

A NEW CONSTRUCTION OF CYCLIC HOMOLOGY

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ABSTRACT. Based on the ideas of Cuntz and Quillen, we give a simple construction of cyclic homology of unital algebras in terms of the noncommutative de Rham complex and a certain differential similar to the equivariant de Rham differential. We describe the Connes exact sequence in this setting.

We define *equivariant Deligne cohomology* and construct, for each $n \geq 1$, a natural map from cyclic homology of an algebra to the GL_n -equivariant Deligne cohomology of the variety of n -dimensional representations of that algebra. The bridge between cyclic homology and equivariant Deligne cohomology is provided by *extended cyclic homology*, which we define and compute here, based on the extended noncommutative de Rham complex introduced previously by the authors.

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1. INTRODUCTION

1.1. There are several definitions of cyclic homology. The original definition, due to A. Connes [Con] (see also Tsygan [Tsy1]), works for algebras over fields of characteristic zero and is based on a cyclic version of the Hochschild complex. Another definition that involves a double complex and that works in arbitrary characteristic is due to Loday and Quillen [LQ] (motivated by Tsygan [Tsy1]). This definition may be reformulated in terms of a slightly different (b, B) -complex, where b and B are the Hochschild and the Connes differentials, respectively. There is yet another definition of cyclic homology as a certain nonabelian derived functor, due to Feigin and Tsygan [FT]. All the above definitions are known to be equivalent (see, e.g., [Lo]), and each one has certain advantages.

In this paper, we give one more definition of (reduced) cyclic homology for unital algebras over fields of characteristic zero (more generally, over commutative rings k containing \mathbb{Q} , when the algebra is k -split). Our approach is partly motivated by a well known analogy, that goes back to Rinehart [Ri], between the Connes differential B and the familiar de Rham differential on differential forms. In the case of commutative algebras, this analogy can be made precise (see, e.g., [Lo, §3.4]). In the general case of not necessarily commutative algebras, Karoubi [Kar82, p. 383], and independently Connes (implicitly in [Con, §II]), introduced a certain complex of

noncommutative differential forms *noncommutative de Rham complex*, that comes equipped with a natural differential d , the Karoubi-de Rham differential. The cohomology of the noncommutative de Rham complex is not directly related to cyclic homology, however. A precise relation between the two is given by the so-called Connes-Karoubi theorem ([Con, p. 334, Theorem 34], [Kar87, Théorème 2.15]); see also (4.1.2) below.

In our approach, cyclic homology is constructed directly in terms of the noncommutative de Rham complex so that the Karoubi-de Rham differential d literally replaces the Connes differential B . There is also another differential involved in the construction, a counterpart of the Hochschild differential b . This new differential i , introduced first in [CBEG, §2.8, §3.1], anti-commutes with d (while b does not anti-commute with d) and is, we believe, more closely related to geometry than b . Specifically, for every (not necessarily commutative) algebra A , there is a canonical derivation $\Delta : A \rightarrow A \otimes A$ given by the formula $\Delta(a) := 1 \otimes a - a \otimes 1$. The differential i may be thought of as a contraction operation of noncommutative differential forms with the derivation Δ ; see *op. cit.*, [GS, Lemma 3.5.1.(i)], and Section 5 below for more details.

Using our construction, we obtain a simple proof of the exact sequence (4.1.2) that does not require double complexes (as Karoubi's proof in [Kar87, §2] does; although Connes' proof in [Con, §I.4, II.1–4] does not use double complexes, the construction there of S involves a cup product of cycles and appears more technically involved than our construction).

There are two additional features of the construction of cyclic homology given in this paper which, we believe, are especially appealing. The first one is its close analogy with equivariant cohomology. A well known result of Goodwillie [Go] says that the cyclic homology of (the de Rham DG algebra of) a manifold X is isomorphic to the S^1 -equivariant cohomology of $L(X)$, the corresponding free loop space. The role of the *extended noncommutative de Rham complex*, introduced in §5 below, is somewhat analogous to that of the equivariant de Rham complex of the free loop space. Specifically, our extended noncommutative de Rham complex comes equipped with a pair of anti-commuting differentials, the Karoubi-de Rham differential d and a differential i_t which is analogous to the S^1 -equivariant differential. Cyclic homology is then related to the cohomology of the extended noncommutative de Rham complex via a map, which we call the 'Tsygan map', that intertwines the differentials b and B , respectively, in the standard (b, B) -complex with the differentials i_t and d , respectively, in the extended noncommutative de Rham complex. The Tsygan map may be viewed as a noncommutative analogue of multiplication by the function $\exp(t)$.

The second feature of our construction is its obvious similarity with *truncated de Rham complexes*. The latter show up in the standard description of cyclic homology of the coordinate ring of an algebraic variety. Thus, our construction provides an analogous description in the noncommutative setting. Some of the above mentioned analogies can be made precise via the notion of representation functor and equivariant Deligne cohomology. This will be discussed in §8.

1.2. Layout of the paper. In §2, we introduce the noncommutative de Rham complex and discuss harmonic decomposition, a powerful technical tool discovered by Cuntz and Quillen [CQ1], [CQ2].

In §3, we give our new construction of cyclic homology, as well as similar constructions for periodic and negative cyclic homology. We then use harmonic decomposition to prove that our construction and the standard one are equivalent.

Section 4 is devoted to the Connes exact sequence. We show how to construct such a sequence entirely within our approach. We also establish the equivalence with the standard construction.

Section 5 contains some of the most important results of the paper. We introduce the extended noncommutative de Rham complex and define extended cyclic homology. Extended cyclic homology groups come equipped with an additional 'weight grading.' We show that the positive weight part is isomorphic to the Karoubi-de Rham homology, while the negative weight part reduces to

the usual cyclic homology. A key role in proving these results is played by the Tsygan map. We have arrived at the definition of the Tsygan map by analyzing Tsygan’s construction [Tsy2] (that construction was motivated, in turn, by our results in [GS]).

Section 6 is devoted to the proof of Theorem 5.3.10, which is perhaps the most difficult result of the paper. In §7, we discuss ‘extended versions’ of periodic and negative cyclic homology.

In the last section we relate our results to geometry. We begin by introducing *equivariant Deligne cohomology*, an equivariant counterpart of Deligne cohomology, i.e., of the cohomology theory based on truncated algebraic de Rham complexes. The *equivariant Deligne cohomology* theory that we define appears to be new.

Next, we give a short review of some standard constructions related to the representation functor. Recall that, associated with a (not necessarily commutative) algebra A and an integer $n \geq 1$, there is an affine scheme $\text{Rep}_n A$ parametrizing n -dimensional representations of A , cf. [CBEG]. This scheme comes equipped with a natural action of the algebraic group GL_n , by base change transformations. An important application of our approach is a construction of a canonical map from the cyclic homology of A to the equivariant Deligne cohomology of the scheme $\text{Rep}_n A$. We should point out that considering equivariant (as opposed to ordinary) cohomology of representation schemes is, in a sense, the only natural thing to do. Indeed, two representations of the algebra A are equivalent if and only they belong to the same GL_n -orbit in $\text{Rep}_n A$. Thus, the object which is most relevant here is the quotient stack $\text{Rep}_n A / \text{GL}_n$ rather than the scheme $\text{Rep}_n A$ itself. The GL_n -equivariant Deligne cohomology of the scheme $\text{Rep}_n A$ may be viewed as the ordinary Deligne cohomology of the stack $\text{Rep}_n A / \text{GL}_n$.

In the special case where $n = 1$ and A is the coordinate ring of a smooth affine algebraic variety X , one has $\text{Rep}_1 A = X$. In this case, our construction reduces to, and provides a more explicit form of, the well known isomorphism between cyclic homology of the coordinate ring and Deligne cohomology of X , cf. [Lo, §3.6].

A different relation between cyclic homology of an algebra and its representation schemes is studied in the recent paper by Berest, Khachatryan, and A. Ramadoss [BKR]. It does not consider equivariant cohomology, however.

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2. NONCOMMUTATIVE DIFFERENTIAL FORMS

2.1. The Karoubi-de Rham complex. Fix \mathbf{k} , a unital commutative \mathbb{Q} -algebra, and put $\otimes = \otimes_{\mathbf{k}}$.

Throughout the paper, we fix a unital associative algebra A over the ground ring \mathbf{k} satisfying the additional assumption that the natural map $\underline{\mathbf{k}} \rightarrow \underline{A}$ is a \mathbf{k} -split injection, i.e., there exists a \mathbf{k} -module direct sum decomposition $A = \mathbf{k} \oplus \bar{A}$, where $\bar{A} = A/\mathbf{k}$. This assumption trivially holds if \mathbf{k} is a field.

From A one constructs ([Kar82, p. 383], [Con, p. 314]¹) the dg algebra $(\Omega A, d)$ of *noncommutative differential forms*. By definition, ΩA is the \mathbf{k} -algebra generated by symbols $a \in A$ and $da, a \in A$, subject to the relations

$$a \cdot b = ab, d(ab) = a \cdot db + (da) \cdot b.$$

The differential is then the derivation defined by $d(a) = da$ and $d(da) = 0$. It evidently has square zero. The grading is given by $|a| = 0, |da| = 1$ for nonzero $a \in A$.

¹In this reference, Connes refers to the earlier papers [Arv, Kar82], but they seem to discuss only commutative differential forms.

Since d kills $\mathbf{k} \subset \Omega^0 A$, it descends to the *reduced complex*, $\overline{\Omega}A$, defined by $\overline{\Omega}A := \Omega A/\mathbf{k}$. As explained in [CQ1, §1], one has a canonical isomorphism of left A -modules

$$A \otimes \overline{A}^{\otimes n} \xrightarrow{\sim} \Omega^n A, \quad a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto a_0 da_1 da_2 \cdots da_n, \quad a_0 \in A, a_1, \dots, a_n \in \overline{A}. \quad (2.1.1)$$

From this and the assumption that $\mathbf{k} \hookrightarrow A$ is split it follows that $(\overline{\Omega}A, d)$ is acyclic; see [CQ1, §1].

For every graded algebra B and graded vector subspace $V \subseteq B$, let $[V, V]$ denote the linear span of (super)commutators $[v_1, v_2]$, $v_1, v_2 \in V$. Following Karoubi [Kar87, 1.3, 1.24], define the noncommutative de Rham complex of A and its reduced version by

$$\mathrm{DR} A := \Omega A / [\Omega A, \Omega A], \quad \text{and} \quad \overline{\mathrm{DR}} A := (\mathrm{DR} A) / \mathbf{k} = \overline{\Omega}A / [\overline{\Omega}A, \overline{\Omega}A].$$

We write f_{\natural} for the image of an element $f \in \Omega A$ under the natural projection $\Omega A \rightarrow \mathrm{DR} A$.

The differential d on ΩA descends to $\mathrm{DR} A$. Let $\mathrm{HD}_n(A)$ and $\overline{\mathrm{HD}}_n(A)$ denote the n -th homology of $(\mathrm{DR} A, d)$ and $(\overline{\mathrm{DR}} A, d)$, respectively. We will call these the *Karoubi-de Rham homology* and its reduced version.

As noticed by Karoubi [Kar82, p. 30] (and also implicit in Connes' work), the Hochschild differential on ΩA (or on $\overline{\Omega}A$) is given by the formula

$$\mathbf{b}(\alpha da) = (-1)^{n-1}[\alpha, a], \quad \alpha \in \Omega^n A, \quad a \in A, \quad \text{and} \quad \mathbf{b}|_{\Omega^0 A} = 0. \quad (2.1.2)$$

A crucial role below will also be played by another map $\mathbf{i} : \Omega A \rightarrow \Omega A$, defined by

$$\mathbf{i}(a_0 da_1 \cdots da_n) = \sum_{\ell=1}^n (-1)^{(\ell-1)(n-1)+1} [a_\ell, da_{\ell+1} \cdots da_n a_0 da_1 \cdots da_{\ell-1}] \in [A, \Omega^{n-1} A]. \quad (2.1.3)$$

It is straightforward to check that the map \mathbf{i} vanishes on $[\Omega A, \Omega A]$ and hence descends to a well defined map $\mathbf{i} : \mathrm{DR}^* A \rightarrow \Omega^{*-1} A$. It follows that $\mathbf{i}^2 = 0$. In addition, one can check that $d\mathbf{i} + \mathbf{i}d = 0$.

The map \mathbf{i} was first introduced in [CBEG, Lemma 3.1.1.(ii)]. A more conceptual definition of this map (see §5) was discovered in [GS, Lemma 3.5.1.(i)]. From that definition the above stated properties become immediate consequences of noncommutative calculus ([GS, Lemma 3.5.1.(ii)]).

2.2. Harmonic decomposition. Since the algebra A is fixed throughout the paper, we will often use simplified notation $\Omega = \Omega A$, $\overline{\Omega} = \overline{\Omega}A$, $\mathrm{DR} = \mathrm{DR} A$, etc.

Following [Kar87], define the Karoubi operator as

$$\kappa : \overline{\Omega} \rightarrow \overline{\Omega}, \quad \kappa(\alpha da) = (-1)^{|\alpha|} da \alpha, \quad \alpha \in \Omega, \quad a \in A; \quad \kappa|_{\overline{\Omega}^0} = 0. \quad (2.2.1)$$

The Karoubi operator is related to the operators \mathbf{b} and d by [Kar87, CQ1]

$$\mathbf{b}d + d\mathbf{b} = \mathrm{Id} - \kappa. \quad (2.2.2)$$

According to [CQ2, §2], there is a direct sum decomposition

$$\overline{\Omega} = P\overline{\Omega} \oplus P^\perp \overline{\Omega}, \quad P\overline{\Omega} := \ker(\mathrm{Id} - \kappa)^2, \quad P^\perp \overline{\Omega} := \mathrm{im}(\mathrm{Id} - \kappa)^2.$$

Let P and P^\perp denote the projections onto the first and second summands of this decomposition, respectively, which we will call the harmonic and antiharmonic parts.

It follows from (2.2.2) that κ commutes with d and \mathbf{b} . Hence, harmonic decomposition is stable under κ , \mathbf{b} , and d , and it induces a similar decomposition on $H(\Omega, \mathbf{b})$ and $H(\Omega, d)$.

Let N be the grading operator on $\overline{\Omega}$, defined by $N|_{\overline{\Omega}^n} = n \cdot \mathrm{Id}$. With the above notation, formula (2.1.3) may be rewritten in the form [GS, Proposition 5.1.1]

$$\mathbf{i}|_{\overline{\Omega}^n} = (\mathrm{Id} + \kappa + \cdots + \kappa^{n-1}) \mathbf{b}|_{\overline{\Omega}^n A} = \mathbf{b}N P|_{\overline{\Omega}^n}. \quad (2.2.3)$$

This is straightforward to check explicitly from (2.1.2), (2.1.3), and (2.2.1).

Finally, following [CQ2], one defines the Connes differential by

$$\mathbf{B}|_{\overline{\Omega}^n} := (\mathrm{Id} + \kappa + \cdots + \kappa^n) d|_{\overline{\Omega}^n} = N d P|_{\overline{\Omega}^n}. \quad (2.2.4)$$

The above formulas show that the operators \mathbf{i} and \mathbf{B} respect harmonic decomposition and one has

$$\mathbf{i} d = \mathbf{b} N d P = \mathbf{b} B, \quad \text{and} \quad d \mathbf{i} = d \mathbf{b} N P = N d \mathbf{b} P = B \mathbf{b}.$$

Thus, the equations $d \mathbf{i} + \mathbf{i} d = 0$ and $B \mathbf{b} + \mathbf{b} B = 0$ are equivalent.

The next lemma collects various technical properties of harmonic decomposition which will be used in various proofs (but not in the statements) in subsequent sections.

Lemma 2.2.5. *One has*

$$P[\overline{\Omega}, \overline{\Omega}] = P \mathbf{b} \overline{\Omega} = \mathbf{i} \overline{\Omega} \quad (2.2.6)$$

$$P^\perp[\overline{\Omega}, \overline{\Omega}] = P^\perp \overline{\Omega} \quad (2.2.7)$$

$$P^\perp d \overline{\Omega} = (\text{Id} - \kappa) d \overline{\Omega} = \overline{[d\Omega, d\Omega]} \quad (2.2.8)$$

$$P^\perp \mathbf{b} \overline{\Omega} = (\text{Id} - \kappa) \mathbf{b} \overline{\Omega}. \quad (2.2.9)$$

Proof. Given a κ -action on a vector space V , let $V^\kappa = \ker(\text{Id} - \kappa)|_V$ denote the spaces of κ -invariants. If the κ -action on V has finite order then $V = V^\kappa \oplus (\text{Id} - \kappa)V$; in particular, the operator $(\text{Id} - \kappa)^2$ acts on V by zero if and only if $V = V^\kappa$.

It was observed in [Kar87, Lemme 2.12] that κ has finite order on $d \overline{\Omega}$ and $\overline{\Omega} / \mathbf{b} \overline{\Omega}$. It follows that κ has finite order also on $\overline{\Omega} / d \overline{\Omega} \cong d \overline{\Omega}$ and on $\mathbf{b} \overline{\Omega} = \mathbf{b}(\overline{\Omega} / \mathbf{b} \overline{\Omega})$. Each of these spaces is κ -stable. The first equation in (2.2.8) and equation (2.2.9) follow. Similarly, we deduce that the operator $\text{Id} - \kappa$ annihilates $P \overline{\Omega} / P \mathbf{b} \overline{\Omega}$, i.e., $(\text{Id} - \kappa)P \overline{\Omega} \subset \mathbf{b} P \overline{\Omega}$.

Formula (2.2.1) readily implies the second equation in (2.2.8); it shows also that $(\text{Id} - \kappa) \overline{\Omega} = \overline{[dA, \Omega]}$. Similarly, from formula (2.1.2) we get that $\mathbf{b} \overline{\Omega} = \overline{[A, \Omega]}$. Thus we obtain (cf. [CQ2]):

$$\overline{[\Omega, \Omega]} = \overline{[A, \Omega]} + \overline{[dA, \Omega]} = \mathbf{b} \overline{\Omega} + (\text{Id} - \kappa) \overline{\Omega}. \quad (2.2.10)$$

Applying P to this equation and using the inclusion $(\text{Id} - \kappa)P \overline{\Omega} \subset \mathbf{b} P \overline{\Omega}$ proved above, we deduce the first equation in (2.2.6). The second equation in (2.2.6) is clear from (2.2.3).

Finally, since $\text{Id} - \kappa$ is invertible on $P^\perp \overline{\Omega}$, formula (2.2.10) shows that $P^\perp \overline{\Omega} \subseteq (\text{Id} - \kappa) \overline{\Omega} \subseteq \overline{[\Omega, \Omega]}$. This yields the inclusion $P^\perp \overline{\Omega} \subseteq P^\perp \overline{[\Omega, \Omega]}$. The opposite inclusion is obvious, proving (2.2.7). \square

3. MAIN CONSTRUCTIONS

3.1. Construction of Hochschild and cyclic homology. Let $\overline{\Omega}A((u))$ be the space of formal Laurent series with coefficients in $\overline{\Omega}A$. We assign the variable u degree -2 , so that the total degree of an element $f \in \overline{\Omega}^p A \cdot u^{-r}$ equals $|f| = p + 2r$. Thus, each of the differentials $u \mathbf{B}$ and \mathbf{b} has degree -1 .

Standard constructions of various versions of cyclic homology involve the $\mathbf{k}[u]$ -modules $\mathbf{k}[[u]] \subset \mathbf{k}((u))$ and $R := \mathbf{k}((u))/uk[[u]]$. Specifically, reduced *relative* (to \mathbf{k}) Hochschild and cyclic homology of A are defined as follows

$$\overline{\text{HH}}_\bullet(A) = H(\overline{\Omega}A, \mathbf{b}), \quad \overline{\text{HC}}_\bullet(A) = H(\overline{\Omega}A \otimes R, \mathbf{b} - u \mathbf{B}), \quad (3.1.1)$$

$$\overline{\text{HC}}_\bullet^{\text{per}}(A) = H(\overline{\Omega}A((u)), \mathbf{b} - u \mathbf{B}), \quad \overline{\text{HC}}_\bullet^-(A) = H(\overline{\Omega}A[[u]], \mathbf{b} - u \mathbf{B}). \quad (3.1.2)$$

A central result of this article is

Theorem 3.1.3. *For a \mathbf{k} -algebra A satisfying our standing assumptions, one has canonical isomorphisms:*

$$\overline{\text{HH}}_\bullet(A) \xrightarrow{\sim} \ker(\mathbf{i} : \overline{\text{DR}}^\bullet \rightarrow \overline{\Omega}^{\bullet-1}). \quad (3.1.4)$$

$$\overline{\text{HC}}_\bullet(A) \xrightarrow{\sim} \ker(\mathbf{i} : \overline{\text{DR}}^\bullet / d \overline{\text{DR}}^{\bullet-1} \rightarrow \overline{\Omega}^{\bullet-1} / d \overline{\Omega}^{\bullet-2}). \quad (3.1.5)$$

Here, the first isomorphism is induced by the map $\overline{\Omega} \rightarrow \overline{\text{DR}}^\bullet$, $f \mapsto f_{\mathfrak{q}}$ and the second isomorphism is induced by the map $\overline{\Omega} \otimes R \rightarrow \overline{\text{DR}}^\bullet$, $\sum_{k \geq 0} f_k u^{-k} \mapsto (f_0)_{\mathfrak{q}}$.

Isomorphism (3.1.4) has been already established in [GS, Theorem 4.1.1]. However, we will reproduce the proof, since most of the intermediate steps will also be used in the proof of (3.1.5).

Proof of Theorem 3.1.3. First of all, from harmonic decomposition for $\overline{\Omega}$ and $[\overline{\Omega}, \overline{\Omega}]$, respectively, using equation (2.2.7), we see that the projection $\overline{\Omega} \rightarrow P\overline{\Omega}$ induces an isomorphism $\overline{\Omega}/[\overline{\Omega}, \overline{\Omega}] \xrightarrow{\sim} P\overline{\Omega}/P[\overline{\Omega}, \overline{\Omega}]$. Further, equation (2.2.6) yields $P\overline{\Omega}/\mathfrak{b}P\overline{\Omega} = P\overline{\Omega}/P[\overline{\Omega}, \overline{\Omega}]$. Also, the map i annihilates the space $P^\perp\overline{\Omega}$, by (2.2.3). Thus, we obtain a commutative diagram

$$\begin{array}{ccccccc}
P\overline{\Omega}/\mathfrak{b}P\overline{\Omega} & \xlongequal{\quad} & P\overline{\Omega}/P[\overline{\Omega}, \overline{\Omega}] & \xlongequal{\quad} & \overline{\Omega}/[\overline{\Omega}, \overline{\Omega}] & \xlongequal{\quad} & \overline{DR} \\
& \searrow \mathfrak{b}N & \downarrow i & & & & \downarrow i \\
& & P\overline{\Omega} & \xrightarrow{\quad} & \overline{\Omega} & &
\end{array} \tag{3.1.6}$$

We see from the diagram that the assignment $f \mapsto f_{\mathfrak{h}}$ maps the space $H(P\overline{\Omega}, \mathfrak{b}) = \ker(\mathfrak{b} : P\overline{\Omega}/\mathfrak{b}P\overline{\Omega} \rightarrow P\overline{\Omega})$ isomorphically onto the space $\ker(i : \overline{DR} \rightarrow \overline{\Omega})$.

Next, since $(\overline{\Omega}, d)$ is acyclic, it follows that its harmonic and anti-harmonic parts, $(P\overline{\Omega}, d)$ and $(P^\perp\overline{\Omega}, d)$, are acyclic as well. Hence also $(P\overline{\Omega}, B)$ is acyclic, by (2.2.4). Cuntz and Quillen observed in [CQ2] that this implies that the cohomology of all of the complexes appearing in (3.1.1)–(3.1.2) are not affected by the replacement of the space $\overline{\Omega}$ by its harmonic part $P\overline{\Omega}$. Specifically, it follows from the above, using equations (2.2.2) and (2.2.4), that the following complexes are acyclic:

$$(P^\perp\overline{\Omega}, \mathfrak{b}), \quad (P^\perp\overline{\Omega}(\mathfrak{u}), \mathfrak{b} - \mathfrak{u}B), \quad (P^\perp\overline{\Omega}[\mathfrak{u}], \mathfrak{b} - \mathfrak{u}B), \quad (P^\perp\overline{\Omega} \otimes R, \mathfrak{b} - \mathfrak{u}B).$$

Thus, the assignment $f \mapsto f_{\mathfrak{h}}$ gives an isomorphism $H(P^\perp\overline{\Omega}, \mathfrak{b}) \xrightarrow{\sim} 0$, and (3.1.4) follows.

To proceed further, we need the following result of Cuntz and Quillen, [CQ2, Proposition 3.1].

Lemma 3.1.7. *The projection modulo u^{-1} and $\text{im}(B)$ induces an isomorphism*

$$\overline{HC} \cdot (A) = H \cdot (\overline{\Omega} \otimes R, \mathfrak{b} - \mathfrak{u}B) \xrightarrow{\sim} H \cdot (\overline{\Omega}/B\overline{\Omega}, \mathfrak{b}), \quad \sum_{k \geq 0} f_k \cdot u^{-k} \mapsto f_0 \text{ mod } B\overline{\Omega}. \tag{3.1.8}$$

Proof. Introduce an increasing filtration on $\overline{\Omega} \otimes R$ as follows:

$$F_j(P\overline{\Omega} \otimes R) := P\overline{\Omega} \otimes (u^{-j}\mathbf{k}[\mathfrak{u}]/\mathfrak{u}\mathbf{k}[\mathfrak{u}]) = P\overline{\Omega} \otimes \text{Span}(u^{-j}, u^{-j+1}, \dots, u^{-1}, 1).$$

The fact that the complex $(P\overline{\Omega} \otimes R, B)$ is acyclic implies that the standard spectral sequence associated to our filtration collapses at the second page to $H(P\overline{\Omega}/B\overline{\Omega}, \mathfrak{b})$. We conclude that the map $\sum_{k \geq 0} f_k \cdot u^{-k} \mapsto f_0$ yields an isomorphism $H(P\overline{\Omega} \otimes R, \mathfrak{b} - \mathfrak{u}B) \xrightarrow{\sim} H(P\overline{\Omega}/B\overline{\Omega}, \mathfrak{b})$, of harmonic components. The same map also gives an isomorphism $H(P^\perp\overline{\Omega} \otimes R, \mathfrak{b} - \mathfrak{u}B) \xrightarrow{\sim} H(P^\perp\overline{\Omega}/B\overline{\Omega}, \mathfrak{b})$, of anti-harmonic components. Since $B(P^\perp\overline{\Omega}) = 0$, the second cohomology group is $H(P^\perp\overline{\Omega}, \mathfrak{b}) = 0$, so these cohomology groups vanish. \square

Remark 3.1.9. As pointed out to us, an alternative proof of the theorem can be given as follows: as mentioned in the proof of Lemma 2.2.5, Karoubi observed that κ has finite order on the quotient $\overline{\Omega}/\mathfrak{b}\overline{\Omega}$. Therefore, the κ -invariants and coinvariants in $\overline{\Omega}/\mathfrak{b}\overline{\Omega}$ are isomorphic. Hence, the maps $i : \overline{DR}^\bullet \rightarrow (\overline{\Omega}^{\bullet-1})^\kappa$ and $\mathfrak{b} : \overline{DR}^\bullet \rightarrow (\overline{\Omega}^{\bullet-1})^\kappa$ are identified, up to scaling. Then, the operator i can be replaced by \mathfrak{b} in the proof, and one can deduce (3.1.4) as in the proof of [Kar87, Lemme 2.12] (see also [Lo, p. 85]).

The isomorphism in (3.1.5) is now obtained as a composition:

$$\begin{aligned}
\overline{HC}(A) &= H(P\overline{\Omega}/B P\overline{\Omega}, \mathfrak{b}) \quad \text{by the proof of Lemma 3.1.7} \\
&= \frac{\{f \in P\overline{\Omega} \mid \mathfrak{b}f \in B P\overline{\Omega}\}}{B P\overline{\Omega} + B P\overline{\Omega}} \\
&= \ker(\mathfrak{b} : P\overline{\Omega}/(B P\overline{\Omega} + B P\overline{\Omega}) \rightarrow P\overline{\Omega}/B P\overline{\Omega}) \\
&= \ker(i : (P\overline{\Omega}/i P\overline{\Omega})/d(P\overline{\Omega}/i P\overline{\Omega}) \rightarrow P\overline{\Omega}/d P\overline{\Omega}) \quad \text{by (2.2.3)–(2.2.4)} \\
&= \ker(i : \overline{DR}/d\overline{DR} \rightarrow \overline{\Omega}/d\overline{\Omega}), \quad \text{by (3.1.6)}. \quad \square
\end{aligned}$$

3.2. Negative and periodic cyclic homology. Following [GS], we consider the bicomplex $(\overline{\Omega}((u)), u d, \mathfrak{i})$, where each of the differentials $u d$ and \mathfrak{i} has degree -1 . The result below provides an interpretation of periodic, negative, and ordinary cyclic homology in terms of the noncommutative de Rham complex (in the latter case, this differs from the description of the previous section).

Note that the space $\overline{[d\Omega, d\Omega]} \subseteq \overline{\Omega}$ is annihilated by each of the differentials d and \mathfrak{i} . We view it also as a subspace of $\overline{\Omega}[[u]]$ (and $\overline{\Omega}((u))$), as constant terms in u .

Theorem 3.2.1. *There are natural isomorphisms*

$$\overline{HC}^{\text{per}}(A) \cong H_*(\overline{\Omega}((u)), \mathfrak{i} - u d) \quad (3.2.2)$$

$$\overline{HC}^-(A) \cong H_*(\overline{\Omega}[[u]], \mathfrak{i} - u d)/\overline{[d\Omega, d\Omega]} \quad (3.2.3)$$

$$\overline{HC}_*(A) \cong H_*(\overline{\Omega} \otimes R, \mathfrak{i} - u d)/\overline{[d\Omega, d\Omega]}. \quad (3.2.4)$$

In the case of (3.2.3), the $\overline{[d\Omega, d\Omega]}$ includes into $H_*(\overline{\Omega}[[u]], \mathfrak{i} - u d)$ via the aforementioned inclusion into $\overline{\Omega}[[u]]$ as constant terms in u . As we will see in the proof of the theorem, the target coincides with the antiharmonic part of $H_*(\overline{\Omega}[[u]], \mathfrak{i} - u d)$, and hence the inclusion is canonically split.

In (3.2.4), the inclusion of $\overline{[d\Omega, d\Omega]}$ into $H_*(\overline{\Omega} \otimes R, \mathfrak{i} - u d)$ is more complicated, but can also be given explicitly (in the proof of the theorem, we will see that the target is the antiharmonic part, so this inclusion is also canonically split). The inclusion in question is defined as follows.

Observe first that the map $\bar{d} : \overline{\Omega}/d\overline{\Omega} \rightarrow d\overline{\Omega}$, induced by d , is an isomorphism since the complex $(\overline{\Omega}, d)$ is acyclic. Write $\bar{d}^{-1} : d\overline{\Omega} \xrightarrow{\sim} \overline{\Omega}/d\overline{\Omega}$ for an inverse isomorphism. Further, let $\tilde{d}^{-1} : d\overline{\Omega} \rightarrow \overline{\Omega}$ be a lift of \bar{d}^{-1} . In other words, we fix an arbitrary (set-theoretic) section, \tilde{d}^{-1} , of the surjection $d : \overline{\Omega} \twoheadrightarrow d\overline{\Omega}$.

With the above notation, the inclusion used in (3.2.4) is given by

$$\overline{[d\Omega, d\Omega]} \hookrightarrow H_*(\overline{\Omega} \otimes R, \mathfrak{i} - u d), \quad f \mapsto (\text{Id} - u^{-1}\tilde{d}^{-1}\mathfrak{i})^{-1}\tilde{d}^{-1}f.$$

One can show that this formula makes sense and the result is independent of the choice of a section \tilde{d}^{-1} , in a manner similar to the proof of Proposition 5.7.1 below.

In discussing Connes sequences involving these groups below, we will not use (3.2.4) but rather the description of (3.1.5). Isomorphism (3.2.2) was already proved in [GS, Theorem 4.2.2]; we included it in the above theorem for completeness only. For a related discussion of negative cyclic homology see also [GS].

Proof of (3.2.3) and (3.2.4). We first deal with (3.2.3). Similarly to the arguments in the proof of Theorem 3.1.3,

$$\overline{HC}^-(A) = H(\overline{\Omega}[[u]], \mathfrak{b} - u B) = H(P\overline{\Omega}[[u]], \mathfrak{b} - u B) = H(P\overline{\Omega}[[u]], \mathfrak{i} - u d).$$

Now, thanks to equation (2.2.8), we have embeddings $\overline{[d\Omega, d\Omega]} \xrightarrow{\sim} P^\perp d\overline{\Omega} \hookrightarrow P^\perp \overline{\Omega} \hookrightarrow P^\perp \overline{\Omega}[[u]]$. To complete the proof, it suffices to show that the composite embedding yields a quasi-isomorphism

of complexes

$$\overline{[d\Omega, d\Omega]} \rightarrow (P^\perp \overline{\Omega}[u], \mathbf{i} - u d), \quad (3.2.5)$$

where the LHS is considered as a complex with zero differential.

To this end, we observe that the operator \mathbf{i} vanishes on $P^\perp \overline{\Omega}$ and the complex $(\overline{\Omega}, d)$ is acyclic. Therefore, we find

$$H_*(P^\perp \overline{\Omega}[u], \mathbf{i} - u d) = H_*(P^\perp \overline{\Omega}[u], -u d) \cong dP^\perp \overline{\Omega}^*.$$

The term on the right equals $\overline{[d\Omega, d\Omega]}$ by (2.2.8), proving that (3.2.5) is a quasi-isomorphism.

For the isomorphism (3.2.4), we can apply the same argument as above to conclude that

$$\overline{HC}(A) = H(P^\perp \overline{\Omega} \otimes R, \mathbf{b} - u B) = H(P^\perp \overline{\Omega} \otimes R, \mathbf{i} - u d).$$

Then, since \mathbf{i} vanishes on $P^\perp \overline{\Omega}$ and the complex $(P^\perp \overline{\Omega}, d)$ is acyclic, we obtain

$$H(P^\perp \overline{\Omega} \otimes R, d) = P^\perp \overline{\Omega} / u d(u^{-1} P^\perp \overline{\Omega}) \cong P^\perp \overline{\Omega} / d P^\perp \overline{\Omega} \xrightarrow{\tilde{d}} d P^\perp \overline{\Omega} \cong \overline{[d\Omega, d\Omega]}.$$

Finally, note that the resulting isomorphism $H(P^\perp \overline{\Omega} \otimes R, d) \cong \overline{[d\Omega, d\Omega]}$ is indeed inverted by the claimed formula $(\text{Id} - u^{-1} \tilde{d}^{-1} \mathbf{i})^{-1} \tilde{d}^{-1}$, provided the chosen section \tilde{d}^{-1} respects harmonic decomposition (or at least preserves antiharmonic forms). In fact, in this case, since \mathbf{i} is zero on antiharmonic forms, the resulting map reduces to $\tilde{d}^{-1} = P^\perp \tilde{d}^{-1}$. On the other hand, as observed above, the formula does not actually depend on the choice of \tilde{d}^{-1} . \square

4. THE CONNES EXACT SEQUENCE

4.1. Connes exact sequence via noncommutative differential forms. We now construct a version of Connes' exact sequence, involving reduced homology groups, using the interpretation for cyclic and Hochschild homology provided by Theorem 3.1.3. As pointed out in §1.1, the proof is simpler than those of Karoubi and Connes, and unlike the former, does not involve double complexes.

Theorem 4.1.1. *There is an exact sequence,*

$$0 \rightarrow \overline{HD}_n(A) \xrightarrow{S_2} \overline{HC}_n(A) \xrightarrow{B} \overline{HH}_{n+1}(A) \xrightarrow{I} \overline{HC}_{n+1}(A) \xrightarrow{S_1} \overline{HD}_{n-1}(A) \rightarrow 0, \quad (4.1.2)$$

where the maps $S_2, B, I,$ and S_1 are defined by the following assignments:

$$S_2 f_{\mathfrak{h}} = f_{\mathfrak{h}}, \quad B f_{\mathfrak{h}} = (df)_{\mathfrak{h}}, \quad I f_{\mathfrak{h}} = f_{\mathfrak{h}}, \quad S_1(f_{\mathfrak{h}}) = (\tilde{d}^{-1} \mathbf{i} f_{\mathfrak{h}})_{\mathfrak{h}}. \quad (4.1.3)$$

Note here that the map $B : \overline{HC}_n(A) \rightarrow \overline{HH}_{n-1}(A)$ is distinguished by its font from the differential B on Ω and $\overline{\Omega}$. We let $S := S_2 \circ S_1 : \overline{HC}_n(A) \rightarrow \overline{HC}_{n-2}(A)$. This is the standard periodicity map which splices the above exact sequence into the more familiar Connes' long exact sequence. The exact sequence of Theorem 4.1.1 incorporates additional information, implying that $\overline{HD}_*(A) = \ker(B)$ (the Connes-Karoubi theorem).

We now prove that the above maps are well-defined. We will prove that they coincide with the usual definitions in §4.2, and give a direct proof of exactness (using the right hand sides of (3.1.4)–(3.1.5) as definitions) in §4.3.

The map S_2 is obtained from the clearly well-defined map $\ker(d|_{\overline{DR}}) / d\overline{DR} \rightarrow \overline{DR} / d\overline{DR}$; we need to check that the image is contained in the kernel of \mathbf{i} . Indeed, if $(df)_{\mathfrak{h}} = 0$, then $d \mathbf{i} f_{\mathfrak{h}} = -\mathbf{i}(df)_{\mathfrak{h}} = 0$, and hence $\mathbf{i} f_{\mathfrak{h}} \in d\overline{\Omega}$, as required.

We now show that B is well-defined. First note that, if $f_{\mathfrak{h}} \in \overline{DR} / d\overline{DR}$, then $(df)_{\mathfrak{h}} \in \overline{DR}$ makes sense independently of the choice of representative element $f \in \overline{\Omega}$. Next, if $\mathbf{i} f_{\mathfrak{h}} \in d\overline{\Omega}$, then $\mathbf{i}(df)_{\mathfrak{h}} = -d \mathbf{i} f_{\mathfrak{h}} = 0$, so $(df)_{\mathfrak{h}} \in \overline{DR}$ defines a class in Hochschild homology.

It is immediate that I is well-defined.

To prove that S_1 is well defined, we compute

$$\bar{d}^{-1} i d \bar{D}\bar{R} = \bar{d}^{-1} d i \bar{D}\bar{R} = (i \bar{D}\bar{R} + d \bar{\Omega}) / d \bar{\Omega}.$$

Since $i \bar{D}\bar{R} \subseteq \overline{[A, \Omega]}$, by (2.1.3), the right most term above projects to zero in $\bar{D}\bar{R} / d \bar{D}\bar{R}$. It follows that the formula for S_1 gives a well defined map $S_1 : \overline{HC}(A) \rightarrow \bar{D}\bar{R} / d \bar{D}\bar{R}$. It remains to check that $d S_1(f) = 0$ for every $f \in \bar{D}\bar{R}$ such that $i f \in d \bar{\Omega}$. Indeed, $d S_1(f) = d(\bar{d}^{-1} i f)_{\natural} = (i f)_{\natural} = 0$, since $i \bar{D}\bar{R} \subseteq \overline{[A, \Omega]}$. \square

4.2. Comparison of maps in (4.1.2) with the usual definitions. We now show that the maps in (4.1.2) coincide with the usual ones up to nonzero integer multiples depending on degree (i.e., a nonvanishing expression involving the aforementioned degree operator N). We will make use of the fact that every homology class can be represented by a harmonic cycle.

Here and later on we will make use of the notation f_0 for the constant term in u of an element $f \in \bar{\Omega}((u))$ (or $f \in \bar{\Omega}[[u]]$).

4.2.1. The map B . If f is a harmonic cycle with respect to the usual definition of Hochschild or cyclic homology, then (2.2.3) implies that $B f = N d f$. Since the usual B is essentially defined by applying B and our version is defined by applying d , this shows that the difference is post-composition with N , which is nonvanishing since d has de Rham degree 1.

4.2.2. The map I . The map I is defined by leaving the harmonic cycle representing a homology class unchanged, and mapping it from one complex to another (throwing out the coefficients of $u^{>0}$ in the case of I acting on the usual definition of cyclic homology). Since the isomorphisms from ordinary versions to our versions of Hochschild and cyclic homology also leave the harmonic cycles unchanged, we obtain the desired compatibility.

4.2.3. The maps S_1 and S_2 . Let us consider S_1 . Suppose that $f \in \bar{\Omega}((u))/u\bar{\Omega}[[u]]$ is a harmonic cycle of degree $n + 1$, i.e., $|f| = n + 1$ under the total degree in which $|\bar{\Omega}^j| = j$ and $|u| = -2$. The usual periodicity operation is $f \mapsto u \cdot f$. We need to show that $i(f_0)_{\natural} = d((u f)_0)_{\natural}$. Since $(i - u d)f = 0$, this is immediate.

Regarding the map S_2 , we only need to observe that the usual formula for S_2 is uniquely determined by the fact that the part mapping to a coefficient of u^0 is the obvious map (leaving harmonic cycles unchanged). (To obtain the coefficients of $u^{<0}$, we can use the usual periodicity map $S = S_2 \circ S_1$, which is multiplication by u . Then, our construction yields the explicit formula $\bar{d}^{-1} i$ for this operation, by the previous paragraph.)

4.3. Direct proof of exactness. Since we showed that the maps in (4.1.2) are compatible with the isomorphisms between the new versions of cyclic and Hochschild homology and the usual ones, exactness follows from the usual construction of (4.1.2) by a commutative diagram of short exact sequences of the complexes computing Hochschild and cyclic homologies (cf. [Lo]).

Nonetheless, here we will prove exactness directly using our formulas (3.1.4)–(3.1.5).

Injectivity of S_2 : By definition, $\overline{HD}(A) \subseteq \bar{D}\bar{R} / d \bar{D}\bar{R}$.

$\text{im}(S_2) = \ker(B)$: By definition, the cycles in $\ker(B)$ are represented by $f_{\natural} \in \bar{D}\bar{R}^n$ such that $d f_{\natural} = 0$ and $i f_{\natural} \in d \bar{\Omega}$. The second condition is equivalent to $d i f_{\natural} = 0$, which follows from the first condition. We deduce that $\ker(B) = \overline{HD}_n(A)$, as desired.

$\text{im}(B) = \ker(I)$: The cycles in $\ker(I)$ are those elements $f_{\natural} \in \bar{D}\bar{R}^n$ such that $i f_{\natural} = 0$ and $f_{\natural} \in d \bar{D}\bar{R}^{n-1}$. Given the second condition, we can assume $f = dg$. Then $i(dg)_{\natural} = i d g_{\natural} = 0$ holds if and only if $i g_{\natural} \in d \bar{\Omega}$, i.e., $g_{\natural} \in \overline{HC}_{n-1}(A)$. Conversely, it is clear that $\text{im}(B) \subseteq \ker(I)$. We conclude that $\ker(I) = \text{im}(B)$.

$\text{im}(I) = \ker(S_1)$: Every cycle in $\ker(S_1)$ is represented by a harmonic element $f_{\natural} \in \overline{\text{DR}}$ such that $i f_{\natural} \in d[\overline{\Omega}, \overline{\Omega}]$. By (2.2.6), $P[\overline{\Omega}, \overline{\Omega}] = \mathbf{i}\overline{\Omega}$. We deduce that

$$i f_{\natural} = P i f_{\natural} \in P d[\overline{\Omega}, \overline{\Omega}] = d P[\overline{\Omega}, \overline{\Omega}] = d \mathbf{i}\overline{\Omega} = \mathbf{i} d \overline{\Omega}.$$

It follows that one can find $g, h \in \overline{\Omega}$ such that $i(g_{\natural}) = 0$ and $f_{\natural} = g_{\natural} + d h_{\natural}$. Thus, in $\overline{\text{HC}}(A)$, we obtain $f_{\natural} \text{ mod } d \overline{\text{DR}} = I(g_{\natural})$, as desired.

Surjectivity of S_1 : Every class in $\overline{\text{HD}}(A)$ is represented by a harmonic element $f \in \overline{\Omega}$ such that $d f_{\natural} = 0$. This means that $d f \in P[\overline{\Omega}, \overline{\Omega}] = \mathbf{i}\overline{\Omega}$, thanks to (2.2.6). Thus, there exists $g \in \overline{\Omega}$ such that, in $\overline{\text{HD}}(A)$, we obtain $f_{\natural} = \overline{d}^{-1} \mathbf{i}(g) = S_1(g_{\natural})$, as desired.

4.4. Connes' exact sequences for periodic and negative cyclic homology. Now, using Theorem 3.2.1 (for periodic and negative cyclic homology) as well as Theorem 3.1.3 (for Hochschild and ordinary cyclic homology), we construct versions of Connes' exact sequences involving periodic and negative cyclic homology.

It will be convenient below to introduce, for each $n = 0, 1, \dots$, the following space of degree n elements in $\overline{\Omega}[[u]]$:

$$\begin{aligned} Z_n(A) &:= \{f \in (\overline{\Omega}[[u]])_n \mid (\mathbf{i} - u d)f \text{ is independent of } u\} \\ &= \{f = f^n + u f^{n+2} + u^2 f^{n+4} + \dots \mid f^j \in \overline{\Omega}^j, d f^j = \mathbf{i}(f^{j+2}), \forall j = n, n+2, \dots\}. \end{aligned} \quad (4.4.1)$$

We observe that, for every element $f = f^n + u f^{n+2} + u^2 f^{n+4} + \dots \in Z_n(A)$, in $\overline{\text{DR}}^{n+1}$, one has $d(f^n)_{\natural} = (\mathbf{i}(f^{n+2}))_{\natural} = 0$. Therefore, the assignment sending $f = f^n + u f^{n+2} + u^2 f^{n+4} + \dots$ to $(f^n)_{\natural} \in \overline{\text{DR}}^n$ gives a well defined map $I' : Z(A) \rightarrow \{\alpha \in \overline{\text{DR}} \mid d\alpha = 0\}$. In particular, one has an induced map

$$I'_{\text{HD}} : Z(A) \rightarrow \overline{\text{HD}}(A) = \{\alpha \in \overline{\text{DR}} \mid d\alpha = 0\} / d \overline{\text{DR}}.$$

Next, it is clear from the definition that, for $f = f^n + u f^{n+2} + u^2 f^{n+4} + \dots \in Z_n(A)$, the condition $(\mathbf{i} - u d)f = 0$ is equivalent to $\mathbf{i}(f^n) = 0$. Also, since $\mathbf{i}\overline{\Omega}$ projects to zero in $\overline{\text{DR}}$, the map I' sends the subspace $(\mathbf{i} - u d)\overline{\Omega}[[u]] \subseteq Z(A)$ of boundaries to zero. Hence, using (3.1.4), we deduce that the map I' gives a well defined morphism

$$I'_{\text{HH}} : H_*(\overline{\Omega}[[u]], \mathbf{i} - u d) \rightarrow \overline{\text{HH}}_*(A).$$

We define

$$\begin{aligned} \overline{\text{HD}}'(A) &:= \text{im}[I'_{\text{HD}} : Z(A) \rightarrow \overline{\text{HD}}(A)] \\ \overline{\text{HH}}'(A) &:= \text{im}[I'_{\text{HH}} : H_*(\overline{\Omega}[[u]], \mathbf{i} - u d) \rightarrow \overline{\text{HH}}(A)]. \end{aligned} \quad (4.4.2)$$

Theorem 4.4.3. *There is a commutative diagram with exact rows,*

$$\begin{array}{ccccccccccc} 0 & \rightarrow & \overline{\text{HH}}'_n & \longrightarrow & \overline{\text{HH}}_n & \xrightarrow{B^-} & \overline{\text{HC}}_{n+1}^- & \xrightarrow{S^-} & \overline{\text{HC}}_{n-1}^- & \xrightarrow{I^-} & \overline{\text{HH}}'_{n-1} & \rightarrow & 0 \\ & & \downarrow J & & \downarrow I & & \parallel & & \searrow p^- & & \downarrow J & & \\ 0 & \rightarrow & \overline{\text{HD}}'_n & \xrightarrow{p_2} & \overline{\text{HC}}_n & \xrightarrow{B'} & \overline{\text{HC}}_{n+1}^- & \xrightarrow{p^-} & \overline{\text{HC}}_{n+1}^{\text{per}} \cong \overline{\text{HC}}_{n-1}^{\text{per}} & \xrightarrow{p_1} & \overline{\text{HD}}'_{n-1} & \rightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow I^- & & \searrow p & & \downarrow & & \\ 0 & \rightarrow & \overline{\text{HD}}_n & \xrightarrow{S_2} & \overline{\text{HC}}_n & \xrightarrow{B} & \overline{\text{HH}}_{n+1} & \xrightarrow{I} & \overline{\text{HC}}_{n+1} & \xrightarrow{S_1} & \overline{\text{HD}}_{n-1} & \rightarrow & 0 \end{array} \quad (4.4.4)$$

The maps in the diagram are defined by the assignments

$$\begin{aligned} p_2 f_{\natural} &= f_{\natural}, & B^-(f_{\natural}) &= df, & B'(f_{\natural}) &= df, & S^-(f) &= uf, & p^-(f) &= f, \\ I^-(f) &= (f_0)_{\natural}, & p_1(f) &= (f_0)_{\natural}, & Jf_{\natural} &= f_{\natural}, & p &= p_2 \circ p_1. \end{aligned}$$

Let us explain why all the maps above are well-defined. For B^- , it is immediate that $B^- = B'I$, so it is enough to show that B' is well-defined. To this end, let $f \in \overline{\Omega}$ satisfy $i f_{\natural} \in d\overline{\Omega}$. Then, $(i - u d) df = 0$, so that df indeed defines a class in negative cyclic homology. We then need to show that $d[\overline{\Omega}, \overline{\Omega}] \subseteq (i - u d)\overline{\Omega}[[u]] + \overline{[d\Omega, d\Omega]}$. To prove this, we write $\overline{[\Omega, \Omega]} = P[\overline{\Omega}, \overline{\Omega}] + P^\perp[\overline{\Omega}, \overline{\Omega}]$. We know that $P[\overline{\Omega}, \overline{\Omega}] = i\overline{\Omega}$, by (2.2.6). Further, we compute

$$P^\perp[\overline{\Omega}, \overline{\Omega}] \stackrel{(2.2.10)}{=} P^\perp \mathfrak{b}\overline{\Omega} + P^\perp(\text{Id} - \kappa)\overline{\Omega} \stackrel{(2.2.9)}{=} (\text{Id} - \kappa) \mathfrak{b}\overline{\Omega} + (\text{Id} - \kappa)P^\perp\overline{\Omega} \subset (\text{Id} - \kappa)\overline{\Omega} = \overline{[dA, \Omega]}.$$

Thus, we have proved the inclusion we need: $\overline{[\Omega, \Omega]} \subset i\overline{\Omega} + \overline{[d\Omega, \Omega]}$.

It is immediate that S^- is well-defined. To show that p^- is well-defined, we only need to show that the space $\overline{[d\Omega, d\Omega]}$ is contained in the image of the differential $i - u d$, on $\overline{\Omega}((u))$. This follows from the fact that $i[\overline{\Omega}, d\overline{\Omega}] = 0$.

For the map I^- , we observe that this map is induced by the map (4.4.2). This makes sense since the map I' annihilates the space $\overline{[d\Omega, d\Omega]} \subseteq Z(A)$.

Finally, it is immediate that p_1 and J are well-defined, and that the diagram commutes.

4.5. Comparison of the maps in (4.4.4) with the usual definitions. The fact that B^- and B' coincide with the usual definitions is the same as the reason we gave for B in §4.2: these maps take harmonic representatives f to df under our definitions, and to Bf in the usual definitions, which differ by a scalar depending on degree. Similarly, for I^- and J , these maps leave harmonic representatives unchanged.

The fact that the periodicity map S^- is compatible with our isomorphism is obvious, since it is defined by multiplication by u in both cases.

Finally, the maps p and p^- , like the case of the maps of type I , send harmonic cycles to the classes represented by the same harmonic cycles (in the case of p , taking the coefficients of $u^{\leq 0}$ or u^0).

4.6. Direct proof of exactness in diagram (4.4.4).

4.6.1. The middle row of (4.4.4). Injectivity of p_2 : This is clear from construction (or from injectivity of S_2).

$\text{im}(p_2) = \ker(B')$: The cycles in $\ker(B')$ are represented by $f_{\natural} \in \overline{\text{HC}}(A)$ such that $df = (i - u d)g + h$ for some $g \in \overline{\Omega}[[u]]$ and some $h \in \overline{[d\Omega, d\Omega]}$. Restricting to harmonic f and g , by (2.2.8) we can also set $h = 0$. Then, $(i - u d)(f + ug) = i f_{\natural} \in d\overline{\Omega}$. So $\ker(B') \subseteq \text{im}(p_2)$. For the opposite inclusion, if $f \in Z(A)$, then $(i - u d)f = i(f_0)_{\natural} \in d\overline{\Omega}$. Thus, $df_0 = (i - u d)u^{-1}(f_0 - f)$, so $(f_0)_{\natural} \in \ker(B')$.

$\text{im}(B') = \ker(p^-)$: The elements in $\ker(p^-)$ are represented by harmonic elements $f \in \overline{\Omega}[[u]]$ such that $f = (i - u d)g$ where $g \in \overline{\Omega}((u))$ is harmonic. Replacing f with $f - (i - u d)\sum_{j \geq 0} g_j u^j$, we can assume that $g_j = 0$ for $j \geq 0$. Then, $f = f_0 = dg_{-1}$, and $i(g_{-1})_{\natural} \in d\overline{\Omega}$. Hence, $(g_{-1})_{\natural} \in \overline{\text{HC}}(A)$ and $f = B'(g_{-1})_{\natural}$. Therefore, $\ker(p^-) \subseteq \text{im}(B')$. For the opposite inclusion, suppose that $f \in \text{im}(B')$. That is, there exists $h \in \overline{\Omega}$ satisfying $f = dh$ and $i h_{\natural} \in d\overline{\Omega}$. Now, we can set $g_{-1} := h$, and inductively we can choose g_{-j} for $j > 1$ such that $d g_{-j} = i g_{1-j}$ for all $j > 1$. We deduce that $-(i - u d)g = dh = f$. Therefore, $\text{im}(B') \subseteq \ker(p^-)$.

$u \text{im}(p^-) = \ker(p_1)$: Elements of $\ker(p_1)$ are $f \in \overline{\Omega}((u))$ such that $f_0 \in d\overline{\Omega} + \overline{[\Omega, \Omega]}$ and $(i - u d)f = 0$. We claim first that $f_j \in d\overline{\Omega} + \overline{[\Omega, \Omega]}$ for all $j \leq 0$. Indeed, $df_j = i(f_{j+1})_{\natural}$ for $j < 0$. So, if

$f_{j+1} \in d\bar{\Omega} + [\bar{\Omega}, \bar{\Omega}]$, then $df_j = i(f_{j+1})_{\natural} \in \mathbf{i}d\bar{\Omega} = d\mathbf{i}\bar{\Omega} \subseteq d[\bar{\Omega}, \bar{\Omega}]$. By acyclicity of $(\bar{\Omega}, d)$, we conclude that $f_j \in d\bar{\Omega} + [\bar{\Omega}, \bar{\Omega}]$ as well, so by induction, we conclude the desired result.

Suppose further that f is harmonic. Then, $f_0 \in dP\bar{\Omega} + \mathbf{i}P\bar{\Omega}$ since $P[\bar{\Omega}, \bar{\Omega}] = \mathbf{i}P\bar{\Omega}$, by (2.2.6). Writing $f_0 = dg + \mathbf{i}h_{\natural}$, we conclude that $f' := f + (\mathbf{i} - ud)(u^{-1}g - h)$ satisfies $f'_0 = 0$, and $f'_j \in d\bar{\Omega} + [\bar{\Omega}, \bar{\Omega}]$ for $j < 0$. Applying the same reasoning to uf' , we eventually conclude that f is homologous to a harmonic cycle $\tilde{f} \in uP\bar{\Omega}[[u]]$. Therefore, $f \in u \cdot \text{im}(p^-)$.

The converse containment, $u \cdot \text{im}(p^-) \subseteq \ker(p_1)$, is obvious.

Surjectivity of p_1 : Suppose that $f \in Z(A)$. Then $(\mathbf{i} - ud)f = i(f_0)_{\natural} \in d\bar{\Omega}$. As observed above, setting $g_0 = f_0$, we can choose g_{-j} for $j \geq 1$ such that $d g_{-j} = \mathbf{i}g_{1-j}$ for all $j \geq 1$. Set $h := g + (f - f_0)$. Then, $(\mathbf{i} - ud)h = -udf_0 - i(f_0)_{\natural} + (udf_0 + i(f_0)_{\natural}) = 0$. Since $(h_0)_{\natural} = (f_0)_{\natural}$, we conclude that $(f_0)_{\natural} = p_1(h)$ is in the image of p_1 .

4.6.2. The top row of (4.4.4). The first map is tautologically injective.

$\ker(B^-) = \overline{\text{HH}}'(A)$: Assume that f is harmonic and $i f_{\natural} = 0$, and suppose that $f_{\natural} \in \overline{\text{HH}}(A)$ satisfies $df = (\mathbf{i} - ud)g$ for some harmonic $g \in \bar{\Omega}[[u]]$. Then, $(\mathbf{i} - ud)(f - ug) = 0$, so $f_{\natural} \in \overline{\text{HH}}'_n(A)$. Thus $\ker(B^-) \subseteq \overline{\text{HH}}'(A)$. Conversely, if $f \in \ker(\mathbf{i} - ud) \subseteq \bar{\Omega}[[u]]$, then $df_0 = (\mathbf{i} - ud)(f - f_0)$, so $(f_0)_{\natural} \in \ker(B^-)$.

$\text{im}(B^-) = \ker(S^-)$: The kernel of S^- is represented by elements of the form $u^{-1}(\mathbf{i} - ud)g$, where $g \in P\bar{\Omega}[[u]]$ satisfies $i(g_0)_{\natural} = 0$. Up to coboundary, this is the same as elements $d g_0$ such that $i(g_0)_{\natural} = 0$, i.e., the image of B^- . Conversely, if $i(g_0)_{\natural} = 0$, then $B^-(g_0)_{\natural} = d g_0 = -u^{-1}(\mathbf{i} - ud)(g_0)$ is in the kernel of S^- .

$\text{im}(S^-) = \ker(I^-)$: The kernel of I^- is represented by elements $f \in \bar{\Omega}[[u]]$ such that $f_0 \in [\bar{\Omega}, \bar{\Omega}]$. If we assume that f is harmonic, then $f_0 \in \mathbf{b}P\bar{\Omega} = \mathbf{i}P\bar{\Omega}$, see (2.2.6). Write $f_0 = \mathbf{i}g_{\natural}$ for some g . Then, $f - (\mathbf{i} - ud)g \in u\bar{\Omega}[[u]]$. Hence, the class of f is in the image of S^- . Conversely, it is clear that the image of S^- maps to zero under I^- .

Surjectivity of I^- : This is immediate from the definition of $\overline{\text{HH}}'(A)$.

5. EXTENDED KAROUBI-DE RHAM COMPLEX AND CYCLIC HOMOLOGY

In this section, we will consider, following [GS], an “extended” version, $\overline{\text{DR}}_t A$, of the noncommutative de Rham complex. One of the reasons for introducing such an extended version is that it maps naturally to the *equivariant* (as opposed to the ordinary) de Rham complex of representation varieties of the algebra in question, see §8.

Another important feature of $\overline{\text{DR}}_t$ is that it comes equipped with natural anti-commuting differentials d and i_t . We will show that the homology of the complex $(\overline{\text{DR}}_t A \otimes R, i_t - ud)$ captures both the cyclic homology $\overline{\text{HC}}(A)$ and the Karoubi-de Rham homology $\overline{\text{HD}}(A)$ at the same time.

5.1. Extended noncommutative de Rham complex. Let $B *_k C$ denote the free product of unital algebras B and C over k . Define

$$\Omega_t = \Omega_t A := \Omega(A *_k \mathbf{k}[t]) / (dt) \cong \Omega A *_k \mathbf{k}[t], \quad (5.1.1)$$

where (dt) denotes the two-sided ideal generated by the element dt . For all p and q , let $\Omega_t^{p,q} \subseteq \Omega_t$ denote the subspace of de Rham degree p and degree q in t (note that this notation differs from that of [GS]). The isomorphism on the right of (5.1.1) respects the bigradings.

Next, we define the commutator quotient

$$\text{DR}_t = \text{DR}_t A := \Omega_t / [\Omega_t, \Omega_t], \quad \text{DR}_t = \bigoplus_{p,q} \text{DR}_t^{p,q} \quad \text{where} \quad \text{DR}_t^{p,q} = (\Omega_t^{p,q})_{\natural}.$$

As observed in [GS], for each $q \geq 1$, there is a canonical isomorphism $\mathrm{DR}_t^{\bullet,q} \cong \Omega^{\otimes q}/(\mathbb{Z}/q)$. This yields a direct sum decomposition

$$\mathrm{DR}_t = \bigoplus_{q \geq 0} \mathrm{DR}_t^{\bullet,q} \cong \mathrm{DR} \oplus \Omega \oplus \Omega^{\otimes 2}/(\mathbb{Z}/2) \oplus \cdots. \quad (5.1.2)$$

In particular, there is a canonical isomorphism $\iota : \mathrm{DR}_t^{\bullet,1} = (t\Omega^\bullet)_{\natural} \xrightarrow{\sim} \Omega^\bullet$.

One has a natural differential $d : \Omega_t^{\bullet,q} \rightarrow \Omega_t^{\bullet+1,q}$ that restricts to the ordinary Karoubi-de Rham differential on Ω , and kills t . One also has a derivation $\frac{d}{dt} : \Omega_t^{p,\bullet} \rightarrow \mathrm{DR}_t^{p,\bullet-1}$ that kills $\Omega \subset \Omega_t$ and sends t to 1. These maps descend to DR_t .

There are also reduced versions of all the above. In particular, $\overline{\Omega}_t = \overline{\Omega}_t A := \Omega_t/\mathbf{k}[t]$ and $\overline{\mathrm{DR}}_t = \overline{\mathrm{DR}}_t A := \overline{\Omega}_t/[\overline{\Omega}_t, \overline{\Omega}_t] = \mathrm{DR}_t/\mathbf{k}[t]$. Further, we put $\overline{\mathrm{DR}}_t^+ := \bigoplus_{q > 0} \overline{\mathrm{DR}}_t^{\bullet,q} = (t\overline{\Omega}_t)_{\natural} \subseteq \overline{\mathrm{DR}}_t$.

Lemma 5.1.3. *The complex $(\overline{\mathrm{DR}}_t^+, d)$ is acyclic.*

Proof. From (5.1.2), we see that $\mathrm{DR}_t^+ = \Omega \oplus \Omega^{\otimes 2}/(\mathbb{Z}/2) \oplus \cdots$. Now, $H(\Omega, d) \cong \mathbf{k}$, spanned by the natural inclusion $\mathbf{k} \subseteq \Omega$. Taking coinvariants with respect to the cyclic group \mathbb{Z}/q is exact for all $q \geq 1$, thanks to our assumption that $\mathbf{k} \supseteq \mathbb{Q}$. Hence, $H(\mathrm{DR}_t^+, d) = \bigoplus_{q \geq 1} \mathbf{k}^{\otimes q}$, spanned by the inclusions $\mathbf{k}^{\otimes q} \cong \mathbf{k} \cdot (t^q)_{\natural} \subseteq \mathrm{DR}_t^+$. The result now follows from the long exact sequence on homology associated to the short exact sequence $0 \rightarrow t \cdot \mathbf{k}[t] \rightarrow \mathrm{DR}_t^+ \rightarrow \overline{\mathrm{DR}}_t^+ \rightarrow 0$. \square

As an immediate consequence of Lemma 5.1.3 we obtain

Corollary 5.1.4. *The natural projection $\overline{\mathrm{DR}}_t \rightarrow \overline{\mathrm{DR}}_t / \overline{\mathrm{DR}}_t^+ = \overline{\mathrm{DR}}$, modulo t , induces an isomorphism $H(\overline{\mathrm{DR}}_t, d) \xrightarrow{\sim} H(\overline{\mathrm{DR}}, d)$, of cohomology.*

5.2. Following [GS], we observe that the derivation $(-\mathrm{ad} t)$ of the algebra $A *_{\mathbf{k}} \mathbf{k}[t]$ naturally gives rise to a contraction operation $\mathbf{i}_t : \Omega_t^{p,q} \rightarrow \Omega_t^{p-1,q+1}$. By definition, the map \mathbf{i}_t is the (super) derivation of the algebra Ω_t that acts on generators by

$$\mathbf{i}_t : t \mapsto 0, \quad \Omega^0 = A \mapsto 0, \quad \Omega^1 \supset dA \ni da \mapsto [a, t] \quad \forall a \in A.$$

One checks that $\mathbf{i}_t^2 = 0$ and that \mathbf{i}_t descends to a map $i_t : \mathrm{DR}_t \rightarrow \mathrm{DR}_t$. The operations \mathbf{i}_t and i_t also descend to the corresponding reduced versions.

We are now able to give, following [GS], a more conceptual definition of the map $i : \mathrm{DR}^\bullet \rightarrow \Omega^{\bullet-1}$ that was introduced in §2 using an explicit formula (2.1.3). Specifically, one easily checks that

$$i = \iota \circ i_t, \quad \text{as maps} \quad \mathrm{DR}^\bullet = \mathrm{DR}_t^{\bullet,0} \longrightarrow \Omega^{\bullet-1} = \mathrm{DR}_t^{\bullet-1,1}. \quad (5.2.1)$$

It follows immediately that the differentials d and \mathbf{i}_t (hence, also i_t) commute with $\frac{d}{dt}$, and that the differentials d and i_t , on $\overline{\mathrm{DR}}_t$, anti-commute (see [CBEG] and [GS] for the latter fact).

5.3. The Tsygan map. Let $\overline{\mathrm{DR}}_t^{\wedge \bullet} = \overline{\mathrm{DR}}_t^{\wedge}(A)^\bullet := \prod_{q \geq 0} \overline{\mathrm{DR}}_t^{\bullet,q}$ be the t -adic completion of $\overline{\mathrm{DR}}_t$. We define the *Tsygan map* as a \mathbf{k} -linear map $E : \overline{\Omega} \rightarrow \overline{\mathrm{DR}}_t^{\wedge \bullet}$ given, for any $a_0, a_1, \dots, a_n \in A$, by

$$E(a_0 da_1 \cdots da_n) = \sum_{k_0, \dots, k_n \geq 0} \frac{1}{(k_0 + \cdots + k_n + n)!} (a_0 t^{k_0} da_1 t^{k_1} \cdots da_n t^{k_n})_{\natural}. \quad (5.3.1)$$

The map $E|_{\overline{\Omega}^n}$ may be viewed as a noncommutative analogue of multiplication by $\frac{1}{n!} \exp(t)$. To explain this, we view the algebra $\Omega \otimes \mathbf{k}[t]$ as a quotient of the algebra $\Omega *_{\mathbf{k}} \mathbf{k}[t]$ by the two-sided ideal generated by $[t, \Omega]$. Thus, one has the natural algebra homomorphism $\Omega_t \rightarrow \Omega[t]$ that ‘makes the variable t central.’ This homomorphism clearly descends to a linear map $\overline{\mathrm{DR}}_t \rightarrow \overline{\mathrm{DR}}[t]$. The latter map can further be extended, by continuity, to a map $c_t : \overline{\mathrm{DR}}_t^{\wedge} \rightarrow \overline{\mathrm{DR}}[[t]]$, between the corresponding t -adic completions. Then, one has

Lemma 5.3.2. For every $f \in \overline{\Omega}^n$, in $\overline{\text{DR}}[[t]]$ we have: $c_t \circ E(f) = \frac{1}{n!} \cdot \exp(t) f_{\natural}$.

Proof. Given positive integers k and n , let $P_n(k)$ denote the number of partitions $k_0 + \dots + k_n = k$, of k into $n + 1$ parts. That number is given by the formula $P_n(k) = \frac{(k+n)!}{k! n!}$. Using this and the definition of the map E , for every $f = a_0 da_1 \dots da_n \in \overline{\Omega}^n$, we compute

$$\begin{aligned} c_t \circ E(f) &= \sum_{k_0, \dots, k_n \geq 0} \frac{1}{(k_0 + \dots + k_n + n)!} \cdot c_t((a_0 t^{k_0} da_1 t^{k_1} \dots da_n t^{k_n})_{\natural}) \\ &= \sum_{k_0, \dots, k_n \geq 0} \frac{1}{(k_0 + \dots + k_n + n)!} \cdot t^{k_0 + \dots + k_n} (a_0 da_1 \dots da_n)_{\natural} = \sum_{k \geq 0} P_n(k) \frac{1}{(k+n)!} \cdot t^k f_{\natural} \\ &= \sum_{k \geq 0} \frac{(k+n)!}{k! n!} \frac{1}{(k+n)!} \cdot t^k f_{\natural} = \sum_{k \geq 0} \frac{1}{k! n!} \cdot t^k f_{\natural} = \frac{1}{n!} \cdot \exp(t) f_{\natural}. \quad \square \end{aligned}$$

Theorem 5.3.3. The map E has the properties:

$$(i) \quad \frac{d}{dt} \circ E = E; \quad (ii) \quad E \circ B = d \circ E; \quad (iii) \quad E \circ b = i_t \circ E. \quad (5.3.4)$$

This theorem will be proved in §5.4.

Remark 5.3.5. We have arrived at the definition of the map E by analyzing a construction of Tsygan [Tsy2] (that construction was, in its turn, motivated by our results [GS]). In particular, equations (5.3.4)(ii)–(iii) are an adaptation of the main result of [Tsy2]. It is likely that a proof of these equations, which is more conceptual than the one given in §5.4 below, can be extracted from [Tsy2], although the arguments in *op. cit.* are written in the setting of the standard (i.e., commutative) equivariant cochain complex rather than the extended noncommutative de Rham complex. However, such a noncommutative adaptation of the argument of [Tsy2] would be less direct since it would have to go through an auxiliary, much larger, complex $\overline{\Omega}(\overline{\Omega}A)$ (this would allow one to write an analogue of E as an exponential of a certain contraction map on this larger complex and to apply the noncommutative calculus developed in [TT]).

Remark 5.3.6. One can use (5.3.4)(i) to find the coefficients of the map E in the summation (5.3.1). Although one might expect that these coefficients should be $\frac{1}{\text{deg}_t!}$, inspecting the LHS of (5.3.4)(i) more carefully one sees that it must be $\frac{1}{(\text{deg}_t + \text{deg}_{\text{DR}})!}$ as indicated.

Remark 5.3.7. Restricting equations (5.3.4)(ii)–(iii) to the harmonic part of $\overline{\Omega}$ and using the chain map $N!$, defined as $N!|_{\overline{\Omega}^n} = n! \text{Id}$, one obtains $E|_{P\overline{\Omega}} N! \circ d = d \circ E|_{P\overline{\Omega}} N!$ and $E|_{P\overline{\Omega}} N! \circ i = i_t \circ E|_{P\overline{\Omega}} N!$.

Curiously, we do not know if there is any explicit formula, without using harmonic decomposition, for a map $E' : \overline{\Omega} \rightarrow \overline{\text{DR}}_t^{\wedge}$ which satisfies $E' \circ d = d \circ E'$ and $E' \circ i = i_t \circ E'$. (To define such a map using harmonic decomposition, one could set $E' = E N! P$.)

Perhaps this is related to the fact that (the antiharmonic part of) the bicomplex $(\overline{\Omega}, d, i)$ is somewhat badly behaved, i.e., the homologies of $(\overline{\Omega} \otimes R, i - u d)$ and $(\overline{\Omega}[[u]], i - u d)$ pick up the huge extra factor $[\overline{d\Omega}, d\overline{\Omega}]$ (see Theorem 3.2.1). On the other hand, we will see in §5.5 and §7 below that the homologies of $(\overline{\text{DR}}_t \otimes R, i_t - u d)$ and $(\overline{\text{DR}}_t[[u]], i_t - u d)$ are nicely expressible in terms of ordinary cyclic and Karoubi-de Rham homology. This suggests that the bicomplex $(\overline{\text{DR}}_t, i_t, d)$ should indeed be more closely related to $(\overline{\Omega}, b, B)$ than to $(\overline{\Omega}, i, d)$.

Next, we recall that the differentials B and b , as well as d and i_t , anti-commute. Therefore, thanks to Theorem 5.3.3, the map E gives a morphism $(\overline{\Omega}, B, b) \rightarrow (\overline{\text{DR}}_t^{\wedge}, d, i_t)$, of bicomplexes. This induces maps $H^*(\overline{\Omega}, b) \rightarrow H^*(\overline{\text{DR}}_t^{\wedge}, i_t)$ and $H^*(\overline{\Omega}/B\overline{\Omega}, b) \rightarrow H^*(\overline{\text{DR}}_t^{\wedge}/d\overline{\text{DR}}_t^{\wedge}, i_t)$ on homology.

Observe that, since the differential i_t is homogeneous in t , each individual homology group of the complexes $(\overline{\mathrm{DR}}_t, i_t)$ and $(\overline{\mathrm{DR}}_t / d\overline{\mathrm{DR}}_t, i_t)$ has an additional grading

$$H^p(\overline{\mathrm{DR}}_t, i_t) = \bigoplus_q H(\overline{\mathrm{DR}}_t, i_t)^{p,q}, \quad \text{and} \quad H^p(\overline{\mathrm{DR}}_t / d\overline{\mathrm{DR}}_t, i_t) = \bigoplus_q H(\overline{\mathrm{DR}}_t / d\overline{\mathrm{DR}}_t, i_t)^{p,q},$$

where p refers to the de Rham degree and q refers to degree in t . Therefore, the map E produces, by separating degrees in t , an infinite collection $\{E^q, q = 0, 1, \dots\}$ of maps

$$H^\bullet(\overline{\Omega}, b) \xrightarrow{E^q} H^\bullet(\overline{\mathrm{DR}}_t, i_t)^{\bullet,q}, \quad \text{and} \quad H^\bullet(\overline{\Omega} / B\overline{\Omega}, b) \xrightarrow{E^q} H(\overline{\mathrm{DR}}_t / d\overline{\mathrm{DR}}_t, i_t)^{\bullet,q}. \quad (5.3.8)$$

For $q = 0$, the above maps reduce to the maps $f \mapsto f_{\natural}$ and $f \bmod (\mathrm{im} B) \mapsto f_{\natural} \bmod (\mathrm{im} d)$.

Observe further that the map $\frac{d}{dt}$ commutes with d and i_t , and hence it induces endomorphisms of the complex $(\overline{\mathrm{DR}}_t, i_t)$ and $(\overline{\mathrm{DR}}_t / d\overline{\mathrm{DR}}_t, i_t)$. On homology, this gives maps

$$H(\overline{\mathrm{DR}}_t, i_t)^{p,\bullet} \xrightarrow{\frac{d}{dt}} H(\overline{\mathrm{DR}}_t, i_t)^{p,\bullet-1}, \quad \text{and} \quad H(\overline{\mathrm{DR}}_t / d\overline{\mathrm{DR}}_t, i_t)^{p,\bullet} \xrightarrow{\frac{d}{dt}} H(\overline{\mathrm{DR}}_t / d\overline{\mathrm{DR}}_t, i_t)^{p,\bullet-1}.$$

It is clear from (5.3.4)(i) that the maps $E^q, q \geq 0$, satisfy a chain of relations

$$\frac{d^k}{dt^k} \circ E^q = E^{q-k}, \quad \text{for every } k \leq q. \quad (5.3.9)$$

Our second important result about the Tsygan map, that will be proved in section 6, reads

Theorem 5.3.10. *Each of the maps in (5.3.8) is an isomorphism, for every $q \geq 0$.*

5.4. Proof of Theorem 5.3.3. Equation (5.3.4)(i) follows from the expansion

$$\frac{d}{dt}(t^{k_0} a_0 t^{k_1} da_1 \dots t^{k_n} da_n) = \sum_j k_j t^{k_0} a_0 \dots t^{k_{j-1}} da_{j-1} t^{k_{j-1}} da_j t^{k_{j+1}} \dots t^{k_n} da_n,$$

generalizing the proof that $\frac{d}{dt} \exp(t) = \exp(t)$, in terms of the expansion $\exp(t) = \sum_{m \geq 0} \frac{1}{m!} t^m$.

To prove (5.3.4)(ii), we compute

$$\begin{aligned} E \circ B(a_0 da_1 \dots da_n) &= \sum_{j=0}^n \sum_{k_0, \dots, k_{n+1}} \frac{1}{(k_0 + \dots + k_{n+1} + (n+1))!} (t^{k_0} da_j t^{k_1} da_{j+1} \dots t^{k_n} da_{j-1} t^{k_{n+1}})_{\natural} \\ &= \sum_{k_0, \dots, k_n} \frac{1}{(k_0 + \dots + k_n + n)!} (da_0 t^{k_0} \dots da_n t^{k_n})_{\natural} = d \circ E(a_0 da_1 \dots da_n). \end{aligned}$$

Here and below, the indices k_0, \dots, k_{n+1} always run over all nonnegative integers.

To prove (5.3.4)(iii) we will use the identities, for every $n \geq 1$:

$$\begin{aligned} \sum_{p,q \geq 0} \frac{t^p [t, a] t^q}{(p+q+n)!} &= \sum_{p,q \geq 0} \frac{t^{p+1} a t^q - t^p a t^{q+1}}{(p+q+n)!} = \quad (\text{taking } k := p+1 \text{ and } \ell := q+1) \\ &= \left(\sum_{k,q \geq 0} \frac{t^k a t^q}{(k-1+q+n)!} - \sum_{q \geq 0} \frac{a t^q}{(q+n-1)!} \right) - \left(\sum_{p,\ell \geq 0} \frac{t^p a t^\ell}{(p+\ell-1+n)!} - \sum_{p \geq 0} \frac{t^p a}{(p+n-1)!} \right) \\ &= \sum_{p \geq 0} \frac{t^p a}{(p+n-1)!} - \sum_{q \geq 0} \frac{a t^q}{(q+n-1)!} = \sum_{j \geq 0} \frac{t^j a - a t^j}{(j+n-1)!}. \end{aligned}$$

Using the above formula, for every $a_0, \dots, a_n \in A$, in $\overline{\Omega}_t$, we find

$$\begin{aligned}
& \sum_{j=1}^n \sum_{k_0, \dots, k_n} \frac{(-1)^j}{(k_0 + \dots + k_n + n)!} a_0 t^{k_0} da_1 t^{k_1} \dots da_{j-1} t^{k_{j-1}} [t, a_j] t^{k_j} da_{j+1} t^{k_{j+1}} \dots da_n t^{k_n} \\
&= \sum_{j=1}^n \sum_{k_0, \dots, \hat{k}_j, \dots, k_n} \frac{(-1)^j}{(k_0 + \dots + \hat{k}_j + k_n + (n-1))!} (a_0 t^{k_0} da_1 t^{k_1} \dots da_{j-1} t^{k_{j-1}} a_j da_{j+1} t^{k_{j+1}} \dots da_n t^{k_n} \\
&\quad - a_0 t^{k_0} da_1 t^{k_1} \dots da_{j-1} a_j t^{k_{j-1}} da_{j+1} t^{k_{j+1}} \dots da_n t^{k_n}) \tag{5.4.1} \\
&= \sum_{j=1}^{n-1} \left(\sum_{k_0, \dots, \hat{k}_j, \dots, k_n} \frac{(-1)^j}{(k_0 + \dots + \hat{k}_j + k_n + (n-1))!} a_0 t^{k_0} da_1 \dots da_{j-1} t^{k_{j-1}} d(a_j a_{j+1}) t^{k_{j+1}} \dots da_n t^{k_n} \right) \\
&\quad + \sum_{k_1, \dots, k_n} \frac{1}{(k_1 + \dots + k_n + (n-1))!} (a_0 a_1 t^{k_1} da_2 t^{k_2} \dots da_n t^{k_n} + (-1)^n a_0 t^{k_1} da_1 t^{k_2} \dots da_{n-1} t^{k_n} a_n).
\end{aligned}$$

Now, by cyclic symmetry, in \overline{DR}_t ,

$$(a_0 t^{k_1} da_1 t^{k_2} \dots da_{n-1} t^{k_n} a_n)_{\natural} = (a_n a_0 t^{k_1} da_1 t^{k_2} \dots da_{n-1} t^{k_n})_{\natural}.$$

Therefore, the image in \overline{DR}_t of the expression in the last two lines of (5.4.1) may be written as

$$\begin{aligned}
& E \left(\sum_{j=1}^{n-1} (-1)^j a_0 da_1 \dots da_{j-1} d(a_j a_{j+1}) da_{j+2} \dots da_n \right) \\
&\quad + E(a_0 a_1 da_2 \dots da_n) + (-1)^n E(a_n a_0 da_1 da_2 \dots da_{n-1}).
\end{aligned}$$

Further, the standard formula for the Hochschild differential written in terms of differential forms [CQ1, formula (10)] reads:

$$\begin{aligned}
b(a_0 da_1 \dots da_n) &= \sum_{j=1}^{n-1} (-1)^j a_0 da_1 \dots da_{j-1} d(a_j a_{j+1}) da_{j+2} \dots da_n \\
&\quad + a_0 a_1 da_2 \dots da_n + (-1)^n a_n a_0 da_1 da_2 \dots da_{n-1}.
\end{aligned}$$

Thus, combining everything together and using that the expression in the top line of (5.4.1) equals $i_t \circ E(a_0 da_1 \dots da_n)$, we deduce that $i_t \circ E(a_0 da_1 \dots da_n) = E \circ b(a_0 da_1 \dots da_n)$. \square

5.5. Extended cyclic homology. It will be convenient to introduce an additional parameter u , and consider the $\mathbf{k}[u]$ -module $\overline{DR}_t \otimes R$. We equip this module with two different gradings. The first grading, referred to as *homological grading*, is defined by assigning the component $\overline{DR}_t^{p,q} \cdot u^{-r}$ homological degree $\ell = p + 2r$. Thus, the u variable has homological degree -2 . The t parameter has homological degree zero, so it does not contribute to the homological grading. The second grading, referred to as *internal grading*, is defined by assigning the component $\overline{DR}_t^{p,q} \cdot u^{-r}$ internal degree $p + q + r$.

On $\overline{DR}_t \otimes R$, we have a pair of anti-commuting differentials, $u d$ and i_t . Each of these differentials has homological degree -1 and internal degree zero. We define $\overline{EHC}(A)$, the *reduced extended cyclic homology* of A , as

$$\overline{EHC}_{\ell}(A) = H^{\ell}(\overline{DR}_t \otimes R, i_t - u d), \tag{5.5.1}$$

where ℓ on the RHS stands for homological degree.

Since the internal grading is preserved by both differentials, it provides a direct sum decomposition of complexes,

$$\overline{\mathrm{DR}}_t \otimes R = \bigoplus_k (\overline{\mathrm{DR}}_t \otimes R)_k, \quad (\overline{\mathrm{DR}}_t \otimes R)_k := \bigoplus_{p+q+r=k} \overline{\mathrm{DR}}^{p,q} \cdot u^{-r},$$

where, $(\overline{\mathrm{DR}}_t \otimes R)_k$, the homogeneous component of internal degree k , is a subcomplex of $\overline{\mathrm{DR}}_t \otimes R$ with the inherited differential $i_t - u d$ and inherited homological grading. We write $w_k : \overline{\mathrm{DR}}_t \otimes R \rightarrow (\overline{\mathrm{DR}}_t \otimes R)_k$ for the projection to that homogeneous component.

For every integer m , we let

$$\overline{\mathrm{EHC}}_\ell(A, m) := H^\ell((\overline{\mathrm{DR}}_t \otimes R, i_t - u d)_{\ell-m}).$$

Thus, one obtains a ‘weight decomposition’: $\overline{\mathrm{EHC}}_\ell(A) = \bigoplus_m \overline{\mathrm{EHC}}_\ell(A, m)$, for $m \in \mathbb{Z}$.

Our main result about extended cyclic homology is the following theorem. Part (i) of the theorem shows that the knowledge of the extended cyclic homology is equivalent to the knowledge of both the usual cyclic homology and the Karoubi-de Rham homology.

Theorem 5.5.2. (i) *One has a canonical $\mathbf{k}[u]$ -module isomorphism*

$$\overline{\mathrm{EHC}}(A) \cong (\overline{\mathrm{HD}}(A) \otimes u^{-1} \mathbf{k}[u^{-1}]) \bigoplus \overline{\mathrm{HC}}(A)[t],$$

where the variable ‘ u ’ annihilates the second summand, and on $\overline{\mathrm{HD}}(A) \otimes u^{-1}$ restricts to the natural inclusion $\overline{\mathrm{HD}}(A) \hookrightarrow \overline{\mathrm{HC}}(A)$ into the second summand in degree zero in t . Furthermore, the group $\overline{\mathrm{EHC}}_\ell(A, m)$ vanishes for all $m > \lfloor \frac{\ell}{2} \rfloor$, and

$$\overline{\mathrm{EHC}}_\ell(A, m) \cong \begin{cases} \overline{\mathrm{HD}}_{\ell-2m}(A) & 0 < m \leq \lfloor \frac{\ell}{2} \rfloor, \\ \overline{\mathrm{HC}}_\ell(A) & m \leq 0. \end{cases} \quad (5.5.3)$$

(ii) *For every $q \geq 0$, there are canonical graded space isomorphisms*

$$H(\overline{\mathrm{DR}}_t, i_t)^{\bullet, q} \cong \overline{\mathrm{HH}}_\bullet(A), \quad (5.5.4)$$

$$H(\overline{\mathrm{DR}}_t / d \overline{\mathrm{DR}}_t, i_t)^{\bullet, q} \cong \overline{\mathrm{HC}}_\bullet(A). \quad (5.5.5)$$

Remark 5.5.6. In the special case $q = 0$, using the identification in (5.2.1), the above becomes

$$H(\overline{\mathrm{DR}}_t, i_t)^{\bullet, 0} = \ker(i : \overline{\mathrm{DR}}^\bullet \rightarrow \overline{\Omega}^{\bullet-1}) \quad (5.5.7)$$

$$H(\overline{\mathrm{DR}}_t / d \overline{\mathrm{DR}}_t, i_t)^{\bullet, 0} = \ker(i : \overline{\mathrm{DR}}^\bullet / d \overline{\mathrm{DR}}^{\bullet-1} \rightarrow \overline{\Omega}^{\bullet-1} / d \overline{\Omega}^{\bullet-2}). \quad (5.5.8)$$

Therefore, for $q = 0$, isomorphisms (5.5.4)–(5.5.5) may be equivalently rewritten as

$$\ker(i : \overline{\mathrm{DR}}^\bullet \rightarrow \overline{\Omega}^{\bullet-1}) \cong \overline{\mathrm{HH}}_\bullet(A),$$

$$\ker(i : \overline{\mathrm{DR}}^\bullet / d \overline{\mathrm{DR}}^{\bullet-1} \rightarrow \overline{\Omega}^{\bullet-1} / d \overline{\Omega}^{\bullet-2}) \cong \overline{\mathrm{HC}}_\bullet(A).$$

These are nothing but our isomorphisms (3.1.4)–(3.1.5). Thus, we see that Theorem 5.5.2(ii) incorporates Theorem 3.1.3, as the special case $q = 0$.

5.6. Theorem 5.3.10 implies Theorem 5.5.2. We consider part (ii) of Theorem 5.5.2 first. The proof of this part involves the Tsygan map E in a crucial way.

In more detail, the Hochschild homology of A is computed by the complex $(\overline{\Omega}, b)$. Thus, the map $H(\overline{\Omega}, b) \rightarrow H^\bullet(\overline{\mathrm{DR}}_t, i_t)^{\bullet, q}$ in (5.3.8), which is an isomorphism thanks to Theorem 5.3.10, provides isomorphism (5.5.4). Similarly, in the cyclic homology case, we use Lemma 3.1.7. Thus, we define isomorphism (5.5.5) to be the composite of the isomorphism $\overline{\mathrm{HC}}_\bullet(A) \xrightarrow{\sim} H^\bullet(\overline{\Omega} / B \overline{\Omega}, b)$, of Lemma 3.1.7, with the map E^q , where the latter is an isomorphism by Theorem 5.3.10.

For part (i), we first prove

Proposition 5.6.1. (i) *The assignments*

$$\pi_m : \sum_{k \geq 0} f_k \cdot u^{-k} \mapsto f_m \bmod (t), \quad \pi_- : \sum_{k \geq 0} f_k \cdot u^{-k} \mapsto f_0 \bmod (\text{im } d),$$

induce isomorphisms

$$\pi_m : \overline{\text{EHC}}_\ell(A, m) \xrightarrow{\sim} \overline{\text{HD}}_{\ell-2m}(A), \quad m > 0; \quad (5.6.2)$$

$$\pi_- : \overline{\text{EHC}}_\ell(A, m) \xrightarrow{\sim} H(\overline{\text{DR}}_t / d \overline{\text{DR}}_t, i_t)^{\ell, -m}, \quad m \leq 0. \quad (5.6.3)$$

(ii) *For every $m \leq 0$, the following diagram commutes:*

$$\begin{array}{ccc} H^\ell(\overline{\Omega} \otimes R, \mathfrak{b} - u \mathfrak{B}) & \xrightarrow{H_\ell(\mathfrak{w}_{\ell-m} \circ \mathfrak{E})} & \overline{\text{EHC}}_\ell(A, m) \\ \text{Lemma 3.1.7} \downarrow \cong & & \downarrow \pi_- \\ H^\ell(\overline{\Omega} / \mathfrak{B} \overline{\Omega}, \mathfrak{b}) & \xrightarrow{\mathfrak{E}^{-m}} & H(\overline{\text{DR}}_t / d \overline{\text{DR}}_t, i_t)^{\ell, -m}. \end{array} \quad (5.6.4)$$

Proof. We view $\overline{\text{DR}}_t \otimes R$ as a bicomplex M of the form $M_{p,q} = \overline{\text{DR}}_t^{p-q} \cdot u^{-q}$. The differentials are $i_t : M_{p,q} \rightarrow M_{p-1,q}$ and $u d : M_{p,q} \rightarrow M_{p,q-1}$. We apply the spectral sequence associated with that bicomplex, by taking cohomology with respect to $u d$ first.

To compute the first page of the spectral sequence, we use the natural direct sum decomposition $\overline{\text{DR}}_t \otimes R = \overline{\text{DR}}_t \oplus u^{-1} \overline{\text{DR}}_t[u^{-1}]$. For the cohomology of the first differential, this gives a direct sum decomposition $H(\overline{\text{DR}}_t \otimes R, u d) = \overline{\text{DR}}_t / d \overline{\text{DR}}_t \oplus u^{-1} H(\overline{\text{DR}}_t, d)[u^{-1}]$. By Corollary 5.1.4, the projection modulo t induces an isomorphism $H(\overline{\text{DR}}_t, d) \xrightarrow{\sim} H(\overline{\text{DR}}, d)$. Therefore, the first page of the spectral sequence takes the form $E_1 = \overline{\text{DR}}_t / d \overline{\text{DR}}_t \oplus u^{-1} H(\overline{\text{DR}}, d)[u^{-1}]$. The differential $i_t : E_1 \rightarrow E_1$, on the first page, clearly increases degree in t by 1. Hence, this differential annihilates the direct summand $u^{-1} H(\overline{\text{DR}}, d)[u^{-1}]$. We conclude that the second page of the spectral sequence reads $H(\overline{\text{DR}}_t / d \overline{\text{DR}}_t, i_t) \oplus u^{-1} H(\overline{\text{DR}}, d)[u^{-1}]$, which is the right-hand side of the isomorphism of the proposition. It is immediate to see that the spectral sequence collapses at the second page. This proves part (i). Part (ii) follows directly from the formula for the map π_- . \square

To complete the proof of Theorem 5.5.2(i), we note that the group that appears in the upper left corner of diagram (5.6.4) equals $\overline{\text{HC}}_\ell(A)$, by (3.1.1). Thus, we may (and will) define isomorphism (5.5.3) by combining isomorphisms (5.5.5) and those of Proposition 5.6.1 together.

We conclude that the proof of Theorem 5.5.2 would be complete once we prove Theorem 5.3.10.

Remark 5.6.5. An analogue involving the maps π_m of the statement that diagram (5.6.4) commutes will be given in Proposition 7.2.1(i) in §7.

5.7. It is possible to write a relatively explicit formula for the inverse of the isomorphisms (5.6.2)–(5.6.3). To this end, recall first that the complex $(\overline{\text{DR}}_t^+, d)$ is acyclic, by Lemma 5.1.3. Therefore, the Karoubi-de Rham differential d induces an isomorphism $\overline{\text{DR}}_t^+ / d \overline{\text{DR}}_t^+ \xrightarrow{\sim} d \overline{\text{DR}}_t^+$.

Let $\tilde{d}^{-1} : d \overline{\text{DR}}_t^+ \rightarrow \overline{\text{DR}}_t^+$ be a (set-theoretic) section of the surjection $d : \overline{\text{DR}}_t^+ \rightarrow d \overline{\text{DR}}_t^+$ (we use the same notation as for the similar map in §3.2). The choice of \tilde{d}^{-1} will not affect anything below, and we need it only to write formulas which are independent of this choice.

Proposition 5.7.1. *The inverses of the isomorphisms (5.6.2) and (5.6.3) of Proposition 5.6.1 are induced respectively by the maps*

$$\ker(d|_{\overline{\text{DR}}}) \rightarrow H(\overline{\text{DR}}_t \otimes R, i_t - u d) \quad \text{and} \quad \ker(i_t|_{\overline{\text{DR}}_t / d \overline{\text{DR}}_t}) \rightarrow H(\overline{\text{DR}}_t \otimes R, i_t - u d)$$

given by the assignments

$$f \mapsto u^m (\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1}(f) \quad \text{and} \quad f \mapsto (\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1}(f).$$

Remark 5.7.2. In view of (4.1.3), the map $\tilde{d}^{-1} i_t$ on $\ker(i_t |_{\overline{DR}_t / d\overline{DR}_t})$ may be thought of as an extended analogue of the periodicity operator S . Via Theorem 5.3.10, it actually becomes S after making the above identification.

Proof. The statement of the proposition is essentially a simple consequence of the construction of differentials in the spectral sequence of a double complex. Below, we explain the case of the map π_- . The case of the maps π_m , $m > 0$, is similar (and even easier).

Let $\ker(d i_t)$ be the kernel of the map $d \circ i_t : \overline{DR}_t \rightarrow \overline{DR}_t$. We claim first that $\tilde{d}^{-1} i_t$ is a well-defined set-theoretical endomorphism of $\ker(d i_t)$. To see this, observe that $i_t(\overline{DR}_t) \subset \overline{DR}_t^+$. Hence, thanks to acyclicity of the complex (\overline{DR}_t^+, d) , see Lemma 5.1.3, we deduce that an element $x \in \overline{DR}_t$ belongs to $\ker(d i_t)$ if and only if $i_t x = d y$, for some $y \in \overline{DR}_t^+$. Thus, for such an x , the element $\tilde{d}^{-1} i_t x = \tilde{d}^{-1} d y$ is indeed well-defined. Furthermore, we compute $d i_t(\tilde{d}^{-1} i_t x) = -i_t d(\tilde{d}^{-1} i_t x) = -(i_t)^2(x) = 0$. Thus, we have proved that $\tilde{d}^{-1} i_t x \in \ker(d i_t)$.

The operator $\tilde{d}^{-1} i_t$ on $\ker(d i_t)$ has de Rham degree -2 , and hence $\text{Id} - u^{-1} \tilde{d}^{-1} i_t$ is invertible. Explicitly, an inverse is given, for every $x \in \ker(d i_t)$, by the formula $(\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1}(x) = \sum_{k \geq 0} u^{-k} (\tilde{d}^{-1} i_t)^k(x)$, where the terms $(\tilde{d}^{-1} i_t)^k(x)$ vanish for all k greater than half the de Rham degree of x . Further, using that $d \tilde{d}^{-1} = \text{Id}$, we compute

$$\begin{aligned} (i_t - u d)(\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1} &= (d \tilde{d}^{-1} i_t - u d)(\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1} \\ &= u d(u^{-1} \tilde{d}^{-1} i_t - \text{Id})(\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1} = -u d. \end{aligned}$$

We see that for every z in the image of the map $(\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1}$, the element $(i_t - u d)(z)$ belongs to $u \overline{DR}_t$. It follows that $z \bmod u \overline{DR}_t[[u]]$ is a cycle in the complex $(\overline{DR}_t \otimes R, i_t - u d)$. Thus, we have constructed the map

$$(\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1} : \ker(d i_t) \longrightarrow H(\overline{DR}_t \otimes R, i_t - u d).$$

Now, it is immediate from the above construction that the above constructed map is a right inverse to the map π_- in the sense that, for every $x \in \ker(d i_t)$, the class of the element x in $H(\overline{DR}_t / d \overline{DR}_t, i_t)$ equals $\pi_-(\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1}(x)$. By definition, one has

$$H(\overline{DR}_t / d \overline{DR}_t, i_t) = \frac{\ker(d i_t)}{\ker d + \text{im } i_t}.$$

Therefore, for every $x \in \ker d + \text{im } i_t$, we must have $\pi_-(\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1}(x) = 0$. Since π_- is an isomorphism, by Proposition 5.6.1, we deduce that the map $(\text{Id} - u^{-1} \tilde{d}^{-1} i_t)^{-1}$ takes $\ker d + \text{im } i_t$ to zero on homology. We conclude that the map given in Proposition 5.7.1 is well defined. Moreover, it is a right inverse, and hence a two-sided inverse, to the isomorphism π_- . \square

6. PROOF OF THEOREM 5.3.10

6.1. First of all, thanks to equation (5.3.9), $\frac{d^q}{dt^q} \circ E^q = E^0$. The maps E^0 in (5.3.8) are isomorphisms by Theorem 3.1.3 together with Lemma 3.1.7. It follows that, for every $q \geq 0$, the map E^q is injective. Therefore, we only have to show that E^q is surjective.

To this end, we introduce a filtration on \overline{DR}_t by the degree in A : precisely, the degree $\leq m$ part of the filtration is spanned by monomials in t, A , and dA with the elements of the latter two appearing at most m times. This descends to filtrations under the surjections $\Omega_t \twoheadrightarrow \overline{\Omega}_t \twoheadrightarrow \overline{DR}_t$. The differentials i_t and d preserve the resulting filtration. So we obtain a filtered bicomplex $(\overline{DR}_t, d, i_t)$.

It is immediate to check that $\text{gr}(\overline{DR}_t, d, i_t)$, the associated graded bicomplex, is naturally isomorphic to $(\overline{DR}_t A', d, i_t)$, where $A' := \overline{A} \oplus \mathbf{k}$ is the algebra with the trivial multiplication $\overline{A} \cdot \overline{A} = 0$. Let E' denote the Tsygan map for the algebra A' . Clearly, one has that $E' = \text{gr } E$.

Thus, we have reduced the proof of the theorem to

Proposition 6.1.1. *The maps $(E')^q : H(\overline{\Omega}A', \mathfrak{b}) \rightarrow H(\overline{\mathrm{DR}}_t A', i_t)^{\bullet, q}$ and $(E')^q : H(\overline{\Omega}A' / \mathfrak{B}\overline{\Omega}A', \mathfrak{b}) \rightarrow H(\overline{\mathrm{DR}}_t A' / \mathfrak{d}\overline{\mathrm{DR}}_t A', i_t)^{\bullet, q}$ are both surjective for every $q \geq 0$.*

The rest of §6 is devoted to the proof of this proposition. The idea of the proof is to reduce the result further to the special case where $\overline{A} = \mathfrak{k}$ (and to lift from cyclic tensors to ordinary tensors). In this case, we compute the cohomology of the complexes $(\overline{\mathrm{DR}}_t A', i_t)$ and $(\overline{\mathrm{DR}}_t A' / \mathfrak{d}\overline{\mathrm{DR}}_t A', i_t)$ explicitly. More precisely, we will define, for each $\ell \geq 1$, a complex C_ℓ equipped with an action of \mathbb{Z}/ℓ , the cyclic group. We identify the complex $(\overline{\mathrm{DR}}_t A', i_t)$ with $\bigoplus_{\ell \geq 1} (C_\ell)_{\mathbb{Z}/\ell}$, a direct sum of complexes of coinvariants of cyclic groups. We then compute explicitly the homology of each complex C_ℓ . The computation is, however, quite technical; it will not be used elsewhere in the paper.

6.2. Linear algebra lemmas. We begin with a few elementary lemmas from linear algebra.

Let V be a vector space over \mathbb{R} equipped with a positive definite inner product $\langle -, - \rangle : V \times V \rightarrow \mathbb{R}$. Recall that a linear operator $T : V \rightarrow V$ is called *positive* if it is self-adjoint and $\langle Tv, v \rangle \geq 0$ for all $v \in V$. For a positive operator T and $v \in V$, the equation $\langle Tv, v \rangle = 0$ implies that $Tv = 0$.

Lemma 6.2.1. *Suppose that T_0, \dots, T_k are positive operators on vector spaces V_0, \dots, V_k . Let $T := \sum_{i=0}^k \mathrm{Id}^{\otimes i} \otimes T_i \otimes \mathrm{Id}^{\otimes k-i}$. Then T is positive, and $\ker(T) = \bigotimes_{i=0}^k \ker(T_i)$.*

Proof. Clearly each $\mathrm{Id}^{\otimes i} \otimes T_i \otimes \mathrm{Id}^{\otimes k-i}$ is positive, and so the sum is positive. Since the kernel of T is the space of vectors v such that $\langle Tv, v \rangle = 0$, it is the intersection of the kernels of $\mathrm{Id}^{\otimes i} \otimes T_i \otimes \mathrm{Id}^{\otimes k-i}$, i.e., $\bigotimes_{i=0}^k \ker(T_i)$. \square

Lemma 6.2.2. *Let $B = (b_{ij}) \in \mathrm{Mat}_n(\mathbb{R})$ be a symmetric $n \times n$ -matrix such that $b_{ii} \geq \sum_{j \neq i} |b_{ij}|$ for all i . Then B is positive. Assume, in addition, that the graph obtained from B with vertex set $\{1, \dots, n\}$ and edges between i and j if $b_{ij} \neq 0$ is connected. Then $\dim \ker(B) \leq 1$. This kernel is nonzero if and only if:*

- The equality $b_{ii} = \sum_{j \neq i} |b_{ij}|$ holds for all i ;
- There exist signs $\lambda_i \in \{1, -1\}$ such that $\mathrm{sign}(b_{ij}) = -\lambda_i \lambda_j$ for all $i \neq j$.

Then the kernel is spanned by the vector $(\lambda_1, \dots, \lambda_n)^\top$.

Here and below, we use the notation $(\lambda_1, \dots, \lambda_n)^\top = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$, where \top stands for ‘‘transpose.’’

Proof. Let $v = (v_1, \dots, v_n)^\top$. Then

$$\langle Bv, v \rangle = \sum_{i < j} b_{ij}(v_i^2 + 2v_i v_j + v_j^2) + \sum_i (b_{ii} - \sum_{j \neq i} b_{ij})v_i^2 \geq 0.$$

Hence B is positive. Now, suppose $Bv = 0$ and $v \neq 0$. Let i be such that $|v_i| \geq |v_j|$ for all j . Then $0 = b_{ii}v_i + \sum_{j \neq i} b_{ij}v_j$ and $b_{ii} \geq \sum_{j \neq i} |b_{ij}|$ implies that the equality holds, and that $|v_j| = |v_i|$ whenever $b_{ij} \neq 0$. Now assume that the graph of B is connected. Then we conclude that $|v_i|$ are all equal. Hence the equality $b_{ii} = \sum_{j \neq i} |b_{ij}|$ holds for all i . Now, $Bv = 0$ if and only if $\mathrm{sign}(b_{ij}) = -v_i/v_j$ whenever $i \neq j$. \square

Remark 6.2.3. Lemma 6.2.2 can be significantly generalized: for arbitrary complex B which is not necessarily symmetric, the condition $b_{ii} \geq \sum_{j \neq i} |b_{ij}|$ guarantees that the real parts of all eigenvalues of B are nonnegative, and zero only for the zero eigenvalue. Let $Q(B)$ be a directed graph with vertex set $\{1, 2, \dots, n\}$ such that there is an edge from vertex i to j whenever $b_{ij} \neq 0$ and $i \neq j$. If the graph $Q(B)$ is strongly connected (i.e., there exists a directed path from every vertex

to every other vertex), then $\dim \ker(B) \leq 1$ and it is nonzero if and only if $b_{ii} = \sum_{j \neq i} |b_{ij}|$ for all i and there exist complex numbers $\lambda_1, \dots, \lambda_n$ of absolute value 1 such that $b_{ij}/|b_{ij}| = -\lambda_i/\lambda_j$ whenever $b_{ij} \neq 0$. In this case the kernel is spanned by $(\lambda_1, \dots, \lambda_n)^\top$.

Lemma 6.2.4. *Suppose that (V^\bullet, d) is a complex of vector spaces with $d : V^\bullet \rightarrow V^{\bullet+1}$. Suppose we are given an operator $h : V^\bullet \rightarrow V^{\bullet-1}$ such that*

$$V = \ker[h, d] \oplus \operatorname{im}[h, d], \quad \ker d = \ker([h, d]|_{\ker d}) \oplus \operatorname{im}([h, d]|_{\ker d}).$$

(E.g., this holds if $[h, d]$ acts semisimply on V or its base change to an algebraically closed field.) Then, the inclusion $(\ker[h, d], d|_{\ker[h, d]}) \rightarrow (V, d)$ is a quasi-isomorphism.

Proof. Since $(V, d) = (\ker[h, d], d|_{\ker[h, d]}) \oplus (\operatorname{im}[h, d], d|_{\operatorname{im}[h, d]})$ is a direct sum of complexes, we only need to show that the second factor is acyclic. Since $\ker(d) = \ker(d)|_{\ker[h, d]} \oplus \ker(d)|_{\operatorname{im}[h, d]}$, and the first factor equals $\ker([h, d]|_{\ker d})$, the assumption implies that $\ker(d)|_{\operatorname{im}[h, d]} = \operatorname{im}([h, d]|_{\ker d}) = \operatorname{im}(dh|_{\ker d})$. So every cycle in $(\operatorname{im}[h, d], d|_{\operatorname{im}[h, d]})$ is a boundary. \square

Remark 6.2.5. More generally, in Lemma 6.2.4 we could replace $\ker[h, d]$ and $\operatorname{im}[h, d]$ by $\ker[h, d]^N$ and $\operatorname{im}[h, d]^N$ for every $N \geq 1$. In particular, if $N \geq 1$ is such that $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a graded vector space with $\dim V_i \leq N$ for all i , then the conditions of the lemma will necessarily be satisfied with this modification.

In particular, Lemma 6.2.4 applies when V^\bullet is a finite-dimensional complex of rational vector spaces and $[h, d]$ is a positive operator.

6.3. An auxiliary complex. Our ultimate goal is to define a complex C_ℓ and compute its homology, as well as that of a certain quotient $C_\ell/d_s C_\ell$; here we begin, as an intermediate step, with a somewhat simpler complex. Consider the supercommutative algebra $S := \mathbf{k}[s]/(s^2)$, with s odd, and the left S -module

$$M := \operatorname{Span}(1, s, t^i, st^i, t^i s, st^i s \mid i \geq 1) \subset \mathbf{k}\langle s, t \rangle / (s^2).$$

Assign M the grading in which 1 is odd, s is even, and for all $i \geq 1$, t^i and $st^i s$ are even and st^i and $t^i s$ are odd. (In other words, we modify the natural grading from the algebra $\mathbf{k}\langle s, t \rangle / (s^2) \supset M$ by shifting the degree of the submodule $S \subset M$.) Moreover, equip M with the right S -module structure \cdot_R ,

$$(f) \cdot_R 1 = f, \quad (f) \cdot_R s = -fs,$$

which differs from the natural multiplication from the algebra $\mathbf{k}\langle s, t \rangle / (s^2)$ by a sign.

From now until the end of §6.7, we will use simplified notation $\otimes = \otimes_S$. For convenience, equip $T_S M$ with a partially defined multiplication,

$$(f_1 \otimes \cdots \otimes f_i) \cdot (g_1 \otimes \cdots \otimes g_j) = f_1 \otimes \cdots \otimes f_i g_1 \otimes \cdots \otimes g_j,$$

where here $f_i g_1$ is the multiplication in the ring $\mathbf{k}\langle s, t \rangle / (s^2)$ (not using \cdot_R). This is defined whenever $f_i g_1 \in M$. Further, put $1_m := 1^{\otimes m} = 1 \otimes 1 \otimes \cdots \otimes 1$ (m factors).

For each $\ell \geq 1$, we define a triple $(\tilde{d}_s, \tilde{\partial}^0, \tilde{\partial}^1)$, of differentials on $M^{\otimes \ell}$, as follows. First, we set

$$\begin{aligned} \tilde{d}_s &: 1_k \mapsto s \cdot 1_k \mapsto 0, \\ \tilde{\partial}^0(1_k) &:= \sum_{j=1}^{k-2} (-1)^{j-1} (1_j \otimes t \otimes 1_{k-j-1}), \\ \tilde{\partial}^0(s \cdot 1_k) &:= \sum_{j=1}^{k-2} (1_j \otimes (st - ts) \otimes 1_{k-j-1}), \\ \tilde{\partial}^1(1_k) &:= 0, \quad \tilde{\partial}^1(s \cdot 1_k) := t \otimes 1_{k-1} + (-1)^k 1_{k-1} \otimes t. \end{aligned}$$

Now, let \tilde{d} stand for one of the three symbols $\tilde{d}_s, \tilde{\partial}^0, \tilde{\partial}^1$. Then, using the above formulas, for every $m \geq 1$ and $h_0, \dots, h_n \in \{1_k, s \cdot 1_k \mid k \geq 1\}$, and $i_1, \dots, i_n \geq 1$, we define the corresponding differential by the formula

$$\tilde{d}(h_0 t^{i_1} h_1 \cdots t^{i_n} h_n) = \sum_{j=0}^n (-1)^{|h_0 t^{i_1} h_1 \cdots t^{i_{j-1}} h_{j-1}|} \cdot h_0 t^{i_1} h_1 \cdots t^{i_{j-1}} h_{j-1} \tilde{d}(h_j) t^{i_{j+1}} \cdots t^{i_n} h_n.$$

The meaning of the above differentials is explained by Lemma 6.3.1 below (which we will not actually need, but we will need the generalization in Lemma 6.8.1 below). This says that the differentials \tilde{d}_s and $\tilde{\partial}^0 + \tilde{\partial}^1$ may be obtained by transporting the natural differentials d and i_t by means of an appropriate isomorphism. In more detail, let $A_\tau = A'_\tau = \mathbf{k} \oplus \mathbf{k} \cdot \tau$ be a \mathbf{k} -algebra with one generator τ such that $\tau^2 = 0$.

Lemma 6.3.1. *The assignment*

$$\zeta : t \mapsto t, \quad s^\epsilon \cdot 1_\ell \mapsto \tau^\epsilon (d\tau)^{\ell-\epsilon} := \tau^\epsilon \underbrace{d\tau \, d\tau \cdots d\tau}_{\ell - \epsilon \text{ times}}, \quad \epsilon \in \{0, 1\}, \quad \ell = 1, 2, \dots$$

yields an isomorphism of bicomplexes

$$\zeta : \left(\bigoplus_{\ell \geq 1} M^{\otimes \ell}, d_s, \tilde{\partial}^0 + \tilde{\partial}^1 \right) \xrightarrow{\sim} (\overline{\Omega}_t(A_\tau), d, i_t).$$

Sketch of Proof. The fact that the assignment of the lemma respects the differentials can be verified directly. Further, isomorphism (2.1.1) shows that the set of elements of the form $\tau^\epsilon (d\tau)^{\ell-\epsilon}$ is a \mathbf{k} -basis of $\overline{\Omega}(A_\tau)$. Hence, this set combined with the set of words in the elements of the form $t^i \tau^\epsilon (d\tau)^{\ell-\epsilon}$, $i \geq 1$, gives a \mathbf{k} -basis of $\overline{\Omega}_t(A_\tau)$. It follows easily that the map ζ is a bijection. We note that the direct summand $M^{\otimes \ell}$ goes, via the bijection, to the subspace $[\overline{\Omega}_t(A_\tau)]_{(\ell)}$ of $\overline{\Omega}_t(A_\tau)$ spanned by the words in which τ appears ℓ times. \square

6.4. We will need to compute the homology of the complex $(M^{\otimes \ell}, \tilde{\partial}^0)$.

Lemma 6.4.1. *A basis of $H(M^{\otimes \ell}, \tilde{\partial}^0)$ as a free \mathbf{k} -module is given by elements of the form*

$$s^{\epsilon_0} \otimes (st + ts)^{\otimes j_0} \otimes t^{i_1} s^{\epsilon_1} \otimes (st + ts)^{\otimes j_1} \otimes \cdots \otimes t^{i_k} s^{\epsilon_k} \otimes (st + ts)^{\otimes j_k}, \quad (6.4.2)$$

for $\epsilon_0, \dots, \epsilon_k \in \{0, 1\}$, $j_0, \dots, j_k \geq 0$, and $i_1, \dots, i_k \geq 2$.

Proof. We define an operation \tilde{D} on $M^{\otimes \ell}$ resembling a derivation which takes st and ts to $\frac{1}{2}s$, t to 1, and everything else to zero (such a map cannot actually be a derivation, because the Leibniz rule on st is not satisfied). Precisely, we define

$$\tilde{D}(t^i) = \delta_{i,1}, \quad \tilde{D}(st^i) = \frac{1}{2}\delta_{i,1}s = \tilde{D}(t^i s), \quad \tilde{D}(st^i s) = 0,$$

then write

$$\tilde{D}(1_p \otimes f \otimes 1_q) = (-1)^p \cdot 1_p \otimes \tilde{D}(f) \otimes 1_q,$$

and finally put

$$\tilde{D}(f_0 t^{i_1} f_1 \cdots t^{i_k} f_k) = \tilde{D}(f_0 t^{i_1} f_1) t^{i_2} f_2 \cdots t^{i_k} f_k + (-1)^{|f_0 t^{i_1}|} f_0 t^{i_1} \tilde{D}(f_1 t^{i_2} f_2 \cdots t^{i_k} f_k),$$

where $f_0, \dots, f_k \in T_S \langle 1, s \rangle \subseteq T_S M$ (i.e., each of these elements can be taken to be either 1_j for some j , or $s \cdot 1_j$ for some j).

Note that both \tilde{D} and $\tilde{\partial}^0$ preserve the subcomplexes which are homogeneous of fixed degrees ≥ 2 in t in specified tensor components: precisely, such a subcomplex is one where, in components $1 \leq c_1 < \cdots < c_p \leq \ell$, t appears with degrees $d_1, \dots, d_p \geq 2$, and t appears with degree ≤ 1 in all other components.

We are going to show that the operator $D_{\square} := [\tilde{D}, \tilde{\partial}^0]$ is close to an isomorphism, and that its kernel includes quasi-isomorphically into $(M^{\otimes \ell}, \tilde{\partial}^0)$. To this end, we will compute D_{\square} explicitly.

By the observation from two paragraphs above, this computation reduces to the subspace $\langle 1, t, st, ts \rangle^{\otimes m}$. More precisely, D_{\square} satisfies the following rule which resembles the derivation condition:

$$D_{\square}(g_1 \otimes t^i \otimes g_2) = D_{\square}(g_1) \otimes t^i \otimes g_2 + g_1 \otimes t^i \otimes D_{\square}(g_2), \quad \forall i \geq 2, g_1, g_2 \in T_S M. \quad (6.4.3)$$

Moreover, it satisfies another rule which resembles an order-two differential operator condition:

$$D_{\square}(g_1 \otimes t \otimes f \otimes t \otimes g_2) = D_{\square}(g_1 \otimes t \otimes f) \otimes t \otimes g_2 + g_1 \otimes t \otimes D_{\square}(f \otimes t \otimes g_2) - g_1 \otimes t \otimes D_{\square}(f) \otimes t \otimes g_2. \quad (6.4.4)$$

This reduces the computation to the six equations

$$\begin{aligned} (1) \quad & D_{\square}(1_m) = m \cdot 1_m, & (2) \quad & D_{\square}(s \cdot 1_m) = ms \cdot 1_m, \\ (3) \quad & D_{\square}(1_m \otimes t \otimes 1_n) = (m+n+1)(1_m \otimes t \otimes 1_n), \\ (4) \quad & D_{\square}(1_m \otimes st \otimes 1_n) = (m+n+\frac{1}{2})1_m \otimes st \otimes 1_n - \frac{1}{2} \sum_{j=0}^n 1_{m+j} \otimes ts \otimes 1_{n-j}, \\ (5) \quad & D_{\square}(1_m \otimes ts \otimes 1_n) = (m+n+\frac{1}{2}) \cdot 1_m \otimes ts \otimes 1_n - \frac{1}{2} \sum_{j=0}^m 1_{m-j} \otimes st \otimes 1_{n+j}, \\ (6) \quad & D_{\square}(s \cdot 1_m \otimes t \otimes s \cdot 1_n) = (m+n)s \cdot 1_m \otimes t \otimes s \cdot 1_n - \frac{1}{2} \sum_{\substack{0 \leq j \leq m+n \\ j \neq m}} s \cdot 1_j \otimes t \otimes s \cdot 1_{m+n-j}. \end{aligned} \quad (6.4.5)$$

Next, we will show that D_{\square} is positive and use this to compute its kernel. More precisely, let us assume without loss of generality that $\mathbf{k} = \mathbb{Q}$ (note that $T_S M$ is the tensor product of its \mathbb{Q} -form by \mathbf{k}). We equip $T_S M$ with the inner product induced by the monomial basis

$$s^{\epsilon_0} \cdot 1_{j_0} \otimes t^{i_1} s^{\epsilon_1} \otimes 1_{j_1} \otimes \cdots \otimes t^{i_r} s^{\epsilon_r} \otimes 1_{j_r}, \quad (6.4.6)$$

for all $r \geq 0$ and $\epsilon_0, \dots, \epsilon_r \in \{0, 1\}$, $j_0, \dots, j_r \geq 0$, and $i_1, \dots, i_r \geq 1$, i.e., the unique inner product for which this basis is orthonormal.

Next, for each $k \geq 0$, we fix $i_1, \dots, i_k \geq 2$, $j_0, \dots, j_k \geq 0$, and $p_0, \dots, p_k \geq 0$ such that $p_r \leq j_r$ for all r , and $q_0, \dots, q_k \geq 0$ such that $q_r \leq p_r + 1$ for all r . Then we define the D_{\square} -invariant subspace, $V_{k, i_{\bullet}, j_{\bullet}, p_{\bullet}, q_{\bullet}}$, spanned by elements of the form

$$f_0 \otimes t^{i_1} \otimes f_1 \otimes \cdots \otimes t^{i_k} \otimes f_k,$$

where, for all $r \in \{0, \dots, k\}$, $f_r \in T_S^{j_r} \langle 1, t, st, ts \rangle$, and it is homogeneous in s and t of degrees q_r and p_r , respectively.

Claim 6.4.7. (a) The operator D_{\square} is positive with kernel of dimension ≤ 1 on $V_{k, i_{\bullet}, j_{\bullet}, p_{\bullet}, q_{\bullet}}$.

(b) This kernel is nonzero if and only if $p_r = j_r$ and $q_r \in \{p_r, p_r + 1\}$, for all r . In this case it is spanned by the element (6.4.2) for $\epsilon_r = q_r - p_r$ where $r = 0, 1, \dots, k$.

Proof. In view of Lemma 6.2.1 and (6.4.3), it suffices to prove (a) and (b) in the case that $k = 0$. We then denote for simplicity $j = j_0, p = p_0$, and $q = q_0$. The fact that D_{\square} is self-adjoint here is immediate from (6.4.4) and (6.4.5)(1)–(6). To show that it is positive, observe again from (6.4.4)–(6.4.5)(5) that the symmetric matrix $B = (b_{ij})$ for D_{\square} on $V_{0, i_{\bullet}, j_{\bullet}, p_{\bullet}, q_{\bullet}}$ in the monomial basis of (6.4.6) satisfies $b_{ii} \geq \sum_{j \neq i} |b_{ij}|$ for all i . Equality holds for all i if and only if the condition of part (b) is satisfied: $p = j$ and $q \in \{p, p + 1\}$, i.e., in those formulas (6.4.5)(3)–(6) which arise, always $m = n = 0$, and also (6.4.5)(2) does not arise (and neither (6.4.5)(1)).

Observe that the (undirected) graph obtained from B , with vertex set $1, \dots, \dim V_{0,i \cdot, j, p, q}$ and with edges between i and j if $b_{ij} \neq 0$, is connected. The claim then follows from Lemma 6.2.2. \square

Now, we complete the proof of Lemma 6.4.1. We showed that D_{\square} is positive with kernel spanned by the elements (6.4.2). By Lemma 6.2.4, this implies that $(T_S M, \tilde{\partial}^0)$ is quasi-isomorphic to the subcomplex spanned by the given elements. It is easy to check that $\tilde{\partial}^0$ has zero differential restricted to this subcomplex, and that the elements appearing in (6.4.2) are linearly independent. \square

6.5. The complex C_ℓ . Write $[s, m] := s \cdot m - (-1)^{|m|} m \cdot_R s$, for $s \in S$, $m \in M^{\otimes \ell}$. We keep the notation $\otimes = \otimes_S$ and put $C_\ell := M^{\otimes \ell} / [S, M^{\otimes \ell}]$. We introduce a triple d_s, ∂^0 , and ∂^1 , of differentials on C_ℓ , as follows.

The definition of the differentials breaks up into two cases: the case of degree zero in t and the case of positive degree in t , respectively. In the case of degree zero in t , we use the formulas

$$\begin{aligned} d_s(1_\ell) &= s \cdot 1_\ell, & d_s(s \cdot 1_\ell) &= 0, \\ \partial^0(1_\ell) &= 0, & \partial^0(s \cdot 1_\ell) &= 0, \\ \partial^1(1_\ell) &= 0, & \partial^1(s \cdot 1_\ell) &= (1 + (-1)^\ell) \cdot \sum_{0 \leq j \leq \ell-1} 1_j \cdot t \cdot 1_{\ell-j}. \end{aligned}$$

In the case of positive degree in t , we observe that the differentials $\tilde{d}_s, \tilde{\partial}^0$ and $\tilde{\partial}^1$ on $M^{\otimes \ell}$, introduced in the previous section, descend to well defined differentials d_s, ∂^0 , and ∂^1 , respectively, modulo commutators with s . This gives the required differentials on $C_\ell = M^{\otimes \ell} / [S, M^{\otimes \ell}]$.

Let the cyclic group \mathbb{Z}/ℓ act on C_ℓ by cyclic permutations of tensor components using the Koszul sign rule, in terms of the $\mathbb{Z}/2$ grading above. We write $(C_\ell)_{\mathbb{Z}/\ell}$ for the corresponding space of \mathbb{Z}/ℓ -coinvariants.

The next lemma is a ‘cyclic counterpart’ of Lemma 6.3.1 (which we will also not actually need).

Lemma 6.5.1. *The map ζ , of Lemma 6.3.1, descends to an isomorphism of bicomplexes*

$$\zeta_{\natural} : \left(\bigoplus_{\ell \geq 1} (C_\ell)_{\mathbb{Z}/\ell}, d_s, \partial^0 + \partial^1 \right) \xrightarrow{\sim} (\overline{\text{DR}}_t A_\tau, d, i_t).$$

Sketch of Proof. The argument is based on the following general result. Let A be an arbitrary algebra, let $V \subset A$ be a \mathbf{k} -submodule, and consider $\alpha = a_0 da_1 \dots da_m$ and $\alpha' = a'_0 da'_1 \dots da'_n$ for arbitrary elements $a_0, a_1, \dots, a_m, a'_0, a'_1, \dots, a'_n \in V$. Then, the following implication holds:

$$V \cdot V = 0 \quad \Rightarrow \quad ((d\alpha) \cdot t \cdot \beta \cdot t \cdot \alpha' + (-1)^{|\alpha|+|\beta|} \alpha \cdot t \cdot \beta \cdot t \cdot (d\alpha'))_{\natural} = 0, \quad \forall \beta \in \overline{\Omega}_t A. \quad (6.5.2)$$

Now, equation $V \cdot V = 0$ obviously holds in the case where $A = A_\tau$ and $V = \mathbf{k} \cdot \tau$. Further, it is straightforward to verify that the map ζ sends the \mathbf{k} -submodule $\bigoplus_{\ell} [S, M^{\otimes \ell}]$, of $\bigoplus_{\ell} M^{\otimes \ell}$, onto the \mathbf{k} -submodule of $\overline{\Omega}_t(A_\tau)$ spanned by the elements of the form $(d\alpha) \cdot t \cdot \beta \cdot t \cdot \alpha' + (-1)^{|\alpha|+|\beta|} \alpha \cdot t \cdot \beta \cdot t \cdot (d\alpha')$, which appear in (6.5.2).

Using this, and the cyclic symmetry of $\overline{\text{DR}}_t A_\tau$, one shows that the map ζ_{\natural} is well-defined. To see it is a bijection, we first observe that $\overline{\text{DR}}_t A_\tau$ can be presented as the quotient of $\overline{\Omega}_t A_\tau$ by the above relation, for $V = \mathbf{k} \cdot \tau$, along with the relation imposing cyclic symmetry. Then, the result follows from the above observations and the fact that ζ is a bijection. \square

6.6. We now compute the homology of the complexes (C_ℓ, ∂^0) and $(C_\ell/d_s C_\ell, \partial^0)$.

Lemma 6.6.1. (i) *A basis for $H(C_\ell, \partial^0)$ as a free \mathbf{k} -module consists of elements of the form*

$$(st + ts)^{\otimes j_0} \otimes t^{i_1} s^{\epsilon_1} \otimes (st + ts)^{\otimes j_1} \otimes \dots \otimes t^{i_k} s^{\epsilon_k} \otimes (st + ts)^{\otimes j_k}, \quad (6.6.2)$$

where $i_1, \dots, i_k \geq 2$, $j_0, \dots, j_k \geq 0$, and $\epsilon_1, \dots, \epsilon_k \in \{0, 1\}$, together with the elements, for all $k \in \{0, \dots, \ell - 1\}$, and $\epsilon \in \{0, 1\}$,

$$\sum_{\substack{1 \leq r \leq k: j_0, \dots, j_k \geq 0: \\ j_0 + \dots + j_k = \ell - k}} (-1)^{(\epsilon - 1)(j_0 + j_1 + \dots + j_{r-1} + r - 1)} \cdot 1_{j_0} \otimes ts \otimes \dots \otimes 1_{j_{r-1}} \otimes ts^\epsilon \otimes \dots \otimes 1_{j_{k-1}} \otimes ts \otimes 1_{j_k}. \quad (6.6.3)$$

(ii) A basis for $H(C_\ell/d_s C_\ell, \partial^0)$ as a free \mathbf{k} -module consists of elements of the form

$$(st + ts)^{\otimes j_0} \otimes t \otimes (st + ts)^{\otimes j_1} \otimes ts^{\epsilon_2} \otimes (st + ts)^{\otimes j_2} \otimes \dots \otimes ts^{\epsilon_k} \otimes (st + ts)^{\otimes j_k}, \quad (6.6.4)$$

where $i_1, \dots, i_k \geq 2$, $j_0, \dots, j_k \geq 0$, and $\epsilon_2, \dots, \epsilon_k \in \{0, 1\}$, together with, for all $k \in \{0, \dots, \ell - 1\}$, the element

$$\sum_{\substack{1 \leq r \leq k: j_0, \dots, j_k \geq 0: \\ j_0 + \dots + j_k = \ell - k}} (-1)^{j_0 + j_1 + \dots + j_{r-1} + r - 1} \cdot 1_{j_0} \otimes ts \otimes \dots \otimes 1_{j_{r-1}} \otimes t \otimes \dots \otimes 1_{j_{k-1}} \otimes ts \otimes 1_{j_k}. \quad (6.6.5)$$

See the next subsection for an explanation of formulas (6.6.3) and (6.6.5) (they will appear as the associated graded expressions of (6.7.1) with respect to the increasing filtration by the degree in s , i.e., the number of times s appears).

Proof. To prove part (i), we will use the proof of Lemma 6.4.1. The key step is to define a homotopy D on C_ℓ similar to the operator \tilde{D} . As before, assume $f_1, \dots, f_k \in T_S\langle 1, s \rangle \subseteq T_S M$. Then, for $i_1, \dots, i_k \geq 1$ and $p_1, p_2 \geq 0$, define

$$\begin{aligned} D(1_{p_1} \otimes s^\epsilon t^{i_1} f_1 \dots t^{i_{k-1}} f_{k-1} t^{i_k} \otimes 1_{p_2}) &= \tilde{D}(1_{p_1} \otimes s^\epsilon t^{i_1} f_1) t^{i_2} f_2 \dots t^{i_{k-1}} f_{k-1} t^{i_k} \otimes 1_{p_2} \\ &+ (-1)^{\epsilon \cdot \deg(t^{i_1} f_1 \dots t^{i_{k-1}} f_{k-1} t^{i_k} \otimes 1_{p_2})} \cdot 1_{p_1} \otimes t^{i_1} \tilde{D}(f_1 \dots t^{i_{k-1}} f_{k-1} t^{i_k} \otimes 1_{p_2} \cdot_R s^\epsilon), \end{aligned}$$

and so that D kills elements of degree zero in t (elements of the form $(f_1)_{\mathfrak{q}}$ with f_1 as above). By definition, D commutes with the \mathbb{Z}/ℓ action.

By the same argument as before, we can assume $\mathbf{k} = \mathbb{Q}$, and then $[D, \partial^0]$ is a positive operator on C_ℓ , viewed as an inner product space with orthonormal basis

$$1_{j_0} \otimes t^{i_1} s^{\epsilon_1} \otimes 1_{j_2} \otimes \dots \otimes t^{i_k} s^{\epsilon_k} \otimes 1_{j_k}, \quad (6.6.6)$$

for $\epsilon_1, \dots, \epsilon_k \in \{0, 1\}$, $j_0, \dots, j_k \geq 0$, and $i_1, \dots, i_k \geq 1$. The only difference comes in computing the kernel, i.e., the statement of Claim 6.4.7(b) has to be modified (with the same proof, i.e., still using Lemma 6.2.2). Here, we just have to add also the case where $k = 0$, p_0 is arbitrary, and $q_0 \in \{p_0, p_0 + 1\}$. One can verify that kernel in this case is spanned by the element (6.6.3) (note that in this formula, there is a different parameter k , which need not be zero).

For part (ii), note that D commutes with d_s , so it descends to an operator on $C_\ell/d_s C_\ell$, which we also denote (abusively) by D . Since $[D, \partial^0]$ is positive on C_ℓ and preserves $d_s C_\ell$, it is positive on $(d_s C_\ell)^\perp$. Now, consider the composition $(d_s C_\ell)^\perp \hookrightarrow C_\ell \twoheadrightarrow C_\ell/d_s C_\ell$, which is an isomorphism. We claim that it commutes with the actions of $[D, \partial^0]$. This will imply that $[D, \partial^0]$ is also positive on $C_\ell/d_s C_\ell$. The only subtlety here is that $(d_s C_\ell)^\perp$ is not preserved by D and ∂^0 . However, by definition, the operators ∂^0 and D on $C_\ell/d_s C_\ell$ are obtained from the projection $C_\ell \twoheadrightarrow C_\ell/d_s C_\ell$, so the claim follows.

As a result, $(C_\ell/d_s C_\ell, \partial^0)$ is quasi-isomorphic to the subcomplex spanned by the projection of the subcomplex appearing in part (b) to $C_\ell/d_s C_\ell$. This is evidently spanned by the elements (6.6.4)–(6.6.5), since this is the collection of elements from (6.6.2)–(6.6.3) which do not project to zero, and it is easy to see that these remaining elements (6.6.4)–(6.6.5) are linearly independent. As before, the differential ∂^0 is zero on this subcomplex, so the result follows. \square

6.7. We continue to keep the notation $\otimes = \otimes_s$. For each $\ell \geq 1$, $q \geq 0$, and $\epsilon \in \{0, 1\}$, we consider the image in C_ℓ of the element

$$\Upsilon_{q,\ell}^\epsilon := \sum_{\substack{k \geq 0, 1 \leq r \leq k, i_1, \dots, i_k \geq 1, j_0, \dots, j_k \geq 0: \\ i_1 + \dots + i_k = q, j_0 + \dots + j_k = \ell - k}} (-1)^{\nu_\epsilon(j_0, \dots, j_{r-1})} \cdot 1_{j_0} \otimes t^{i_1} s \otimes \dots \otimes 1_{j_{r-1}} \otimes t^{i_r} s^\epsilon \otimes \dots \otimes 1_{j_{k-1}} \otimes t^{i_k} s \otimes 1_{j_k}, \quad (6.7.1)$$

where $\nu_\epsilon(j_0, \dots, j_{r-1}) := (\epsilon - 1)(j_0 + \dots + j_{r-1} + r - 1)$.

Lemma 6.7.2. (1) For all $q \geq 0$ and all $\epsilon \in \{0, 1\}$, we have that $(\partial^0 + \partial^1)\Upsilon_{q,\ell}^\epsilon = 0$.

(2) Using the notation of Lemma 6.6.1, one has

(i) The homology $H(C_\ell, \partial^0 + \partial^1)$ is spanned by $\Upsilon_{q,\ell}^\epsilon$, $q \geq 0$, $\epsilon \in \{0, 1\}$.

(ii) The homology $H(C_\ell/d_s C_\ell, \partial^0 + \partial^1)$ is spanned by $\Upsilon_{q,\ell}^0$, $q \geq 0$.

Proof. Part (1) is an explicit (and straightforward) computation, which we omit.

To prove (2), we introduce an ascending filtration on C_ℓ by the degree in s (a nonnegative integer). This descends to a filtration on $C_\ell/d_s C_\ell$ as well. The differentials ∂^0 and ∂^1 respect the filtration. The associated graded complexes are

$$\begin{aligned} \text{gr}_s(C_\ell, \partial^0 + \partial^1) &= (\text{gr}_s C_\ell, \partial^0) \cong (C_\ell, \partial^0), \\ \text{gr}_s(C_\ell/d_s C_\ell, \partial^0 + \partial^1) &= (\text{gr}_s(C_\ell/d_s C_\ell), \partial^0) \cong (C_\ell/d_s C_\ell, \partial^0). \end{aligned} \quad (6.7.3)$$

One has the standard spectral sequence associated with the filtered complex $(C_\ell, \partial^0 + \partial^1)$. From the isomorphism in (6.7.3), we see that the first page of the spectral sequence is given by the homology of the complex $(C_\ell/d_s C_\ell, \partial^0)$. This homology is described by Lemma 6.7.2. The second page is obtained from the first page by taking the homology of the differential induced by ∂^1 .

We now prove (i); the proof of (ii) is similar. Since ∂^0 and ∂^1 are homogeneous of degree one in t , we can prove the statement in each fixed degree q in t . Let $C_{\ell,q} \subset C_\ell$ be the subset of elements of degree q in t . The proof rests on comparing $C_{\ell,q}$ with the part of the complex $(H(\text{gr}_s C_\ell, \partial^0), \partial^1)$ in degree $q + \ell$.

Let $(T_S^\ell M)_q$ denote the part of $T_S^\ell M$ of degree q in t , and $(T_S^\ell M)'_q$ the subspace spanned by (6.4.2). Let $C'_{\ell,q} \subseteq H(\text{gr}_s C_\ell, \partial^0)$ be the subspace of degree q in t , which is spanned by the elements (6.6.2)–(6.6.3). Let $C'_\ell = \bigoplus_{q \geq 0} C'_{\ell,q}$. The differential ∂' on C'_ℓ , in terms of the basis (6.6.2)–(6.6.3), is given by $\partial' = pr \circ \partial^1$, where ∂^1 is given by the usual formula on C_ℓ , and $pr : C_\ell \rightarrow C'_\ell$ is the orthogonal projection in the basis (6.6.6): more precisely, one first projects to $\ker(\partial^0)$ orthogonally and then takes the quotient by $\text{im}(\partial^0)$; equivalently, one projects orthogonally to the span of the elements (6.6.2)–(6.6.3). The reason why this is the differential is that C'_ℓ is obtained from C_ℓ as the kernel of the positive operator $[D, \partial^0]$, and so the formula for ∂' follows from the proof of Lemma 6.2.4 (namely, from the fact that $\text{im}([D, \partial^0])$, the orthogonal complement to the span of (6.6.2)–(6.6.3), is an acyclic direct summand of the complex (C_ℓ, ∂^0)).

A key step is to construct a map ψ_ℓ , essentially multiplication by $t^{\otimes \ell}$ with some constants, from $C_{\ell,q}$ to $C'_{\ell,q+\ell}$ which commutes with the differential d_s , and sends $\partial^0 + \partial^1$ to ∂' . The precise definition of ψ_ℓ is given on (6.4.6) by

$$\begin{aligned} \psi_\ell(s^{\epsilon_0} \cdot 1_{j_0} \otimes t^{i_1} s^{\epsilon_1} \otimes 1_{j_1} \otimes \dots \otimes t^{i_k} s^{\epsilon_k} \otimes 1_{j_k}) \\ = ((j_0 + 1)^{-1} s)^{\epsilon_0} \cdot (st + ts)^{\otimes j_0} \otimes t^{i_1+1} ((j_1 + 1)^{-1} s)^{\epsilon_1} \otimes (st + ts)^{\otimes j_1} \otimes \\ \dots \otimes t^{i_k+1} ((j_k + 1)^{-1} s)^{\epsilon_k} \otimes (st + ts)^{\otimes j_k}. \end{aligned} \quad (6.7.4)$$

Claim 6.7.5. The map ψ_ℓ is an isomorphism $C_{\ell,q} \rightarrow C'_{\ell,q+\ell}$ for all $q \geq 0$ which intertwines d_s with d_s and intertwines $\partial^0 + \partial^1$ on C_ℓ with ∂' on C'_ℓ .

To prove the claim, one has to verify that ψ_ℓ is an isomorphism, that it preserves d_s , and that it sends $\partial^0 + \partial^1$ to ∂' . These facts all follow from straightforward explicit computations, which we omit.

We now complete the proof of Lemma 6.7.2. We do this by induction on q . The base case is where $q < \ell$. In this case, $C'_{\ell,q}$ is already spanned by the elements $\text{gr}_s \Upsilon_{q,\ell}^\epsilon$, $q \geq 0, \epsilon \in \{0, 1\}$, which are exactly (6.6.3) up to nonzero constant factors, and, moreover, these elements are killed by ∂' .

For the inductive step, assume that, in degree q , $H(C_\ell, \partial^0 + \partial^1)$ is spanned by the elements $\Upsilon_{q,\ell}^\epsilon$ for $\epsilon \in \{0, 1\}$. By Claim 6.7.5, this implies that, in degree $q + \ell$, $H(C'_\ell, \partial')$ is spanned by $\psi_\ell(\Upsilon_{q,\ell}^\epsilon)$. We claim that

$$\psi_\ell(\Upsilon_{q,\ell}^\epsilon) = \text{gr}_s \Upsilon_{q+\ell,\ell}^\epsilon. \quad (6.7.6)$$

Then, by the spectral sequence for $(C_\ell, \partial^0 + \partial^1)$, this implies that the homology of $(C_\ell, \partial^0 + \partial^1)$ in degree $q + \ell$ is spanned by the classes $\Upsilon_{q+\ell,\ell}^\epsilon$, completing the induction.

It remains to prove (6.7.6). Clearly, the RHS is given by restricting the sum in (6.7.1) to the summands where $k = \ell$. In this case, each such summand is obtained from the summand of element $\Upsilon_{q,\ell}^\epsilon$. This proves (6.7.6). \square

6.8. Proof of Proposition 6.1.1. The proof amounts to a generalization of the preceding material from A_τ to A' . Here we revert to the notation $\otimes = \otimes_{\mathbf{k}}$ used before (and as we will use in the remainder of the paper).

Below, we will use, for every $p, q \geq 0$, a direct sum decomposition $\overline{\Omega}_t^{p,q} A' = \bigoplus_{\ell \geq 1} [\overline{\Omega}_t^{p,q} A']_{(\ell)}$, where $[\overline{\Omega}_t^{p,q} A']_{(\ell)}$ denotes the span of all elements in which \overline{A} appears ℓ times. This induces a similar decomposition $\overline{\text{DR}}_t^{p,q} A' = \bigoplus_{\ell \geq 1} [\overline{\text{DR}}_t^{p,q} A']_{(\ell)}$. Also, let a superscript of q denote the degree q part in t .

For each $\ell \geq 1$, we define a map $\zeta : \overline{A}^{\otimes \ell} \otimes M^{\otimes s\ell} \longrightarrow [\overline{\Omega}_t A']_{(\ell)}$ as follows. In positive degree in t , the map is given, for every $f_1, \dots, f_\ell \in \{1, t^i, t^i s \mid i \geq 1\} \subset M$ not all equal to 1, and $a_1, \dots, a_\ell \in \overline{A}$, by the formula

$$(a_1 \otimes \cdots \otimes a_\ell) \otimes (f_1 \otimes_s \cdots \otimes_s f_\ell) \mapsto (f_1 \star a_1)(f_2 \star a_2) \cdots (f_\ell \star a_\ell),$$

where we put

$$1 \star a = da, \quad t^i \star a = t^i a, \quad t^i s \star a = t^i da, \quad i \geq 1.$$

In degree zero in t , we use the formulas

$$(a_1 \otimes \cdots \otimes a_\ell) \otimes (1_\ell) \mapsto a_1 da_2 \cdots da_\ell, \quad (a_1 \otimes \cdots \otimes a_\ell) \otimes (s \cdot 1_\ell) \mapsto da_1 da_2 \cdots da_\ell.$$

Recall next that the cyclic group \mathbb{Z}/ℓ acts on C_ℓ by cyclic permutations of tensor components using the Koszul sign rule. We let \mathbb{Z}/ℓ act on $\overline{A}^{\otimes \ell}$ by ordinary cyclic permutations and we equip $\overline{A}^{\otimes \ell} \otimes C_\ell$ with \mathbb{Z}/ℓ -diagonal action. Further, we extend each of the three differentials $d = d_s, \partial^0$, and ∂^1 on C_ℓ to the differentials $d_A := \text{Id} \otimes d$ on $\overline{A}^{\otimes \ell} \otimes C_\ell$, where $\text{Id} : \overline{A}^{\otimes \ell} \rightarrow \overline{A}^{\otimes \ell}$ is the identity. Similarly, we extend the tilde versions \tilde{d} to $\tilde{d}_A := \text{Id} \otimes \tilde{d}$ on $\overline{A}^{\otimes \ell} \otimes M^{\otimes s\ell}$. Finally, for every $q \geq 0$, let $(\overline{A}^{\otimes \ell} \otimes M^{\otimes s\ell})^q$ and $(\overline{A}^{\otimes \ell} \otimes C_\ell)_{\mathbb{Z}/\ell}^q$ be the homogeneous component of $\overline{A}^{\otimes \ell} \otimes M^{\otimes s\ell}$ and $(\overline{A}^{\otimes \ell} \otimes C_\ell)_{\mathbb{Z}/\ell}$ of degree q in t , respectively.

Lemma 6.8.1. *For $\ell \geq 1$, (i) For every $q \geq 0$, the map ζ induces isomorphisms of complexes*

$$((\overline{A}^{\otimes \ell} \otimes M^{\otimes s\ell})^q, d_{s,A}) \xrightarrow{\sim} ([\overline{\Omega}_t^{\bullet,q} A']_{(\ell)}, d), \quad \text{and} \quad ((\overline{A}^{\otimes \ell} \otimes C_\ell)_{\mathbb{Z}/\ell}^q, d_{s,A}) \xrightarrow{\sim} ([\overline{\text{DR}}_t^{\bullet,q} A']_{(\ell)}, d).$$

(ii) For $\epsilon \in \{0, 1\}$ and $\ell \geq 1$, put $\Upsilon_\ell^\epsilon := \sum_{q \geq 0} \frac{1}{(q+\ell-1)!} \cdot \Upsilon_{q,\ell}^\epsilon$. Then, for every $\alpha \in \overline{A}^{\otimes \ell}$, the following diagram commutes:

$$\begin{array}{ccc} ((\overline{A}^{\otimes \ell} \otimes M^{\otimes s\ell})^0, \partial_A^0 + \partial_A^1) & \xrightarrow{\alpha \otimes s^\epsilon \cdot 1_\ell \mapsto \alpha \otimes \Upsilon_\ell^\epsilon} & (\prod_{q \geq 0} (\overline{A}^{\otimes \ell} \otimes C_\ell)_{\mathbb{Z}/\ell}^q, \partial_A^0 + \partial_A^1) \\ \cong \downarrow \zeta & & \cong \downarrow \zeta \\ ([\overline{\Omega}A']_{(\ell)}, b) & \xrightarrow{E'} & (\prod_{q \geq 0} [\overline{\text{DR}}_t^{\bullet, q} A']_{(\ell)}, i_t). \end{array}$$

Sketch of proof. In the special case where $\overline{A} = \mathbf{k}$ is a rank one free \mathbf{k} -module, $A' = A_\tau = \mathbf{k} \oplus \mathbf{k}\tau$. In this case, the isomorphisms of part (i) of the lemma, as well as fact that the map ζ respects the differentials, reduce to Lemmas 6.3.1 and 6.5.1. The proof in the general case is similar. Note that implication (6.5.2), used in the proof of Lemma 6.5.1, is applicable in the general case as well, since in the algebra A' it follows that $\overline{A} \cdot \overline{A} = 0$.

Finally, commutativity of the diagram of part (ii) is verified by direct computation using the explicit formula for the Tsygan map and for the map ζ . We omit the details. \square

Remark 6.8.2. In the special case $A = A_\tau$, i.e., $\overline{A} = \mathbf{k}$, the commutative diagram of Lemma 6.8.1(ii) reduces to the diagram:

$$\begin{array}{ccc} ((M^{\otimes s\ell})^0, \tilde{\partial}^0 + \tilde{\partial}^1) & \xrightarrow{s^\epsilon \cdot 1_\ell \mapsto \Upsilon_\ell^\epsilon} & (\prod_{q \geq 0} (C_\ell)_{\mathbb{Z}/\ell}^q, \partial^0 + \partial^1) \\ \cong \downarrow \zeta & & \cong \downarrow \zeta_t \\ ([\overline{\Omega}A_\tau]_{(\ell)}, b) & \xrightarrow{E'} & (\prod_{q \geq 0} [\overline{\text{DR}}_t^{\bullet, q} A_\tau]_{(\ell)}, i_t). \end{array}$$

We remark that it is immediate from commutativity of this diagram that $(\partial^0 + \partial^1)\Upsilon_\ell^\epsilon$ maps to zero in $(C_\ell)_{\mathbb{Z}/\ell}$. This way, one obtains the equation of Lemma 6.7.2(1), but only modulo cyclic permutations.

Proposition 6.1.1 is a direct consequence of the above lemma and Lemma 6.7.2.

7. PERIODIC AND NEGATIVE EXTENDED CYCLIC HOMOLOGY

7.1. We define extended periodic and negative cyclic homology of A as follows

$$\overline{\text{EHC}}^{\text{per}}(A) := H(\overline{\text{DR}}_t((u)), i_t - u d), \quad \text{and} \quad \overline{\text{EHC}}^-(A) := H(\overline{\text{DR}}_t[[u]], i_t - u d).$$

Similarly, let us define versions using the t -adic completion, $\overline{\text{EHC}}^{\text{per}, \wedge} := H(\overline{\text{DR}}_t^\wedge((u)), i_t - u d)$ and $\overline{\text{EHC}}^{-, \wedge}(A) := H(\overline{\text{DR}}_t^\wedge[[u]], i_t - u d)$.

Remark 7.1.1. The complexes above can be viewed as bicomplexes with differentials $u d$ and i_t . Each of the two differentials preserves ‘internal degree,’ defined as the sum of de Rham degree and degree in t minus degree in u . Separating homogeneous components with respect to internal degree gives decompositions of $(\overline{\text{DR}}_t^\wedge((u)), i_t - u d)$ and $(\overline{\text{DR}}_t^\wedge[[u]], i_t - u d)$ into infinite products of subcomplexes. (Unlike for $(\overline{\text{DR}}_t \otimes R, i_t - u d)$, the uncompleted versions here are not direct sums of subcomplexes in each internal degree, which is why we restrict our attention in this section to the t -adically completed versions, which are better behaved.)

Proposition 7.1.2. *The projection $(\overline{\text{DR}}_t^\wedge((u)), i_t - u d) \rightarrow (\overline{\text{DR}}((u)), -u d)$, modulo t , is a morphism of complexes, inducing an isomorphism on homology $\overline{\text{EHC}}^{\text{per}, \wedge}(A) \xrightarrow{\sim} \overline{\text{HD}}(A)((u))$.*

Remark 7.1.3. The above isomorphism, as well as all other isomorphisms which appear in this section, preserves the ‘homological grading,’ which is defined to be de Rham degree minus twice the degree in u .

Proof of Proposition. Exactly as in the proof of Proposition 5.6.1, we can take homology first with respect to $u d$ and use a spectral sequence. This spectral sequence is obviously convergent on each homogeneous component with respect to the internal degree, cf. Remark 7.1.1; moreover, it collapses on the first page (the homology with respect to $u d$). \square

Remark 7.1.4. One can similarly compute $\overline{\text{EHC}}^{-,\wedge}(A)$. First of all, observe that there is a subcomplex isomorphic to $\ker_{\overline{\text{HH}}(A)}(B \circ I)$ (recall (4.1.3) for the definitions of B and I), with zero differential, sitting in degree zero in both t and u ; this maps isomorphically to a subspace of $\overline{\text{EHC}}^{-,\wedge}(A)$ since the differential lands only in the sum of elements of positive degree in t or in u . In other words, there is a noncanonical splitting of $\ker_{\overline{\text{HH}}(A)}(B \circ I)$ off of $\overline{\text{EHC}}^{-,\wedge}(A)$ obtained by choosing a closed vector space complement which contains everything which has positive degree in either t and u . As in the literature, let us make the abuse of notation $B := B \circ I : \overline{\text{HH}}(A) \rightarrow \overline{\text{HH}}(A)$ for the induced differential on $\overline{\text{HH}}(A)$.

Claim 7.1.5. There is a natural exact sequence

$$0 \rightarrow \prod_{q \geq 1} \frac{H(\overline{\text{DR}}_t / d \overline{\text{DR}}_t, i_t)^{\bullet, q}}{(\tilde{d}^{-1} i_t)^q (\overline{\text{HD}}(A))} \xrightarrow{d} \frac{\overline{\text{EHC}}^{-,\wedge}(A)}{\ker_{\overline{\text{HH}}(A)}(B)} \xrightarrow{\pi} \prod_{m \geq 1} u^m \cdot \ker_{\overline{\text{HD}}(A)}((\tilde{d}^{-1} i_t)^{m+1}) \rightarrow 0.$$

Here, the map d is induced by the obvious map $\bar{d} : \overline{\text{DR}}_t^\wedge / d \overline{\text{DR}}_t^\wedge \rightarrow \overline{\text{DR}}_t^\wedge$, and the map π mods by the constant term in u and by positive degree in t . Further, $(\tilde{d}^{-1} i_t)^q : \overline{\text{HD}}(A) \rightarrow H(\overline{\text{DR}}_t / d \overline{\text{DR}}_t, i_t)^{\bullet, q}$ is the map considered in §5.7 (see the proof of Proposition 5.7.1). More explicitly, the map $\tilde{d}^{-1} i_t$ induces an endomorphism of $H(\overline{\text{DR}}_t / d \overline{\text{DR}}_t, i_t)$ as follows. Given an element of $H(\overline{\text{DR}}_t / d \overline{\text{DR}}_t, i_t)$, one first lifts it to $\overline{\text{DR}}_t$, then applies i_t . After that, one applies \tilde{d}^{-1} , and projects to $\overline{\text{DR}}_t / d \overline{\text{DR}}_t$, which lands back in $\ker_{\overline{\text{DR}}_t / d \overline{\text{DR}}_t}(i_t)$. Finally, we precompose with the inclusion $\overline{\text{HD}}(A) \subseteq \overline{\text{HC}}(A) \cong H(\overline{\text{DR}}_t / d \overline{\text{DR}}_t, i_t)^{\bullet, 0} \subseteq H(\overline{\text{DR}}_t / d \overline{\text{DR}}_t, i_t)$.

By definition, $(\tilde{d}^{-1} i_t)^m|_{\overline{\text{HD}}_k(A)} = 0$ when $k < 2m$. Thus, for fixed k and $m > k/2 - 1$, the component $u^m \cdot \ker_{\overline{\text{HD}}_k(A)}((\tilde{d}^{-1} i_t)^{m+1})$ of the above product is nothing but $u^m \cdot \overline{\text{HD}}_k(A)$.

Further, using isomorphism (5.5.5), the short exact sequence of Claim 7.1.5 becomes

$$0 \rightarrow \prod_{q \geq 1} \frac{\overline{\text{HC}}(A) \cdot t^q}{\text{im}(S^{q+1})} \rightarrow \frac{\overline{\text{EHC}}^{-,\wedge}(A)}{\ker_{\overline{\text{HH}}(A)}(B)} \rightarrow \prod_{m \geq 1} u^m \cdot \ker_{\overline{\text{HD}}(A)}((S_1 S_2)^{m+1}) \rightarrow 0, \quad (7.1.6)$$

where we note that the q became $q + 1$ in the first factor since the periodicity operator S is acting on $\overline{\text{HC}}(A)$ rather than on $\overline{\text{HD}}(A) = \text{im}(S_1)$.

Proof of Claim 7.1.5. We use the spectral sequence for the bicomplex $(\overline{\text{DR}}_t^\wedge[[u]], u d, i_t)$, taking homology first with respect to $u d$. This yields the first page,

$$(u \cdot \overline{\text{HD}}(A)[[u]] \oplus \ker_{\overline{\text{DR}}_t^\wedge}(d), 0 \oplus i_t).$$

Using the isomorphism $d : (\overline{\text{DR}}_t^\wedge)^+ / d(\overline{\text{DR}}_t^\wedge)^+ \xrightarrow{\sim} d(\overline{\text{DR}}_t^\wedge)^+$, which commutes with i_t , the second page becomes

$$u \cdot \overline{\text{HD}}(A)[[u]] \oplus H^{>0}(\overline{\text{DR}}_t^\wedge / d \overline{\text{DR}}_t^\wedge, i_t) / (\tilde{d}^{-1} i_t \overline{\text{HD}}(A)) \oplus \ker_{\overline{\text{HH}}(A)}(d).$$

The terms on the left and right of the exact sequence of Claim 7.1.5 then appear on the $(m + 2)$ -nd and $(q + 1)$ -st pages. One sees that the spectral sequence converges to this, i.e., there are no other

nontrivial differentials that appear. Therefore, the associated graded (where by this we mean an infinite product of homogeneous components) of $\overline{\text{EHC}}^{-,\wedge}(A)$ with respect to the t -adic filtration is the above, and one can check that the maps in the exact sequence of Claim 7.1.5 induce the required isomorphism on the level of associated graded, so the sequence there must be exact (with injective first arrow and surjective third arrow). \square

Remark 7.1.7. According to (5.5.2), Proposition 7.1.2, and (7.1.6), the extended versions of all cyclic homology groups $\overline{\text{EHC}}(A)$, $\overline{\text{EHC}}^{\text{per},\wedge}(A)$, and $\overline{\text{EHC}}^{-,\wedge}(A)$ are expressible in terms of the *original* Connes sequence (4.1.2), and do not require $\overline{\text{HC}}^{-}(A)$ or $\overline{\text{HC}}^{\text{per}}(A)$, which use infinite series in their definition. In other words, taking the extended version breaks the complex up into a direct product of complexes only involving finite sums, i.e., elements of $\overline{\text{DR}}_t[u, u^{-1}]$ (and via Theorem 5.3.10, elements of $\overline{\Omega}[u, u^{-1}]$), owing to the internal grading. As a consequence, the natural morphisms we will describe in §8 from the various flavors of cyclic homology of A to the (equivariant) cohomology of the representation varieties of A all factor through (infinite products of) subquotients of $\overline{\text{HC}}(A)$ and $\overline{\text{HH}}(A)$.

7.2. The result below describes the maps induced by E on the versions of cyclic homology.

Let $\pi' : \overline{\text{EHC}}(A) \rightarrow \overline{\text{HD}}(A) \otimes u^{-1}\mathbf{k}[u^{-1}] \subseteq \overline{\text{HD}}(A) \otimes R$ be the map given by $f = \sum_{m \geq 0} f_m u^{-m} \mapsto \sum_{m \geq 1} f_m u^{-m} \bmod (t)$, cf. Proposition 5.6.1. Let $\overline{\text{EHC}}^{\wedge}(A) = H(\overline{\text{DR}}_t^{\wedge} \otimes R, i_t - u d) \cong \overline{\text{HC}}(A)[[t]] \oplus u^{-1} \overline{\text{HD}}(A)[u^{-1}]$, and define π' in the same way on it.

Proposition 7.2.1. (i) *The composite map $\overline{\text{HC}}(A) \xrightarrow{E} \overline{\text{EHC}}^{\wedge}(A) \xrightarrow{\pi'} \overline{\text{HD}}(A) \otimes R$ can be expressed as $\pi' \circ E = \sum_{m \leq 1} u^{-m} \cdot S_1 S^{m-1}$.*

(ii) *The induced map $\overline{\text{HC}}^{\text{per}}(A) \rightarrow \overline{\text{EHC}}^{\text{per},\wedge}(A) \cong \overline{\text{HD}}(A)((u))$ is $\sum_{m \in \mathbb{Z}} u^{-m} \cdot p_1 u^m$.*

(iii) *The induced map $\overline{\text{HC}}^{-}(A) \rightarrow \overline{\text{EHC}}^{-,\wedge}(A)$, in terms of (7.1.5), is the sum of I^- to $\ker_{\overline{\text{HH}}(A)}(d)$ with the map which is $\sum_{m \geq 1} u^m (p_1 u^{-m} p^-)$ projected to the subspace of $\prod_{m \geq 1} u^m \cdot \ker_{\overline{\text{HD}}(A)}((\check{d}^{-1} i_t)^{m+1})$, and, restricted to the kernel of this map, maps to the zero fiber $\prod_{q \geq 1} H(\overline{\text{DR}}_t / d \overline{\text{DR}}_t, i_t)^{\bullet, q} / \text{im}((\check{d}^{-1} i_t)^q)$ by $\sum_{q \geq 1} E^q \circ p \circ p^-$.*

The proof is a direct consequence of formula (5.3.9), Propositions 5.6.1 and 7.1.2 and Remark 7.1.4.

7.3. Extended version of the Connes exact sequence. There is an extended analogue of diagram (4.4.4). One can obtain this diagram by applying E to the diagram of short exact sequences of complexes that induce (4.4.4) (in the usual proof). This yields

$$\begin{array}{ccccccc}
0 & \longrightarrow & \overline{\text{DR}}_t^{\wedge}[[u]] & \xrightarrow{\cdot u} & \overline{\text{DR}}_t^{\wedge}[[u]] & \longrightarrow & \overline{\text{DR}}_t^{\wedge} \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & \overline{\text{DR}}_t^{\wedge}[[u]] & \xrightarrow{\cdot u} & \overline{\text{DR}}_t^{\wedge}((u)) & \longrightarrow & \overline{\text{DR}}_t^{\wedge} \otimes R \longrightarrow 0 \\
& & \downarrow & & \downarrow \cdot u^{-1} & & \parallel \\
0 & \longrightarrow & \overline{\text{DR}}_t^{\wedge} & \longrightarrow & \overline{\text{DR}}_t^{\wedge} \otimes R & \xrightarrow{\cdot u} & \overline{\text{DR}}_t^{\wedge} \otimes R \longrightarrow 0.
\end{array}$$

This induces the diagram (4.4.4) of short exact sequences, with E put in front of every nonzero term and t -adic completions taken everywhere, where $\overline{\text{EHH}}$, $\overline{\text{EHH}}'$, $\overline{\text{EHD}}$, and $\overline{\text{EHD}}'$ are given by:

$$\begin{aligned}\overline{\text{EHH}}(A) &:= H(\overline{\text{DR}}_t, i_t), & \overline{\text{EHH}}'(A) &:= \bigoplus_{q \geq 0} [(\tilde{d}^{-1} i_t)^q (\ker_{\overline{\text{DR}}}(\text{d}) \cap \ker(i_t))] \subseteq \overline{\text{EHH}}(A), \\ \overline{\text{EHD}}(A) &:= \overline{\text{HD}}(A)[u^{-1}] \oplus \bigoplus_{q \geq 1} \ker(\text{d} : H(\overline{\text{DR}}_t / \text{d} \overline{\text{DR}}_t, i_t) \rightarrow H(\overline{\text{DR}}_t, i_t)), \\ \overline{\text{EHD}}'(A) &:= \overline{\text{HD}}(A)[u^{-1}] \oplus \bigoplus_{q \geq 1} (\tilde{d}^{-1} i_t)^q (\overline{\text{HD}}(A)) \subseteq \overline{\text{EHD}}(A),\end{aligned}$$

and the t -adically completed versions are obtained by replacing $\overline{\text{DR}}_t$ with $\overline{\text{DR}}_t^\wedge$, and the direct sums above by direct products. Then, E maps (4.4.4) commutatively onto the resulting diagram.

Remark 7.3.1. By Theorem 5.3.10, one has isomorphisms $E^q : \overline{\text{HH}}(A) \xrightarrow{\sim} \overline{\text{EHH}}(A)^{\bullet, q}$, $q \geq 0$. We deduce that $\overline{\text{EHH}}'(A)^{\bullet, 0} \cong \ker_{\overline{\text{HH}}(A)}(B) = I^{-1}(\text{im } S)$, where $S : \overline{\text{HC}}(A) \rightarrow \overline{\text{HC}}(A)$ is the periodicity operator. Then one obtains:

$$\begin{aligned}\overline{\text{EHH}}(A) &\cong \overline{\text{HH}}(A)[t] \supseteq \overline{\text{EHH}}'(A) \cong \bigoplus_{q \geq 0} I^{-1}(\text{im}(S^{q+1})) \cdot t^q, \\ \overline{\text{EHD}}(A) &\cong \overline{\text{HD}}(A)[u^{-1}] \bigoplus t \overline{\text{HD}}(A)[t] \supseteq \overline{\text{EHD}}'(A) \cong \overline{\text{HD}}(A)[u^{-1}] \bigoplus (\bigoplus_{q \geq 1} \text{im}(S_1 S_2)^q \cdot t^q).\end{aligned}$$

The t -adically completed versions of $\overline{\text{EHH}}(A)$, $\overline{\text{EHH}}'(A)$, $\overline{\text{EHD}}(A)$, and $\overline{\text{EHD}}'(A)$ are all obtained by t -adically completing the above formulas (replacing sums by products and polynomials in t by power series in t).

Remark 7.3.2. One can explicitly compute the extended, t -adically completed version of (4.4.4) in terms of the preceding remark along with (5.5.2), Proposition 7.1.2, and (7.1.6), and verify that the rows are exact using only exactness of (4.4.4). We describe just the horizontal maps in these terms:

The bottom row:

ES_2 : This is the identity on $u^{-1} \overline{\text{HD}}(A)[u^{-1}]$ and the inclusion S_2 on $\overline{\text{HD}}(A)[t] \subseteq \overline{\text{HC}}(A)[t]$, preserving degree in t .

EB : $\overline{\text{EHC}}^\wedge(A) \cong u^{-1} \overline{\text{HD}}(A)[u^{-1}] \oplus \overline{\text{HC}}(A)[t] \rightarrow \overline{\text{EHH}}^\wedge(A) \cong \overline{\text{HH}}(A)[t]$ is zero on the first factor of the source and B on the second factor.

EI : $\overline{\text{EHH}}^\wedge(A) \cong \overline{\text{HH}}(A)[t] \rightarrow \overline{\text{EHC}}^\wedge(A) = u^{-1} \overline{\text{HD}}(A)[u^{-1}] \oplus \overline{\text{HC}}(A)[t]$ is zero on the first factor of the target, and I on the second factor of the target.

ES_1 is the tautological map (preserving the degree in t).

The middle row:

EB' : $\overline{\text{EHC}}^\wedge(A) \rightarrow \overline{\text{EHC}}^{-, \wedge}(A)$: in terms of (7.1.5), this kills $u^{-1} \overline{\text{HD}}[u^{-1}]$, maps by B on $\overline{\text{HC}}(A)$ in degree $q = 0$ of t , and for positive degree $q \geq 1$ in t , it is the projection $\overline{\text{HC}}(A) \rightarrow \overline{\text{HC}} / \text{im } S^{q+1} \subseteq \overline{\text{EHC}}^{-, \wedge}(A)$ by (7.1.5).

Ep^- : $\overline{\text{EHC}}^{-, \wedge}(A) \rightarrow \overline{\text{EHC}}^{\text{per}, \wedge}(A)$ projects onto $\ker_{\overline{\text{HH}}(A)}(B) \oplus \prod_{m \geq 1} u^m \ker_{\overline{\text{HD}}(A)}(S_1 S_2)^{m+1}$, and then applies I to the first factor and the natural inclusion on the second factor into $u \overline{\text{HD}}[u]$, preserving degree in u .

Ep_1 is the identity on $\overline{\text{HD}}(A)[u^{-1}]$ and sends $u^m \overline{\text{HD}}(A)$ to the factor for $q = m$ by the map $S^m S_2 u^{-m} t^m$.

Ep_2 is the tautological inclusion (preserving degree in t).

The first row:

The first map is the tautological inclusion.

EB^- maps $\overline{\text{HH}}(A)$ via B to $\ker_{\overline{\text{HH}}(A)}(B)$ in degree $q = 0$ of t , and in higher degrees $q \geq 1$ maps $\overline{\text{HH}}(A)$ to $\overline{\text{HC}}(A)$ via I and projects to $\overline{\text{HC}}(A) / \text{im } S^{q+1}$.

ES^- applies $t \cdot S$ to $\overline{\text{HC}}(A)t^q / \text{im } S^{q+1}$ for $q \geq 1$, increasing degree in t by one. Modulo these factors (the kernel of the sequence (7.1.5)), it multiplies by u : on the quotient of (7.1.5) these are

the inclusions $\ker_{\overline{\text{HD}}(A)}(S_1 S_2)^{m+1} \subseteq \ker_{\overline{\text{HD}}(A)}(S_1 S_2)^{m+2}$, and on $\ker_{\overline{\text{HH}}(A)}(B)$ this is the natural map $\ker_{\overline{\text{HH}}(A)}(B) \rightarrow \ker_{\overline{\text{HD}}(A)}(S_1 S_2)$ (this follows because $\ker_{\overline{\text{HH}}(A)}(B) = \ker_{\overline{\text{DR}}}(d) \cap \ker_{\overline{\text{DR}}}(it)$, so projecting mod $d \overline{\text{DR}}$ we land in $\ker_{\overline{\text{HD}}(A)}(S_1 S_2) \subseteq \ker_{\overline{\text{HD}}(A)}(S_1 S_2)^2$).

EI^- is the identity on $\ker_{\overline{\text{HH}}(A)}(B) = I^{-1} \text{im}(S)$ and applies B to the kernel of (7.1.5), which gives a well-defined map to $\overline{\text{HH}}(A)$ since $B \circ S^q = 0$ for $q \geq 1$. The image of the latter map is the subspace $\ker(I) = \text{im}(B)$ of $\overline{\text{EHH}}^\wedge(A) = \bigoplus_{q \geq 0} I^{-1} \text{im} S^{q+1}$. Then, on the quotient (7.1.5), EI^- applies $(S_1 S_2)^m u^{-m}$ to $u^m \ker(S_1 S_2)^{m+1}$ for $m \geq 1$, which gives a well-defined map to $\overline{\text{HH}}(A)/\ker(I) \cong \text{im}(I) \subseteq \overline{\text{HC}}(A)$, with image equal to $\text{im}_{\overline{\text{HD}}(A)}(S_1 S_2)^m \cap \ker_{\overline{\text{HD}}(A)}(S_1 S_2)$, which is $\text{im}(S^{m+1}) \cap \ker(S) = \text{im}(S^{m+1}) \cap \text{im}(I)$ as a subspace of $\text{im}(I) \subseteq \overline{\text{HC}}(A)$.

8. THE REPRESENTATION FUNCTOR AND EQUIVARIANT DELIGNE COHOMOLOGY

8.1. Equivariant Deligne cohomology. Let X be an affine² variety equipped with an action of a connected reductive algebraic group G with Lie algebra \mathfrak{g} . Let $\text{act} : \mathfrak{g} \rightarrow \text{Vect}(X)$ be the infinitesimal action and let Ω_X^1 be the space of Kähler differentials on X . Thus, the algebra $\Omega_X = \wedge_{\mathbf{k}[X]}(\Omega_X^1)$ of algebraic differential forms on X acquires a natural structure of \mathfrak{g} -module. One also has a natural \mathfrak{g} -action on $\mathbf{k}[\mathfrak{g}]$, a polynomial algebra, induced by the adjoint \mathfrak{g} -action on \mathfrak{g} itself. We let \mathfrak{g} act diagonally on the algebra $\Omega_X[\mathfrak{g}] := \Omega_X \otimes \mathbf{k}[\mathfrak{g}]$.

Recall that the Cartan model of the equivariant (algebraic) de Rham complex of X is defined as

$$((\Omega_X[\mathfrak{g}])^{\mathfrak{g}}, d_X - i_{\mathfrak{g}}). \quad (8.1.1)$$

where d_X is the de Rham differential on X and $i_{\mathfrak{g}}$ is the equivariant differential, given by

$$i_{\mathfrak{g}}(df)(x) = \text{act}(x)(f), \quad i_{\mathfrak{g}}|_{\Omega_X^q} = 0 = i_{\mathfrak{g}}|_{\mathbf{k}[\mathfrak{g}]}$$

By analogy with cyclic homology, we introduce three new versions of the Cartan model, all equipped with the differential $i_{\mathfrak{g}} - u d_X$, where u is an extra parameter:

$$\text{CC}_{\mathfrak{g}}^{\text{per}}(X) := (\Omega_X[\mathfrak{g}])^{\mathfrak{g}}((u)), \quad \text{CC}_{\mathfrak{g}}(X) := (\Omega_X[\mathfrak{g}])^{\mathfrak{g}} \otimes R, \quad \text{CC}_{\mathfrak{g}}^-(X) := (\Omega_X[\mathfrak{g}])^{\mathfrak{g}}[[u]]. \quad (8.1.2)$$

Remark 8.1.3. The complex $(\text{CC}_{\mathfrak{g}}^{\text{per}}(X), i_{\mathfrak{g}} - u d_X)$ is isomorphic, up to sign in the differential, to the standard version (8.1.1) tensored by $\mathbf{k}((u))$, via the map $\mathfrak{g}^* \xrightarrow{\sim} u^{-1} \mathfrak{g}^*, g \mapsto u^{-1} g$.

We define the *homological grading* $\text{CC}_{\mathfrak{g}}(X) = \bigoplus_n \text{CC}_{\mathfrak{g}}^n(X)$ by placing the space $(\Omega_X^p \otimes \mathbf{k}[\mathfrak{g}])^{\mathfrak{g}} \cdot u^{-r}$ in degree $n = p + 2r$, for every p and r . Thus, the tensor factor $\mathbf{k}[\mathfrak{g}]$, is assigned homological degree zero, and u has homological degree $|u| = -2$. Each of the differentials $u d_X$ and $i_{\mathfrak{g}}$, and hence also $i_{\mathfrak{g}} - u d_X$, has homological degree (-1) . We define *equivariant Deligne cohomology*, which will serve as the natural receptacle of cyclic homology under the representation functor, by

$$\mathcal{D}H_{\mathfrak{g}}^{\bullet}(X) := H(\text{CC}_{\mathfrak{g}}^{\bullet}(X), i_{\mathfrak{g}} - u d).$$

Periodic and negative versions are defined in an obvious way.

Similarly to §5.5, we also introduce an *internal grading* $\text{CC}_{\mathfrak{g}}(X) = \bigoplus_{k \in \mathbb{Z}} \text{CC}_{\mathfrak{g}}(X)_k$, by assigning the space $(\Omega_X^p \otimes \mathbf{k}^q[\mathfrak{g}])^{\mathfrak{g}} \cdot u^{-r}$ internal degree $p + q + r$, for every p, q , and r . Here, we use the notation $\mathbf{k}^q[\mathfrak{g}]$ for the space of degree q homogeneous polynomials on \mathfrak{g} . Each of the differentials $u d_X$ and $i_{\mathfrak{g}}$ preserves the internal grading. Therefore, the complex $(\text{CC}_{\mathfrak{g}}(X), i_{\mathfrak{g}} - u d_X)$ breaks up into a direct sum over all $k \in \mathbb{Z}$ of the subcomplexes $(\text{CC}_{\mathfrak{g}}(X)_k, i_{\mathfrak{g}} - u d_X)$.

For $\ell \geq 0$ and $m \in \mathbb{Z}$, we define the *weight* m component of $\mathcal{D}H_{\mathfrak{g}}^{\ell}(X)$, cf. §5.5, by

$$\mathcal{D}H_{\mathfrak{g}}^{\ell}(X, m) := H^{\ell}(\text{CC}_{\mathfrak{g}}(X)_{\ell-m}, i_{\mathfrak{g}} - u d_X).$$

²It is possible to generalize everything below to the case of not necessarily affine varieties by replacing the space of global differential forms on X by the sheaf of differential forms.

The natural multiplication in the algebra $\Omega_X((u)) \otimes \mathbf{k}[\mathfrak{g}]$ induces a canonical $(\mathbf{k}[\mathfrak{g}]^{\mathfrak{g}})[u]$ -module structure on the equivariant Deligne cohomology. The module structure and the gradings are related by

$$(\mathbf{k}^q[\mathfrak{g}]^{\mathfrak{g}})^{u^r} \otimes \mathcal{D}H_{\mathfrak{g}}^{\ell}(X, m) \longrightarrow \mathcal{D}H_{\mathfrak{g}}^{\ell-2r}(X, m - q - r), \quad \forall q, r \geq 0. \quad (8.1.4)$$

There is also an algebra structure on the equivariant Deligne cohomology, similar to the one on the ordinary Deligne cohomology. In the equivariant case, the product is defined by the formula $\omega \cdot \omega' = \omega \cup u^{-1}(i_{\mathfrak{g}} - u d_X)\omega'$. Here $u^{-1}(i_{\mathfrak{g}} - u d_X)\omega'$ should be computed by first lifting ω' to $(\Omega_X[\mathfrak{g}])^{\mathfrak{g}} \otimes \mathbf{k}((u))$, then after performing the operation, projecting back to $(\Omega_X[\mathfrak{g}])^{\mathfrak{g}} \otimes R$. Compare this with [Lo, §3.6.2] regarding usual Deligne cohomology. (Note that, when ω' is a cocycle, i.e., $(i_{\mathfrak{g}} - u d_X)\omega' = 0$ in $CC_{\mathfrak{g}}^*(X)$, then the above formula can be rewritten as $\omega \cup -d_X \omega'_0$, where ω'_0 is the constant term in u of ω' .) The algebra structure will not play a role below.

Equivariant Deligne cohomology is functorial with respect to G -equivariant maps.

Example 8.1.5. If \mathfrak{g} acts trivially on X , we obtain $CC_{\mathfrak{g}}(X) \cong (\Omega_X \otimes R, u d) \otimes \mathbf{k}[\mathfrak{g}]^{\mathfrak{g}}$. The cohomology of the first term on the RHS is

$$H(\Omega_X \otimes R, u d) \cong \Omega_X / d\Omega_X \oplus (H_{\text{DR}}(X) \otimes u^{-1} \cdot \mathbf{k}[u^{-1}]).$$

We find that the group $\mathcal{D}H_{\mathfrak{g}}^{\ell}(X, m)$ vanishes for $m > \lfloor \frac{\ell}{2} \rfloor$, and

$$\mathcal{D}H_{\mathfrak{g}}^{\ell}(X, m) = \begin{cases} \left(\bigoplus_{1 \leq j \leq \lfloor \ell/2 \rfloor} H_{\text{DR}}^{\ell-2j}(X) \otimes \mathbf{k}^{j-m}[\mathfrak{g}]^{\mathfrak{g}} \right) \oplus \left(\Omega_X^{\ell} / d\Omega_X^{\ell-1} \otimes \mathbf{k}^{-m}[\mathfrak{g}]^{\mathfrak{g}} \right), & m \leq 0 \\ \bigoplus_{m \leq j \leq \lfloor \ell/2 \rfloor} H_{\text{DR}}^{\ell-2j}(X) \otimes \mathbf{k}^{j-m}[\mathfrak{g}]^{\mathfrak{g}}, & 0 < m \leq \lfloor \frac{\ell}{2} \rfloor. \end{cases}$$

In total degree ℓ , we therefore obtain $\mathcal{D}H_{\mathfrak{g}}^{\ell}(X) \cong \mathcal{D}H^{\ell}(X) \otimes \mathbf{k}[\mathfrak{g}]^{\mathfrak{g}}$, where we write

$$\mathcal{D}H^{\ell}(X) := \bigoplus_{1 \leq j \leq \lfloor \ell/2 \rfloor} H_{\text{DR}}^{\ell-2j}(X) \oplus \Omega_X^{\ell} / d\Omega_X^{\ell-1}$$

for the even or odd-degree part of the usual Deligne cohomology of X for the integer $\ell + 1$ shifted down in degree by one. The Deligne cohomology is denoted $\mathcal{D}\mathbb{H}(X, \mathbb{Z}(n+1))$ in [Lo, §3.6].

Example 8.1.6. Let $p : X \rightarrow Y$ be a principal G -bundle on an affine variety Y , the latter being viewed as a G -variety with the trivial G -action. Then, analogously to the case of usual equivariant cohomology, there is a canonical isomorphism

$$\mathcal{D}H_{\mathfrak{g}}^{\bullet}(X) \cong \mathcal{D}H^{\bullet}(Y).$$

To prove this, one observes first that a G -bundle on an affine variety admits an algebraic connection. Using a connection, one shows that the pull-back morphism $p^* : \Omega_Y \rightarrow \Omega_X$ induces an isomorphism $\mathbf{k}[X] \otimes_{\mathbf{k}[Y]} \Omega_Y \xrightarrow{\sim} H(\Omega_X[\mathfrak{g}], i_{\mathfrak{g}})$. Taking G -invariants on each side and using that G is connected and reductive, we deduce an isomorphism $\Omega_Y = (\mathbf{k}[X] \otimes_{\mathbf{k}[Y]} \Omega_Y)^{\mathfrak{g}} \xrightarrow{\sim} H((\Omega_X[\mathfrak{g}])^{\mathfrak{g}}, i_{\mathfrak{g}})$. The required isomorphism $\mathcal{D}H_{\mathfrak{g}}^{\bullet}(X) \cong \mathcal{D}H^{\bullet}(Y)$ then follows from the spectral sequence of a double complex, by taking homology of the differential $i_{\mathfrak{g}}$ first.

8.2. The representation functor. Associated with an algebra A and $n \geq 1$, there is an affine scheme $\text{Rep}_n A$ that parametrizes n -dimensional representations of A . Its closed points are algebra homomorphisms $\rho : A \rightarrow \text{Mat}_n(\mathbf{k})$, $a \mapsto \rho(a)$. The scheme $\text{Rep}_n A$ comes equipped with a natural action of GL_n , the general linear group, by base change transformations. We write $\mathfrak{gl}_n = \text{Lie GL}_n$ for the corresponding Lie algebra.

Below, we will use simplified notation: $\text{Rep}_n = \text{Rep}_n A$, $\text{DR}_t = \text{DR}_t A$, etc. The connection between DR_t and the GL_n -equivariant de Rham complex of Rep_n is through the evaluation map, which is a ‘tautological’ homomorphism of algebras,

$$\text{ev} : A \rightarrow (\mathbf{k}[\text{Rep}_n] \otimes \text{Mat}_n(\mathbf{k}))^{\mathfrak{gl}_n}, \quad a \mapsto (\rho \mapsto \rho(a)).$$

The above can be defined scheme-theoretically by extending this functorially to representations with coefficients in arbitrary \mathbf{k} -algebras; see, e.g., [Gi, §12].

The evaluation map extends uniquely to a homomorphism of dg algebras,

$$\text{ev} : \Omega A \rightarrow (\Omega_{\text{Rep}_n} \otimes \text{Mat}_n(\mathbf{k}))^{\mathfrak{gl}_n}, \quad \text{ev}(da) = d \text{ev}(a).$$

Finally, to extend this to $\Omega_t A$, let $\text{ev}(t) : \mathfrak{gl}_n \rightarrow \text{Mat}_n(\mathbf{k})$ be the identity map, viewed as a linear function on \mathfrak{gl}_n valued in $\text{Mat}_n(\mathbf{k})$, i.e., as an element of the algebra $\mathbf{k}[\mathfrak{gl}_n] \otimes \text{Mat}_n(\mathbf{k})$. Put together, we obtain a dg algebra map $\text{ev} : \Omega_t A \rightarrow (\Omega_{\text{Rep}_n}[\mathfrak{gl}_n] \otimes \text{Mat}_n(\mathbf{k}))^{\mathfrak{gl}_n}$. Further, taking a trace gives a chain of maps

$$\Omega_t A \xrightarrow{\text{ev}} (\Omega_{\text{Rep}_n}[\mathfrak{gl}_n] \otimes \text{Mat}_n(\mathbf{k}))^{\mathfrak{gl}_n} \xrightarrow{\text{Id} \otimes \text{trace}} (\Omega_{\text{Rep}_n}[\mathfrak{gl}_n] \otimes \mathbf{k})^{\mathfrak{gl}_n} = (\Omega_{\text{Rep}_n}[\mathfrak{gl}_n])^{\mathfrak{gl}_n}.$$

The resulting composite map descends to a linear map

$$\text{tr} \circ \text{ev} : \text{DR}_t \rightarrow (\Omega_{\text{Rep}_n}[\mathfrak{gl}_n])^{\mathfrak{gl}_n}.$$

Then, it is not difficult to verify

Theorem 8.2.1 ([GS], Theorem 6.2.5). *For all $n \geq 1$, the map $\text{tr} \circ \text{ev}$ induces a map of bicomplexes*

$$\text{tr} \circ \text{ev} : (\text{DR}_t, d, i_t) \rightarrow ((\Omega_{\text{Rep}_n}[\mathfrak{gl}_n])^{\mathfrak{gl}_n}, d_{\text{Rep}_n}, i_{\mathfrak{gl}_n}).$$

8.3. From cyclic to equivariant Deligne cohomology. Let us define reduced versions of the previous equivariant complexes and cohomology groups of X by replacing Ω_X by $\overline{\Omega}_X = \Omega_X/\mathbf{k}$ (where $\mathbf{k} \subseteq \Omega_X$ are the scalars in degree zero), and leaving everything else the same. We will put lines over everything to denote the reduced versions.

By Theorem 8.2.1 and construction, we deduce that the map $\text{tr} \circ \text{ev}$ induces, for all $n \geq 1$, a canonical morphism of $\mathbf{k}[\mathfrak{gl}_n]^{\mathfrak{gl}_n}[u]$ -modules:

$$\overline{\text{EHC}}(A) = H(\overline{\text{DR}}_t \otimes R, i_t - u d) \rightarrow {}^{\mathcal{D}}\overline{H}_{\mathfrak{gl}_n}(\text{Rep}_n) = H((\overline{\Omega}_{\text{Rep}_n}[\mathfrak{gl}_n])^{\mathfrak{gl}_n} \otimes R, i_{\mathfrak{gl}_n} - u d_{\text{Rep}_n}). \quad (8.3.1)$$

The main result of this section is an analogue of the above for the usual cyclic homology:

Theorem 8.3.2. (i) *For every $n \geq 1$, one has canonical maps given by the composition*

$$\Psi^m : \overline{\text{HC}}.(A) \xrightarrow{w_{\bullet, -m} \circ E} \overline{\text{EHC}}.(A, m) \xrightarrow{\text{tr} \circ \text{ev}} {}^{\mathcal{D}}\overline{H}_{\mathfrak{gl}_n}^{\bullet}(\text{Rep}_n, m), \quad m \in \mathbb{Z}.$$

(ii) *For $m = 0$ and every $\ell \geq 0$, the composite map*

$$\Psi'_{\ell} : \ker(i : \overline{\text{DR}}^{\ell} / d \overline{\text{DR}}^{\ell-1} \rightarrow \overline{\Omega}^{\ell-1} / d \overline{\Omega}^{\ell-2}) \xrightarrow{(3.1.5)} \overline{\text{HC}}_{\ell}(A) \xrightarrow{\Psi_{\ell}^0} {}^{\mathcal{D}}\overline{H}_{\mathfrak{gl}_n}^{\bullet}(\text{Rep}_n, 0)$$

is given by the explicit formula

$$\Psi'_{\ell}(f) = \frac{1}{\ell!} \sum_{0 \leq j \leq \lfloor \ell/2 \rfloor} u^{-j} \cdot \text{tr} \circ \text{ev} \circ (\tilde{d}^{-1} i_t)^j(f). \quad (8.3.3)$$

Remark 8.3.4. (1) In the above formula, $\text{tr} \circ \text{ev} \circ (\tilde{d}^{-1} i_t)^j(f) \in \overline{\Omega}_{\text{Rep}_n}^{\ell-2j} \otimes \mathbf{k}^j[\mathfrak{gl}_n]$. Thus, the sum in the RHS of (8.3.3) *a priori* lives in

$$\left(\bigoplus_{1 \leq j \leq \lfloor \ell/2 \rfloor} u^{-j} \cdot \overline{\Omega}_{\text{Rep}_n}^{\ell-2j} \otimes \mathbf{k}^j[\mathfrak{gl}_n] \oplus \overline{\Omega}_{\text{Rep}_n}^{\ell} / d \overline{\Omega}_{\text{Rep}_n}^{\ell-1} \right)^{\mathfrak{gl}_n}.$$

One checks further that this sum is, in fact, annihilated by the differential $i_{\mathfrak{gl}_n} - u d_{\text{Rep}_n}$, and hence gives a well defined class in $\mathcal{D}\overline{H}_{\mathfrak{gl}_n}^\bullet(\text{Rep}_n, 0)$.

(2) Note that formula (8.3.3), unlike the definition of the map Ψ^m in the general case of an arbitrary m , does not involve the Tsygan map E .

Proof of Theorem 8.3.2. Part (i) follows from (8.3.1), by separating individual homogeneous components and using isomorphism (5.5.3). To prove (ii), we use diagram (5.6.4) and Proposition 5.7.1. In more detail, we consider the sequence of maps

$$\begin{aligned} \ker(i : \overline{\text{DR}}^\ell / d\overline{\text{DR}}^{\ell-1} \rightarrow \overline{\Omega}^{\ell-1} / d\overline{\Omega}^{\ell-2}) &\xrightarrow{(3.1.5)} H^\ell(\overline{\Omega} \otimes R, \mathfrak{b} - u\mathfrak{B}) \xrightarrow{(3.1.8)} H^\ell(\overline{\Omega} / \mathfrak{B}\overline{\Omega}, \mathfrak{b}) \\ &\xrightarrow{\cong} H(\overline{\text{DR}}_\ell / d\overline{\text{DR}}_\ell, i_t)^{\ell,0} \xrightarrow{\eta} H^\ell((\overline{\text{DR}}_\ell, i_t - u d)_\ell) \xrightarrow{\text{tr} \circ \text{ev}} \mathcal{D}\overline{H}_{\mathfrak{gl}_n}^\ell(\text{Rep}_n, 0), \end{aligned} \quad (8.3.5)$$

where the map η is given by the formula $\eta = (\text{Id} - u^{-1}\tilde{d}^{-1}i_t)^{-1} = \sum_j u^{-j} \cdot (\tilde{d}^{-1}i_t)^j$.

Observe first that, for $f \in \overline{\Omega}^\ell$, the component of $E(f)$ of degree zero in t equals $\frac{1}{\tilde{d}} \cdot f_{\tilde{d}}$ (one can see this either directly from formula (5.3.1) or from Lemma 5.3.2). Therefore, the restriction of the map E^0 to $H^\ell(\overline{\Omega} / \mathfrak{B}\overline{\Omega}, \mathfrak{b})$ is essentially the map $f \mapsto \frac{1}{\tilde{d}} \cdot f_{\tilde{d}}$. It follows that the composition of the first four isomorphisms in (8.3.5) is the isomorphism from (5.5.8) times $\frac{1}{\tilde{d}}$. We deduce that the composite of all the maps in (8.3.5) equals the map given by formula (8.3.3).

On the other hand, the map η provides, thanks to Proposition 5.7.1, an inverse to the isomorphism $\overline{\text{EHC}}_\ell(A, 0) \xrightarrow{\sim} H(\overline{\text{DR}}_\ell / d\overline{\text{DR}}_\ell, i_t)^{\ell,0}$, in (5.5.3). Therefore, from the commutativity of diagram (5.6.4) we deduce that the composite of the second, third, and fourth maps in (8.3.5) is equal to the map $w_\ell \circ E : H^\ell(\overline{\Omega} \otimes R, \mathfrak{b} - u\mathfrak{B}) \rightarrow \overline{\text{EHC}}_\ell(A, 0)$. We conclude that the composite of the last four maps in (8.3.5) equals the map Ψ_ℓ^0 , completing the proof. \square

One can also restrict the map (8.3.1) to the Karoubi-de Rham cohomology $\overline{\text{HD}}(A) \subseteq \overline{\text{HC}}(A)$ and $\overline{\text{EHD}}(A) \subseteq \overline{\text{EHC}}(A)$. This will land in $\mathcal{D}\overline{H}_{\text{cl}, \mathfrak{gl}_n}(\text{Rep}_n(A))$, the cohomology of the complex

$$\overline{\text{CC}}_{\text{cl}, \mathfrak{gl}_n}(\text{Rep}_n) := (\overline{\Omega}_{\text{cl}, \text{Rep}_n}^{\mathfrak{gl}_n} \oplus (\overline{\Omega}_{\text{Rep}_n} \otimes \mathbf{k}[\mathfrak{gl}_n]_+)^{\mathfrak{gl}_n} \otimes R, i_{\mathfrak{gl}_n} - u d_{\text{Rep}_n}),$$

where $\overline{\Omega}_{\text{cl}, \text{Rep}_n}^{\mathfrak{gl}_n}$ denotes the subspace of closed forms and $\mathbf{k}[\mathfrak{gl}_n]_+ \subset \mathbf{k}[\mathfrak{gl}_n]$ denotes the augmentation ideal. Let $\overline{\text{CC}}_{\text{cl}, \mathfrak{gl}_n}^\wedge(\text{Rep}_n)$ be the corresponding $\mathbf{k}[\mathfrak{gl}_n]_+$ -adic completion.

Corollary 8.3.6. *By restriction, we obtain maps*

$$\overline{\text{EHD}}(A) \rightarrow H(\overline{\text{CC}}_{\text{cl}, \mathfrak{gl}_n}(\text{Rep}_n)), \quad \text{and} \quad \overline{\text{HD}}(A) \rightarrow H(\overline{\text{CC}}_{\text{cl}, \mathfrak{gl}_n}^\wedge(\text{Rep}_n)). \quad \square$$

Remark 8.3.7. There are also ‘periodic’ and ‘negative’ versions of the above constructions, which produce maps

$$\overline{\text{EHC}}^{\text{per}, \wedge}(A) \rightarrow H(\overline{\text{CC}}_{\mathfrak{gl}_n}^{\text{per}, \wedge}(\text{Rep}_n A)) \cong \overline{H}_{\mathfrak{gl}_n}(\text{Rep}_n A)((u)), \quad \overline{\text{EHC}}^{-, \wedge}(A) \rightarrow H(\overline{\text{CC}}_{\mathfrak{gl}_n}^{-, \wedge}(\text{Rep}_n A)).$$

Composing with E , we obtain maps

$$\overline{\text{HC}}^{\text{per}}(A) \rightarrow H(\overline{\text{CC}}_{\mathfrak{gl}_n}^{\text{per}, \wedge}(\text{Rep}_n A))((u)), \quad \overline{\text{HC}}^{-}(A) \rightarrow H(\overline{\text{CC}}_{\mathfrak{gl}_n}^{-, \wedge}(\text{Rep}_n A)).$$

Moreover, we obtain commutative diagrams of commutative squares with exact rows, from (4.4.4) to the version with E in front of all groups (and with t -adic completions), to the commutative square connecting the $\mathbf{k}[\mathfrak{gl}_n]_+$ -adic completed versions of the above equivariant cohomology groups (and versions one can similarly define analogous to $\overline{\text{HH}}'$, $\overline{\text{HH}}$, and $\overline{\text{HD}}'$; note that the equivariant analogue of $\overline{\text{HD}}$, $\mathcal{D}\overline{H}_{\text{cl}, \mathfrak{g}}(\text{Rep}_n)$, was defined above).

8.4. Special cases. *The case of $n = 1$:* If $n = 1$, then GL_1 is a 1-dimensional torus, and $\mathbf{k}[\mathfrak{gl}_1] \cong \mathbf{k}[t]$. Furthermore, since the algebra $\mathrm{Mat}_1(\mathbf{k})$ is commutative, the evaluation map for $\overline{\Omega}_t A$ may be factored as a composition

$$\overline{\Omega}_t A \xrightarrow{f \mapsto f_{\mathfrak{h}}} \overline{\mathrm{DR}}_t \xrightarrow{c_t} \overline{\mathrm{DR}}[t] \xrightarrow{\mathrm{ev} \otimes \mathrm{Id}_{\mathbf{k}[t]}} \overline{\Omega}_{\mathrm{Rep}_1 A}^\bullet[t] \otimes \mathrm{Mat}_1(\mathbf{k}),$$

where c_t is the map that makes t a central variable, cf. Lemma 5.3.2. Given $f \in \overline{\Omega}A$, we will use simplified notation $\bar{f} := \mathrm{tr} \circ \mathrm{ev}(f_{\mathfrak{h}}) \in \overline{\Omega}_{\mathrm{Rep}_1 A}$. Thus, using the formula of Lemma 5.3.2 and the fact that the trace map yields an algebra isomorphism $\mathrm{tr} : \mathrm{Mat}_1(\mathbf{k}) \xrightarrow{\sim} \mathbf{k}$, we deduce

$$\mathrm{tr} \circ \mathrm{ev} \circ \mathbf{E}(f) = \frac{1}{k!} \cdot \bar{f} \exp(t), \quad \forall f \in \overline{\Omega}^k A. \quad (8.4.1)$$

One can use the above formula to obtain an explicit description of the map of Theorem 8.3.2 in the special case that $n = 1$ and $m = 0$. To this end, observe first that the group GL_1 acts trivially on the scheme $\mathrm{Rep}_1 A$. Therefore, we are in the setting of Example 8.1.5. According to the isomorphism given there, in the special case where $\mathfrak{g} = \mathfrak{gl}_1$ and $m = 0$, Theorem 8.3.2 yields a map

$$\Psi_\ell^0 : \overline{\mathrm{HC}}_\ell(A) \rightarrow \mathcal{D}\overline{H}^\ell(\mathrm{Rep}_1, 0) = \overline{\Omega}_{\mathrm{Rep}_1 A}^\ell / d\overline{\Omega}_{\mathrm{Rep}_1 A}^{\ell-1} \oplus \left(\bigoplus_{j \geq 1} \overline{H}^{\ell-2j}(\mathrm{Rep}_1 A) t^j u^{-j} \right) \quad (8.4.2)$$

To compute this map explicitly, we plug formula (8.4.1) for the map \mathbf{E} into (8.3.1) and separate the relevant homogeneous components. Then, one finds that the map (8.4.2) is given, for every $\sum_{0 \leq j \leq \lfloor \ell/2 \rfloor} f_{\ell-2j} u^{-j}$ representing a class in $\overline{\mathrm{HC}}_\ell(A) = H^\ell(\overline{\Omega}A \otimes R, \mathfrak{b} - u\mathfrak{B})$, by the formula

$$\sum_{0 \leq j \leq \lfloor \ell/2 \rfloor} f_{\ell-2j} u^{-j} \mapsto \sum_{0 \leq j \leq \lfloor \ell/2 \rfloor} \frac{1}{(\ell-2j)!(\ell-2j)!} \cdot \bar{f}_{\ell-2j} t^j u^{-j}, \quad f_{\ell-2j} \in \overline{\Omega}^{\ell-2j} A. \quad (8.4.3)$$

Next, let $A_{\mathrm{ab}} := A/([A, A])$ be the *abelianization* of A , a quotient of the algebra A by the two-sided ideal generated by the set $[A, A]$. A well known construction in the theory of cyclic homology of commutative algebras produces a map

$$\overline{\mathrm{HC}}_\ell(A_{\mathrm{ab}}) = H^\ell(\overline{\Omega}A_{\mathrm{ab}} \otimes R, \mathfrak{b} - u\mathfrak{B}) \rightarrow \bigoplus_{j \geq 1} H^{\ell-2j}(\mathrm{Spec} A_{\mathrm{ab}}) \oplus \overline{\Omega}_{\mathrm{Spec} A_{\mathrm{ab}}}^\ell / d\overline{\Omega}_{\mathrm{Spec} A_{\mathrm{ab}}}^{\ell-1}.$$

Specifically, according to [Lo, Proposition 2.3.7], this map is defined by the assignment

$$\Psi_\ell^{\mathrm{classical}} : \sum_{0 \leq j \leq \lfloor \ell/2 \rfloor} f_{\ell-2j} u^{-j} \mapsto \bigoplus_{0 \leq j \leq \lfloor \ell/2 \rfloor} \frac{1}{(\ell-2j)!} \cdot \bar{f}_{\ell-2j}. \quad (8.4.4)$$

Let $\mathrm{ab} : A \rightarrow A_{\mathrm{ab}}$ be the abelianization homomorphism and write $\mathrm{HC}(\mathrm{ab}) : \overline{\mathrm{HC}}(A) \rightarrow \overline{\mathrm{HC}}(A_{\mathrm{ab}})$ for the induced map on cyclic homology. The abelianization map $\mathrm{HC}(\mathrm{ab})$ clearly intertwines the Tsygan maps for the algebras A and A_{ab} , respectively. Note further that Hilbert's Nullstellensatz yields canonical isomorphisms $\mathrm{Rep}_1 A = \mathrm{Rep}_1 A_{\mathrm{ab}} = \mathrm{Spec} A_{\mathrm{ab}}$. With these observations in mind, comparing formulas (8.4.3) and (8.4.4) yields the comparison result

Corollary 8.4.5. *One has $\mathbf{N}! \circ \Psi_\ell^0 = \Psi_\ell^{\mathrm{classical}} \circ \mathrm{HC}(\mathrm{ab})$, where $\mathbf{N}!$ is the map that acts by scalar multiplication by $k!$ in de Rham degree k .*

Restricting the map (8.4.2) to $\overline{\mathrm{HD}}_\ell(A)$ via the embedding S_2 from the Connes exact sequence, cf. Theorem 4.1.1, we obtain a map

$$\Psi_\ell^0 \circ S_2 : \overline{\mathrm{HD}}_\ell(A) \longrightarrow \bigoplus_{j \geq 0} \overline{H}^{\ell-2j}(\mathrm{Rep}_1 A). \quad (8.4.6)$$

Remark 8.4.7. In the case where A is a smooth commutative algebra the map $\Psi_\ell^{\mathrm{classical}}$ is known to be an isomorphism, see [Lo, Theorem 3.4.12]. It follows that, in this case, each of the maps (8.4.2) and (8.4.6) is an isomorphism as well.

Restriction of \mathfrak{gl}_n to scalars: For general $n \geq 1$, the subalgebra of scalar matrices $\mathfrak{gl}_1 \subseteq \mathfrak{gl}_n$ still acts trivially on $\text{Rep}_n A$. Thus, restricting the general construction to \mathfrak{gl}_1 produces a map

$$\overline{\text{HC}}_\ell(A) \rightarrow \overline{\Omega}_{\text{Rep}_n A}^\ell / d \overline{\Omega}_{\text{Rep}_n A}^{\ell-1} \oplus \left(\bigoplus_{j \geq 1} \overline{H}^{\ell-2j}(\text{Rep}_n A) \right). \quad (8.4.8)$$

There is also a comparison result, similar to Corollary 8.4.5, saying that the map (8.4.8) is equal, up to a twist by the automorphism $N!$, to the composition of the evaluation map

$$\overline{\text{HC}}_\ell(A) \rightarrow \overline{\text{HC}}_\ell(\mathbf{k}[\text{Rep}_n A] \otimes \text{Mat}_n(\mathbf{k})) \cong \overline{\text{HC}}_\ell(\mathbf{k}[\text{Rep}_n A])$$

with isomorphism $\Psi^{\text{classical}}$.

If we don't restrict to scalars, the maps $\overline{\text{HC}}(A) \rightarrow \mathcal{D}\overline{H}_{\mathfrak{gl}_n}(\text{Rep}_n A)$ should capture finer information, having to do with the equivariant geometry of the representation scheme.

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