Approximation property and nuclearity on mixed-norm L^p, modulation and Wiener amalgam spaces

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ABSTRACT

In this paper, we first prove the metric approximation property for weighted mixed-norm $L_w^{(p_1,\ldots,p_n)}$ spaces. Using Gabor frame representation, this implies that the same property holds in weighted modulation and Wiener amalgam spaces. As a consequence, Grothendieck's theory becomes applicable, and we give criteria for nuclearity and r-nuclearity for operators acting on these spaces as well as derive the corresponding trace formulae. Finally, we apply the notion of nuclearity to functions of the harmonic oscillator on modulation spaces.

1. Introduction

Approximation properties of Banach spaces constitute the fundamental properties of the geometry of Banach spaces; see, for example, Figiel *et al.* [15] for a recent review of the subject, as well as Pietsch's book [31, Paragraph 5.7.4] for a survey of different approximation properties and relations among them, as well as for the historical perspective.

Indeed, one of the important aspects of this particular property is that once a Banach space is known to have it, the Grothendieck theory of nuclear operators becomes applicable, leading to numerous further developments. Overall, the topic finds itself closely related to a wide range of analysis: spectral analysis, operator theory, functional analysis, harmonic analysis and partial differential equations. On one hand, the question of a space having an approximation property is important for general spaces of functional analysis; see, for example, [38] or [39]. On the other hand, it is important to know this also for a range of particular spaces. For example, Alberti *et al.* [1] established the bounded approximation property (BAP) for functions of bounded variations, and Roginskaya and Wojciechowski [34] for Sobolev spaces $W^{1,1}$.

In this paper, this property is established for three scales of spaces that are of importance in different applications. First, the mixed Lebesgue spaces provide for a basic tool for harmonic analysis and evolution partial differential equations (for example, through Strichartz estimates). The approximation property of such spaces may give rise to an introduction of further spectral methods (following Grothendieck) to questions of harmonic analysis and partial differential equations. Thus, a part of the paper is also devoted to the development of some of these ideas. Second, Wiener amalgam spaces are a central object of the time–frequency analysis, another area with links to several mathematical subjects as well as

its applications. Finally, the approximation property in the scale of modulation spaces gives rise to the introduction of further spectral analysis to partial differential equations of very different type, these spaces become more and more effective (in addition to Besov spaces) in many types of equations, including such equations as the Navier–Stokes equation; see, for example, [**23**].

There are other links between this subject and other mathematical areas through the study of Fredholm determinants (and this is also one of the implications of the paper). More specifically, determinants of operators of the form $I + A$ are an important tool in the study of certain differential equations, a known fact that goes back in a rigorous shape to Poincaré in his work on Hill's equation [**32**], where the Banach space ℓ^1 is relevant. A point of view to define the determinant of $I + A$ consists of considering A as an operator belonging to a class endowed with a trace. This was the idea adopted by Grothendieck [**22**] in the setting of Banach spaces. In general, there are several approaches to define traces and determinants in the setting of Banach spaces, two of them being that of embedded algebras introduced by Gohberg *et al.* [**17**] and the other one of operator ideals introduced by Pietsch [**30**]. These points of view agree, when we consider the ideal of nuclear operators on Banach spaces in the sense of Ruston–Grothendieck satisfying the approximation property, and the underlying Banach space is fixed in the point of view of Pietsch. The study of Fredholm determinants is an active field of research, in particular due to its applications in the analysis of differential equations; see, for example, [**4**, **5**, **16**, **26**, **44**]. The interest in such applications has recently also attracted attention towards the numerical analysis of such determinants. A systematic study of numerical computations for Fredholm determinants was initiated by Bornemann [**3**]. We refer to these papers for further references and motivations.

In the present paper, we prove the metric approximation property (which, in particular, implies the BAP because the control of the constant is explicit) for weighted mixed-norm Lebesgue spaces, modulation and Wiener amalgam spaces, and apply this property further to derive spectral information about operators acting on these spaces.

The modulation spaces were introduced in 1983 by Feichtinger [**12**], and have been intensively investigated in the last decades. We refer the reader to the survey [**13**] by Feichtinger for a historical account of the development of such spaces and a good account of the literature. We also refer to Gröchenig's book [**18**] for the basic definitions and properties of modulation spaces. Modulation spaces start finding numerous applications in various problems in linear and nonlinear partial differential equations;see [**36**] for a recent survey.

The analysis of the Schatten properties of pseudo-differential operators acting on L^2, but with symbols of low regularity, has been a subject of intensive recent research too; see, for example, Toft [**42**, **43**], Sobolev [**37**] or the authors' papers [**7**, **8**]. In particular, Schatten properties of pseudo-differential operators with symbols in modulation spaces have been analysed and established by Toft [**40**, **41**], Gröchenig and Toft [**20**]; see also [**19**].

One purpose of this paper is to analyse the analogous properties of operators, but this time acting on modulation spaces. Since these are in general Banach spaces, the Schatten properties are replaced by the notion of nuclearity (or r-nuclearity) introduced by Grothendieck [**21**]. In order for this theory to become effective, we prove that the modulation spaces (and hence also Wiener amalgam spaces) have the approximation property. This is done by proving the same property for weighted mixed-norm Lebesgue spaces, and then using the Gabor frame description of modulation spaces, reducing them to weighted mixed-norm sequence spaces. Consequently, we derive criteria for nuclearity and r-nuclearity of operators acting on modulation spaces with the subsequent trace formulae. The obtained results are applied to study functions of the harmonic oscillator on modulation spaces and the corresponding trace formula of Lidskii type, relating the operator trace to the sums of eigenvalues for these operators.

To formulate the concepts more precisely, we now recall the notion of modulation spaces and that of nuclear operators on Banach spaces. For a suitable weight w on $\mathbb{R}^{2d}$, $1 \leqslant p, q < \infty$ and a window $g \in \mathcal{S}(\mathbb{R}^d)$, the modulation space $\mathcal{M}_w^{p,q}(\mathbb{R}^d)$ consists of the temperate distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{\mathcal{M}_w^{p,q}} := \|V_g f\|_{L_w^{p,q}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p w(x,\xi)^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty, \qquad (1.1)$$

where

$$V_g f(x,\xi) = \int_{\mathbb{R}^d} f(y) \overline{g(y-x)} \, e^{-iy\cdot\xi} \, dy$$

denotes the short-time Fourier transform of f with respect to g at the point (x,ξ). The modulation space $\mathcal{M}_w^{p,q}(\mathbb{R}^d)$ endowed with the above norm becomes a Banach space, independent of $g \neq 0$.

We now recall the required basic conditions on w for the development of the theory of modulation spaces $\mathcal{M}_w^{p,q}$, and we refer the reader to the Chapter 11 of [18] for a detailed exposition. A *weight function* is a non-negative, locally integrable function on $\mathbb{R}^{2d}$. A weight function v on $\mathbb{R}^{2d}$ is called *submultiplicative* if

$$v(x+y) \leqslant v(x)v(y) \quad \text{for all } x, y \in \mathbb{R}^{2d}. \qquad (1.2)$$

A weight function w on $\mathbb{R}^{2d}$ is *v-moderate* if

$$w(x+y) \leqslant v(x)w(y) \quad \text{for all } x, y \in \mathbb{R}^{2d}. \qquad (1.3)$$

In particular, the weights of polynomial type play an important role. They are of the form

$$v_s(x,\xi) = (1 + |x|^2 + |\xi|^2)^{s/2}. \qquad (1.4)$$

The v_s-moderated weights (for some s) are called *polynomially moderated*.

On the other hand, the approximation property on a Banach space is crucial to define the concept of trace of nuclear operators, and in particular for the study of trace formulae such as the Grothendieck–Lidskii formula. Let $\mathcal{B}$ be a Banach space; a linear operator T from $\mathcal{B}$ to $\mathcal{B}$ is called *nuclear* if there exist sequences (x_n') in $\mathcal{B}'$ and (y_n) in $\mathcal{B}$ such that

$$Tx = \sum_{n=1}^{\infty} \langle x, x_n' \rangle y_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|x_n'\|_{\mathcal{B}'} \|y_n\|_{\mathcal{B}} < \infty.$$

This definition agrees with the concept of trace class operator in the setting of Hilbert spaces. The set of nuclear operators from $\mathcal{B}$ into $\mathcal{B}$ forms the ideal of nuclear operators $\mathcal{N}(\mathcal{B})$ endowed with the norm

$$N(T) = \inf \left\{ \sum_{n=1}^{\infty} \|x_n'\|_{\mathcal{B}'} \|y_n\|_{\mathcal{B}} : T = \sum_{n=1}^{\infty} x_n' \otimes y_n \right\}.$$

It is natural to attempt to define the trace of $T \in \mathcal{N}(\mathcal{B})$ by

$$\text{Tr}(T) := \sum_{n=1}^{\infty} x_n'(y_n), \qquad (1.5)$$

where $T = \sum_{n=1}^{\infty} x_n' \otimes y_n$ is a representation of T. Grothendieck [21] discovered that $\text{Tr}(T)$ is well defined for all $T \in \mathcal{N}(\mathcal{B})$ if and only if $\mathcal{B}$ has the *approximation property* (cf. Pietsch [30] or Defant and Floret [6]), that is, if for every compact set K in $\mathcal{B}$ and for every $\epsilon > 0$, there exists $F \in \mathcal{F}(\mathcal{B})$ such that

$$\|x - Fx\| < \epsilon \quad \text{for every } x \in K,$$

where we have denoted by $\mathcal{F}(\mathcal{B})$ the space of all finite rank bounded linear operators on $\mathcal{B}$. We denote by $\mathcal{L}(\mathcal{B})$ the C^*-algebra of bounded linear operators on $\mathcal{B}$. There are more

related approximation properties, for example, if in the definition above the operator F satisfies $\|F\| \leqslant 1$ one says that $\mathcal{B}$ possesses the *metric approximation property*. This is closely related to the BAP; see, for example, Lindenstrauss and Tzafriri [25, Definition 1.e.11].

It is well known that the classical spaces $C(X)$, where X is a compact topological space, as well as $L^p(\mu)$ for $1 \leqslant p < \infty$ for any measure μ satisfy the metric approximation property (cf. Pietsch [29]). In [11], Enflo constructed a counterexample to the approximation property in Banach spaces. A more natural counterexample was then found by Szankowski [38], who proved that $B(H)$ does not have the approximation property.

An important feature on Banach spaces even endowed with the approximation property is that the Lidskii formula does not hold in general for nuclear operators, as it was proved by Lidskiĭ [24] in Hilbert spaces showing that the operator trace is equal to the sum of the eigenvalues of the operator counted with multiplicities. Thus, in the setting of Banach spaces, Grothendieck [21] introduced a more restricted class of operators where the Lidskii formula holds, this fact motivating the following definition.

Let $\mathcal{B}$ be a Banach space and $0 < r \leqslant 1$; a linear operator T from $\mathcal{B}$ into $\mathcal{B}$ is called *r-nuclear* if there exist sequences (x'_n) in $\mathcal{B}'$ and (y_n) in $\mathcal{B}$ so that

$$Tx = \sum_{n=1}^{\infty} \langle x, x'_n \rangle y_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|x'_n\|_{\mathcal{B}'}^r \|y_n\|_{\mathcal{B}}^r < \infty. \tag{1.6}$$

We associate a quasi-norm $n_r(T)$ by

$$n_r(T)^r := \inf \left\{ \sum_{n=1}^{\infty} \|x'_n\|_{\mathcal{B}'}^r \|y_n\|_{\mathcal{B}}^r \right\},$$

where the infimum is taken over the representations of T as in (1.6). When $r = 1$, the 1-nuclear operators agree with the nuclear operators; in that case, this definition agrees with the concept of trace class operator in the setting of Hilbert spaces ($\mathcal{B} = H$). More generally, Oloff proved in [28] that the class of r-nuclear operators coincides with the Schatten class $S_r(H)$ when $\mathcal{B} = H$ and $0 < r \leqslant 1$. Moreover, Oloff proved that

$$\|T\|_{S_r} = n_r(T), \tag{1.7}$$

where $\| \cdot \|_{S_r}$ denotes the classical Schatten quasi-norms in terms of singular values.

In [21], Grothendieck proved that if T is $\frac{2}{3}$-nuclear from $\mathcal{B}$ into $\mathcal{B}$ for a Banach space $\mathcal{B}$ endowed with the approximation property, then

$$\text{Tr}(T) = \sum_{j=1}^{\infty} \lambda_j, \tag{1.8}$$

where λ_j $(j = 1, 2, \ldots)$ are the eigenvalues of T with multiplicities taken into account, and $\text{Tr}(T)$ is as in (1.5). Nowadays the formula (1.8) is referred to as Lidskii's formula, proved by Lidskiĭ [24] in the Hilbert space setting. Grothendieck also established its applications to the distribution of eigenvalues of operators in Banach spaces. We refer the reader to [8] for several conclusions in the setting of compact Lie groups concerning summability and distribution of eigenvalues of operators on L^p-spaces, once we have information on their r-nuclearity. Kernel conditions on compact manifolds have been investigated in [7, 9].

For our purposes it is convenient to consider first mixed-norm spaces. In Section 2, we establish the metric approximation property for the weighted mixed-norm $L_w^{(p_1, \ldots, p_n)}$ spaces, and in Section 3 for modulation spaces $\mathcal{M}_w^{p,q}$ and Wiener amalgam spaces $\mathcal{W}_w^{p,q}$, also recalling the definition of the latter. In Section 4, we characterize r-nuclear operators acting between weighted mixed-norm spaces L_w^P. In Section 5, we apply it to the questions of r-nuclearity, and trace formulae in modulation spaces and functions of the harmonic oscillator in that setting.

2. $L_w^{(p_1,\ldots,p_n)}$ has the metric approximation property

In this section, we prove that mixed-norm spaces $L_w^{(p_1,\ldots,p_n)}$, and consequently the modulation spaces $\mathcal{M}_w^{p,q}$ satisfy the metric approximation property. Through the Fourier transform, also the Wiener amalgam spaces $\mathcal{W}_w^{p,q}$ will have the same property.

We start the analysis of the approximation property by recalling a basic lemma which simplifies the proof of the metric approximation property (cf. [29, Lemma 10.2.2]).

LEMMA 2.1. *A Banach space $\mathcal{B}$ satisfies the metric approximation property if, given $x_1,\ldots,x_m \in \mathcal{B}$ and $\epsilon > 0$, there exists an operator $F \in \mathcal{F}(\mathcal{B})$ such that $\|F\| \leqslant 1$ and*

$$\|x_i - Fx_i\| \leqslant \epsilon \quad \text{for } i = 1,\ldots,m.$$

We shall now establish the metric approximation property for weighted mixed-norm spaces. We first briefly recall its definition, and we refer the reader to [2] for the basic properties of these spaces.

Let (Ω_i, S_i, μ_i), for $i = 1,\ldots,n$, be given σ-finite measure spaces. We write $x = (x_1,\ldots,x_n)$, and let $P = (p_1,\ldots,p_n)$ be a given n-tuple with $1 \leqslant p_i < \infty$. We say that $1 \leqslant P < \infty$ if $1 \leqslant p_i < \infty$ for all $i = 1,\ldots,n$. Let w be a strictly positive measurable function. The norm $\|\cdot\|_{L_w^P}$ of a measurable function $f(x_1,\ldots,x_n)$ on the corresponding product measure space is defined by

$$\|f\|_{L_w^P} := \left(\int_{\Omega_n} \cdots \left(\int_{\Omega_2} \left(\int_{\Omega_1} |f(x)|^{p_1} w(x) \, d\mu_1(x_1) \right)^{p_2/p_1} d\mu_2(x_2) \right)^{p_3/p_2} \cdots d\mu_n(x_n) \right)^{1/p_n}.$$

As should be observed, the order of integration on these spaces is crucial. The L_w^P-spaces endowed with the $\|\cdot\|_{L_w^P}$-norm become Banach spaces and the dual $(L_w^P)'$ of L_w^P is $L_{w^{-1}}^{P'}$, where $P' = (p_1',\ldots,p_n')$. In view of our application to the modulation spaces $\mathcal{M}_w^{p,q}$, we will consider in particular the case of the index of the form $(P,Q) = (p_1,\ldots,p_d,q_1,\ldots,q_d)$, where $p_i = p, q_i = q$ and $\Omega_i = \mathbb{R}$ endowed with the Lebesgue measure. In this case, the weight is taken in the form $w = w(x,\xi)$, where $x \in \mathbb{R}^d, \xi \in \mathbb{R}^d$, with some special conditions on w that we briefly recall at the end of this section.

We first establish a useful lemma for the proof of the metric approximation property. We will require some notations. For $n \in \mathbb{N}$ and $P = (p_1,\ldots,p_n)$, we will denote by $\ell^P(I)$ the L^P mixed-norm space corresponding to I^n, where I is a countable set of indices endowed with the counting measure. We note that such ℓ^P-norm is given by

$$\|h\|_{\ell^P} = \left(\sum_{k_n \in I} \cdots \left(\sum_{k_2 \in I} \left(\sum_{k_1 \in I} |h(k_1,\ldots,k_n)|^{p_1} \right)^{p_2/p_1} \right)^{p_3/p_2} \cdots \right)^{1/p_n}.$$

Given a Banach space $\mathcal{B}$ and $u \in \mathcal{B}, z \in \mathcal{B}'$, we will also denote by $\langle u, z \rangle_{\mathcal{B},\mathcal{B}'}$, or simply by $\langle u, z \rangle$, the valuation $z(u)$.

LEMMA 2.2. *Let $\mathcal{B}$ be a Banach space and $1 \leqslant Q < \infty$. Let I be a countable set. Let $(u_i)_{i \in I}, (v_i)_{i \in I}$ be sequences in $\mathcal{B}', \mathcal{B}$, respectively, such that*

$$\|\langle x, u_i \rangle\|_{\ell^Q(I)}, \|\langle v_i, z \rangle\|_{\ell^{Q'}(I)} \leqslant 1 \quad \text{for } \|x\|_{\mathcal{B}}, \|z\|_{\mathcal{B}'} \leqslant 1.$$

Then the operator $T = \sum_{i \in I} u_i \otimes v_i$ from $\mathcal{B}$ into $\mathcal{B}$ is well defined, bounded and satisfies $\|T\|_{\mathcal{L}(\mathcal{B})} \leqslant 1$.

Proof. Let $N \subset I$ be a finite subset of I. Let us write $T_N := \sum_{i \in N} u_i \otimes v_i$. It is clear that T_N is well defined. Moreover, T_N is a bounded finite rank operator.

Now, since $T_N x = \sum_{i \in N} \langle x, u_i \rangle v_i$, we observe that, for $x \in \mathcal{B}, z \in \mathcal{B}'$ such that $\|x\|_{\mathcal{B}}$, $\|z\|_{\mathcal{B}'} \leqslant 1$, we have

$$|\langle T_N x, z \rangle| \leqslant \sum_{i \in N} |\langle x, u_i \rangle| |\langle v_i, z \rangle| \leqslant \|\langle x, u_i \rangle\|_{\ell^Q} \|\langle v_i, z \rangle\|_{\ell^{Q'}} \leqslant 1.$$

We have applied the Hölder inequality in ℓ^Q mixed-norm spaces (cf. [**2**]) for the second inequality. Hence $T = \lim_N T_N$ exists in $\mathcal{L}(\mathcal{B})$ and $\|T\|_{\mathcal{L}(\mathcal{B})} \leqslant 1$. $\qquad \square$

We will apply the lemma above for the particular case of the single index $Q = p_n$ and the Banach space L^P. In the rest of this section, we will assume that the weights w satisfy the following condition for all $x \in \Omega$:

$$w(x_1, \ldots, x_n) \leqslant w_1(x_1) \cdots w_n(x_n), \tag{2.1}$$

where w_j is a weight on Ω_j (that is, a strictly positive locally integrable function). In particular, the condition holds for polynomially moderate weights on $\mathbb{R}^n$ satisfying, for a suitable n-tuple $(\beta_1, \ldots, \beta_n)$, the condition

$$w(x_1, \ldots, x_n) \leqslant \langle x_1 \rangle^{\beta_1} \cdots \langle x_n \rangle^{\beta_n}, \tag{2.2}$$

where $\langle x_j \rangle = 1 + |x_j|$.

THEOREM 2.3. *The weighted mixed-norm spaces $L_w^P = L_w^{(p_1, \ldots, p_n)}$ with w satisfying (2.1) have the metric approximation property.*

Proof. (Step 1) We will first prove the metric approximation property for the constant weight $w = 1$. Let $f_1, \ldots, f_N \in L^{(p_1, \ldots, p_n)}$ and let $\epsilon > 0$. By Lemma 2.1, it is enough to construct an operator $L \in \mathcal{F}(L^P, L^P)$ such that $\|L\|_{\mathcal{L}(L^P)} \leqslant 1$ and

$$\|f_i - L f_i\|_{L^P} \leqslant \epsilon \quad \text{for } i = 1, \ldots, N. \tag{2.3}$$

We consider first elementary functions of the form

$$f_i^0(x_1, \ldots, x_n) := \sum_{k_1, \ldots, k_n = 1}^{l} \alpha_{k_1, \ldots, k_n}^i \prod_{j=1}^{n} 1_{\Omega_{k_j}^j}(x_j),$$

where the sets $\prod_{j=1}^{n} \Omega_{k_j}^j$ are disjoint, $1_{\Omega_{k_j}^j}$ denotes the characteristic function of the set $\Omega_{k_j}^j$,

$$\mu_j(\Omega_{k_j}^j) < \infty$$

and

$$\|f_i - f_i^0\|_{L^{(p_1, \ldots, p_n)}} \leqslant \frac{\epsilon}{2}.$$

For the density of simple functions in L^P, see [**2**]. Since we are excluding the index $p = \infty$ in the multi-index P, the density always holds in our context.

We will define

$$\left(\prod_{j=1}^{n} 1_{\Omega_{k_j}^j} \right)(x_1, \ldots, x_n) := \prod_{j=1}^{n} 1_{\Omega_{k_j}^j}(x_j).$$

We define

$$u_{(k)} := \frac{\prod_{j=1}^n 1_{\Omega_{k_j}^j}}{\prod_{j=1}^n \mu_j(\Omega_{k_j}^j)^{1/p_j'}}, v_{(k)} := \frac{\prod_{j=1}^n 1_{\Omega_{k_j}^j}}{\prod_{j=1}^n \mu_j(\Omega_{k_j}^j)^{1/p_j}},$$

where $(k) = (k_1, \ldots, k_n)$.

Set

$$L := \sum_{k_1,\ldots,k_n=1}^l \frac{\prod_{j=1}^n 1_{\Omega_{k_j}^j}}{\prod_{j=1}^n \mu_j(\Omega_{k_j}^j)^{1/p_j'}} \otimes \frac{\prod_{j=1}^n 1_{\Omega_{k_j}^j}}{\prod_{j=1}^n \mu_j(\Omega_{k_j}^j)^{1/p_j}}$$

$$= \sum_{(k)=k_1,\ldots,k_n=1}^l v_{(k)} \otimes u_{(k)}$$

$$= \sum_{(k)=k_1,\ldots,k_n=1}^l \frac{1}{\prod_{j=1}^n \mu_j(\Omega_{k_j}^j)} \left(\prod_{j=1}^n 1_{\Omega_{k_j}^j} \otimes \prod_{j=1}^n 1_{\Omega_{k_j}^j} \right).$$

In order to prove that $\|L\|_{\mathcal{L}(L^P)} \leqslant 1$, we will apply Lemma 2.2 for $\mathcal{B} = L^P$, the families $u(k), v(k)$ and $Q = p_n$. The special role of the power p_n will become clear later. Let $f \in L^P$, $g \in L^{P'}$ be such that $\|f\|_{L^P}, \|g\|_{L^{P'}} \leqslant 1$. Then we have to show that

$$\|\langle f, u_{(k)} \rangle\|_{\ell^{p_n}} \leqslant 1 \quad \text{and} \quad \|\langle v_{(k)}, g \rangle\|_{\ell^{p_n'}} \leqslant 1.$$

In order to verify the corresponding property for $f \in L^P$, it is enough to consider an elementary function $f \in L^P$ such that $\|f\|_{L^P} \leqslant 1$. The general case follows then by approximation. By redefining partitions, we can assume that f can be written in the form

$$f(x_1, \ldots, x_n) = \sum_{k_1,\ldots,k_n=1}^l \lambda_{k_1,\ldots,k_n} \prod_{j=1}^n 1_{\Omega_{k_j}^j}(x_j).$$

We note that

$$\langle f, u_{(k)} \rangle = \lambda_{k_1,\ldots,k_n} \prod_{j=1}^n \mu_j(\Omega_{k_j}^j)^{1/p_j},$$

and

$$\|\langle f, u_{(k)} \rangle\|_{\ell^{p_n}} = \left(\sum_{(k)} |\lambda_{(k)}|^{p_n} \prod_{j=1}^n \mu_j(\Omega_{k_j}^j)^{p_n/p_j} \right)^{1/p_n}.$$

On the other hand, a straightforward but long calculation shows that

$$\|f\|_{L^P(\mu)} = \left(\sum_{k_1,\ldots,k_n=1}^l |\lambda_{k_1,\ldots,k_n}|^{p_n} \prod_{j=1}^n \mu_j(\Omega_{k_j}^j)^{p_n/p_j} \right)^{1/p_n}.$$

Since $\|f\|_{L^P} \leqslant 1$, we have shown that $\|\langle f, u_{(k)} \rangle\|_{\ell^{p_n}} \leqslant 1$. The proof for $\|\langle v_{(k)}, g \rangle\|_{\ell^{p_n'}}$ is similar and we omit it.

Therefore, $\|L\|_{\mathcal{L}(L^P)} \leqslant 1$.

Now we obtain (2.3) in view of

$$\|f_i - Lf_i\|_{L^P} \leqslant \|f_i - f_i^0\|_{L^P} + \|Lf_i^0 - Lf_i\|_{L^P} \leqslant \epsilon$$

since $Lf_i^0 = f_i^0$. Indeed, one has

$$
\begin{aligned}
L\left(\prod_{\ell=1}^{n} 1_{\Omega_{k_\ell}^\ell}\right) &= \sum_{k_1,\ldots,k_n=1}^{l} \left(\frac{\prod_{j=1}^{n} 1_{\Omega_{k_j}^j}}{\prod_{j=1}^{n} \mu_j(\Omega_{k_j}^j)^{1/p_j'}} \otimes \frac{\prod_{j=1}^{n} 1_{\Omega_{k_j}^j}}{\prod_{j=1}^{n} \mu_j(\Omega_{k_j}^j)^{1/p_j}}\right) \prod_{\ell=1}^{n} 1_{\Omega_{k_\ell}^\ell} \\
&= \sum_{k_1,\ldots,k_n=1}^{l} \frac{\langle \prod_{\ell=1}^{n} 1_{\Omega_{k_\ell}^\ell}, \prod_{j=1}^{n} 1_{\Omega_{k_j}^j} \rangle_{P',P}}{\prod_{j=1}^{n} \mu_j(\Omega_{k_j}^j)} \prod_{j=1}^{n} 1_{\Omega_{k_j}^j} \\
&= \frac{\prod_{\ell=1}^{n} \mu_\ell(\Omega_{k_\ell}^\ell)}{\prod_{\ell=1}^{n} \mu_\ell(\Omega_{k_\ell}^\ell)} \prod_{\ell=1}^{n} 1_{\Omega_{k_\ell}^\ell} \\
&= \prod_{\ell=1}^{n} 1_{\Omega_{k_\ell}^\ell}.
\end{aligned}
$$

For the third equality, we have used the fact that the sets $\prod_{\ell=1}^{n} 1_{\Omega_{k_\ell}^\ell}$ are disjoint.

(Step 2) We will now prove the metric approximation property for an elementary weight w (in this case we ask for weights to be non-negative and locally integrable). We can write such w in the form

$$
w(x_1,\ldots,x_n) = \sum_{k_1,\ldots,k_n=1}^{l} \gamma_{k_1,\ldots,k_n} \prod_{j=1}^{n} 1_{\Omega_{k_j}^j}(x_j), \tag{2.4}
$$

where the sets $\prod_{j=1}^{n} \Omega_{k_j}^j$ are disjoint, $\mu_j(\Omega_{k_j}^j) < \infty$ and $\gamma_{k_1,\ldots,k_n} > 0$ for all (k).

Let $f_1,\ldots,f_N \in L_w^{(p_1,\ldots,p_n)}$ and let $\epsilon > 0$. With a slight modification of (Step 1) we will find an operator $L \in \mathcal{F}(L_w^P, L_w^P)$ such that $\|L\|_{\mathcal{L}(L_w^P)} \leqslant 1$ and

$$
\|f_i - Lf_i\|_{L_w^P} \leqslant \epsilon \quad \text{for } i = 1,\ldots,N. \tag{2.5}
$$

We consider again elementary functions; by redefining partitions, they can be written in the form

$$
f_i^0(x_1,\ldots,x_n) := \sum_{k_1,\ldots,k_n=1}^{l} \alpha_{k_1,\ldots,k_n}^i \prod_{j=1}^{n} 1_{\Omega_{k_j}^j}(x_j)
$$

and

$$
\|f_i - f_i^0\|_{L_w^{(p_1,\ldots,p_n)}} \leqslant \frac{\epsilon}{2}.
$$

We define

$$
u_{(k)} := \frac{\prod_{j=1}^{n} \gamma_k^{1/p_j} 1_{\Omega_{k_j}^j}}{\prod_{j=1}^{n} \mu_j(\Omega_{k_j}^j)^{1/p_j'}}, \quad v_{(k)} := \frac{\prod_{j=1}^{n} \gamma_k^{1/p_j'} 1_{\Omega_{k_j}^j}}{\prod_{j=1}^{n} \mu_j(\Omega_{k_j}^j)^{1/p_j}},
$$

and set

$$
L := \sum_{(k)=k_1,\ldots,k_n=1}^{l} v_{(k)} \otimes u_{(k)}.
$$

Again, in order to prove that $\|L\|_{\mathcal{L}(L_w^P)} \leqslant 1$, we will apply Lemma 2.2 for $\mathcal{B} = L^P$ and the families $u(k), v(k)$ and $Q = p_n$. We note that w^{-1}, defined as $1/w$ on the support of w, is also an elementary weight. Let $f \in L_w^P$, $g \in L_{w^{-1}}^{P'}$ be such that $\|f\|_{L_w^P}, \|g\|_{L_{w^{-1}}^{P'}} \leqslant 1$.

In order to verify the corresponding property for $f \in L_w^P$, it is enough to consider an elementary function $f \in L^P$ such that $\|f\|_{L_w^P} \leqslant 1$. By redefining partitions, we can assume

that f can be written in the form

$$f(x_1, \ldots, x_n) = \sum_{k_1, \ldots, k_n = 1}^{l} \lambda_{k_1, \ldots, k_n} \prod_{j=1}^{n} 1_{\Omega_{k_j}^j} (x_j).$$

We note that

$$\langle f, u_{(k)} \rangle = \lambda_{k_1, \ldots, k_n} \prod_{j=1}^{n} \gamma_{(k)}^{1/p_j} \mu_j(\Omega_{k_j}^j)^{1/p_j},$$

and

$$\|\langle f, u_{(k)} \rangle\|_{\ell^{p_n}} = \left(\sum_{(k)} |\lambda_{(k)}|^{p_n} \prod_{j=1}^{n} \gamma_{(k)}^{p_n/p_j} \mu_j(\Omega_{k_j}^j)^{p_n/p_j} \right)^{1/p_n}.$$

We also have

$$\|f\|_{L_w^P(\mu)} = \left(\sum_{k_1, \ldots, k_n = 1}^{l} |\lambda_{k_1, \ldots, k_n}|^{p_n} \prod_{j=1}^{n} \gamma_{(k)}^{p_n/p_j} \mu_j(\Omega_{k_j}^j)^{p_n/p_j} \right)^{1/p_n}.$$

Since $\|f\|_{L_w^P} \leqslant 1$, we have shown that $\|\langle f, u_{(k)} \rangle\|_{\ell^{p_n}} \leqslant 1$. The proof for $\|\langle v_{(k)}, g \rangle\|_{\ell^{p'_n}}$ is similar.

Therefore, $\|L\|_{\mathcal{L}(L_w^P)} \leqslant 1$. The rest of the proof follows as in (Step 1).

(Step 3) We now suppose that the weight w belongs to the mixed-norm space $L^P(\mu)$ for some $1 \leqslant P < \infty$.

Then there exists an increasing sequence w_m of elementary weights which can be written in the form (2.4) and $\sup_m w_m = w$. We set

$$w_m(x_1, \ldots, x_n) = \sum_{k_1, \ldots, k_n = 1}^{l} \gamma_{(k)}^{(m)} \prod_{j=1}^{n} 1_{\Omega_{k_j}^{j,m}}(x_j), \tag{2.6}$$

where the sets $\prod_{j=1}^{n} \Omega_{k_j}^{j,m}$ are disjoint, $\mu_j(\Omega_{k_j}^{j,m}) < \infty$ and $\gamma_{k_1, \ldots, k_n}^{(m)} > 0$ for all (k).

We consider

$$u_{(k)}^m := \frac{\prod_{j=1}^{n} (\gamma_k^{(m)})^{1/p_j} 1_{\Omega_{k_j}^{j,m}}}{\prod_{j=1}^{n} \mu_j(\Omega_{k_j}^{j,m})^{1/p'_j}}, \quad v_{(k)} := \frac{\prod_{j=1}^{n} (\gamma_k^{(m)})^{1/p'_j} 1_{\Omega_{k_j}^{j,m}}}{\prod_{j=1}^{n} \mu_j(\Omega_{k_j}^{j,m})^{1/p_j}},$$

$$L_m := \sum_{(k) = k_1, \ldots, k_n = 1}^{l} v_{(k)}^m \otimes u_{(k)}^m.$$

The desired operator L can be obtained by defining $L := \lim_m L_m$ in $\mathcal{L}(L_w^P)$. We note that since $\|L_m\|_{\mathcal{L}(L_{w_m}^P)} \leqslant 1$ one gets $\|L\|_{\mathcal{L}(L_w^P)} \leqslant 1$.

(Step 4) Now if a general weight w satisfies (2.1), then we have the estimate

$$w(x_1, \ldots, x_n) \leqslant w_1(x_1) \cdots w_n(x_n)$$

for some positive functions w_j. Let $\psi_j(x_j)$ be a positive function of a variable x_j only such that $1/\psi_j \in L^{\widetilde{p}}(\mu_j)$ for some $1 \leqslant \widetilde{p} < \infty$. We define $\widetilde{w}_j := w_j \psi_j$. Then we observe that by writing

$$w(x_1, \ldots, x_n) = w(x_1, \ldots, x_n) \widetilde{w}_1(x_1)^{-1} \cdots \widetilde{w}_n(x_n)^{-1} \widetilde{w}_1(x_1) \cdots \widetilde{w}_n(x_n),$$

we obtain

$$\|f\|_{L_w^P(\mu)} = \|f\|_{L_{\widetilde{w}}^P(\widetilde{\mu})},$$

where

$$\widetilde{w} = w(x_1, \ldots, x_n) \widetilde{w}_1(x_1)^{-1} \cdots \widetilde{w}_n(x_n)^{-1} \in L^{\widetilde{P}}(\mu)$$

for $1 \leqslant \widetilde{P} = (\tilde{p}, \ldots, \tilde{p}) < \infty$ and

$$\widetilde{\mu} = \widetilde{w}_1(x_1)\mu_1 \otimes \cdots \otimes \widetilde{w}_n(x_n)\mu_n.$$

Since $\widetilde{w} \in L^{\widetilde{P}}(\mu)$ for some $1 \leqslant \widetilde{P} < \infty$, by (Step 3) applied to $L^P_{\widetilde{w}}(\widetilde{\mu})$, the approximation property follows for $L^P_w(\mu)$.

This concludes the proof of the theorem. $\hfill\square$

3. $\mathcal{M}^{p,q}_w$ and $\mathcal{W}^{p,q}_w$ have the metric approximation property

It is also important to consider the special case of discrete weighted mixed-norm spaces. Given $\alpha, \beta > 0$, a strictly positive function $\widetilde{w}$ on the lattice $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, we denote by $\ell^{p,q}_{\widetilde{w}}(\mathbb{Z}^{2d})$ the set of sequences $a = (a_{kl})_{k,l \in \mathbb{Z}^d}$ for which the norm

$$\|a\|_{\ell^{p,q}_{\widetilde{w}}} = \left(\sum_{l \in \mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d} |a_{kl}|^p \widetilde{w}(\alpha k, \beta l)^p \right)^{q/p} \right)^{1/q}$$

is finite. The main example arises from restrictions of weights on $\mathbb{R}^{2d}$ to a lattice $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ and will be crucial in the next Corollary 3.1.

Let us recall now the definition of the Wiener amalgam spaces $\mathcal{W}^{p,q}_w(\mathbb{R}^d)$. There are several definitions possible for the spaces $\mathcal{W}^{p,q}_w$, in particular involving the short-time Fourier transform similarly to the definition of the modulation spaces in (1.1). To make an analogy with modulation spaces, we can reformulate their definition (1.1) in terms of the mixed-norm Lebesgue spaces, by saying that

$$f \in \mathcal{M}^{p,q}_w(\mathbb{R}^d) \quad \text{if and only if} \quad V_g f \cdot w \in L^{(p,q)}(\mathbb{R}^d \times \mathbb{R}^d). \tag{3.1}$$

Now, for a function $F \in L^1_{\text{loc}}(\mathbb{R}^{2d})$, we define $\mathcal{R}F(x,\xi) := F(\xi, x)$. Then we can define

$$f \in \mathcal{W}^{p,q}_w(\mathbb{R}^d) \quad \text{if and only if} \quad \mathcal{R}(V_g f \cdot w) \in L^{(q,p)}(\mathbb{R}^d \times \mathbb{R}^d). \tag{3.2}$$

However, for our purposes the following description through the Fourier transform will be more practical. For a review of different definitions we refer the reader to [**35**]. So, in what follows, we will always assume that the weights in modulation and Wiener amalgam spaces are submultiplicative and polynomially moderate (But we do not need to assume this when talking about weighted mixed-norm L^P-spaces.) as in (1.2)–(1.4). Then, because of the identity

$$|V_g f(x, \xi)| = (2\pi)^{-d} |V_{\widehat{g}} \widehat{f}(\xi, -x)|,$$

the Wiener amalgam space $\mathcal{W}^{p,q}_w$ and the modulation spaces are related through the Fourier transform by the formula

$$\|f\|_{\mathcal{W}^{p,q}_w} \simeq \|\widehat{f}\|_{\mathcal{M}^{q,p}_{w_0}}, \tag{3.3}$$

where $w(x, \xi) = w_0(\xi, -x)$.

As a consequence of Theorem 2.3, we now obtain the following corollary.

COROLLARY 3.1. *Let $1 \leqslant p, q < \infty$, and w be a submultiplicative polynomially moderate weight. Then $\mathcal{M}^{p,q}_w$ has the metric approximation property. Consequently, also the Wiener amalgam space $\mathcal{W}^{p,q}_w$ has the metric approximation property.*

Proof. We first observe that the polynomially moderate weights satisfy conditions (2.1) and (2.2) by choosing $\beta_j \geqslant s$. Also, we have the (topological) equivalence $\mathcal{M}^{p,q}_w \cong \ell^{p,q}_{\widetilde{w}}(\mathbb{Z}^{2d})$ with equivalence of norms, by [**18**, Theorems 12.2.3, 12.2.4], where $\widetilde{w}$ is the restriction of w to

the lattice $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, that is $\widetilde{w}(j,k) = w(\alpha j, \beta k)$. The result now follows from Theorem 2.3 by taking $\Omega_1, \ldots, \Omega_{2d} = \mathbb{Z}$, the weight $\widetilde{\omega}$, $\mu_1 = \cdots = \mu_{2d} = \mu$ (the counting measure). This proves the metric approximation property for $\mathcal{M}_w^{p,q}$ since that property is preserved under isomorphism.

The metric approximation property for the Wiener amalgam spaces now follows from that in modulation spaces in view of the relation (3.3). □

It was observed by Feichtinger and Gröchenig [14] that for the metric approximation property for a space, it is enough to establish it for the sequence space obtained through the atomic decompositions should they exist. Our approach is, however, different: for the spaces $\mathcal{M}_w^{p,q}$ and $\mathcal{W}_w^{p,q}$, we have obtained it immediately as a direct consequence of the established property for the mixed-norm Lebesgue spaces. The method of proof in this paper has a certain advantage from the point of view of being applicable to spaces that are not necessarily translation invariant; see [10] for its application to the Lebesgue spaces $L^{p(\cdot)}$ with variable exponent.

4. r-nuclearity on weighted mixed-norm spaces L_w^P

Since the metric approximation property is now established for the spaces of our interest, it is now relevant to consider nuclear operators on weighted mixed-norm spaces L_w^P. In this section, we will characterize nuclear operators on L_w^P, and present some applications to the study of the harmonic oscillator on modulation spaces $\mathcal{M}_w^{p,q}$.

We first formulate a basic lemma for special measures and weights. We will consider $1 \leqslant P, Q < \infty$. The multi-index P will be associated to the measures μ_i $(i = 1, \ldots, l)$, and Q will correspond to the measures ν_j $(j = 1, \ldots, m)$. We will also denote by $\mu := \mu_1 \otimes \cdots \otimes \mu_l$ and $\nu := \nu_1 \otimes \cdots \otimes \nu_m$ the corresponding product measures on the product spaces $\Omega = \prod_{i=1}^l \Omega_i, \Xi = \prod_{j=1}^m \Xi_j$. For a weight w we will define $w_P(\Omega) := \|1_\Omega\|_{L_w^P(\mu)}$. The additional property (2.1) will be only required for the formulation of trace relations.

LEMMA 4.1. Let $(\Omega_i, \mathcal{M}_i, \mu_i)$ $(i = 1, \ldots, l)$, $(\Xi_j, \mathcal{M}_j', \nu_j)$ $(j = 1, \ldots, m)$ be measure spaces. Let $w, \widetilde{w}$ be weights on Ω, Ξ, respectively, such that $w_P(\Omega), \widetilde{w}_{Q'}^{-1}(\Xi) < \infty$. Let $f \in L_w^P(\mu)$, and $(g_n)_n, (h_n)_n$ be sequences in $L_{\widetilde{w}}^Q(\nu)$ and $L_{w^{-1}}^{P'}(\mu)$, respectively, such that $\sum_{n=1}^\infty \|g_n\|_{L_{\widetilde{w}}^Q(\nu)} \|h_n\|_{L_{w^{-1}}^{P'}(\mu)} < \infty$. Then the following conditions are satisfied:

(a) the series $\sum_{j=1}^\infty g_j(x)h_j(y)$ converges absolutely for almost every (a.e.) (x, y) and, consequently, $\lim_n \sum_{j=1}^n g_j(x)h_j(y)$ is finite for a.e. (x, y);

(b) for $k(x, y) := \sum_{j=1}^\infty g_j(x)h_j(y)$, we have $k \in L^1(\nu \otimes \mu)$;

(c) if $k_n(x, y) = \sum_{j=1}^n g_j(x)h_j(y)$, then $\|k_n - k\|_{L^1(\nu \otimes \mu)} \to 0$;

(d) $\lim_n \int_\Omega (\sum_{j=1}^n g_j(x)h_j(y))f(y)\, d\mu(y) = \int_\Omega (\sum_{j=1}^\infty g_j(x)h_j(y))f(y)\, d\mu(y)$

for a.e. x.

Proof. Let $k_n(x, y) := \sum_{j=1}^n g_j(x)h_j(y)f(y)$. Applying the Hölder inequality, we obtain

$$\iint_{\Omega\Xi} |k_n(x,y)|\, d\nu(x)\, d\mu(y) \leqslant \iint_{\Omega\Xi} \sum_{j=1}^n |g_j(x)h_j(y)f(y)|\, d\nu(x)\, d\mu(y)$$

$$\leqslant \sum_{j=1}^n \int_\Xi |g_j(x)|\, d\nu(x) \int_\Omega |h_j(y)||f(y)|\, d\mu(y)$$

$$\leqslant \widetilde{w}_{Q'}^{-1}(\Xi) \|f\|_{L_w^P(\mu)} \sum_{j=1}^{n} \|g_j\|_{L_{\widetilde{w}}^Q(\nu)} \|h_j\|_{L_{w^{-1}}^{P'}(\mu)}$$

$$\leqslant M < \infty \text{ for all } n.$$

Hence $\|k_n\|_{L^1(\nu \otimes \mu)} \leqslant M$ for all n.

On the other hand, the sequence (s_n), with $s_n(x,y) = \sum_{j=1}^{n} |g_j(x)h_j(y)f(y)|$, is increasing in $L^1(\nu \otimes \mu)$ and verifies

$$\sup_n \int\int |s_n(x,y)| \, d\mu(x) \, d\mu(y) \leqslant M < \infty.$$

Using the Levi monotone convergence theorem, the limit $s(x,y) = \lim_n s_n(x,y)$ is finite for a.e. (x,y). Moreover, $s \in L^1(\nu \otimes \mu)$. Choosing $f = 1$ which belongs to $L_w^P(\mu)$ and from the fact that $|k(x,y)| \leqslant s(x,y)$, we deduce (a) and (b). Part (c) can be deduced using the Lebesgue dominated convergence theorem applied to the sequence (k_n) dominated by $s(x,y)$. For the part (d), we observe that letting $k_n(x,y) = \sum_{j=1}^{n} g_j(x)h_j(y)f(y)$, we have $|k_n(x,y)| \leqslant s(x,y)$ for all n and every (x,y). From the fact that $s \in L^1(\nu \otimes \mu)$ we obtain that $s(x,\cdot) \in L^1(\mu)$ for a.e x. Then (d) is obtained from the Lebesgue dominated convergence theorem.   □

REMARK 4.2.   In the case of a single measure space $(l = 1)$, the condition $w_P(\Omega) < \infty$ is equivalent to the fact that $w\mu$ is a finite measure. In particular, if $w = 1$, then we have $w_P(\Omega) < \infty$ if and only if μ is a finite measure.

We establish below a characterization of nuclear operators on the weighted mixed-norm spaces L_w^P for weights satisfying the assumptions of Lemma 4.1.

THEOREM 4.3.   Let $0 < r \leqslant 1$. Let $(\Omega_i, \mathcal{M}_i, \mu_i)(i = 1, \ldots, l)$, $(\Xi_j, \mathcal{M}'_j, \nu_j)(j = 1, \ldots, m)$ be measure spaces. Let $w, \widetilde{w}$ be weights on Ω, Ξ, respectively, satisfying conditions $w_P(\Omega), \widetilde{w}_{Q'}^{-1}(\Xi) < \infty$. Then T is r-nuclear operator from $L_w^P(\mu)$ into $L_{\widetilde{w}}^Q(\nu)$ if and only if there exist a sequence (g_n) in $L_{\widetilde{w}}^Q(\nu)$ and a sequence (h_n) in $L_{w^{-1}}^{P'}(\mu)$ such that $\sum_{n=1}^{\infty} \|g_n\|_{L_{\widetilde{w}}^Q(\nu)}^r \|h_n\|_{L_{w^{-1}}^{P'}(\mu)}^r < \infty$, and such that, for all $f \in L_w^P(\mu)$,

$$Tf(x) = \int_\Omega \left(\sum_{n=1}^{\infty} g_n(x)h_n(y) \right) f(y) \, d\mu(y) \quad \text{for a.e } x.$$

Proof.   It is enough to consider the case $r = 1$. The case $0 < r < 1$ follows by inclusion. Let T be a nuclear operator from $L_w^P(\mu)$ into $L_{\widetilde{w}}^Q(\nu)$. Then there exist sequences (g_n) in $L_{\widetilde{w}}^Q(\nu)$, (h_n) in $L_{w^{-1}}^{P'}(\mu)$ such that $\sum_{n=1}^{\infty} \|g_n\|_{L_{\widetilde{w}}^Q(\nu)} \|h_n\|_{L_{w^{-1}}^{P'}(\mu)} < \infty$ and

$$Tf = \sum_n \langle f, h_n \rangle g_n.$$

Now

$$Tf = \sum_n \langle f, h_n \rangle g_n = \sum_n \left(\int_\Omega h_n(y) f(y) \, d\mu(y) \right) g_n,$$

where the sums converges in the $L_{\widetilde{w}}^Q(\nu)$-norm. There exist (cf. [2, Theorem 1(a)]) two subsequences $(\widetilde{g}_n)$ and $(\widetilde{h}_n)$ of (g_n) and (h_n), respectively, such that

$$(Tf)(x) = \sum_n \langle f, \widetilde{h}_n \rangle \widetilde{g}_n(x) = \sum_n \left(\int_\Omega \widetilde{h}_n(y) f(y) \, d\mu(y) \right) \widetilde{g}_n(x), \quad \text{a.e } x.$$

Since the couple $((\widetilde{g}_n), (\widetilde{h}_n))$ satisfies

$$\sum_{n=1}^{\infty} \|\widetilde{g}_n\|_{L^Q_{\widetilde{w}}(\nu)} \|\widetilde{h}_n\|_{L^{P'}_{w^{-1}}(\mu)} < \infty,$$

by applying Lemma 4.1(d), it follows that

$$\sum_{n=1}^{\infty} \left(\int_{\Omega} \widetilde{h}_n(y) f(y)\, d\mu(y) \right) \widetilde{g}_n(x) = \lim_n \sum_{j=1}^{n} \left(\int_{\Omega} \widetilde{h}_j(y) f(y)\, d\mu(y) \right) \widetilde{g}_j(x)$$

$$= \lim_n \int_{\Omega} \left(\sum_{j=1}^{n} \widetilde{g}_j(x) \widetilde{h}_j(y) f(y) \right) d\mu(y)$$

$$= \int_{\Omega} \left(\sum_{n=1}^{\infty} \widetilde{g}_n(x) \widetilde{h}_n(y) \right) f(y)\, d\mu(y), \quad \text{a.e } x.$$

Conversely, assume that there exist sequences $(g_n)_n$ in $L^Q_{\widetilde{w}}(\nu)$, and $(h_n)_n$ in $L^{P'}_{w^{-1}}(\mu)$ so that $\sum_{n=1}^{\infty} \|g_n\|_{L^Q_{\widetilde{w}}(\nu)} \|h_n\|_{L^{P'}_{w^{-1}}(\mu)} < \infty$, and, for all $f \in L^P_w(\mu)$,

$$Tf(x) = \int_{\Omega} \left(\sum_{n=1}^{\infty} g_n(x) h_n(y) \right) f(y)\, d\mu(y), \quad \text{a.e } x.$$

By Lemma 4.1(d), we have

$$\int_{\Omega} \left(\sum_{n=1}^{\infty} g_n(x) h_n(y) \right) f(y)\, d\mu(y) = \lim_n \int_{\Omega} \left(\sum_{j=1}^{n} g_j(x) h_j(y) f(y) \right) d\mu(y)$$

$$= \lim_n \sum_{j=1}^{n} \left(\int_{\Omega} h_j(y) f(y)\, d\mu(y) \right) g_j(x)$$

$$= \sum_n \left(\int_{\Omega} h_n(y) f(y)\, d\mu(y) \right) g_n(x)$$

$$= \sum_n \langle f, h_n \rangle g_n(x) = (Tf)(x), \quad \text{a.e. } x.$$

To prove that $Tf = \sum_n \langle f, h_n \rangle g_n$ in $L^Q_{\widetilde{w}}(\nu)$, we let $s_n := \sum_{j=1}^{n} \langle f, h_j \rangle g_j$; then $(s_n)_n$ is a sequence in $L^Q_{\widetilde{w}}(\nu)$ and

$$|s_n(x)| \leqslant \|f\|_{L^P_w(\mu)} \sum_{j=1}^{n} \|h_j\|_{L^{P'}_{w^{-1}}(\mu)} |g_j(x)|$$

$$\leqslant \|f\|_{L^P_w(\mu)} \sum_{j=1}^{\infty} \|h_j\|_{L^{P'}_{w^{-1}}(\mu)} |g_j(x)| =: \gamma(x), \quad \text{for all } n.$$

Moreover, γ is well defined and $\gamma \in L^Q_{\widetilde{w}}(\nu)$, since it is the increasing limit of the sequence $(\gamma_n)_n = (\|f\|_{L^P_w(\mu)} \sum_{j=1}^{n} \|h_j\|_{L^{P'}_{w^{-1}}(\mu)} |g_j(x)|)_n$ of $L^Q_{\widetilde{w}}(\nu)$ functions and

$$\|\gamma_n\|_{L^Q_{\widetilde{w}}(\nu)} \leqslant \|f\|_{L^P_w} \sum_{j=1}^{\infty} \|h_j\|_{L^{P'}_{w^{-1}}(\mu)} \|g_j\|_{L^Q_{\widetilde{w}}(\nu)} \leqslant M < \infty.$$

By the Levi monotone convergence theorem, we see that $\gamma \in L^Q_{\widetilde{w}}(\nu)$. Finally, applying the Lebesgue dominated convergence theorem, we deduce that $s_n \to Tf$ in $L^Q_{\widetilde{w}}(\nu)$. $\qquad\square$

Before establishing a characterization of r-nuclear operators for more general measures and weights, we first generalize Lemma 4.1. The following definition will be useful.

DEFINITION 4.4. Let $(\Omega_i, \mathcal{M}_i, \mu_i)$ $(i = 1, \ldots, l)$ be measure spaces and $\mu := \mu_1 \otimes \cdots \otimes \mu_l$ be the corresponding product measure on $\Omega = \prod_{i=1}^{l} \Omega_i$. We will also call $\Lambda \in \mathcal{M} := \bigotimes_{i=1}^{l} \mathcal{M}_i$ a *box* if it is of the form $\Lambda = \prod_{i=1}^{l} \Lambda_i$. For a measure μ, a weight w on Ω and a multi-index P, we will say that the triple (μ, w, P) is σ-finite if there exists a family of disjoint boxes Ω^k such that $\mu(\Omega^k) < \infty$, $\bigcup_{k=1}^{\infty} \Omega^k = \Omega$ and

$$w_P(\Omega^k) = \|1_{\Omega^k}\|_{L^P_w(\mu)} < \infty.$$

REMARK 4.5. We observe that, for the case of a single measure space $(l = 1)$, a triple (μ, w, p) is σ-finite if and only if $w\mu$ is σ-finite. If in addition we restrict to consider weights such that $0 < w(x) < \infty$, then triple (μ, w, p) is σ-finite if and only if the measure μ is σ-finite.

LEMMA 4.6. Let $(\Omega_i, \mathcal{M}_i, \mu_i)$ $(i = 1, \ldots, l)$, $(\Xi_j, \mathcal{M}'_j, \nu_j)$ $(j = 1, \ldots, m)$ be measure spaces. Let $1 \leqslant P, Q < \infty$. Let $w, \widetilde{w}$ be weights on Ω, Ξ, respectively, such that the triples $(\mu, w, P), (\nu, \widetilde{w}^{-1}, Q')$ are σ-finite. Let $f \in L^P_w(\mu)$, and $(g_n)_n, (h_n)_n$ be sequences in $L^Q_{\widetilde{w}}(\nu)$ and $L^{P'}_{w^{-1}}(\mu)$, respectively, such that $\sum_{n=1}^{\infty} \|g_n\|_{L^Q_{\widetilde{w}}(\nu)} \|h_n\|_{L^{P'}_{w^{-1}}(\mu)} < \infty$. Then the parts (a) and (d) of Lemma 4.1 hold.

Proof. Since the triples $(\mu, w, P), (\nu, \widetilde{w}^{-1}, Q')$ are σ-finite, there exist two sequences $(\Omega^k)_k$ and $(\Xi^j)_j$ of disjoint subsets of Ω and Ξ, respectively, such that $\bigcup_k \Omega^k = \Omega$, $\bigcup_j \Xi^j = \Xi$ and, for all j, k,

$$w_P(\Omega^k), \widetilde{w}^{-1}_{Q'}(\Xi^j) < \infty.$$

We now consider the measure spaces $(\Omega^k, \mathcal{M}^k, \mu^k)$ and $(\Xi^j, \mathcal{M}'^j, \nu^j)$ that we obtain by restricting Ω to Ω^k, and Ξ to Ξ^j for every k, j, and restricting the functions g_n to Ξ^j, and h_n to Ω^k. Then, for all k, j,

$$\sum_{n=1}^{\infty} \|g_n\|_{L^Q_{\widetilde{w}}(\nu^j)} \|h_n\|_{L^{P'}_{w^{-1}}(\mu^k)} < \infty.$$

By Lemma 4.1(a), it follows that $\sum_{j=1}^{\infty} g_j(x) h_j(y)$ converges absolutely for a.e $(x, y) \in \Xi^j \times \Omega^k$. Hence $\sum_{j=1}^{\infty} g_j(x) h_j(y)$ converges absolutely for almost every $(x, y) \in \Xi \times \Omega$. This proves part (a).

From part (a), the series $\sum_{j=1}^{\infty} g_j(x) h_j(y) f(y)$ converges absolutely for a.e. $(x, y) \in \Xi \times \Omega$; part (d) follows from the Lebesgue dominated convergence theorem applied as in the 'only if' part of the proof of Theorem 4.3 (use of γ_n, and γ). $\qquad\square$

We can now formulate a characterization of r-nuclear operators on weighted mixed-norm spaces and a trace formula.

THEOREM 4.7. Let $0 < r \leqslant 1$. Let $(\Omega_i, \mathcal{M}_i, \mu_i)$ $(i = 1, \ldots, l)$, $(\Xi_j, \mathcal{M}'_j, \nu_j)$ $(j = 1, \ldots, m)$ be measure spaces. Let $1 \leqslant P, Q < \infty$. Let $w, \widetilde{w}$ be weights on Ω, Ξ, respectively, such that the triples $(\mu, w, P), (\nu, \widetilde{w}^{-1}, Q')$ are σ-finite. Then T is r-nuclear operator from $L^P_w(\mu)$ into $L^Q_{\widetilde{w}}(\nu)$ if and only if there exist a sequence (g_n) in $L^Q_{\widetilde{w}}(\nu)$ and a sequence (h_n) in $L^{P'}_{w^{-1}}(\mu)$ such that

$\sum_{n=1}^{\infty} \|g_n\|_{L^Q_{\widetilde{w}}(\nu)}^r \|h_n\|_{L^{P'}_{w^{-1}}(\mu)}^r < \infty$, and such that, for all $f \in L^P_w(\mu)$,

$$Tf(x) = \int_{\Omega} \left(\sum_{n=1}^{\infty} g_n(x) h_n(y) \right) f(y) \, d\mu(y) \quad \text{for a.e } x.$$

Moreover, if $w = \widetilde{w}$ satisfies (2.1), $\mu = \nu$, $P = Q$ and T is r-nuclear on $\mathcal{L}(L^P_w(\mu))$ with $r \leqslant \frac{2}{3}$, then

$$\mathrm{Tr}(T) = \sum_{j=1}^{\infty} \lambda_j,$$

where λ_j $(j = 1, 2, \ldots)$ are the eigenvalues of T with multiplicities taken into account, and $\mathrm{Tr}(T) = \sum_{j=1}^{\infty} \langle u_j, v_j \rangle$.

Proof. Again, for the proof of the characterization it is enough to consider the case $r = 1$. But that characterization now follows from the same lines of the proof of Theorem 4.3 by replacing the references to part (d) of Lemma 4.1 by the part (d) of Lemma 4.6. On the other hand, since w additionally satisfies (2.1), the metric approximation property holds. If T is r-nuclear with $r \leqslant \frac{2}{3}$ on L^P_w, then the trace formula follows from the aforementioned Grothendieck's theorem in the introduction. □

5. *r-nuclearity on modulation spaces and the harmonic oscillator*

In this section, we describe the r-nuclearity and a trace formula in modulation spaces. We restrict our attention to modulation spaces, but note that the same conclusions hold also in the Wiener amalgam spaces. Thus, as an immediate consequence of Theorem 4.7 and Corollary 3.1, we have the following corollary.

COROLLARY 5.1.   *Let $0 < r \leqslant 1$, $1 \leqslant p, q < \infty$ and w be a submultiplicative polynomially moderate weight. An operator $T \in \mathcal{L}(\mathcal{M}^{p,q}_w, \mathcal{M}^{p,q}_w)$ is r-nuclear if and only if its kernel $k(x, y)$ can be written in the form*

$$k(x, y) = \sum_{j=1}^{\infty} u_j \otimes v_j,$$

with $u_j \in \mathcal{M}^{p,q}_w, v_j \in \mathcal{M}^{p',q'}_{w^{-1}}$ and

$$\sum_{j=1}^{\infty} \|u_j\|_{\mathcal{M}^{p,q}_w}^r \|v_j\|_{\mathcal{M}^{p',q'}_{w^{-1}}}^r < \infty.$$

Moreover, if T is r-nuclear with $r \leqslant \frac{2}{3}$, then

$$\mathrm{Tr}(T) = \sum_{j=1}^{\infty} \lambda_j, \tag{5.1}$$

where λ_j $(j = 1, 2, \ldots)$ are the eigenvalues of T with multiplicities taken into account.

In principle, in Corollary 5.1, the order $r \leqslant \frac{2}{3}$ in the r-nuclearity is sharp for the validity of the trace formula (5.1) in the context of general Banach spaces (with approximation property). However, for the traces in, for example, the L^p-spaces, the trace formula (5.1) may hold for r-nuclear operators also with larger values of r. In fact, the condition $r \leqslant \frac{2}{3}$ may be relaxed to the condition $r \leqslant r_0(p)$ with the index $r_0(p) \geqslant \frac{2}{3}$ depending on p; see Reinov and Laif [**33**], as

well as the authors' paper [**8**]. The same property may be expected also for general operators in L^P weighted mixed-norm spaces and consequently in modulation spaces.

However, for some special operators, the trace formula (5.1) may be valid for even larger values of r, for example, even for simply nuclear operators (that is, for $r = 1$).

We shall now consider an application of such nuclearity concepts to the study of the harmonic oscillator $A = -\Delta + |x|^2$ on $\mathbb{R}^d$. We will consider in particular the modulation space $\mathcal{M}_s^{p,q}$ corresponding to the weight $w(x, \xi) = (1 + |\xi|)^s$. If

$$A\phi_j \equiv (-\Delta + |x|^2)\phi_j = \lambda_j \phi_j,$$

then the eigenvalues λ_j can be enumerated in the form $\lambda = \lambda_{(k)} = \sum_{i=1}^d (2k_i + 1)$, $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$; see, for example, [**27**, Theorem 2.2.3]. For the corresponding sequence of orthonormal eigenfunctions ϕ_j in $L^2(\mathbb{R}^d)$, we can write with convergence in $L^2(\mathbb{R}^d)$:

$$f = \sum_{j=1}^\infty \langle f, \overline{\phi}_j \rangle \phi_j.$$

Hence, formally the kernel of A can be written as

$$k(x, y) = \sum_{j=1}^\infty A\phi_j(x)\overline{\phi}_j(y) = \sum_{j=1}^\infty \lambda_j \phi_j(x)\overline{\phi}_j(y).$$

We note that this can be justified by taking negative powers of the harmonic oscillator $(-\Delta + |x|^2)^{-N}$ for $N > 0$ large enough, so that we start with the decomposition for the corresponding kernel in the form

$$k_N(x, y) = \sum_{j=1}^\infty \lambda_j^{-N} \phi_j(x)\overline{\phi}_j(y).$$

More generally, for functions of the harmonic oscillator, defined by

$$F(-\Delta + |x|^2)\phi_j = F(\lambda_j)\phi_j, \quad j = 1, 2, \ldots, \tag{5.2}$$

we have the following theorem.

THEOREM 5.2.   *Let $0 < r \leqslant 1$, $s \in \mathbb{R}$ and $1 \leqslant p, q < \infty$. The operator $F(-\Delta + |x|^2)$ is r-nuclear on $\mathcal{M}_s^{p,q}(\mathbb{R}^d)$, provided that*

$$\sum_{j=1}^\infty |F(\lambda_j)|^r \|\phi_j\|_{\mathcal{M}_s^{p,q}}^r \|\phi_j\|_{\mathcal{M}_{-s}^{p',q'}}^r < \infty. \tag{5.3}$$

Moreover, if (5.3) holds with $r = 1$, then we have the trace formula

$$\mathrm{Tr}F(-\Delta + |x|^2) = \sum_{j=1}^\infty F(\lambda_j), \tag{5.4}$$

with the absolutely convergent series.

Proof.   The first part follows from Corollary 5.1. Moreover, while formula (5.4) can be expected from the general Grothendieck's theory (at least for $r \leqslant \frac{2}{3}$), in this case it follows in an elementary way due to the smoothness property of Hermite functions. Indeed, from (5.2), the integral kernel of $F(-\Delta + |x|^2)$ is given by

$$k(x, y) = \sum_{j=1}^\infty F(\lambda_j)\phi_j(x)\overline{\phi}_j(y). \tag{5.5}$$

At the same time, we know that functions ϕ_j are all smooth and fast decaying, implying in particular that $\phi_j \in \mathcal{M}_s^{p,q}$ (actually, this also follows if the assumption (5.3) holds with $r = 1$).

Consequently, by (1.5), we obtain

$$\mathrm{Tr}F(-\Delta + |x|^2) = \sum_{j=1}^{\infty} F(\lambda_j)\langle \phi_j, \overline{\phi}_j \rangle_{\mathcal{M}_s^{p,q}, \mathcal{M}_{-s}^{p',q'}} = \sum_{j=1}^{\infty} F(\lambda_j),$$

in view of the equality

$$\langle \phi_j, \overline{\phi}_j \rangle_{\mathcal{M}_s^{p,q}, \mathcal{M}_{-s}^{p',q'}} = (\phi_j, \phi_j)_{L^2} = 1.$$

The series in (5.4) converges absolutely in view of

$$\sum_{j=1}^{\infty} |F(\lambda_j)| = \sum_{j=1}^{\infty} |F(\lambda_j)|(\phi_j, \phi_j)_{L^2} = \sum_{j=1}^{\infty} |F(\lambda_j)|\langle \phi_j, \overline{\phi}_j \rangle_{\mathcal{M}_s^{p,q}, \mathcal{M}_{-s}^{p',q'}}$$

$$\leqslant \sum_{j=1}^{\infty} |F(\lambda_j)| \|\phi_j\|_{\mathcal{M}_s^{p,q}} \|\phi_j\|_{\mathcal{M}_{-s}^{p',q'}} < \infty,$$

which is finite by the assumption. This completes the proof. $\qquad\square$

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