Distinguished unipotent elements and multiplicity-free subgroups of simple algebraic groups

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Abstract

For G a simple algebraic group over an algebraically closed field of characteristic 0, we determine the irreducible representations $\rho : G \to I(V)$, where $I(V)$ denotes one of the classical groups $SL(V)$, $Sp(V)$, $SO(V)$, such that ρ sends distinguished unipotent elements of G to distinguished elements of $I(V)$. We also settle a base case of the general problem of determining when the restriction of ρ to a simple subgroup of G is multiplicity-free.

1 Introduction

Let G be a simple algebraic group of rank at least 2 defined over an algebraically closed field of characteristic 0 and let $\rho : G \to I(V)$ be an irreducible representation, where $I(V)$ denotes one of the classical groups $SL(V), Sp(V)$, or $SO(V)$. In this paper we consider two closely related problems. We determine those representations for which distinguished unipotent elements of G are sent to distinguished elements of $I(V)$. Also we settle a base case of the general problem of determining when the restriction of ρ to a simple subgroup of G is multiplicity-free.

A unipotent element of a simple algebraic group is said to be *distinguished* if it is not centralized by a nontrivial torus. Let $u \in G$ be a unipotent element. If $\rho(u)$ is distinguished in $I(V)$ then u must be distinguished in G. The distinguished unipotent elements of $I(V)$ can be decomposed into Jordan blocks of distinct sizes. Indeed they are a single Jordan block, the sum of blocks of distinct even sizes, or the sum of blocks of distinct odd sizes, according as $I(V) = SL(V), Sp(V)$ or $SO(V)$ (see [5, 3.5]).

Now u can be embedded in a subgroup A of G of type A_1 by the Jacobson-Morozov theorem; given u, the subgroup A is unique up to conjugacy in G. If $\rho(u)$ is distinguished, then $\rho(A)$ acts on V with irreducible summands of the same dimensions as the Jordan blocks of u, and hence the restriction $V \downarrow \rho(A)$ is multiplicity-free – that is, each irreducible summand appears with multiplicity 1. Indeed, $V \downarrow \rho(A)$ is either irreducible, or the sum of irreducibles of distinct even dimensions or of distinct odd dimensions.

Our main result determines those situations where $V \downarrow \rho(A)$ is multiplicity-free. In order to state it, we recall that a subgroup of G is said to be *G-irreducible* if it is contained in no proper parabolic subgoup of G. It follows directly from the definition that an A_1 subgroup of G is G-irreducible if and only if its non-identity unipotent elements are distinguished in G. If these unipotent elements are regular in G, we call the subgroup a *regular* A_1 in G.

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Theorem 1 *Let G be a simple algebraic group of rank at least 2 over an algebraically closed field K of characteristic zero, let $A \cong A_1$ be a G-irreducible subgroup of G, let $u \in A$ be a non-identity unipotent element, and let V be an irreducible KG-module of highest weight λ. Then $V \downarrow A$ is multiplicity-free if and only if λ and u are as in Tables 1 or 2, where λ is given up to graph automorphisms of G. Table 1 lists the examples where u is regular in G, and Table 2 lists those with u non-regular.*

Theorem 1 is the base case of a general project in progress, which aims to determine all irreducible KG-modules V and G-irreducible subgroups X of G for which $V \downarrow X$ is multiplicity-free.

The answer to the original question on distinguished unipotent elements is as follows.

Corollary 2 *Let G be as in the theorem, and let $\rho : G \to I(V)$ be an irreducible representation with highest weight λ, where $I(V)$ is $SL(V)$, $Sp(V)$ or $SO(V)$. Let $u \in G$ be a non-identity unipotent element, and suppose that $\rho(u)$ is a distinguished element of $I(V)$.*

(i) If $I(V) = SL(V)$, then $G = A_n$, B_n, C_n or G_2 and $\lambda = \omega_1$ (or ω_n if $G = A_n$), and u is regular in G.

(ii) If $I(V) = Sp(V)$ or $SO(V)$ then λ and u are as in one of the cases in Table 1 or 2 for which $V = V_G(\lambda)$ is a self-dual module (equivalently, $\lambda = -w_0(\lambda)$ where w_0 is the longest element of the Weyl group of G). Conversely, for each such case in the tables, $\rho(u)$ is distinguished in $I(V)$.

The layout of the paper is as follows. Section 2 consists of notation and preliminary lemmas. This is followed by Sections 3,4 and 5 where we prove Theorem 1 in the special case where A is a regular A_1 subgroup of G. Then in Section 6 we consider the remaining cases where A is non-regular. There are far fewer examples in that situation. Finally Section 7 contains the proof of the corollary.

For many of the proofs we need to calculate dimensions of weight spaces in various G-modules. When the rank of G is small, such dimensions can be computed using Magma [1], and we make occasional use of this facility.

2 Preliminary Lemmas

Continue to let G be a simple algebraic group over an algebraically closed field K of characteristic zero. Let $A \cong A_1$ be a G-irreducible subgroup of G, let u be a non-identity unipotent element of A, and let $T < A$ be a 1-dimensional torus such that the conjugates of u under T form the non-identity elements of a maximal unipotent group of A.

We fix some notation that will be used throughout the paper. Let $T \leq T_G$, where T_G is a maximal torus of G and let $\Pi_G = \{\alpha_1, \cdots, \alpha_n\}$ denote a fundamental system of roots. We label the nodes of the Dynkin diagram of G with these roots as in [2, p.250]. Write s_i for the reflection in α_i, an element of the Weyl group $W(G)$. When $G = D_n$ we assume that $n \geq 4$ (and regard D_3 as the group A_3).

The torus T determines a labelling of the Dynkin diagram by 0's and 2's (see 3.18 and Table 13.2 of [5]) which gives the weights of T on fundamental roots. When u is regular in G these labels are all 2's.

Table 1: $V \downarrow A$ multiplicity-free, $u \in G$ regular in G

G	λ
A_n	ω_1, ω_2, $2\omega_1$, $\omega_1 + \omega_n$,
	ω_3 $(5 \leq n \leq 7)$,
	$3\omega_1$ $(n \leq 5)$, $4\omega_1 (n \leq 3)$, $5\omega_1 (n \leq 3)$
A_3	110
A_2	$c1, c0$
B_n	ω_1, ω_2, $2\omega_1$
	ω_n $(n \leq 8)$
B_3	101, 002, 300
B_2	$b0$, $0b$ $(1 \leq b \leq 5)$, 11, 12, 21
C_n	ω_1, ω_2, $2\omega_1$,
	ω_3 $(3 \leq n \leq 5)$
	ω_n $(n = 4, 5)$
C_3	300
C_2	$b0$, $0b(1 \leq b \leq 5)$, 11, 12, 21
D_n $(n \geq 4)$	ω_1, ω_2 $(n = 2k+1)$, $2\omega_1$ $(n = 2k)$
	ω_n $(n \leq 9)$
E_6	ω_1, ω_2
E_7	ω_1, ω_7
E_8	ω_8
F_4	ω_1, ω_4
G_2	10, 01, 11, 20, 02, 30

Denote by $\omega_1, \cdots, \omega_n$ the fundamental dominant weights of G. For a dominant weight $\lambda = \sum c_i\omega_i$, let $V_G(\lambda)$ be the irreducible KG-module of highest weight λ. For $A \cong A_1$ and a non-negative integer r, we abbreviate the irreducible module $V_A(r)$ by V_r or just r. More generally we frequently denote the module $V_G(\lambda)$ by just the weight λ, or the string $c_1 \cdots c_l$ (where l is the rank).

Let $V = V_G(\lambda)$ and let λ afford weight r when restricted to T. Since all weights of V can be obtained by subtracting roots from the highest weight, the restriction of each weight to T has the form $r - 2k$ for some non-negative integer k. If $V \downarrow A$ is multiplicity-free, then $V \downarrow A = V_{r_1} + V_{r_2} + V_{r_3} + \cdots$, where $r = r_1 > r_2 > r_3 > \cdots$. Then the T-weights on V are $(r_1, r_1 - 2, \ldots, -r_1), (r_2, r_2 - 2, \ldots, -r_2), (r_3, r_3 - 2, \ldots, -r_3), \ldots$. Noting that all the r_i have the same parity, it follows that the weight r_i appears with multiplicity i for all $i \geq 1$. Note that weight $r - 2$ arises as the restriction of $\lambda - \alpha_i$ for those i with $c_i > 0$. Therefore, there can be at most 2 such values of i.

We often use the following short hand notation. Rather than writing $\lambda - x\alpha_i - y\alpha_j - z\alpha_k - \cdots$, we simply write $\lambda - i^x j^y k^z \cdots$.

Lemma 2.1 *If $V \downarrow A$ is multiplicity-free, then* $\dim V \leq (\frac{r}{2} + 1)^2$ *or* $(\frac{r+1}{2})(\frac{r+3}{2})$, *according as r is even or odd, respectively.*

Proof If $V \downarrow A$ is multiplicity-free, then $V \downarrow A$ is a direct summand of the module $r + (r-2) + (r-4) + \cdots$. The assertion follows by taking dimensions. ∎

Table 2: $V \downarrow A$ multiplicity-free, $u \in G$ distinguished but not regular

G	λ	class of u in G
B_n, C_n, D_n	ω_1	any
$D_n\,(5 \le n \le 7)$	ω_n	regular in $B_{n-2}B_1$
F_4	ω_4	$F_4(a_1)$
E_6	ω_1	$E_6(a_1)$
E_7	ω_7	$E_7(a_1)$ or $E_7(a_2)$
E_8	ω_8	$E_8(a_1)$

Lemma 2.2 *Assume $V \downarrow A$ is multiplicity-free.*

(i) If $c \ge 1$ then the T-weight $r - 2c$ occurs with multiplicity at most one more than the multiplicity of T-weight $r - 2(c-1)$.

(ii) For $c \ge 1$, the T-weight $r - 2c$ occurs with multiplicity at most $c + 1$.

(iii) If T-weight $r-2$ occurs with multiplicity 1 (e.g. if all labels are 2 and $\lambda = b\omega_i$) and if $c \ge 1$, then T-weight $r - 2c$ occurs with multiplicity at most c.

Proof Suppose i is maximal with $r - 2c$ in the weight string $r_i, \cdots, -r_i$. Then T-weight $r - 2c$ occurs with the same multiplicity as does T-weight r_i. And weight r_i occurs with multiplicity at most one more than weight r_{i-1} as otherwise there would be two direct summands of highest weight r_i. Now (i) follows as does (ii). Part (iii) also follows, since the assumption rules out a summand of highest weight $r - 2$. ∎

Lemma 2.3 *Assume $V \downarrow A$ is multiplicity-free and that $\lambda = b\omega_i$ with $b > 1$.*

(i) Then α_i is an end-node of the Dynkin diagram.

(ii) If G has rank at least 3, then the node adjacent to α_i has label 2.

Proof (i) Suppose that $\alpha_j \ne \alpha_k$ both adjoin α_i in the Dynkin diagram. If both these roots have label 0, then T-weight $r - 2$ is afforded by each of $\lambda - i, \lambda - ij, \lambda - ik, \lambda - ijk$, contradicting 2.2(ii). Next assume α_j has label 2 and α_k has label 0. Here we consider $r - 4$ which is afforded by $\lambda - i^2, \lambda - i^2k, \lambda - i^2k^2, \lambda - ij$, again contradicting 2.2(ii). If both labels are 2, then $r-4$ is afforded by $\lambda - i^2, \lambda - ij, \lambda - ik$. But here $r - 2$ only occurs from $\lambda - \alpha_i$, so this contradicts 2.2(iii).

(ii) Assume G has rank at least 3. By (i) α_i is an end-node. Let α_j be the adjoining node. We must show α_j has label 2. Suppose the label is 0 and let α_k be another node adjoining α_j. If α_k has label 0, then $r - 2$ is afforded by each of $\lambda - i, \lambda - ij, \lambda - ijk$, a contradiction. Therefore α_k has label 2. But then $r - 4$ is afforded by each of $\lambda - i^2, \lambda - i^2j, \lambda - i^2j^2, \lambda - ijk$, a contradiction. ∎

The next lemma will be frequently used, often implicitly, in what follows.

Lemma 2.4 *If $c \ge d$ are nonnegative integers, then the tensor product of A_1-modules $c \otimes d = (c + d) \oplus (c + d - 2) \oplus \cdots \oplus (c - d)$.*

Proof This follows from a consideration of weights in the tensor product. ■

Lemma 2.5 *Suppose that $\lambda = \omega_i + \omega_j$ with $j > i$ and that the subdiagram with base $\{\alpha_i, \cdots, \alpha_j\}$ is of type A, or is of rank at most 3, or is of type F_4. Then the T_G-weight $\lambda - i(i+1)\cdots j$ occurs with multiplicity $j - i + 1$.*

Proof Since the weight space lies entirely within the corresponding irreducible for the Levi factor with base $\{\alpha_i, \cdots, \alpha_j\}$, we may assume that G is equal to this Levi factor; that is, $i = 1$ and $j = n$. Then the hypothesis of the lemma implies that G is A_n, B_2, B_3, C_2, C_3, G_2 or F_4. For all but the first case the conclusion follows by computation using Magma.

Now suppose $G = A_n$. Then $\omega_1 \otimes \omega_n = \lambda \oplus 0$. In the tensor product we see precisely $n+1$ times the weight $\lambda - \alpha_1 - \cdots - \alpha_n$ by taking weights of the form $(\omega_1 - 1\cdots j) \otimes (\omega_n - (j+1)\cdots n)$ for $1 \leq j \leq n-1$, together with the weights $\omega_1 \otimes (\omega_n - 1\cdots n)$ and $(\omega_1 - 1\cdots n) \otimes \omega_n$. Each occurs with multiplicity 1, so the conclusion follows, as $\lambda - \alpha_1 - \cdots - \alpha_n = 0$. ■

Lemma 2.6 *Assume that there exist $i < j$ with $c_i \neq 0 \neq c_j$ and that $V \downarrow A$ is multiplicity-free.*

(i) Then $c_k = 0$ for $k \neq i, j$.

(ii) Nodes adjoining α_i and α_j have label 2.

(iii) Either $c_i = 1$ or $c_j = 1$. Moreover $c_i = c_j = 1$ unless α_i and α_j are adjacent.

(iv) Either α_i or α_j is an end-node.

(v) If either $c_i > 1$ or $c_j > 1$, then G has rank 2.

(vi) If α_i, α_j are non-adjacent and if all nodes have label 2, then both α_i and α_j are end-nodes.

Proof (i) This is immediate, as otherwise $\lambda - i, \lambda - j, \lambda - k$ all afford T-weight $r-2$, contradicting 2.2(ii).

(ii) Suppose (ii) is false. By symmetry we can assume α_k adjoins α_i and has label 0. Then $\lambda - i, \lambda - j, \lambda - ik$ all afford $r-2$, a contradiction.

(iii) By (ii), nodes adjacent to α_i and α_j have label 2. Consider T-weight $r-4$ which has multiplicity at most 3 by 2.2. Suppose $c_k > 1$ for $k = i$ or j. Then $\lambda - k^2$ and $\lambda - ij$ both afford weight $r-4$. Assume α_i and α_j are not adjacent. We give the argument when the diagram has no triality node. The other cases require only a slight change of notation. With this assumption we also get $r-4$ from $\lambda - i(i+1)$ and $\lambda - (j-1)j$, a contradiction. So $c_k > 1$ implies that α_i, α_j are adjacent. If both $c_i > 1$ and $c_j > 1$, then we again have a contradiction, since $r-4$ is afforded by $\lambda - i^2, \lambda - j^2$ and $\lambda - ij$, and the latter appears with multiplicity 2 by [8, 1.35].

(iv) Suppose neither α_i nor α_j is an end-node. We give details assuming there is no triality node. The remaining cases just require a slight change of notation. Consider weight $r-4$. This is afforded by $\lambda - ij$, $\lambda - (i-1)i$ and $\lambda - j(j+1)$. If $c_i > 1$ then $\lambda - i^2$ also affords $r-4$. This forces $c_i = 1$, and similarly $c_j = 1$. If $j = i+1$, then $\lambda - ij$ has multiplicity 2 by 2.5, again a contradiction. And if

$j > i+1$, then $\lambda - i(i+1)$ and $\lambda - (j-1)j$ afford weight $r-4$. In either case $r-4$ appears with multiplicity at least 4, contradicting 2.2.

(v) Suppose $c_k > 1$ for $k = i$ or j. By (iv) we can assume α_i is an end-node. If G has rank at least 3, let α_l adjoin α_j, where $l \neq i$. Then (ii) implies that $r-4$ is afforded by $\lambda - ij, \lambda - k^2, \lambda - jl$. If α_j is adjacent to α_i then the first weight occurs with multiplicity 2 by [8, 1.35]. Otherwise there is another node α_m adjacent to α_i and $\lambda - im$ affords $r-4$. In either case we contradict 2.2.

(vi) As above we treat the case where the Dynkin diagram has no triality node. By (iv) and symmetry we can assume α_i is an end-node. Suppose $j < n$. Then $r-4$ is afforded by each of $\lambda - i(i+1), \lambda - (j-1)j, \lambda - j(j+1), \lambda - ij$, contradicting 2.2. Therefore, $j = n$. ∎

Lemma 2.7 *Suppose $\lambda = \omega_i$ and the Dynkin diagram has a string $\alpha_{i-3}, \ldots, \alpha_{i+3}$ for which each node has T-label 2. Then $r-8$ occurs with multiplicity at least 5. In particular $V \downarrow A$ is not multiplicity-free.*

Proof The T-weight $r-8$ arises from each of the following weights:

$$\lambda - i(i+1)(i+2)(i+3), \lambda - (i-1)i(i+1)(i+2), \lambda - (i-2)(i-1)i(i+1),$$
$$\lambda - (i-3)(i-2)(i-1)i, \lambda - (i-1)i^2(i+1)$$

(the last is a weight as it is equal to $(\lambda - (i-1)i(i+1))^{s_i}$). This proves the first assertion and the second assertion follows from 2.2(iii). ∎

The final lemma is an inductive tool. Let L be a Levi subgroup of G in our fixed system of roots, and let μ be the corresponding highest weight of L'. Namely, $\mu = \sum c_j\omega_j$, where the sum runs just over those fundamental weights corresponding to simple roots in the subsystem determined by L.

Lemma 2.8 *Fix $c \geq 1$ and and let s denote the sum of the dimensions of all weight spaces of $V_{L'}(\mu)$ for all weights of form $\mu - \sum d_j\alpha_j$ such that $\sum d_j = c$ and each α_j with nonzero coefficient has label 2.*

(i) If $s > c+1$, then $V \downarrow A$ is not multiplicity-free.

(ii) If T-weight $r-2$ occurs with multiplicity 1 (e.g. if all labels are 2 and $\lambda = b\omega_i$) and $s > c$, then $V \downarrow A$ is not multiplicity-free.

Proof This is immediate from 2.2, since $T \leq L$ and the weight $\mu - \sum d_j\alpha_j$ corresponds to a weight $\lambda - \sum d_j\alpha_j$ which affords T-weight $r-2c$. ∎

3 The case where A is regular and $\lambda \neq c\omega_i$

As in the hypothesis of Theorem 1, let G be a simple algebraic group of rank at least 2, let $A \cong A_1$ be a G-irreducible subgroup, and let $V = V_G(\lambda)$, where $\lambda = \sum c_i\lambda_i$. This section and the next two concern the case of Theorem 1 where A is a regular A_1 of G (recall that this means that unipotent elements of A are regular in G). In this case all the T-labels of the Dynkin diagram of G are equal to 2. In this section we handle situations where $c_i > 0$ for at least two values of i.

If $V \downarrow A$ is multiplicity-free, $\lambda \neq c\omega_i$ and G has rank at least 3, then Lemma 2.6 implies that $\lambda = \omega_i + \omega_j$, where either α_i, α_j are both end-nodes, or one is an end-node and the other is adjacent to it.

Proposition 3.1 *Assume $V \downarrow A$ is multiplicity-free. Then there exist at least two values of i for which $c_i > 0$ if and only G and λ are in the following table, up to graph automorphisms.*

G	λ
A_2	$c1$
A_3	110
B_2, C_2	$11, 12, 21$
G_2	11
B_3	101
A_n	$10\cdots01$

The proof will be in a series of lemmas.

Lemma 3.2 *Suppose $G = A_2$ and $\lambda = c1$ for $c \geq 1$. Then $V \downarrow A$ is multiplicity-free.*

Proof Assume $G = A_2$. The weight $c1 - \alpha_1 - \alpha_2 = (c-1)0$ occurs with multiplicity 2 in the module $c1$ and multiplicity 3 in $c0 \otimes 01$. A dimension comparison shows that $c0 \otimes 01 = c1 + (c-1)0$.

Now $c0 = S^c(10)$, so weight considerations show that for c even, $S^c(10) \downarrow A = 2c \oplus (2c-4) \oplus (2c-8) \oplus \cdots \oplus 0$ and $S^{c-1}(10) = (2c-2) \oplus (2c-6) \oplus \cdots \oplus 2$. Therefore 2.4 implies that

$$(c0 \otimes 01) \downarrow A = ((2c+2)+2c+(2c-2))+((2c-2)+(2c-4)+(2c-6))+\cdots+(6+4+2)+2,$$

and it follows from the first paragraph that $V \downarrow A$ is multiplicity free. A similar argument applies for c odd. ■

Lemma 3.3 *(i) If $G = C_2$ and $V = V_G(\lambda)$ with $\lambda = c1$ or $1c$ for $c \geq 1$, then $V \downarrow A$ is multiplicity-free if and only if $\lambda = 11, 21$, or 12.*

(ii) If $G = G_2$ and $V = V_G(\lambda)$ with $\lambda = c1$ or $1c$ for $c \geq 1$, then $V \downarrow A$ is multiplicity-free if and only if $\lambda = 11$.

Proof (i) Let $G = C_2$. We first settle the cases which are multiplicity-free. A Magma computation shows that $10 \otimes 01 = 11 + 10$, and hence $11 \downarrow A = 7 + 5 + 1$, which is multiplicity-free. Next consider $\lambda = 12$. First note that $10 \otimes 02 = 12 + 11$ and $02 = S^2(01) - 00$. It follows that $12 \downarrow A = 3 \otimes (S^2(4) - 0) - (7 + 5 + 1) = 3 \otimes (8 + 4) - (7 + 5 + 1) = (11 + 9 + 7 + 5) + (7 + 5 + 3 + 1) - (7 + 5 + 1) = 11 + 9 + 7 + 5 + 3$ and $V \downarrow A$ is multiplicity-free. Finally, consider $\lambda = 21$. In this case $20 \otimes 01 = 21 + 20 + 01$. Now $20 \downarrow A = S^2(3) = 6 + 2$, so that $(20 \otimes 01) \downarrow A = (6+2) \otimes 4 = (10+8+6+4+2)+(6+4+2)$. It follows that $21 \downarrow A = 10+8+6+4+2$ and $V \downarrow A$ is multiplicity-free.

If $\lambda = 1b$ or $b1$ for $b \geq 3$, then $r = 3 + 4b$ or $3b + 4$, and $\dim V = \frac{1}{3}(b+1)(b+3)(2b+4)$ or $\frac{1}{3}(b+1)(b+3)(b+5)$, respectively. Now Lemma 2.1 shows that $V \downarrow A$ cannot be multiplicity-free.

(ii) Let $G = G_2$. First consider $\lambda = 11$. A Magma computation yields $10 \otimes 01 = 11 + 20 + 10$. Also, $10 \downarrow A = 6$ and $01 \downarrow A = 10 + 2$. Using the fact that

$S^2(10) = 20+00$, we find that $V \downarrow A = 16+14+10+8+6+4$, which is multiplicity-free.

Now consider $\lambda = c1$ or $1c$ with $c > 1$. Then $r = 6c+10$ or $10c+6$ and $\dim V = \frac{1}{60}(c+1)(c+3)(c+5)(c+7)(2c+8)$ or $\frac{1}{60}(c+1)(c+3)(2c+4)(3c+5)(3c+7)$, respectively. In either case, 2.1 shows that $V \downarrow A$ is not multiplicity-free. ■

Lemma 3.4 *Suppose G has rank at least 3 and $\lambda = \omega_i + \omega_j$, where α_i, α_j are adjacent and one of them is an end-node. Then $V \downarrow A$ is multiplicity-free if and only if $G = A_3$.*

Proof First assume that $G = A_n, B_n, C_n$ or D_n and $\lambda = \omega_1 + \omega_2$. If $n \geq 4$, then the weights $\lambda - 123 = (\lambda - 12)^{s_3}, \lambda - 234, \lambda - 1^2 2 = (\lambda - 2)^{s_1}, \lambda - 12^2 = (\lambda - 1)^{s_2}$ occur with multiplicities 2,1,1,1 and all afford T weight $r-6$. Hence this weight occurs with multiplicity at least 5, and 2.2 shows that $V \downarrow A$ is not multiplicity-free. If $G = B_3$ or C_3, then of the above weights only $\lambda - 234$ does not occur; however the weight $\lambda - 23^2 = (\lambda - 2)^{s_3}$ or $\lambda - 2^2 3 = (\lambda - 23)^{s_2}$ occurs, respectively, affording T weight $r-6$, which again gives the conclusion by 2.2. And if $G = A_3$, then $100 \otimes 010 = 110 + 001$, and restricting to A we have $3 \otimes (4+0) = (7+5+3+1)+3$. Therefore, $110 \downarrow A = 7+5+3+1$ which is multiplicity-free, as in the conclusion.

Next consider $G = B_n$ or C_n with $\lambda = \omega_{n-1} + \omega_n$. For B_n, the weight $r-6$ is afforded by $\lambda - (n-2)(n-1)n$, $\lambda - (n-1)n^2 = (\lambda - (n-1)n)^{s_n}$ and $(\lambda - (n-1)^2 n) = (\lambda - n)^{s_{n-1}}$. Moreover the first two weights occur with multiplicity 2, and so $r-6$ appears with multiplicity 5, so that $V \downarrow A$ is not multiplicity-free. A similar argument applies for C_n.

For $G = F_4$, the conclusion follows by using Lemma 2.8, applied to a Levi subgroup B_3 or C_3. Likewise, for $D_n\,(n \geq 5)$ with $\lambda = \omega_n + \omega_{n-2}$ or $\omega_{n-1} + \omega_{n-2}$, or for $G = E_n$, we use a Levi subgroup A_r with $r \geq 4$. Finally, for D_4 the result follows from the first paragraph using a triality automorphism. ■

Lemma 3.5 *Assume $n \geq 3$ and $G = A_n, B_n, C_n$, or D_n and $\lambda = \omega_i + \omega_j$, where α_i, α_j are end-nodes. Then $V \downarrow A$ is multiplicity-free if and only if $\lambda = \omega_1 + \omega_n$ and $G = A_n$ or B_3.*

Proof First consider $G = A_n, B_n, C_n$. By 2.6(vi) we have $\lambda = \omega_1 + \omega_n$. If $G = B_n$ with $n \geq 4$, then $\lambda - 123, \lambda - (n-2)(n-1)n, \lambda - 1(n-1)n, \lambda - 12n$ and $\lambda - (n-1)n^2 = (\lambda - (n-1)n)^{s_n}$ all restrict to $r-6$ on T, so $V \downarrow A$ is not multiplicity-free by 2.2. We argue similarly for $G = C_n$ with $n \geq 4$, replacing the last weight by $\lambda - (n-1)^2 n = (\lambda - (n-1)n)^{s_{n-1}}$. And if $G = A_n$, then $V \downarrow A$ is just $(n \otimes n) - 0$ and hence is multiplicity-free.

Now suppose $n = 3$ and $\lambda = 101$. If $G = B_3$, then Magma gives $100 \otimes 001 = 101 + 001$. Restricting to A the left side is $6 \otimes (6+0)$ and we find that $101 \downarrow A = 12 + 10+8+6+4+2$, multiplicity-free. For $G = C_3$, Magma yields $100 \otimes 001 = 101 + 010$, $\wedge^2(100) = 010 + 000$ and $\wedge^3(100) = 001 + 100$. Restricting to A and considering weights we have $101 \downarrow A = 14+12+10+8+6^2+4+2$ which is not multiplicity-free.

Finally, consider $G = D_n$ with $n \geq 4$. First consider $\lambda = \omega_1 + \omega_{n-1}$. The T-weight $r - 2(n-1)$ is afforded by $\lambda - 1 \cdots (n-1), \lambda - 2 \cdots n, \lambda - 1 \cdots (n-2)n$, which, using 2.5, occur with multiplicities $n-1, 1, 1$ respectively, giving the conclusion by 2.2. A similar argument applies if $\lambda = \omega_1 + \omega_n$. Finally assume $\lambda = \omega_{n-1} + \omega_n$. Here, T-weight $r-6$ is afforded by $\lambda - (n-2)(n-1)n, \lambda - (n-3)(n-2)(n-1), \lambda - (n-3)(n-2)n$ with multiplicities $3, 1, 1$ so again 2.2 applies. ■

Lemma 3.6 *Assume $G = E_6, E_7, E_8$ or F_4 and $\lambda = \omega_i + \omega_j$, where α_i, α_j are end-nodes. Then $V \downarrow A$ is not multiplicity-free.*

Proof First assume $G = F_4$. Then $\lambda = 1001$ and we consider T-weight $r - 8$ which is afforded by weights $\lambda - 1234, \lambda - 123^2 = (\lambda - 12)^{s_3}, \lambda - 23^24 = (\lambda - 234)^{s_3}$, occurring with multiplicities 4,1,1, respectively, giving the result by 2.2.

So now assume $G = E_n$. If $\lambda = \omega_1 + \omega_n$ then the weights $\lambda - 134 \cdots n, \lambda - 1234 \cdots (n-1), \lambda - 23 \cdots n$ all afford T-weight $r - 2(n-1)$ and (by 2.5) occur with multiplicities $n-1, 1, 1$ respectively, and now we apply 2.2. If $\lambda = \omega_1 + \omega_2$, we argue similarly using weights $\lambda - 1234, \lambda - 1345, \lambda - 2345$. And if $\lambda = \omega_2 + \omega_n$, use weights $\lambda - 245 \cdots n, \lambda - 345 \cdots n, \lambda - 23 \cdots (n-1)$. ∎

This completes the proof of Proposition 3.1.

4 The case where A is regular and $\lambda = b\omega_i$, $b \geq 2$

Continue to assume that G is a simple algebraic group, A is a regular A_1 in G, and $V = V_G(\lambda)$. In this section we prove Theorem 1 in the case where $\lambda = b\omega_i$ for some i and some $b \geq 2$. In this case, the T-weight $r - 2$ appears in V with multiplicity 1 and 2.2(iii) applies. Also 2.3 implies that if $V \downarrow A$ is multiplicity-free then α_i is an end-node.

Proposition 4.1 *Assume $\lambda = b\omega_i$ with $b > 1$. Then $V \downarrow A$ is multiplicity-free if and only if G and λ are as in the following table, up up to graph automorphisms of A_n or D_4.*

λ	G
$2\omega_1$	A_n, B_n, C_n, D_n $(n = 2k)$, G_2
$3\omega_1$	A_n $(n \leq 5)$, B_n $(n = 2, 3)$, $C_n (n = 2, 3)$, G_2
$4\omega_1, 5\omega_1$	A_n $(n = 2, 3)$, B_2, C_2
$b\omega_1$ $(b \geq 6)$	A_2
$b\omega_1$ $(b \leq 5)$	C_2
$2\omega_3$	B_3
$2\omega_2$	G_2

The proof is carried out in a series of lemmas.

Lemma 4.2 *Assume that $\lambda = 2\omega_1$. If $G = A_n, B_n$, or C_n, then $V \downarrow A$ is multiplicity-free. If $G = D_n$, then $V \downarrow A$ is multiplicity-free if and only if n is even.*

Proof If $G = A_n$, then $V \downarrow A$ is just $S^2(n)$ and a consideration of weights shows that this is $2n + (2n-4) + (2n-8) + \cdots$, hence is multiplicity-free. If $G = B_n$ or C_n we can embed G in A_{2n} or A_{2n-1}, respectively. In each case A acts irreducibly on the natural module with highest weight $2n$ or $2n-1$, respectively, and the conclusion follows from the first sentence.

Now consider $G = D_n$. In this case A acts on the natural module ω_1 for G, as $(2n-2) + 0$. Now $S^2(\omega_1) = V + 0$ and hence $V \downarrow A = S^2(2n-2) + (2n-2) = ((4n-4) + (4n-8) + \cdots) + (2n-2)$. If n is odd, we find that $2n-2$ appears with multiplicity 2, while if n is even, $V \downarrow A$ is multiplicity-free. ∎

Lemma 4.3 *Assume that $G = B_n\,(n \geq 3)$, $C_n\,(n \geq 3)$ or $D_n\,(n \geq 4)$ and that $\lambda = b\omega_i$ with $b > 1$ and $i > 1$. Then $V \downarrow A$ is multiplicity-free if and only if $G = B_3$ and $\lambda = 2\omega_3$ or $G = D_4$ and $\lambda = 2\omega_i$ for $i = 3$ or 4.*

Proof By 2.3 we can assume that α_i is an end-node, so we may take $i = n$. First consider C_n. If $b \geq 3$, then the weight $r-6$ occurs with multiplicity at least 4 (from $\lambda-(n-2)(n-1)n, \lambda-(n-1)n^2, \lambda-n^3, \lambda-(n-1)^2n = (\lambda-n)^{s_{n-1}}$) and so $V \downarrow A$ is not multiplicity-free. For $b = 2$ first consider $G = C_3$. We have $S^2(001) = V+200$. As $001 \downarrow A = 9+3$, it follows that $V \downarrow A$ contains $6^2 (= (r-12)^2)$. Next suppose that $G = C_n$ with $n \geq 4$ and $b = 2$. This case essentially follows from the C_3 result. We need only show that there are at least two more weights $r-12$ than weights $r-10$. For $n = 4$ the only weights $r-10$ that do not arise from the C_3 Levi, are $\lambda-123^24, \lambda-1234^2$. Correspondingly there are new $r-12$ weights, $\lambda-12^23^24, \lambda-123^24^2$. Similar reasoning applies for C_5, where $\lambda-12345$ is the only weight $r-10$ not appearing for C_4 and we conjugate by s_4 to get a new weight $r-12$. And for $n \geq 6$ there are no $r-10$ weights that were not present in a C_5 Levi factor.

Now let $G = B_n$. If $b \geq 3$ we find that T weight $r-6$ appears with multiplicity at least 4. Indeed, for the B_2 Levi the module $0b = S^b(01)$ and this yields weights $\lambda-n^3$, $\lambda-(n-1)n^2$, the latter with multiplicity 2. Also $\lambda-(n-2)(n-1)n$ affords T-weight $r-6$, which yields the assertion.

Now assume $b = 2$. First consider $G = B_3$, so that $\lambda = 002$. The module 001 for B_3 is the spin module where A acts as $6+0$. We have $S^2(001) = 002+000$, and it follows that $V \downarrow A = 12+8+6+4+0$, which is multiplicity-free. Now assume $n > 3$. Here we show that T-weight $r-8$ occurs with multiplicity 5. The above shows that $r-8$ occurs with multiplicity 4 just working in the B_3 Levi. As $\lambda-(n-3)(n-2)(n-1)n$ affords $r-8$ the assertion follows.

Finally, consider $G = D_n$. If $b \geq 3$ then T-weight $r-6$ occurs with multiplicity 4 (from $\lambda-n^3, \lambda-(n-2)n^2, \lambda-(n-1)(n-2)n, \lambda-(n-3)(n-2)(n)$), and so $V \downarrow A$ is not multiplicity-free by 2.2(iii). Now assume $b = 2$. Applying a graph automorphism if necessary, we can assume $n \geq 5$ (the conclusion allows for D_4 using 4.2). Then T-weight $r-8$ occurs with multiplicity at least 5 (from $\lambda-(n-4)(n-3)(n-2)n, \lambda-(n-3)(n-2)(n-1)n, \lambda-(n-3)(n-2)n^2, \lambda-(n-1)(n-2)n^2, \lambda-(n-2)^2n^2$). Therefore $V \downarrow A$ is not multiplicity-free. ■

Lemma 4.4 *Assume that $G = A_n$, $B_n\,(n \geq 3)$, $C_n\,(n \geq 3)$ or $D_n\,(n \geq 4)$, and that $\lambda = b\omega_1$ with $b \geq 3$. Then $V \downarrow A$ is multiplicity-free only for the cases listed in rows $2-4$ of the table in Proposition 4.1.*

Proof First let $G = A_n$, so $V = V_G(b\omega_1) = S^b(\omega_1)$. First consider $b = 3$, so that $r = 3n$. If $n \geq 6$, then T-weight $3n-12$ occurs with multiplicity at least 7 and $V \downarrow A$ cannot be multiplicity-free. Indeed, independent vectors of weight $3n-12$ occur as tensor symmetric powers of vectors of weights (i,j,k), where (i,j,k) is one of $(n,n,n-12)$, $(n,n-2,n-10)$, $(n,n-4,n-8)$, $(n,n-6,n-6)$, $(n-2,n-2,n-8)$, $(n-2,n-4,n-6)$, $(n-4,n-4,n-4)$. On the other hand for $n \leq 5$ the restriction is multiplicity-free.

Next consider $b = 4$, so that $r = 4n$. If $n \geq 4$, then $4n-8$ appears with multiplicity at least 5 and hence $V \downarrow A$ is not multiplicity-free. Indeed, independent vectors arise from symmetric powers of vectors of weights $(n,n,n,n-8)$, $(n,n,n-$

$2, n-6), (n, n, n-4, n-4), (n, n-2, n-2, n-4), (n-2, n-2, n-2, n-2)$. And for $n \leq 3$ a direct check shows that $S^b(\omega_1) \downarrow A$ is multiplicity-free. If $b \geq 5$, $n \geq 3$ and $(b, n) \neq (5, 3)$ then a similar argument shows that weight $bn - 12$ occurs with multiplicity at least two more than does $bn - 10$; hence $V \downarrow A$ is not multiplicity-free in these cases. And if $(b, n) = (5, 3)$ one checks that $V \downarrow A = S^5(3) = 15 + 11 + 9 + 7 + 5 + 3$, which is multiplicity-free.

The final case for $G = A_n$ is when $n = 2$. We first note that the multiplicity of weight $2j$ in $S^b(2)$ is precisely the multiplicity of weight 0 in $S^{b-j}(2)$. Indeed, if we write $2^c 0^d (-2)^e$ to denote a symmetric tensor of c vectors of weight 2, d vectors of weight 0 and e vectors of weight -2, then a basis for the $2j$-weight space is given by vectors $2^j 0^{b-j}(-2)^0, 2^{j+1} 0^{b-j-2}(-2)^1, 2^{j+2} 0^{b-j-4}(-2)^2, \cdots$ and ignoring the first j terms in each tensor we obtain the assertion. The multiplicity of weight 0 in $S^{b-j}(2)$ is easily seen to be $\frac{b-j+1}{2}$ if $b - j$ is odd and $\frac{b-j+2}{2}$ if $b - j$ is even. From this information we see that $S^b(2) = 2b + (2b - 4) + (2b - 8) + \cdots$ and hence $V \downarrow A$ is multiplicity-free.

Now consider $G = B_n, C_n$, or D_n. The C_n case follows from the A_{2n-1} case since $V = S^b(\omega_1)$ (see [6]). If $G = D_n$ with $n \geq 4$, then $A \leq B_{n-1} < G$. If the corresponding module for this subgroup is not multiplicity-free, then the same holds for G since it appears as a direct summand of V.

So assume $G = B_n$. If $b \geq 4$, then T-weight $r - 8$ occurs with multiplicity at least 4. Indeed, if $n \geq 4$ this weight arises from $\lambda - 1234, \lambda - 1^2 23, \lambda - 1^2 2^2, \lambda - 1^3 2, \lambda - 1^4$, whereas if $n = 3$ replace the first of these weights by $\lambda - 123^2 = (\lambda - 12)^{s_3}$. Now consider $b = 3$. If $n = 4$, then $S^3(\lambda_1) = 3000 + 1000$ and one checks that T-weight $r - 12 = 12$ occurs with multiplicity 7, and so $V \downarrow A$ is not multiplicity-free. And for $n > 4$ we apply Lemma 2.8 to get the same conclusion. Finally, if $n = 3$ then $S^3(\lambda_1) = V + 100$, and a direct check of weights shows that $S^3(\lambda_1) \downarrow A = 18 + 14 + 12 + 10 + 8 + 6^2 + 2$, which implies that $V \downarrow A$ is multiplicity-free.

The only remaining case is when $G = D_4$ and $b = 3$, since here the module $300 \downarrow A$ for B_3 is multiplicity-free. As a module for G we have $S^3(\omega_1) = 3\omega_1 \oplus \omega_1$, so that $V \downarrow A = S^3(6 + 0) - (6 + 0)$, which one easily checks is not multiplicity-free. ■

Lemma 4.5 *Assume that $G = B_2, C_2$ or G_2 and $\lambda = b\omega_i$ (with $b \geq 2$). Then $V \downarrow A$ is multiplicity-free if and only if one of the following holds:*

(i) $G = B_2$ or C_2 and $\lambda = b0$, $0b$ ($b \leq 5$).

(ii) $G = G_2$ and $\lambda = 20, 30$ or 02.

Proof (i) Let $G = B_2$. Then the module $0b = S^b(01)$ which restricts to A as $S^b(3)$. Therefore the assertion follows from the A_3 result which has already been established.

Now assume $\lambda = b0$. Here $\dim(b0) = (b+1)(b+2)(2b+3)/6$ and the highest weight of $V \downarrow A$ is $4b$. If the restriction were multiplicity-free, then weight $4b - 2$ would only occur with multiplicity 1, and the restriction with largest possible dimension would have composition factors $4b + (4b - 4) + (4b - 6) + \cdots + 2 + 0$ which totals $4b^2 + 2$. For $b \geq 7$, this is less than the above dimension of $b0$ and so the restriction cannot be multiplicity-free. And for $b \leq 3$, V is a summand of $S^b(4)$ which we have already seen to be multiplicity-free. This leaves the cases $b = 4, 5, 6$.

A computation gives the following decompositions of symmetric powers of the the G-module 10:

$$\begin{aligned} S^6(10) &= 60+40+20+00,\\ S^5(10) &= 50+30+10,\\ S^4(10) &= 40+20+00,\\ S^3(10) &= 30+10,\\ S^2(10) &= 20+00. \end{aligned}$$

It follows that $40 \downarrow A = 16+12+10+8+4$ and $50 \downarrow A = 20+16+14+12+10+8+4$, so these are both multiplicity-free. Also $S^6(4) = 24+20+18+16^2+14+12^3+\cdots$. This and the above imply that $60 \downarrow A$ is not multiplicity-free. This completes the proof of (i).

(ii) It follows from [6] that $V_{B_3}(b00)$ is irreducible upon restriction to G_2, with highest weight $b0$, and also a regular A in B_3 lies in a subgroup G_2. So for $i=1$ the assertion follows from our results for B_3. Now assume $i=2$. Then

$$\dim(0b) = \frac{1}{120}(b+1)(b+2)(2b+3)(3b+4)(3b+5),$$

and the highest T-weight is $10b$. First let $b=2$. Then $V \downarrow A$ is a direct summand of $S^2(01) \downarrow A = 20+16+12^2+10+8^2+4^2+0^2$. We have $S^2(01) = V \oplus 20 \oplus 00$ and hence $V \downarrow A = 20+16+12+10+8+4+0$, which is multiplicity-free. On the other hand if $b \geq 3$, then 2.1 implies that $V \downarrow A$ is not multiplicity-free. ■

Lemma 4.6 *If $G = E_n$ and $\lambda = b\omega_i$ with $b > 1$, then $V \downarrow A$ is not multiplicity-free.*

Proof By Lemma 2.3, we can take α_i to be an end-node. First assume $i=1$. If $b=2$ one checks that $r-6$ is only afforded by $\lambda-134, \lambda-1^23$, while $r-8$ is afforded by $\lambda-1234, \lambda-1345, \lambda-1^234, \lambda-1^23^2$, so that $V \downarrow A$ is not multiplicity-free by 2.2(ii). Similarly for $b \geq 3$ as T-weight $r-6$ appears with multiplicity 3 (from $\lambda-134, \lambda-1^23, \lambda-1^3$), but $r-8$ appears with multiplicity at least 5 (from $\lambda-1345, \lambda-1234, \lambda-1^234, \lambda-1^22^2, \lambda-1^33$).

If $i=2$, we see that weight $r-8$ appears with multiplicity at least 5, since it is afforded by each of $\lambda-2345, \lambda-1234, \lambda-2456, \lambda-2^234, \lambda-2^245$. So $V \downarrow A$ is not multiplicity-free by 2.2(iii).

Finally, assume that $i=n$. For $n=6$, V is just the dual of $V_G(\lambda_1)$, so suppose $G = E_7$ or E_8. If $b \geq 4$ it is easy to list weights and verify that T-weight $r-8$ appears with multiplicity at least 5, so 2.2(iii) shows that $V \downarrow A$ is not multiplicity-free. And if $b=2$ or 3, we see that T-weight $r-12$ appears with multiplicity at least 2 more than T-weight $r-10$. ■

Lemma 4.7 *If $G = F_4$ and $\lambda = b\omega_i$ with $b > 1$, then $V \downarrow A$ is not multiplicity-free.*

Proof As usual we can take α_i to be an end-node. First assume $i=1$. If $b=2$, then T weight $r-6$ occurs with multiplicity 2 (from $\lambda-123, \lambda-1^22$) whereas $r-8$ occurs with multiplicity 4 (from $\lambda-1234, \lambda-123^2 = (\lambda-12)^{s_3}, \lambda-1^223, \lambda-1^22^2$). If $b \geq 3$, then the weight $r-6$ appears with multiplicity 3 due to the additional weight $\lambda-1^3$. But we also get an additional weight $r-8$ from $\lambda-1^32$. In either case 2.2 implies that $V \downarrow A$ is not multiplicity-free.

Now assume $i=4$. First assume $b=2$. Then $S^2(0001) = V+0001+0000$. Moreover, a consideration of weights shows that $0001 \downarrow A = 16+8$ and we conclude that $V \downarrow A$ is not multiplicity-free as there is a summand 20^2.

Finally, assume $b \geq 3$. The T-weight $r-6$ occurs with multiplicity 3 (from $\lambda - 234, \lambda - 34^2, \lambda - 4^3$), whereas T-weight $r-8$ occurs with multiplicity at least 5 (from $\lambda - 1234, \lambda - 23^24 = (l-234)^{s_3}, \lambda - 234^2, \lambda - 3^24^2, \lambda - 34^3$). ■

This completes the proof of Proposition 4.1.

5 The case where A is regular and $\lambda = \omega_i$

Continue to assume that G is a simple algebraic group, A is a regular A_1 in G, and $V = V_G(\lambda)$. In this section we prove Theorem 1 in the case where $\lambda = b\omega_i$ for some i.

Proposition 5.1 *Assume that $\lambda = \omega_i$ for some i. Then $V \downarrow A$ is multiplicity-free if and only if G and λ are as in the following table, up to graph automorphisms.*

λ	G
ω_1, ω_2	A_n, B_n, C_n, D_n $(n = 2k+1)$, G_2
ω_3	A_n $(n \leq 7)$, C_n $(n \leq 5)$
ω_n	C_4, C_5
ω_n	B_n $(n \leq 8)$, D_n $(n \leq 9)$
ω_1, ω_2	$G = E_6$
ω_1, ω_7	E_7
ω_8	E_8
ω_1, ω_4	F_4

The proof is carried out in a series of lemmas.

Lemma 5.2 *Assume that $\lambda = \omega_i$.*

(i) Then $V \downarrow A$ is not multiplicity-free if $G = A_n$, B_n, C_n or D_n and $4 \leq i \leq n-3$.

(ii) If $i = 3$ and $G = A_n$ with $n \geq 5$, then $V \downarrow A$ is multiplicity-free if and only if $n \leq 7$.

(iii) If $G = A_n, B_n, C_n, D_n$ or G_2 and $i = 1$ or 2, then $V \downarrow A$ is multiplicity-free except when $G = D_n$, $i = 2$, and n even.

Proof (i) This follows from 2.7.

(ii) Assume $i = 3$ and $G = A_n$ with $n \geq 5$. Then $V = \wedge^3(\omega_1)$ and a computation using Magma shows that $V \downarrow A$ is multiplicity-free for $n = 5, 6, 7$. If $n \geq 8$ one checks that T-weight $r - 12$ occurs with multiplicity at least 7. Indeed, here $r = 3n-6$, and $r - 12 = 3n - 18$ is afforded by the wedge of tensors of weight vectors for each of the following weights: $n(n-2)(n-16), n(n-4)(n-14), n(n-6)(n-12), n(n-8)(n-10), (n-2)(n-4)(n-12), (n-2)(n-6)(n-10), (n-4)(n-6)(n-8)$. Hence $V \downarrow A$ is not multiplicity-free for $n \geq 8$ by 2.2(iii).

(iii) If $G = A_n$ then A is irreducible on the natural module (i.e. ω_1) for G with highest weight n. And if $i = 2$, then $V \downarrow A = \wedge^2(n)$ is a direct summand of $n \otimes n = 2n + (2n-2) + (2n-4) + \cdots + 0$, and hence $V \downarrow A$ is multiplicity-free. Now consider $G = B_n, C_n, D_n$ embedded in $X = A_{2n}, A_{2n-1}, A_{2n-1}$. In the first two cases A acts irreducibly on the natural module, $V_X(\omega_1)$, and in the third case A acts as $(2n-2)+0$. So $V \downarrow A$ is obviously multiplicity-free for $i=1$. Now consider $i=2$. Then $V_X(\omega_2) \downarrow G = V$ if $G = B_n$ or D_n ([6]) and equals $V+0$ if $G = C_n$ (the fixed space corresponds to a fixed alternating form). Therefore $V \downarrow A = \wedge^2(2n), \wedge^2((2n-2)+0)$ or $\wedge^2(2n-1)-0$, respectively. So $V \downarrow A$ is multiplicity-free if $G = B_n$ or C_n. But if $G = D_n$, then $V \downarrow A = \wedge^2((2n-2)+0) = (2n-2)+(4n-6)+(4n-10)+\cdots$ and this is multiplicity-free only if n is odd. Finally consider $G = G_2$ viewed as a subgroup of A_6. Then A is irreducible on the natural 7-dimensional module $V_G(\omega_1)$. Also $V_G(\omega_2)$ is a direct summand of $\wedge^2(V_G(\omega_1))$. So $V \downarrow A$ is multiplicity-free in both cases. ■

Lemma 5.3 *Suppose that $G = B_n$, C_n or D_n, that $\lambda = \omega_i$ for $i \geq 3$ and that V is not a spin module for B_n or D_n. Then $V \downarrow A$ is multiplicity-free if and only if one of the following holds:*

(i) $i = n$ and $G = C_4$ or C_5.

(ii) $i = 3$ and $G = C_n$ for $n = 3, 4, 5$.

Proof If $G = B_n$ or D_n, then $V = \wedge^i(\omega_1)$ and the result follows from the A_{2n} or A_{2n-1} part of 5.2. Indeed, if $G = B_n$, then A is regular in A_{2n} while if $G = D_n$, $A < B_{n-1} < D_n$. Therefore we may assume that $G = C_n$. If $4 \leq i \leq n-3$ then $V \downarrow A$ is not multiplicity-free by 5.2.

Suppose $i \geq 4$. By the previous paragraph we can assume that $i > n-3$. If $i = n-2$, then T-weight $r-8$ occurs with multiplicity at least 5 as it is afforded by $\lambda - (i-3)(i-2)(i-1)i$, $\lambda - (i-2)(i-1)i(i+1)$, $\lambda - (i-1)i(i+1)(i+2)$, $\lambda - (i-1)i^2(i+1)$, $\lambda - i(i+1)^2(i+2) = (\lambda - i(i+1)(i+2))^{s_{i+1}}$, so $V \downarrow A$ is not multiplicity-free by 2.2(iii).

Next assume $i = n-1$. First consider $n = 5$, where $\wedge^4(\omega_1) = \omega_4 + \omega_2 + 0$. Here $r = 24$ and a computation shows that $r - 12 = 12$ occurs with multiplicity 9 in $\wedge^4(\omega_1)$ but it only occurs twice in $\wedge^2(\omega_1) = \omega_2 + 0$. Therefore this weight occurs with multiplicity 7 in V and hence $V \downarrow A$ is not multiplicity-free by 2.2(iii). Now return to the general case with $i = n-1$. Then an application of 2.8(ii) to a C_5 Levi subgroup shows that T-weight $r-12$ appears with multiplicity at least 7, against 2.2.

A similar argument settles the case where $n = i$. If $n = 4$ or 5, then a Magma computation shows that $V \downarrow A$ is multiplicity-free. If $n = 6$, weights $24 = r-12$ and $26 = r-10$ occur with multiplicities 6 and 4 respectively, and so 2.2(i) implies that $V \downarrow A$ is not multiplicity-free. For $n > 6$ we also compare weights $r-10$ and $r-12$. These must already be weights of the C_6 Levi subgroups, so again this contradicts 2.2(i).

Now assume $i = 3$ with $G = C_n$. Then $\wedge^3(\omega_1) = V + \omega_1$. Also A is irreducible on the natural module for A_{2n-1}. In the proof of 5.2(ii) we saw that for $n \geq 5$ the weight $r - 12 = 6n - 21$ occurs in $\wedge^3(\omega_1)$ with multiplicity at least 7. If $n \geq 6$, then all these weights occur within V, so $V \downarrow A$ is not multiplicity-free. This leaves $n = 3, 4, 5$. In these cases a simple check of weights shows that $V \downarrow A$ is multiplicity-free. ■

Lemma 5.4 *Assume V is a spin module for B_n or D_n. Then $V \downarrow A$ is multiplicity-free if and only if $n \leq 8$ for B_n and $n \leq 9$ for D_n.*

Proof If $G = D_n$, then $A \leq B_{n-1} < G$ and B_{n-1} is irreducible on V, so it will suffice to settle the $G = B_n$ case. In terms of roots, $\omega_n = \sum(i\alpha_i)/2$, so that $r = n(n+1)/2$. As $\dim(V) = 2^n$, Lemma 2.1 shows that $V \downarrow A$ is not multiplicity-free if $n \geq 10$. If $n = 9$ then $\dim V = 2^9 = 512$ while the sum in 2.1 is 552. However, $V \downarrow A$ does not contain a summand of highest weight $r - 2 = 43$, so $\dim V \leq 552 - 44 = 508$. So here too $V \downarrow A$ fails to be multiplicity-free. This leaves the case $n \leq 8$.

Consider the restriction $V \downarrow L$, where $L = GL_n$ is a Levi subgroup. One checks (see [5, 11.15]) that the restriction to SL_n consists of the natural module and all its wedge powers together with two trivial modules. For example, when $n = 8$ the restriction to A of the weights $\lambda, \lambda - 8, \lambda - 78^2 = (\lambda - 8)^{s_7 s_8}, \lambda - 67^2 8^3 = (\lambda - 78^2)^{s_6 s_7 s_8}, \cdots$ afford the modules $0, \omega_7, \omega_6, \omega_5, \cdots$ for the A_7 factor. However, the T-weights are shifted in accordance with the the number of fundamental roots subtracted. In the above example, the T-weight of 0 is just that of λ, namely 36 and the T-weights of ω_7 are $34, 32, \cdots, 20$, etc.

Carrying out the above we obtain the conclusion. We indicate below some of the decompositions for $V \downarrow A$ as they will be needed later.

$n = 8:$ $36 + 30 + 26 + 24 + 22 + 20 + 18 + 16 + 14 + 12 + 10 + 8 + 6 + 0$
$n = 7:$ $28 + 22 + 18 + 16 + 14 + 10 + 8 + 4$
$n = 6:$ $21 + 15 + 11 + 9 + 3$
$n = 5:$ $15 + 9 + 5$
$n = 4:$ $10 + 4$
$n = 3:$ $6 + 0.$

■

Lemma 5.5 *Assume that $G = E_n$ or F_4. Then $V \downarrow A$ is multiplicity-free if and only if λ is as in the following table.*

G	λ
E_6	$\omega_1, \omega_2, \omega_6$
E_7	ω_1, ω_7
E_8	ω_8
F_4	ω_1, ω_4

Proof First assume $G = F_4$ and $\lambda = \omega_4$. It is straightforward to list the first few weights and see that $V \downarrow A = 16 + 8$. Propositions 2.4 and 2.5 of [4] show that $V \downarrow A$ is multiplicity-free for each of the remaining cases listed in the table.

It remains to show that all other possibilities fail to be multiplicity-free. To do this, we use 2.1 along with the dimensions of $V = V(\omega_i)$, which can be found using Magma; the values of r can be calculated using the expressions for ω_i in terms of roots, given in [2, p.250]. ■

This completes the proof of Proposition 5.1

6 The case where A is non-regular

Assume that G is a simple algebraic group, and $A \cong A_1$ is a G-irreducible subgroup of G. Recall from the Introduction that this means that a non-identity unipotent element u of A is distinguished in G. In this section we prove Theorem 1, classifying G-modules $V = V_G(\lambda)$ such that $V \downarrow A$ is multiplicity-free, in the case where u is distinguished, but not a regular element of G. Such elements exist for G of type $B_n\,(n \geq 4)$, $C_n\,(n \geq 3)$, $D_n\,(n \geq 4)$, E_6, E_7, E_8, F_4 or G_2. We shall see that there are relatively few examples; they are listed in Table 2 of Section 1.

We begin with the analysis of the classical groups.

Proposition 6.1 *Assume that $G = B_n$, C_n or D_n and u is distinguished but not regular. Then up to graph automorphisms of D_n, $V_G(\lambda) \downarrow A$ is multiplicity-free if and only if one of the following holds:*

(i) $\lambda = \omega_1$.

(ii) $G = D_n$ *with* $5 \leq n \leq 7$, $\lambda = \omega_n$, *and* $A < B_{n-2}B_1$ *projecting to a regular* A_1 *in each factor.*

For the next four lemmas assume the hypotheses of 6.1. The natural G-module, when restricted to A, is a direct sum of irreducible modules of distinct highest weights, and we first discuss the corresponding T-labelling of the Dynkin diagram of G. A full description can be found in [5, 3.18]. As an example, consider $G = C_{15}$ with A acting as $15 + 9 + 3$. The T-weights are $15, 13, 11, 9^2, 7^2, 5^2, 3^3, 1^3$ plus negatives. The corresponding labelling of the Dynkin diagram is 222020202002002. So the labelling begins with an *initial string* of 2's, then a number of terms 20, several of type 200, and so on. For C_n, the end-node α_n has label 2, and for B_n it has label 0. For D_n both of α_{n-1}, α_n have the same label; it is 2 or 0, according to whether there are just two summands for A or more than two, respectively.

As in previous sections, let $V = V_G(\lambda)$, of highest weight $\lambda = \sum c_i\omega_i$ affording T-weight r.

Lemma 6.2 *Assume $V \downarrow A$ is multiplicity-free. Then the following hold.*

(i) $c_i = 0$ *if* α_i *has label* 0.

(ii) $c_i = 0$ *if* α_i *has label* 2 *and* α_i *is adjacent to two nodes having label* 0.

(iii) $\lambda = b\omega_i$ *for some* i.

(iv) If $\lambda = b\omega_i$ *with* $b > 1$, *then* $i = 1$.

(v) $\lambda \neq \omega_n$ *if* $G = B_n$ *or* C_n.

Proof (i) Assume α_i has label 0 but $c_i \neq 0$. Then $\lambda - \alpha_i$ is a weight affording T-weight r, which implies that r^2 is a summand of $V \downarrow A$, a contradiction.

(ii) Next suppose that α_i has label 2 but nodes on either side have label 0. If we label these nodes $\alpha_i, \alpha_j, \alpha_k$, then $\lambda - i, \lambda - ij, \lambda - ik$ all afford T-weight $r - 2$, contradicting 2.2.

(iii) Assume $c_i \neq 0 \neq c_j$. Then $\lambda - i$ and $\lambda - j$ afford the only T-weights $r - 2$. This implies that neither α_i nor α_j can be adjacent to a node with 0 label, as otherwise $r - 2$ would occur with multiplicity at least 3. Therefore both occur in the initial string of 2's, and within this string we can argue exactly as in the regular case. Indeed, the argument of 2.6(iv),(v) implies that $i = 1, j = 2$, and $c_i = c_j = 1$. Then the first paragraph of the proof of Lemma 3.4 implies that the initial string of 2's has length 3. But then T-weight $r-4$ is afforded by $\lambda - 12$ (multiplicity 2), $\lambda - 23$ and $\lambda - 234$, contradicting 2.2.

(iv) Assume $\lambda = b\omega_i$ with $b > 1$. By 2.3(i), α_i is an end-node. Suppose $i = n$. Then $G \neq B_n$, as otherwise α_n has label 0, against (i). If $G = C_n$, then $\lambda - n, \lambda - n(n-1)$, $\lambda - n(n-1)^2 = (\lambda - n(n-1))^{s_{n-1}}$ all afford $r-2$. And for D_n, $r-4$ is afforded by $\lambda - n^2$, $\lambda - n^2(n-2)$, $\lambda - n^2(n-2)^2, \lambda - n(n-2)(n-1)$. This is a contradiction. A similar argument applies if $G = D_n$ and $i = n-1$.

(v) Suppose $\lambda = \omega_n$. The last argument of the previous paragraph also shows that $V \downarrow A$ is not multiplicity-free if $G = C_n$. And if $G = B_n$ then α_n has label 0, contradicting (i). ∎

Lemma 6.3 *Suppose $G = D_n$ for $n \geq 5$ and $\lambda = \omega_n$. Then $V \downarrow A$ is multiplicity-free if and only if $n \leq 7$ and $A < B_{n-2}B_1$, projecting to a regular A_1 in each factor.*

Proof Assume $G = D_n$ and $\lambda = \omega_n$. Then the labels of α_{n-1} and α_n are both 2, and A has two irreducible summands on the natural G-module. The label of α_{n-2} is 0.

Suppose $V \downarrow A$ is multiplicity-free. If α_{n-3} also has label 0, then $\lambda - n$, $\lambda - (n-2)n$, $\lambda - (n-3)(n-2)n$ all afford $r-2$, a contradiction. Therefore α_{n-3} has label 2. Next consider α_{n-4}. If α_{n-4} has label 0 then $n \geq 6$ and α_{n-5} must have label 2. Hence $r-6$ is afforded by each of $\lambda - (n-3)(n-2)(n-1)n$, $\lambda - (n-4)(n-3)(n-2)(n-1)n$, $\lambda - (n-3)(n-2)^2(n-1)n$, $\lambda - (n-4)(n-3)(n-2)^2(n-1)n$, $\lambda - (n-5)(n-4)(n-3)(n-2)n$, again a contradiction. Therefore, α_{n-4} has label 2. This forces the full labelling to be $22\cdots22022$.

Hence A acts on the natural G-module as $(2n-4)+2$ and so lies in a subgroup $B_{n-2}B_1$, which acts on V as the tensor product of spin modules for the factors. That is, $V \downarrow A = X \otimes 1$ where X is the restriction of the spin module of B_{n-1} to a regular A_1. As we are assuming $V \downarrow A$ to be multiplicity-free, this forces X to be multiplicity-free. Applying 5.4 we see that this implies $n - 2 \leq 8$. Moreover, at the end of the proof of 5.4 we listed the decompositions of X when this occurs. Tensoring these with 1 it is immediate from 2.4 that the V is multiplicity-free if and only if $n \leq 7$. ∎

Lemma 6.4 *(i) Assume $\lambda = b\omega_1$ with $b > 1$. Then $V \downarrow A$ is not multiplicity-free.*

(ii) Assume $\lambda = \omega_2$. Then $V \downarrow A$ is not multiplicity-free.

Proof (i) First suppose $b = 2$. Note that $S^2(\omega_1) = V$ if $G = C_n$, while $S^2(\omega_1) = V + 0$ if $G = B_n$ or D_n. Let A act on the natural module for G as $c + d + \cdots$, where $c > d > \cdots$. Note that if $d = 0$, then u is a regular element of B_{n-1} and is hence regular in $G = D_n$, which we are assuming is not the case. Hence $d > 0$.

Now $S^2(\omega_1) \downarrow A$ contains $S^2(c) = 2c + (2c-4) + \cdots$ and $c \otimes d = (c+d) + (c+d-2) + \cdots$ as direct summands. If $c - d = 4k$, then $2c - 4k = c + d$ is common to both summands. And if $c - d = 4k - 2$, then $2c - 4k = c + d - 2$ is common to both summands. In either case we see that $V \downarrow A$ is not multiplicity-free.

Now assume that $b \geq 3$ and that $V \downarrow A$ is multiplicity-free. We first settle some special cases. If the T- labelling is $202\cdots$, then $r-4$ is afforded by $\lambda - 1^2$, $\lambda - 1^22$, $\lambda - 1^22^2$, $\lambda - 123$, a contradiction. Similarly, if the labelling is $2202\cdots$, then $r-4$ is afforded by $\lambda-12$, $\lambda-123$, $\lambda-1^2$, which contradicts 2.2(iii). And if the labelling is $22202\cdots$, then $r-8$ is afforded by $\lambda-12345$, $\lambda-1^223$, $\lambda-1^2234$, $\lambda-1^22^2$, $\lambda-1^32$, again contradicting 2.2(iii).

Now suppose that the initial string of 2's has length at least 4. If $b\geq 4$, the weights $\lambda-1234$, $\lambda-1^223$, $\lambda-1^22^2$, $\lambda-1^32$, $\lambda-1^4$ all afford $r-8$, against 2.2(iii). So assume $b=3$. Then $S^3(\omega_1)=V$ or $V+\omega_1$ according to whether or not $G=C_n$. One checks $S^3(\omega_1)$ to see that $r-12$ occurs with multiplicity at least 7 in $V\downarrow A$, and hence $V\downarrow A$ is not multiplicity-free.

(ii) The argument is similar to the $b=2$ case in (i). Assume A acts on the natural module as $c+d+\cdots$, where $c>d>\cdots$. Note that $d>0$, as otherwise u would be a regular element of $G=D_n$. Then $\wedge^2(\omega_1)=V$ or $V+0$ according to whether or not G is an orthogonal group. So $\wedge^2(\omega_1)\downarrow A$ contains $\wedge^2(c)=(2c-2)+(2c-6)+\cdots$, as well as $c\otimes d=(c+d)+(c+d-2)+\cdots$, as direct summands. If $c-d=4k+2$, then $2c-2-4k=c+d$ and if $c-d=4k$, then $2c-2-4k=c+d-2$. In either case $V\downarrow A$ is not multiplicity-free. ∎

Lemma 6.5 *Assume $\lambda=\omega_i$ for $3\leq i<n$ and V is not a spin module for D_n. Then $V\downarrow A$ is not multiplicity-free .*

Proof Assume $V\downarrow A$ is multiplicity-free. By 6.2(ii) we know that α_i is in the initial string of 2's. Suppose the end of this string is at α_j. First assume $i\geq 4$. If in addition, $i\leq j-3$, then the result follows from 2.7. So we now consider situations where $i>j-3$ (still with $i\geq 4$).

Suppose $i=j$. Then α_{i+1} has label 0. If $n=i+1$, then $G=B_n$ and each of $\lambda-i$, $\lambda-i(i+1)$, $\lambda-i(i+1)^2=(\lambda-i(i+1))^{s_{i+1}}$ afford $r-2$, a contradiction. Therefore $n>i+1$. If α_{i+2} has label 0 we obtain the same contradiction from $\lambda-i$, $\lambda-i(i+1)$, $\lambda-i(i+1)(i+2)$. So suppose α_{i+2} has label 2. Then $r-4$ is afforded by each of $\lambda-(i-1)i$, $\lambda-(i-1)i(i+1)$, $\lambda-i(i+1)(i+2)$, which is not yet a contradiction. If $n=i+2$, then $G=C_n$ and we also get $r-4$ from $\lambda-i(i+1)^2(i+2)=(\lambda-i(i+1)(i+2))^{s_{i+2}}$. And if $n>i+2$, either α_{i+3} has label 0 or else $G=D_{n+3}$. In either case we get an extra weight affording $r-4$, which does contradict 2.2.

Therefore $i<j$. Then $r-2$ appears with multiplicity 1 and 2.2(iii) applies. By assumption, α_{j+1} has label 0. Suppose $i=j-1$. Then $r-4$ is afforded by each of $\lambda-(i-1)i$, $\lambda-ij$, $\lambda-ij(j+1)$ a contradiction. And if $i=j-2$, then $r-8$ is afforded by each of $\lambda-(i-3)(i-2)(i-1)i$, $\lambda-(i-2)(i-1)i(i+1)$, $\lambda-(i-1)i(i+1)(i+2)$, $\lambda-(i-1)i(i+1)(i+2)(i+3)$, $\lambda-(i-1)i^2(i+1)$, contradicting 2.2(iii).

Now assume $i=3$. Then $\wedge^3(\omega_1)$ equals V or $V+\omega_1$ depending on whether or not G is an orthogonal group. Write $\omega_1\downarrow A=a+b+\cdots$ with $a>b>\cdots$. We know that α_3 is in the intial string of 2's, and this forces $a-b\geq 6$ so that $r=3a-6$. If G is an orthogonal group, then $a,b,\cdots$ are even and so $a\geq 8$ (note that $b>0$ as A is not regular). Then $V\downarrow A$ contains $\wedge^3(a)$ as a direct summand which is not multiplicity-free by 5.2(ii). Indeed, there is a direct summand of highest weight $r-12=3a-18$ appearing with multiplicity 2. Now consider $G=C_n$. The same argument applies provided $3a-18>a$. So it remains to consider $a\leq 9$. The cases are $(a,b)=(7,1),(9,3),(9,1)$. Then $\wedge^3(\omega_1)\downarrow A$ contains $\wedge^3(a)$ and $\wedge^2(a)\otimes b$ as direct summands. As $\wedge^3(a)=(3a-6)+(3a-10)+\cdots$ and $\wedge^2(a)\otimes b=(2a-2+b)+(2a-4+b)+\cdots$, it follows that in each case, $3a-10$ occurs with multiplicity at least 2 and is not present in ω_1. ∎

This completes the proof of Proposition 6.1.

It remains to consider the exceptional groups. Here we label the distinguished non-regular classes as in [5]. For convenience we reproduce the list in Table 3.

Table 3: Distinguished non-regular classes in exceptional groups

G	classes	labellings
G_2	$G_2(a_1)$	02
F_4	$F_4(a_1)$, $F_4(a_2)$, $F_4(a_3)$	2202, 0202, 0200
E_6	$E_6(a_1)$, $E_6(a_3)$	222022, 200202
E_7	$E_7(a_1)$, $E_7(a_2)$, $E_7(a_3)$, $E_7(a_4)$, $E_7(a_5)$	2220222, 2220202, 2002022, 2002002, 0002002
E_8	$E_8(a_1)$, $E_8(a_2)$, $E_8(a_3)$, $E_8(a_4)$, $E_8(a_5)$, $E_8(a_6)$, $E_8(a_7)$, $E_8(b_4)$, $E_8(b_5)$, $E_8(b_6)$	22202222, 22202022, 20020222, 20020202, 20020020, 00020020, 00002000, 20020022, 00020022, 00020002

Proposition 6.6 *Assume G is an exceptional group and u is distinguished but not regular. Then up to graph automorphisms of E_6, $V_G(\lambda) \downarrow A$ is multiplicity-free if and only if λ and u are as in the following table.*

G	u	λ
F_4	$F_4(a_1)$	ω_4
E_6	$E_6(a_1)$	ω_1
E_7	$E_7(a_1)$ *or* $E_7(a_2)$	ω_7
E_8	$E_8(a_1)$	ω_8

Lemma 6.7 *Proposition* 6.6 *holds if $G = G_2$ or F_4.*

Proof First consider $G = F_4$. Suppose $V \downarrow A$ is multiplicity-free. If there exist $i \neq j$ with $c_i \neq 0 \neq c_j$, then either α_i or α_j is adjacent to a node with label 0, contradicting 2.6(ii). Therefore $\lambda = b\omega_i$ for some i. From the diagrams in Table 3, and considering the multiplicity of $r-2$ using 6.2(ii), we see that u cannot be in the class $F_4(a_3)$, and that if $u = F_4(a_2)$ then $i = 4$. But then $\lambda - 234$, $\lambda - 1234$, $\lambda - 23^24$, $\lambda - 123^24$ all afford $r-4$, contradicting 2.2.

Now consider u in class $F_4(a_1)$. If $i = 2$, then $\lambda - 2$, $\lambda - 23$, $\lambda - 23^2$ all afford $r-2$, a contradiction. If $i = 1$, then $r-2$ appears with multiplicity 1, but $\lambda - 12$, $\lambda - 123$, $\lambda - 123^2$ all afford $r-4$, contradicting 2.2(i). Therefore $i = 4$. If $b > 1$, $r-4$ appears with multiplicity 4, which is impossible. And if $\lambda = \omega_4$ it follows from [7, Table A, p.65] and the tables at the end of [4] that $A < B_4$, and $\omega_4 \downarrow B_4 = 1000 + 0001 + 0000$. Using the information at the end of the proof of 5.4, we find that $V \downarrow A = 8 + (10 + 4) + 0$ and hence $V \downarrow A$ is multiplicity-free.

Finally consider G_2 where the only labelling is 02. Hence $\lambda = b\omega_2$. Then $\lambda - 2$, $\lambda - 12$, $\lambda - 1^32$ all afford $r-2$, a contradiction. ■

Lemma 6.8 *Proposition* 6.6 *holds if $G = E_n$.*

Proof Assume $G = E_n$ and $V \downarrow A$ is multiplicity-free. First suppose that there exist $i > j$ with $c_i \neq 0 \neq c_j$. Lemma 2.6 shows these are the only two such nodes, that neither can adjoin a node with label 0, that at least one must be an end-node, and that $c_i = c_j = 1$. Suppose $j = 1$. Then α_3 must be labelled 2 and from the list of possible labellings in Table 3 we see that α_4 has label 0. This forces $i \geq 6$. But then $r - 4$ is afforded by $\lambda - 13$, $\lambda - 134$, $\lambda - 1i$, $\lambda - (i-1)i$, a contradiction. Therefore, $j \neq 1$ and hence $i = n$. If $j \neq n - 1$, then we must have $G = E_8, j = 6$, and $u = E_8(a_1)$. But here we see that $r - 4$ occurs with multiplicity at least 5, a contradiction.

Suppose $i = n, j = n-1$. If α_{n-3} has label 2, then $r - 6$ occurs with multiplicity at least 5 from $\lambda - (n-2)(n-1)n$ (multiplicity 2), $\lambda - (n-1)^2 n = (\lambda - n)^{s_{n-1}}$, $\lambda - (n-1)n^2 = (\lambda - (n-1))^{s_n}$, $\lambda - (n-3)(n-2)(n-1)$. We get the same contradiction if α_{n-3} has label 0, by replacing the last weight with $\lambda - (n-3)(n-2)(n-1)n$, (it even appears with multiplicity 2).

Hence $\lambda = b\omega_i$ for some i. Suppose $b > 1$. Then 2.3 implies that α_i is an end-node with label 2 and that the adjacent node has label 2. Therefore $i = 1$ or $i = n$. If $i = 1$, then $r - 6$ is afforded by $\lambda - 1234$, $\lambda - 1345$, $\lambda - 1^2 3$, $\lambda - 1^2 34$, contradicting 2.2(iii).

Next consider $i = n$ where we can assume $n = 7$ or 8 since the E_6 case follows from the above via a graph automorphism. If α_{n-2} has label 0, then $r - 4$ is afforded by $\lambda - (n-1)n$, $\lambda - (n-2)(n-1)n$, $\lambda - n^2$, contradicting 2.2(iii). Therefore α_{n-2} has label 2. The only possibilities satisfying these conditions are $u = E_7(a_1)$, $E_8(a_1)$, $E_8(a_3)$. If $u = E_8(a_1)$, then $r - 12$ arises from $\lambda - 1345678$, $\lambda - 2345678$, $\lambda - 234^2 5678$, $\lambda - 345678^2$, $\lambda - 245678^2$, $\lambda - 567^2 8^2$, $\lambda - 6^2 7^2 8^2$, a contradiction. A similar argument applies to $E_7(a_1)$ and $E_8(a_3)$, using the weight $r - 8$.

At this point we have $\lambda = \omega_i$. As in the proof of 5.5, we use 2.1 to reduce to the cases $(G; i) = (E_6; 1, 2, 6)$, $(E_7; 1, 7)$ and $(E_8; 8)$. The action of A on $L(G)$ is given in [7] (see Table A, p.65 and Table 1, p.193). This settles all but the 27 dimensional modules ω_1, ω_6 for E_6 and the 56 dimensional module ω_7 for E_7.

Suppose $G = E_6$. From [7, p.65] we see that u is a regular element in C_4 or $A_1 A_5$ according to whether $u = E_6(a_1)$ or $E_6(a_3)$. Then [4, 2.3,2.5] shows that only the first case is multiplicity-free.

Finally assume $G = E_7$ and $\lambda = \omega_7$. Lemma 2.5 of [4] shows that $V \downarrow A$ is multiplicity-free if $u = E_7(a_1)$. If $u = E_7(a_2)$, then $A \leq A_1 F_4$ by [7, p.65], and [4, 2.5] shows that $V \downarrow A = (1 \otimes (16 + 8)) + 3$, which is multiplicity-free. If $u = E_7(a_4)$ or $E_7(a_5)$, then both α_5 and α_6 have label 0 so that $r - 2$ occurs with multplicity 3, a contradiction. This leaves $u = E_7(a_3)$, in which case [7, p.65] shows that $A < A_1 B_5 < A_1 D_6$. Then [4, 2.3] shows that $V \downarrow A_1 D_6 = 1 \otimes \omega_1 + 0 \otimes \omega_5$. Applying the decomposition at the end of the proof of 5.4, we see that this is not multiplicity-free.

■

This completes the proof of Theorem 1.

7 Proof of Corollary 2

Now we prove Corollary 2. Let G be a simple algebraic group of rank at least 2, let $u \in G$ be a distinguished unipotent element and let A be an A_1 subgroup of G containing u. Let $\rho : G \to I(V)$ is an irreducible representation with highest weight λ.

If $I(V) = SL(V)$, then $\rho(u)$ is distinguished in $I(V)$ if and only if $V \downarrow \rho(A)$ is irreducible, so the conclusion goes back to Dynkin [3], but see also [6, Theorem 7.1] where the result is given explicitly. Alternatively it is easy to check the Tables 1 and 2 of Theorem 1, except for ω_1 for A_n, B_n, C_n and 10 for G_2, the subgroup acts reducibly on $V_G(\lambda)$.

Now suppose $I(V) = Sp(V)$ or $SO(V)$. If $\rho(u)$ is distinguished in $I(V)$, then $V \downarrow \rho(A)$ is multiplicity-free, and so λ is as in Table 1 or 2 of Theorem 1. Moreover V is self-dual, so that $\lambda = -w_0(\lambda)$. Conversely, for all such λ in the tables, $V \downarrow \rho(A)$ is multiplicity-free, and so $\rho(u)$ has Jordan blocks on V of distinct sizes, hence is distinguished. This completes the proof.

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