ASYMPTOTICALLY CYLINDRICAL CALABI-YAU MANIFOLDS

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ABSTRACT. Let X be a compact connected Ricci-flat manifold and let M be a complete Kähler Ricci-flat manifold with one end which converges to $[0,\infty)\times X$ at an exponential rate. We prove general structure theorems for M. In particular we show that there is no loss of generality in assuming that M is simply-connected and irreducible. If $\dim_{\mathbb{C}} M > 2$, we then show that there exists a projective orbifold $\overline{M}$ and an orbifold divisor $\overline{D} \in |-K_{\overline{M}}|$ such that M is biholomorphic to $\overline{M} \setminus \overline{D}$. We give examples where $\overline{M}$ is not smooth; the existence of such examples appears not to have been noticed previously. If X splits off a circle, then $\overline{M}$ and $\overline{D}$ are both smooth and we show that the linear system $|\overline{D}|$ defines a fibration $\overline{M} \to \mathbb{P}^1$. Conversely for any such fibred manifold $\overline{M}$ we give a short direct and self-contained proof of the existence and uniqueness of (exponentially) asymptotically cylindrical Calabi-Yau metrics with split cross-section on $\overline{M}\setminus\overline{D}$. As a consequence, *all* asymptotically cylindrical Calabi-Yau manifolds with split cross-section must arise from this particular construction.

1. Introduction

Background and overview. In one of their foundational papers on complete Kähler Ricci-flat metrics Tian and Yau proved the existence of such metrics with linear volume growth on particular smooth noncompact quasiprojective varieties M [40, Corollary 5.1]. In fact, their estimates establish that M is uniformly bi-Lipschitz at infinity to half of a metric cylinder $M_\infty = \mathbb{R} \times X$ with a given smooth compact Ricci-flat cross-section X. The current paper has two principal goals:

(i) To give a short and self-contained proof of a refined version of the Tian-Yau existence theorem that in particular establishes exponential convergence of M to M_∞.
(ii) To understand to what extent every Kähler Ricci-flat manifold that is exponentially asymptotically cylindrical in this sense arises from the given construction.

(i) is important because the exponential convergence is used in an essential way in the so-called *twisted connected sum* construction of compact Riemannian 7-manifolds with holonomy group equal to the compact exceptional Lie group G_2 [9, 10, 23], first suggested by Donaldson and then pioneered by Kovalev in [23]. At present no complete proof of the existence of exponentially asymptotically cylindrical Calabi-Yau manifolds exists in the literature: see also the introduction to Section 4. Moreover, the existence proof of [40] is difficult and much more general; we will show that the asymptotically cylindrical case allows for a fairly short and direct treatment.

(ii) fits naturally into the broader framework of complex analytic compactifications of complete Kähler Ricci-flat manifolds—a topic Yau raised in his 1978 ICM Address [45]. Under the assumption of finite topology all currently known complete Kähler Ricci-flat manifolds are biholomorphic to a quasi-projective variety (the examples of [1] show that even complete hyper-Kähler 4-manifolds need not have finite topology though). However, we are not aware of any general results in the Kähler Ricci-flat case that establish quasiprojectivity even under additional hypotheses. One strategy which has had some success in Kähler manifolds with $\mathrm{Ric} < 0$ and finite volume [31] (or with $\mathrm{Ric} > 0$ and Euclidean volume growth [30]) relies on studying the section ring of the (anti-)canonical bundle. In this paper we adopt a completely different approach to proving compactification results.

Let M be asymptotically cylindrical Kähler Ricci-flat and let $n = \dim_{\mathbb{C}} M$. We show that there is no loss of generality in assuming that M is simply-connected of holonomy $\mathrm{SU}(n)$. If $n > 2$, we will then prove that the asymptotic cylinder M_∞ of M has a finite cover that splits as a Kähler product $\mathbb{R} \times \mathbb{S}^1 \times D$, where D is compact Kähler Ricci-flat. The cylinder M_∞ now admits a natural orbifold compactification, so we can try to use the fact that M is asymptotic to M_∞ to build a (projective) orbifold compactification of M. We emphasise that a considerable amount of extra work is required to pass from a compactification of M_∞ to a compactification of M; see Section 3.

Basic terminology. Before proceeding to a more detailed description of the main results and the organization of the paper, we begin with a few basic definitions and remarks.

Definition 1.1. A complete Riemannian manifold (M, g) is called *asymptotically cylindrical* (ACyl) if there exist a bounded domain $U \subset M$, a closed (not necessarily connected) Riemannian manifold (X, h), and a diffeomorphism $\Phi : [0,\infty) \times X \to M \setminus U$ such that $|\nabla^k(\Phi^* g - g_\infty)| = O(e^{-\delta t})$ in terms of $g_\infty := dt^2 + h$ for some $\delta > 0$ and all $k \in \mathbb{N}_0$. Here t denotes projection onto the $[0,\infty)$ factor; we often extend $t \circ \Phi^{-1}$ by zero and refer to this as a *cylindrical coordinate function* on M. We call the connected components of $M_\infty := \mathbb{R} \times X$ endowed with the product metric g_∞ the *asymptotic cylinders* (or sometimes the *cylindrical ends*) and (X, h) the *cross-section* of (M, g).

We will often suppress the map Φ in our notation, or tacitly replace it by $\Phi \circ [(t, x) \mapsto (t + t_0, x)]$ for some large constant t_0. Also, it will be irrelevant whether we measure norms of tensors on $M \setminus U$ with respect to g or g_∞. Finally, we remark that exponential asymptotics are a priori more natural than polynomial or even weaker ones because solutions to linear elliptic equations on cylinders tend to behave exponentially. The Calabi-Yau condition is not linear, but we obtain a consistent theory within the exponential setting; see also the Concluding Remarks at the end of this section.

Remark 1.2. We will mainly be interested in ACyl manifolds that are *Ricci-flat*. In this case:

(i) M has only a single end except when it is isometric to a product cylinder. This is an immediate consequence of the splitting theorem [6], and holds even if we assume only $\mathrm{Ric} \geq 0$. From now on in this remark, assume M is not a product cylinder.

(ii) The end M_∞ is a Ricci-flat cylinder, so the cross-section X is compact connected and Ricci-flat. We recall a basic structure result: there exists a flat torus $\mathbb{T}$ with $\dim \mathbb{T} \geq b^1(X)$, a simply-connected compact Ricci-flat manifold X', and a finite Riemannian covering $\mathbb{T} \times X' \to X$ [11, Thm 4.5]. This is deduced from a more general theorem for $\mathrm{Ric} \geq 0$ [6, Thm 3], but uses the inequality $\mathrm{Ric} \leq 0$ in an essential way to ascertain that all Killing fields are parallel.

We also need to recall some terminology related to holonomy groups and explain what we will mean by a Calabi-Yau manifold in this paper. We say that (M, g) is *locally irreducible* if the representation of the restricted holonomy group $\mathrm{Hol}_0(M)$ is irreducible; by de Rham's theorem, this is equivalent to M being locally irreducible in the sense of isometric product decompositions. We call (M^{2n}, g) *Calabi-Yau* if $\mathrm{Hol}(M) \subseteq \mathrm{SU}(n)$; this implies M is Ricci-flat Kähler. Conversely, if M is Ricci-flat Kähler then $\mathrm{Hol}_0(M) \subseteq \mathrm{SU}(n)$, so if additionally M is simply-connected then it is Calabi-Yau. If M is Ricci-flat Kähler and locally irreducible then, by Berger's classification, either $\mathrm{Hol}_0(M) = \mathrm{SU}(n)$, or n is even and $\mathrm{Hol}_0(M) = \mathrm{Sp}(\frac{n}{2})$. In general, if $\mathrm{Hol}(M) \subseteq \mathrm{Sp}(\frac{n}{2})$, we say M is *hyper-Kähler*.

A final point of notation: $\mathbb{S}^k$ will denote a round k-sphere and $\mathbb{T}^k$ a flat k-torus (not necessarily a product of k circles). Thus $\mathbb{S}^1 = \mathbb{T}^1$ is a circle but we do not specify its radius. However, we always identify $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ topologically and denote the resulting angular coordinate on $\mathbb{S}^1$ by θ.

Killing the fundamental group. Our first main result gives an ACyl analogue of the structure theorem for compact Ricci-flat manifolds of Remark 1.2(ii). This is again an easy consequence of a structure result for (ACyl) manifolds with nonnegative Ricci curvature (Theorem 2.13).

Theorem A. *Every Ricci-flat* ACyl *manifold has a finite normal covering space that splits as the isometric product of a flat torus and a simply-connected Ricci-flat* ACyl *manifold.*

Theorem A allows us to reduce to the case that the Ricci-flat ACyl manifold in question is simply-connected and irreducible. In particular, if the manifold is in addition Kähler—as it will be in most of our results—we can assume without loss that its full holonomy is either $\mathrm{SU}(n)$ or $\mathrm{Sp}(\frac{n}{2})$.

Holonomy and the asymptotic cylinder. If M is ACyl Kähler, then naively one might expect that the orbits of the parallel vector field $J\partial_t$ on the cross-section X of the asymptotic cylinder M_∞ necessarily split off as isometric circle factors. But, even if $\mathrm{Ric} \geq 0$, this may not be true even up to a finite cover. However, in the Ricci-flat case, we will prove that such pathologies never occur once $\dim_{\mathbb{C}} M > 2$. This is important for the compactification problem in that the splitting gives us a

> *Compactification ansatz*: A complex product cylinder $\mathbb{R} \times \mathbb{S}^1 \times D \cong \mathbb{C}^* \times D$ can be compactified as $\mathbb{C} \times D$. If D has a holomorphic volume form Ω_D, then $(dt + id\theta) \wedge \Omega_D$ extends to a meromorphic volume form with a simple pole along $\{0\} \times D$.

However, we point out that this is really only an ansatz, or in other words a sufficient condition—a complex compactification may exist even if the metric is "irregular" in the sense that the $J\partial_t$-orbits do *not* split off as $\mathbb{S}^1$-factors in any finite cover. But the construction of a complex compactification would then require a new ansatz and could be much more complicated; see Remark 1.6.

The following theorem is proved in Section 2.3 based on holonomy considerations. The structure theorem for compact Ricci-flat manifolds of Remark 1.2(ii) plays a role too.

Theorem B. *Let M be simply-connected irreducible* ACyl *Calabi-Yau with $n = \dim_{\mathbb{C}} M > 2$.*

(i) *M is not hyper-Kähler, or in other words $\mathrm{Hol}(M) = \mathrm{SU}(n)$.*
(ii) *There exists a compact Calabi-Yau manifold D with a Kähler isometry ι of finite order m such that the cross-section X of M can be written as $X = (\mathbb{S}^1 \times D)/\langle\iota\rangle$, where ι acts on the product via $\iota(\theta, x) = (\theta + \frac{2\pi}{m}, \iota(x))$. Moreover, ι preserves the holomorphic volume form on D but no other holomorphic forms of positive degree. In particular, $b^1(X) = 1$.*

In particular, ACyl hyper-Kähler manifolds exist only in real dimension 4. But their asymptotic cylinders need not be finite quotients of a product $\mathbb{R} \times \mathbb{S}^1 \times D$; see again Remark 1.6.

The order m of the Kähler isometry ι of Theorem B(ii) really can be greater than 1 even though $\pi_1(M) = 0$; see Examples 1.5 and 1.8, both of which are 3-dimensional. This possibility seems not to have been realized before. In particular, such spaces do not fit within the remit of the Tian-Yau construction [40, Corollary 5.1]. However, we regard $m > 1$ as a nongeneric case because only a small fraction of the currently known ACyl Calabi-Yau manifolds have $m > 1$; Examples 1.5 and 1.8 are essentially the only series of examples with $m > 1$ known to us.

Remark 1.3. Theorem B(ii) severely restricts the possible shapes of M_∞.

(i) If $n = 3$ then D could be $\mathbb{T}^4$ or $K3$, but not a finite quotient of either; in Examples 1.5 and 1.8 we show that both occur (with $m > 1$). In both cases there are strong a priori restrictions on the possible values of m: if $D = \mathbb{T}^4$ then $m \in \{2, 3, 4, 6\}$ by [13, Lemma 3.3], while if $D = K3$ then $m \leq 8$ (and the number of fixed points of ι depends only on m) by [32, §0.1] or [34].
(ii) If $m = 1$, then $h^{p,0}(D) = 1$ for $p \in \{0, n-1\}$ but $h^{p,0}(D) = 0$ otherwise. Thus, if $n = 3$ then $D = K3$; alternatively if $\pi_1(D) = 0$ then $\mathrm{Hol}(D) = \mathrm{SU}(n-1)$. However, D could be reducible: $D = (K3 \times K3)/\mathbb{Z}_2$ is not a priori ruled out if $\mathbb{Z}_2$ acts anti-symplectically on each factor, *i.e.* as a holomorphic involution of $K3$ that changes the sign of the holomorphic volume form.

Theorem B(ii) implies that M_∞ is biholomorphic to the complement of $D/\mathbb{Z}_m$ in $(\mathbb{C} \times D)/\mathbb{Z}_m$. This compactification could be smooth even if $m > 1$, but if n is odd and if D has no holomorphic forms except in degrees 0 and $n - 1$, then the holomorphic Lefschetz formula tells us that ι must have fixed points, so in this case the compactification is definitely not smooth if $m > 1$.

A compactification theorem. In Section 3 we will prove that any ACyl Kähler manifold M that satisfies the conclusion of Theorem B(ii) has an orbifold holomorphic compactification $\overline{M}$ modeled on the obvious holomorphic compactification of M_∞. Moreover, $\overline{M}$ is in fact Kähler, and if M is Calabi-Yau, then $\overline{M}$ must necessarily be projective. Thus, our results are most comprehensive if M satisfies the assumptions of Theorem B; for simplicity we give the statement only in this case.

Theorem C. *Let M be simply-connected irreducible* ACyl *Calabi-Yau of complex dimension* > 2. *Let X, D, $\iota \in \mathrm{Isom}(D)$, and m be as in Theorem* B(ii) *and define $\overline{D} = D/\langle\iota\rangle$. Then with respect to either of the two parallel complex structures on M we have:*

(i) *There exists a projective orbifold $\overline{M}$ with $h^{p,0}(\overline{M}) = 0$ for all $p > 0$ such that $\overline{D} \in |-K_{\overline{M}}|$ is an admissible*[1] *divisor and M is biholomorphic to $\overline{M} \setminus \overline{D}$. If $m = 1$ then $\overline{M}$ is smooth.*

(ii) *The* ACyl *Kähler form is cohomologous to the restriction to M of a Kähler form on $\overline{M}$.*

(iii) *If $b^1(D) = 0$ then the linear system $|m\overline{D}|$ is a pencil on $\overline{M}$, defining a fibration $\overline{M} \to \mathbb{P}^1$ with $\overline{D}$ as an m-fold fibre. In particular this holds for $m = 1$ since $b^1(X) = 1$ by Theorem* B(ii).

Before discussing the statement in more detail, we describe the basic motivation for our proof:

> Let $\overline{D}$ be a smooth divisor with trivial holomorphic normal bundle in some complex manifold $\overline{M}$. Then typically no tubular neighborhood of $\overline{D}$ is biholomorphic to the product of the unit disk with $\overline{D}$. However, there still exists an exponential map that sends the fibres of the normal bundle to holomorphic disks in $\overline{M}$.

In proving Theorem C (at least when $m = 1$), we first construct a "punctured version" of such an exponential map purely within M, with M_∞ playing the role of the normal bundle of the putative divisor $\overline{D}$ at infinity. Studying $\bar{\partial}$-type equations along the resulting punctured holomorphic disks in M allows us to prove that the complex structure of M is sufficiently regular at infinity to admit a holomorphic compactification $\overline{M}$. The normal bundle to $\overline{D}$ in $\overline{M}$ is then necessarily trivial.

Remark 1.4. We make some basic comments about the fibration in Theorem C(iii).

(i) No compact complex manifold with finite fundamental group can fibre over a Riemann surface with non-zero genus, since then the lift of the fibering map to the universal cover would be a non-constant holomorphic function from a compact complex manifold to $\mathbb{C}$.

(ii) We can compare the conclusions of Theorems B(i) and C(iii) with the following observation of Matsushita's [26, Lemma 1(2)]: If M is a *compact* Kähler manifold with holonomy $\mathrm{Sp}(\frac{n}{2})$, $n = \dim_{\mathbb{C}} M$, and if $f : M \to B$ is a surjective holomorphic map onto a Kähler manifold B of complex dimension $0 < b < n$, then $b = \frac{n}{2}$. (In this situation, a much more difficult result due to Hwang [18] then asserts that B is projective space if both M and B are algebraic.)

(iii) We do not know whether or not $|m\overline{D}|$ still defines a fibration of $\overline{M}$ over $\mathbb{P}^1$ if $b^1(D) > 0$ (hence necessarily $m > 1$). See also the discussion following Example 1.5.

We now describe a simply-connected ACyl Calabi-Yau 3-fold where $m = 2$ and D is a torus. This space is closely related to a Kummer construction due to Joyce; see [35, 7.3.3(iv)].

Example 1.5. Let E be an elliptic curve and $\overline{M}_0 = (\mathbb{P}^1 \times E \times E)/\langle\alpha, \beta\rangle$, where α and β act on $\mathbb{P}^1$ as the commuting holomorphic involutions $z \mapsto \frac{1}{z}$ and $z \mapsto -\frac{1}{z}$, and on $E \times E$ as $(-1, 1)$ and $(1, -1)$. Let $\overline{M}$ be the blow-up of $\overline{M}_0$ at the fixed sets of α and β (these have complex codimension 2). The fixed points of $\iota = \alpha\beta$ become orbifold singularities in $\overline{M}$ contained in the image $\overline{D} \cong (E{\times}E)/\{\pm 1\}$ of $\{0, \infty\} \times E \times E$. Since $\{0, \infty\}$ is an anticanonical divisor on $\mathbb{P}^1$ and the blow-up is crepant, $\overline{D}$ is an anticanonical orbifold divisor on $\overline{M}$ ("two cylindrical ends folded into one").

We can deduce that $M = \overline{M} \setminus \overline{D}$ admits ACyl Calabi-Yau metrics from a slight generalization of Theorem D; see Remark 1.7(ii). However, we can also think of M as a blow-up of the flat orbifold

$$M_0 = (\mathbb{R} \times \mathbb{S}^1 \times E \times E)/\langle\alpha, \beta\rangle$$

and obtain ACyl Calabi-Yau metrics by a generalised Kummer construction [35, 7.3.3(iv)]. Because $\langle\alpha, \beta\rangle$ is generated by elements with fixed points, the argument of [20, §12.1.1] can be applied to show that $\pi_1(\mathbb{R} \times \mathbb{S}^1 \times E \times E) \to \pi_1(M_0)$ is surjective, and that M_0 and M are simply-connected. This model for M also makes it easy to see that the cross-section X is the quotient of $\mathbb{S}^1 \times E \times E$ by the fixed-point free involution $(\theta, x, y) \mapsto (\theta + \pi, -x, -y)$; in particular, $b^1(X) = 1$ in accordance with Theorem B(ii) since the only $\mathbb{Z}_2$-invariant parallel 1-form upstairs is $d\theta$.

[1] This notion was introduced in [41, Definition 1.1(iii)] to capture a closely related phenomenon.

Example 1.5 is also interesting in view of the discussion of Theorem C(iii) in Remark 1.4(iii). By composing the projection $\mathbb{P}^1 \times E \times E \to \mathbb{P}^1$ with a degree 4 map $\mathbb{P}^1 \to \mathbb{P}^1$ invariant under $\langle \alpha, \beta \rangle$, we obtain a fibration $\overline{M} \to \mathbb{P}^1$; this must correspond to the linear system $|2\overline{D}|$. (The reader's intuition may be aided by observing that the special fibres are $I_0^* \times E$, $E \times I_0^*$, and $2\overline{D}$ in Kodaira's notation [22].) However, one can show that M admits nontrivial ACyl Calabi-Yau deformations with the same cylindrical end as M; it is not clear to us whether or not these are still fibred by $|2\overline{D}|$.

Remark 1.6. The compactification question for $n = 2$ is more subtle. To begin with, we have $X = \mathbb{T}^3$ since $\mathrm{Hol}(\mathbb{R} \times X) \not\subseteq \mathrm{SU}(2)$ if X is a proper quotient of $\mathbb{T}^3$ (but all orientable proper quotients of $\mathbb{T}^3$ occur as cross-sections of *locally* hyper-Kähler ACyl 4-manifolds with nontrivial π_1 [4, Thm 0.2]). By [17, Thm 1.10], X need not be an isometric product $\mathbb{S}^1 \times \mathbb{T}^2$, and by extending the construction of [17] one can show that *every* flat torus $\mathbb{T}^3$ occurs as a cross-section. Thus, for a generic choice of hyper-Kähler metric or parallel complex structure J, the orbits of $J\partial_t$ do not split off as isometric $\mathbb{S}^1$-factors in any finite cover of X, and our compactification ansatz does not apply.

It is nevertheless possible to compactify M_∞ holomorphically, strongly suggesting that M itself can be compactified so that $\overline{M}$ is $\mathbb{P}^2$ blown up in 9 general points, $\overline{D}$ is the proper transform of the *unique* cubic passing through these points, and $|\overline{D}|$ is trivial. By contrast, the construction in [17] is based on *pencils* of cubics in $\mathbb{P}^2$. We plan to discuss the details of this picture elsewhere.

Existence and uniqueness of ACyl Calabi-Yau metrics. Next we prove—at least in the split case ($m = 1$)—that every pair $(\overline{M}, \overline{D})$ satisfying the conclusions of Theorem C gives rise to ACyl Calabi-Yau metrics on $M = \overline{M} \setminus \overline{D}$. This is the content of Theorem D below. Moreover, we state a uniqueness result (Theorem E) that in particular implies that if M is simply-connected ACyl Calabi-Yau and if $(\overline{M}, \overline{D})$ is the compactification of M obtained in Theorem C, then the ACyl Calabi-Yau structure on M provided by Theorem D recovers the given one on M.

Theorem D. *Let $\overline{M}$ be a projective manifold of dimension $n \geq 2$. Assume that the linear system $|-K_{\overline{M}}|$ defines a map $\overline{M} \to \mathbb{P}^1$ with connected fibres. Let $\overline{D}$ be a smooth divisor in $|-K_{\overline{M}}|$ and let Ω be a rational n-form on $\overline{M}$ with a simple pole along $\overline{D}$. Then for every Kähler class $\mathfrak{k}$ on $\overline{M}$ there exists an* ACyl *Calabi-Yau metric ω on $M = \overline{M} \setminus \overline{D}$ such that $\omega \in \mathfrak{k}|_M$ and $\omega^n = i^{n^2} \Omega \wedge \bar{\Omega}$.*

Remark 1.7.

(i) Theorem D always yields ACyl Calabi-Yau manifolds with split cross-section $X = \mathbb{S}^1 \times \overline{D}$, *i.e.* $m = 1$ in Theorems B and C. Here $\overline{D}$, which has trivial canonical bundle by adjunction, carries the unique Ricci-flat Kähler metric representing $\mathfrak{k}|_{\overline{D}}$ provided by the Calabi-Yau theorem [44]. Also, Ω is unique only up to a scalar factor, so we get a 1-*parameter family* of ACyl Calabi-Yau metrics representing $\mathfrak{k}|_M$; this parameter is the length of the $\mathbb{S}^1$-factor.

(ii) We can construct examples with $m > 1$ proceeding from projective *orbifolds* $\overline{M}$ provided that we assume both that the orbifold divisor $\overline{D} \in |-K_{\overline{M}}|$ is a global quotient $\overline{D} = D/\langle \iota \rangle$ and that $\overline{M}$ is fibred over $\mathbb{P}^1$ by the linear system $|m\overline{D}|$ (as in Theorem C(iii)); in that case only minor modifications of the proof of Theorem D are needed. If $m > 1$, then we currently do not know that $\overline{M}$ is fibred except if $b^1(D) = 0$; *cf.* Remark 1.4(iii) and the discussion after Example 1.5. On the other hand, while the fibration assumption greatly simplifies the proof of Theorem D, it may ultimately only be a technical convenience.

Examples of projective manifolds $\overline{M}$ satisfying the hypotheses of Theorem D were first constructed by Kovalev [23] as blow-ups of Fano 3-folds; this construction yields around one hundred families of split ACyl Calabi-Yau 3-folds. In [9] so-called *weak* Fano manifolds are used instead; the weak Fano construction yields hundreds of thousands of families of split ACyl Calabi-Yau 3-folds.

Kovalev-Lee [24] describe a different class of manifolds $\overline{M}$ satisfying the hypotheses of Theorem D (therefore necessarily with $m = 1$) proceeding from $K3$ surfaces with anti-symplectic involutions. This leads to around 70 further families of split ACyl Calabi-Yau 3-folds. By modifying [24], we can find admissible orbifolds $\overline{M}$ with $m > 1$; the cross-section of the resulting non-split ACyl Calabi-Yau 3-fold will be the mapping torus of a finite order symplectic automorphism of $K3$.

Example 1.8. Let D be a $K3$ surface with a group $G = \langle\iota, \tau\rangle$ of holomorphic automorphisms where ι is symplectic of order m and τ is an anti-symplectic involution with non-empty fixed set such that $\tau\iota\tau = \iota^{-1}$; in particular, G is isomorphic to the dihedral group with $2m$ elements.

Let ι act on $\mathbb{P}^1$ by $z \mapsto e^{2\pi i/m}z$, and τ by $z \mapsto \frac{1}{z}$. Let $\overline{M}_0 = (\mathbb{P}^1 \times D)/G$ and let $\overline{M}$ be the blow-up at the fixed sets of the reflections $\tau\langle\iota\rangle \subset G$ (which are disjoint). $\overline{M}$ has orbifold singularities from the fixed points of the rotations $\langle\iota\rangle$, which all lie in the image $\overline{D} = D/\mathbb{Z}_m$ of $\{0, \infty\} \times D$.

We claim that $M = \overline{M} \setminus \overline{D}$ admits ACyl Calabi-Yau metrics with cross-section $X = (\mathbb{S}^1 \times D)/\mathbb{Z}_m$. This follows from the fibred orbi-version of Theorem D discussed in Remark 1.7(ii). Moreover, we can construct a fibration $\overline{M} \to \mathbb{P}^1$ with $\overline{D}$ as an m-fold fibre as in Example 1.5, but in this case the existence of the fibration is also guaranteed by Theorem C(iii) since $b^1(D) = 0$.

Here we choose not to pursue a systematic study of such examples and instead content ourselves with exhibiting a few concrete ones. As in Remark 1.3(i) we have the a priori bound $m \leq 8$. [21, §3] describes a $K3$ surface with automorphism group $A_6 \rtimes \mathbb{Z}_4$ containing G of the required kind for $2 \leq m \leq 6$; see also [12, §7]. For $m = 2, 3, 4$ one can also use Kummer surface constructions.

We now state our uniqueness result. Given some facts from ACyl Hodge theory, the proof is fairly straightforward; *cf.* Section 4.4. See also [17, Thm 1.9] and the surrounding discussion.

Theorem E. *Let M be an open complex manifold with only one end and let ω_1, ω_2 be* ACyl *Kähler metrics on M such that $\omega_1 - \omega_2$ is exponentially decaying with respect to either ω_1 or ω_2. If ω_1, ω_2 represent the same class in $H^2(M)$ and have the same volume form, then $\omega_1 = \omega_2$.*

Our main reason for including Theorem E here is that it allows us to see that Theorems C and D are inverse to each other—at least in the simply-connected, split ($m = 1$), complex dimension $n > 2$ setting, which we assume throughout the following discussion.

Let $(\overline{M}, \overline{D})$ be as in Theorem D and let $M = \overline{M} \setminus \overline{D}$ be the resulting ACyl Calabi-Yau manifold. The cylindrical end structure $\Phi : \mathbb{R}^+ \times \mathbb{S}^1 \times \overline{D} \to M$ of M is obtained by trivializing the fibration $\overline{M} \to \mathbb{P}^1$ smoothly in a tubular neighborhood of $\overline{D}$ and identifying the unit disk in $\mathbb{C}$ with $\mathbb{R}^+ \times \mathbb{S}^1$ via the complex exponential function. It is clear that applying Theorem C to M recovers $(\overline{M}, \overline{D})$.

On the other hand, if we start from an ACyl Calabi-Yau n-fold M with metric ω, apply Theorem C to compactify it to $\overline{M}$, and apply Theorem D to $\overline{M}$ to construct another ACyl Calabi-Yau metric ω' on M in the same Kähler class as ω, then $\omega - \omega'$ will be exponentially decaying—independent of all the choices involved in constructing ω'. Then Theorem E implies that $\omega = \omega'$.

Concluding remarks. We have now come full circle in our theory of exponentially asymptotically cylindrical Calabi-Yau manifolds, at least if $n > 2$ and $m = 1$: there exists a simple Tian-Yau type construction, and we have proved that this construction exhausts all possible examples. If $n = 2$, or if $n > 2$ and $m > 1$, then basic questions remain open as discussed in Remarks 1.6 and 1.7.

Even in the split case it remains to understand the purely complex projective geometry question of classifying the possible projective manifolds $\overline{M}$ satisfying the hypotheses of Theorem D. In three dimensions the vast majority of known examples (but not all [24]) arise by blowing up the base loci of smooth anticanonical pencils in smooth weak Fano 3-folds. This produces a very large but finite number of deformation families of split ACyl Calabi-Yau 3-folds. Is it possible to prove that there are only finitely many deformation families of split ACyl Calabi-Yau 3-folds?

Another (metric) question that remains is whether there exist asymptotically cylindrical Calabi-Yau manifolds with *slower* than exponential convergence. However, the methods of Cheeger-Tian [8] would seem to rule this out—if the Gromov-Hausdorff distance of a complete Calabi-Yau manifold to a cylinder goes to zero at infinity, then the convergence should automatically be exponential in C^∞ because the cross-section of the cylinder is always integrable as an Einstein manifold.

For a potentially more interesting analytic question, recall that complete Riemannian manifolds of nonnegative Ricci curvature always have at least linear volume growth. The case of *precisely* linear volume growth would therefore seem to be somewhat rigid; but examples due to Sormani show that numerous pathologies can occur [39]. Does the Calabi-Yau condition impose further restrictions? Is a complete Calabi-Yau of linear volume growth necessarily Gromov-Hausdorff asymptotic to $\mathbb{R} \times X$ for some geodesic metric space X? If so, then could X be non-compact or singular?

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2. Basic properties of ACyl Calabi-Yau manifolds

This section discusses the basic analysis, geometry, and topology of ACyl Calabi-Yau manifolds. In particular, it provides the technical tools necessary for the rest of the paper. The results stated in Theorems A and B will be proved as we go along: see Corollary 2.15 for A and §2.3 for B.

2.1. **Linear analysis and Hodge theory on ACyl manifolds.** We review some analytic facts for elliptic operators on manifolds with cylindrical ends from Lockhart-McOwen [25], with applications to the scalar and Hodge Laplacians and the Dirac operator on ACyl manifolds.

Suppose that $M = U \cup ([0,\infty) \times X)$ topologically for a bounded domain $U \subset M$ and a compact (but not necessarily connected) manifold X. A differential operator $\mathcal{A} : \Gamma(E) \to \Gamma(F)$ on sections of tensor bundles on M is called *asymptotically translation-invariant* if there is a translation-invariant operator $\mathcal{A}_\infty$ on sections of the corresponding bundles on $\mathbb{R}_t \times X$ such that the difference between the coefficients of $\mathcal{A}$ and $\mathcal{A}_\infty$ goes to zero in C^∞ uniformly as $t \to \infty$. Now even if $\mathcal{A}$ is elliptic, then since M is noncompact we cannot expect $\mathcal{A}$ to induce a Fredholm operator on ordinary Hölder or Sobolev spaces. To fix this, it is helpful to introduce Hölder norms with exponential weights.

Definition 2.1. Extend t smoothly to the whole of M. For $u \in C_0^\infty(E)$ define

$$\|u\|_{C^{k,\alpha}_\delta(E)} \equiv \|e^{\delta t} u\|_{C^{k,\alpha}(E)}, \tag{2.2}$$

and let $C^{k,\alpha}_\delta(E)$ denote the associated Banach space completion of $C_0^\infty(E)$. Thus, $C^{k,\alpha}_\delta$ sections are exponentially decaying for $\delta > 0$, and at worst exponentially growing for $\delta < 0$. We will occasionally use the notation $C^\infty_\delta(E) \equiv \bigcap C^{k,\alpha}_\delta(E)$.

We now assume that $\mathcal{A}$ is elliptic, *i.e.* that the principal symbol of $\mathcal{A}$ is an isomorphism in every cotangent direction. Then δ is called a *critical weight* if there exists a non-zero solution of

$$\mathcal{A}_\infty(e^{i\lambda t} u) = 0, \tag{2.3}$$

where $\operatorname{Im}\lambda = \delta$ and u is a section of $E \to \mathbb{R} \times X$ that is polynomial in t. The set of critical weights is a discrete subset of $\mathbb{R}$. We then have the following basic result [25, Thm 6.2]:

Proposition 2.4. *Let $\mathcal{A} : \Gamma(E) \to \Gamma(F)$ be asymptotically translation-invariant elliptic of order r. If δ is not a critical weight then the induced linear map $\mathcal{A} : C^{k+r,\alpha}_\delta(E) \to C^{k,\alpha}_\delta(F)$ is Fredholm.*

We mention some ingredients of the proof—partly because the result is stated for Sobolev rather than Hölder spaces in [25], and partly because we will need Remark 2.6 repeatedly in Section 3. The first step is to invert $\mathcal{A}$ along the cylindrical end.

Proposition 2.5. *If δ is not critical then there exists $\mathcal{R} : C^{k,\alpha}_\delta(F) \to C^{k+r,\alpha}_\delta(E)$ linear and bounded such that $\mathcal{A} \circ \mathcal{R} = \mathrm{id}$ on the complement of a bounded subset of M.*

Proof. Maz'ya-Plamenevskiĭ [27, Thm 5.1] show that $\mathcal{A}_\infty : C^{k+r,\alpha}_\delta(E) \to C^{k,\alpha}_\delta(F)$ is an isomorphism by using the Fourier transform. The condition on δ ensures that if $v \in \Gamma(F)$ is translation-invariant and $\operatorname{Im}\lambda = \delta$, then $\mathcal{A}_\infty(e^{i\lambda t}u) = e^{i\lambda t} v$ has a unique translation-invariant solution $u \in \Gamma(E)$.

Let $t_0 \gg 1$ and let $\rho : \mathbb{R}^+ \to \mathbb{R}$ be a cut-off function that is 0 for $t < t_0 - 1$ and 1 for $t > t_0$. Set $\mathcal{A}' \equiv (1-\rho)\mathcal{A}_\infty + \rho\mathcal{A}$ on $X \times \mathbb{R}$. Then $\mathcal{A}'$ is close to $\mathcal{A}_\infty$ in operator norm, so it has an inverse $\mathcal{R}' : C^{k,\alpha}_\delta(E) \to C^{k+r,\alpha}_\delta(E)$. If we define $\mathcal{R}(u) \equiv \mathcal{R}'(\rho u)$ on M, then $\mathcal{A}(\mathcal{R}(u)) = u$ for $t > t_0$. ☐

Remark 2.6. What is proved here is that $\mathcal{A}$ has a right inverse defined on $C^{k,\alpha}_\delta(F)$ over $[t_0,\infty)\times X$ provided that t_0 is large enough depending on k,α,δ. Since such right inverses are not unique, it is not immediately clear from the statement whether or not the right inverse given by Proposition 2.5 is independent of k,α, *i.e.* compatible with the obvious inclusions $C^{\ell,\beta}_\delta \subseteq C^{k,\alpha}_\delta$ for $\ell \geq k$ and $\beta \geq \alpha$. But this is clear from the proof, provided that the same cut-off function ρ is used.

Now let $\psi \in C^\infty_0(M)$ be a cut-off function which is equal to 1 for $t < t_0$. Proposition 2.4 can be deduced from Proposition 2.5 together with local Schauder theory and the fact that multiplication by ψ and the commutator $[\mathcal{A},\psi]$ define compact maps $C^{k+r,\alpha}_\delta(E) \to C^{k,\alpha}_\delta(E)$; see [25, §2].

In [25, Thm 6.2], Lockhart-McOwen also provide a formula to compute the change in index of $\mathcal{A}$ as δ passes a critical weight, by counting the number of solutions of (2.3). In [25, Thm 7.4], this is used to compute the indices of formally self-adjoint operators for $|\delta| \ll 1$. One application is

Proposition 2.7. *If X is connected and $\delta > 0$ is smaller than the square root of the first eigenvalue of the scalar Laplacian on X, then the scalar Laplacian on M maps $C^{k+2,\alpha}_\delta(M)$ isomorphically onto the subspace $C^{k+2,\alpha}_\delta(M)_0$ of functions of mean value zero.*

Proof. Integration by parts shows that the kernel of $\Delta : C^{k+2,\alpha}_\delta(M) \to C^{k,\alpha}_\delta(M)$ is trivial, and that functions in the image have mean value zero. But the index of Δ on these spaces is -1. □

The proof of the index formula uses asymptotic expansions for the elements in the kernel of $\mathcal{A}$. If we assume that $\mathcal{A}$ is asymptotic to $\mathcal{A}_\infty$ at an exponential (rather than just uniform) rate, these can be described more simply. This often makes it possible to imitate Hodge theoretic arguments on compact manifolds that are based on integration by parts and Weitzenböck formulas.

For example, if M is ACyl in the sense of Definition 1.1, then every bounded harmonic form α on M has an asymptotic limit α_∞, which is itself a harmonic form on M_∞, such that $\alpha - \alpha_\infty \in C^{k,\alpha}_\delta$ on M_∞ for all k,α and some $\delta > 0$. The bounded harmonic forms with $\alpha_\infty = 0$ are precisely the L^2-integrable ones. We denote the space of all bounded harmonic k-forms by $\mathcal{H}^k_{\mathrm{bd}}(M)$.

Proposition 2.8. *Let M be an* ACyl *Riemannian manifold.*

(i) *The natural map $\mathcal{H}^k_{\mathrm{bd}}(M) \to H^k(M)$ to the de Rham cohomology of M is surjective.*
(ii) *If M has a single end then $\mathcal{H}^1_{\mathrm{bd}}(M) \to H^1(M)$ is an isomorphism.*
(iii) *If M has nonnegative Ricci curvature then any bounded harmonic* 1*-form is parallel.*
(iv) *If M has nonpositive Ricci curvature then any Killing vector field is parallel.*

Proof. For (i), see Melrose [28, Thm 6.18]. For (ii), see [36, Cor 5.13]. (iii) is proved by the Bochner method. For (iv), first note that every Killing field of M converges exponentially to a Killing field of M_∞ [36, Prop 6.22]. Thus, the Bochner method applies again. □

Another application, which will be very significant for us, is to the Dirac operator of an ACyl spin manifold M. Let $\mathcal{H}^S_\infty$ be the space of translation-invariant solutions of the Dirac equation $\partial\!\!\!/ s = 0$ on M_∞, and let $\mathcal{H}^S_{\mathrm{bd}}$ and $\mathcal{H}^S_{L^2}$ denote the bounded and L^2 solutions on M. In analogy with harmonic forms, every element of $\mathcal{H}^S_{\mathrm{bd}}$ is asymptotic at an exponential rate to an element of $\mathcal{H}^S_\infty$.

Proposition 2.9. *Let M be an* ACyl *spin manifold.*

(i) $\dim(\mathcal{H}^S_{\mathrm{bd}}/\mathcal{H}^S_{L^2}) = \frac{1}{2}\dim \mathcal{H}^S_\infty$.
(ii) *If M has nonnegative scalar curvature, then every element of $\mathcal{H}^S_{\mathrm{bd}}$ is parallel.*

Proof. (i) is essentially an instance of (3.25) in Atiyah-Patodi-Singer [2]. It can also be deduced from the previously mentioned index formula [25, Thm 7.4]; see [35, §2.3.5] for details. (ii) follows from the Lichnerowicz formula and integration by parts. □

The strength of Proposition 2.9 is well-illustrated by the following "positive mass theorem", which is an immediate consequence by [43] (but will not be used in the rest of this paper).

Corollary 2.10. *Let M be an* ACyl *spin manifold of nonnegative scalar curvature. If the end M_∞ is Ricci-flat of special holonomy, then so is M.*

2.2. **Structure of Ricci-flat ACyl manifolds.** The goal here is to extend the structure theorem for compact Ricci-flat manifolds of Remark 1.2(ii) to the ACyl setting, proving Theorem A. As in the compact case, this will be a relatively easy consequence of a more general result (Theorem 2.13) for manifolds with $\mathrm{Ric} \geq 0$. At the end of this section, we also collect some closely related remarks that will not be used in rest of this paper, but are useful in [9, §2] and [10, §3]. All coverings in this section will be Riemannian, and all deck transformations are isometries.

The theory in the compact case rests on a subtle observation due to Cheeger-Gromoll in the proof of [6, Thm 3]. The following proposition states a slight extension of their idea that we require for our ACyl structure theorem. We give the proof for convenience.

Proposition 2.11. *A complete Riemannian manifold Z with* $\mathrm{Ric} \geq 0$ *admits a cocompact isometric group action if and only if Z splits as the isometric product of $\mathbb{R}^k$ and some compact manifold. In this case, every cocompact and discrete subgroup $\Gamma \subset \mathrm{Iso}(Z)$ contains a normal subgroup Ψ of finite index such that $[\Psi, \Psi]$ is finite and $\Psi/[\Psi, \Psi]$ is a free abelian group of rank k.*

Proof. By the splitting theorem, $Z = \mathbb{R}^k \times Z'$, where Z' contains no lines, and we must show that Z' is necessarily compact. Notice that $\mathrm{Iso}(Z) = \mathrm{Iso}(\mathbb{R}^k) \times \mathrm{Iso}(Z')$ because Z' is line-free. Since $\mathrm{Iso}(Z)$ acts cocompactly on Z, there exists a compact set $F' \subset Z'$ whose translates under $\mathrm{Iso}(Z')$ cover Z'. If Z' itself was noncompact, then there would exist a nontrivial ray $\gamma : [0, \infty) \to Z'$. For each $n \in \mathbb{N}$ there exists $g_n \in \mathrm{Iso}(Z')$ with $g_n(\gamma(n)) \in F'$. We can assume that $g_n(\gamma(n))$ has a limit as $n \to \infty$ because F' is compact. But then the shifted rays $\gamma_n : [-n, \infty) \to Z'$ defined by $\gamma_n(t) = g_n(\gamma(t + n))$ subconverge to a line locally uniformly in t, which contradicts the definition of Z'.

Let Γ' be the kernel of the projection of Γ to $\mathrm{Iso}(\mathbb{R}^k)$. Then Γ' is a discrete subgroup of $\mathrm{Iso}(Z')$, hence finite. On the other hand, the *image* Γ'' of the projection of Γ to $\mathrm{Iso}(\mathbb{R}^k)$ acts cocompactly on $\mathbb{R}^k$, and is discrete because $\mathrm{Iso}(Z')$ is compact and Γ is discrete. Thus Γ'' is a Bieberbach group. In other words, we have an exact sequence $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ with Γ' finite, and a split exact sequence $1 \to \mathbb{Z}^k \to \Gamma'' \to \Gamma''' \to 1$ with Γ''' a finite subgroup of $\mathrm{O}(k)$ acting on $\mathbb{Z}^k$ in the standard fashion. The preimage Ψ of $\mathbb{Z}^k$ under $\Gamma \to \Gamma''$ is then normal of finite index in Γ. Also, we have an exact sequence $1 \to \Psi' \to \Psi \to \mathbb{Z}^k \to 1$, so that $[\Psi, \Psi] \subset \Psi' \subset \Gamma'$ must be finite. □

Remark 2.12. Given a finitely generated group Γ with a finite index normal subgroup Ψ such that $[\Psi, \Psi]$ is finite, the rank $k < \infty$ of the abelian group $\Psi/[\Psi, \Psi]$ only depends on Γ; in fact, k is equal to the volume growth exponent of the Cayley graph of Γ.

By applying Proposition 2.11 to various normal covers of the cross-section of an ACyl manifold and bringing in some ACyl Hodge theory from Section 2.1, we will prove the following key

Theorem 2.13. *Let M be* ACyl *with* $\mathrm{Ric} \geq 0$ *and a single end. Then either M is a $\mathbb{Z}_2$-quotient of a cylinder, or its universal cover is isometric to $\mathbb{R}^k \times M'$, where M' is* ACyl *with a single end.*

Remark 2.14. We will see in the proof that $k \geq b^1(M)$, but the inequality can be strict; this already happens in the compact case if M is any compact flat k-manifold other than $\mathbb{T}^k$. However, k equals b^1 of a certain finite normal cover of M whose fundamental group has finite derived group.

The structure theorem for Ricci-flat ACyl manifolds (Theorem A) follows from this.

Corollary 2.15. *Every Ricci-flat* ACyl *manifold has a finite normal cover that splits isometrically as the product of a flat torus and a simply-connected Ricci-flat* ACyl *manifold.*

Proof. If M is a cylinder or a $\mathbb{Z}_2$-quotient of one, then the claim follows from Remark 1.2(ii) applied to the cross-section. If not, then Theorem 2.13 shows that the universal cover $\tilde{M}$ of M splits as an isometric product $\mathbb{R}^k \times M'$, where M' is ACyl with a single end. Thus, $\mathrm{Iso}(\tilde{M}) = \mathrm{Iso}(\mathbb{R}^k) \times \mathrm{Iso}(M')$. As M' has a single end, the orbits of $\mathrm{Iso}(M')$ are bounded, which implies that $\mathrm{Iso}(M')$ is compact. Therefore the projection of $\pi_1(M)$ to $\mathrm{Iso}(\mathbb{R}^k)$ is discrete, hence a Bieberbach group, so its projection to $\mathrm{SO}(k) = \mathrm{Iso}(\mathbb{R}^k)/\mathbb{R}^k$ is finite. Since M' is simply-connected Ricci-flat, Proposition 2.8(iv) tells us that $\mathrm{Iso}(M')$ is discrete, hence finite. The kernel Γ of the projection $\pi_1(M) \to \mathrm{SO}(k) \times \mathrm{Iso}(M')$ is therefore a finite index normal subgroup of $\pi_1(M)$ whose image in $\mathrm{Iso}(\mathbb{R}^k)$ acts on $\mathbb{R}^k$ as a full rank lattice of translations. Thus $(\mathbb{R}^k/\Gamma) \times M'$ is a cover of the required form. □

Example 2.16. To appreciate the role that the Ricci-flat condition plays in this proof, it is helpful to consider the following (compact) example [7, p. 440]. Let M be the mapping torus of a rotation of $\mathbb{S}^2$ by an irrational angle. Then M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, $\mathrm{Ric}\, M \geq 0$, but no finite cover of M splits isometrically as $\mathbb{S}^1 \times \mathbb{S}^2$. The proof of Corollary 2.15 fails at the point where one uses that the isometry group of M' is finite: the kernel of $\pi_1(M) \to \mathrm{SO}(k) \times \mathrm{Iso}(M')$ is trivial here.

We preface the proof of Theorem 2.13 with a simple lemma that will be applied twice.

Lemma 2.17. *Let Y be a connected manifold and $i : X \to Y$ the inclusion of a connected open set. Let G be a subgroup of $\pi_1(Y)$ and $p : \widetilde{Y} \to Y$ the covering space with characteristic group G. Then the number of connected components of $p^{-1}(X)$ is equal to the index of $\langle G, i_*(\pi_1(X))\rangle$ in $\pi_1(Y)$, and each such connected component is a covering of X with characteristic group $i_*^{-1}(G) \subset \pi_1(X)$.*

The first application deserves separate mention since it will itself be applied repeatedly.

Lemma 2.18. *If M is* ACyl *with* $\mathrm{Ric} \geq 0$ *and a single end, then either $\pi_1(M_\infty) \to \pi_1(M)$ is onto and every finite cover of M has a single end, or else the image has index* 2 *and $M = M_\infty/\mathbb{Z}_2$.*

Proof. If $\pi_1(M_\infty) \to \pi_1(M)$ is not surjective, consider the cover $\tilde{M} \to M$ with characteristic group equal to the image. By Lemma 2.17, $\tilde{M}$ has at least two cylindrical ends on which the covering map is a diffeomorphism onto M_∞. Thus, by the splitting theorem, $\tilde{M} = M_\infty$, and $M = M_\infty/\mathbb{Z}_2$. □

Proof of Theorem 2.13. Write $M_\infty = \mathbb{R} \times X$ for the end of M. By Lemma 2.18, we can assume that $\pi_1(M_\infty) \to \pi_1(M)$ is surjective. By Proposition 2.11 applied to the universal cover of X, $\pi_1(M_\infty)$ contains a finite index normal subgroup whose derived group is finite. Since $\pi_1(M_\infty)$ surjects onto $\pi_1(M)$, the image Ψ of this subgroup in $\pi_1(M)$ is still normal of finite index and has finite derived group. Replacing M by its finite normal cover with characteristic group Ψ, which is still ACyl with a single end, we can thus assume without loss that $\pi_1(M)$ itself has finite derived group.

Let $k \in \mathbb{N}_0$ denote the rank of the abelianization of $\pi_1(M)$. Then in particular $b^1(M) = k$, and so Proposition 2.8(ii)-(iii) tells us that k is also the number of parallel vector fields on M. Thus, by de Rham's theorem, the universal cover $\tilde{M}$ splits as an isometric product $\tilde{M} = \mathbb{R}^k \times M'$, where M' is complete and simply-connected. A priori M' could split off further line factors, but our goal is to show that this does not happen and moreover that M' is ACyl with a single end.

The parallel vector fields on M form a k-dimensional abelian Lie algebra $\mathfrak{a}$ of Killing fields on M. Sending each element of $\mathfrak{a}$ to its asymptotic limit under the ACyl diffeomorphism Φ^{-1} of Definition 1.1, we obtain an isomorphism $\phi : \mathfrak{a}_\infty \to \mathfrak{a}$ with an abelian Lie algebra $\mathfrak{a}_\infty$ of parallel Killing fields on $M_\infty = \mathbb{R} \times X$. The elements of $\mathfrak{a}_\infty$ have no ∂_t-components—or in other words, can be regarded as parallel Killing fields on X—since otherwise $\mathrm{Iso}(M)$ would have unbounded orbits, which is not possible since M has only one end. Notice also that Φ is asymptotically ϕ-equivariant: we have

$$\mathrm{dist}_M(\Phi(t, \exp(a)x),\ \exp(\phi(a))\Phi(t, x)) \leq C|a|e^{-\delta t} \tag{2.19}$$

for all $a \in \mathfrak{a}_\infty$, simply by how ϕ was defined.

Elements of $\mathfrak{a}$ pull back to parallel Killing fields on $\tilde{M}$. By construction, the Lie algebra $\tilde{\mathfrak{a}}$ of all such pull-backs consists of the parallel vector fields tangent to the $\mathbb{R}^k$ factor in $\tilde{M} = \mathbb{R}^k \times M'$. We can assume that the domain U of Definition 1.1 is $\mathfrak{a}$-invariant. Put $E \equiv M \setminus U$ and let $\tilde{E}$ be the preimage of E under the covering map $\tilde{M} \to M$. By $\tilde{\mathfrak{a}}$-invariance, we have $\tilde{E} = \mathbb{R}^k \times E'$ with $E' \subset M'$.

Lemma 2.17 tells us that $\tilde{E}$ is a connected normal covering space of E with characteristic group $\ker(\pi_1(M_\infty) \to \pi_1(M))$ and deck group $\pi_1(M)$. There certainly exists a connected normal covering space $\tilde{X} \to X$ such that there exists a diffeomorphism $\tilde{\Phi} : [0, \infty) \times \tilde{X} \to \tilde{E}$ covering Φ. Let $\tilde{\mathfrak{a}}_\infty$ be the pull-back of $\mathfrak{a}_\infty$ to $\tilde{X}$. Then $\tilde{\mathfrak{a}}_\infty$ is an abelian Lie algebra of parallel Killing fields on $\tilde{X}$, ϕ induces an isomorphism $\tilde{\phi} : \tilde{\mathfrak{a}}_\infty \to \tilde{\mathfrak{a}}$, and (2.19) implies that

$$\mathrm{dist}_{\tilde{M}}(\tilde{\Phi}(t, \exp(\tilde{a})\tilde{x}),\ \exp(\tilde{\phi}(\tilde{a}))\tilde{\Phi}(t, \tilde{x})) \leq C|\tilde{a}|e^{-\delta t} \tag{2.20}$$

for all $\tilde{a} \in \tilde{\mathfrak{a}}_\infty$; to prove (2.20), fix $N \gg 1$ depending only on $\tilde{a}$ such that, for every $\tilde{y} \in \tilde{X}$, $\exp(\frac{\tilde{a}}{N})\tilde{y}$ is closer to $\tilde{y}$ than any deck group translate of $\tilde{y}$, and then apply (2.19) N times.

We now wish to use these preparations to argue that $\tilde{X} = \mathbb{R}^k \times X'$ with X' *compact*, and that $\tilde{\Phi}$ induces an ACyl diffeomorphism $\Phi' : [0,\infty) \times X' \to E'$ in the sense of Definition 1.1. The key point of this argument is the following: $\pi_1(M)$ acts isometrically on $\tilde{X}$ with compact quotient X. Thus, Proposition 2.11 tells us that $\tilde{X} = \mathbb{R}^\ell \times X'$ with X' compact for some $\ell \in \mathbb{N}_0$, and that $\pi_1(M)$ has a finite index normal subgroup with finite derived group whose abelianization has rank ℓ. But recall that we arranged for $\pi_1(M)$ itself to have finite derived group; thus, $\ell = k$ by Remark 2.12.

Now the basic idea for splitting off Φ' from $\tilde{\Phi}$ is as follows. Since $\tilde{\Phi}$ is an almost isometry, it sends lines to almost lines. But the lines in $\tilde{M}$ are $\tilde{\mathfrak{a}}$-orbits and $\tilde{\Phi}$ is almost equivariant, so the lines in $\tilde{X}$ are $\tilde{\mathfrak{a}}_\infty$-orbits (approximately—hence precisely) even though a priori we only knew that $\tilde{\mathfrak{a}}_\infty$ consisted of parallel vector fields and X' might have parallel vector fields too. Using the approximate isometry and equivariance properties of $\tilde{\Phi}$ again, it quickly follows that $\tilde{\Phi}$ acts as an almost isometry on the $\mathbb{R}^k$ factor and as an ACyl diffeomorphism on the $[0,\infty) \times X'$ factor.

In fact we will argue slightly differently. If $\tilde{a} \in \tilde{\mathfrak{a}}_\infty$ had a nontrivial X'-component, the curves $\gamma_t(s) \equiv (t, \exp(s\tilde{a})\tilde{x})$ would not be lines, *i.e.* there exist $s_0 > 0$ and $\theta < 1$ independent of t such that the distance between $\gamma_t(0)$ and $\gamma_t(s_0)$ is θs_0. But $\tilde{\mathfrak{a}}$ is tangent to the $\mathbb{R}^k$ factor in $\tilde{E}$, so (2.20) shows that $\tilde{\Phi} \circ \gamma_t : [0, s_0] \to \tilde{E}$ remains $O(s_0 e^{-\delta t})$ close to a line segment of length s_0. This means that if σ is any other curve in $\tilde{X}$ connecting $\gamma_t(0)$ and $\gamma_t(s_0)$, then $\tilde{\Phi} \circ \sigma$ has length at least $s_0 - O(s_0 e^{-\delta t})$. Now $\tilde{\Phi}^* g_{\tilde{M}} = dt^2 + g_{\tilde{X}} + O(e^{-\delta t})$, so the length of σ itself is at least $s_0 - O(s_0 e^{-\delta t})$. Taking σ to be distance minimizing and t sufficiently large relative to θ and s_0, we get a contradiction.

Now we know that the $\tilde{\mathfrak{a}}_\infty$-orbits are the lines in $\tilde{X} = \mathbb{R}^k \times X'$. Fixing linear coordinates y on $\mathbb{R}^k$ and writing x for points in X' for simplicity, (2.20) then implies that

$$\tilde{\Phi}(t, y, x) = (\tilde{\Phi}(t, 0, x)_{\mathbb{R}^k} + \tilde{\phi}(y), \tilde{\Phi}(t, 0, x)_{M'}) + O(|y|e^{-\delta t}). \tag{2.21}$$

Here we have decomposed the target $\tilde{M} = \mathbb{R}^k \times M'$. Notice that (2.20) provides $O(|y|e^{-\delta t})$ control on the errors only in a distance sense; we will take it for granted that if $|y| \ll 1$ and $t \gg 1$ then this can be upgraded to C^∞ control in local charts (alternatively we could arrange for $\tilde{\Phi}$ to be *precisely* equivariant but this requires similar technical work to make precise). It then follows from (2.21) and the almost isometry property $\tilde{\Phi}^*[dy^2 + g_{M'}] = [dt^2 + dy^2 + g_{X'}] + O(e^{-\delta t})$ that

$$\tilde{\Phi}(t, 0, x)_{\mathbb{R}^k} = const + O(e^{-\delta t}), \;\; (\Phi')^*[g_{M'}] = [dt^2 + g_{X'}] + O(e^{-\delta t}), \tag{2.22}$$

where we have defined $\Phi'(t, x) \equiv \tilde{\Phi}(t, 0, x)_{M'}$.

To conclude that M' is an ACyl manifold in the sense of Definition 1.1, it remains to prove that $M' \setminus E'$ is bounded. If not, then M' would be a cylinder by the splitting theorem, *i.e.* there exists a function $t' : M' \to \mathbb{R}$ with $\nabla^2 t' = 0$ which is exponentially asymptotic to $t : E' \to [0,\infty)$ on E'. Notice that the trivial extension of t' to $\tilde{M} = \mathbb{R}^k \times M'$ is deck group invariant because $\tilde{E}$ and t are. But then t' pushes down to an unbounded Lipschitz function on the bounded region $U \subset M$. (This whole argument crucially exploits that $\tilde{E}$ is connected by our initial reductions.) □

With the proof of the main theorem of this section out of the way, we now explain some related but more elementary observations that are needed in [9, §2] and [10, §3].

Proposition 2.23. *Let M be* ACyl *Calabi-Yau and let $n = \dim_{\mathbb{C}} M$.*

(i) *If $\pi_1(M)$ is finite then M has a single end and $\pi_1(M_\infty) \to \pi_1(M)$ is surjective.*
(ii) *If $\pi_1(M)$ is finite and $n = 3$ then M has holonomy* SU(3).
(iii) *If $M_\infty = \mathbb{R} \times \mathbb{S}^1 \times D$ with $\pi_1(D)$ finite then either $\pi_1(M)$ is finite or $M = M_\infty/\mathbb{Z}_2$.*

Proof. (i) This follows from Lemma 2.18 if we can show that every cover $\tilde{M} \to M$ has a single end. But otherwise $\tilde{M}$ would be a Calabi-Yau cylinder $\mathbb{R} \times \tilde{X}$ by the splitting theorem, and $b^1(\tilde{X}) = 0$ since $\pi_1(\tilde{M})$ is finite, whereas Jdt is a nontrivial harmonic 1-form on $\tilde{X}$.

(ii) Let $\tilde{M}$ be the universal cover of M. By (i), this is ACyl with a single end. If $\mathrm{Hol}(\tilde{M})$ were a proper subgroup of SU(3) then by the de Rham theorem $\tilde{M}$ would be a product of simply-connected lower-dimensional submanifolds with even smaller holonomies, so at least one of these factors would be $\mathbb{C}$, contradicting that $\tilde{M}$ is ACyl. Now $\mathrm{Hol}(\tilde{M}) = \mathrm{SU}(3)$ implies $\mathrm{Hol}(M) = \mathrm{SU}(3)$ by [35, 4.1.10].

(iii) If $\pi_1(M)$ is infinite then Corollary 2.15 shows that M has a finite cover $\tilde{M} = \mathbb{T}^k \times M'$ with $k \geq 1$ and M' simply-connected ACyl Calabi-Yau. Let X' denote the cross-section of M'; this may not be connected. Then $\mathbb{T}^k \times X'$ covers $\mathbb{S}^1 \times D$, so $\pi_1(D)$ finite implies $k = 1$. Since $\tilde{M}$ is Kähler, the space of parallel 1-forms on $\tilde{M}$ inherits a complex structure and therefore has even dimension. Hence M' has a parallel 1-form. Since $b^1(M') = 0$, M' must have more than one end by Proposition 2.8(ii), hence split as a cylinder, and so Lemma 2.18 tells us that $M = M_\infty/\mathbb{Z}_2$. □

The simplest example of an ACyl Calabi-Yau manifold $M = M_\infty/\mathbb{Z}_2$ as in Proposition 2.23(iii) is $M = (\mathbb{R} \times \mathbb{S}^1 \times D)/(-1, -1, \tau)$ with D a $K3$ surface and τ a free anti-symplectic involution of D; see Remark 1.3. There is exactly one deformation family of such pairs (D, τ) ("Enriques surfaces"), so this is essentially the unique M of this kind with $n \leq 3$.

2.3. **Holonomy considerations.** The main content of this section is the proof of Theorem B but first we need to recall some background material.

The first ingredient is the well-known relation between special holonomy and parallel spinors [43]. If Z is a Riemannian spin manifold, then we write $s(Z)$ for the number of parallel spinors on Z. A Kähler manifold Z with trivial canonical bundle is spin and its spinor bundle is naturally identified with the total bundle of $(0, p)$-forms [3, 1.156], so that parallel spinors correspond to parallel $(0, p)$-forms and we always have $s(Z) \geq 1$ from $p = 0$. Let $d = \dim_{\mathbb{C}} Z$. If $\mathrm{Hol}(Z) \subseteq \mathrm{SU}(d)$ then $s(Z) \geq 2$ from the conjugate holomorphic volume form except if Z is a point. If Z is even hyper-Kähler, *i.e.* $\mathrm{Hol}(Z) \subseteq \mathrm{Sp}(\frac{d}{2})$, then $s(Z) \geq \frac{d}{2} + 1$ from the powers of the conjugate holomorphic symplectic form. If $\mathrm{Hol}(Z)$ is *equal* to $\mathrm{SU}(d)$ or $\mathrm{Sp}(\frac{d}{2})$, then $s(Z) = 2$ if $d > 0$ and $s(Z) = \frac{d}{2} + 1$, respectively [43]; this is a purely representation-theoretic fact. (The converse is false—in Remark 1.3(ii) we mentioned a Kähler 4-fold with holonomy $(\mathrm{SU}(2) \times \mathrm{SU}(2)) \rtimes \mathbb{Z}_2$ and $s = 2$.) Finally, it is helpful to keep in mind that *all* holomorphic forms on a compact Kähler manifold with $\mathrm{Ric} \geq 0$ are parallel by the Bochner method; this still holds for all bounded holomorphic forms in the ACyl case.

The second ingredient is the following structure theorem for compact Ricci-flat manifolds.

Proposition 2.24 (Calabi, Fischer-Wolf)**.** *Let X be compact connected Ricci-flat and set $k = b^1(X)$. There exists a flat torus $\mathbb{T}^k$ and a finite normal Riemannian covering $\mathbb{T}^k \times X' \to X$ such that:*

(i) *The deck group can be written as $\{(h(\psi), \psi) : \psi \in \Psi\}$, where Ψ is a finite group of isometries of X' and h is an injective homomorphism of Ψ into the translation group of $\mathbb{T}^k$.*

(ii) *X' is compact connected Ricci-flat and carries no Ψ-invariant parallel vector fields.*

This could be deduced from Remark 1.2(ii) (*i.e.* [11, Thm 4.5]) but is also proved directly in [11, Thm 4.1] without relying on the splitting theorem of [6]. The proposition generalizes an earlier result for compact flat manifolds due to Calabi; according to [11], Calabi was independently aware of this extension to the compact Ricci-flat case, but had only published the result for X Kähler.

Proof of Theorem B. Since M is simply-connected irreducible, either $\mathrm{Hol}(M) = \mathrm{SU}(n)$ or n is even and $\mathrm{Hol}(M) = \mathrm{Sp}(\frac{n}{2})$. The proof proceeds by analyzing these two cases separately but in parallel, based on the facts reviewed above and on the following consequence of Proposition 2.9:

$$s(M) = \frac{1}{2} s(M_\infty). \tag{2.25}$$

The main aim is to rule out the $\mathrm{Sp}(\frac{n}{2})$ case and show that in the $\mathrm{SU}(n)$ case, $b^1(X)$ (which is always at least 1 because of the parallel 1-form Jdt) has to be exactly 1. This already implies a significant part of the statement of Theorem B(ii) by applying Proposition 2.24 for $k = 1$.

The analysis in fact relies on the conclusion of Proposition 2.24, *i.e.* that we have a finite normal Riemannian covering $\mathbb{T}^k \times X' \to X$ whose deck group Ψ is a finite group of isometries of X' acting effectively on $\mathbb{T}^k$ by translations, and that Ψ does not preserve any parallel vector fields on X'. We will use this to construct parallel spinors on M_∞—almost always more than (2.25) allows.

Case 1: Holonomy $\mathrm{SU}(n)$. Then M has exactly two parallel holomorphic forms, so (2.25) tells us that $s(M_\infty) = 4$. Now since M_∞ is Kähler with respect to J_∞, the parallel vector fields on M_∞ are closed under J_∞, and so both $\mathbb{R} \times \mathbb{T}^k$ and X' are Ψ-invariantly Kähler. Thus, $k = 2\ell + 1$ for some $\ell \in \mathbb{N}_0$ and $\mathbb{R} \times \mathbb{T}^k$ is Ψ-invariantly Calabi-Yau. But this implies that X' is not just Ricci-flat and Ψ-invariantly Kähler, but Ψ-invariantly Calabi-Yau—by contracting the holomorphic n-form pulled back from M_∞ with the holomorphic $(\ell + 1)$-form on $\mathbb{R} \times \mathbb{T}^k$. We see that $\mathbb{R} \times \mathbb{T}^k$ has $2^{\ell+1}$ parallel holomorphic Ψ-invariant forms, and X' has at least 2 unless X' is a point, when there is only one. Thus, $s(M_\infty) \geq 2^{\ell+2}$ if X' is not a point, and $s(M_\infty) \geq 2^{\ell+1}$ if X' is a point. But $s(M_\infty) = 4$, and hence $\ell = 0$, $k = 1$, unless $\ell = 1$, $k = 3$, $n = 2$; we explicitly excluded the latter case.

If $k = 1$, then Ψ is a finite subgroup of $\mathrm{U}(1)$, so $\Psi = \langle \iota \rangle$ for some finite order isometry ι of X'. Moreover, we already know that ι preserves the complex structure and holomorphic volume form. Now X' can have more parallel $(p, 0)$-forms with $p > 0$ (*e.g.* parallel vector fields), but if any of those were Ψ-invariant, this would immediately contradict the above counting inequalities.

Case 2: Holonomy $\mathrm{Sp}(\frac{n}{2})$. In this case, $s(M_\infty) = n + 2$. Since M_∞ is hyper-Kähler, the parallel vector fields on M_∞ are closed under $I_\infty, J_\infty, K_\infty$, so $\mathbb{R} \times \mathbb{T}^k$ and X' are themselves Ψ-invariantly hyper-Kähler. In particular, $k = 4\ell + 3$ for some $\ell \in \mathbb{N}_0$, and there are now even more Ψ-invariant parallel holomorphic forms than before (though also more on M_∞ to begin with): $2^{2\ell+2}$ on the $\mathbb{R}\times\mathbb{T}^k$ factor and at least $\frac{n}{2} - \ell$ on the X' factor (which equals 1 if X' is a point). As before we deduce that $n + 2 \geq 2^{2\ell+2}(\frac{n}{2} - \ell)$. We now argue that this leaves no possibility except for $\ell = 0$, $k = 3$, $n = 2$; but this is the excluded case. If the inequality fails for some ℓ and n then it also fails for the same ℓ and all larger n. But $n \geq 2\ell + 2$, and the inequality does fail for $n = 2\ell + 2$ unless $\ell = 0$. If $\ell = 0$ then $k = 3$, and the inequality clearly holds for $n = 2$ but fails for all larger n. □

Remark 2.26. A similar argument of counting parallel spinors was used in [35, Thm 4.1.19] to give a criterion for an ACyl 8-manifold to have holonomy Spin(7).

3. Quasiprojectivity

3.1. **Proof of Theorem C modulo technical results.** Let M be simply-connected irreducible ACyl Calabi-Yau of complex dimension $n > 2$. By Theorem B(i), M has holonomy $\mathrm{SU}(n)$; hence there exists precisely one parallel complex structure J on M up to sign. Theorem B(ii) tells us that the cylindrical end M_∞ has a finite cover $\tilde{M}_\infty$ biholomorphic to $\mathbb{C}^* \times D$ for some compact Ricci-flat Kähler manifold D. Thus, $\tilde{M}_\infty$ can be compactified as $\mathbb{C} \times D$. One would then expect that M itself has a holomorphic compactification $\overline{M}$. This is true, but not obvious; it is also not obvious that $\overline{M}$ is Kähler. However, once we know this, Theorem C follows reasonably quickly.

We begin by stating the technical compactification results. This requires some terminology. Let Δ denote the unit disc in $\mathbb{C}$ and put $\Delta^* = \Delta \setminus \{0\}$. Let D be a compact complex manifold and g_D an arbitrary Hermitian metric on D. Let $M^+_\infty = \mathbb{R}^+ \times \mathbb{S}^1 \times D$ with product complex structure J_∞ and Hermitian metric $g_\infty = dt^2 + d\theta^2 + g_D$, where $\theta \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $J_\infty(\partial_t) = \partial_\theta$.

Theorem 3.1. *Let J be an integrable complex structure on M^+_∞ such that $J - J_\infty = O(e^{-\delta t})$ with respect to g_∞ as $t \to +\infty$, including all covariant derivatives, for some $\delta > 0$. Then there exists a diffeomorphism $\Psi : M^+_\infty \to \Delta^* \times D$ such that Ψ_*J extends as an integrable complex structure on $\Delta \times D$. Moreover, the submanifold $\{0\} \times D$ is complex and biholomorphic to D with respect to this extension, and its normal bundle is trivial as a holomorphic line bundle on D.*

Theorem 3.2. *In the setting of Theorem 3.1, assume in addition that there exists a J-Kähler form ω on M^+_∞ such that $\omega - \omega_\infty = O(e^{-\delta t})$ as $t \to +\infty$. Then $\Delta \times D$ admits a Ψ_*J-Kähler form which coincides with $\Psi_*\omega$ on $\{\frac{1}{2} < |w| < 1\} \times D$, where w denotes a complex coordinate on Δ.*

Let us first see how the full statement of Theorem C now follows.

Proof of Theorem C. We are given an m-sheeted covering $\tilde{M}_\infty$ of M_∞ such that $\tilde{M}_\infty = \mathbb{R} \times \mathbb{S}^1 \times D$ for some compact Ricci-flat Kähler manifold D. We can assume that the circle factor has length 2π.

Pulling back J from M to M_∞^+ by the ACyl diffeomorphism and further pulling back by the covering map $\tilde{M}_\infty^+ \to M_\infty^+$, we obtain a complex structure $\tilde{J}$ on $\tilde{M}_\infty^+$. Theorem 3.1 applies and produces a $\tilde{J}$-holomorphic compactification $\tilde{\Psi} : \tilde{M}_\infty^+ \hookrightarrow \Delta \times D$. The action of the deck group of the covering $\tilde{M}_\infty \to M_\infty$ extends and preserves the divisor D at infinity, so that M itself can be compactified as an orbifold $\overline{M}$ by adding a suborbifold $\overline{D} = D/\langle\iota\rangle$. Averaging the Kähler form on $\Delta \times D$ provided by Theorem 3.2 under the given holomorphic $\mathbb{Z}_m$-action, passing to the quotient, and joining it to the ACyl Kähler form on M, we obtain an orbifold Kähler form on $\overline{M}$.

Following [24, Prop 2.2], we can now easily see that $\overline{M}$ must even be projective: As in the smooth case, it suffices to prove that $\overline{M}$ does not admit any holomorphic $(2,0)$-forms. But any holomorphic $(p,0)$-form on $\overline{M}$ restricts to an asymptotically translation-invariant holomorphic $(p,0)$-form on M, and since $\mathrm{Hol}(M) = \mathrm{SU}(n)$, the usual Bochner argument then shows that there are no such forms if $0 < p < n$ (the unique bounded holomorphic n-form on M is Ω, which has a pole along $\overline{D}$).

As for the fibration of $\overline{M}$ by $|m\overline{D}|$, observe that we have a short exact sequence

$$0 \to \mathcal{O}_{\overline{M}} \to \mathcal{O}_{\overline{M}}(m\overline{D}) \to \mathcal{O}_{m\overline{D}}(m\overline{D}) \to 0. \tag{3.3}$$

The cokernel sheaf $\mathcal{O}_{m\overline{D}}(m\overline{D})$ is the sheaf of sections of the restriction of the line bundle $m\overline{D}$ to the scheme $m\overline{D}$, *i.e.* an infinitesimal "thickening" of $\overline{D}$. This yields a long exact sequence

$$0 \to H^0(\mathcal{O}_{\overline{M}}) \to H^0(\mathcal{O}_{\overline{M}}(m\overline{D})) \to H^0(\mathcal{O}_{m\overline{D}}(m\overline{D})) \to H^1(\mathcal{O}_{\overline{M}}).$$

Notice that $H^{0,1}(\overline{M}) = 0$. Thus, if we knew that $\mathcal{O}_{m\overline{D}}(m\overline{D})$ had a section, then we would find that $h^0(\mathcal{O}_{\overline{M}}(m\overline{D})) = 2$, so $|m\overline{D}|$ is a pencil. Now the line bundle $\ell\overline{D}$ is trivial on $\overline{D}$ for all $\ell \in m\mathbb{Z}$, but this does not imply that it is trivial on $m\overline{D}$ except if $m = 1$ (on the other hand, if $m = 1$, it is then also clear that $|\overline{D}|$ has no base locus). However, we have a general "lifting" sequence

$$0 \to \mathcal{O}_{k\overline{D}}(\ell\overline{D}) \to \mathcal{O}_{(k+1)\overline{D}}((\ell+1)\overline{D}) \to \mathcal{O}_{\overline{D}}((\ell+1)\overline{D}) \to 0 \tag{3.4}$$

for every $k \in \mathbb{N}_0$ and $\ell \in \mathbb{Z}$. Setting $k = \ell = m-1$ and taking cohomology yields

$$H^0(\mathcal{O}_{m\overline{D}}(m\overline{D})) \to H^0(\mathcal{O}_{\overline{D}}(m\overline{D})) \to H^1(\mathcal{O}_{(m-1)\overline{D}}((m-1)\overline{D})). \tag{3.5}$$

Thus, if the H^1 term vanishes (*e.g.* if $m = 1$), then our trivializing section extends from $\overline{D}$ to $m\overline{D}$. We can get a handle on this H^1 by taking cohomology in the *upstairs counterpart* to (3.4):

$$H^1(\mathcal{O}_{kD}(\ell D)) \to H^1(\mathcal{O}_{(k+1)D}((\ell+1)D)) \to H^1(\mathcal{O}_D((\ell+1)D)).$$

Now suppose that $b^1(D) = 0$ (which in fact follows from $m = 1$ in our setting). Since ℓD is trivial on D for all $\ell \in \mathbb{Z}$, the third term vanishes, and so induction on $k \in \mathbb{N}_0$ yields $H^1(\mathcal{O}_{kD}(\ell D)) = 0$ for all $k \in \mathbb{N}_0$ and $\ell \in \mathbb{Z}$. In particular, setting $k = \ell = m-1$ and taking $\mathbb{Z}_m$-invariants, we find that the obstruction space in (3.5) vanishes and the trivializing section of $\mathcal{O}_{\overline{D}}(m\overline{D})$ does extend. □

Remark 3.6. In Example 1.5, we have $m = 2$, so the formal obstruction space in (3.5) coincides with the $\mathbb{Z}_2$-invariants in $H^1(\mathcal{O}_D(D))$. To compute these, it is helpful to identify this H^1 with the space of constant $(0,1)$-forms on D taking values in the normal bundle. The two standard generators are then $d\bar{x} \otimes \frac{\partial}{\partial w}$ and $d\bar{y} \otimes \frac{\partial}{\partial w}$, with $w = re^{-i\theta}$, as in Example 1.5. But these are obviously $\mathbb{Z}_2$-invariant and so the formal obstruction space to fibering $\overline{M}$ by $|2\overline{D}|$ is 2-dimensional.

It remains to prove Theorems 3.1–3.2. This will be done in the following two subsections.

3.2. **Holomorphic compactification.** We begin with a discussion of the main difficulties and an outline of the argument. For $(t,\theta) \in \mathbb{R}^+ \times \mathbb{S}^1$ let $w = e^{-t-i\theta}$. Then the diffeomorphism

$$\Psi_\infty : M_\infty^+ \to \Delta^* \times D,\ \ (t,\theta,x) \mapsto (w,x), \tag{3.7}$$

pushes J_∞ forward to the product complex structure J_Δ on $\Delta^* \times D$, which is clearly compactifiable. However, $(\Psi_\infty)_*J$ may not even be *uniformly bounded* with respect to $g_\Delta = |dw|^2 + g_D$ as $w \to 0$. Specifically, for any section s of $(T^*\Delta)^a \otimes (T^*D)^b \otimes (T\Delta)^c \otimes (TD)^d$ over $\Delta^* \times D$ we have that

$$|\Psi_\infty^* s|_{g_\infty} = O(e^{-\delta t}) \iff |s|_{g_\Delta} = O(|w|^{\delta+c-a}). \tag{3.8}$$

Thus, in terms of the decomposition $T\Delta \oplus TD$, the off-diagonal $T^*\Delta \otimes TD$ components of $(\Psi_\infty)_*J$ can be expected to blow up like $|w|^{-1+\delta}$ as $|w| \to 0$; all the remaining components of $(\Psi_\infty)_*J$ are at least $C^{0,\delta}$ Hölder continuous along $\{0\} \times D$, but not—a priori—smooth.

The key point in resolving this problem is to exploit that the integrability of J is equivalent to a nonlinear first-order differential equation: the vanishing of the Nijenhuis torsion. This equation is not elliptic, but the lack of ellipticity can be traced back to diffeomorphism invariance. In other words, there is hope that a suitable improvement of Ψ_∞ *will* map J to a smooth complex structure on $\Delta \times D$.

The proof of Theorem 3.1 now follows in three steps. Step 1 shows how to construct a gauge in which J coincides with J_∞ in directions tangent to $\mathbb{R}^+ \times \mathbb{S}^1 \times \{x\}$ $(x \in D)$. This already fixes the discontinuity of $(\Psi_\infty)_*J$ at infinity. Based on this, Step 2 then uses an elliptic regularity argument along these cylinders to show that the pushforward of J is actually smooth at infinity; this involves the $C^{1,\alpha}$ Newlander-Nirenberg theorem of [33]. Step 3 deals with the normal bundle.

Step 1: Gauge fixing. The pushforward $(\Psi_\infty)_*J$ fails to be continuous at $\{0\} \times D$ if and only if the J_∞-holomorphic cylinders $\mathbb{R}^+ \times \mathbb{S}^1 \times \{x\}$ are not J-holomorphic. This suggests replacing Ψ_∞ by $\Psi_\infty \circ F^{-1}$, where $F \in \mathrm{Diff}(M^+_\infty)$ maps each $\mathbb{R}^+ \times \mathbb{S}^1 \times \{x\}$ onto a J-holomorphic curve exponentially asymptotic to it. For this, it suffices to find (J_∞, J)-holomorphic maps $F_x : \mathbb{R}^+ \times \mathbb{S}^1 \times \{x\} \to M^+_\infty$ that are exponentially asymptotic to the identity and depend smoothly on $x \in D$.

To solve this problem, it is helpful to invoke some of the usual formalism for the construction of holomorphic curves. Given $x \in D$ and the tautological map $f_{0,x} : \mathbb{R}^+ \times \mathbb{S}^1 \to \mathbb{R}^+ \times \mathbb{S}^1 \times \{x\} \subset M^+_\infty$, let $\mathcal{E}_x$ denote the space of all smooth embeddings $f : \mathbb{R}^+ \times \mathbb{S}^1 \to M^+_\infty$ exponentially asymptotic to $f_{0,x}$, and let $\mathcal{V}_x \to \mathcal{E}_x$ denote the natural vector bundle whose fibre at $f \in \mathcal{E}_x$ is the vector space of all exponentially decaying vector fields along f. With a very slight abuse of notation, we then have a section $\bar\partial \in \Gamma(\mathcal{E}_x, \mathcal{V}_x)$ whose value at f is given by $\bar\partial f \equiv \frac{\partial f}{\partial t} + J\frac{\partial f}{\partial \theta}$. Restricting to the region $t \gg 1$, we can assume that $\|\bar\partial f_{0,x}\| \ll 1$ uniformly in x, and our goal is to construct a genuine zero $f_x \in \mathcal{E}_x$ of the section $\bar\partial$ which, as an embedding of $\mathbb{R}^+ \times \mathbb{S}^1$ into M^+_∞, depends smoothly on x.

We begin by choosing a chart for $\mathcal{E}_x$ near $f_{0,x}$ (modelled on a definite neighborhood of the origin in $T_{f_{0,x}}\mathcal{E}_x$), as well as a trivialization for $\mathcal{V}_x$ over it. There are no canonical choices for either, but a natural and useful way is to apply the exponential map and parallel transport with respect to g_∞. This now allows us to view $\bar\partial \in \Gamma(\mathcal{E}_x, \mathcal{V}_x)$ as a nonlinear first-order differential operator $\bar\partial_x$ acting on some definite open neighborhood of the origin in $C^{k,\alpha}_\delta(\mathbb{R}^+ \times \mathbb{S}^1, f^*_{0,x}TM^+_\infty)$. We have $\|\bar\partial_x(0)\| \ll 1$, and the linearization $\mathcal{L}_x$ of $\bar\partial_x$ at 0 satisfies $\mathcal{L}_x = \mathcal{L} + \mathcal{U}_x$, where

$$\mathcal{L}V \equiv \frac{\partial V}{\partial t} + J_\infty\left(\frac{\partial V}{\partial \theta}\right), \ \ \|\mathcal{U}_x\|_{\mathrm{op}} \ll 1.$$

Also, $\mathcal{U}_x$ varies smoothly with x if we use parallel transport with respect to the Chern connection of (M^+_∞, g_∞) in order to identify $C^{k,\alpha}_\delta(\mathbb{R}^+ \times \mathbb{S}^1, f^*_{0,x}TM^+_\infty)$ with $C^{k,\alpha}_\delta(\mathbb{R}^+ \times \mathbb{S}^1, f^*_{0,y}TM^+_\infty)$ for different points $x, y \in D$. Notice that these identifications do not affect the operator $\mathcal{L} \equiv \bar\partial_{J_\infty}$.

Using Remark 2.6, we can construct a bounded right inverse $\mathcal{R}$ to $\mathcal{L}$ (since the $\bar\partial$-equation in one complex variable with values in a complex vector space is elliptic). The desired holomorphic maps f_x are then obtained by an elementary fixed point argument—specifically, by iterating the contraction mappings $\mathcal{R} \circ (\mathcal{L} - \bar\partial_x)$ on some neighborhood of the origin.

Step 2: Elliptic regularity. If we define $\Psi \equiv \Psi_\infty \circ F^{-1}$ with $F \in \mathrm{Diff}(M^+_\infty)$ as in Step 1, then we know that Ψ_*J is equal to the standard complex structure J_Δ on the horizontal subbundle $T\Delta$ of $T(\Delta \times D)$. In particular, by (3.8), Ψ_*J extends $C^{0,\delta}$ across $\{0\} \times D$. We will now first explain how the vanishing of the Nijenhuis torsion of J implies that Ψ_*J automatically extends $C^{1,\alpha}$.

Since $\tilde J \equiv F^*J$ satisfies $\tilde J\partial_t = \partial_\theta$, the vanishing of the torsion of J (or $\tilde J$) implies that

$$\frac{\partial \tilde J}{\partial t} + \tilde J \circ \frac{\partial \tilde J}{\partial \theta} = 0. \tag{3.9}$$

Thus, the endomorphism field $K \equiv \tilde{J} - J_\infty$, which is exponentially decaying, satisfies the following quadratic perturbation of the $\mathcal{L}$- or $\bar{\partial}_{J_\infty}$-equation:

$$\mathcal{L}K + K \circ \frac{\partial K}{\partial \theta} = 0. \tag{3.10}$$

Using the right inverse $\mathcal{R}$ to $\mathcal{L}$ of Remark 2.6, we can therefore write $K = \tilde{K} - \mathcal{R}(K \circ \partial_\theta K)$ with $\tilde{K}$ in the kernel of $\mathcal{L}$, which consists of Laurent series in w with constant coefficients on $\mathbb{R}^+ \times \mathbb{S}^1 \times \{x\}$. Since K already decays exponentially and $\mathcal{R}$ preserves the decay rate, an iteration shows that

$$K = \tilde{J} - J_\infty = w\tilde{K}_1 + O(|w|^{1+\alpha}) \tag{3.11}$$

for every $\alpha \in (0,1)$. Here $\tilde{K}_1 = \tilde{K}_1(x)$ denotes a constant section of $\mathrm{End}_{\mathbb{R}}(f_{0,x}^* TM_\infty^+)$ that depends smoothly on x, and the product with w is again understood in the sense that $iA \equiv J_\infty \circ A$ for any endomorphism A. Since $(\Psi_\infty)_* K$ vanishes on the horizontal subbundle $T\Delta$, the same is true for the slicewise constant section $(\Psi_\infty)_* \tilde{K}_1$, which therefore extends smoothly to $\Delta \times D$. It then follows from (3.11) and (3.8) that $\Psi_* J$ extends $C^{1,\alpha}$ to $\Delta \times D$ for every $\alpha < 1$.

The version of the Newlander-Nirenberg theorem of [33, Thm II] now tells us that there exists a complex analytic atlas on $\Delta \times D$ whose coordinate functions are $\Psi_* J$-holomorphic and $C^{1,\frac{\alpha}{n}}$ with respect to g_Δ. (Thus, in our main application—Theorem C—we would now already know that M is holomorphically compactifiable by adding a divisor.) However, we are claiming more: $\Psi_* J$ in fact extends smoothly *as a tensor field*, not just modulo $C^{1,\frac{\alpha}{n}}$ local diffeomorphisms.

To prove this, note that [33, Thm II] in particular tells us that there exist sufficiently many local $\Psi_* J$-holomorphic functions so that $\Psi_* J$ can be recovered from their differentials as a tensor field. It therefore suffices to check that $\Psi_* J$-holomorphic functions are smooth: Let z be $\Psi_* J$-holomorphic on a neighborhood of a point in $\{0\} \times D$. Since $\Psi_* J$ coincides with J_Δ on $T\Delta$, we immediately find that z is J_Δ-holomorphic on each horizontal slice. In other words, we have

$$z = z_0 + wz_1 + w^2 z_2 + \cdots, \tag{3.12}$$

and the Cauchy integral formula expresses the coefficients $z_i = z_i(x)$ in terms of $z(w,x)$ with $w \neq 0$. But we already know that z is smooth for $w \neq 0$ because $\Psi_* J$ is.

Remark 3.13. It is conceivable that a similar (but more difficult) argument could work for the endomorphism K itself, by refining the partial expansion (3.11) to a complete one based on (3.10). We will not need that much regularity for $\Psi_* J$, but this is certainly a natural question.

Step 3: Normal bundle to the compactifying divisor. We identify J and $\Psi_* J$ for convenience. It is clear that $\{0\} \times D$ is a J-complex submanifold of $\Delta \times D$, biholomorphic to D. It remains only to prove that the normal bundle N_D is holomorphically trivial with respect to J. Since every slice $\Delta \times \{x\}$ is a J-complex submanifold by construction, the complex tangent vector field $\frac{\partial}{\partial w}$ is of type $(1,0)$ with respect to J. We show that the section of N_D that it induces is J-holomorphic.

For every $x \in D$ there is a J-holomorphic function z on a neighbourhood U of $(0,x)$ in $\Delta \times D$ which vanishes to order 1 along D. Let $U' \equiv U \cap (\{0\} \times D)$. Then dz is a trivializing holomorphic section of N_D^* over U', and so $\frac{\partial}{\partial w}$ will be a holomorphic section of N_D if and only if $dz(\frac{\partial}{\partial w}) = \frac{\partial z}{\partial w}$ is a holomorphic function on U'. Now if we expand z as a power series in w as in (3.12),

$$z = wz_1 + w^2 z_2 + \cdots, \tag{3.14}$$

then $\frac{\partial z}{\partial w} = z_1$ on U'. On the other hand, applying the $\bar{\partial}$-operator of J to (3.14) yields

$$0 = \bar{\partial}_J z = w\bar{\partial}_J z_1 + (z_1 + 2wz_2)\bar{\partial}_J w + O(|w|^2).$$

In order to conclude from this that $\bar{\partial}_J z_1 = 0$ along U', we need to know that $\bar{\partial}_J w = o(|w|)$ in terms of g_Δ. But w is J_Δ-holomorphic, so $\bar{\partial}_J w = \frac{i}{2} dw \circ (J - J_\Delta)$. Now the only components of $J - J_\Delta$ not annihilated by dw are the $T^*D \otimes T\Delta$ ones, whose g_Δ-length is $|w|$ times their g_∞-length, and the g_∞-length of $J - J_\Delta$ certainly goes to zero; in fact, by (3.11), it is even $O(|w|)$. □

3.3. **Kähler compactification.** We have found two different proofs of Theorem 3.2, both of which will be explained in this section. We assume the conclusion of Theorem 3.1 and will be ignoring the diffeomorphism Ψ throughout. Both proofs begin by writing the ACyl Kähler form on M^+_∞ as

$$\omega = i\partial\bar{\partial}t^2 + \omega_D + O(e^{-\delta t}). \tag{3.15}$$

Here $i\partial\bar{\partial}$ is with respect to J, and ω_D is pulled back from the D factor in $M^+_\infty = \mathbb{R}^+ \times \mathbb{S}^1 \times D$; in particular, ω_D is closed, but not necessarily $(1,1)$ with respect to J. The most intuitive approach to "compactifying" ω may be to replace t^2 by the Kähler potential of a half-cylinder with a spherical cap attached, but there are two (related) problems with this: (1) The $O(e^{-\delta t})$ terms have no reason to extend smoothly to the complex compactification. (2) The capped-off potential will be $O(e^{-2t})$, so the $O(e^{-\delta t})$ errors may dominate and the modified form may not be positive.

Our first proof uses ideas from Section 3.2 to fix (1) and, by consequence, (2). Specifically, recall that the cylinders $\mathbb{R}^+ \times \mathbb{S}^1 \times \{x\}$ are J-holomorphic by the construction of Ψ. Solving $\bar{\partial}$-equations along these cylinders, we will be able to construct $u = O(e^{-\delta t})$ supported far out in M^+_∞ such that the exponential errors of the *corrected* Kähler form $\omega + i\partial\bar{\partial}u$ do extend smoothly. It then follows immediately from this that we can cap off the $i\partial\bar{\partial}t^2$ part without losing positivity.

The second proof will emphasize positivity over smoothness. We back up one step and cap off the infinite end of the cylinder metric on $\mathbb{R}^+ \times \mathbb{S}^1$ by a *cone* of angle $2\pi\varepsilon$ ($\varepsilon \ll \delta$) rather than a disk or hemisphere. This amounts to replacing t^2 in (3.15) by $e^{-2\varepsilon t}$ rather than e^{-2t} at infinity. Then (2) is not a problem to begin with, but (1) now looks worse. However, geometrically, we have created an *edge singular* Kähler metric on the compactified space. We will prove that this "edge metric" has continuous local Kähler potentials. It can therefore be regularized using the method of [42].

First proof of Theorem 3.2. By translating t, we can assume without loss that (3.15) holds on all of $M^+_\infty = \mathbb{R}^+ \times \mathbb{S}^1 \times D$ and that the exponential errors are bounded by $\varepsilon e^{-\delta t}$, where ε is as small as we like. The moral point of the proof is to correct ω by $i\partial\bar{\partial}u$, with u exponentially decaying and small (obtained by solving $\bar{\partial}$-equations on each horizontal slice), in order to arrange that the exponential errors of $\omega + i\partial\bar{\partial}u$ have a power series expansion in w, or are at least smooth at infinity.

Let ψ denote the $O(e^{-\delta t})$ error terms in (3.15). We begin by noting that $\psi = d(\eta + \bar{\eta})$ for some $(0,1)$-form $\eta = O(e^{-\delta t})$. Indeed, we can write $\psi = dt \wedge \psi_1 + \psi_2$, where $\psi_i = O(e^{-\delta t})$ is a 1-parameter family of i-forms on X; the closedness of ψ implies that $\xi(t,x) \equiv -\int_t^\infty \psi_1(s,x)\,ds$ is a primitive for ψ and we let η be the $(0,1)$-part of ξ. Next, we solve $\bar{\partial}f_x = \eta|_{C_x}$ along $C_x = \mathbb{R}^+ \times \mathbb{S}^1 \times \{x\} \subset M^+_\infty$ for each $x \in D$ in such a way that the f_x depend smoothly on x with $|f_x| \leq C\varepsilon e^{-\delta t}$. In particular, we obtain a smooth complex-valued function f on M^+_∞, and we now put $u \equiv -2\,\mathrm{Im} f$.

It is immediate that

$$\omega + i\partial\bar{\partial}u = i\partial\bar{\partial}t^2 + \omega_D + d(\kappa + \bar{\kappa}) > 0, \;\; \kappa \equiv \eta - \bar{\partial}f = O(e^{-\delta t}), \tag{3.16}$$

and the restriction of κ to each of the usual J-holomorphic cylinders C_x vanishes by construction. Thus, for all (t,θ,x), we can view $\kappa|_{(t,\theta,x)}$ as an element of $V_x \equiv T^*_xD \otimes \mathbb{C}$, which we in turn view as a real vector space (with an obvious complex structure, but this will not be relevant). Now V_x has a natural family of complex structures $\mathcal{J}_x(t,\theta)$ defined by the pullback action of $-J$, which leaves $T^*D \subset T^*M^+_\infty$ invariant because the action of J on vectors preserves $T\Delta \subset TM^+_\infty$. Given any fixed x, we then view κ as a function on $\mathbb{R}^+ \times \mathbb{S}^1$ taking values in V_x, and we claim that

$$\frac{\partial\kappa}{\partial t} + \mathcal{J}_x\frac{\partial\kappa}{\partial\theta} = 0. \tag{3.17}$$

To see this, first note that $\partial_t\kappa + \mathcal{J}_x\partial_\theta\kappa = (\partial_t + i\partial_\theta) \llcorner \bar{\partial}\kappa$, where $\bar{\partial}\kappa$ means the $\bar{\partial}$-derivative of κ as a $(0,1)$-form on M^+_∞; this is proved using that $\bar{\partial}\kappa = \frac{1}{2}(d\kappa - J^*d\kappa)$, that κ is vertical, and that $T\Delta$ is J-invariant. On the other hand, $\bar{\partial}\kappa$ is equal to the $(0,2)$-part of $-\omega_D$ by (3.16), and

$$\omega_D^{0,2}(X,Y) = \frac{1}{4}(\omega_D(X,Y) - \omega_D(JX,JY) + i(\omega_D(JX,Y) + \omega_D(X,JY))),$$

so if X is horizontal then this vanishes for every Y since JX is horizontal as well.

We now exploit the $\bar{\partial}$-type equation (3.17), together with the smoothness at infinity of $\mathcal{J}_x$ from Section 3.2, to deduce that κ is itself smooth at infinity. For this we pass to the disk picture, writing $w = u + iv \in \Delta$ with $u = e^{-t}\cos\theta$ and $v = -e^{-t}\sin\theta$. Then (3.17) yields $\partial_u\kappa + \mathcal{J}_x\partial_v\kappa = 0$ on Δ^*, where the function $\kappa : \Delta \to V_x$ is $C^{0,\delta}$ Hölder continuous, smooth away from the origin, and zero at the origin itself, and the function $\mathcal{J}_x : \Delta \to \mathrm{End}_{\mathbb{R}}(V_x)$ is smooth with $\mathcal{J}_x^2 = -\mathrm{id}_{V_x}$. Smoothness of κ at $w = 0$ now follows from elementary elliptic regularity; for example, by applying $\partial_u - \mathcal{J}_x\partial_v$ we can deduce that $\Delta\kappa + \mathcal{K}_x(\partial_v\kappa) = 0$, where $\mathcal{K}_x \equiv \partial_u\mathcal{J}_x - \mathcal{J}_x\partial_v\mathcal{J}_x$ is smooth (but *not* always zero, despite (3.9); *cf.* Remark 3.20), and using $\kappa = O(|w|^\delta)$ and $d\kappa = O(|w|^{\delta-1})$ one checks that $\kappa \in W^{1,2}$ solves this equation weakly at $w = 0$. Smooth dependence of $\kappa = \kappa_x(u, v)$ on x is then standard.

To conclude the proof, we will now verify that the closed $(1,1)$-form

$$\omega_D + d(\kappa + \bar{\kappa}) + i\partial\bar{\partial}((1-\chi)t^2 + \chi\phi) \tag{3.18}$$

on M_∞^+ is positive and extends to a smooth Kähler form on $\Delta \times D$, where $\chi(t)$ is a cutoff function with $\chi \equiv 0$ on $\{t < 1\}$ and $\chi \equiv 1$ on $\{t > 2\}$, and $\phi(t)$ is a convex function with

$$\phi(t) = \begin{cases} t^2 + C_1 t + C_2 & \text{for } t \in (0,3), \\ C_3 e^{-2t} & \text{for } t \in (5,\infty), \end{cases}$$

the absolute constants C_1, C_2, C_3 being chosen so that the two branches of the definition match up at $t = 4$ including first and second derivatives. This is understood in the sense that we have already shifted t so that $|J - J_\infty| + |\kappa| \le \varepsilon e^{-\delta t}$ on the whole of M_∞^+, with ε as small as necessary.

Since we already know that J, κ extend smoothly, and since $e^{-2t} = |w|^2$ is smooth on $\Delta \times D$, it is clear that the form in (3.18) extends smoothly. Positivity for $t \in (0,3)$ is also clear, given that we can assume that $|i\partial\bar{\partial}t| \le \varepsilon$. For $t \in (3,\infty)$, we would be stuck if all we knew was that $\kappa = O(e^{-\delta t})$ for some $\delta > 0$ (even $\delta = 1$) because such terms can easily swamp $i\partial\bar{\partial}\phi$. But $d(\kappa + \bar{\kappa}) + \omega_D^{2,0} + \omega_D^{0,2}$ extends smoothly and vanishes along D, while $i\partial\bar{\partial}\phi + \omega_D^{1,1}$ is smooth and positive near D. ☐

Remark 3.19. Unlike κ of (3.16), the $(0,1)$-form η describing the exponential errors in (3.15) has no reason to be smooth at infinity even though $(\partial_t + i\partial_\theta) \lrcorner \bar{\partial}\eta = 0$. Of course we expect that κ really is more regular than η, but there is a subtle point here: formally, (3.17), which gives regularity for κ, is derived from $(\partial_t + i\partial_\theta) \lrcorner \bar{\partial}\kappa = 0$, which also holds for η, using *only* that κ is vertical.

Remark 3.20. We also mention an alternative approach to regularity for κ. In the disk picture, pick a $\mathbb{C}$-basis $\{\kappa_i\}$ for $(V_x, \mathcal{J}_x(0))$, so that $\{\kappa_i\}$ still is a $\mathbb{C}$-basis for $(V_x, \mathcal{J}_x(w))$ if $|w|$ is small. Each κ_i trivially solves (3.17), and using (3.9) one can compute that $\mathcal{J}_x\kappa_i$ solves (3.17) too. We now expand $\kappa = \sum f_i\kappa_i$ with $f_i : \Delta^* \to \mathbb{C}$, again in the sense that $i \in \mathbb{C}$ acts on V_x by $\mathcal{J}_x$. Then κ solves (3.17) if and only if all the f_i are holomorphic, so we can apply the removable singularities theorem.

We can interpret this argument as follows. By (3.9), the $(0,1)$-part of the trivial connection ∇ on the complex vector bundle $(V_x, \mathcal{J}_x)$ is a $(0,1)$-connection, *i.e.* $\nabla^{0,1}(f\kappa) = \bar{\partial}f \otimes \kappa + f\nabla^{0,1}\kappa$. We could have worked in any local frame $\{\kappa_i\}$ with $\nabla^{0,1}\kappa_i = 0$. Such frames exist for every $(0,1)$-connection over the disk (*i.e.* the $(0,1)$-connection is integrable, defining a holomorphic structure).

Second proof of Theorem 3.2. We again assume that all $O(e^{-\delta t})$ error terms are uniformly as small as necessary on the whole cylinder M_∞^+, and we write our ACyl Kähler form as $\omega = i\partial\bar{\partial}t^2 + \omega_D + \psi$ with $\psi = O(e^{-\delta t})$. We then construct the following closed $(1,1)$ modification $\tilde{\omega}$ of ω:

$$\tilde{\omega} = i\partial\bar{\partial}((1-\chi)t^2 + \chi\phi) + \omega_D + \psi, \tag{3.21}$$

where $\chi(t)$ is a cutoff with $\chi \equiv 0$ on $\{t < 1\}$ and $\chi \equiv 1$ on $\{t > 2\}$, and $\phi(t)$ is convex with

$$\phi(t) = \begin{cases} t^2 + C_1 t + C_2 & \text{for } t \in (0,3), \\ C_3 e^{-2\varepsilon t} & \text{for } t \in (5,\infty). \end{cases}$$

Here $\varepsilon > 0$ is fixed but *strictly smaller* than $\frac{\delta}{2}$, and C_1, C_2, C_3 are determined by ε so that the two branches match up at $t = 4$ including first and second derivatives. This construction is similar to

(3.18), except that now the reason why (3.21) defines a *positive* form on M^+_∞ is that the good term $i\partial\bar\partial\phi + \omega_D^{1,1} > 0$ swallows the error $\psi + \omega_D^{2,0} + \omega_D^{0,2}$ by Cauchy-Schwarz because ε is small.

Now $\tilde\omega$ does not extend smoothly, but the Riemannian metric associated with $\tilde\omega$ only has a fairly mild (conical with cone angle $2\pi\varepsilon$) singularity along the compactifying divisor $\{0\} \times D$. We pursue this idea by proving that $\tilde\omega$ has local potentials that remain continuous at the divisor. For this, we first cover a neighborhood of $\{0\} \times D$ by holomorphic coordinate charts isomorphic to $\Delta \times \mathbb{B}$, where $\mathbb{B}$ denotes the unit ball in $\mathbb{C}^{n-1}$, such that $(\{0\} \times D) \cap (\Delta \times \mathbb{B}) = \{0\} \times \mathbb{B}$. It is then easy to see that Proposition 3.22 applies to $\eta \equiv \tilde\omega - p^*\omega_D$, where p denotes projection onto $z_2, \ldots, z_n$. This produces a smooth potential ϕ for $\tilde\omega$ on $\Delta^* \times \mathbb{B}$ such that ϕ extends as a $C^{0,2\varepsilon}$ function to the full domain $\Delta \times \mathbb{B}$ with $d\phi = O(|z_1|^{2\varepsilon-1})$.

We now apply the (elementary but clever) Varouchas method [42] for smoothing singular Kähler forms with continuous local potentials; the presentation in Perutz [37] is particularly convenient. In order to do so, we first need to check that ϕ is strictly plurisubharmonic in the sense of currents on the whole of $\Delta \times \mathbb{B}$. By definition, we must prove that $\phi' \equiv \phi - \lambda|z|^2$ is *weakly* plurisubharmonic in the sense of currents for some $\lambda > 0$. Now if λ is small enough, then surely $\tilde\omega' \equiv \tilde\omega - i\partial\bar\partial(\lambda|z|^2) \geq 0$ on $\Delta^* \times \mathbb{B}$. We then pick any test form $\zeta \in C^\infty_0(\wedge^{n-1,n-1}(\Delta \times \mathbb{B}))$ with $\zeta \geq 0$ and compute

$$\int_{|z_1|>\delta} \phi' dd^c\zeta = \int_{|z_1|>\delta} \tilde\omega' \wedge \zeta + \int_{|z_1|=\delta} (\phi' d^c\zeta - d^c\phi' \wedge \zeta);$$

the first term is nonnegative, and the second term goes to zero as $\delta \to 0$ because $d\phi' = O(|z_1|^{2\varepsilon-1})$. We are now in a position to apply [37, Lemma 7.5] to the Kähler cocycle $(U_i, \phi_i)_{i\in I}$ thus obtained, where $X = \Delta \times D$, $X_1 = \Delta^* \times D$, and X_2 is the union of all our $\Delta \times \mathbb{B}$ coordinate charts. ☐

It remains to prove the $i\partial\bar\partial$-lemma with estimates that was crucially used in the above. The result is perhaps most conveniently stated by identifying $\Delta^* \times \mathbb{B}$ with the cylinder $\mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{B}$ and using weighted Hölder spaces $C^{k,\alpha}_\varepsilon$ on this cylinder. Also, we will again write $z_1, \ldots, z_n$ for the standard holomorphic coordinates on $\Delta \times \mathbb{B}$, and we will use indices with respect to those.

Proposition 3.22. *Fix $\varepsilon > 0$ small enough. Let $\eta \in C^\infty_\varepsilon$ be a closed real $(1,1)$-form on $\Delta^* \times \mathbb{B}$. Then $\eta = i\partial\bar\partial\xi$ for some real-valued function $\xi \in C^\infty_\varepsilon$. In particular, $\xi = O(|z_1|^\varepsilon)$ extends as a $C^{0,\varepsilon}$ Hölder function to the full domain $\Delta \times \mathbb{B}$ and $d\xi = O(|z_1|^{\varepsilon-1})$.*

Proof. The proof consists of a reduction to known analytic results on the two factors. We make no pretense of optimality in the analysis. Let us begin by stating the results that we need.

(i) The operators $\partial, \partial\bar\partial$ acting on weighted Hölder spaces $C^{k,\alpha}_\varepsilon$ on $\Delta^* = \mathbb{R}^+ \times \mathbb{S}^1$ admit bounded right inverses $\mathcal{R}^h_z, \mathcal{R}^h_{z\bar z}$ (the h means "horizontal" and the subscripts are mnemonics) that are compatible with the obvious inclusions of Hölder spaces. See Remark 2.6 for this.

(ii) The operators $\bar\partial, \partial\bar\partial$ acting on smooth functions on $\mathbb{B}$ have right inverses $\mathcal{R}^v_{\bar z}, \mathcal{R}^v_{z\bar z}$ defined on the spaces of smooth $\bar\partial$-closed $(0,1)$-forms and smooth d-closed $(1,1)$-forms, respectively, that extend to bounded operators $C^k \to C^k$. For $\bar\partial$ this is proved in [38]. For $\partial\bar\partial$ let $\mathcal{P}$ denote the usual Poincaré operator on star-shaped domains [19, §11.5], so that $d\mathcal{P}\eta = \eta$ for all closed forms η. Then $\mathcal{R}^v_{z\bar z}\eta \equiv 2i\mathrm{Im}\,\mathcal{R}^v_{\bar z}((\mathcal{P}\eta)^{0,1})$ works because $\mathcal{P}$ is clearly bounded $C^k \to C^k$.

(iii) Since these right inverses $\mathcal{R}$ are all linear and bounded with respect to C^k type norms, they commute with partial differentiation of C^∞ forms with respect to C^∞ parameters.

We now define $\xi \equiv \mathrm{Re}(\xi^{(1)} + \xi^{(2)} + \xi^{(3)})$, where the $\xi^{(i)}$ are constructed as follows. First,

$$\xi^{(1)} \equiv \mathcal{R}^h_{z\bar z}(\eta_{1\bar 1})$$

on each horizontal slice. Next, we construct a vertical $(0,1)$-form ζ by setting

$$\zeta_{\bar k} \equiv \mathcal{R}^h_z(\eta_{1\bar k} - \xi^{(1)}_{,1\bar k}) \ \ (k > 1).$$

Then (iii) above and the closedness of η imply that ζ is $\bar\partial$-closed on each vertical fibre; hence we can set $\xi^{(2)} \equiv \mathcal{R}^v_{\bar z}(\zeta)$ fibrewise. Again using (iii) and the closedness of η, one checks that

$$\xi^{(2)}_{,1\bar 1} = 0, \ \ \xi^{(2)}_{,1\bar k} = \eta_{1\bar k} - \xi^{(1)}_{,1\bar k} \ \ (k > 1).$$

With $\xi^{(3)} \equiv \mathcal{R}^v_{z\bar{z}}(\eta_{j\bar{k}} - \xi^{(1)}_{,j\bar{k}} - \xi^{(2)}_{,j\bar{k}})$, where again $j, k > 1$, a similar computation shows that $\xi^{(3)}_{,1} = 0$. The proposition now follows easily from the stated identities. □

4. Existence and uniqueness

4.1. **Discussion and overview.** The main aim of this section is to prove Theorem D. This refines an existence result for complete Ricci-flat Kähler metrics of linear volume growth due to Tian-Yau [40, Corollary 5.1]. At the end of the section we also quickly explain the proof of Theorem E. As we already discussed in Section 1, Theorem E shows that Theorem D exhausts all possible examples of ACyl Calabi-Yau manifolds with $n = \dim_{\mathbb{C}} M > 2$ and split cross-section ($m = 1$).

Theorem D can be proved (although this proof is not written down anywhere) by combining the *proof* of the Tian-Yau theorem [40] with a new idea concerning asymptotics of solutions to complex Monge-Ampère equations from [17]. However, the ingredients from [40] needed in this approach are in fact much more general and, correspondingly, technically quite formidable. The proof in the ACyl case that we give here will be significantly easier. We achieve this by using weighted function spaces throughout, and by retooling the decay argument from [17, Prop 2.9(i)] as an a priori estimate.

Joyce already employed weighted spaces to treat certain examples of *maximal* volume growth—ALE and QALE Kähler manifolds, see [20, §8.5, §9.6]—but his weighted nonlinear estimates break down in our *minimal* volume growth situation. This issue is related to an error in the construction of ACyl Calabi-Yau manifolds with exponential asymptotics in [23], where the analysis is based [23, p. 132] on an estimate for the maximal volume growth case [41, p. 52]—this is incorrect because the estimate from [41] crucially relies on a Euclidean type Sobolev inequality that definitely *fails* for any volume growth rate less than the maximal one. See Proposition 4.6 below for comparison.

The following is our main analytic existence theorem.

Theorem 4.1 (ACyl version of the Calabi conjecture)**.** *Let (M, g, J) be an* ACyl *Kähler manifold of complex dimension n with Kähler form ω. If $0 < \varepsilon \ll 1$ and if $f \in C^\infty_\varepsilon(M)$ satisfies*

$$\int_M (e^f - 1)\omega^n = 0, \tag{4.2}$$

then there exists a unique $u \in C^\infty_\varepsilon(M)$ such that $\omega + i\partial\bar{\partial}u > 0$ and $(\omega + i\partial\bar{\partial}u)^n = e^f\omega^n$.

Remark 4.3. Integration by parts shows that (4.2) is indeed necessary in order for u to exist. This is a nonlinear version of the mean-value-zero assumption of Proposition 2.7. As in the linear case, if $f \in C^\infty_\varepsilon(M)$ but (4.2) is *not* satisfied, then there may still exist solutions that grow at infinity since the Green's function on M is asymptotically pluriharmonic (in fact, asymptotically linear).

We will prove Theorem 4.1 in Section 4.3, after having deduced Theorem D from it in Section 4.2. The proof of Theorem E is essentially independent of this and will be given in Section 4.4. It may be worth advertising that our proof of Theorem 4.1 will be *self-contained* with only two exceptions: (1) We use Proposition 2.7 without proof, but no other facts from linear analysis on ACyl manifolds. (2) We assume that the reader is familiar with Yau's proof of the Calabi conjecture on compact Kähler manifolds [44]; see Błocki [5] for a detailed and readable exposition.

4.2. **The analytic existence theorem implies the geometric one.** In order to prove Theorem D we need to construct an ACyl Kähler metric $\tilde{\omega}$ on $M = \overline{M} \setminus \overline{D}$ such that Theorem 4.1 applies to the pair $(M, \tilde{\omega})$ and the function f defined by

$$e^f\tilde{\omega}^n = i^{n^2}\Omega \wedge \bar{\Omega}. \tag{4.4}$$

Notice that f is always C^∞ because Ω has neither poles nor zeros in M. If u is the solution provided by Theorem 4.1, the desired Calabi-Yau metric ω is then given by $\omega = \tilde{\omega} + i\partial\bar{\partial}u$.

To construct $\tilde{\omega}$, we begin with an arbitrary Kähler form ω_0 in the chosen Kähler class $\mathfrak{k}$ on $\overline{M}$. The first step is to find a Kähler form $\tilde{\omega}_0$ on $\overline{M}$ that is cohomologous to ω_0 when restricted to M and Ricci-flat when restricted to $\overline{D}$. For this, we first of all observe that $K_{\overline{D}}$ is trivial by adjunction.

Thus, by the Calabi-Yau theorem, there exists $\phi \in C^\infty(\overline{D})$ such that $\omega_0|_{\overline{D}} + i\partial\bar\partial\phi > 0$ is Ricci-flat. Fix a C^∞ trivialization of the anticanonical fibration $\overline{M} \to \mathbb{P}^1$ near $\overline{D}$, thus smoothly identifying a tubular neighborhood of $\overline{D}$ with $\Delta \times \overline{D}$, where Δ again denotes the unit disk $\{|z| < 1\}$. Extend ϕ to be constant along the Δ factor and multiply this extension by a cutoff function pulled back from Δ to further extend ϕ to the whole of $\overline{M}$. If the tubular neighborhood was sufficiently small, then the restriction of $\omega_0 + i\partial\bar\partial\phi$ to any fibre will be positive definite. Any negative components of $\omega_0 + i\partial\bar\partial\phi$ on the total space $\overline{M}$ can be compensated by adding the pullback of a sufficiently positive "bump 2-form" on Δ supported in an annulus containing the cutoff region; such a pullback is automatically closed $(1,1)$ on $\overline{M}$ and exact on M. This creates the desired Kähler form $\tilde\omega_0$.

We now modify $\tilde\omega_0$ to become asymptotically cylindrical with the correct volume form at infinity. Notation: Define $\Delta(r) = \{|z| < r\}$, fix parameters $s \ll r \ll 1$ to be chosen later, and pick a cutoff function $\chi : \Delta \to \mathbb{R}$ with $\chi = 1$ on $\Delta(r-s)$, $\chi = 0$ away from $\Delta(r+s)$, and $s|\chi_z| + s^2|\chi_{z\bar z}| \leq C$. Fix a bump 2-form $\beta \geq 0$ on Δ with support contained in $\Delta(r+2s) \setminus \Delta(r-2s)$ such that $\beta = \frac{i}{2} dz \wedge d\bar z$ on $\Delta(r+s) \setminus \Delta(r-s)$, and identify β with its pullback to $\overline{M}$ under the anticanonical map.

The Kähler potentials of the cylindrical metric $\frac{i}{2}|z|^{-2} dz \wedge d\bar z$ are given by $u(z) = (\log|z|)^2 + h(z)$ with h a harmonic function. We use these potentials to define closed $(1,1)$-forms on M:

$$\tilde\omega_t := \tilde\omega_0 + \lambda i\partial\bar\partial(\chi u) + t\beta.$$

Being compactly supported, the $t\beta$ term does not change the asymptotics of the metric at infinity, but the extra degree of freedom $t > 0$ is needed to deal with the integral condition (4.2). Also, $\lambda > 0$ is a fixed real number determined by the condition that

$$(\tilde\omega_0|_{\overline{D}})^{n-1} = \frac{2}{n\lambda} i^{(n-1)^2} R \wedge \bar R,$$

where $R = \mathrm{Res}_{\overline{D}}\Omega$ is the holomorphic volume form on $\overline{D}$ specified by $\Omega = \frac{dz}{z} \wedge R + O(1)$ as $z \to 0$. The forms $\tilde\omega_t$ are then positive definite except possibly over $\Delta(r+s) \setminus \Delta(r-s)$. Moreover, if $\tilde\omega_t$ is in fact positive definite globally, then the associated Riemannian metric on M is ACyl and the volume form $\tilde\omega_t^n$ is exponentially asymptotic to $i^{n^2}\Omega \wedge \bar\Omega$ at infinity.

We complete the construction by choosing $h(z) = (\log r)^2 - (2\log r)\log|z|$. This implies that

$$|u| + s|u_z| \leq C\frac{|\log r|}{r^2} s^2 \tag{4.5}$$

in the gluing region $\Delta(r+s) \setminus \Delta(r-s)$, by making a Taylor expansion centered at $|z| = r$.

Claim. Given any fixed choice of $r \ll 1$ and $s \ll r$, there exists a unique value of $t > 0$ such that we have $\tilde\omega_t > 0$ globally and $\int_M (\tilde\omega_t^n - i^{n^2}\Omega \wedge \bar\Omega) = 0$.

Thus for any choice of $s \ll r \ll 1$ we obtain an ACyl Kähler metric $\tilde\omega = \tilde\omega_t$ such that the function $f \in C^\infty_\varepsilon(M)$ associated with $\tilde\omega$ by (4.4) satisfies (4.2) with respect to $(M, \tilde\omega)$. Then Theorem 4.1 can be applied. (The resulting Calabi-Yau metric ω is independent of r, s, by Theorem E.)

Proof of the claim. Using (4.5), positivity quickly reduces to $t \gg \frac{1}{r^2}|\log r|$. The integral condition is equivalent to the following linear equation for t:

$$\int_M (\tilde\omega_0^n + n\lambda i\partial\bar\partial(\chi u) \wedge \tilde\omega_0^{n-1} - i^{n^2}\Omega \wedge \bar\Omega) + nt\int_M \beta \wedge \tilde\omega_0^{n-1} = 0.$$

The t-coefficient is positive and $\sim rs$. The constant term can be split as a sum of three contributions: $O(r)$ from $\Delta(r-s)$ since due to our choice of λ the integrand is $O(|z|^{-1}\tilde\omega_0^n)$ there; $O(|\log r|\frac{s}{r})$ from the gluing region, using (4.5) again; and a negative part $\sim \log r$ from the rest of M. We see that the solution $t \sim \frac{1}{rs}|\log r|$ if $s \ll r \ll 1$, which is well within the positivity constraint.

4.3. **Proof of the analytic existence theorem.** The proof of Theorem 4.1 requires a nontrivial technical preliminary: the proof of a global Sobolev inequality on M. Such inequalities are sensitive to the volume growth at infinity, and need to take rather different shapes depending on whether the growth rate is slower or faster than quadratic. Our proof follows the strategy expounded in [14]; see also [16, 29] for closely related results and applications.

Proposition 4.6. *Let (M^n, g) be an* ACyl *manifold as in Definition 1.1. Then for all $\mu > 0$ there exists a piecewise constant positive function $\psi_\mu = O(e^{-2\mu t})$ with $\int_M \psi_\mu \,\mathrm{dvol} = 1$ such that*

$$\|e^{-\mu t}(u - \bar{u}_\mu)\|_{2\sigma} \leq C_{M,\mu,\sigma} \|\nabla u\|_2 \tag{4.7}$$

holds for all $\sigma \in [1, \frac{n}{n-2}]$ and all $u \in C_0^\infty(M)$, where $\bar{u}_\mu \equiv \int_M u\psi_\mu \,\mathrm{dvol}$.

The subtraction of an average on the left-hand side of (4.7) is inevitable because M has less than quadratic volume growth. In [40], the relation (4.2) is directly applied to compensate this.

Proof of Proposition 4.6. We have $M = \bigcup \mathrm{clos}(A_i)$, where $A_0 = U$ and $A_i = (i-1, i) \times X$ for $i \in \mathbb{N}$, and we begin by discretizing the left-hand side of (4.7) accordingly:

$$\|e^{-\mu t}(u - \bar{u}_\mu)\|_{2\sigma}^2 \leq C \sum \|\chi_i(u - \bar{u}_i)\|_{2\sigma}^2 + C \sum e^{-2\mu i} |\bar{u}_i - \bar{u}_\mu|^2, \tag{4.8}$$

where χ_i is the characteristic function of A_i and $\bar{u}_i$ is the average of u over A_i. Since the A_i have uniformly bounded geometry, $\|\chi_i(u - \bar{u}_i)\|_{2\sigma} \leq C\|\chi_i \nabla u\|_2$ by the usual Sobolev inequality. Thus, it suffices to estimate the second sum in (4.8). This involves defining the weight function ψ_μ. In order for our argument to go through, we require that $\sum e^{-2\mu i}(\bar{u}_i - \bar{u}_\mu) = 0$ for all test functions u, and so we define $\psi_\mu \equiv \phi_\mu / \int_M \phi_\mu \,\mathrm{dvol}$, where ϕ_μ is constant equal to $e^{-2\mu i}/|A_i|$ on A_i. Then

$$\sum e^{-2\mu i} |\bar{u}_i - \bar{u}_\mu|^2 \leq C \sum_{i<j} e^{-2\mu(i+j)} |\bar{u}_i - \bar{u}_j|^2 \leq C \sum_{i<j} e^{-2\mu(i+j)} |i - j| \sum_{k=i}^{j-1} |\bar{u}_k - \bar{u}_{k+1}|^2.$$

Next, we define $B_k \equiv \mathrm{int}(\mathrm{clos}(A_k \cup A_{k+1}))$ and observe that

$$|\bar{u}_k - \bar{u}_{k+1}|^2 \leq \frac{1}{|A_k||A_{k+1}|} \int_{A_k \times A_{k+1}} |u(x) - u(y)|^2 \,dx\,dy \leq \frac{2|B_k|}{|A_k||A_{k+1}|} \int_{B_k} |u - \bar{u}_{B_k}|^2,$$

where $\bar{u}_{B_k}$ denotes the average of u over B_k. Since B_k is connected, we can now apply the standard Poincaré inequality on B_k, which completes the proof. □

Proof of Theorem 4.1. The uniqueness claim is proved independently in Section 4.4 and really only requires that $u \in C_\varepsilon^2(M)$. Thus, it suffices to prove the existence of a solution $u \in C_\varepsilon^{k+2,\alpha}(M)$ for any given $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$. For this we take $\varepsilon \in (0, \delta]$ to be smaller than the square root of the first eigenvalue of the Laplacian on the cross-section X, and set up a continuity method. Let

$$\mathcal{X} = \{u \in C_\varepsilon^{k+2,\alpha}(M) : \omega_u = \omega + i\partial\bar{\partial}u > 0\}, \;\; \mathcal{Y} = \{f \in C_\varepsilon^{k,\alpha}(M) : \textstyle\int_M (e^f - 1)\omega^n = 0\}.$$

Then $\mathcal{X}$ is an open set, $\mathcal{Y}$ is a hypersurface, and the complex Monge-Ampère operator $\mathcal{F}$ given by $(\omega + i\partial\bar{\partial}u)^n = e^{\mathcal{F}(u)}\omega^n$ induces a map $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$. For $u \in \mathcal{X}$, the metric g_u associated with ω_u is again asymptotically cylindrical (though only of regularity $C_\varepsilon^{k,\alpha}$) with respect to Φ and X.

Given f as in the statement of the theorem, we wish to solve the family of equations $\mathcal{F}(u_\tau) = f_\tau$ for $u_\tau \in \mathcal{X}$, with $f_\tau \equiv \log\big(1 + \tau(e^f - 1)\big) \in \mathcal{Y}$ for $\tau \in [0, 1]$. We have a trivial solution $u_0 = 0$. Next, we need to show that the set of all τ for which a solution $u_\tau \in \mathcal{X}$ exists is open. For $u \in \mathcal{X}$,

$$T_u\mathcal{F} = \frac{1}{2}\Delta_{g_u} : T_u\mathcal{X} = C_\varepsilon^{k+2,\alpha}(M) \to T_{\mathcal{F}(u)}\mathcal{Y} = C_\varepsilon^{k,\alpha}(M)_{0,g_u},$$

the subscripts $0, g_u$ indicating mean value zero with respect to g_u, and we must show that this is an isomorphism if $u = u_\tau$. But if $u = u_\tau$, then $\mathcal{F}(u_\tau) = f_\tau$, which implies $u_\tau \in C_\varepsilon^\infty(M)$ by a standard bootstrapping argument, and so g_u is regular enough to apply Proposition 2.7 as written.

It remains to prove a *quantitative* a priori bound on the $C_\varepsilon^{k+2,\alpha}$-norm of u_τ, using the *qualitative* information that $u_\tau \in C_\varepsilon^\infty(M)$. We proceed in a sequence of four partial a priori estimates. We will write $u = u_\tau$ and $f = f_\tau$, but all constants are understood to be independent of τ.

Step 1: C^0 from Moser iteration. We apply Moser iteration as in [16, §3.1] or [40, Lemma 3.5] to derive an a priori bound on the sup norm of u. First let us recall the basic underlying computation. To this end, fix $T > 0$ and define an auxiliary form $\eta \equiv \sum_{k=0}^{n-1} \omega^k \wedge \omega_u^{n-1-k}$. Then we have

$$\int_{t<T} |\nabla |u|^{\frac{p}{2}}|^2 \omega^n \leq -\frac{np^2}{2(p-1)} \left[\int_{t<T} u|u|^{p-2}(e^f - 1)\omega^n - \frac{1}{2} \int_{t=T} u|u|^{p-2} d^c u \wedge \eta \right] \tag{4.9}$$

for all $p > 1$. See [5, p. 212] for this, although in [5] there are of course no boundary terms. Notice that (4.9) still holds with u replaced by $u - \lambda$ for any constant $\lambda \in \mathbb{R}$, and also that the boundary term goes to zero as $T \to \infty$ (no matter what λ we subtract) because $d^c(u - \lambda) = O(e^{-\varepsilon t})$.

We begin the iteration process by setting $p = 2$ and $\lambda = \bar{u}_\mu$ (as in Proposition 4.6), with μ to be determined as we go along. If $\mu < \varepsilon$, then (4.7) and (4.9) imply that

$$\|e^{-\mu t}(u - \bar{u}_\mu)\|_{2\sigma}^2 \leq C\|\nabla u\|_2^2 \leq C\|e^{-\varepsilon t}(u - \bar{u}_\mu)\|_1 \leq C\|e^{-\mu t}(u - \bar{u}_\mu)\|_{2\sigma}.$$

To continue the iteration, we are now going to prove that

$$\left\| e^{-\mu t}|u - \bar{u}_\mu|^{\sigma^{k+1}} \right\|_{2\sigma}^2 \leq C\sigma^k \max\left\{1, \left\| e^{-\mu t}|u - \bar{u}_\mu|^{\sigma^k} \right\|_{2\sigma}^{2\sigma}\right\} \tag{4.10}$$

for all $k \in \mathbb{N}_0$, provided that $\sigma < 2$ and $2\mu\sigma < \varepsilon$. Given this, a standard argument [5, p. 212] shows that the $L^{2\sigma^k}$-norm of $u - \bar{u}_\mu$ with respect to the measure $e^{-2\mu\sigma t}d\mathrm{vol}$ is bounded uniformly in k, so that $\|u - \bar{u}_\mu\|_\infty \leq C$. Since $u = O(e^{-\varepsilon t})$, we deduce that $|\bar{u}_\mu| \leq C$, hence $\|u\|_\infty \leq C$ as desired.

In order to prove (4.10), we first apply (4.9) with $p = 2\sigma^{k+1}$ and with u replaced by $u - \bar{u}_\mu$, and then (4.7). Abbreviating $u_k \equiv |u - \bar{u}_\mu|^{\sigma^k}$, this yields the following inequalities:

$$\|e^{-\mu t}(u_{k+1} - \overline{u_{k+1}}_{,\mu})\|_{2\sigma}^2 \leq C\|\nabla u_{k+1}\|_2^2 \leq C\sigma^k \|e^{-\varepsilon t}|u - \bar{u}_\mu|^{2\sigma^k - 1}\|_1.$$

Proceeding on the right-hand side, Hölder's inequality tells us that

$$\|e^{-\varepsilon t}|u - \bar{u}_\mu|^{2\sigma^k - 1}\|_1 \leq C\|e^{(2\mu\sigma - \varepsilon)t}\|_{2\sigma^{k+1}} \max\{1, \|e^{-\mu t}u_k\|_{2\sigma}^{2\sigma}\},$$

and if $2\mu\sigma < \varepsilon$ then the prefactor converges to 1 as $k \to \infty$. On the other hand,

$$\|e^{-\mu t}\overline{u_{k+1}}_{,\mu}\|_{2\sigma}^2 = \|e^{-\mu t}\|_{2\sigma}^2 \|\psi_\mu u_{k+1}\|_1^2 \leq C\|e^{(\sigma-2)\mu t}\|_2^2 \|e^{-\mu t}u_k\|_{2\sigma}^{2\sigma},$$

which is finite if $\sigma < 2$, and of the required form. All in all, this proves (4.10).

Step 2: C^0 implies C^∞. We do not need to say very much here. Given that functions in the space $\mathcal{X}$ attain their extrema on M and that M has uniformly bounded geometry at infinity, the classical arguments proving Step 2 in the compact case [5, §5.5, §5.6] go through verbatim.

Step 3: C^∞ implies $C^\infty_{\varepsilon'}$ for some uniform $\varepsilon' \in (0, \varepsilon]$. This is a special case of an energy decay argument from [17, Prop 2.9(i)], which we apply as an priori estimate here. We begin by writing out the counterpart of the $p = 2$ case of (4.9) for the outer domain $\{t > T\}$:

$$\int_{t>T} |\nabla u|^2 \omega^n \leq -2n \left[\int_{t>T} u(e^f - 1)\omega^n + \frac{1}{2} \int_{t=T} u\, d^c u \wedge \eta \right]. \tag{4.11}$$

This is proved by repeating the standard computation on $\{T < t < T'\}$ and sending $T' \to \infty$. Also, (4.11) again holds with u replaced by $u - \lambda$ for any constant $\lambda \in \mathbb{R}$; we take λ to be the average of u over $\{t = T\}$. Defining Q_T to be the quantity on the left-hand side of (4.11), this yields

$$Q_T \leq Ce^{-\varepsilon T} + C\int_{t=T} |u - \lambda||\nabla u| \leq Ce^{-\varepsilon T} + C\int_{t=T} |\nabla u|^2 \leq Ce^{-\varepsilon T} - C\frac{dQ_T}{dT},$$

where we have used our C^2 a priori estimate from Steps 1 and 2, Cauchy-Schwarz, and the Poincaré inequality. It is elementary to deduce from this that $Q_T \leq Ce^{-\varepsilon' T}$ for some uniform $\varepsilon' \in (0, \varepsilon]$.

Now define $A_T \equiv \{T < t < T + 1\}$ and let u_T denote the average of u on A_T. Then our estimate for Q_T and the Poincaré inequality imply that $\|u - u_T\|_{L^2(A_T)} \leq Ce^{-\varepsilon' T}$. On the other hand, simply by rewriting the Monge-Ampère equation, we have

$$\mathcal{L}(u - u_T) = e^f - 1 = O(e^{-\varepsilon T}) \text{ on } A_T,$$

where

$$(\mathcal{L}v)\omega^n = i\partial\bar{\partial}v \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega_u + \cdots + \omega_u^{n-1}) \tag{4.12}$$

as in [23, p. 137]. Since $\mathcal{L}$ is uniformly elliptic with respect to g by Step 2, Moser iteration now tells us that $|u - u_T| \leq Ce^{-\varepsilon' T}$ on a slightly smaller domain; see [15, Thm 4.1] for this type of estimate. Then Schauder theory gives $|\nabla^k u| \leq C_k e^{-\varepsilon' t}$ for all $k > 0$. Thus, eventually, $|u| \leq Ce^{-\varepsilon' t}$ for some uniform constant C, by integrating up the exponentially decaying bound on ∇u.

Step 4: $C^\infty_{\varepsilon'}$ implies C^∞_ε. We are assuming that $u \in C^\infty_\varepsilon(M)$ with ineffective bounds, and Step 3 yields $u \in C^\infty_{\varepsilon'}(M)$ with effective bounds for some uniform $\varepsilon' \in (0, \varepsilon]$. To upgrade from ε' to ε in the effective bounds, we first rewrite the complex Monge-Ampère equation as

$$\frac{1}{2}\Delta_g u = (e^f - 1) - \mathcal{Q}(u), \;\; \mathcal{Q}(u)\omega^n = \binom{n}{2}(i\partial\bar{\partial}u)^2 \wedge \omega^{n-2} + \cdots + (i\partial\bar{\partial}u)^n. \tag{4.13}$$

If $u \in C^\infty_\delta(M)$ for some $\delta \in (0, \varepsilon]$, then the right-hand side of (4.13) is in $C^\infty_{\delta'}(M)_{0,g}$, $\delta' = \min\{2\delta, \varepsilon\}$, and so Proposition 2.7 yields $u \in C^\infty_{\delta'}(M)$, effective estimates understood throughout. We then put $\delta = \varepsilon'$ and iterate a bounded number of times to obtain the desired conclusion. □

Remark 4.14. Let us quickly review how we used that $\int_M (e^f - 1)\omega^n = 0$. Unlike in [40, Lemma 3.4], this played no direct role in the nonlinear estimates. However, we needed to drop boundary terms at infinity in (4.9) and (4.11). This was possible because we were working in a space of functions with exponential decay, which the linear analysis allowed us to do because $\int_M (e^f - 1)\omega^n = 0$.

4.4. **Uniqueness.** Finally, let us explain why the Ricci-flat ACyl metric produced by Theorem D is unique among metrics that are ACyl with respect to the same diffeomorphism Φ. This follows from Hodge theory arguments as in Section 2.1.

Proof of Theorem E. First we deduce an ACyl $i\partial\bar{\partial}$-lemma, showing that the exact decaying $(1,1)$-form $\omega = \omega_2 - \omega_1$ can be written as $i\partial\bar{\partial}u$ for some function u of linear growth.

Since ω is exact and decaying, it can according to [35, Thm 2.3.27] be written as $\omega = d\alpha$, where α is asymptotic to a translation-invariant harmonic 1-form on M_∞. In particular, $\bar{\partial}^*\alpha^{0,1}$ is a decaying function and can therefore be written as $\bar{\partial}^*\bar{\partial}\gamma$ for a function γ of linear growth. The form $\bar{\partial}\gamma - \alpha^{0,1}$ is bounded harmonic, hence closed. Thus, if we set $u = 2\,\mathrm{Im}\,\gamma$, then $i\partial\bar{\partial}u = \partial\alpha^{0,1} + \bar{\partial}\alpha^{1,0} = \omega$.

Now $\omega_1^n = \omega_2^n$ implies that $\mathcal{L}u = 0$, where $\mathcal{L}v = i\partial\bar{\partial}v \wedge \eta$ with

$$\eta = \omega_1^{n-1} + \omega_1^{n-2} \wedge \omega_2 + \cdots + \omega_2^{n-1}$$

as in (4.12). The $(n{-}1, n{-}1)$-form η is positive in the sense that $\eta \wedge \alpha \wedge \bar{\alpha} > 0$ for every nonzero $(1,0)$-form α. It follows that there is a Hermitian metric ω such that $\omega^{n-1} = \eta$. This is not typically Kähler, but the "balanced" condition that $d\omega^{n-1} = 0$ implies that $\mathcal{L}$ is exactly the Laplacian with respect to the Riemannian metric associated with ω. Since any subexponentially growing harmonic function h defines a direction in the cokernel of the Laplacian on exponentially decaying functions (because $\int(\Delta v)h = 0$ if v is decaying), and since this cokernel is 1-dimensional by Proposition 2.7, the only subexponential harmonic functions are constants. Hence u is a constant. □

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