, 1[$ where this filtration jumps form an interesting set of invariants of A; we will simply refer to these numbers as the jumps of A. Note that, by definition, these jumps are *real* numbers. The question whether they are always rational is one of the main open problems about these invariants.

Let K^s be a separable closure of K. By Grothendieck's Semi-Stable Reduction Theorem [SGA7-I, IX.3.6] there exists a smallest finite extension L of K in K^s such that the abelian variety $A \times_K L$ has semi-stable reduction; this means that the identity component of the special fiber of the Néron model of $A \times_K L$ is an extension of an abelian k-variety by an algebraic k-torus. We say that A is tamely ramified if L is a tame extension of K, and wildly ramified otherwise. When A is tamely ramified, Edixhoven explained how the jumps of A can be computed from the action of the Galois group $\mathrm{Gal}(L/K)$ on the Néron model of $A \times_K L$. This description implies in

particular that the jumps are rational and that the degree of L over K is the least common multiple of their denominators.

In this paper we are mainly interested in the wildly ramified case. We will study the jumps of A under the assumption that A is the Jacobian of a K-curve C. In this case, it was already proven by the second author in [Ha10b] that the jumps are rational; see also [HN14, §5.3]. In fact, he proved a much stronger result, namely that the jumps of A only depend on the combinatorial reduction data of C, and not on the characteristic of k. A result of Winters [Wi74] guarantees that we can always find a curve D over the field of complex Laurent series $\mathbb{C}((t))$ with the same reduction data, and thus the same jumps, as C. Since D is automatically tame, it follows that the jumps of C are rational. By Corollary 3.1.5 in Chapter 5 of [HN14], the least common multiple of their denominators is the so-called *stabilization index* of C, a combinatorial invariant whose definition we will recall in (2.3.3).

This result is quite powerful, but the proof of the rationality of the jumps is somewhat indirect, since one uses the combinatorial nature of the jumps and Winters's result to reduce to the case where the residue field k has characteristic zero. The aim of the present paper is to give an interpretation of the jumps of A in terms of lattices of logarithmic differential forms on R-models of C (Theorems 4.3.5 and 4.4.1), and to deduce a direct proof of their rationality (Theorem 4.4.5). Very roughly, we show that the jumps are controlled by the so-called *saturation* of a log regular model of C. The saturation is essentially equivalent to a semi-stable model in the tamely ramified case, but not in general; it can be viewed as a kind of combinatorial (characteristic-free) approximation of a semi-stable model. We expect that this approach can be generalized to arbitrary abelian K-varieties, provided that one finds a good notion of logarithmic Néron model of A. To our knowledge, such a construction has not yet appeared in the literature. Using our logarithmic interpretation of the jumps, we then establish an explicit and compact formula for the jumps of A in terms of the combinatorial reduction data of C (Theorem 5.4.1). Such a formula was not known before. It allows to prove directly that the stabilization index $e(C)$ is the least common multiple of the denominators of the jumps (Corollary 5.5.7), without relying on Winters's result. We also deduce some interesting new properties of the jumps that make it easy to compute them in concrete examples (Propositions 5.5.2 and 5.5.6). Our methods give a conceptual explanation of the role that is played by the stabilization index of C: it is the smallest possible saturation index of a log regular model of C (see Corollary 5.5.8). Our formula and the properties we deduce from it are somewhat reminiscent of jumping numbers of multiplier ideals of divisors on surfaces, but we do not know if there are any direct connections.

Notation. Throughout the paper, we will let R denote a henselian discrete valuation ring, with maximal ideal $\mathfrak{m}$, quotient field K and residue field k. We will assume that k is algebraically closed and we denote its characteristic exponent by $p \geq 1$. We set $S = \operatorname{Spec} R$. Given a finite extension K' of K, we will denote by R' the integral closure of R in K' (which is again a henselian discrete valuation ring) and we set $S' = \operatorname{Spec} R'$.

We fix a separable closure K^s of K, and we denote by K^t the tame closure of K in K^s. The integral closures of R in K^t and K^s will be denoted by R^t and R^s, respectively.

For every ring A we denote by (Sch/A) the category of A-schemes. For every A-algebra B we denote by

$$(\cdot)_B : (\operatorname{Sch}/A) \to (\operatorname{Sch}/B) : \mathscr{X} \mapsto \mathscr{X}_B = \mathscr{X} \times_A B$$

the base change functor.

Throughout the paper, C will always denote a smooth, proper and geometrically connected K-curve of genus $g > 0$. We also assume that C has a zero divisor of degree one. The Jacobian variety of C is denoted $\operatorname{Jac}(C)$.

If P is a monoid, then we write P^{gp} for its groupification and P^{sat} for its saturation.

2. A few reminders on Edixhoven's filtration and Néron models of Jacobians

2.1. **Chai's base change conductor.**

(2.1.1) Let A be an abelian K-variety of dimension g and let $\mathscr{A}$ denote its Néron model over R. Let moreover K'/K be a finite separable field extension and denote by R' the integral closure of R in K' and by $\mathfrak{m}'$ its maximal ideal. We denote by $\mathscr{A}'$ the Néron model of $A \times_K K'$ over R'. Since $\mathscr{A} \times_R R'$ is smooth and $\mathscr{A}'$ is a Néron model, there exists a unique morphism

$$h : \mathscr{A} \times_R R' \to \mathscr{A}'$$

extending the canonical isomorphism between the generic fibers. We shall refer to this morphism as the *base change morphism*.

(2.1.2) On the level of Lie algebras, the base change morphism induces an injective homomorphism

$$\operatorname{Lie}(h) : \operatorname{Lie}(\mathscr{A}) \otimes_R R' \to \operatorname{Lie}(\mathscr{A}')$$

of free R'-modules of rank $g = \dim(A)$.

Definition 2.1.3.
  (1) *The tuple of K'-elementary divisors of A is the unique non-decreasing tuple*

$$(c_1(A, K'), \ldots, c_g(A, K'))$$

  in $\mathbb{N}^g$ such that

$$\operatorname{coker}(\operatorname{Lie}(h)) \cong \bigoplus_{i=1}^{g} \left(R'/(\mathfrak{m}')^{c_i(A,K')} \right).$$

(2) *The tuple $(c_1(A), \ldots, c_g(A))$ of elementary divisors of A is defined by*

$$c_i(A) = \frac{1}{[K' : K]} c_i(A, K')$$

where K' is any finite separable extension of K such that $A \times_K K'$ has semi-stable reduction. The base change conductor $c(A)$ of A is defined by

$$c(A) = \sum_{i=1}^{g} c_i(A) = \frac{1}{[K' : K]} \mathrm{length}_{R'} \mathrm{coker}(\mathrm{Lie}(h)).$$

(2.1.4) It follows from [SGA7-I, IX.3.3] that the definition of $c_i(A)$ and $c(A)$ is independent of choice of the extension K'/K over which A has semi-stable reduction. The base change conductor $c(A)$ and the elementary divisors $c_i(A)$ were defined by Chai and Yu for algebraic tori in [CY01]. Chai generalized this definition to semi-abelian varieties in [Ch00]. The base change conductor measures the defect of semi-stable reduction of A; in particular, it vanishes if and only if A has semi-stable reduction.

2.2. **Edixhoven's filtration and the tame base change conductor.**

(2.2.1) In [Ed92], Edixhoven defined a filtration $F^\bullet \mathscr{A}_k$ on the special fiber $\mathscr{A}_k$ of the Néron model $\mathscr{A}$ of A, by closed subgroups $F^i \mathscr{A}_k$ indexed by rational numbers $i \in [0, 1[$. This filtration measures the behaviour of the Néron model under tame finite extensions of K. One can define the jumps in this filtration by looking at the indices where $F^i \mathscr{A}_k$ changes. These jumps are real numbers in $[0, 1[$, and each of them has a natural multiplicity; see [HN14, 5.1.3.6]. The number of jumps (counted with multiplicities) is equal to the dimension g of the abelian variety A.

(2.2.2) Edixhoven also proposed an alternative way to compute the jumps and their multiplicities, which we can reformulate as follows. Let $K_0 \subset K_1 \subset \ldots$ be a tower of finite extensions of K in K^t that is cofinal in the set of all finite extensions of K in K^t, ordered by inclusion. Then for every index i in $\{1, \ldots, g\}$, the sequence

$$(c_i(A, K_n)/[K_n : K])_{n \in \mathbb{N}}$$

is a non-decreasing sequence of rational numbers; one can show that the elements of this sequence are strictly bounded by 1. We denote by $j_i(A)$ the limit of this sequence. By [HN14, 5.1.3.7], the non-decreasing tuple of real numbers

$$(j_1(A), \ldots, j_g(A))$$

is precisely the tuple of jumps of A, counted with multiplicities.

(2.2.3) In [HN14, 5.1.3.6], the last two authors defined the *tame base change conductor* $c_{\text{tame}}(A)$ of A as the sum of the jumps:

$$c_{\text{tame}}(A) = \sum_{i=1}^{g} j_i(A).$$

If A is tamely ramified, then $j_i(A) = c_i(A)$ for all i and $c_{\text{tame}}(A) = c(A)$ by [HN11, 4.18], but these invariants differ in general.

(2.2.4) Whereas the elementary divisors $c_i(A)$ are rational numbers by definition, it is an open problem whether the jumps $j_i(A)$ are always rational numbers (this question was already raised by Edixhoven in [Ed92, 5.4.5]). It is not even known whether their sum $c_{\text{tame}}(A)$ is always rational. Assuming that all the jumps are rational, we define the stabilization index $e(A)$ of A as the smallest positive integer e such that $e \cdot j_i(A)$ is rational for every i. If A is tamely ramified, it follows from the equalities $c_i(A) = j_i(A)$ that the jumps are rational numbers, and one can show that $e(A)$ is equal to the degree of the minimal extension L of K in K^s such that $A \times_K L$ has semi-stable reduction [HN14, 5.1.3.12].

(2.2.5) The stabilization index of A (whose definition depends on the rationality of the jumps) seems to capture important information about the behaviour of the Néron model of A under tame base change: the idea is that the Néron model should change "as little as possible" if the degree of the base extension is prime to $e(A)$. This principle is made precise in section 1 of Part 4 in [HN14]; it is the key to understanding the so-called motivic zeta function of A.

2.3. **Regular models of curves.**

(2.3.1) Let C be a smooth, proper and geometrically connected K-curve of genus $g > 0$. We assume that C has a zero divisor of degree one. An R-model of C is a flat and proper R-scheme $\mathscr{C}$, endowed with an isomorphism of K-schemes

$$\mathscr{C} \times_R K \to C.$$

Morphisms of models are defined in the obvious way. An *ncd*-model (resp. *sncd*-model) of C is a regular R-model $\mathscr{C}$ such that the special fiber $\mathscr{C}_k$ is a divisor with normal crossings (resp. strict normal crossings, i.e. the reduced irreducible components are smooth). Since we assume that the genus of C is at least one, the curve C has a unique minimal regular model, a unique minimal *ncd*-model and a unique minimal *sncd*-model; we refer to section 2.2 of [Ni13] for a brief summary of the literature with detailed references. We say that $\mathscr{C}$ has semi-stable reduction if the special fiber of the minimal *ncd*-model of C is reduced. By the Semi-Stable Reduction Theorem for curves [DM69], there exists a smallest finite extension L of K in K^s such that $C \times_K L$ has semi-stable reduction. Moreover, since we assume that C has a zero divisor of degree one, C has semi-stable reduction if and only if its Jacobian variety has semi-stable reduction (this does not hold for genus one curves without a rational point, but such a curve never has a zero divisor of degree one).

(2.3.2) If $\mathscr{C}$ is an *sncd*-model of C with special fiber $\mathscr{C}_k = \sum_{i=1}^{r} N_i E_i$, then the *combinatorial reduction data* of $\mathscr{C}$ consist of the dual graph of $\mathscr{C}_k$, where we label the vertex corresponding to E_i with the multiplicity N_i and the genus $g(E_i)$ of E_i. We define the combinatorial reduction data of C to be those of the minimal *sncd*-model of C. The condition that C has a zero divisor of degree one is equivalent to the condition that the greatest common divisor of the multiplicities N_i is equal to one, by [Ra70, 7.1.6]. This implies that the structural morphism $\mathscr{C} \to S$ is cohomologically flat [Ra70, 8.2.1], so that $H^1(\mathscr{C}, \mathcal{O}_{\mathscr{C}})$ is a free R-module.

(2.3.3) If $\mathscr{C}$ is an *sncd*-model of C, then we say that an irreducible component E of the special fiber $\mathscr{C}_k$ is *principal* if the genus of E is at least one or E is a rational curve meeting the other components of $\mathscr{C}_k$ in at least three points. The *stabilization index* of C is defined as the least common multiple of the multiplicities of the principal components in the special fiber of the minimal *sncd*-model of C; see Definition 2.2.2 in Chapter 3 of [HN14]. Let L be the minimal finite extension of K in K^s such that $C \times_K L$ has semi-stable reduction. If this extension is tame, then its degree is equal to $e(C)$ by [Ni13, 3.4.4], but this is false in general. As we have already mentioned in the introduction, it was proven by the last two authors that the stabilization index of C is equal to that of its Jacobian in the sense of (2.2.5) (see Corollary 3.1.5 in Chapter 5 of [HN14]). We will give a more direct and conceptual argument in Corollary 5.5.7.

2.4. Néron models of Jacobians.

(2.4.1) Let $\mathscr{C}$ be a regular model of C, let $\mathscr{J}$ be the Néron model of $J = \mathrm{Jac}(C)$ and let $\mathscr{J}^0$ be its identity component. Recall that we assume that the curve C has a zero divisor of degree one. By a fundamental theorem of Raynaud [BLR90, 9.5.4], there is a natural isomorphism

$$\mathrm{Pic}^0_{\mathscr{C}/R} \cong \mathscr{J}^0.$$

Via this description of $\mathscr{J}^0$, it is possible to reduce many computations concerning Néron models to computations on regular models of curves, something which is often very useful. We will see that this is true, in particular, for the jumps in Edixhoven's filtration. To this end, we will need the following well known interpretation of the invariant differential forms on $\mathscr{J}$.

(2.4.2) Let $e_{\mathscr{J}} : S \to \mathscr{J}$ be the unit section of the Néron model of J. Then we define $\Omega(J)$ to be the module of translation-invariant relative differential forms on $\mathscr{J}$ (see [BLR90, §4.2]); thus

$$\Omega(J) = e_{\mathscr{J}}^* \Omega^1_{\mathscr{J}/S}.$$

This is an R-lattice in the g-dimensional K-vector space

$$H^0(J, \omega_{J/K}) \cong H^0(C, \omega_{C/K}).$$

Proposition 2.4.3. *Let $\mathscr{C}$ be a regular R-model of C, and denote by $\omega_{\mathscr{C}/R}$ the relative canonical sheaf. Then one has*

$$\Omega(J) = H^0(\mathscr{C}, \omega_{\mathscr{C}/R})$$

as lattices in $H^0(C, \omega_{C/K})$.

Proof. By [BLR90, 8.4.1], the canonical isomorphism

$$\mathrm{Lie}(J) \to H^1(C, \mathcal{O}_C)$$

can be extended to an isomorphism of R-modules

$$\mathrm{Lie}(\mathscr{J}) \to H^1(\mathscr{C}, \mathcal{O}_{\mathscr{C}}).$$

Since $\omega_{\mathscr{C}/R}$ is a dualizing sheaf for the structural morphism $\mathscr{C} \to S$, Grothendieck duality yields an isomorphism

$$\mathrm{Lie}(\mathscr{J})^{\vee} \cong \Omega(J) \to H^0(\mathscr{C}, \omega_{\mathscr{C}/R}).$$

$\square$

3. Log schemes and differential forms

In this section we will prove some basic results on sheaves of differentials on logarithmic schemes. The standard introduction to logarithmic geometry is [Ka89]. Our log structures are defined with respect to the étale topology, and our definition of log regularity is the one from [Ni06, 2.2]; see [Ni06, 2.3] for a comparison with Kato's definition for the Zariski topology in [Ka94].

3.1. **Base change of fs log schemes.**

(3.1.1) We will denote by S^+ the scheme $S = \mathrm{Spec}\,(R)$ with its standard log structure, that is, the log structure induced by the morphism of monoids $R \setminus \{0\} \to R$. If $\mathscr{X}$ is a flat R-scheme, we will denote by $\mathscr{X}^+$ the log scheme we obtain by endowing $\mathscr{X}$ with the divisorial log structure associated with $\mathscr{X}_k$. Similar notation will be used for $S' = \mathrm{Spec}\,(R')$ and schemes over R', if K' is a finite extension of K and R' is its valuation ring. The fiber product in the category of fine and saturated (fs) log schemes will be denoted by $\times^{fs}$. Beware that it does not commute with the forgetful functor to the category of schemes.

(3.1.2) Let $\mathscr{C}$ be an R-model of C such that $\mathscr{C}^+$ is log regular. Let K' be a finite extension of K. Set

$$\mathscr{D}^+ = \mathscr{C}^+ \times^{fs}_{S^+} (S')^+.$$

Let $\mathscr{D}$ be the underlying scheme of $\mathscr{D}^+$. Then the log structure on $\mathscr{D}^+$ is the divisorial log structure induced by $\mathscr{D}_k$, so that our notation is consistent. The scheme $\mathscr{D}$ is canonically equipped with a finite morphism

$$\mathscr{D} \to \mathscr{C} \times_R R'$$

which is an isomorphism on the generic fibers since there the log structure is trivial.

(3.1.3) If $\mathscr{C}^+$ is log smooth over S^+, then $\mathscr{D}^+$ is log smooth over $(S')^+$, because log smoothness is preserved by base change in the category of fs log schemes. Likewise, if K' is a tame extension of K, then $(S')^+$ is log étale over S^+ and $\mathscr{D}^+$ is log étale over $\mathscr{C}^+$. In both cases, $\mathscr{D}^+$ is log smooth over a log regular scheme, and thus itself log regular [Ka94, 8.2], which implies that the underlying scheme $\mathscr{D}$ is normal [Ka94, 4.1]. Therefore,

$$\mathscr{D} \to \mathscr{C} \times_R R'$$

must be a normalization map.

3.2. **Differential forms on log regular schemes.**

(3.2.1) Let $\mathscr{X}$ be a flat R-scheme of finite type such that the associated log scheme $\mathscr{X}^+$ is log regular and let x be a closed point of $\mathscr{X}_k$. The aim of this section is to construct a free resolution of the module of germs of log differential forms $\Omega^1_{\mathscr{X}^+/S^+,x}$. This result will be important for the study of logarithmic canonical sheaves in the following sections. To simplify the notations, we introduce the following abbreviations: we write $\mathcal{O}$ for the étale-local ring of $\mathscr{X}$ at x (the henselization of $\mathcal{O}_{\mathscr{X},x}$), $\mathcal{M} = \mathcal{O} \cap (\mathcal{O} \otimes_R K)^\times$ for the monoid $\mathcal{M}_{\mathscr{X}^+,x}$, $\overline{\mathcal{M}} = \mathcal{M}/\mathcal{O}^\times$ for the characteristic monoid and Ω for the $\mathcal{O}$-module $\Omega^1_{\mathscr{X}^+/S^+,x}$ (where the stalk at x is taken in the étale topology). Recall that the monoid $\overline{\mathcal{M}}$ is toric (that is, sharp, fine and saturated). We fix a section $\overline{\mathcal{M}} \to \mathcal{M}$ for the projection morphism $\mathcal{M} \to \overline{\mathcal{M}}$, so that we can view $\overline{\mathcal{M}}$ as a submonoid of the multiplicative monoid $(\mathcal{O}, \cdot)$. Denote by I the ideal of $\mathcal{O}$ generated by $\overline{\mathcal{M}} \setminus \{1\}$. By definition of log regularity, the local ring $\mathcal{O}/I$ is regular, and its dimension r is equal to the dimension of $\mathcal{O}$ minus the rank of the free abelian group $\overline{\mathcal{M}}^{\mathrm{gp}}$. We choose elements $t_1, \ldots, t_r$ in $\mathcal{O}$ whose reductions modulo I form a regular system of local parameters in $\mathcal{O}/I$, and we denote by M the $\mathcal{O}$-module

$$M = \mathcal{O} \otimes_{\mathbb{Z}} (\overline{\mathcal{M}}^{\mathrm{gp}} \oplus \mathbb{Z}^r).$$

Proposition 3.2.2. *Denote by*

$$\gamma : M \to \Omega$$

the unique morphism of $\mathcal{O}$-modules that sends $a \otimes (m \oplus n)$ to

$$a\left(\mathrm{dlog}(m) + \sum_{i=1}^r n_i dt_i\right)$$

for all a in $\mathcal{O}$, all m in $\overline{\mathcal{M}}$ and all n in $\mathbb{Z}^r$. Then γ is surjective, and its kernel is a free $\mathcal{O}$-module of rank one.

Proof. By faithful flatness of the completion morphism $\mathcal{O} \to \widehat{\mathcal{O}}$, it is enough to prove the statement after base change to $\widehat{\mathcal{O}}$. We denote by $\widehat{R}$ the completion of R and by $\mathfrak{X}$ the formal $\widehat{R}$-scheme $\mathrm{Spf}\,\widehat{\mathcal{O}}$, and we define a log structure on $\mathfrak{X}$ by means of the chart $\overline{\mathcal{M}} \to \widehat{\mathcal{O}}$. The resulting log formal scheme will be denoted by $\mathfrak{X}^+$, and we write $\mathfrak{S}^+$ for the formal scheme $\mathrm{Spf}\,\widehat{R}$ with its standard log structure. Then we can identify $\Omega \otimes_{\mathcal{O}} \widehat{\mathcal{O}}$ with the module of log differentials $\Omega^1_{\mathfrak{X}^+/\mathfrak{S}^+}$. Set $A = \widehat{R}[[\overline{\mathcal{M}}]][[T_1, \ldots, T_r]]$ and denote by $\mathfrak{Y}^+$ the log formal scheme $\mathrm{Spf}\,A$ with chart $\overline{\mathcal{M}} \to A$. By the local

description of toric singularities in [Ka94, 3.2], we know that we can view $\mathfrak{X}^+$ as a strict closed log formal subscheme of $\mathfrak{Y}^+$ defined by a principal ideal J such that t_i is the restriction of T_i to $\mathfrak{X}^+$ for every i (in the notation of [Ka94, 3.2(2)] the ring R is the ring of Witt vectors $W(k)$, but the proof can be adapted in an obvious way). Let $m_1, \ldots, m_s$ be a basis for $\overline{\mathcal{M}}^{\mathrm{gp}}$. Then $\Omega^1_{\mathfrak{Y}^+/\mathfrak{S}^+}$ is free with basis $dT_1, \ldots, dT_r, \mathrm{dlog}(m_1), \ldots, \mathrm{dlog}(m_s)$. Thus the base change of γ to $\widehat{\mathcal{O}}$ fits into the fundamental short exact sequence of $\widehat{\mathcal{O}}$-modules

$$0 \to J/J^2 \to \Omega^1_{\mathfrak{Y}^+/\mathfrak{S}^+} \otimes_A (A/J) = M \otimes_{\mathcal{O}} \widehat{\mathcal{O}} \to \Omega^1_{\mathfrak{X}^+/\mathfrak{S}^+} = \Omega \otimes_{\mathcal{O}} \widehat{\mathcal{O}} \to 0.$$

$\square$

3.3. Logarithmic canonical sheaves.

(3.3.1) Let X be a Noetherian scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Recall that the *reflexive hull* of $\mathcal{F}$ is the double dual $\mathcal{F}^{\vee\vee}$ of $\mathcal{F}$, and that $\mathcal{F}$ is called *reflexive* if the natural morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism. We recall a few basic properties of reflexive sheaves:

- If X is regular, then every reflexive rank one sheaf on X is a line bundle [Ha80, 1.9].
- If X is normal, then every reflexive sheaf $\mathcal{F}$ on X has the S_2 property [Ha94, 1.9]. This implies that, for every closed subscheme Z of X of codimension at least two, the restriction map

$$\mathcal{F}(X) \to \mathcal{F}(X \setminus Z)$$

  is an isomorphism.
- Assume that X is normal and let Z be a closed subscheme of X of codimension at least two. Let $\mathcal{G}$ be a reflexive sheaf on $U = X \setminus Z$. If we denote by i the open immersion $U \to X$, then $i_*\mathcal{G}$ is a reflexive sheaf on X, and it is the unique extension of $\mathcal{G}$ to a reflexive sheaf on X.

(3.3.2) Let $\mathscr{C}$ be a normal R-model of C. Denote by $\mathscr{U} = \mathscr{C}^{\mathrm{reg}}$ the open subscheme of regular points of $\mathscr{C}$ and by $i : \mathscr{U} \to \mathscr{C}$ the open immersion. We denote by $\omega_{\mathscr{U}/R}$ the canonical line bundle of the morphism $\mathscr{U} \to \mathrm{Spec}\,R$ and we define the canonical sheaf of the R-scheme $\mathscr{C}$ by

$$\omega_{\mathscr{C}/R} = i_*\omega_{\mathscr{U}/R}.$$

This is a reflexive rank one sheaf on $\mathscr{C}$, whose restriction to C is naturally isomorphic to the canonical line bundle $\omega_{C/K}$. If the structural morphism $f : \mathscr{C} \to \mathrm{Spec}\,R$ is l.c.i., then $\omega_{\mathscr{C}/R}$ is canonically isomorphic to the relative canonical bundle of f, i.e., the determinant of $\Omega^1_{\mathscr{C}/R}$. We say that $\mathscr{C}$ has canonical singularities if, for every morphism $g : \mathscr{C}' \to \mathscr{C}$ of R-models of C such that $\mathscr{C}'$ is regular, we have $g^*\omega_{\mathscr{C}/R} \subset \omega_{\mathscr{C}'/R}$ as subsheaves of $j_*\omega_{C/K}$, where j denotes the open immersion $C \to \mathscr{C}'$.

(3.3.3) Likewise, we define the logarithmic canonical sheaf on $\mathscr{C}$ by

$$\omega^{\log}_{\mathscr{C}/R} = i_*(\det \Omega^1_{\mathscr{U}^+/S^+}).$$

This is again a reflexive rank one sheaf on $\mathscr{C}$ whose restriction to C is naturally isomorphic to the canonical line bundle $\omega_{C/K}$. Its relation to $\omega_{\mathscr{C}/R}$ is explained in the following proposition.

Proposition 3.3.4. *Let $\mathscr{C}$ be a normal model of C, and denote by j the open immersion $j : C \to \mathscr{C}$. Then*

$$\omega^{\log}_{\mathscr{C}/R} = \omega_{\mathscr{C}/R}(\mathscr{C}_{k,\mathrm{red}} - \mathscr{C}_k)$$

as subsheaves of $j_*\omega_{C/K}$. In particular, if $\mathscr{C}_k$ is reduced, then $\omega_{\mathscr{C}/S}$ and $\omega^{\log}_{\mathscr{C}/R}$ coincide.

Proof. Since both sheaves are reflexive, it suffices to prove that they coincide on the complement of a finite set of closed points, so that we can replace $\mathscr{C}$ by a regular open subscheme $\mathscr{U}$ such that $\mathscr{U}_k$ has strict normal crossings. Then the statement follows from [KS04, 5.3.4] by taking determinants. $\square$

(3.3.5) The logarithmic canonical sheaf $\omega^{\log}_{\mathscr{C}/R}$ behaves well under fs base change, in the following sense. Assume either that $\mathscr{C}^+$ is log smooth over S^+, or that $\mathscr{C}^+$ is log regular and K' is a tame finite extension of K. If we set

$$\mathscr{D}^+ = \mathscr{C}^+ \times^{fs}_{S^+} (S')^+$$

then $\omega^{\log}_{\mathscr{D}/R'}$ is canonically isomorphic to the pullback of $\omega^{\log}_{\mathscr{C}/R}$ to $\mathscr{D}$. This follows from the fact that $\mathscr{D}$ is normal (cf. 3.1.3), $\Omega^1_{\mathscr{D}^+/(S')^+}$ is the pullback of $\Omega^1_{\mathscr{C}^+/S^+}$ and $\mathscr{D} \to \mathscr{C}$ is flat at every point of codimension ≤ 1 (recall that taking determinants commutes with flat base change).

Proposition 3.3.6. *Let $\mathscr{C}$ be an R-model of C such that $\mathscr{C}^+$ is log regular. Then the logarithmic canonical sheaf $\omega^{\log}_{\mathscr{C}/R}$ is the determinant line bundle of the perfect coherent sheaf $\Omega^1_{\mathscr{C}^+/S^+}$. If $h : \mathscr{D} \to \mathscr{C}$ is a morphism of models of C such that the morphism of log schemes $\mathscr{D}^+ \to \mathscr{C}^+$ is log étale, and if we denote by j the open immersion $\mathscr{D}_K \to \mathscr{D}$, then we have $\omega^{\log}_{\mathscr{D}/R} = h^*\omega^{\log}_{\mathscr{C}/R}$ as subsheaves of $j_*\omega_{C/K}$.*

Proof. The entire statement is local with respect to the étale topology on $\mathscr{C}$. Thus we may assume, by Proposition 3.2.2, that there exists a resolution of $\Omega^1_{\mathscr{C}^+/S^+}$ by free coherent sheaves of the form

$$(3.3.7) \qquad\qquad 0 \to \mathcal{O}_{\mathscr{C}} \to \mathcal{F} \to \Omega^1_{\mathscr{C}^+/S^+} \to 0.$$

The determinant line bundle $\det(\Omega^1_{\mathscr{C}^+/S^+})$ is equal to $\omega^{\log}_{\mathscr{C}/R}$ because these reflexive sheaves coincide on the regular locus of $\mathscr{C}$ and $\mathscr{C}$ is normal. Since h is log étale, the pullback $h^*\Omega^1_{\mathscr{C}^+/S^+}$ is isomorphic to $\Omega^1_{\mathscr{D}^+/S^+}$. Applying the right exact functor h^* to our resolution (3.3.7) yields a right exact sequence

$$\mathcal{O}_{\mathscr{D}} \to h^*\mathcal{F} \to \Omega^1_{\mathscr{D}^+/S^+} \to 0.$$

This sequence is also exact on the left because this holds on the generic fiber C of $\mathscr{D}$ and $\mathscr{D}$ is flat over R. Thus

$$\omega^{\log}_{\mathscr{D}/R} = \det(\Omega^1_{\mathscr{D}^+/S^+}) = h^*\omega^{\log}_{\mathscr{C}/R}.$$

$\square$

4. A LOGARITHMIC INTERPRETATION OF THE JUMPS IN EDIXHOVEN'S FILTRATION

4.1. **Comparing lattices over discrete valuation rings.**

(4.1.1) Let V be a vector space of dimension g over K. For every pair of R-lattices $L_0 \subset L_1$ in V, we define the tuple of *elementary divisors* of this pair as the unique non-decreasing tuple

$$(c_1(L_1/L_0), \ldots, c_g(L_1/L_0))$$

in $\mathbb{N}^g$ such that

$$L_1/L_0 \cong \oplus_{i=1}^g R/\mathfrak{m}^{c_i(L_1/L_0)}.$$

The *conductor* $c(L_1/L_0)$ of the pair of lattices is defined by

$$c(L_1/L_0) = \sum_{i=1}^g c_i(L_1/L_0) = \text{length}_R(L_1/L_0).$$

(4.1.2) Denote by R^s the valuation ring of K^s. Let $L_0 \subset L_1$ be a pair of R^s-lattices in $V \otimes_K K^s$. Then we can choose a finite extension K' of K in K^s and R'-lattices $L'_0 \subset L'_1$ in $V \otimes_K K'$ such that $L_i = L'_i \otimes_{R'} R^s$ for $i = 0, 1$. We define the elementary divisors $c_1(L_1/L_0), \ldots, c_g(L_1/L_0)$ and the conductor $c(L_1/L_0)$ of the pair (L_0, L_1) by

$$\begin{aligned}
c_i(L_1/L_0) &= \frac{1}{[K':K]} c_i(L'_1/L'_0), \\
c(L_1/L_0) &= \frac{1}{[K':K]} c(L'_1/L'_0).
\end{aligned}$$

It is straightforward to check that these definitions do not depend on the choice of K'. We will make use of the following elementary property.

Proposition 4.1.3. *Let $L_0 \subset L_1 \subset L_2$ be lattices in $V \otimes_K K^s$, and let a be a non-zero element of R^s such that the R-module L_1/L_0 is killed by a. If we denote by N the valuation of a in R^s (with respect to the unique valuation on R^s that extends the normalized discrete valuation on R), then*

$$c_i(L_2/L_1) \leq c_i(L_2/L_0) \leq c_i(L_2/L_1) + N$$

for every i in $\{1, \ldots, g\}$.

Proof. We can assume that the lattices L_0, L_1 and L_2 are defined over R, and that a is an element of R. The proof is based on the following elementary observation: for every positive integer M, the number of elementary divisors of the pair (L_0, L_2) greater than or equal to M is equal to the dimension of the k-vector space

$$V_{0,2}^M = (\mathfrak{m}^{M-1}L_2/L_0) \otimes_R k$$

and the analogous statement holds for (L_1, L_2). The projection $L_2/L_0 \to L_2/L_1$ gives rise to a surjection $V_{0,2}^M \to V_{1,2}^M$ for every M, which means that $c_i(L_2/L_1) \leq c_i(L_2/L_0)$ for all i. On the other hand, multiplication with a defines a morphism of R-modules $L_2/L_1 \to L_2/L_0$ which induces a surjection $V_{1,2}^M \to V_{0,2}^{M+N}$ for every M, so that we also have $c_i(L_2/L_0) \leq c_i(L_2/L_1) + N$ for all i. $\square$

4.2. **Lattices of differential forms.**

Proposition 4.2.1.
Let $\mathscr{C}$ be a normal model of C over R.

    (1) *If $\mathscr{C}^+$ is log regular, then the R-lattice*

$$H^0(\mathscr{C}, \omega_{\mathscr{C}/R}^{\log})$$

       in $H^0(C, \omega_{C/K})$ only depends on C, and not on the choice of $\mathscr{C}$.

    (2) *If we assume that $\mathscr{C}$ has at worst canonical singularities, then the R-lattice*

$$H^0(\mathscr{C}, \omega_{\mathscr{C}/R})$$

       in $H^0(C, \omega_{C/K})$ only depends on C, and not on the choice of $\mathscr{C}$.

Proof. (1) Let $\mathscr{C}_1$ be an R-model of C such that $\mathscr{C}_1^+$ is log regular. Let $f : \mathscr{C}_2 \to \mathscr{C}_1$ be a morphism of R-models of C that is obtained by blowing up $\mathscr{C}_1$ at a closed point x of its special fiber. Since $\mathscr{C}_1$ has only rational singularities, the scheme $\mathscr{C}_2$ is normal [Li69, 1.5]. It suffices to show that $\mathscr{C}_2^+$ is log regular and

$$H^0(\mathscr{C}_1, \omega_{\mathscr{C}_1/R}^{\log}) = H^0(\mathscr{C}_2, \omega_{\mathscr{C}_2/R}^{\log}),$$

since any pair of log regular R-models can be connected by a chain of such point blow-ups.

It is clear that

$$H^0(\mathscr{C}_2, \omega_{\mathscr{C}_2/R}^{\log}) \subset H^0(\mathscr{C}_1, \omega_{\mathscr{C}_1/R}^{\log})$$

because f is an isomorphism over $\mathscr{C}_1 \setminus \{x\}$ and $\omega_{\mathscr{C}_1/R}^{\log}$ is reflexive. Thus it is enough to prove that

(4.2.2) $$f^* \omega_{\mathscr{C}_1/R}^{\log} \subset \omega_{\mathscr{C}_2/R}^{\log}$$

as subsheaves of the pushforward of $\omega_{C/K}$ to $\mathscr{C}_2$.

First, assume that x is a regular point of the reduced special fiber $(\mathscr{C}_{1,k})_{\mathrm{red}}$ of $\mathscr{C}_1$. Then $\mathscr{C}_1$ is also regular at x by [Ni06, 5.2]. Thus $\mathscr{C}_2^+$ is log regular and (4.2.2) is a straightforward consequence of the analogous inclusion for the canonical sheaves $\omega_{\mathscr{C}_i/R}$, together with Proposition 3.3.4.

Now assume that x is a singular point of $(\mathscr{C}_{1,k})_{\mathrm{red}}$. Then the morphism $\mathscr{C}_2^+ \to \mathscr{C}_1^+$ is a log blow-up by [Ni06, 4.3], and hence log-étale, so that $\mathscr{C}_2^+$ is log regular and (4.2.2) follows from Proposition 3.3.6.

Property (2) can be proven in a similar way, replacing $\mathscr{C}_1$ by a normal model of C with at worst canonical singularities and $f : \mathscr{C}_2 \to \mathscr{C}_1$ by a resolution of singularities of $\mathscr{C}_1$; then the inclusion

$$f^* \omega_{\mathscr{C}_1/R} \subset \omega_{\mathscr{C}_2/R}$$

follows from the definition of a canonical singularity. $\square$

Definition 4.2.3. *Let $\mathscr{C}$ be a normal R-model of C. If $\mathscr{C}^+$ is log regular, then we call the R-lattice*

$$H^0(\mathscr{C}, \omega_{\mathscr{C}/R}^{\log}) \subset H^0(C, \omega_{C/K})$$

the logarithmic lattice associated with C, and we denote it by $\Omega_{\log}(C)$. If $\mathscr{C}$ has at worst canonical singularities, then we call the R-lattice

$$H^0(\mathscr{C}, \omega_{\mathscr{C}/R}) \subset H^0(C, \omega_{C/K})$$

the canonical lattice associated with C, and denote it by $\Omega_{\mathrm{can}}(C)$.

(4.2.4) We can always find an R-model $\mathscr{C}$ of C such that $\mathscr{C}^+$ is log regular (for instance, an *ncd*-model). By Proposition 4.2.1, Definition 4.2.3 does not depend on the choice of $\mathscr{C}$. Note that

$$\Omega_{\log}(C) \subset \Omega_{\mathrm{can}}(C)$$

by Proposition 3.3.4, and that they are equal when C has semi-stable reduction.

Proposition 4.2.5. *Let K' be a finite separable extension of K and denote by R' the integral closure of R in K'. Set $C' = C \times_K K'$.*

(1) *We have*

$$\Omega_{\mathrm{can}}(C) \otimes_R R' \supset \Omega_{\mathrm{can}}(C')$$

as lattices in $H^0(C', \omega_{C'/K'})$, with equality if C has semi-stable reduction.

(2) *Assume either that C has an R-model $\mathscr{C}$ such that $\mathscr{C}^+$ is log smooth over S^+, or that K' is a tame extension of K. Then we have*

$$\Omega_{\log}(C) \otimes_R R' \subset \Omega_{\log}(C')$$

as lattices in $H^0(C', \omega_{C'/K'})$.

Proof. (1) The inclusion follows from from Proposition 2.4.3. If $\mathscr{C}$ is a semi-stable R-model of C, then $\mathscr{C} \times_R R'$ is a normal R'-model of C' with canonical singularities (rational double points of type A_n) so that

$$\Omega_{\mathrm{can}}(C') = \Omega_{\mathrm{can}}(C) \otimes_R R'.$$

(2) This follows from (3.3.5). $\square$

(4.2.6) In other words, the logarithmic lattice grows under tame extensions of K, and the canonical lattice shrinks under arbitrary extensions of K. It will be convenient to summarize all of the above inclusions in the following diagram; here K' is a finite extension of K in K^t, K'' is a finite extension of K' in K^s, and R' and R'' denote their respective valuation rings.

$$\Omega_{\mathrm{can}}(C) \otimes_R R^s \supset \Omega_{\mathrm{can}}(C \times_K K') \otimes_{R'} R^s \supset \Omega_{\mathrm{can}}(C \times_K K'') \otimes_{R''} R^s$$

$$\cup \qquad\qquad\qquad \cup$$

$$\Omega_{\log}(C) \otimes_R R^s \subset \Omega_{\log}(C \otimes_K K') \otimes_{R'} R^s$$

Now we can associate with the curve C two further lattices.

(4.2.7) The *semi-stable lattice* is defined by

$$\Omega_{\mathrm{ss}}(C) = \Omega_{\mathrm{can}}(C \times_K K') \otimes_{R'} R^s \subset H^0(C, \omega_{C/K}) \otimes_K K^s,$$

where K' is any finite extension of K in K^s such that $C \times_K K'$ has semi-stable reduction. It follows from Proposition 4.2.5 that this definition does not depend on the choice of K', and that $\Omega_{\mathrm{ss}}(C)$ is the intersection of the lattices

$$\Omega_{\mathrm{can}}(C \times_K K') \otimes_{R'} R^s$$

where K' ranges over all the finite extensions of K in K^s.

(4.2.8) The *saturated lattice* $\Omega_{\mathrm{sat}}(C)$ is defined by

$$\Omega_{\mathrm{sat}}(C) = \bigcap_{K'} (\Omega_{\mathrm{can}}(C \times_K K') \otimes_{R'} R^s)$$

where K' runs through the finite extensions of K in K^t. We will prove in Theorem 4.3.5 that this is indeed a lattice in $H^0(C, \omega_{C/K}) \otimes_K K^s$.

(4.2.9) By (4.2.6), the lattices we have defined are related as follows:

$$\Omega_{\mathrm{ss}}(C) \subset \Omega_{\mathrm{sat}}(C) \subset \Omega_{\mathrm{can}}(C) \otimes_R R^s.$$

We will prove in Theorem 4.4.1 that Edixhoven's jumps and Chai's elementary divisors of the Jacobian variety $\mathrm{Jac}(C)$ measure precisely the difference between these lattices.

(4.2.10) The saturated (resp. semi-stable) lattice is invariant under base change of C to a finite extension of K in K^t (resp. in K^s). If C is tamely ramified and K' is a finite extension of K in K^t such that $C \times_K K'$ has semi-stable reduction, then

$$\Omega_{\mathrm{ss}}(C) = \Omega_{\mathrm{sat}}(C) = \Omega_{\mathrm{can}}(C \times_K K') \otimes_{R'} R^s.$$

However, if C is wildly ramified, then the intersection in the definition of the saturated lattice *never* stabilizes for large K'[1] makes the definition difficult to work with; it is not even clear from this definition that the saturated lattice is indeed a lattice. We will now give a more convenient description, which also explains our choice of terminology.

4.3. **Saturated models.**

---

[1] One way to see this is to combine Theorem 4.4.1 and Corollary 5.5.7 with [HN14, III.2.2.4]: if C is wildly ramified then the least common multiple of the denominators of the jumps of $\mathrm{Jac}(C)$ is divisible by p, which implies that the saturated lattice is not defined over a tame extension of R.

(4.3.1) We will need a few results on saturated morphisms of log schemes that have been established by T. Tsuji in an unpublished 1997 preprint; a published account of the properties we need can be found at the beginning of [Vi04, §1.3]. A morphism of saturated monoids $P \to Q$ is called saturated if, for every morphism of saturated monoids $P \to P'$, the coproduct $P' \oplus_P Q$ is still saturated. A morphism of fs log schemes $f : X \to Y$ is called saturated if for every geometric point x on X, the morphism of characteristic monoids $\overline{\mathcal{M}}_{Y,f(x)} \to \overline{\mathcal{M}}_{X,x}$ is saturated. We will only use this notion for morphisms of the form $f : \mathscr{C}^+ \to S^+$, where $\mathscr{C}$ is an R-model of C. If f is saturated, then for every finite separable extension K' of K, the fs base change $\mathscr{D}^+ = \mathscr{C}^+ \times_{S^+}^{fs} (S')^+$ coincides with the base change in the category of log schemes. In particular, the underlying scheme of $\mathscr{D}^+$ is simply the fiber product $\mathscr{C} \times_R R'$.

(4.3.2) Let $\mathscr{C}$ be an R-model of C. The *saturation index* of $f : \mathscr{C}^+ \to S^+$ is defined on page 993 of [Vi04, §1.3]. It is a positive integer m such that, for every finite separable extension K' of K of degree m, the morphism

$$\mathscr{D}^+ = \mathscr{C}^+ \times_{S^+}^{fs} (S')^+ \to (S')^+$$

is saturated; we have $m = 1$ if and only if f is itself saturated. The saturation index m is easy to compute if $\mathscr{C}^+$ is log regular, which is the only case we will need: it is precisely the least common multiple of the multiplicities of the prime components in the divisor $\mathscr{C}_k$.

Lemma 4.3.3. *Let P be a fine and saturated monoid, and let $(\mathbb{N}, +) \to P$ be a morphism of monoids. Then for every integer $d > 0$, the image of the natural morphism*

$$Q = P \oplus_{\mathbb{N}} (1/d)\mathbb{N} \to Q^{\mathrm{sat}}$$

contains $(0, 1) + Q^{\mathrm{sat}}$.

Proof. For every rational number a, we write $\lfloor a \rfloor$ for its integral part (the largest integer smaller than or equal to a) and $\{a\} = a - \lfloor a \rfloor$ for its fractional part. We denote by e the image of 1 under the morphism $\mathbb{N} \to P$.

Using criterion (iv) in [Ka89, 4.1], it is straightforward to verify that the morphism $\mathbb{N} \to (1/d)\mathbb{N}$ is integral. Thus Q is integral, and we can view it as the submonoid of the amalgamated sum $P^{\mathrm{gp}} \oplus_{\mathbb{Z}} (1/d)\mathbb{Z}$ generated by P and $(0, 1/d)$.

Let q be an element of Q^{sat}. Then we can write q as $(p_1, n_1/d) - (p_2, n_2/d)$ with p_1, p_2 in P and n_1, n_2 non-negative integers, and we know that there exists an integer $N > 0$ such that

$$N(p_1, n_1/d) - N(p_2, n_2/d)$$
$$= \quad (N(p_1 - p_2) + \lfloor N(n_1 - n_2)/d \rfloor e, \{N(n_1 - n_2)/d\})$$

lies in Q. This is only possible if

$$N(p_1 - p_2) + \lfloor N(n_1 - n_2)/d \rfloor e$$

lies in P, which implies that

$$N(p_1 - p_2 + (\lfloor (n_1 - n_2)/d \rfloor + 1)e)$$

lies in P because
$$N\lfloor n/d \rfloor + N \geq \lfloor nN/d \rfloor$$
for all integers n. Since P is saturated, it follows that
$$p_1 - p_2 + (\lfloor (n_1 - n_2)/d \rfloor + 1)e$$
belongs to P. Thus
$$\begin{aligned} q + (0,1) &= q + (e,0) \\ &= (p_1 - p_2 + (\lfloor (n_1 - n_2)/d \rfloor + 1)e, \{(n_1 - n_2)/d\}) \end{aligned}$$
lies in Q. $\qquad\square$

Lemma 4.3.4. *We have*
$$\mathfrak{m}\Omega_{\mathrm{can}}(C) \subset \Omega_{\log}(C)$$
where $\mathfrak{m}$ denotes the maximal ideal in R.

Proof. Let $\mathscr{C}$ be an *sncd*-model of C. We denote by D the divisor $\mathscr{C}_k - \mathscr{C}_{k,\mathrm{red}}$ on $\mathscr{C}$ and by i the closed immersion $D \to \mathscr{C}$. By Proposition 3.3.4, we have a short exact sequence of coherent $\mathcal{O}_{\mathscr{C}}$-modules
$$0 \to \omega_{\mathscr{C}/R}^{\log} \to \omega_{\mathscr{C}/R} \to i_* i^* \omega_{\mathscr{C}/R} \to 0.$$
Thus the cokernel of the inclusion of R-lattices
$$\Omega_{\log}(C) \to \Omega_{\mathrm{can}}(C)$$
is a submodule of $H^0(D, i^*\omega_{\mathscr{C}/R})$. It is killed by $\mathfrak{m}$, since every element of $\mathfrak{m}$ vanishes on D. $\qquad\square$

Theorem 4.3.5. *Let $\mathscr{C}$ be an R-model of C such that $\mathscr{C}^+$ is log regular. Let K' be a finite extension of K in K^s with valuation ring R' such that the log scheme*
$$\mathscr{D}^+ = \mathscr{C}^+ \times_{S^+}^{fs} (S')^+$$
is saturated over $(S')^+$, and denote by h the morphism $\mathscr{D} \to \mathscr{C}$. Then

$$(4.3.6) \qquad \Omega_{\mathrm{sat}}(C) = H^0(\mathscr{D}, h^*\omega_{\mathscr{C}/R}^{\log}) \otimes_{R'} R^s.$$

In particular, $\Omega_{\mathrm{sat}}(C)$ is a lattice in $H^0(C, \omega_{C/K}) \otimes_K K^s$.

Proof. Denote by K_0 the tame closure of K in K', by R_0 its valuation ring, and by S_0^+ the spectrum of R_0 with its standard log structure. Set $C_0 = C \times_K K_0$ and $\mathscr{C}_0^+ = \mathscr{C}^+ \times_{S^+}^{fs} S_0^+$ and denote by $\mathfrak{m}_0$ the maximal ideal of R_0.

It follows from (3.3.5) and Lemma 4.3.4 that
$$(\mathfrak{m}_0 \Omega_{\mathrm{can}}(C_0)) \otimes_{R_0} R^s \subset \Omega_{\log}(C_0) \otimes_{R_0} R^s \subset H^0(\mathscr{D}, h^*\omega_{\mathscr{C}/R}^{\log}) \otimes_{R'} R^s.$$

The right hand side of this expression does not change if we replace K' by a finite extension of K' in K^s, because $\mathscr{D}^+$ is saturated over $(S')^+$ so that fs base change coincides with base change in the category of log schemes and commutes with the forgetful functor to the category of schemes. Since we can dominate any finite extension of K in K^t by a finite extension of K' in

K^s, and the maximal ideal $\mathfrak{m}^s$ of R^s is generated by the uniformizers in the finite extensions of K in K^t, we find that

$$\mathfrak{m}^s\Omega_{\mathrm{sat}}(C) \subset H^0(\mathscr{D}, h^*\omega^{\log}_{\mathscr{C}/R}) \otimes_{R'} R^s.$$

This is only possible if

$$\Omega_{\mathrm{sat}}(C) \subset H^0(\mathscr{D}, h^*\omega^{\log}_{\mathscr{C}/R}) \otimes_{R'} R^s.$$

Now we prove the converse inclusion. We claim that

(4.3.7) $$\mathfrak{m}_0 H^0(\mathscr{D}, h^*\omega^{\log}_{\mathscr{C}/R}) \subset H^0(\mathscr{C}_0, \omega^{\log}_{\mathscr{C}_0/R_0}) \otimes_{R_0} R'.$$

This implies that

$$(\mathfrak{m}_0 H^0(\mathscr{D}, h^*\omega^{\log}_{\mathscr{C}/R})) \otimes_{R'} R^s \subset \Omega_{\mathrm{sat}}(C)$$

because we have

$$\Omega_{\log}(C \times_K K_0) \otimes_{R_0} R^s \subset \Omega_{\log}(C \times_K K_0') \otimes_{R_0'} R^s \subset \Omega_{\mathrm{can}}(C \times_K K_0') \otimes_{R_0'} R^s$$

for every finite extension K_0' of K_0 in K^t with valuation ring R_0'. As in the first part of the proof, we see by replacing K' by its finite tame extensions (and letting K_0 grow accordingly) that

$$\mathfrak{m}^s(H^0(\mathscr{D}, h^*\omega^{\log}_{\mathscr{D}/R}) \otimes_{R'} R^s) \subset \Omega_{\mathrm{sat}}(C),$$

and thus

$$H^0(\mathscr{D}, h^*\omega^{\log}_{\mathscr{D}/R}) \otimes_{R'} R^s \subset \Omega_{\mathrm{sat}}(C),$$

which is what we wanted to show.

It remains to prove our claim (4.3.7). The schemes $\mathscr{D}$ and $\mathscr{C}_0$ are related by a finite morphism

$$g: \mathscr{D} \to \mathscr{C}_0 \times_{R_0} R',$$

which is an isomorphism on the generic fibers. Moreover, $h^*\omega^{\log}_{\mathscr{C}/R}$ is isomorphic to the pullback of the line bundle $\omega^{\log}_{\mathscr{C}_0/R_0}$ to $\mathscr{D}$. Thus it suffices to show that the cokernel of the morphism

$$\mathcal{O}_{\mathscr{C}_0 \times_{R_0} R'} \to g_*\mathcal{O}_{\mathscr{D}}$$

is killed by $\mathfrak{m}_0$. This property is local with respect to the étale topology, so that we can assume that the morphism of log schemes $\mathscr{C}_0^+ \to S_0^+$ has a chart $\mathbb{N} \to P$, with P a fine and saturated monoid, sending $1 \in \mathbb{N}$ to a uniformizer π_0 in R_0. If we denote by d the degree of K' over K_0, then the morphism $(S')^+ \to S_0^+$ has a chart of the form

$$\mathbb{N} \to \frac{1}{d}\mathbb{N} \oplus \mathbb{Z}$$

sending $1 \in \mathbb{N}$ to π_0, since we can write π_0 as the d-th power of a uniformizer in R' times a unit u in R'. Thus the fiber product $\mathscr{C}_0^+ \times_{S_0^+} (S')^+$ in the category of log schemes has a chart

$$Q = (P \oplus \mathbb{Z}) \oplus_{\mathbb{N}} \frac{1}{d}\mathbb{N} \to \mathcal{O}(\mathscr{C}_0 \times_{S_0} S')$$

that sends $1 \in \mathbb{N}$ to $u^{-1}\pi_0$. The scheme $\mathscr{D}$ is given by

$$(\mathscr{C}_0 \times_{R_0} R') \times_{\mathbb{Z}[Q]} \mathbb{Z}[Q^{\mathrm{sat}}],$$

so that our claim follows from Lemma 4.3.3.   □

4.4. **Relation with Edixhoven's filtration and Chai's elementary divisors.**

Theorem 4.4.1.

(1) *The tuple of jumps in Edixhoven's filtration for* $\mathrm{Jac}(C)$ *is equal to the tuple of elementary divisors of the lattices*

$$\Omega_{\mathrm{sat}}(C) \subset \Omega_{\mathrm{can}}(C) \otimes_R R^s$$

in $H^0(C, \omega_{C/K}) \otimes_K K^s$. *In particular, all the jumps are rational numbers and*

$$c_{\mathrm{tame}}(\mathrm{Jac}(C)) = c(\Omega_{\mathrm{can}}(C) \otimes_R R^s / \Omega_{\mathrm{sat}}(C)).$$

(2) *The tuple of elementary divisors for* $\mathrm{Jac}(C)$ *is equal to the tuple of elementary divisors of the lattices*

$$\Omega_{\mathrm{ss}}(C) \subset \Omega_{\mathrm{can}}(C) \otimes_R R^s$$

in $H^0(C, \omega_{C/K}) \otimes_K K^s$. *In particular,*

$$c(\mathrm{Jac}(C)) = c(\Omega_{\mathrm{can}}(C) \otimes_R R^s / \Omega_{\mathrm{ss}}(C)).$$

Proof. Point (2) is essentially the definition of the elementary divisors, modulo the identification $\Omega(J) = \Omega_{\mathrm{can}}(C)$ in Proposition 2.4.3 and the duality between $\Omega(J)$ and $\mathrm{Lie}(\mathscr{J})$. The same arguments show that the tuple of K'-elementary divisors of $\mathrm{Jac}(C)$ is equal to the tuple of elementary divisors of the lattices

$$\Omega_{\mathrm{can}}(C \times_K K') \subset \Omega_{\mathrm{can}}(C) \otimes_R R'$$

for every finite extension K' of K in K^t with valuation ring R'. If we denote by $\mathfrak{m}'$ the maximal ideal of R', then it follows from (4.2.6) and Lemma 4.3.4 that

$$\mathfrak{m}'\Omega_{\mathrm{can}}(C \times_K K') \otimes_{R'} R^s \subset \Omega_{\mathrm{sat}}(C) \subset \Omega_{\mathrm{can}}(C \times_K K') \otimes_{R'} R^s.$$

Now we apply Proposition 4.1.3 to the chain of lattices

$$\Omega_{\mathrm{sat}}(C) \subset \Omega_{\mathrm{can}}(C \times_K K') \otimes_{R'} R^s \subset \Omega_{\mathrm{can}}(C) \otimes_R R^s.$$

This yields

$$\frac{c_i(\mathrm{Jac}(C), K')}{[K' : K]} \leq c_i(\Omega_{\mathrm{can}}(C) \otimes_R R^s / \Omega_{\mathrm{sat}}(C)) \leq \frac{c_i(\mathrm{Jac}(C), K') + 1}{[K' : K]}$$

for every i in $\{1, \ldots, g\}$. Letting K' range through the finite extensions of K in K^t, we find that

$$j_i(\mathrm{Jac}(C)) = c_i(\Omega_{\mathrm{can}}(C) \otimes_R R^s / \Omega_{\mathrm{sat}}(C))$$

for every i. $\qquad\square$

Remark 4.4.2. Theorem 4.4.1 suggests to define a new set of invariants, the *wild* elementary divisors, as the tuple of elementary divisors of the lattices

$$\Omega_{\mathrm{ss}}(C) \subset \Omega_{\mathrm{sat}}(C)$$

in $H^0(C, \omega_{C/K}) \otimes_K K^s$. If we define the wild base change conductor $c_{\mathrm{wild}}(Jac(C))$ as the sum of the wild elementary divisors, then

$$c_{\mathrm{wild}}(Jac(C)) = c(\Omega_{\mathrm{sat}}(C) / \Omega_{\mathrm{ss}}(C)) = c(\mathrm{Jac}(C)) - c_{\mathrm{tame}}(\mathrm{Jac}(C)).$$

The wild elementary divisors and the wild base change conductor form an interesting measure for the wild ramification of C; if C is tamely ramified then these invariants all vanish. If C is an elliptic curve, then $c_{\mathrm{wild}}(C)$ can be computed from the Swan conductor of C; see [HN14, Ch.5, 2.2.4].

(4.4.3) Observe that Theorem 4.4.1 yields a new proof of the rationality of Edixhoven's jumps for Jacobians. We will now give a more conceptual explanation for the role of the stabilization index. In order to do this, we first prove an elementary lemma.

Lemma 4.4.4. *Let $\mathscr{C}$ be an sncd-model of C. Let E_0 be an irreducible component of $\mathscr{C}_k$ of multiplicity $N_0 > 1$, and suppose that E_0 is a rational curve that meets the other components of $\mathscr{C}_k$ in precisely one point. Then there exists a principal component in $\mathscr{C}_k$ whose multiplicity N is divisible by N_0. If $\mathscr{C}$ is a minimal sncd-model, then we can find such a component with the additional property $N > N_0$.*

Proof. Let E_1 be the unique component of $\mathscr{C}_k$ intersecting E_0. From the intersection formula $\mathscr{C}_k \cdot E_0 = 0$ we obtain that $N_1 = -N_0(E_0 \cdot E_0)$; in particular, N_0 divides N_1. If E_1 is not principal, then it is a rational curve that meets exactly one irreducible component E_2 of $\mathscr{C}_k$ different from E_0 (it has to meet another component because, by our assumption that C has a zero divisor of degree one, the greatest common divisor of the multiplicities of the components of $\mathscr{C}_k$ must be equal to one; see (2.3.2)). The intersection formula $\mathscr{C}_k \cdot E_1 = 0$ tells us that $N_2 = -N_0 - N_1(E_1 \cdot E_1)$, so that N_0 divides N_2. Repeating the argument, we eventually find a principal component E_t of $\mathscr{C}_k$ whose multiplicity N_t is divisible by N_0. If $\mathscr{C}$ is minimal, then every rational component intersecting the rest of the special fiber in at most two points has self-intersection number at most -2, because otherwise it would be contractible by Castelnuovo's criterion, contradicting the minimality of $\mathscr{C}$. The above computations now easily yield $N_t > N_0$. □

Theorem 4.4.5. *If we denote by $e(C)$ the stabilization index of the curve C, then for every jump j in Edixhoven's filtration for $\mathrm{Jac}(C)$, the product $e(C) \cdot j$ is an integer.*

Proof. Let $\mathscr{C}$ be an R-model of C such that $\mathscr{C}^+$ is log regular, and denote by m the saturation index of the morphism $\mathscr{C}^+ \to S^+$. By Theorem 4.3.5, the lattice $\Omega_{\mathrm{sat}}(C)$ is defined over an extension of K of degree m. Thus it follows from Theorem 4.4.1 that the product $m \cdot j$ is integer for every jump j of $\mathrm{Jac}(C)$. Therefore, it suffices to show that we can choose $\mathscr{C}$ such that m is equal to the stabilization index $e(C)$.

Let $\mathscr{C}'$ be the minimal *sncd*-model of C. By Lipman's generalization of Artin's contractibility criterion [Li69, 27.1], any chain of rational curves in $\mathscr{C}'_k$ can be contracted to a rational singularity. In particular, there exists a morphism $h : \mathscr{C}' \to \mathscr{C}$ of normal R-models of C that contracts precisely the rational components of $\mathscr{C}'_k$ that meet the rest of the special fiber in exactly two points. The special fiber of $\mathscr{C}$ has étale locally two distinct branches at any point in the image of the exceptional locus of h, so we can deduce from [IS14, §3] that $\mathscr{C}^+$ is log regular. By Lemma 4.4.4, the saturation index of

$\mathscr{C}^+ \to S^+$ is equal to $e(C)$, because the non-principal components of $\mathscr{C}'_k$ are either contracted by h or do not contribute to the saturation index. This concludes the proof. $\square$

5. A formula for the jumps

5.1. The basic formula.

(5.1.1) We will now use our logarithmic interpretation of the jumps of $\mathrm{Jac}(C)$ in Theorem 4.4.1 to deduce an explicit formula for the jumps in terms of the combinatorial reduction data of C. Our starting point is the chain of lattices

$$\Omega_{\log}(C) \otimes_R R^s \subset \Omega_{\mathrm{sat}}(C) \subset \Omega_{\mathrm{can}}(C) \otimes_R R^s$$

in $H^0(C, \omega_{C/K}) \otimes_K K^s$. Recall that, by Theorem 4.4.1, the jumps of $\mathrm{Jac}(C)$ are precisely the elementary divisors of the pair of lattices

$$\Omega_{\mathrm{sat}}(C) \subset \Omega_{\mathrm{can}}(C) \otimes_R R^s.$$

We will compute these elementary divisors by computing those of the other inclusions in the chain:

$$(5.1.2) \qquad\qquad \Omega_{\log}(C) \quad\subset\quad \Omega_{\mathrm{can}}(C),$$
$$(5.1.3) \qquad\qquad \Omega_{\log}(C) \otimes_R R^s \quad\subset\quad \Omega_{\mathrm{sat}}(C).$$

The tuples of elementary divisors do not behave additively in chains, in general, but they do in special cases, as is explained in the following easy lemma.

Lemma 5.1.4. *Let V be a vector space over K of finite dimension g and let $\Omega_1 \subset \Omega_2 \subset \Omega_3$ be a chain of lattices in $V \otimes_K K^s$. Suppose that the tuple of elementary divisors of $\Omega_1 \subset \Omega_3$ is of the form*

$$v = (0, \dots, 0, a, \dots, a)$$

for some positive rational number a, and denote by $n \leq g$ the number of entries equal to 0. Then the tuple of elementary divisors

$$w = (w_1, \dots, w_g)$$

of $\Omega_1 \subset \Omega_2$ satisfies $0 \leq w_i \leq v_i$ for all i in $\{1, \dots, g\}$, and the tuple of elementary divisors of $\Omega_2 \subset \Omega_3$ is given by

$$(0, \dots, 0, a - w_g, \dots, a - w_{n+1}).$$

Proof. We may assume that all the lattices are defined over R. Then a is a positive integer and the statement boils down to the following simple fact: if N is a submodule of $M = (R/\mathfrak{m}^a)^{g-n}$ for some positive integer a, then N is isomorphic to $\oplus_{i=1}^{g-n} R/\mathfrak{m}^{a_i}$ for some non-negative integers $a_i \leq a$, and the quotient M/N is isomorphic to $\oplus_{i=1}^{g-n} R/\mathfrak{m}^{a-a_i}$. $\square$

The following proposition will be quite useful in our computations.

Proposition 5.1.5. *If $\mathscr{C}$ is an sncd-model of C, then the R-modules $H^1(\mathscr{C}, \omega_{\mathscr{C}/R})$ and $H^1(\mathscr{C}, \omega_{\mathscr{C}/R}^{\log})$ have no torsion.*

Proof. We write D for the divisor $\mathscr{C}_k - \mathscr{C}_{k,\mathrm{red}}$ on $\mathscr{C}$. We know by Proposition 3.3.4 that

$$\omega_{\mathscr{C}/R}^{\log} = \omega_{\mathscr{C}/R}(-D).$$

The cohomology module $H^1(\mathscr{C}, \mathcal{O}_{\mathscr{C}})$ is free by cohomological flatness of the structural morphism $\mathscr{C} \to S$ (see (2.3.2)). The short exact sequence of $\mathcal{O}_{\mathscr{C}}$-modules

$$0 \to \mathcal{O}_{\mathscr{C}}(D - \mathscr{C}_k) \to \mathcal{O}_{\mathscr{C}} \to \mathcal{O}_{\mathscr{C}_{k,\mathrm{red}}} \to 0$$

gives rise to an injection

$$H^1(\mathscr{C}, \mathcal{O}_{\mathscr{C}}(D)) \to H^1(\mathscr{C}, \mathcal{O}_{\mathscr{C}})$$

by surjectivity of the map

$$R = H^0(\mathscr{C}, \mathcal{O}_{\mathscr{C}}) \to H^0(\mathscr{C}, \mathcal{O}_{\mathscr{C}_{k,\mathrm{red}}}) = k.$$

Thus we see that $H^1(\mathscr{C}, \mathcal{O}_{\mathscr{C}}(D))$ is free, as well. Now Grothendieck-Serre duality provides us with isomorphisms

$$H^1(\mathscr{C}, \omega_{\mathscr{C}/R}) \to H^0(\mathscr{C}, \mathcal{O}_{\mathscr{C}})^{\vee}$$

and

$$H^1(\mathscr{C}, \omega_{\mathscr{C}/R}^{\log}) \to H^0(\mathscr{C}, \mathcal{O}_{\mathscr{C}}(D))^{\vee}.$$

In particular, $H^1(\mathscr{C}, \omega_{\mathscr{C}/R})$ and $H^1(\mathscr{C}, \omega_{\mathscr{C}/R}^{\log})$ have no torsion. $\square$

5.2. **Computation of $\Omega_{\mathrm{can}}(C)/\Omega_{\log}(C)$.**

(5.2.1) We know that $\mathfrak{m}\Omega_{\mathrm{can}}(C) \subset \Omega_{\log}(C)$ by Lemma 4.3.4. Thus the quotient $\Omega_{\mathrm{can}}(C)/\Omega_{\log}(C)$ is isomorphic to $(R/\mathfrak{m})^u$ for some $0 \leq u \leq g$ and the tuple of elementary divisors of $\Omega_{\log}(C) \subset \Omega_{\mathrm{can}}(C)$ is given by $(0, \ldots, 0, 1, \ldots, 1)$ where the number of zeroes is equal to $g - u$. It only remains to determine the value of u, that is, the dimension of the k-vector space $\Omega_{\mathrm{can}}(C)/\Omega_{\log}(C)$. To compute this dimension, we rewrite it as

$$\dim_k(\Omega_{\mathrm{can}}(C)/\Omega_{\log}(C)) = g - \dim_k(\Omega_{\log}(C)/\mathfrak{m}\Omega_{\mathrm{can}}(C)).$$

Let $\mathscr{C}$ be an *sncd*-model of C. By Proposition 3.3.4, we have a short exact sequence of $\mathcal{O}_{\mathscr{C}}$-modules

$$(5.2.2) \qquad 0 \to \mathfrak{m}\omega_{\mathscr{C}/R} = \omega_{\mathscr{C}/R}^{\log}(-\mathscr{C}_{k,\mathrm{red}}) \to \omega_{\mathscr{C}/R}^{\log} \to \iota_*\iota^*\omega_{\mathscr{C}/R}^{\log} \to 0$$

where ι denotes the closed immersion $\mathscr{C}_{k,\mathrm{red}} \to \mathscr{C}$. By Proposition 5.1.5, the R-module $H^1(\mathscr{C}, \omega_{\mathscr{C}/R})$ is torsion free. Therefore, the sequence

$$0 \to H^0(\mathscr{C}, \mathfrak{m}\omega_{\mathscr{C}/R}) \to H^0(\mathscr{C}, \omega_{\mathscr{C}/R}^{\log}) \to H^0(\mathscr{C}_{k,\mathrm{red}}, \iota^*\omega_{\mathscr{C}/R}^{\log}) \to 0$$

is still exact; the start of this sequence is precisely the inclusion of lattices $\mathfrak{m}\Omega_{\mathrm{can}}(C) \to \Omega_{\log}(C)$. Hence,

$$\dim_k(\Omega_{\mathrm{can}}(C)/\Omega_{\log}(C)) = g - \dim_k H^0(\mathscr{C}_{k,\mathrm{red}}, \iota^*\omega_{\mathscr{C}/R}^{\log}).$$

(5.2.3) Now we compute $\dim_k H^0(\mathscr{C}_{k,\mathrm{red}}, \iota^*\omega^{\mathrm{log}}_{\mathscr{C}/R})$. Denote by

$$a_1 : \widetilde{\mathscr{C}}_{k,\mathrm{red}} \to \mathscr{C}_{k,\mathrm{red}}$$

the normalization morphism of $\mathscr{C}_{k,\mathrm{red}}$. If we write $\mathscr{C}_{k,\mathrm{red}} = \sum_{i=1}^r E_i$ then $\widetilde{\mathscr{C}}_{k,\mathrm{red}}$ is simply the disjoint union $\sqcup_i E_i$. We denote by Σ the singular locus of $\mathscr{C}_{k,\mathrm{red}}$ and by a_2 the closed immersion $\Sigma \to \mathscr{C}_{k,\mathrm{red}}$. For notational convenience, we write $\mathcal{L}$ for the line bundle $\omega^{\mathrm{log}}_{\mathscr{C}/R}$ on $\mathscr{C}$, and $\mathcal{L}_i$ for the pullback of $\mathcal{L}$ to E_i. Then we can construct the usual short exact sequence

$$0 \to \iota^*\mathcal{L} \to (a_1)_*(a_1)^*\iota^*\mathcal{L} \to (a_2)_*(a_2)^*\iota^*\mathcal{L} \to 0$$

which tells us that

$$\chi(\iota^*\mathcal{L}) = \sum_{i=1}^r \chi(\mathcal{L}_i) - |\Sigma|.$$

By Proposition 3.3.4 and the adjunction formula, $\mathcal{L}_i$ is isomorphic to the sheaf

$$\omega_{E_i/k}(E_i \cap \Sigma)$$

of differential forms on E_i with logarithmic poles along $E_i \cap \Sigma$. Thus

$$\chi(\mathcal{L}_i) = g(E_i) - 1 + |E_i \cap \Sigma|$$

where $g(E_i)$ denotes the genus of E_i. Since every point of Σ lies on precisely two irreducible components of $\mathscr{C}_{k,\mathrm{red}}$, we find

$$\chi(\iota^*\mathcal{L}) = \sum_{i=1}^r g(E_i) - r + |\Sigma|$$

and hence

$$\dim_k H^0(\mathscr{C}_{k,\mathrm{red}}, \iota^*\mathcal{L}) = \sum_{i=1}^r g(E_i) - r + |\Sigma| + \dim_k H^1(\mathscr{C}_{k,\mathrm{red}}, \iota^*\mathcal{L}).$$

(5.2.4) The short exact sequence (5.2.2) also yields an exact sequence

$$0 \to H^1(\mathscr{C}, \omega^{\mathrm{log}}_{\mathscr{C}/R}(-\mathscr{C}_{k,\mathrm{red}})) \to H^1(\mathscr{C}, \omega^{\mathrm{log}}_{\mathscr{C}/R}) \to H^1(\mathscr{C}_{k,\mathrm{red}}, \iota^*\mathcal{L}) \to 0.$$

By Proposition 3.3.4 and Grothendieck-Serre duality, the second arrow can be identified with the map

$$H^0(\mathscr{C}, \mathcal{O}_{\mathscr{C}}(\mathscr{C}_k))^\vee \to H^0(\mathscr{C}, \mathcal{O}_{\mathscr{C}}(\mathscr{C}_k - \mathscr{C}_{k,\mathrm{red}}))^\vee$$

whose cokernel is clearly isomorphic to k. Thus

$$\dim_k H^1(\mathscr{C}_{k,\mathrm{red}}, \iota^*\mathcal{L}) = 1.$$

Putting all these pieces together, we arrive at the following result.

Proposition 5.2.5. *Let $\mathscr{C}$ be an sncd-model of C. Denote by Γ the dual graph of $\mathscr{C}_k$ and by $\beta(\Gamma)$ its first Betti number. We write $u(C)$ for the unipotent rank of $\mathrm{Jac}(C)$, that is, the dimension of the unipotent radical of*

the identity component of the special fiber of the Néron model of $\mathrm{Jac}(C)$. *Then*

$$\dim_k(\Omega_{\mathrm{can}}(C)/\Omega_{\mathrm{log}}(C)) \;=\; g - \sum_{i=1}^{r} g(E_i) - \beta(\Gamma)$$
$$=\; u(C).$$

Proof. The above computations yield the formula

$$\dim_k(\Omega_{\mathrm{can}}(C)/\Omega_{\mathrm{log}}(C)) = g - \sum_{i=1}^{r} g(E_i) + r - |\Sigma| - 1.$$

Since r is equal to the number of vertices of Γ and $|\Sigma|$ equals the number of edges, we have $r - |\Sigma| - 1 = -\beta(\Gamma)$. The equality

$$g - \sum_{i=1}^{r} g(E_i) - \beta(\Gamma) = u(C)$$

is well-known; see for instance [Lo90, p.148]. $\qquad\qquad\square$

Remark 5.2.6. If $\mathscr{C}_k$ is reduced, then $\Omega_{\mathrm{can}}(C) = \Omega_{\mathrm{log}}(C)$ and our formula in Proposition 5.2.5 boils down to the classical expression

$$g = \sum_{i=1}^{r} g(E_i) + \beta(\Gamma).$$

5.3. Computation of $\Omega_{\mathrm{sat}}(C)/(\Omega_{\mathrm{log}}(C) \otimes_R R^s)$.

(5.3.1) We still denote by $\mathscr{C}$ an *sncd*-model of C and by $\mathcal{L}$ the line bundle $\omega_{\mathscr{C}/R}^{\mathrm{log}}$ on $\mathscr{C}$. Denote by m the saturation index of $\mathscr{C}^+ \to S^+$, that is, the least common multiple of the multiplicities of the prime components in $\mathscr{C}_k$. Let K' be a degree m extension of K in K^s and denote by R' its valuation ring. Set $\mathscr{C}' = \mathscr{C} \times_R R'$ and denote by $\mathscr{D}$ the underlying scheme of $\mathscr{C}^+ \times_{S^+}^{fs} (S')^+$. We write $\mathcal{L}'$ for the pullback of $\mathcal{L}$ to $\mathscr{C}'$ and h for the natural morphism $\mathscr{D} \to \mathscr{C}'$. By flat base change, we have

$$H^0(\mathscr{C}', \mathcal{L}') = \Omega_{\mathrm{log}}(C) \otimes_R R'.$$

By Theorem 4.3.5, we know that

$$\Omega_{\mathrm{sat}}(C) = H^0(\mathscr{D}, h^*\mathcal{L}') \otimes_{R'} R^s.$$

Therefore, we need to compute the cokernel of

$$H^0(\mathscr{C}', \mathcal{L}') \to H^0(\mathscr{D}, h^*\mathcal{L}').$$

(5.3.2) Denote by $\mathcal{F}$ the cokernel of the morphism of $\mathcal{O}_{\mathscr{C}'}$-modules

$$\mathcal{O}_{\mathscr{C}'} \to h_*\mathcal{O}_{\mathscr{D}}.$$

Note that $\mathcal{F}$ is trivial on the generic fiber $\mathscr{C}'$ because the morphism $\mathscr{D}_{K'} \to \mathscr{C}'_{K'}$ is an isomorphism. We have a short exact sequence of $\mathcal{O}_{\mathscr{C}'}$-modules

$$0 \to \mathcal{L}' \to h_*h^*\mathcal{L}' \to \mathcal{L}' \otimes_{\mathcal{O}_{\mathscr{C}'}} \mathcal{F} \to 0$$

which gives rise to a short exact sequence of R'-modules

$$0 \to H^0(\mathscr{C}, \mathcal{L}') \to H^0(\mathscr{D}, h^*\mathcal{L}') \to H^0(\mathscr{C}', \mathcal{L}' \otimes_{\mathcal{O}_{\mathscr{C}'}} \mathcal{F}) \to 0$$

because $H^1(\mathscr{C}', \mathcal{L}') = H^1(\mathscr{C}, \mathcal{L}) \otimes_R R'$ has no torsion (see Proposition 5.1.5). As we explained in the proof of Theorem 4.3.5, it follows from Lemma 4.3.3 that $\mathcal{F}$ is killed by $\mathfrak{m}$, so that we can also view it as a coherent sheaf on the k-scheme

$$X = \mathscr{C}' \times_{R'} (R'/\mathfrak{m}R') \cong \mathscr{C}_k \times_k (R'/\mathfrak{m}R').$$

Thus, denoting by $\mathcal{L}_k$ the pullback of $\mathcal{L}$ to $\mathscr{C}_k$ and by κ the projection morphism $X \to \mathscr{C}_k$, we find that the cokernel of

$$H^0(\mathscr{C}', \mathcal{L}') \to H^0(\mathscr{D}, h^* \mathcal{L}')$$

is isomorphic to the $R'/\mathfrak{m}R'$-module

$$M = H^0(\mathscr{C}_k, \mathcal{L}_k \otimes_{\mathcal{O}_{\mathscr{C}_k}} \kappa_* \mathcal{F}).$$

(5.3.3) Denote by $\mathfrak{m}'$ the maximal ideal of R'. For every i in $\{0, \dots, m\}$, we set

$$\mathcal{F}_i = \kappa_*((\mathfrak{m}')^i \mathcal{F}/(\mathfrak{m}')^{i+1} \mathcal{F})$$

and $V_i = (\mathfrak{m}')^i M/(\mathfrak{m}')^{i+1} M$. Then $\mathcal{F}_i$ is a coherent $\mathcal{O}_{\mathscr{C}_k}$-module and V_i is a k-vector space, and they both vanish if $i = m$. The dimensions of the vector spaces V_i completely determine the R'-module structure of M: for every i in $\{0, \dots, m-1\}$, the multiplicity of $R'/(\mathfrak{m}')^{i+1}$ as a direct summand in M is equal to $\dim_k V_i - \dim_k V_{i+1}$.

From the spectral sequence for the hypercohomology of a filtered complex [De71, 1.4.5], we deduce that

$$(5.3.4) \qquad V_i \cong \ker(d_i : H^0(\mathscr{C}_k, \mathcal{L}_k \otimes \mathcal{F}_i) \to H^1(\mathscr{C}_k, \mathcal{L}_k \otimes \mathcal{F}_{i+1}))$$

for every i in $\{0, \dots, m-1\}$, where d_i is the connecting homomorphism in the long exact cohomology sequence associated to the short exact sequence of $\mathcal{O}_{\mathscr{C}_k}$-modules

$$0 \to \mathcal{L}_k \otimes \mathcal{F}_{i+1} \to \mathcal{L}_k \otimes \kappa_*((\mathfrak{m}')^i \mathcal{F}/(\mathfrak{m}')^{i+2} \mathcal{F}) \to \mathcal{L}_k \otimes \mathcal{F}_i \to 0.$$

(5.3.5) We will now give an explicit description of the $\mathcal{O}_{\mathscr{C}_k}$-modules $\mathcal{F}_i$. It will be convenient to use the following notation. For every effective $\mathbb{R}$-divisor D on $\mathscr{C}$, we denote by $\lfloor D \rfloor$ the integral part of D, obtained by rounding down the coefficients, and we write $\langle D \rangle$ for the divisor obtained from D by putting all the coefficients in $\mathbb{R} \setminus \mathbb{Z}$ equal to zero. Moreover, we write $\langle D \rangle_{\mathrm{red}}$ for the reduction of $\langle D \rangle$ and $\mathcal{J}(D)$ for the pullback of $\mathcal{O}_{\mathscr{C}}(\lfloor D \rfloor)$ to $\langle D \rangle_{\mathrm{red}}$. The letter $\mathcal{J}$ stands for "jump": if we let the coefficients of D grow continuously in $\mathbb{R}$, then $\mathcal{J}(D)$ detects how $\lfloor D \rfloor$ changes.

Proposition 5.3.6. *For every i in $\{0, \dots, m-1\}$, the $\mathcal{O}_{\mathscr{C}_k}$-module $\mathcal{F}_i$ is isomorphic to*

$$\bigoplus_{j=1}^{m-i-1} \mathcal{J}((j/m)\mathscr{C}_k).$$

Proof. The proof is based on a local study of the morphism $h : \mathscr{D} \to \mathscr{C}'$.

Case 1. First, let x be a regular point of $\mathscr{C}_{k,\mathrm{red}} \cong \mathscr{C}'_{k,\mathrm{red}}$. Since $\mathscr{C}$ is an *sncd*-model, the morphism $\mathscr{C}^+ \to S^+$ has Zariski-locally at x a chart of the form

$$\mathbb{N} \to \frac{1}{a}\mathbb{N} \oplus \mathbb{Z} : 1 \mapsto (1,1)$$

where a is the multiplicity of $\mathscr{C}_k$ at x, the generator of $\mathbb{N}$ is sent to a uniformizer π in R, and the generator of $(1/a)\mathbb{N}$ is sent to a local defining function f for $\mathscr{C}_{k,\mathrm{red}}$ in $\mathscr{C}$ at x. As we've already explained in the proof of Theorem 4.3.5, the morphism $(S')^+ \to S^+$ has a chart of the form

$$\mathbb{N} \to \frac{1}{m}\mathbb{N} \oplus \mathbb{Z} : 1 \mapsto (1,1)$$

sending the generator of $\mathbb{N}$ to π and the generator of $(1/m)\mathbb{N}$ to a uniformizer π' in R'. Thus, locally at x, $(\mathscr{C}')^+$ has a chart of the form

$$Q = (\mathbb{Z} \oplus \frac{1}{a}\mathbb{N}) \oplus_{\mathbb{N}} \frac{1}{m}\mathbb{N} \to \mathcal{O}_{\mathscr{C}',x}$$

sending the generator of $(1/a)\mathbb{N}$ to f and the generator of $(1/m)\mathbb{N}$ to π'. If we denote by $Q \to Q^{\mathrm{sat}}$ the natural morphism from Q to its saturation, then over some open neighbourhood of x in $\mathscr{C}'$, $\mathscr{D}$ is given by

$$\mathscr{C}' \times_{\mathbb{Z}[Q]} \mathbb{Z}[Q^{\mathrm{sat}}].$$

Thus we must understand the exact shape of the morphism $Q \to Q^{\mathrm{sat}}$.

The groupification Q^{gp} of Q is the coproduct

$$(\mathbb{Z} \oplus \frac{1}{a}\mathbb{Z}) \oplus_{\mathbb{Z}} \frac{1}{m}\mathbb{Z} \cong (\mathbb{Z} \oplus \frac{1}{a}\mathbb{Z} \oplus \frac{1}{m}\mathbb{Z})/\langle (1,1,-1) \rangle.$$

An element $(u, v/a, w/m)$ of Q^{gp} belongs to Q^{sat} if and only if $v/a + w/m$ is non-negative, that is, $mv + aw \geq 0$. Thus the $\mathbb{Z}[Q]$-module $\mathbb{Z}[Q^{\mathrm{sat}}]/\mathbb{Z}[Q]$ is generated by the elements $(0, -\lfloor ja/m \rfloor, j)$ with $j \in \{1, \ldots, m-1\}$. This means that the stalk $\mathcal{F}_x$ is generated as an $\mathcal{O}_{\mathscr{C},x}$-module by the elements $(\pi')^j/f^{\lfloor ja/m \rfloor}$ with $j \in \{1, \ldots, m-1\}$. An element of the form $(\pi')^s/f^t$ is divisible by $(\pi')^i$ in $\mathcal{F}_x$ if and only if $a(s-i) - mt \geq 0$. In particular, if $(\pi')^s/f^t$ is divisible by $(\pi')^i$ then $(\pi')^s/f^{t-1}$ is divisible by $(\pi')^{i+1}$. From these computations, we deduce that locally at x, the morphism

$$\bigoplus_{j=1}^{m-1-i} \mathcal{O}_{\mathscr{C}}(\lfloor (j/m)\mathscr{C}_k \rfloor) \to \mathcal{F}_i : (c_1, \ldots, c_{m-1-i}) \mapsto \sum_{j=1}^{m-1-i} c_j (\pi')^{j+i}$$

factors through an isomorphism

$$\bigoplus_{j=1}^{m-i-1} \mathcal{J}((j/m)\mathscr{C}_k) \to \mathcal{F}_i,$$

for every $i \in \{0, \ldots, m-1\}$.

Case 2. Now we treat the case where x is a singular point of $\mathscr{C}_{k,\mathrm{red}}$. The morphism $\mathscr{C}^+ \to S^+$ has Zariski-locally at x a chart of the form

$$\mathbb{N} \to \mathbb{Z} \oplus \frac{1}{a}\mathbb{N} \oplus \frac{1}{b}\mathbb{N} : 1 \mapsto (1,1,1)$$

where a and b are the multiplicities of $\mathscr{C}_k$ along the irreducible components E_1 and E_2 passing through at x, the generator of $\mathbb{N}$ is sent to a uniformizer π in R, and the generators $(1/a)\mathbb{N}$ and $(1/b)\mathbb{N}$ are sent to local defining functions f_1 and f_2 for E_1 and E_2 in $\mathscr{C}$ at x. In a similar way as in Case 1, we can produce a local chart

$$Q = (\mathbb{Z} \oplus \frac{1}{a}\mathbb{N} \oplus \frac{1}{b}\mathbb{N}) \oplus_{\mathbb{N}} \frac{1}{m}\mathbb{N} \to \mathcal{O}_{\mathscr{C}',x}$$

sending the generators of $(1/a)\mathbb{N}$ and $(1/b)\mathbb{N}$ to f_1 and f_2, respectively, and the generator of $(1/m)\mathbb{N}$ to π'. Now Q^{gp} is given by

$$(\mathbb{Z} \oplus \frac{1}{a}\mathbb{Z} \oplus \frac{1}{b}\mathbb{Z}) \oplus_{\mathbb{Z}} \frac{1}{m}\mathbb{Z} \cong (\mathbb{Z} \oplus \frac{1}{a}\mathbb{Z} \oplus \frac{1}{b}\mathbb{Z} \oplus \frac{1}{m}\mathbb{Z})/\langle(1,1,1,-1)\rangle$$

and an element $(t, u/a, v/b, w/m)$ of Q^{gp} belongs to Q^{sat} if and only if $mu + aw \geq 0$ and $mv + bw \geq 0$. It follows that the stalk $\mathcal{F}_x$ is generated over $\mathcal{O}_{\mathscr{C},x}$ by the elements $(\pi')^j/(f_1^{\lfloor ja/m\rfloor} f_2^{\lfloor jb/m\rfloor})$ with $j \in \{1, \ldots, m-1\}$ and $j \geq \min\{m/a, m/b\}$. We deduce as in Case 1 that locally at x, the morphism

$$\bigoplus_{j=1}^{m-i-1} \mathcal{O}_{\mathscr{C}}(\lfloor (j/m)\mathscr{C}_k\rfloor) \to \mathcal{F}_i : (c_1, \ldots, c_{m-1-i}) \mapsto \sum_{j=1}^{m-1-i} c_j(\pi')^{j+i}$$

factors through an isomorphism

$$\bigoplus_{j=1}^{m-i-1} \mathcal{J}((j/m)\mathscr{C}_k) \to \mathcal{F}_i,$$

for every $i \in \{0, \ldots, m-1\}$.

As x varies, all of these local isomorphisms glue to an isomorphism as required in the statement. $\qquad\square$

(5.3.7) We can now use Proposition 5.3.6 to finish our computations. We will need the following vanishing result. Recall that we write $\mathscr{C}_k = \sum_{i=1}^{r} N_i E_i$ and that we denote by $\mathcal{L}$ the line bundle $\omega_{\mathscr{C}/R}^{\log}$ on $\mathscr{C}$, by ι the closed immersion $\mathscr{C}_{k,\mathrm{red}} \to \mathscr{C}_k$, and by Σ the set of singular points of $\mathscr{C}_{k,\mathrm{red}}$.

Lemma 5.3.8. *Let E be a connected, smooth and proper curve over k and let D and D' be divisors on E of degrees d and d', respectively. Assume that D' is reduced. Then $H^1(E, \omega_{E/k} \otimes \mathcal{O}_E(D))$ vanishes if $d > 0$, and the restriction map*

$$H^0(E, \omega_{E/k} \otimes \mathcal{O}_E(D)) \to \bigoplus_{x \in D'} \iota_x^*(\omega_{E/k} \otimes \mathcal{O}_E(D))$$

is surjective if $d > d'$. Here we wrote ι_x for the closed immersion of x in E.

Proof. These are standard applications of Serre duality: if $d > 0$ then

$$H^1(E, \omega_{E/k} \otimes \mathcal{O}_E(D)) \cong H^0(E, \mathcal{O}_E(-D))^{\vee} = 0.$$

Moreover, the cokernel of

$$H^0(E, \omega_{E/k} \otimes \mathcal{O}_E(D)) \to \bigoplus_{x \in D'} \iota_x^*(\omega_{E/k} \otimes \mathcal{O}_E(D))$$

is isomorphic to

$$H^1(E, \omega_{E/k} \otimes \mathcal{O}_E(D - D')),$$

and thus vanishes if $d > d'$. □

Proposition 5.3.9. *For every j in $\{1, \ldots, m-1\}$ we have*

$$H^1(\mathscr{C}_{k,\mathrm{red}}, \iota^*\mathcal{L} \otimes \mathcal{J}((j/m)\mathscr{C}_k)) = 0.$$

Proof. We denote by I_j the set of indices i in $\{1, \ldots, r\}$ such that j is a multiple of m/N_i. The components E_i with $i \in I_j$ are precisely the prime components of $\langle (j/m)\mathscr{C}_k \rangle$. For every $i \in I_j$ we denote by $\mathcal{J}_{j,i}$ the restriction of $\mathcal{J}((j/m)\mathscr{C}_k)$ to E_i. This is a line bundle on E_i whose degree is equal to the intersection number $(E_i \cdot \lfloor (j/m)\mathscr{C}_k \rfloor)$. Writing $\{(j/m)\mathscr{C}_k\}$ for the fractional part $(j/m)\mathscr{C}_k - \lfloor (j/m)\mathscr{C}_k \rfloor$ of the $\mathbb{Q}$-divisor $(j/m)\mathscr{C}_k$, we compute:

$$0 = (E_i \cdot (j/m)\mathscr{C}_k) = (E_i \cdot \lfloor (j/m)\mathscr{C}_k \rfloor) + (E_i \cdot \{(j/m)\mathscr{C}_k\}).$$

The prime components of the divisor $\{(j/m)\mathscr{C}_k\}$ are precisely the components of $\mathscr{C}_k$ that are not contained in $\langle (j/m)\mathscr{C}_k \rangle$. We write $\sigma_{j,i}$ for the number of intersection points of E_i with the support of $\{(j/m)\mathscr{C}_k\}$. Note that these are singular points of $\mathscr{C}_{k,\mathrm{red}}$, so that $\sigma_{j,i} \leq |\Sigma \cap E_i|$. Since the coefficients of $\{(j/m)\mathscr{C}_k\}$ are strictly contained between 0 and 1, we find that

$$(E_i \cdot \lfloor (j/m)\mathscr{C}_k \rfloor) \geq \min\{-\sigma_{j,i} + 1, 0\}.$$

We have already observed in (5.2.3) that the restriction $\mathcal{L}_i$ of $\mathcal{L}$ to E_i is isomorphic to the sheaf of differential forms with logarithmic poles at the points of $(\Sigma \cap E_i)$. Thus Lemma 5.3.8 implies that

$$H^1(E_i, \mathcal{L}_i \otimes \mathcal{J}_{j,i}) = 0$$

for every $i \in I_j$. Now we can compute

$$H^1(\mathscr{C}_{k,\mathrm{red}}, \iota^*\mathcal{L} \otimes \mathcal{J}((j/m)\mathscr{C}_k))$$

using the standard resolution of the line bundle $\iota^*\mathcal{L} \otimes \mathcal{J}((j/m)\mathscr{C}_k)$ on the reduced strict normal crossings divisor $\langle (j/m)\mathscr{C}_k \rangle_{\mathrm{red}}$ as in (5.2.3). The associated long exact cohomology sequence contains the exact subsequence

$$\bigoplus_{i \in I_j} H^0(E_i, \mathcal{L}_i \otimes \mathcal{J}_{j,i}) \to \bigoplus_{x \in \Sigma_j} \iota_x^*(\mathcal{L} \otimes \mathcal{J}((j/m)\mathscr{C}_k)) \to H^1(\mathscr{C}_{k,\mathrm{red}}, \iota^*\mathcal{L} \otimes \mathcal{J}((j/m)\mathscr{C}_k)) \to 0$$

where Σ_j denotes the set of singular points of $\langle (j/m)\mathscr{C}_k \rangle_{\mathrm{red}}$ and we write ι_x for the closed immersion of x in $\mathscr{C}_{k,\mathrm{red}}$. We will prove that the first arrow in this sequence is surjective, which means that

$$H^1(\mathscr{C}_{k,\mathrm{red}}, \iota^*\mathcal{L} \otimes \mathcal{J}((j/m)\mathscr{C}_k)) = 0.$$

We say that a point x_0 in Σ_j is *good* if the characteristic function of $\{x_0\}$ lies in the image of

$$\rho : \bigoplus_{i \in I_j} H^0(E_i, \mathcal{L}_i \otimes \mathcal{J}_{j,i}) \to \bigoplus_{x \in \Sigma_j} \iota_x^*(\mathcal{L} \otimes \mathcal{J}((j/m)\mathscr{C}_k)).$$

For every $i \in I_j$, the restriction map

$$H^0(E_i, \mathcal{L}_i \otimes \mathcal{J}_{j,i}) \to \bigoplus_{x \in E_i \cap \Sigma_j} \iota_x^*(\mathcal{L}_i \otimes \mathcal{J}_{j,i})$$

is surjective if $\sigma_{j,i} \neq 0$, because the number of points in $E_i \cap \Sigma_j$ is precisely equal to $|\Sigma \cap E_i| - \sigma_{j,i}$ so that we can apply Lemma 5.3.8. Thus if if $\sigma_{j,i} \neq 0$ then every point of $\Sigma_j \cap E_i$ is good. On the other hand, if i is any element of I_j such that $\sigma_{j,i} = 0$ and E_i contains a good point x_0 of Σ_j, then any point $x_1 \neq x_0$ of $\Sigma_j \cap E_i$ is good, because we can always find an element of

$$H^0(E_i, \mathcal{L}_i \otimes \mathcal{J}_{j,i})$$

that is non-zero at x_0 and x_1 and vanishes at all other points of $\Sigma_j \cap E_i$ (this is again a consequence of Lemma 5.3.8, applied to $D' = \{x_0\}$ and $D' = \{x_1\}$). Note that I_j cannot be the whole set $\{1, \ldots, r\}$ because this contradicts our overall assumption that the multiplicities N_i are coprime. Thus every connected component of $\sum_{i \in I_j} E_i$ contains at least one prime component E_i with $\sigma_{j,i} \neq 0$, and we may conclude that every point in Σ_j is good, so that ρ is surjective. $\qquad\square$

5.4. An explicit formula for the jumps.

Theorem 5.4.1. *Let $\mathscr{C}$ be an sncd-model of C. Write $\mathscr{C}_k = \sum_{i=1}^r N_i E_i$ and denote by m the least common multiple of the multiplicities N_i of the irreducible components E_i in $\mathscr{C}_k$. For each i, we denote by $g(E_i)$ the genus of E_i. For every j in $\{0, \ldots, m\}$, we denote by I_j the set of indices i in $\{1, \ldots, r\}$ such that j is a multiple of m/N_i, and we write σ_j for the number of singular points of $\mathscr{C}_{k,\mathrm{red}}$ that lie on at least one component E_i with $i \in I_j$.*

The jumps of $\mathrm{Jac}(C)$ are contained in the set $\{0, \ldots, (m-1)/m\}$. For every element j in $\{0, \ldots, m-1\}$, the multiplicity of j/m as a jump of $\mathrm{Jac}(C)$ is equal to

$$\left(\sum_{i \in I_j} E_i \cdot \lfloor (j/m)\mathscr{C}_k \rfloor\right) + \sum_{i \in I_j} g(E_i) - |I_j| + \sigma_j + \delta_{j,0}$$

where δ is the Kronecker symbol.

Proof. We already know by Theorem 4.4.5 that the jumps are multiples of $1/m$, and it follows from Edixhoven's original definition in [Ed92] that they are always strictly smaller than one (for arbitrary abelian K-varieties). Thus each jump is of the form j/m with $j \in \{0, \ldots, m-1\}$. First, assume that $j \neq 0$. By (5.1.1) and Lemma 5.1.4, the multiplicity of j/m as a jump of $\mathrm{Jac}(C)$ is equal to the number of occurrences of the value $(m-j)/m$ in the tuple of elementary divisors of the inclusion of lattices

$$\Omega_{\log}(C) \otimes_R R^s \subset \Omega_{\mathrm{sat}}(C).$$

We will denote this number by γ_j. As we have explained in (5.3.3), the number γ_j is equal to

$$\dim_k V_{m-j-1} - \dim_k V_{m-j}.$$

Propositions 5.3.6 and 5.3.9 imply that the target of the morphism d_i from (5.3.4) vanishes for every i in $\{0, \ldots, m-1\}$, so that we can identify the k-vector space V_i with

$$H^0(\mathscr{C}_k, \mathcal{L}_k \otimes \mathcal{F}_i).$$

Using the description of the sheaves $\mathcal{F}_i$ in Propositions 5.3.6, we can write

$$\gamma_j = \dim_k H^0(\mathscr{C}_{k,\mathrm{red}}, \iota^* \mathcal{L} \otimes \mathcal{J}((j/m)\mathscr{C}_k)),$$

which is also equal to the Euler characteristic

$$\chi(\mathscr{C}_{k,\mathrm{red}}, \iota^*\mathcal{L} \otimes \mathcal{J}((j/m)\mathscr{C}_k))$$

by the vanishing result in Proposition 5.3.9. Recall that the sheaf $\mathcal{J}(j/m)\mathscr{C}_k$ was defined as the pullback of the line bundle $\mathcal{O}_{\mathscr{C}}(\lfloor (j/m)\mathscr{C}_k \rfloor)$ to the reduced strict normal crossings divisor $\sum_{i \in I_j} E_i$. Computing the above Euler characteristic as in (5.2.3), we find that

$$\gamma_j = (\sum_{i \in I_j} E_i \cdot \lfloor (j/m)\mathscr{C}_k \rfloor) + \sum_{i \in I_j} g(E_i) - |I_j| + \sigma_j.$$

It remains to compute the multiplicity of 0 as a jump of $\mathrm{Jac}(C)$. By (5.1.1) and Lemma 5.1.4, it is equal to the number of occurrences of the value 0 in the tuple of elementary divisors of the inclusion of lattices $\Omega_{\log}(C) \subset \Omega_{\mathrm{can}}(C)$ (which is $g - u(C)$ by Proposition 5.2.5) plus the number of occurrences of the value m in the tuple of elementary divisors of the inclusion of lattices

$$\Omega_{\log}(C) \otimes_R R^s \subset \Omega_{\mathrm{sat}}(C).$$

The latter number is equal to the dimension of the k-vector space

$$V_{m-1} \cong H^0(\mathscr{C}_k, \mathcal{L}_k \otimes \mathcal{F}_{m-1}),$$

but it follows from Proposition 5.3.6 that the sheaf $\mathcal{F}_{m-1}$ vanishes. Thus the multiplicity of 0 as a jump of $\mathrm{Jac}(C)$ is equal to $g - u(C)$. By Proposition 5.3.6, $g - u(C)$ is given by the expression in the statement for $j = 0$ because $I_0 = \{1, \ldots, r\}$. $\square$

Corollary 5.4.2. *The jumps of C only depend on the combinatorial reduction data of C, and not on the characteristic of k.*

Proof. This is obvious, since all the terms in the formula in Theorem 5.4.1 only depend on the combinatorial reduction data. $\square$

Corollary 5.4.3. *The number of non-zero jumps of $\mathrm{Jac}(C)$ (counted with multiplicities) is equal to the unipotent rank of $\mathrm{Jac}(C)$.*

Proof. We have already remarked in the proof of Theorem 5.4.1 that the multiplicity of zero as a jump is $g - u(C)$, and the total number of jumps is g. $\square$

5.5. **Jumps and principal components.**

(5.5.1) Let $\mathscr{C}$ be the minimal *sncd*-model of C and write $\mathscr{C}_k = \sum_{i=1}^r N_i E_i$ and $m = \mathrm{lcm}\{N_1, \ldots, N_r\}$ as before. It is clear from Theorem 5.4.1 that all the jumps of $\mathrm{Jac}(C)$ are of the form $j/m = a/N_i$ for some i in $\{1, \ldots, r\}$ and some a in $\{0, \ldots, N_i - 1\}$ (otherwise, the set I_j is empty). However, we will now show that one can deduce much more precise information about which components E_i can give rise to a jump. We keep the notations I_j and σ_j from Theorem 5.4.1.

Proposition 5.5.2. *Let j be an element of $\{1, \ldots, m-1\}$. If we denote by Γ_j the dual graph of $\sum_{i \in I_j} E_i$ and by $\beta(\Gamma_j)$ its first Betti number, then the multiplicity of j/m as a jump of $\mathrm{Jac}(C)$ is at least*

$$\beta(\Gamma_j) + \sum_{i \in I_j} g(E_i).$$

Proof. We write

$$(j/m)\mathscr{C}_k = \lfloor (j/m)\mathscr{C}_k \rfloor + \{(j/m)\mathscr{C}_k\}$$

as in the proof of Proposition 5.3.9. The support of the $\mathbb{Q}$-divisor $\{(j/m)\mathscr{C}_k\}$ is precisely the union of the components E_i with $i \notin I_j$, and the coefficients in this $\mathbb{Q}$-divisor are strictly contained between 0 and 1.

Let D be a connected component of $\sum_{i \in I_j} E_i$. Then $D \cdot \lfloor (j/m)\mathscr{C}_k \rfloor$ is strictly bigger than the negative of the number of intersection points of D with $\sum_{i \notin I_j} E_i$. Summing over the connected components, we find that

$$(\sum_{i \in I_j} E_i \cdot \lfloor (j/m)\mathscr{C}_k \rfloor)$$

is at least $c_j - \sigma'_j$ where c_j is the number of connected components of the divisor $\sum_{i \in I_j} E_i$ and σ'_j the number of intersection points of $\sum_{i \in I_j} E_i$ with the remainder of the special fiber. Using the formula in Theorem 5.4.1 we find that the multiplicity of j/m as a jump of $\mathrm{Jac}(C)$ is at least

$$c_j - |I_j| + \sigma_j - \sigma'_j + \sum_{i \in I_j} g(E_i).$$

The value $\sigma_j - \sigma'_j$ is precisely the number of edges of the dual graph Γ_j of $\sum_{i \in I_j} E_i$. Since $|I_j|$ is the number of vertices of Γ_j and c_j its number of connected components, we see that $c_j - |I_j| + \sigma_j - \sigma'_j$ equals the first Betti number of Γ_j. $\qquad\square$

Corollary 5.5.3. *If E is an irreducible component in $\mathscr{C}_k$ of multiplicity N and the genus of E is at least one, then a/N is a jump of $\mathrm{Jac}(C)$ for every a in $\{1, \ldots, N-1\}$.*

Proof. We set $j = am/N$ so that $j/m = a/N$. By our assumption, there is at least one element $i \in I_j$ such that $g(E_i) > 0$, so that j/m is a jump of $\mathrm{Jac}(C)$ by Proposition 5.5.2. $\qquad\square$

Lemma 5.5.4. *Let j be an element of $\{1, \ldots, N-1\}$ and let i_0 be an element of I_j. The intersection of E_{i_0} with $\sum_{i \notin I_j} E_i$ is either empty or consists of at least two points.*

Proof. We once more use the equality

(5.5.5) $\qquad (E_{i_0} \cdot \lfloor (j/m)\mathscr{C}_k \rfloor) = -(E_{i_0} \cdot \{(j/m)\mathscr{C}_k\}).$

Since the left hand side of (5.5.5) is an integer, we see that the intersection of E_{i_0} with $\sum_{i \notin I_j} E_i$ cannot consist of a single point. $\qquad\square$

Proposition 5.5.6. *Each non-zero jump of $\mathrm{Jac}(C)$ is of the form a/N where N is the multiplicity of a principal component in $\mathscr{C}_k$ and a is an element of $\{1, \ldots, N-1\}$. Conversely, for every principal component of $\mathscr{C}_k$ of multiplicity N, there exist a multiple N' of N and an element a in $\{1, \ldots, N'-1\}$ such that a is prime to N' and a/N' is a jump of $\mathrm{Jac}(C)$.*

Proof. Assume that j is an element of $\{1, \ldots, m-1\}$ such that the components E_i with $i \in I_j$ are all non-principal. We will show that j/m cannot be a jump of $\mathrm{Jac}(C)$. A component E_i with $i \in I_j$ cannot be a rational curve intersecting the rest of $\mathscr{C}_k$ in precisely one point, since

otherwise, there would be a principal component E_ℓ with $\ell \in I_j$ by Lemma 4.4.4. Thus each component E_i with $i \in I_j$ is a rational curve intersecting the rest of $\mathscr{C}_k$ in precisely two points. By Lemma 5.5.4 this is only possible if each connected component of $\sum_{i \in I_j} E_i$ consists of a single curve E_i (note that $\sum_{i \in I_j} E_i$ cannot be the entire reduced special fiber $\mathscr{C}_{k,\mathrm{red}}$ by our overall assumption that the multiplicities N_i are coprime). By Theorem 5.4.1, the multiplicity of j/m as a jump of $\mathrm{Jac}(C)$ equals

$$(\sum_{i \in I_j} E_i \cdot \lfloor (j/m)\mathscr{C}_k \rfloor) - |I_j| + \sigma_j.$$

Again using the equality (5.5.5) we see that the intersection product

$$(\sum_{i \in I_j} E_i \cdot \lfloor (j/m)\mathscr{C}_k \rfloor)$$

is at most $-|I_j|$. On the other hand, $-|I_j| + \sigma_j$ is equal to $|I_j|$. Thus

$$(\sum_{i \in I_j} E_i \cdot \lfloor (j/m)\mathscr{C}_k \rfloor) - |I_j| + \sigma_j = 0$$

and j/m is not a jump of C.

Now we prove the converse statement. Let i_0 be an element of $\{1, \ldots, r\}$ such that E_{i_0} is principal; we will prove that there exist a multiple N' of N_{i_0} and an element a in $\{1, \ldots, N' - 1\}$ such that a is prime to N' and a/N' is a jump of $\mathrm{Jac}(C)$. We may assume that there does not exist a principal component E_{i_1} in $\mathscr{C}_k$ such that N_{i_1} is a multiple of N_{i_0} and $N_{i_1} > N_{i_0}$ (otherwise we can simply replace i_0 by i_1). By Corollary 5.5.3 we may also assume that there does not exist a component of multiplicity N_{i_0} with positive genus. Finally, if we set $j = m/N_{i_0}$, then by Proposition 5.5.2, we can also suppose that the divisor $\sum_{i \in I_j} E_i$ does not contain a loop. Otherwise, $\beta(\Gamma_j)$ is positive and $j/m = 1/N_{i_0}$ is a jump.

Assume that $\ell \in I_j$ and that E_ℓ is a non-principal component of $\mathscr{C}_k$. Then E_ℓ must intersect the rest of $\mathscr{C}_k$ in exactly two points, because otherwise there would exist a principal component in $\mathscr{C}_k$ whose multiplicity is a strict multiple of N_{i_0} by Lemma 4.4.4, contradicting the maximality of N_{i_0}. If E and E' are the components of $\mathscr{C}_k$ intersecting E_ℓ (where possibly $E = E'$) then Lemma 5.5.4 implies that either E and E' are both contained in $\sum_{i \in I_j} E_i$, or neither of them is. From these observations we deduce that some connected component of $\sum_{i \in I_j} E_i$ must intersect $\sum_{i \notin I_j} E_i$ in at least three points. We denote by M_1, M_2 and M_3 the multiplicities of the components of $\sum_{i \notin I_j} E_i$ intersecting $\sum_{i \in I_j} E_i$ for an arbitrary choice of three such intersection points. Then for at least one of both elements a in $\{1, N_{i_0} - 1\}$ we have

$$\{aM_1/N_{i_0}\} + \{aM_2/N_{i_0}\} + \{aM_3/N_{i_0}\} < 2,$$

where $\{q\} = q - \lfloor q \rfloor$ denotes the fractional part of a rational number q. In the notation of the proof of Proposition 5.5.2, this implies that

$$(\sum_{i \in I_{aj}} E_i \cdot \lfloor (aj/m)\mathscr{C}_k \rfloor)$$

is at least $c_j + 1 - \sigma'_j$, and the arguments in that proof show that $aj/m = a/N_{i_0}$ is a jump of $\mathrm{Jac}(C)$. $\qquad\square$

Corollary 5.5.7. *The stabilization index $e(C)$ of C is equal to least common multiple of the denominators of the jumps of $\mathrm{Jac}(C)$, that is, the smallest positive integer e such that $e \cdot j$ is an integer for every jump j of $\mathrm{Jac}(C)$. Thus the stabilization index of C is equal to the stabilization index of $\mathrm{Jac}(C)$ in the sense of (2.2.5).*

Proof. This is an immediate consequence of Proposition 5.5.6. $\qquad\square$

Corollary 5.5.8. *The stabilization index $e(C)$ is equal to the smallest possible saturation index of an R-model $\mathscr{C}$ of C such that $\mathscr{C}^+$ is log regular.*

Proof. If m is the saturation index of such a model $\mathscr{C}$, then it follows from Theorems 4.3.5 and 4.4.1 that $m \cdot j$ is an integer for every jump j of C (this was already observed in the proof of Theorem 4.4.5). Thus by Corollary 5.5.7, m must be divisible by $e(C)$. On the other hand, we have constructed in the proof of Theorem 4.4.5 a model $\mathscr{C}$ such that $\mathscr{C}^+$ is log regular and such that the saturation index of $\mathscr{C}^+ \to S^+$ is precisely $e(C)$. $\qquad\square$

5.6. **Examples.**

(5.6.1) We illustrate with a few examples how the formula in Theorem 5.4.1 can be used to compute the jumps in practice. We start with the case where C is an elliptic curve. Then C has precisely one jump, which is zero if and only if C has semi-stable reduction. If this is not the case, the unipotent rank $u(C)$ equals one, and the unique jump is non-zero.

Let $\mathscr{C}$ denote the minimal *sncd*-model of C. If C has reduction type $I^*_{\geq 0}$, the multiplicity of any irreducible component of $\mathscr{C}_k$ is either 1 or 2, and the jump is therefore $1/2$. In all other remaining cases, the special fiber $\mathscr{C}_k$ has precisely one principal component, which is a rational curve intersecting the other components in precisely three points. We denote its multiplicity by N. Then we know by Proposition 5.5.6 that the jump of C is of the form a/N. The proof of Proposition 5.5.6 even tells us how to find a: it is the unique element in $\{1, N-1\}$ that satisfies the property that

$$\{aM_1/N\} + \{aM_2/N\} + \{aM_3/N\} < 2$$

where M_1, M_2 and M_3 are the multiplicities of the components of $\mathscr{C}_k$ intersecting the principal component. In this way, we immediately recover the table from [Ed92, 5.4.5] and [Ha10b, Table 8.1] giving the jump for each of the Kodaira-Néron reduction types (see Table 5.6.1).

Type	$I_{\geq 0}$	II	III	IV	$I^*_{\geq 0}$	IV^*	III^*	II^*
jump	0	1/6	1/4	1/3	1/2	2/3	3/4	5/6

TABLE 5.6.1. The jump of an elliptic curve

(5.6.2) Now we consider the case where C has genus 2. Let $\mathscr{C}$ be the minimal *sncd*-model of C and assume that $\mathscr{C}_k$ has a component of genus one and multiplicity $N > 1$. Then the unipotent rank of $\mathrm{Jac}(C)$ is one, so that 0 is a jump of C of multiplicity one. On the other hand, Corollary 5.5.3 tells us that a/N is a jump of C for every a in $\{1, \ldots, N-1\}$. Since the total number of jumps of C is equal to 2, we must have $N = 2$, and the jumps of C are 0 and $1/2$. With similar arguments one can quickly reproduce the tables in [Ha10b, §8.4] which give the jumps of C for each of the reduction types in the Namikawa-Ueno classification of degenerations of genus 2 curves.

References

[SGA7-I] *Groupes de monodromie en géométrie algébrique. I.* Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I). Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D.S. Rim. Volume 288 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1972.

[BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models.* Volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, 1990.

[Ch00] C.-L. Chai. Néron models for semiabelian varieties: congruence and change of base field. *Asian J. Math.*, 4(4):715–736, 2000.

[CY01] C.-L. Chai and J.-K. Yu. Congruences of Néron models for tori and the Artin conductor (with an appendix by E. de Shalit). *Ann. Math. (2)*, 154:347–382, 2001.

[De71] P. Deligne. Théorie de Hodge II. *Publ. Math. Inst. Hautes Étud. Sci.* 40:5–57, 1971.

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Publ. Math. Inst. Hautes Étud. Sci.*, 36:75–109, 1969.

[Ed92] B. Edixhoven. Néron models and tame ramification. *Compos. Math.*, 81:291–306, 1992.

[Ha10a] L.H. Halle. Stable reduction of curves and tame ramification. *Math. Z.*, 265(3):529–550, 2010.

[Ha10b] L.H. Halle. Galois actions on Néron models of Jacobians. *Ann. Inst. Fourier*, 60(3):853–903, 2010.

[HN11] L.H. Halle and J. Nicaise. Motivic zeta functions of abelian varieties, and the monodromy conjecture. *Adv. Math.*, 227:610–653, 2011.

[HN11b] L.H. Halle and J. Nicaise. Jumps and monodromy of abelian varieties. *Doc. Math.*, 16:937–968, 2011.

[HN14] L.H. Halle and J. Nicaise. Néron models and base change. Submitted, arXiv:1209.5556.

[Ha80] R. Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.

[Ha94] R. Hartshorne. Generalized divisors on Gorenstein schemes. *K-Theory*, 8(3):287–339, 1994.

[IS14] H. Ito and S. Schröer. Wild quotient surface singularities whose dual graphs are not star-shaped. Preprint, arXiv:1209.3605.

[Ka89] K. Kato. Logarithmic structures of Fontaine-Illusie. In: *Algebraic analysis, geometry, and number theory.* Johns Hopkins Univ. Press, Baltimore, MD, pages 191–224, 1989.

[Ka94] K. Kato. Toric singularities. *Am. J. Math.*, 116(5):1073–1099, 1994.

[KS04] K. Kato and T. Saito. On the conductor formula of Bloch. *Publ. Math. Inst. Hautes Études Sci.*, 100:5–151, 2004.

[Li69] J. Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Publ. Math. Inst. Hautes Études Sci.*, 36:195–279, 1969.

[Lo90] D. Lorenzini. Groups of components of Néron models of Jacobians. *Compos. Math.*, 73(2):145–160, 1990.

[Ni13] J. Nicaise. Geometric criteria for tame ramification. *Math. Z.*, 273(3):839–868, 2013.

[Ni06] W. Niziol. Toric singularities: log-blow-ups and global resolutions. *J. Algebraic Geom.* 15(1):1–29, 2006.

[Ra70] M. Raynaud. Spécialisation du foncteur de Picard. *Publ. Math. Inst. Hautes Étud. Sci.*, 38:27–76, 1970.

[Vi04] I. Vidal. Monodromie locale et fonctions zêta des log schémas. In: *Geometric aspects of Dwork theory. Vol. II.* Walter de Gruyter GmbH & Co. KG, Berlin, pages 983–1038, 2004.

[Wi74] G. B. Winters. On the existence of certain families of curves. *Am. J. Math.*, 96:215–228, 1974.

GOTHENBURG UNIVERSITY AND CHALMERS INSTITUTE FOR TECHNOLOGY, DEPARTMENT OF MATHEMATICS, 412 96 GOTHENBURG, SWEDEN
  E-mail address: `dener@chalmers.se`

UNIVERSITY OF STAVANGER, DEPARTMENT OF MATHEMATICS AND NATURAL SCIENCES, 4036 STAVANGER, NORWAY
  E-mail address: `lars.h.halle@uis.no`

KU LEUVEN, DEPARTMENT OF MATHEMATICS, CELESTIJNENLAAN 200B, 3001 HEVERLEE, BELGIUM
  E-mail address: `johannes.nicaise@wis.kuleuven.be`