Odd order Brauer–Manin obstruction on diagonal quartic surfaces

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Abstract

We determine the odd order torsion subgroup of the Brauer group of diagonal quartic surfaces over the field of rational numbers. We show that a non-constant Brauer element of odd order always obstructs weak approximation but never the Hasse principle.

1 Introduction

This paper is devoted to the arithmetic of diagonal quartic surfaces over the field of rational numbers. Let $D = [a, b, c, d] \subset \mathbb{P}_{\mathbb{Q}}^3$ be the surface given by

$$ax^4 + by^4 = cz^4 + dw^4,$$

where $a, b, c, d \in \mathbb{Q}^*$. The set of rational points $D(\mathbb{Q})$ is a subset of the set of adelic points $D(\mathbb{A}_{\mathbb{Q}}) = \prod D(\mathbb{Q}_p)$, where the product is over all completions of $\mathbb{Q}$ including $\mathbb{R}$. The closure of $D(\mathbb{Q})$ in the product topology is contained in the Brauer–Manin set $D(\mathbb{A}_{\mathbb{Q}})^{Br}$ defined as the set of adelic points orthogonal to the Brauer group $\text{Br}(D)$ with respect to the pairing provided by class field theory, see [16, Ch. 6]. The image $\text{Br}_0(D)$ of the natural map $\text{Br}(\mathbb{Q}) \to \text{Br}(D)$ pairs trivially with $D(\mathbb{A}_{\mathbb{Q}})$. The group $\text{Br}(D)/\text{Br}_0(D)$ is finite [17]. However, the computation of this group for an arbitrary diagonal quartic $D$ over $\mathbb{Q}$ is still an open problem. Results obtained in [9, 10] allow one to prove that $\text{Br}(D)/\text{Br}_0(D)$ is zero in certain particular cases. Recall that $\text{Br}_1(D)$ denotes the kernel of the natural map $\text{Br}(D) \to \text{Br}(\overline{D})$. Martin Bright [2, 3] determined the group structure of $\text{Br}_1(D)/\text{Br}_0(D)$ for any diagonal quartic surface $D$ over $\mathbb{Q}$, in particular, he showed that the order of this group is a power of 2.

In this paper we compute the odd order torsion subgroup $(\text{Br}(D)/\text{Br}_0(D))^\text{odd}$ of $\text{Br}(D)/\text{Br}_0(D)$, improving on the estimate obtained previously by the authors jointly with Yuri Zarhin [10, Cor. 3.3, Cor. 4.6]. Let $\Gamma_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the Galois group of $\mathbb{Q}$, and let $\overline{\Gamma} = D \times_{\mathbb{Q}} \overline{\mathbb{Q}}$. Let $\langle a \rangle \subset \mathbb{Q}^*$ be the cyclic subgroup generated by $a \in \mathbb{Q}^*$.
Theorem 1.1 Let $D = [a, b, c, d]$ be a diagonal quartic surface over $\mathbb{Q}$. Then

$$(\text{Br}(D)/\text{Br}_0(D))_{\text{odd}} = \text{Br}(D)_{\text{inv}} \simeq \begin{cases} 
\mathbb{Z}/3 & \text{if } -3abcd \in \langle -4 \rangle \mathbb{Q}^\times, \\
\mathbb{Z}/5 & \text{if } 125abcd \in \langle -4 \rangle \mathbb{Q}^\times, \\
0 & \text{otherwise.}
\end{cases}$$

For an abelian group $A$ we denote by $A_\ell$ the $\ell$-torsion subgroup of $A$ and by $A_{\ell\infty}$ the $\ell$-primary torsion subgroup of $A$. For an odd prime $\ell$ the results of Bright and the divisibility of every element of $\text{Br}_0(D)$ by $\ell$ imply the surjectivity of the natural map $\text{Br}(D)_\ell \to (\text{Br}(D)/\text{Br}_0(D))_\ell$. Thus for $\ell = 3$ or $\ell = 5$ under the conditions of Theorem 1.1 we have an element of $\text{Br}(D)_\ell$ which is not in $\text{Br}_0(D)$. The first example of a diagonal quartic surface $D$ with a non-constant 3-torsion element in $\text{Br}(D)$ was found by Thomas Preu [13]. Preu’s surface $D = [1, 3, 4, 9]$ has an obvious rational point, and he shows that for any rational point on his surface one has $3 |yw$ while there are $\mathbb{Q}_3$-points not satisfying this condition. This failure of weak approximation is explained by a 3-torsion element in $\text{Br}(D)$. We show that this situation is quite general.

Let $\text{inv}_p : \text{Br}(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ be the local invariant isomorphism of class field theory. For an element $A \in \text{Br}(D)_n$ we denote by $\text{ev}_{A,p} : D(\mathbb{Q}_p) \to \mathbb{Z}/\mathbb{Z}$ the evaluation map at the prime $p$, defined as $\text{ev}_{A,p}(P) = \text{inv}_p(A(P))$.

Theorem 1.2 Let $D$ be a diagonal quartic surface over $\mathbb{Q}$ such that $D(\mathbb{Q}_p) \neq \emptyset$. If $A \in \text{Br}(D)_\ell \setminus \text{Br}_0(D)$, where $\ell$ is an odd prime, then

(i) the map $\text{ev}_{A,p}$ is constant for $p \neq \ell$,

(ii) the map $\text{ev}_{A,\ell}$ is surjective.

Proof. By Theorem 1.1 we only need to consider $\ell = 3$ and $\ell = 5$. Statement (i) is proved in Proposition 5.5, and statement (ii) is proved in Proposition 5.12 for $\ell = 5$ and in Proposition 5.19 for $\ell = 3$. □

Corollary 1.3 If $D$ is a diagonal quartic surface over $\mathbb{Q}$ such that $D(\mathbb{A}_Q)$ is non-empty, then $D(\mathbb{A}_Q)^{\text{Br}(D)_{\text{odd}}}$ is non-empty too. If $\text{Br}(D)_{\text{odd}}$ is not contained in $\text{Br}_0(D)$, then $D(\mathbb{A}_Q)^{\text{Br}(D)_{\text{odd}}} \neq D(\mathbb{A}_Q)$.

Proof. By Theorem 1.1 the group $\text{Br}(D)_{\text{odd}}$ is generated by $\text{Br}_0(D)$ and at most one more element $A$ of prime order $\ell \neq 2$. Since $\text{Br}(\mathbb{R}) \cong \mathbb{Z}/2$, evaluating $A$ at any point of $X(\mathbb{R})$ gives zero. Theorem 1.2 now implies that the Brauer–Manin pairing with $A$ is a surjective function $D(\mathbb{A}_Q) \to \mathbb{Z}/\mathbb{Z}$. Since $D(\mathbb{A}_Q)^{\text{Br}(D)_{\text{odd}}} = D(\mathbb{A}_Q)^{A}$ is the zero set of this function, we obtain the corollary. □

The fact that $\text{Br}(D)_{\text{odd}}$ never obstructs the Hasse principle and always obstructs weak approximation on diagonal quartic surfaces over $\mathbb{Q}$ came to us as a surprise.
It seems that all known counterexamples to the Hasse principle on K3 surfaces are given by elements of even order, see [2, 3, 8] and references in these papers. This prompts the following general question.

**Question.** Does there exist a K3 surface $X$ over a number field $k$ with $X(\mathbb{A}_k) \neq \emptyset$ such that $X(\mathbb{A}_k)^{\text{Br}(X)_{\text{odd}}}$ is empty?

Note that when $X(\mathbb{A}_k) \neq \emptyset$ and the degree of the polarisation of $X$ (which is always even) is a power of 2, e.g. when $X$ is a quartic K3 surface, there is a 0-cycle of degree 1 on $X$ over each completion of $k$ such that this collection is orthogonal to $\text{Br}(D)_{\text{odd}}$ with respect to the Brauer–Manin pairing. If, moreover, $\text{Br}(D)/\text{Br}_0(D)$ has odd order, this collection is orthogonal to all of $\text{Br}(D)$. In this case a conjecture of Colliot-Thélène states the existence of a 0-cycle of degree 1 on $X$. In contrast, $X(\mathbb{A}_k)^{\text{Br}(X)_{\text{odd}}} = \emptyset$ implies that $X$ has no $k$-point.

Any K3 surface $X$ has a 0-cycle of degree 24, namely the second Chern class of $X$. In this connection one can ask if an element of $\text{Br}(X)$ of order prime to 6 can obstruct the Hasse principle\(^1\).

For everywhere locally soluble diagonal quartic surfaces $D$ over $\mathbb{Q}$ whose coefficients $a, b, c, d$ are general enough, Bright [4] has shown using [9, 10] that $\text{Br}(D)/\text{Br}_0(D) = \text{Br}_1(D)/\text{Br}_0(D) \simeq \mathbb{Z}/2$ but the Brauer group does not obstruct the Hasse principle. Conditional results on the existence of rational points on diagonal quartic surfaces can be found in [19] and [20, Thm. 1.51].

The failure of weak approximation in Corollary 1.3 can be illustrated in explicit examples. The following result is an immediate corollary of the proof of Proposition 5.12.

**Corollary 1.4** Let $D = [a, b, c, d]$, where $a, b, c, d$ are fourth power free integers such that $5^3abcd \in (-4)^{\mathbb{Q}^4}$. Let $x, y, z, w$ be integers such that $(x, y, z, w) \in D(\mathbb{Q})$.

1. If $a, b, c \in \mathbb{Z}_5^*$ and $a + b \equiv 0 \mod 5$, then $25|xyzw$.
2. If $a, b \in \mathbb{Z}_5^*$ and $c \equiv d \equiv 0 \mod 5$, then $5|zw$.

We would like to emphasise that these divisibility conditions cannot be obtained by solving corresponding congruences modulo powers of 5. Indeed, any surface satisfying the assumption of (1) has a solution $(\alpha, 1, 5, 1)$ for some $\alpha \in \mathbb{Z}_5^*$, and any surface satisfying the assumption of (2) has a solution $(\beta, 1, 1, 1)$ for some $\beta \in \mathbb{Z}_5^*$. For example, for any $n \in \mathbb{Z}$ coprime to 5 and any $\epsilon \in \{0, 1\}$ the surfaces in the family

$$x^4 - y^4 = nz^4 - 5n^3(-4)^\epsilon w^4$$

satisfy the condition in (1), hence all integral solutions of these equations satisfy $25|xyzw$. The surfaces

$$x^4 - y^4 = 5^2nz^4 - 5^3n^3(-4)^\epsilon w^4$$

\(^1\)This observation and this question were suggested by the referee.
satisfy the condition in (2), so in this case we have $5|zw$.

Similarly, the next statement is a corollary of the proof of Proposition 5.19.

**Corollary 1.5** Let $D = [a, b, c, d]$, where $a, b, c, d$ are fourth power free integers such that $-3abcd \in (-4)\mathbb{Q}^{*4}$. Let $x, y, z, w$ be integers such that $(x, y, z, w) \in D(\mathbb{Q})$.

1. If $a, b, c \in \mathbb{Z}_3^*$ and $a \equiv b \equiv c \mod 3$, then $9|xyw$.
2. If $a, b \in \mathbb{Z}_3^*$ and $c \equiv d \equiv 0 \mod 3$, then $3|zw$.

In the same way as before these divisibility conditions cannot be obtained by solving corresponding congruences modulo powers of 3. For any $n \equiv 1 \mod 3$ and any $\epsilon \in \{0, 1\}$ the surfaces in the family

$$x^4 + ny^4 = z^4 - 27(-4)^\epsilon n^3 w^4$$

satisfy the condition in (1), hence all integral solutions of these equations satisfy $9|xyw$. (For any $n \equiv 2 \mod 3$ we have $9|yzw$.) The surfaces

$$x^4 - y^4 = 3nz^4 + 9(-4)^\epsilon n^3 w^4$$

satisfy the condition in (2), so in this case we have $3|zw$. The same holds for the surface $x^4 - 4y^4 = -3z^4 + 9w^4$ considered in [13].

The proof of Theorem 1.1 uses a geometric step which is a reduction to a certain Kummer surface and then to a product of two elliptic curves $E^{m_1} \times E^{m_2}$, see Section 2. These curves have complex multiplication by $\sqrt{-1}$ and the Weierstraß equation $y^2 = x^3 - mx$, so they are quartic twists of each other. The Galois module structure of $\text{Br}(E^{m_1} \times E^{m_2})$ is then explicitly described in Section 3 leading to a proof of Theorem 1.1 in Section 4. The proof of Theorem 1.2 is carried out in Section 5. Our main tool is the standard theory of the formal group attached to an elliptic curve; it provides a simple way to decide if a point on an elliptic curve over $\mathbb{Q}_p$ is divisible by $\ell$. The case when $p \neq \ell$ is easy, but the case $p = \ell$, although it is essentially a finite calculation, requires a subtle analysis of some local pairings.

We expect the methods of this paper to be applicable to more general surfaces dominated by products of curves, such as the surfaces in $\mathbb{P}^3$ given by the equation $P(x, y) = Q(z, w)$, where $P$ and $Q$ are homogeneous polynomials of the same degree.

This paper started as an attempt to understand the calculations in [13], and we are grateful to Thomas Preu for a very useful explanation of his work at the conference “Arithmetic of surfaces” at the Lorentz Center (Leiden). The work on this paper continued during the second author’s visits to the University of Cyprus (Nicosia), the Centre Interfacultaire Bernoulli (Lausanne) and the Hausdorff Research Institute for Mathematics (Bonn). He is deeply grateful to these institutions for their hospitality. We thank Rachel Newton for her interest in this paper and Anthony Várilly-Alvarado for useful discussions.
2 Isogenies between K3 and abelian surfaces

Let $k$ be a field of characteristic 0 with an algebraic closure $\overline{k}$ and absolute Galois group $\Gamma_k = \text{Gal}(\overline{k}/k)$. For a variety $X$ over $k$ we write $X = X \times_k \overline{k}$.

A rational dominant map of K3 or abelian surfaces will be called an isogeny. It is well known that the ranks of the lattices of transcendental cycles of isogenous surfaces are equal. This implies that the geometric Brauer group of a K3 or abelian surface, as an abelian group, depends only on the isogeny class of the surface. Indeed, it is isomorphic to $(\mathbb{Q}/\mathbb{Z})^t$, where $t$ is the rank of the lattice of transcendental cycles.

We now consider isogenies between certain quartic K3 surfaces in $\mathbb{P}^3$ and products of two curves of genus 1. Let $p(t)$ and $q(t)$ be separable polynomials of degree 4. We denote the genus 1 curves $y_1^2 = p(x_1)$, $y_2^2 = q(x_2)$ by $C$ and $C'$, respectively. Let $J_1 = \text{Jac}(C)$ and $J_2 = \text{Jac}(C')$. It is a classical fact that the Jacobian $J$ of $C$ is given by $u^2 = f(t)$, where $f(t)$ is the (monic) resolvent cubic polynomial of $p(x)$, see [1, Lemma 3] or [16, Prop. 3.3.6 (a)].

Let $P(x, y)$ and $Q(z, w)$ be the quartic forms such that $p(t) = P(t, 1)$ and $q(t) = Q(t, 1)$, and let $X$ be the quartic surface in $\mathbb{P}^3$ given by

$$P(x, y) = Q(z, w).$$

It is smooth, hence a K3 surface. Let $Y$ be the Kummer surface $\text{Kum}(C \times C')$ defined as the minimal desingularisation of the quotient of $C \times C'$ by the involution simultaneously changing the signs of $y_1$ and $y_2$. Finally, let $Z = \text{Kum}(J \times J')$.

**Proposition 2.1** There is a commutative diagram of isogenies whose degrees are powers of 2:

$$
\begin{array}{ccc}
C \times C' & \rightarrow & J \times J' \\
\downarrow & & \downarrow \\
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\end{array}
$$

**Proof** The vertical maps are isogenies of degree 2 that come from the definitions of $Y$ and $Z$. The isogeny $X \rightarrow Y$ of degree 2 is constructed as follows. The surface $Y$ is birationally equivalent to the affine surface $v^2 p(t) = q(z)$. Setting $v = y^2$, $yt = x$, we obtain $P(x, y) = Q(z, 1)$, which is an affine equation of $X$.

Choosing a root of $p(x) = 0$ as the origin of the group law on $C$ we define an isomorphism $C \cong J$ that identifies the action of $J$ on $C$ with the action of $J$ on $J$ by translations. This isomorphism identifies the involution $x \mapsto -x$ on $J$ with the involution on $C$ that changes the sign of $y_1$. The action of $J_2$ on $C$ gives rise to the étale morphism $C \rightarrow J = C/J_2$. It is compatible with the involutions on $C$ and
Thus we have a finite étale morphism $C \times C' \to J \times J'$ of degree 16, which is a torsor with structure group $J_2 \times J'_2$. Since this morphism is compatible with the involutions, it induces an isogeny $Y \sim Z$ of degree 16. □

**Corollary 2.2** Diagram (2) induces the following commutative diagram of homomorphisms of $\Gamma_k$-modules

\[
\begin{array}{cccc}
\text{Br}(\overline{J} \times \overline{J'}) & \longrightarrow & \text{Br}(\overline{C} \times \overline{C'}) \\
\uparrow\cong & & \uparrow\cong \\
\text{Br}(\overline{Z}) & \longrightarrow & \text{Br}(\overline{Y}) & \longrightarrow & \text{Br}(\overline{X})
\end{array}
\]

All the homomorphisms in this diagram induce bijections on the subgroups of elements of odd order.

**Proof.** In [10, Prop. 1.1] it is proved that if $V$ and $W$ are geometrically integral smooth varieties over $k$, and $f : V \to W$ is a dominant, generically finite morphism of degree $d$, then the kernel of $f^* : \text{Br}(W) \to \text{Br}(V)$ is killed by $d$. If $f : X \dashrightarrow Y$ is an isogeny of K3 or abelian surfaces of degree $d$, then there is a birational morphism $\varphi : X' \to X$ and a finite surjective morphism $f' : X' \to Y$ of degree $d$ such that $f = f'\varphi^{-1}$. Since $\varphi^* : \text{Br}(\overline{X}) \to \text{Br}(\overline{X}')$ is an isomorphism, there is an induced map of $\Gamma_k$-modules $f^* : \text{Br}(\overline{Y}) \to \text{Br}(\overline{X})$. This map is surjective with the kernel annihilated by $d$. In the particular case when $A$ is an abelian surface, the natural isogeny $f : A \dashrightarrow Y$ induces an isomorphism $f^* : \text{Br}(\overline{Y}) \to \text{Br}(\overline{A})$, by [18, Prop. 1.3]. Thus the vertical arrows are isomorphisms. The last statement follows from the fact that the degrees of all the isogenies in (2) are powers of 2. □

**Corollary 2.3** Let $n$ be an odd positive integer. Then the $\Gamma_k$-module $\text{Br}(\overline{X})_n$ is canonically isomorphic to the $\Gamma_k$-module $\text{Br}(\overline{J} \times \overline{J'})_n = \text{Hom}(J_n, J'_n)/(\text{Hom}(\overline{J}, \overline{J'})/n)$.

**Proof** This follows from the last sentence of Corollary 2.2 and [18, Prop. 3.3] which gives the equality in the displayed formula. □

**Remark.** Examples of elliptic curves $J, J'$ over $\mathbb{Q}$ such that $\text{Br}(\overline{J} \times \overline{J'})^0_{\text{odd}} = 0$ were constructed in [18, Section 4]. By Corollary 2.3 these examples give rise to quartic surfaces $X \subset \mathbb{P}^3_{\mathbb{Q}}$ with $\text{Br}(X)_{\text{odd}} \subset \text{Br}_0(X)$.

### 3 Twists of lemniscata

Let us denote by $E^c$ the elliptic curve $y^2 = x^3 - cx$, where $c \in k^*$. 

Corollary 3.1 Let $D = [a, b, c, d]$ be a diagonal quartic surface over $k$. If $n$ is odd, then the $\Gamma_k$-module $\text{Br}(D)_n$ is canonically isomorphic to

$$\text{Hom}(E_n^{4ab}, E_n^{4cd})/(\text{Hom}(E_n^{4ab}, E_n^{4cd})/n).$$

Proof. In our previous notation we have $p(t) = at^4 + b$ and $q(t) = ct^4 + d$. From [1, Lemma 3] or [16, Prop. 3.3.6 (a)] we see that $J$ is given by $y^2 = x^3 - 4abx$ and $J'$ is given by $y^2 = x^3 - 4cdx$, so that $J = E^{4ab}, J' = E^{4cd}$. It remains to apply Corollary 2.3. □

We can multiply $a, b, c, d$ by a common non-zero rational number without changing $D$. In particular, in the statement of Corollary 3.1 we can replace $4ab$ and $4cd$ by $ab$ and $cd$, respectively.

The curve $E^c$ has complex multiplication by $\mathcal{O} = \mathbb{Z}[\sqrt{-1}]$, so we need to consider in detail elliptic curves with such complex multiplication. The lemniscata is the elliptic curve $E$ with the equation $y^2 = x^3 - x$. Note that $\mathcal{O}$ is the full endomorphism ring $\text{End}(E)$ of $E$. The normalised action of $\mathcal{O}$ on $E$ is the action such that the induced action of $z \in \mathcal{O}$ on the differential form $y^{-1}dx$ is the multiplication by $z$. Then $i = \sqrt{-1}$ sends $(x, y)$ to $(-x, iy)$. For $z \in \mathcal{O}$ we denote the normalised action of $z$ by $[z] \in \text{End}(E^c)$.

Since $\mathcal{O}^* = \{\pm 1, \pm i\} = \mu_4$, we see that the group $k$-scheme $\mu_4$ acts on $E$. The following well known lemma is checked directly from the definition of twisting.

Lemma 3.2 The curve $E^c$ is the quartic twist of $E$ corresponding to the class of $c^{-1}$ in $H^1(k, \mu_4) = k^*/k^{*4}$.

Let $n$ be an odd positive integer. If $\phi : E_n \to E_n$ is an endomorphism, and $x, y \in \mathcal{O}^*$, then $y\phi x^{-1}$ is also an endomorphism of $E_n$. Thus $\text{End}(E_n)$ has a natural structure of a $\mu_4 \times \mu_4$-module. Consider the action of $[i]$ on $\text{End}(E_n)$ by conjugation, that is, via the diagonal map $\mu_4 \to \mu_4 \times \mu_4$. Since all endomorphisms commute with $[-1]$, and $n$ is odd, $\text{End}(E_n)$ is the direct sum of the eigenspaces of $[i]$ with eigenvalues 1 and $-1$:

$$\text{End}(E_n) = \text{End}(E_n)^+ \oplus \text{End}(E_n)^-. \tag{4}$$

This is clearly a direct sum of $\mu_4 \times \mu_4$-modules. It is easy to see that $\text{rk}(\text{End}(E_n)^+) = \text{rk}(\text{End}(E_n)^-) = 2$. It follows that the inclusion $\mathcal{O}/n \subset \text{End}(E_n)^+$ is an equality. The action of the Galois group $\Gamma_k$ preserves this decomposition, hence we have canonical isomorphisms of $\Gamma_k$-modules, and also of $\mu_4 \times \mu_4$-modules:

$$\text{End}(E_n)^- = \text{End}(E_n)/(\mathcal{O}/n). \tag{5}$$

By Corollary 3.1 this is canonically isomorphic to the $\Gamma_k$-module $\text{Br}(X)_n$, where $X = [1, 1, 1, 1]$ is the ‘untwisted’ diagonal quartic surface. We note that the actions
of $\mu_4$ on $\text{End}(E_n)^-$ coming from the action of $\mu_4$ on the source $E_n$ and on the target $E_n$, coincide. (Indeed, $\varphi = \varphi[i]^{-1}$ for any $\varphi \in \text{End}(E_n)^-$. We shall always consider $\text{End}(E_n)^-$ with this structure of a $\mu_4$-module.

**Proposition 3.3** Let $D = [a, b, c, d]$ be a diagonal quartic surface over $k$. If $n$ is odd, then the $\Gamma_k$-module $\text{Br}(\overline{D})_n$ is canonically isomorphic to the twist of $\text{End}(E_n)^-$ by a cocycle whose class in $H^1(k, \mu_4) = k^*/k^4$ is represented by $(abcd)^{-1}$.

**Proof.** We use Corollary 3.1. By Lemma 3.2 the $\Gamma_k$-module $\text{Hom}(E_n^{ab}, E_n^{cd})$ is the twist of $\text{End}(E_n)$ by a cocycle with the class $(4ab, 4cd)^{-1} \in H^1(k, \mu_4) \times H^1(k, \mu_4)$, with respect to its natural $\mu_4 \times \mu_4$-module structure. Now the statement follows from the isomorphism (5), since the right and left actions of $\mu_4$ on $\text{End}(E_n)^-$ coincide. □

**Remark** Any quartic twist of $E$ has complex multiplication by $\mathcal{O}$, so similarly to (4) we can consider a decomposition of $\Gamma_k$-modules

$$\text{Hom}(E_n^{ab}, E_n^{cd}) = \text{Hom}(E_n^{ab}, E_n^{cd})^+ \oplus \text{Hom}(E_n^{ab}, E_n^{cd})^-.$$ 

Then Proposition 3.3 and Corollary 2.2 give canonical isomorphisms of $\Gamma_k$-modules

$$\text{Br}(\overline{D})_n = \text{Br}(\overline{Z})_n = \text{Hom}(E_n^{ab}, E_n^{cd})^- = \text{Hom}(E_n^{cd}, E_n^{ab})^-.$$ (6)

**Proposition 3.4** Let $D = [a, b, c, d]$ be a diagonal quartic surface over $k$. If $n$ is odd, then $(\text{Br}(D)/\text{Br}_0(D))_n = \text{Br}(\overline{D})_{\Gamma_k}^\Gamma$.

**Proof.** Consider the natural map $\text{Br}(D)/\text{Br}_0(D) \to \text{Br}(\overline{D})_{\Gamma_k}^\Gamma$. By Example 1 after Prop. 5.1 in [6, Thm. 4.3] the cokernel of this map is a finite abelian 2-group. On the other hand, the calculations in Bright’s thesis [2] show that the kernel is also a finite abelian 2-group. □

4 Proof of Theorem 1.1

It is known that in the proof of Theorem 1.1 we only need to consider torsion subgroups of order 3 and 5. Indeed, we have the following

**Lemma 4.1** For any odd prime $\ell$ we have $\text{Br}(\overline{D})_{\ell^\infty}^{\Gamma_0} = \text{Br}(\overline{D})_{\ell^\infty}^{\Gamma_0}$. If $\ell \geq 7$, then $\text{Br}(\overline{D})_{\ell^\infty}^{\Gamma_0} = 0$.

**Proof.** [10, Prop. 4.1, Thm. 3.2]. □

By Lemma 4.1 we can assume that $\ell$ is 3 or 5. This assumption will be in force until the end of this paper.
The abelian group $O/\ell$ has a natural structure of a $\Gamma_Q$-module, where $\Gamma_Q$ acts through the quotient $\text{Gal}(Q(i)/Q) \cong \mathbb{Z}/2$ whose generator sends $z$ to $\bar{z}$. The natural left action of $O^* = \mu_4$ on $O/\ell$ is compatible with the actions of $\Gamma_Q$ on $\mu_4$ and $O/\ell$. For $a \in \mathbb{Q}^*$, having fixed a 4-th root of $a$, the function sending $\gamma \in \Gamma_Q$ to $\gamma(\sqrt[a]{\bar{a}})/\sqrt[a]{a}$ is a cocycle $\Gamma_Q \to \mu_4$ representing the class of $a$ in $H^1(Q, \mu_4) = \mathbb{Q}^*/\mathbb{Q}^*4$. We denote the resulting twisted form of the $\Gamma_Q$-module $O/\ell$ by $(O/\ell)^a$. Since $\mu_4$ is abelian, $(O/\ell)^a$ is naturally a left $\mu_4$-module.

**Lemma 4.2** Let $a = 4 \cdot 3^3$ if $\ell = 3$ and $a = 5$ if $\ell = 5$. Then there is an isomorphism of $\Gamma_Q$-modules and left $\mu_4$-modules $\text{End}(E_\ell)^- \cong (O/\ell)^a$.

**Proof.** Let us start by recalling a well known structure of the $\Gamma_Q$-module $E_\ell$. An irreducible element $\pi \in O$ is called a primary prime if $\pi \equiv 1 \mod (1 + i)^3$. For a primary prime $\pi$ let $\text{Frob}_\pi \in \text{Gal}(Q(i)^{ab}/Q(i))$ be the image under the Artin map of the idèle whose component at the prime ideal $(\pi)$ is $\pi$, and all other components are 1. The Hecke character associated to $E/Q(i)$ takes the value $\pi$ at $(\pi)$, see [15, Exercise 2.34]. Hence we get the classical fact (Gauss, Deuring) that the action of $\Gamma_Q$ on $E_\ell$ factors through the surjective homomorphism $\Gamma_Q \to (O/\ell)^*$ that sends $\text{Frob}_\pi$, for $\pi$ prime to $\ell$, to the class of $\pi$ in $(O/\ell)^*$. We note that $(O/\ell)^*$ is a maximal torus in $GL(E_\ell)$; it is split for $\ell = 5$ and non-split for $\ell = 3$.

The Galois group $\Gamma_Q = \text{Gal}(\overline{Q}/Q)$ contains the subgroup $\text{Gal}(\overline{Q}/\mathbb{R} \cap Q) \cong \mathbb{Z}/2$ generated by the complex conjugation $c$. It is clear that $c$ maps to the generator of $\text{Gal}(Q(i)/Q)$. Let $w \in GL(E_\ell)$ be the linear transformation by which $c$ acts on $E_\ell$. We have $w^2 = \text{id}$. Since $c \text{Frob}_\pi c^{-1} = \text{Frob}_\pi$ for any primary prime $\pi$, we also have $wzw^{-1} = \bar{z}$ for any $z \in O/\ell$. Thus the image of $\Gamma_Q$ in $GL(E_\ell)$ is the normaliser of $(O/\ell)^*$ in $GL(E_\ell)$, isomorphic to $(O/\ell)^* \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ is generated by $w$.

Since $GL(E_\ell)$ acts on $\text{End}(E_\ell)$ through $\text{PGL}(E_\ell)$, the Galois group $\Gamma_Q$ acts on $\text{End}(E_\ell)$, and hence on $\text{End}(E_\ell)^-$, through a surjective homomorphism

$$\alpha : \Gamma_Q \longrightarrow ((O/\ell)^*/\mathbb{F}_5^*) \times \mathbb{Z}/2.$$ 

Explicitly, $z \in (O/\ell)^*$ acts on $\text{End}(E_\ell)^-$ by sending an endomorphism $\varphi$ to $z\varphi z^{-1} = z\bar{z}^{-1}\varphi$. If $\ell = 3$, then $\bar{z}$ is congruent to $z^3$ modulo 3, and so the map sending $z$ to $z\bar{z}^{-1} = z^{-2}$ is a surjective homomorphism $(O/\ell)^* \to \mu_4$ whose kernel is $\mathbb{F}_5^*$. The same conclusion holds for $\ell = 5$. Indeed, we have $5 = (2 + i)(2 - i)$, and hence $O/5 \cong O/(2 + i) \oplus O/(2 - i) \cong (\mathbb{F}_5)^2$. The group $\mu_4$ is the subgroup of the diagonal torus $(\mathbb{F}_5^*)^2$ consisting of the matrices with determinant 1. The conjugation swaps the coordinates, hence the map sending $z$ to $z\bar{z}^{-1}$ is a surjective homomorphism $(O/5)^* \to \mu_4$ whose kernel is the subgroup of scalar matrices $\mathbb{F}_5^*$. This allows us to rewrite the homomorphism $\alpha$ through which $\Gamma_Q$ acts on $\text{End}(E_\ell)^-$, as follows:

$$\alpha : \Gamma_Q \longrightarrow D_8 = \mu_4 \times \mathbb{Z}/2,$$
where $D_8$ is the dihedral group of order 8 generated by $\mu_4$ and the image of $w$. A straightforward check shows that the composition of $\alpha$ and the first projection $D_8 \to \mu_4$ is a cocycle $\tilde{\alpha} : \Gamma_\mathbb{Q} \to \mu_4$. Let $a \in \mathbb{Q}^*$ be a representative of the class of $\tilde{\alpha}$ in $H^1(\mathbb{Q}, \mu_4) = \mathbb{Q}^*/\mathbb{Q}^{*4}$.

Since $w$ normalises $(\mathcal{O}/\ell)^*$, but $w \notin (\mathcal{O}/\ell)^*$, we have a direct sum decomposition $\text{End}(E_\ell) = \mathcal{O}/\ell \oplus (\mathcal{O}/\ell)w$, hence $\text{End}(E_\ell)^- = (\mathcal{O}/\ell)w$. The action of $D_8$ on $\text{End}(E_\ell)^-$ translates into the following action of $D_8$ on $(\mathcal{O}/\ell)w$: the subgroup $\mu_4$ acts through its natural left action (which coincides with its right action), and $w$ acts by conjugation, so $w$ sends $zw$ to $\bar{z}w$. The natural action of $\Gamma_\mathbb{Q}$ on $\mathcal{O}/\ell$ is through the quotient $\text{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ whose generator sends $z$ to $\bar{z}$. By the definition of twisting, the action of $\gamma \in \Gamma_\mathbb{Q}$ on $(\mathcal{O}/\ell)^a$ is this action followed by the multiplication by $\bar{\alpha} (\gamma) \in \mu_4$. This shows that the map $\mathcal{O}/\ell \to (\mathcal{O}/\ell)w$ sending $z$ to $zw$ induces an isomorphism of $\mathcal{O}$-modules $(\mathcal{O}/\ell)^a \to (\mathcal{O}/\ell)w = \text{End}(E_\ell)^-$, which is also an isomorphism of left $\mu_4$-modules. Moreover, by construction $\hat{\alpha}(c) = 1$, so the restriction of $\hat{\alpha}$ to $\text{Gal}(\mathbb{Q}/\mathbb{Q} \cap \mathbb{Q})$ is trivial, hence $a > 0$.

The kernel of the restriction map $H^1(\mathbb{Q}, \mu_4) \to H^1(\mathbb{Q}(i), \mu_4)$, which is the natural map $\mathbb{Q}^*/\mathbb{Q}^{*4} \to \mathbb{Q}(i)^*/\mathbb{Q}(i)^{*4}$, is the group of order 2 generated by $-4$. This implies that a cohomology class in $H^1(\mathbb{Q}, \mu_4)$ given by a positive element of $\mathbb{Q}^*$ is uniquely determined by its restriction to $\mathbb{Q}(i)$. Thus to finish the proof it is enough to show that the restriction of $\hat{\alpha}$ to $\Gamma_{\mathbb{Q}(i)}$ is the homomorphism $\Gamma_{\mathbb{Q}(i)} \to \mu_4$ sending $\gamma$ to $\gamma(\sqrt{a})/\sqrt{a}$, where $a = 4 \cdot 3^3$ when $\ell = 3$ and $a = 5$ when $\ell = 5$. It is enough to check this when $\gamma$ is the Frobenius element $\text{Frob}_\pi$ attached to a primary prime $\pi \in \mathcal{O}$ which is coprime to $\ell$ and $a$. In terms of the biquadratic symbol we need to show that

$$\hat{\alpha}(\text{Frob}_\pi) = \frac{\text{Frob}_\pi(\sqrt{a})}{\sqrt{a}} = \left(\frac{a}{\pi}\right)_4 \in \mu_4.$$  

We have seen that $\hat{\alpha}(\text{Frob}_\pi) = \pi \bar{\pi}^{-1} \in \mu_4$. If $\ell = 3$ this equals $\pi^{-2}$. Since $(N_{\mathbb{Q}(i)/\mathbb{Q}}(3)-1)/4 = 2$, the Gauss biquadratic reciprocity law [7, Thm. 4.21, Exercise 5.14] gives

$$\left(\frac{4 \cdot 3^3}{\pi}\right)_4 = \left(\frac{-3^3}{\pi}\right)_4 = \left(\frac{-3}{\pi}\right)_4^{-1} = \left(\frac{\pi}{3}\right)_4^{-1} = \pi^{-2} \in \mu_4 \subset (\mathcal{O}/3)^*,$$

which is exactly what we need.

When $\ell = 5$ we have $5 = \theta \bar{\theta}$, where $\theta = -1 + 2i$ and $\bar{\theta} = -1 - 2i$ are conjugate primary primes. Since $(N_{\mathbb{Q}(i)/\mathbb{Q}}(\theta) - 1)/4 = 1$, the biquadratic reciprocity gives

$$\left(\frac{5}{\pi}\right)_4 = \left(\frac{\pi}{\theta}\right)_4 \left(\frac{\pi}{\bar{\theta}}\right)_4 = \left(\frac{\pi}{\theta}\right)_4 \left(\frac{\pi}{\bar{\theta}}\right)_4^{-1} = \pi \bar{\pi}^{-1} \in \mu_4 \subset (\mathcal{O}/5)^*.$$  

This gives the desired identity and so finishes the proof. □
Proposition 4.3 The $\Gamma_Q$-module $\text{Br}(\overline{D})_3$ is isomorphic to the quartic twist of the natural $\Gamma_Q$-module $O/3$ by a cocycle whose class in $H^1(Q, \mu_4) = \mathbb{Q}^* / \mathbb{Q}^{*4}$ is given by $(2^3abcd)^{-1}$. If $-3abcd \notin \langle -4 \rangle \mathbb{Q}^{*4}$, we have $\text{Br}(\overline{D})_{3}^{\Gamma_5} = \text{Br}(\overline{D})_{3}^{\mathbb{Q}(i)} = 0$. If $-3abcd \in \langle -4 \rangle \mathbb{Q}^{*4}$, we have $\text{Br}(\overline{D})_{3}^{\mathbb{Q}(i)} \cong (\mathbb{Z}/3)^2$ and $\text{Br}(\overline{D})_{3}^{\Gamma_5} \cong \mathbb{Z}/3$.

Proof. The first statement follows from Lemma 4.2 and Proposition 3.3. If $-3abcd$ is not in $\langle -4 \rangle \mathbb{Q}^{*4}$, then $-3abcd$ is not a fourth power in $\mathbb{Q}(i)$, so that $\Gamma_Q(i)$ acts non-trivially on $\text{Br}(\overline{D})_3$. The only element of $O/3 = \mathbb{F}_9$ fixed by the multiplication by a non-trivial element of $\mu_4$ is zero, hence $\text{Br}(\overline{D})_{3}^{\mathbb{Q}(i)} = 0$. If $-3abcd \in \langle -4 \rangle \mathbb{Q}^{*4}$, then $-3abcd$ is a fourth power in $\mathbb{Q}(i)$. Then the action of $\Gamma_Q(i)$ on $\text{Br}(\overline{D})_3$ is trivial, so that $\text{Br}(\overline{D})_{3}^{\mathbb{Q}(i)} \cong (\mathbb{Z}/3)^2$. In this case $\Gamma_Q$ acts on $\text{Br}(\overline{D})_3$ through $\text{Gal}(\mathbb{Q}(i)/\mathbb{Q})$, so that the $\Gamma_Q$-module $\text{Br}(\overline{D})_3$ is the quadratic twist of the $\Gamma_Q$-module $O/3$ by a cocycle that factors through a cocycle $\text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \to \mu_4$. Thus in the twisted module structure the generator of $\text{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ sends $z \in O/3$ to $\pm \bar{z}$ when the cocycle is a coboundary, and to $\pm i\bar{z}$ otherwise. In all cases, the invariant subgroup is a 1-dimensional $\mathbb{F}_3$-vector space. \qed

Proposition 4.4 The $\Gamma_Q$-module $\text{Br}(\overline{D})_5$ is isomorphic to the quartic twist of the natural $\Gamma_Q$-module $O/5$ by a cocycle whose class in $H^1(Q, \mu_4) = \mathbb{Q}^* / \mathbb{Q}^{*4}$ is given by $5(abc)\overline{d})^{-1}$. If $5^3abcd \notin \langle -4 \rangle \mathbb{Q}^{*4}$, we have $\text{Br}(\overline{D})_{5}^{\Gamma_5} = \text{Br}(\overline{D})_{5}^{\mathbb{Q}(i)} = 0$. If $5^3abcd \in \langle -4 \rangle \mathbb{Q}^{*4}$, then we have $\text{Br}(\overline{D})_{5}^{\mathbb{Q}(i)} \cong (\mathbb{Z}/5)^2$ and $\text{Br}(\overline{D})_{5}^{\Gamma_5} \cong \mathbb{Z}/5$.

Proof. This is proved in the same way as Proposition 4.3. \qed

End of proof of Theorem 1.1. The statement of the theorem is a formal consequence of Lemma 4.1 and Propositions 3.4, 4.3 and 4.4. \qed

5 Proof of Theorem 1.2

5.1 Preliminaries

Without loss of generality we assume that $a, b, c, d$ are fourth power free non-zero integers such that $\gcd(a, b, c, d) = 1$. In Proposition 2.1 we constructed a rational map of degree 32 from the diagonal quartic surface $D$ given by (1) to the Kummer surface $Z = \text{Kum}(E^{ab} \times E^{cd})$ with an affine equation

$$Y^2 = (X^3 - 4abX)(T^3 - 4cdT).$$

If we denote $f = ax^4 + by^4$, then this map can be written as follows:

$$X = -4abxy^2f^{-1}, \ T = -4cdz^2w^2f^{-1}, \ Y = 16abcdxyzw(ax^4 - by^4)(cz^4 - dw^4)f^{-3}. \ (7)$$
Let $k$ be a field of characteristic zero, and let $L = (x, y, z, w) \in D(k)$ such that $xyzwf \neq 0$. Define $E'$ as the quadratic twist of $E^{4ab}$ by $f/x^2y^2$. Then $E'$ is the quartic twist of $E$ by $4ab(f/x^2y^2)^2$, that is
\[
E' : \quad u_1^2 = t_1(t_1^2 - 4ab(f/x^2y^2)^2).
\]
Then $P = (-4ab, 4abx^{-2}y^{-2}(ax^4 - by^4))$ is a $k$-point in $E'$. The quartic twist
\[
E'' : \quad u_2^2 = t_2(t_2^2 - 4cd(f/z^2w^2)^2)
\]
is an elliptic curve over $k$ with a $k$-point $Q = (-4cd, 4cdz^{-2}w^{-2}(cz^4 - dw^4))$. Note that $E' \times E''$ is a quartic twist of $E^{4ab} \times E^{4cd}$ defined over $k$, and hence there is a natural rational map $E' \times E'' \to Z \times Q k$ of degree 2. This map sends $(P, Q)$ to the point $M \in Z(k)$ which is the image of $L \in D(k)$. Explicitly this map is
\[
X = x^2y^2f^{-1}t_1, \quad T = z^2w^2f^{-1}t_2, \quad Y = (xyzw)^3f^{-3}u_1u_2.
\]\(8\)

Theorem 1.1 implies, by (6), that if $Br(D)^P_{\ell} \neq 0$, then there is a non-zero homomorphism of $\Gamma_Q$-modules $\varphi : E^{4cd}_{\ell} \to E^{4ab}_{\ell}$ such that $[\bar{t}]\varphi = -\varphi[\bar{t}]$, which is unique up to multiplication by an element of $\mathbb{F}^\ell$. Let us check that $\varphi$ is an isomorphism. Indeed, $\text{Ker}(\varphi)$ is a $\mu_4$-invariant. For $\ell = 3$ the $\mu_4$-module $E^{4ab}_{\ell} \simeq O/\ell$ is irreducible, so $\varphi$ is an isomorphism in this case. For $\ell = 5$ this $\mu_4$-module is the direct sum $\text{Ker}[1 + 2\bar{t}] \oplus \text{Ker}[1 - 2\bar{t}]$, however, neither factor is a $\Gamma_Q$-submodule of $E^{4ab}_{\ell}$.

We need to recall from [18, Section 3] how an element of $\text{Br}(Z)_{\ell}$ is constructed from $\varphi$. The multiplication by $\ell$ turns $E^{4ab}$ into an $E^{4ab}_{\ell}$-torsor with structure group $E^{4ab}_{\ell}$. Let us call this torsor $T_1$, and similarly for the $E^{4cd}_{\ell}$-torsor $T_2$. The class $[T_1]$ is an element of the étale cohomology group $H^1(E^{4ab}_{\ell}, E^{4ab}_{\ell})$. Similarly, $[T_2] \in H^1(E^{4cd}_{\ell}, E^{4cd}_{\ell})$. The homomorphism $\varphi$ gives rise to the $E^{4cd}_{\ell}$-torsor $\varphi_*T_2$ with structure group $E^{4ab}_{\ell}$. The Galois-equivariant Weil pairing
\[
E^{4ab}_{\ell} \times E^{4ab}_{\ell} \to \mu_\ell
\]
gives rise to the cup-product pairing of étale cohomology groups
\[
H^1(E^{4ab}_{\ell} \times E^{4cd}_{\ell}, E^{4ab}_{\ell}) \times H^1(E^{4ab}_{\ell} \times E^{4cd}_{\ell}, E^{4ab}_{\ell}) \to \text{Br}(E^{4ab}_{\ell} \times E^{4cd}_{\ell})_{\ell}.
\]
Let us denote by $C \in \text{Br}(E^{4ab}_{\ell} \times E^{4cd}_{\ell})_{\ell}$ the cup-product of the pullback of $T_1$ via the projection to $E^{4ab}_{\ell}$ and the pullback of $\varphi_*T_2$ via the projection to $E^{4cd}_{\ell}$. According to [18, Lemma 3.1, Prop. 3.3] the natural map
\[
\text{Br}(E^{4ab}_{\ell} \times E^{4cd}_{\ell})_{\ell} \to \text{Br}(E^{4ab}_{\ell} \times E^{4cd}_{\ell})_{\ell} \simeq \text{Hom}_Q(E^{4cd}_{\ell}, E^{4ab}_{\ell})
\]
sends $C$ to $\varphi$. By the proof of [18, Thm. 2.4], in particular, by diagram (16), the natural map $\text{Br}(Z)_{\ell} \to \text{Br}(E^{4ab}_{\ell} \times E^{4cd}_{\ell})_{\ell}$ identifies $\text{Br}(Z)_{\ell}$ with the subgroup of
Br(\(E^{4ab} \times E^{4cd}\)_\ell) consisting of the elements fixed by the antipodal involution \([-1]\) on \(E^{4ab} \times E^{4cd}\). Let \(B \in \text{Br}(Z)_\ell\) be the element corresponding to \(C\).

The quadratic twists of \(E^{4ab}\) and \(E^{4cd}\) by the same element \(e \in k^*\) are elliptic curves \(E^{4abe^2}\) and \(E^{4cd e^2}\) over \(k\). We then obtain a homomorphism of \(\Gamma_k\)-modules \(E^{4abe^2}_\ell \to E^{4abe^2}_\ell\). It is easy to see that the resulting element of \(\text{Br}(Z \times Q k)\) is precisely the image of \(B\) in \(\text{Br}(Z \times Q k)\).

Let \(A \in \text{Br}(D)_\ell\) be the image of \(B\) under the natural map \(\text{Br}(Z)_\ell \to \text{Br}(D)_\ell\).

**Proposition 5.1** If \(\nu\) is the automorphism of \(D\) that alters the sign of one of the coordinates and does not change the other three, then \(\nu^*A = -A\). Up to multiplication by an element of \(\mathbb{F}_\ell^*\), any permutation of variables in the equation of \(D\) gives rise to the same element \(A \in \text{Br}(D)_\ell\).

**Proof.** Without loss of generality we can assume that \(\nu(x, y, z, w) = (-x, y, z, w)\). Let \([-1, 1]\) be the automorphism of \(E^{4ab} \times E^{4cd}\) that acts by \(-1\) on \(E^{4ab}\) and as the identity on \(E^{4cd}\). Since \([-1]^*\{T_i\} = \{-T_i\}\) we see that \([-1, 1]^*\mathcal{C} = -\mathcal{C}\). The involution \([-1, 1]\) on \(E^{4ab} \times E^{4cd}\) is compatible with the involution \(\kappa : Z \to Z\) that alters the sign of \(Y\) and does not change \(X\) and \(T\), hence \(\kappa^*\mathcal{B} = -\mathcal{B}\). Finally, (7) shows that the involutions \(\nu\) and \(\kappa\) are compatible as well, and therefore \(\nu^*A = -A\). Any element of \(\text{Br}(D)_\ell\) that we construct after permuting the variables maps to a generator of \(\text{Br}(\overline{D})_{\Gamma_\ell}^\circ \simeq \mathbb{F}_\ell\), so, up to multiplication by an element of \(\mathbb{F}_\ell^*\), the difference of these elements is some \(\alpha \in \text{Br}_0(D)_\ell\). We have \(\alpha = \nu^*\alpha = -\alpha\), hence \(\alpha = 0\). □

**Corollary 5.2** The subset \(\text{ev}_{\mathcal{A}, p}(D(Q_p)) \subset \frac{1}{\ell} \mathbb{Z}/\mathbb{Z}\) is stable under the map \(x \mapsto -x\). In particular, if \(\text{ev}_{\mathcal{A}, p}\) is a constant map, it is identically zero.

**Proof.** By the first claim of Proposition 5.1 we have \(\mathcal{A}(\nu(L)) = -\mathcal{A}(L) \in \text{Br}(Q_p)_\ell\) for any \(L \in D(Q_p)\). The second statement follows since \(\ell\) is odd. □

**Remark** In a similar way, using (7) one checks that when \(\mu_4\) is contained in \(k\), the image of \((ix, y, z, w) \in D(k)\) in \(Z(k)\) coincides with the image of \([i]P, Q\).

Let \(\chi\) be the homomorphism \(E(k) \to H^1(k, E_\ell')\), and apply the same notation to the twists of \(E\). Thus the point \(P \in E'(k)\) defines a \(k\)-torsor \(\chi_P\) of \(E_\ell'\), and similarly \(Q\) defines a \(k\)-torsor \(\chi_Q\) of \(E_\ell''\). By the construction of \(\mathcal{A}, \mathcal{B}, \mathcal{C}\) we have

\[\mathcal{A}(L) = \mathcal{B}(M) = \chi_P \cup \varphi_\ell(\chi_Q) \in \text{Br}(k)_\ell,\]

where

\[\cup : H^1(k, E_\ell') \times H^1(k, E_\ell') \to \text{Br}(k)_\ell\]

is the non-degenerate pairing induced by the Weil pairing \(E_\ell' \times E_\ell' \to \mu_\ell\).

Write \(L \in D(Q_p)\) as \((x, y, z, w) \in (\mathbb{Z}_p)^4\), where \(p\) does not divide all four coordinates. By the implicit function theorem, up to replacing \(L\) by a small deformation
in \( D(\mathbb{Q}_p) \) we can assume that \( xyzw \neq 0 \) and \( f(L) \neq 0 \). Then \( L \) goes to a well defined point \( M \in Z(\mathbb{Q}_p) \), and \( M \) lifts to the \( \mathbb{Q}_p \)-point \((P,Q)\) in \( E' \times E'' \), where \( E' \) and \( E'' \) are elliptic curves over \( \mathbb{Q}_p \).

Given four non-zero elements \( a, b, x, y \) of \( \mathbb{Z}_p \) such that \( f \neq 0 \) we denote the statement \( "P \in \ell E'(\mathbb{Q}_p)" \) by \( S_p(a, b : x, y) \). Explicitly, it says that the point with \( X \)-coordinate \(-4ab \) is divisible by \( \ell \) on the elliptic curve \( Y^2 = X^3 - 4ab(f/x^2y^2)^2X \).

We shall use this statement in the following equivalent form:

\[
S_p(a, b : x, y) \iff \text{"the point } (-4abx^2y^2, 4abxy(ax^4 - by^4)) \text{ is divisible by } \ell \text{ in the group of } \mathbb{Q}_p\text{-points of the elliptic curve } Y^2 = X^3 - 4abf^2X''. \tag{11}
\]

By (9) either \( S_p(a, b : x, y) \) or \( S_p(c, d : z, w) \) implies \( A(L_p) = 0 \).

### 5.2 Proof of (i)

Let \( m \in \mathbb{Z}_p, m \neq 0 \), and let \( E^m \) be the elliptic curve \( y^2 = x^3 - mx \) over \( \mathbb{Q}_p \). This Weierstraß equation is minimal if \( \text{val}_p(m) \leq 3 \), see [14, Remark VII.1.1]. Let \( \bar{E}^m \) be the (possibly, singular) curve given by the equation \( y^2 = x^3 - mx \) reduced modulo \( p \). To study the divisibility in \( E^m(\mathbb{Q}_p) \) we recall the well known filtration

\[
\ldots \subset E^m(\mathbb{Q}_p)_2 \subset E^m(\mathbb{Q}_p)_1 \subset E^m(\mathbb{Q}_p)_0 \subset E^m(\mathbb{Q}_p),
\]

where \( E^m(\mathbb{Q}_p)_0 \) is the subgroup of points that reduce to a point in \( \bar{E}^m_{\text{smooth}} \), and \( E^m(\mathbb{Q}_p)_1 \) is the kernel of the reduction map \( E^m(\mathbb{Q}_p)_0 \to \bar{E}^m_{\text{smooth}}(\mathbb{F}_p) \). One can also define \( E^m(\mathbb{Q}_p)_n \) for \( n \geq 1 \) as the subgroup of \( E^m(\mathbb{Q}_p) \) consisting of 0 and the points \((x, y)\) such that \( \text{val}_p(x) \leq -2n \).

An application of Tate’s algorithm [15, IV.9.4] shows that at any prime \( p \) the curve \( E^m \) can only have good or additive reduction, and the order of the quotient \( E^m(\mathbb{Q}_p)/E^m(\mathbb{Q}_p)_0 \) can only be 1, 2 or 4. Thus for any prime \( p \) we have an isomorphism

\[
E^m(\mathbb{Q}_p)_0/\ell = E^m(\mathbb{Q}_p)/\ell. \tag{12}
\]

The quotient \( E^m(\mathbb{Q}_p)_0/E^m(\mathbb{Q}_p)_1 \) is naturally isomorphic to \( \bar{E}^m_{\text{smooth}}(\mathbb{F}_p) \) [14, Prop. VII.2.1]. Pannekoek [12, Thm. 1, Lemma 9], using the well known theory of the formal group attached to an elliptic curve [14, Ch. IV], proves that in the case of additive reduction \( E^m(\mathbb{Q}_p)_0 \) is topologically isomorphic to \( \mathbb{Z}_p \) or to \( p\mathbb{Z}_p \times \mathbb{F}_p \), where the last case occurs if and only if \( p = 5 \) and \( m \equiv 15 \text{ mod } 25 \).

**Lemma 5.3** The group \( E^m(\mathbb{Q}_2) \) is divisible by any odd prime. Thus \( S_2(a, b : x, y) \) holds for any non-zero \( a, b, x, y \in \mathbb{Z}_2 \).

**Proof.** Tate’s algorithm shows that \( E^m \) has additive reduction for any \( m \in \mathbb{Q}_2^\ast \). The lemma follows from (12) and the above remarks. □
From now on assume $p > 2$. Then the theory of the formal group attached to $E^m$ gives compatible isomorphisms $E^m(\mathbb{Q}_p)_n \simeq p^n\mathbb{Z}_p$ for $n \geq 1$, see [14, Thm. IV.6.4(b)]. We shall use this fact repeatedly in the rest of the argument.

**Lemma 5.4** Let $p \neq \ell$ be an odd prime. If $a, b, x, y$ are in $\mathbb{Z}_p^*$ and $\text{val}_p(f) \geq 1$, then $S_p(a, b : x, y)$ holds.

**Proof.** If $\text{val}_p(f)$ is odd, then $E'$ has additive reduction at $p$. By (12) the group $E'(\mathbb{Q}_p)$ is divisible by $\ell$.

Suppose $f = up^{2n}$, where $u \in \mathbb{Z}_p^*$ and $n \geq 1$. The coordinate change $X' = p^{-2n}X$, $Y' = p^{-3n}Y$ reduces the equation of $E'$ to an equation the reduction modulo $p$ of which defines a smooth curve $\tilde{E}'$ over $\mathbb{F}_p$. The $X'$-coordinate of $P$ has valuation $-2n$, and so $P$ is in $E'(\mathbb{Q}_p)_1$. This is a pro-$p$-group, hence $P \in \ell E'(\mathbb{Q}_p)$. □

**Proposition 5.5** Let $D = [a, b, c, d]$, where $a, b, c, d \in \mathbb{Q}^*$. If $p \neq \ell$, then the map $ev_{A,p}$ is zero.

**Proof.** By Corollary 5.2 it is enough to show that the map $ev_{A,p}$ is constant. If $p$ does not divide $2abcd$, this was proved in [5, Cor. 3.3]. Since $S_2(a, b : x, y)$ holds by Lemma 5.3, we have $A(L) = 0$ for any $L \in D(\mathbb{Q}_2)$.

Now let $p$ be an odd prime. Since $p \neq \ell$, we see from Theorem 1.1 that 4 divides $\text{val}_p(a) + \text{val}_p(b) + \text{val}_p(c) + \text{val}_p(d)$. In the proof that $ev_{A,p}$ is a constant map we are allowed to rename the variables in the equation of $D$. By doing so and by multiplying $a, b, c, d$ by elements of $\mathbb{Q}^{*4}$ we can assume that $r = [\text{val}_p(a), \text{val}_p(b), \text{val}_p(c), \text{val}_p(d)]$ is either $[0, 0, 2, 2]$ or $[0, 0, 1, 3]$. If $r = [0, 0, 2, 2]$, then either both $x$ and $y$ are units, or both $z$ and $w$ are units. By Lemma 5.4 either $S_p(a, b : x, y)$ or $S_p(c/p^2, d/p^2 : z, w)$ holds, and the last property is equivalent to $S_p(c, d : z, w)$. If $r = [0, 0, 1, 3]$, then $S_p(a, b : x, y)$ holds by the same lemma. □

Theorem 1.2 (i) is now proved, because $A$ generates $\text{Br}(D)_\ell$ modulo $\text{Br}_0(D)_\ell$.

The following corollary follows from Proposition 5.5 and the reciprocity law; it will be used in the next section.

**Corollary 5.6** Let $D = [a, b, c, d]$, where $a, b, c, d \in \mathbb{Q}^*$. Then for any $P \in D(\mathbb{Q})$ we have $ev_{A,\ell}(P) = 0$.

### 5.3 Proof of (ii) for $\ell = 5$

#### 5.3.1 Computation of the pairing

In this section we write $\Gamma = \Gamma_{\mathbb{Q}_5}$ and denote by $E^m$ the elliptic curve $y^2 = x^3 - mx$ where $m \in \mathbb{Q}^*_{\mathbb{Q}_5}$. 15
Since $-1 \in \mathbb{Z}_5^2$ we can define $i \in \mathbb{Z}_5$ by the property that $1 + 2i$ is a generator of the maximal ideal $(5) \subset \mathbb{Z}_5$, or, equivalently, $i \equiv 2 \mod 5$. Then $1 - 2i \in \mathbb{Z}_5^*$. We have a natural decomposition of $\Gamma$-modules $E_5^m = E_{1+2i}^m \oplus E_{1-2i}^m$, and the induced decomposition

$$H^1(\mathbb{Q}_5, E_5^m) = H^1(\mathbb{Q}_5, E_{1+2i}^m) \oplus H^1(\mathbb{Q}_5, E_{1-2i}^m).$$

Since the restriction of the skew-symmetric Weil pairing to $E_{1+2i}^m$ is trivial, each of the subspaces $H^1(\mathbb{Q}_5, E_{1+2i}^m)$ is isotropic. By the non-degeneracy of the $\cup$-product, these subspaces are maximal isotropic subspaces of $H^1(\mathbb{Q}_5, E_5^m)$, each of dimension $\frac{1}{2} \dim H^1(\mathbb{Q}_5, E_5^m)$. We conclude that (10) induces a non-degenerate pairing

$$H^1(\mathbb{Q}_5, E_{1+2i}^m) \times H^1(\mathbb{Q}_5, E_{1-2i}^m) \longrightarrow \text{Br}(\mathbb{Q}_5)_5 \cong \frac{1}{5} \mathbb{Z}/\mathbb{Z}. \quad (13)$$

Recall that $E^m(\mathbb{Q}_5)/5$ is a maximal isotropic subspace of $H^1(\mathbb{Q}_5, E_5^m)$. The Chinese remainder theorem gives an isomorphism

$$E^m(\mathbb{Q}_5)/5 = E^m(\mathbb{Q}_5)/[1 + 2i] \oplus E^m(\mathbb{Q}_5)/[1 - 2i].$$

We have the following commutative square of inclusions:

$$\begin{array}{ccc}
E^m(\mathbb{Q}_5)/[1 + 2i] & \longrightarrow & H^1(\mathbb{Q}_5, E_{1+2i}^m) \\
\downarrow{[1-2i]} & & \downarrow \\
E^m(\mathbb{Q}_5)/5 & \longrightarrow & H^1(\mathbb{Q}_5, E_5^m)
\end{array}$$

where the horizontal maps come from the corresponding Kummer sequences and the right vertical map is induced by the inclusion $E_{1+2i}^m \to E_5^m$.

**Proposition 5.7** If $m \in \mathbb{Q}_5^*$ is not in $(1 \pm 2i)\mathbb{Q}_5^4$, then $E^m(\mathbb{Q}_5)/[1 - 2i] = 0$ and

$$E^m(\mathbb{Q}_5)/5 = E^m(\mathbb{Q}_5)/[1 + 2i] = H^1(\mathbb{Q}_5, E_{1+2i}^m)$$

is a 1-dimensional $\mathbb{F}_5$-vector subspace of the 2-dimensional space $H^1(\mathbb{Q}_5, E_5^m)$.

**Proof.** Let $\mathcal{F}$ be the formal group defined by $E^m$, see [14, Example IV.2.2.3]. One can consider the abelian group $\mathcal{F}(5\mathbb{Z}_5)$ on the set $5\mathbb{Z}_5$ with the group law defined by $\mathcal{F}$, see [14, Example IV.3.1.3]. By [14, VII.2.2] the map sending $(x, y)$ to $-x/y$ is an isomorphism $E^m(\mathbb{Q}_5)_1 \to \mathcal{F}(5\mathbb{Z}_5)$. This isomorphism is clearly compatible with the action of $[i]$ on $E^m(\mathbb{Q}_5)_1$ and the action of $i \in \mathbb{Z}_5^*$ on the set $5\mathbb{Z}_5$. Since the residual characteristic is not 2 we can apply [14, Thm. IV.6.4 (b)] which says that the formal logarithm defines an isomorphism of abelian groups $\mathcal{F}(5^n\mathbb{Z}_5) \to 5^n\mathbb{Z}_5$ for any $n \geq 1$. The composed isomorphism $E^m(\mathbb{Q}_5)_1 \to 5\mathbb{Z}_5$ translates the action of $[i]$ on $E^m(\mathbb{Q}_5)_1$ into the action of $i \in \mathbb{Z}_5^*$ on $5\mathbb{Z}_5$.

Write $m = 5^nu$, where $n \in \mathbb{Z}$, $0 \leq n \leq 3$, and $u \in \mathbb{Z}_5^*$. First suppose that $m \in \mathbb{Z}_5^*$, then $E^m$ has good reduction. Our assumption $m \notin (1 - 2i)\mathbb{Z}_5^4$ implies that $m$ is not
congruent to 2 modulo 5. Let us denote by $\tilde{E}^m$ the elliptic curve over $\mathbb{F}_5$ which is the reduction of $E^m$. An elementary calculation shows that for such values of $m$, the 5-torsion of $\tilde{E}^m(\mathbb{F}_5)$ is trivial, hence $[1 \pm 2i]$-torsion is trivial too. Since $1 - 2i \in \mathbb{Z}_5^*$, the endomorphism $[1 - 2i]$ is invertible on $E^m(\mathbb{Q}_5)$.

Now let $m \in 5^*\mathbb{Z}_5^*$, where $n = 1, 2, 3$. Then $E^m$ has additive reduction. R. Pannekoek shows in [12, Def. 10] that in this situation the formal group $\mathcal{F}$ defines an abelian group $\mathcal{F}(\mathbb{Z}_5)$ on the set $\mathbb{Z}_5$. He then proves [12, Prop. 11 (1)] that the map sending $(x, y)$ to $-x/y$ is an isomorphism $E^m(\mathbb{Q}_5)_0 \to \mathcal{F}(\mathbb{Z}_5)$. As before, this isomorphism is obviously compatible with the action of $[i]$ on $E^m(\mathbb{Q}_5)_0$ and the action of $i \in \mathbb{Z}_5^*$ on $\mathcal{F}(\mathbb{Z}_5)$. By [12, Prop. 17] there is a natural isomorphism between $\mathcal{F}(\mathbb{Z}_5)$ and $\mathbb{Z}_5$ with its usual group structure, unless $m \equiv 15 \equiv 1 + 2i \bmod 25$. This congruence easily implies $m \in (1 + 2i)\mathbb{Z}_5^*$ which is excluded by our assumption. The number of connected components of the Néron model of an elliptic curve with additive reduction is at most 4. So the endomorphism $[1 - 2i]$ is invertible on $E^m(\mathbb{Q}_5)$ in this case as well.

Thus in all cases we have $E^m(\mathbb{Q}_5)/[1 - 2i] = 0$ and $E^m(\mathbb{Q}_5)/5 = E^m(\mathbb{Q}_5)/[1 + 2i]$ is a 1-dimensional vector space over $\mathbb{F}_5$. Since $E^m(\mathbb{Q}_5)/[1 + 2i] \subset H^1(\mathbb{Q}_5, E^m_{1+2i})$ and the dimension of the last space equals the dimension of $E^m(\mathbb{Q}_5)/5$, we see that the injective image of $E^m(\mathbb{Q}_5)/5$ in $H^1(\mathbb{Q}_5, E^m_5)$ is $H^1(\mathbb{Q}_5, E^m_{1+2i})$. $\square$

If $\varphi : E^m_{52} \to E^m_{51}$ is an isomorphism of $\Gamma$-modules which anti-commutes with $i$, then $\varphi = \varphi' + \varphi''$, where $\varphi'$ is an isomorphism of $\Gamma$-modules $E^m_{1+2i} \to E^m_{1-2i}$, and $\varphi''$ is an isomorphism of $\Gamma$-modules $E^m_{1-2i} \to E^m_{1+2i}$. Therefore, the induced isomorphism $\varphi_* : H^1(\mathbb{Q}_5, E^m_{52}) \to H^1(\mathbb{Q}_5, E^m_{51})$ is the sum $\varphi'_* + \varphi''_*$, where

$$\varphi'_* : H^1(\mathbb{Q}_5, E^m_{1+2i}) \to H^1(\mathbb{Q}_5, E^m_{1-2i}),$$

and similarly for $\varphi''_*$. Now the non-degeneracy of (13) implies the non-degeneracy of the pairing

$$H^1(\mathbb{Q}_5, E^m_{1+2i}) \times H^1(\mathbb{Q}_5, E^m_{1-2i}) \to \text{Br}(\mathbb{Q}_5)[5] \cong \frac{1}{5}\mathbb{Z}/\mathbb{Z}$$

(14)
given by $x \cup \varphi'_*(y)$.

**Corollary 5.8** Let $m_1, m_2 \in \mathbb{Q}_5^*$ be such that $m_1m_2 \in 5\mathbb{Q}_5^4$, where $m_1$ and $m_2$ are not in $(1 \pm 2i)\mathbb{Q}_5^4$. Let $\varphi : E^m_{52} \to E^m_{51}$ be an isomorphism of $\Gamma_{\mathbb{Q}_5}$-modules that anti-commutes with $[i]$. For any $P \in E^m_{51}(\mathbb{Q}_5)$ and $Q \in E^m_{52}(\mathbb{Q}_5)$ we have

1. $\chi_P \cup \varphi_*(\chi_Q) = 0$ if and only if $P \in 5E^m_{51}(\mathbb{Q}_5)$ or $Q \in 5E^m_{52}(\mathbb{Q}_5)$;
2. $\chi_{[i]P} \cup \varphi_*(\chi_Q) = \chi_P \cup \varphi_*(\chi_{[i]Q}) = 2\chi_P \cup \varphi_*(\chi_Q)$.

**Proof.** (1) Applying Proposition 5.7 to $m_1$ and $m_2$, we obtain $\chi_P \cup \varphi_*(\chi_Q) = x \cup \varphi'_*(y)$ for some non-zero elements $x \in H^1(\mathbb{Q}_5, E^m_{1+2i})$ and $y \in H^1(\mathbb{Q}_5, E^m_{1-2i})$. The last pairing is a non-degenerate pairing of 1-dimensional spaces, hence we prove (1).

(2) The automorphism $[i]$ acts on $E^m_{1+2i}$, and hence also on $H^1(\mathbb{Q}_5, E^m_{1+2i})$, as the multiplication by 2. We conclude by Proposition 5.7. $\square$
**Corollary 5.9** Let \( D = [a, b, c, d] \), where \( a, b, c, d \in \mathbb{Q}_5 \) are subject to conditions \( 5^3abcd \in \mathbb{Q}_5^{*4} \) and \( ab \in \mathbb{Q}_5^{*2} \). If a point \((x, y, z, w) \in D(\mathbb{Q}_5)\) is such that neither \( S_5(a, b : x, y) \) nor \( S_5(c, d : z, w) \) holds, then \( ev_{A,5}(x, y, z, w) \neq 0 \).

**Proof.** Since \( ab \in \mathbb{Q}_5^{*2} \), we have \( 4abf^2 \notin (1 \pm 2i)\mathbb{Q}_5^{*2} \), hence \( 4cdfs \notin (1 \pm 2i)\mathbb{Q}_5^{*4} \). An application of Corollary 5.8 (1) gives the result. \( \square \)

### 5.3.2 Diagonal quartic surfaces over \( \mathbb{Q}_5 \)

Consider the following diagonal quartic surfaces defined over \( \mathbb{Q} \):

\[
A_n = [1, -1, n, -5n^3], \quad B_n = [1, -1, 5^3n, -5^3n^3], \quad C = [2, 2, 4, 5],
\]

where \( n \) is an integer not divisible by 5. Each of these surfaces has an obvious \( \mathbb{Q} \)-point and satisfies the property \( ab \equiv -1 \mod 5 \), hence \( ab \in \mathbb{Q}_5^{*2} \). In the following lemma and thus everywhere in this section it is enough to consider \( n = 1 \) or 2, though we shall not use this.

**Lemma 5.10** Let \( a, b, c, d \in \mathbb{Q}_5^{*} \) be such that \( abcd \in 5\mathbb{Q}_5^{*4} \) and \( D(\mathbb{Q}_5) \neq \emptyset \). By permuting the variables and multiplying the coefficients \( a, b, c, d \) by a common constant and by elements of \( \mathbb{Q}_5^{*4} \), the surface \( D \) can be reduced to one of the surfaces \( A_n \), \( B_n \) or \( C \). In particular, we can assume without loss of generality that \( ab \in \mathbb{Q}_5^{*2} \).

**Proof.** Let \( r = [\text{val}_5(a), \text{val}_5(b), \text{val}_5(c), \text{val}_5(d)] \). We can assume that \( r \) is \([0, 0, 0, 1]\) or \([0, 0, 2, 3]\). If \( r = [0, 0, 0, 1] \), then the congruence \( ax^4 + by^4 - cz^4 \equiv 0 \mod 5 \) has a non-trivial solution \((x_0, y_0, z_0)\). If \( 5|x_0y_0z_0 \), by \( 5 \) we reduce \( D \) to \( A_n \); otherwise we reduce it to \( C \). When \( r = [0, 0, 2, 3] \) we have \( ab \equiv 0 \mod 5 \), and then \( D \) reduces to \( B_n \). \( \square \)

**Lemma 5.11** Assume that \( ab \in \mathbb{Z}_5^{*2} \). If \( \text{val}_5(f) = 0 \) and \( r = \text{val}_5(xy) > 0 \), then \( S_5(a, b : x, y) \) holds if and only if \( r > 1 \).

**Proof.** In this case the curve \( E' \) given by \( Y^2 = X^3 - 4abf^2X \) has good reduction. The point \( P \) with \( X \)-coordinate \(-4abx^2y^2 \) reduces to the 2-torsion point \((0, 0)\). Since \( (0, 0) = 5(0, 0) \), the point \( P \) is divisible by 5 in \( E'(\mathbb{Q}_5) \) if and only if \( P_1 = P + (0, 0) \) is. The \( X \)-coordinate of \( P_1 \) is \( f^2/x^2y^2 \), and \( \text{val}_5(X(P_1)) = -2r \), so that \( P_1 \) is in \( E'(\mathbb{Q}_5)_1 \simeq 5\mathbb{Z}_5 \). The assumption \( ab \in \mathbb{Z}_5^{*2} \) implies that the group of points on the reduced curve has no 5-torsion. Thus \( P_1 \in 5E'(\mathbb{Q}_5) \) if and only if \( P_1 \in 5E'(\mathbb{Q}_5)_1 \). Since \( \text{val}_5(X(P_1)) = -2r \), we conclude that \( P_1 \in 5^r\mathbb{Z}_5 \), \( P_1 \notin 5^{r+1}\mathbb{Z}_5 \). In other words, \( P_1 \) is divisible by \( 5^{r-1} \) but not by \( 5^r \) in \( E'(\mathbb{Q}_5) \). \( \square \)

**Proposition 5.12** Let \( D = [a, b, c, d] \), where \( a, b, c, d \in \mathbb{Q}_5^{*} \) are such that \( 5^3abcd \in \mathbb{Q}_5^{*4} \) and \( D(\mathbb{Q}_5) \neq \emptyset \). Then for any element in \( \text{Br}(D)_5 \) that is not in \( \text{Br}_0(D) \) the evaluation map \( D(\mathbb{Q}_5) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z} \) is surjective.
Proof. By Lemma 5.10 we may assume that $D$ is one of the surfaces $A_n$, $B_n$ or $C$, in particular, $ab \in \mathbb{Q}_5^2$. It is enough to prove the surjectivity of $\text{ev}_{A,5}$ where $A \in \text{Br}(D)$ is constructed in Section 5.1. Each of our surfaces is defined over $\mathbb{Q}$ and contains a $\mathbb{Q}$-point. By Corollary 5.6 for any $\mathbb{Q}$-point $P$ we have $\text{ev}_{A,5}(P) = 0$. We shall exhibit a $\mathbb{Q}_5$-point $L = (x, y, z, w)$ on each of our surfaces such that neither $S_5(a, b : x, y)$ nor $S_5(c, d : z, w)$ holds. Then $A(L) \neq 0$ by Corollary 5.9. Corollary 5.8 (2) and the remark after Corollary 5.2 imply that the values of $\text{ev}_{A,5}$ at the $\mathbb{Q}_5$-points $(i^n x, y, z, w)$ for $n = 0, 1, 2, 3$ give all the non-zero elements of $\frac{1}{5} \mathbb{Z}/\mathbb{Z}$. This will prove the proposition.

$A_n$. In this case we prove a stronger statement: if 25 does not divide $xyzw$, then neither $S_5(a, b : x, y)$ nor $S_5(c, d : z, w)$ holds. An example of such a point is $(\alpha, 1, 5, 1)$ for an appropriate $\alpha \in \mathbb{Z}_5$.

Indeed, suppose that $x, y, z, w \in \mathbb{Z}_5$ are the coordinates of a point in $D(\mathbb{Q}_5)$ such that $\text{val}_5(xyzw) \leq 1$. Then $\text{val}_5(z) = 0$ or 1.

Let us first assume that $\text{val}_5(z) = 1$. This implies $w \in \mathbb{Z}_5$, and hence $\text{val}_5(f) = 1$. Then the curve $E'$ given by $Y^2 = X^3 - 4abf^2X$ has additive reduction. The point $P$ with $X$-coordinate $-4abz^2y^2 \in \mathbb{Z}_5^*$ is in $E'(\mathbb{Q}_5)_0$ but not in $E'(\mathbb{Q}_5)_1$, so it is not divisible by 5 in $E'(\mathbb{Q}_5)$. Thus $S_5(a, b : x, y)$ does not hold, see (11). The curve $E''$ given by $Y^2 = X^3 - 4cdz^2X$ also has additive reduction. The point $Q$ has $X$-coordinate $-4cdz^2w^2$ of valuation 3, hence it reduces to the singular point $(0, 0)$ of the reduction. The $X$-coordinate of $Q = (0, 0)$ is $f^2/z^2w^2 \in \mathbb{Z}_5^*$, so $Q + (0, 0)$ is not divisible by 5 in $E''(\mathbb{Q}_5)$. Since $(0, 0) = (0, 0)$, the property $S_5(c, d : z, w)$ does not hold.

Now let $\text{val}_5(z) = 0$. This rules out the possibility that $x, y \in \mathbb{Z}_5$, so the valuation of $x$ or $y$ is 1, and hence $w \in \mathbb{Z}_5^*$. We now have $\text{val}_5(f) = 0$. Since $-4abf^2 \in \mathbb{Z}_5^*$ the curve $E'$ has good reduction, and by Lemma 5.11 the property $S_5(a, b : x, y)$ does not hold. The curve $E''$ given by $Y^2 = X^3 - 4cdz^2X$ has additive reduction. The point $Q$ with $X$-coordinate $-4cdz^2w^2$ reduces to $(0, 0)$. The point $Q + (0, 0)$ has $X$-coordinate $f^2/z^2w^2 \in \mathbb{Z}_5^*$, so $Q + (0, 0)$, and hence also $Q$, is not divisible by 5 in $E''(\mathbb{Q}_5)$, so the property $S_5(c, d : z, w)$ does not hold. This finishes the proof of our claim.

$B_n$. In this case we prove a stronger statement: if 5 does not divide $zw$, then neither $S_5(a, b : x, y)$ nor $S_5(c, d : z, w)$ holds. An example of such a point is $(\alpha, 1, 1, 1)$ for an appropriate $\alpha \in \mathbb{Z}_5^*$.

Suppose $x, y, z, w \in \mathbb{Z}_5$ are the coordinates of a point in $D(\mathbb{Q}_5)$ such that $\text{val}_5(z) = \text{val}_5(w) = 0$. Then $\text{val}_5(f) = 2$. The curve $E'$ has good reduction, but its Weierstraß equation is not minimal. Changing variables to make the equation minimal we see that $P$ is in $E'(\mathbb{Q}_5)_1$ but not in $E'(\mathbb{Q}_5)_2 = 5E'(\mathbb{Q}_5)_1$. Since the group of points on the reduced curve has no 5-torsion we conclude that $S_5(a, b : x, y)$ does not hold. The equation $Y^2 = X^3 - 4cdf^2X$ of the curve $E''$ is not minimal. Write $X = 5^aX'$, $Y = 5^bY'$, then the equation becomes $Y'^2 = X'^3 - 4cdf^25^{-a-b}X'$. The reduction is
additive, and the \(X'\)-coordinate of \(Q\) is \(-4cdz^2w^2 - 4\), so \(Q\) reduces to the singular point \((0,0)\). The \(X'\)-coordinate of \(Q + (0,0)\) is \(5^{-4}f^2/z^2w^2\), so this point is in \(E''(\mathbb{Q}_5)_0\) but not in \(E''(\mathbb{Q}_5)_1\). Thus \(Q + (0,0)\), and hence also \(Q\), is not divisible by 5 in \(E''(\mathbb{Q}_5)\), so \(S_5(c, d : z, w)\) does not hold.

\[ \text{C. Let } L = (1, 2, z, 1), \text{ where } z \in \mathbb{Z}^6. \text{ The equation of } E' \text{ is } Y^2 = X^3 - 16 \cdot (34)^2 X, \text{ so } E' \text{ has good reduction. We need to prove that } P = (-64, -960) \text{ is not divisible by } 5 \text{ in } E'(\mathbb{Q}_5). \text{ By writing } X = 4X', Y = 8Y' \text{ we reduce the equation to } Y'^2 = X'^{10} - (34)^2 X'. \text{ In new coordinates } P \text{ is the point } (-16, -120). \text{ The reduction of this point coincides with the reduction of the 2-torsion point } M = (34, 0). \text{ Using sage one sees immediately that the valuation of the } X\text{-coordinate of } P + M \text{ is } -2. \text{ Hence } P - M \in E'(\mathbb{Q}_5)_1, \text{ but } P - M \text{ is not in } 5E'(\mathbb{Q}_5)_1. \text{ The number of } \mathbb{F}_5\text{-points on the reduction of } E' \text{ is 8, hence is prime to } 5. \text{ It follows that } P \in 5E'(\mathbb{Q}_5) \text{ if and only if } P - M \in 5E'(\mathbb{Q}_5)_1. \text{ Thus } P \text{ is not divisible by } 5 \text{ in } E'(\mathbb{Q}_5), \text{ so } S_5(2, 2 : 1, 2) \text{ does not hold.}

\text{The curve } E'' \text{ with equation } Y^2 = X^3 - 4df^2X \text{ has additive reduction. The valuation of the } X\text{-coordinate of the point } Q + (0,0) \text{ is } \text{val}_5(f^2/z^2w^2) = 0, \text{ so this point is not in } E''(\mathbb{Q}_5)_1. \text{ Hence } Q \text{ is not divisible by } 5, \text{ so that } S_5(4, 5 : z, 1) \text{ does not hold. We conclude that } \mathcal{A}(L) \neq 0. \square \]

This finishes the proof of Theorem 1.2 (ii) for \(\ell = 5\).

### 5.4 Proof of (ii) for \(\ell = 3\)

#### 5.4.1 Computation of the pairing

Let \(E''\) be the elliptic curve \(y^2 = x^3 - mx\), where \(m \in \mathbb{Q}_3^*\).

**Proposition 5.13** Suppose that \(-3m_1m_2 \in (-4)\mathbb{Q}_3^4\) and let \(\varphi : E''^{m_2} \to E''^{m_1}\) be an isomorphism of \(\Gamma_{\mathbb{Q}_3}\)-modules. For any \(P \in E''^{m_1}(\mathbb{Q}_3)\) and \(T \in E''^{m_2}(\mathbb{Q}_3)\) we have \(\chi_P \cup \varphi^*(\chi_T) = 0\) if and only if \(P \in 3E''^{m_1}(\mathbb{Q}_3)\) or \(T \in 3E''^{m_2}(\mathbb{Q}_3)\).

**Proof.** One implication being trivial, we suppose that \(\chi_P \neq 0\) and \(\chi_T \neq 0\). The image of \(E(\mathbb{Q}_3)/3\) under the Kummer map is a 1-dimensional maximal isotropic subspace of \(H^1(\mathbb{Q}_3, E_3)\). Therefore it is enough to show that \(\chi_P \neq \pm \varphi(\chi_T)\) in \(H^1(\mathbb{Q}_3, E''^{m_1})\).

The elliptic curves \(E = E''^{m_1}\) and \(C = E''^{m_2}\) are isomorphic over \(F = \mathbb{Q}_3(i, \sqrt{3})\), but not over \(\mathbb{Q}_3(i, \sqrt{3})\). Let \(\lambda : C \to E\) be an isomorphism over \(F\), and let \(Q = \lambda(T)\). Write \(L = \mathbb{Q}_3(E_3) = \mathbb{Q}_3(C_3)\). Then \(L/F\) is a ramified quadratic extension and \(L/\mathbb{Q}_3\) is a Galois extension.

We now prove a series of lemmas.

**Lemma 5.14** If \(P \notin 3E''^{m_1}(\mathbb{Q}_3)\) and \(T \notin 3E''^{m_2}(\mathbb{Q}_3)\), then the images of \(P, [i]P, Q, [i]Q\) in \(E(L)/3\) are linearly independent.
Proof. Since the restriction map $H^1(F, E_3) \to H^1(L, E_3)$ is injective, it suffices to show linear independence in $E(F)/3$. The map $H^1(Q_3, E_3) \to H^1(F, E_3)$ is also injective, so $P$ does not go to zero in $E(F)/3$.

If $[i]P = \pm P$ in $E(F)/3$, then $2P = 0$ which is impossible. Therefore $P$ and $[i]P$ are linearly independent. The same argument shows that $Q$ and $[i]Q$ are linearly independent in $E(F)/3$.

Note that $\lambda$ identifies $C(F)$ with $E(F)$ twisted by $-1$ as $\text{Gal}(F/Q_3(i, \sqrt{3}))$-modules. Therefore the non-trivial element of $\text{Gal}(F/Q_3(i, \sqrt{3}))$ acts by $1$ on the span of $\{P, [i]P\}$ and by $-1$ on the span of $\{Q, [i]Q\}$. The result follows. □

For a field extension $k$ of $Q_3$ and a point $R = (x_R, y_R) \in E^m(k)$ such that $6R \neq 0$, consider the polynomial

$$g_R(t) = t^9 + 12mt^7 + 30m^2t^5 - 36m^3t^3 + 9m^4t - x_R(3t^4 - 6mt^2 - m^2)^2.$$ 

Let $k_R$ be the splitting field of $g_R(t)$. The roots of $g_R(t)$ are the $x$-coordinates of the points $R'$ such that $3R' = R$. Note that $R'$ is defined over $k(x_R)$, because otherwise $2R = 0$.

Lemma 5.15 Let $k$ be a field extension of $Q_3$. Let $R \in E(k)$, $6R \neq 0$.

1. If an extension $k$ contains two roots of $g_R(t)$, then $L \subset k(i)$.

2. Suppose $k \subset L$. Then $L_R/k$ is a Galois extension. For any root $r$ of $g_R(t)$ we have $L_R = L(r)$. Moreover, $[L_R : L]$ divides 9.

3. If $[L_R : L] = 3$, then $\text{Gal}(L_R/F) \cong D_6$.

Proof. (1) If two roots of $g_R(t)$ are in $k$, then a non-zero element of $E_3$ is defined over $k$ and so $E_3 \subset E(k(i))$.

(2) The extension $L_R/k$ is Galois because it is the composite of the Galois extensions $L/k$ and $k_R/k$. Since $E_3 \subset E(L)$, we also have the second assertion. The last assertion follows because the irreducible factors of $g_R(t)$ in $L[t]$ must have the same degree.

(3) By part (2) we see that $L_R/F$ is a Galois extension of degree 6. Therefore it suffices to show that it is not abelian. It is easy to see that at least one irreducible factor of $g_R(t)$ in $F[t]$ must have degree 3, call it $p(t)$. If $L_R/F$ were abelian, then every subextension would be Galois. Hence by adjoining a root of $p(t)$ we would obtain an extension of $F$ of degree 3 containing two roots of $g_R(t)$. By part (1) this extension would contain $L$, which is a contradiction. □

Lemma 5.16 We have $L_Q \neq L_P$.

Proof. Assume $L_Q = L_P$. The $F_3$-vector space

$$H^1(\text{Gal}(L_P/L), E_3) = \text{Hom}(\text{Gal}(L_P/L), E_3) = \text{Ker}[H^1(L, E_3) \to H^1(L_P, E_3)]$$
has dimension at least 4 by Lemma 5.14. Since $|\text{Gal}(L_P/L)|$ divides 9 by Lemma 5.15, we deduce that $\text{Gal}(L_P/L) \cong \mathbb{Z}/3 \times \mathbb{Z}/3$.

Hence the kernel of the map $H^1(L,E_3) \to H^1(L_P,E_3)$ is the 4-dimensional linear span of $P, [i]P, Q, [i]Q$ in $E(L)/3$. Let $k$ be any subextension of $L_P/L$ of degree 3. The kernel of $H^1(L,E_3) \to H^1(k,E_3)$ is the 2-dimensional span of $R, [i]R$ for some $R$ in the span of $P, [i]P, Q, [i]Q$. Thus $k = L_R$. By Lemma 5.15, $k$ is a Galois extension of $F$ with the Galois group isomorphic to $D_6$. But then $\text{Gal}(L_P/F)$ contradicts the following fact from group theory: there does not exist a group of order 18 with at least 4 normal subgroups of order 3 each of which has a quotient isomorphic to $D_6$. The easy proof of this fact is left to the reader. □

End of proof of Proposition 5.13. Let us choose $P'$ such that $3P' = P$ and $Q'$ such that $3Q' = Q$. Since $L_Q \neq L_P$ by Lemma 5.16, we can find an element $g \in \Gamma_L$ that fixes exactly one of $P'$ and $Q'$. Therefore exactly one of $\varphi(\chi_T)(g)$ and $\chi_P(g)$ is non-zero. Hence $\varphi(\chi_T) \neq \pm \chi_P$ in $H^1(L,E_3) = \text{Hom}(\Gamma_L,E_3)$. The proposition is proved. □

In the following immediate corollary $A$ is the element of $\text{Br}(D)$ constructed in Section 5.1.

**Corollary 5.17** Let $D = [a,b,c,d]$, where $a, b, c, d \in \mathbb{Q}^*$ are such that $-3abcd$ is in $(-4)\mathbb{Q}^{*4}$. If a point $(x, y, z, w) \in D(\mathbb{Q}_3)$ is such that neither $S_3(a,b : x, y)$ nor $S_3(c,d : z, w)$ holds, then $\text{ev}_{A,3}((x, y, z, w)) \neq 0$.

### 5.4.2 Diagonal quartic surfaces over $\mathbb{Q}_3$

**Lemma 5.18** Any diagonal quartic surface $D = [a, b, c, d]$ over $\mathbb{Q}_3$ such that $-3abcd$ is in $(-4)\mathbb{Q}_3^{*4}$ and $D(\mathbb{Q}_3) \neq \emptyset$, can be reduced by a permutation of variables and by multiplication of the coefficients $a, b, c, d$ by elements of $\mathbb{Q}_3^{*4}$ and by a common constant in $\mathbb{Q}_3^*$, to one of the following surfaces:

$$A = [1, 1, 1, \pm 27], \quad B = [1, -1, 3, \pm 9], \quad C = [1, 1, 2, \pm 27].$$

**Proof.** We can assume without loss of generality that $[\text{val}_3(a), \text{val}_3(b), \text{val}_3(c), \text{val}_3(d)]$ equals $[0, 0, 0, 3]$ or $[0, 0, 1, 2]$. In the first case if $a, b, -c$ are equal modulo 3, then we reduce to $C$; otherwise our surface reduces to $A$. In the second case the solubility in $\mathbb{Q}_3$ implies $a + b \equiv 0 \mod 3$ and we can take $a = 1$. This gives $B$. □

**Proposition 5.19** Let $D = [a, b, c, d]$, where $a, b, c, d \in \mathbb{Q}^*$ are such that $-3abcd$ is in $(-4)\mathbb{Q}^{*4}$ and $D(\mathbb{Q}_3) \neq \emptyset$. Then for any element of $\text{Br}(D)_3$ which is not in $\text{Br}_0(D)$, the evaluation map $D(\mathbb{Q}_3) \to \frac{1}{3}\mathbb{Z}/\mathbb{Z}$ is surjective.

**Proof.** By Lemma 5.18 it is enough to assume that $D$ is one of the surfaces $A$, $B$, $C$, and then assume that our element is $A \in \text{Br}(D)_3$ constructed in Section 5.1.
Each of our surfaces has a $\mathbb{Q}$-point, and at any such point the value of $A$ is zero by Lemma 5.6. For each surface we shall exhibit a point $L = (x, y, z, w)$ in $D(\mathbb{Q}_3)$, where $x, y, z, w$ are 3-adic integers such that neither $S_3(a, b : x, y)$ nor $S_3(c, d : z, w)$ holds, and use Corollary 5.17. The surjectivity of $\ev_{A, 3}$ then follows from Corollary 5.2.

We remind the reader that $E^m$ has additive reduction if and only if 4 does not divide $\val_3(m)$.

In the case of good reduction the group of points on the reduced curve has no 3-torsion, hence in this case we have $E^m(\mathbb{Q}_3)/3 = E^m(\mathbb{Q}_3)_1/E^m(\mathbb{Q}_3)_2$.

A. In this case we prove a stronger statement: if 9 does not divide $xyw$, then neither $S_3(a, b : x, y)$ nor $S_3(c, d : z, w)$ holds. An example of such a point is $(\alpha, 3, 1, 1)$ for an appropriate $\alpha \in \mathbb{Z}_3^*$.

Indeed, suppose that $\val_3(xyw) \leq 1$. Then an easy argument shows that $z$ is a unit and so is exactly one of $x$ and $y$. Thus $\val_3(xy) = 1$ and hence $w \in \mathbb{Z}_3^*$. Since $f \in \mathbb{Z}_3^*$ the curve $E'$ with equation $Y^2 = X^3 - 4abf^2X$ has good reduction. The point $P$ is in $E^m(\mathbb{Q}_3)_1$ but not in $E^m(\mathbb{Q}_3)_2$, so the property $S_3(a, b : x, y)$ does not hold. The curve $E''$ with equation $Y^2 = X^3 - 4cdf^2X$ has additive reduction. The point $Q + (0, 0)$ with X-coordinate $f^2/z^2w^2$ is in $E''(\mathbb{Q}_3)_0$ but not in $E''(\mathbb{Q}_3)_1$. Because of (12) this point is not in $3E''(\mathbb{Q}_3)$. Since $3(0, 0) = (0, 0)$, we conclude that $Q \notin 3E''(\mathbb{Q}_3)$.

B. In this case we prove a stronger statement: if 3 does not divide $zw$, then neither $S_3(a, b : x, y)$ nor $S_3(c, d : z, w)$ holds. An example of such a point is $(\alpha, 1, 1, 1)$ for an appropriate $\alpha \in \mathbb{Z}_3^*$.

Indeed, suppose that $z, w \in \mathbb{Z}_3^*$. Then $x, y \in \mathbb{Z}_3^*$ and $\val_3(f) = 1$. The curve $E'$ with equation $Y^2 = X^3 - 4abf^2X$ has additive reduction. The X-coordinate of $P$ is $-4abx^2y^2$, so $P$ is in $E'(\mathbb{Q}_3)_0$ but not in $E'(\mathbb{Q}_3)_1$, thus $S_3(a, b : x, y)$ does not hold.

The curve $E''$ with equation $Y^2 = X^3 - 4cdf^2X$ has additive reduction. To get the minimal model we write $X = 9X', Y = 27Y'$, then the point $Q$ has $X'$-coordinate $\pm 12z^2w^2$. The point $Q + (0, 0)$ has $X'$-coordinate $\pm f^2/9z^2w^2$, and so $Q + (0, 0)$ is in $E''(\mathbb{Q}_3)_0$ but not in $E''(\mathbb{Q}_3)_1$. Thus $Q + (0, 0)$, and hence $Q$, is not in $3E''(\mathbb{Q}_3)$, so that $S_3(c, d : z, w)$ does not hold.

C. Take $x = w = 1$ and $y = 2$. Then there exists $z \in \mathbb{Z}_3^*$ such that $(x, y, z, w) \in D(\mathbb{Q}_3)$. We have $f = 17$, so the curve $E'$ with equation $Y^2 = X^3 - (34)X$ has good reduction. The point $P$ has X-coordinate $-16$, so it reduces to the same point as the 2-torsion point $M = (-34, 0)$. Using sage we check that the valuation of the X-coordinate of $P + M$ is $-2$, hence $P + M$ is in $E'(\mathbb{Q}_3)_1$ but not in $E'(\mathbb{Q}_3)_2 = 3E'(\mathbb{Q}_3)_1$. The number of $\mathbb{F}_3$-points on the reduction is 4, thus

$$E'(\mathbb{Q}_3)/3 = E'(\mathbb{Q}_3)_1/3 = E'(\mathbb{Q}_3)_1/E'(\mathbb{Q}_3)_2.$$ Therefore, $S_3(a, b : x, y)$ does not hold.
Next, the curve $E''$ with equation $Y^2 = X^3 - 4cdf^2X$ has additive reduction. The point $Q + (0,0)$ with $X$-coordinate $f^2/z^2w^2$ is in $E''(\mathbb{Q}_3)_0$ but not in $E''(\mathbb{Q}_3)_1$. Therefore, this point, and hence also $Q$, is not in $3E''(\mathbb{Q}_3)$. Thus $S_3(c,d : z, w)$ does not hold. □

The proof of Theorem 1.2 is now complete.

References


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