Hyperbolic Braneworld Backgrounds in Supergravity

Benedict Crampton

September 30th, 2013

Submitted in part fulfillment of the requirements for the degree of Doctor of Philosophy in Theoretical Physics of Imperial College London and the Diploma of Imperial College London
Abstract

The manifolds $\mathcal{H}^{p,q}$ are a family of non-compact hyperboloids carrying inhomogeneous Euclidean metrics. In supergravity they appear as an interesting class of dimensional reductions, related to the well known sphere reductions by a simple analytic continuation. The spectrum of lower dimensional modes in these backgrounds is still poorly understood. In this thesis, we construct the complete Pauli reduction of type IIA supergravity on $\mathcal{H}^{2,2}$. We carefully analyse the spectrum of gravitational waves in the resulting Salam-Sezgin background, and identify the boundary conditions needed to render these modes normalisable. We give these boundary conditions a codimension-2 braneworld interpretation. We then exhibit a supersymmetric braneworld geometry based on the NS5-brane. In the remainder of this thesis we apply holographic methods to the problem of the fractionalisation transition in condensed matter theory. We exhibit a phase transition between a superconducting and a fractionalised phase in a bottom-up Einstein-Maxwell-Dilaton theory, and discuss the importance of entropy scaling in achieving this.
Declaration of Originality

I hereby declare that the work contained within this thesis is my own, except where there are explicit references to others works, or to works done as part of a collaboration. In particular, the results comprising chapters 2 and 3 are taken from the work [1], done in collaboration with my supervisor, Kellogg Stelle, and with Chris Pope, and the results of chapter 4 are taken from the work [2], done in collaboration with Alexander Adam, Julian Sonner, and Benjamin Withers. The zero temperature analysis that forms the bulk of chapter 4 was carried out in collaboration with Alexander Adam, while the finite temperature numerical results presented in figures 6, 7, and 8 were supplied by Julian Sonner and Benjamin Withers.

No part of this work has previously been submitted in part fulfillment for any other degree.

Acknowledgments

First and foremost, I would like to thank my supervisor Kellogg Stelle for all the time, advice, and ideas he has given me during my time at Imperial College. I would also like to thank those with whom I have collaborated or had useful discussions, namely Alexander Adam, Carl Bender, Chris Pope, Tom Pugh, Julian Sonner, Toby Wiseman, and Benjamin Withers.

I would also like to thank my friends and colleagues in the Theoretical Physics group for making the past four years so enjoyable. I would especially like to thank Silvia Nagy for her enduring support while I wrote this thesis.

Finally I would like to thank the Science and Technologies Facilities Council for financially supporting me during much of my PhD.

Copyright Declaration

The copyright of this thesis rests with the author and is made available under a Creative Commons Attribution Non-Commercial No Derivatives license. Researchers are free to copy, distribute or transmit the thesis on the condition that they attribute it, that they do not use it for commercial purposes, and that they do not alter, transform or build upon it. For any reuse or redistribution, researchers must make clear to others the license terms of this work.
## Contents

1 Introduction ............................................................................... 7
  1.1 Supersymmetry ................................................................. 11
  1.2 Supergravity ..................................................................... 13
    1.2.1 Maximal Supergravity in $D = 10$ and $11$ .................. 15
    1.2.2 $D = 10$ Supergravity as a Limit of String Theory ........ 17
    1.2.3 Extended Objects ...................................................... 20
  1.3 Extra Dimensions ............................................................. 23
    1.3.1 Kaluza-Klein Theory ............................................... 24
  1.4 Holography ....................................................................... 28
  1.5 Pauli Reductions on Spheres and Hyperboloids ..................... 31
    1.5.1 Group Manifolds ...................................................... 31
    1.5.2 The $S^4$ Reduction ................................................ 33
    1.5.3 Inhomogeneous Hyperboloids .................................... 37

2 The Pauli Reduction on $H^{2,2}$ .................................................. 41
  2.1 The Wigner-Inönü Contraction .......................................... 42
  2.2 Supersymmetry Transformations ........................................ 45
    2.2.1 $H(3)$ as a Field Strength ....................................... 48
  2.3 The $SO(p, 4 - p)$-Gauged Theory ...................................... 51
    2.3.1 Vacua and Stability ................................................. 53
    2.3.2 Orbifolds and Truncations ......................................... 55
  2.4 The Reduction Ansatz ....................................................... 57

3 Kaluza-Klein Modes of the Salam-Sezgin Background ............... 61
  3.1 Harmonic Functions on the Salam-Sezgin Background ............ 61
    3.1.1 Introduction to Hilbert Spaces ................................... 65
    3.1.2 The $H^{2,2}$ Potential ............................................ 68
    3.1.3 Generic $H^{p,q}$ Backgrounds ................................... 70
  3.2 Localisation of Modes ..................................................... 72
    3.2.1 An Effective Newton’s Law ...................................... 75
    3.2.2 Discussion ............................................................ 78
  3.3 Irregularities at the Origin ................................................. 80
3.3.1 Codimension-2 Branes ........................................ 80
3.3.2 Fixing the Self-Adjoint Extension .......................... 82
3.4 A Backreacted 5-Brane Solution ............................... 83
  3.4.1 Killing Spinors of the Salam-Sezgin Background .......... 85
  3.4.2 Kappa Symmetry ........................................... 87
  3.4.3 The Backreacted Geometry ................................ 90
  3.4.4 Kaluza-Klein Modes and Boundary Conditions .......... 95
3.5 Conclusions .................................................... 97

4 A Holographic Treatment of Fractionalisation ............... 99
  4.1 Hyperscaling .................................................... 101
  4.2 A Bottom-Up Model of Bosonic Fractionalisation .......... 102
    4.2.1 UV Geometries ........................................... 105
    4.2.2 Zero Temperature IR Geometries ........................ 106
    4.2.3 Finite Temperature IR Geometries ...................... 111
    4.2.4 Numerical Results .................................... 112
  4.3 Conclusions .................................................... 117
# List of Figures

1. Effective potential for the $H^{2,2}$ harmonics ........................................ 64
2. Numerical results for the self-adjoint extension $\theta$ as a function of the bound state eigenvalue $\lambda$ ................................................................. 70
3. Logarithmic behaviour of the function $I(s, \rho)$ against $\rho$, for various values of $\rho = \rho_c$ ................................................................. 77
4. Ratio of fractionalised to total charge in each of the three $T = 0$ phases as a function of $\Phi_1/\mu$ ......................................................... 114
5. The free energy of each of the free phases as a function of $\Phi_1/\mu$ ........ 115
6. Finite temperature solutions in the broken phase as a function of $\Phi_1/\mu$ 116
7. Finite temperature solutions as a function of the leading order value $\Phi_0$ of the dilaton in the IR, for various values of $\Phi_1/\mu$ .................. 117
8. Density plot of the entropy scaling parameter $\alpha$ as a function of $T/\mu$ and $\Phi_1/\mu$ ................................................................. 118
1 Introduction

The existence of extra dimensions in string theory presents both a challenge and an opportunity to theoretical physicists. Kaluza-Klein theory allows one to identify lower dimensional fields with some truncated set of higher dimensional degrees of freedom \[3, 4\], and thus embed a whole host of different lower dimensional theories (and particularly supergravity theories) in a renormalisable theory of quantum gravity. The challenge is to identify which of these dimensional reductions are physically justifiable and which are not. One common criterion is that making the Kaluza-Klein truncation corresponds to taking some low energy limit of the higher dimensional theory.

When the extra dimensions form a compact manifold, the degrees of freedom we retain are separated from the rest by a mass gap inversely proportional to the radius of curvature. Consequently these truncations are always justified from a low energy effective action perspective. In this case the lowest mass ‘Kaluza-Klein modes’ are smeared evenly across the extra dimensions. When this transverse manifold is non-compact however, this smearing leads to a vanishing mass gap; if we are to justify a dimensional reduction on such a space we must somehow localise the modes in these directions. In the first of our two topics we will study dimensional reductions on non-compact hyperboloids, and see whether or not we can localise the Kaluza-Klein modes. These will actually be examples of Pauli reductions: a special class of reduction in which one retains the full set of lowest modes.

Dimensional reductions on positively curved manifolds typically lead to lower dimensional actions with anti-de Sitter (AdS) solutions, \emph{i.e.} maximally symmetric, negatively curved, Lorentzian geometries. An astonishing property of string theory is that it is dual to strongly coupled conformal field theories \[5–7\]. Solutions which are only asymptotically AdS correspond to field theories described by a UV fixed point but with more interesting low energy behaviour, such as superconductivity. In the second part of this thesis we apply this AdS/CFT correspondence to the problem of fractionalisation in condensed matter physics. It is worth noting that this ‘holographic’ property of string theory is independent of its status as a ‘theory of everything’. Even if experimental results cause string theory to become disfavoured, it may very well live on as an alternative approach to field theory problems.

Hyperbolic Braneworld Backgrounds

In appropriately warped geometries, Kaluza-Klein modes can be localised in a non-
compact direction [8], resulting in a lower dimensional effective theory. The earliest such examples were constructed over a decade ago [9, 10], and since then there has been a succession of increasingly sophisticated and phenomenologically acceptable toy models, \textit{e.g.} [11–20].

The building blocks of these so-called braneworld scenarios are familiar and unremarkable to string theorists: standard Einstein-Hilbert gravity; extended objects as mild spacetime defects; and minimally coupled bulk scalars or p-form fluxes. Consequently one might presume that it would be a comparatively straightforward exercise to embed a braneworld model in the full string theory, but this has not proven to be the case. Even the simple $\text{AdS}_5/\mathbb{Z}_2$ geometry of Randall and Sundrum [9] uplifts to a pathological type IIB supergravity solution [21]. More promising, recent work [22] has focused on the type IIB Janus solutions as a possible realisation of the Karch-Randall model [10], but has yet to yield a positive result.

In a linearised solution, much of the Kaluza-Klein structure is encoded in the Laplace-Beltrami operator. Indeed perhaps the most mathematically appealing feature of braneworld physics is the universality of the $D = 4$ spin-2 modes: subject to the (usually desirable) constraint of lower dimensional Poincaré, de Sitter, or anti-de Sitter symmetry, the spin-2 spectrum is determined entirely in terms of the admissible harmonic functions on the full geometry [22,23]. Thus, given a metric, we can temporarily dispense with the messy details of the supporting stress-energy, and recast the problem of the gravitational waves as an abstract functional analysis problem. We then have available to us all the powerful machinery of that subject. We can go further and map the eigenfunction equation onto an equivalent Schrödinger problem [8] - \textit{i.e.} the one-dimensional problem of a quantum mechanical particle in some potential - to give a simple and intuitive picture of our system.

A viable approach then, in the quest to embed localised gravity in a UV-complete theory, is to take a solution or family of solutions of string theory with a non-compact extra dimension, systematically convert the metrics to quantum mechanical potentials, and thus read off the harmonic spectrum. In practice we can start with a single, representative solution of a family, with the assumption that the qualitative features of the spin-2 modes will be typical. In this thesis we study the spectrum of the Laplacian on one of the most interesting known families of type IIA string theory/supergravity solutions - the Pauli reduction on the inhomogeneous hyperboloid $\mathcal{H}^{2,2}$ [24,25]. The geometries $\mathcal{H}^{p,q}$ are algebraically the simplest family of hyperboloids we can construct; they are defined as hypersurfaces in $\mathbb{R}^{p+q}$ with metric the induced Euclidean metric.
The Pauli reductions on $\mathcal{H}^{p,q}$ are obtained from the sphere reductions $S^{p+q-1}$ via simple analytic continuations [29]. Normally with dimensional reductions we think of the lower dimensional theory as being a low energy effective action, i.e. we can use the resulting action to study dynamical issues. In particular we can perturb a solution and calculate how the perturbation evolves. This interpretation is justified by a mass gap in the Kaluża-Klein spectrum. The analytic continuation method bypasses all discussion of the higher Kaluża-Klein modes, so the validity of this interpretation in the $\mathcal{H}^{p,q}$ case has remained an open question for some time. In this thesis we demonstrate that the boundary conditions implicit in the $\mathcal{H}^{p,q}$ Pauli reductions are incompatible with even the very idea of a small dynamical perturbation. This is reflected in the vanishing of the lower dimensional Newton’s constant [25]. We find that there exist alternative boundary conditions that lead to a more sensible lower dimensional theory.

We shall see that the Pauli reduction on $\mathcal{H}^{2,2}$ is a consistent truncation of type IIA supergravity. Despite its problematic dynamics then, the lower dimensional action remains a useful tool in generating interesting and exact non-compact backgrounds in $D=10$, about which we can consider our alternative dynamics. The $D=7$ action inherits an $SO(2,2)$ gauge symmetry from the global symmetries of $\mathcal{H}^{2,2}$. Thanks to the inhomogeneity\footnote{Here and throughout this thesis, an inhomogeneous manifold is one that is inhomogeneous with respect to the isometry group of its metric.} of the hyperboloid, this is realised non-linearly and thus does not suffer the ghosts usually associated with non-compact gaugings [29,30]. Two important $D=6$ constructions have been embedded as domain wall solutions of the $D=7$ theory, and hence embedded in string theory: the oft-studied Salam-Sezgin model [31]; and models with gauged R-symmetry [32]. Crucially both these constructions rely on the non-compactness of $\mathcal{H}^{2,2}$: the positive cosmological constant of the Salam-Sezgin model would otherwise be forbidden by a powerful no-go theorem [33]; the gauged R-symmetry can only be obtained via a non-compact gauging [32].

The preponderance of papers on the Salam-Sezgin model justifies our singling it out for special consideration here. It has been a popular basis both for studies of the cosmological consequences of extra dimensions and, coincidentally, for toy models of braneworlds [34–36]. It is also notable for reducing to a chiral theory in $D=4$, though this is a surprisingly subtle point [37]. We shall take as our background of choice the non-compact uplift to ten dimensions of the $\mathbb{R}^{1,3} \times S^2$ Salam-Sezgin vacuum. Of course, with our alternative boundary conditions in the extra dimensions, the dynamical theory
on this background is not *a priori* the Salam-Sezgin model.

The outline of this part of the thesis is as follows. In the remainder of this chapter we shall introduce the necessary background material, summarising the importance of extra dimensions in supergravity and the principles of Kaluża-Klein theory. In particular we emphasise the physical distinction between consistent truncations and reductions with a Kaluža-Klein mass gap. We then describe the sphere reductions and their analytic continuation to the $\mathcal{H}^{p,q}$ reductions. In chapter 2 we derive the Pauli reduction on $\mathcal{H}^{2,2}$ from the $S^4$ reduction via a Wigner-Inönü contraction, *i.e.* by a rescaling of the generators of the gauge symmetry. We extend the works [24, 25] to the maximally supersymmetric case, and expand upon the issue of self-duality in odd dimensions. We also discuss possible orbifolds of the geometry. Finally in chapter 3 we examine the Kaluža-Klein spectrum. We work at the linearised level, solving the Laplace-Beltrami operator on the Salam-Sezgin background using concepts from functional analysis, and interpret the resulting boundary conditions as a conical singularity at the $\mathcal{H}^{2,2}$ origin. Finally we exploit kappa symmetry techniques to identify a braneworld geometry based on an NS5-brane, and discuss how one might solve for the backreaction.

**Fractionalisation**

Fractionalisation is a bizarre but experimentally verified phenomenon [38] in certain condensed matter systems in which a system of ‘electrons’ undergoes a phase transition at low energy, after which it behaves as a system of two separate particles - one carrying spin and the other charge. It is very different from the usual behaviour of condensed matter systems, in that the low energy quasiparticles do not appear to be made up of the elementary particles of the parent theory: in fact, the reverse is true. This splitting phase transition is believed to be described by a quantum critical point - that is, a phase transition at zero temperature driven by quantum fluctuations. Various field theory models have been proposed [39–42], but these are strongly coupled and the usual loop expansion is not under control. A fractionalised phase is one example of a non-Fermi liquid, *i.e.* a fermionic system not modeled at low energies by Fermi liquid theory.

In the past few years the power of the $AdS/CFT$ correspondence has been brought to bare on a variety of strongly coupled condensed matter systems, including superconductors [43–45] and electron stars [46]. Not surprisingly there has been recent work applying the duality to the problem of fractionalisation, and particularly to the phase
transition between fractionalised and non-fractionalised phases [47,48]. In the dual gravitational picture one is interested in a phase transition between a charged black hole background and a background with charged bulk matter. Previous authors have studied gravitational models with Lifschitz scaling solutions, and found that the fractionalised phase is unstable [49–52]. The inclusion of a divergent scalar field results in a more general hyperscaling symmetry [53–56], allowing one to avoid this instability and achieve fractionalisation.

In section 1.4 we shall give a brief presentation of the AdS/CFT correspondence, stressing the equivalence between the radial coordinate in the bulk and the energy cutoff in the field theory. We very briefly sketch the AdS/CMT dictionary, with black holes representing temperature and abelian gauge fields a finite density of particles. The main part of our work can be found in chapter 4, where we consider a class of gravitational models with both fractionalised and superconducting solutions. We demonstrate the existence of a fractionalisation transition numerically, and discuss the rôle of entropy scaling and the third law of thermodynamics.

1.1 Supersymmetry

Locally, $D$-dimensional spacetime is described by the Minkowski metric

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu,$$  \hspace{1cm} (1.1)

which is invariant under translations and Lorentz transformations. Together these symmetries form the Poincaré group $\mathbb{R}^{1,D-1} \rtimes O(1,D-1)$. This is a Lie group, and has a corresponding Lie algebra

$$[P_\mu, P_\nu] = 0,$$

$$[P_\mu, M_{\nu\rho}] = i (\eta_{\mu\rho} P_\nu - \eta_{\mu\nu} P_\rho),$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i (\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}).$$  \hspace{1cm} (1.2)

Here the $P_\mu$ generate the $D$ translations and the $M_{\mu\nu} = -M_{\nu\mu}$ the $D(D-1)/2$ Lorentz transformations, which together make up the full $D(D+1)/2$-dimensional algebra.

This local description is sufficient for many applications of quantum field theory (QFT), e.g. calculating scattering amplitudes, and indeed QFT is best understood in Minkowski space. In addition to the Poincaré symmetry, QFTs commonly have some
gauge symmetry under a Lie group $G$. The generators $\mathfrak{g}$ of these internal symmetries commute with the Poincaré algebra:

$$[\mathfrak{g}, P_\mu] = 0, \quad [\mathfrak{g}, M_{\mu\nu}] = 0.$$  \hspace{1cm} (1.3)

As part of a general program of unification, various authors have proposed QFTs with large gauge groups $G$, containing the $SU(3) \times SU(2) \times U(1)$ gauge group of the standard model (SM) as a subgroup, and realizing it through spontaneous symmetry breaking. One can ask if there is an alternative way to increase the symmetry of a QFT: is there an algebra that mixes Poincaré and internal symmetries? The Coleman-Mandula theorem [57,58] places severe limitations on such a structure:

**Theorem 1** Consider a QFT with a continuous symmetry group $G$ with subgroup the Poincaré group. Then either $G$ is locally isomorphic to a direct product of the Poincaré group and an internal symmetry group, or the $S$ matrix is trivial.

This powerful no-go theorem was originally formulated in $D = 4$; an extension to higher dimensions has since been developed [59]. Fortunately this obstruction can be sidestepped [60–62] by allowing $G$ to be a supergroup, rather than a group: that is, a set of symmetries which infinitesimally describe a Lie superalgebra. Analogously to the Coleman-Mandula theorem, the most general structure of this superalgebra has been characterised via a study of the $S$ matrix [63].

In $D$-dimensions, supersymmetry adds fermionic generators $Q^a_\alpha$ called supercharges (with $a = 1, \ldots, \mathcal{N}$) to the bosonic generators of the Poincaré algebra (1.2). Schematically the superalgebra has the form [64]

$$\{Q^a_\alpha, Q^b_\beta\} \sim \delta^{ab} (\Gamma^\mu \Sigma)_{\alpha\beta} P_\mu,$$

$$[Q^a_\alpha, P_\mu] = 0,$$

$$[Q^a_\alpha, M_{\mu\nu}] \sim (\Gamma_{\mu\nu} Q)^i_\alpha.$$

where the gamma matrices $\Gamma_\mu$ satisfy the Clifford algebra

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}.$$  \hspace{1cm} (1.5)

---

2The conformal algebra extends the Poincaré algebra in this way. All general statements in this section will implicitly exclude QFTs with conformal invariance; equivalently, we will always assume that the spectrum has a mass gap.

3The Coleman-Mandula theorem makes certain technical - but desirable - assumptions concerning the spectrum of the QFT. A more precise statement of the theorem can be found in [58,59].
the $\Gamma_{\mu \nu}$ are antisymmetrised products of the gamma matrices, and $C$ is a charge conjugation matrix satisfying\(^4\)

$$C \Gamma^\mu C^{-1} = \pm (\Gamma^\mu)^T.$$  \(1.6\)

Additionally antisymmetric tensors $Z_{\mu \cdots \nu}$ may appear on the right hand side of the $\{Q, Q\}$ anticommutator, contracted with factors of $\Gamma^{\mu \nu} C$. These are central charges, which by definition commute with all other generators and themselves.

We build representations of the superalgebra in the usual way, defining some highest weight state (in this case highest spin) and acting on it with all possible combinations of the operators $Q$ to generate a multiplet. Because the $Q$ operators anticommute, this process must terminate. We label these multiplets by their highest spin field, e.g. vector multiplets have highest spin 1.

1.2 Supergravity

Mathematically, gravity is formulated in terms of a smooth, real, $D$-dimensional manifold $M$, called spacetime. For a review, see [65]. By definition a manifold comes equipped with a tangent bundle $TM$ - whose elements are called vector fields - which we can equip with some connection $\Gamma$. There exists a dual bundle $T^*M \cong TM$, whose elements are called covector fields. We define a $(p, q)$ tensor to be a linear map

$$T(p, q) : T^*M \times \cdots \times T^*M \times TM \times \cdots \times TM \rightarrow \mathbb{R},$$  \(1.7\)

where the direct product includes $p$ copies of $T^*M$ and $q$ copies of $TM$. An antisymmetric $(0, q)$ tensor is called a $q$-form. The gravitational degrees of freedom are carried by a symmetric $(0, 2)$ tensor called the metric. Taking coordinates $x^M$ on $M$, we can introduce a basis $\{\partial/\partial x^M\}$ for $TM$ and a dual basis $\{dx^M\}$ for $T^*M$. Using the latter, we can write the metric in terms of a line element

$$ds^2 = g_{MN} dx^M dx^N.$$  \(1.8\)

We require that the matrix $g_{MN}$ be invertible and that it have Lorentzian signature $(-, +, \ldots, +)$. One can then use $g_{MN}$ and its inverse to raise or lower covector or vector

\(^4\)Whether we take the plus sign or the minus sign in our definition of $C$ depends on the dimension $D$. Both are valid possibilities when $D$ is even.
indices, providing a preferred map between $TM$ and its dual.

With these definitions we can succinctly state the theory of general relativity. The trajectories (worldlines) of test masses in spacetime travel are geodesics with respect to the line element (1.8). The dynamics of the metric are in turn determined by the Einstein-Hilbert action

$$S = \frac{1}{G_N^{(D)}} \int d^D x \, R \star 1 ,$$

(1.10)

plus some covariantly coupled matter action. Here $G_N^{(D)}$ is Newton’s constant and the Ricci scalar $R$ is the trace of the Ricci tensor $R_{MN}$, which is itself a function of the connection $\Gamma$ and its derivative. Finally we require that the covariant derivative of the metric vanishes, which (in the absence of torsion) fixes $\Gamma$ to be the so-called Levi-Civita connection.

Predictably, the extension of gravity to the supersymmetric case is called supergravity. For a review, see [66]. The metric degrees of freedom define a supergravity multiplet, with highest spin 2. The supersymmetric partner of the metric is a spin-3/2 field $\Psi$, called the gravitino. The inclusion of fermions in the theory necessitates the introduction of vielbeins $e^a_M$, defined via

$$g_{MN} = e^a_M e^b_M \eta_{ab} ,$$

(1.11)

where $\eta_{ab}$ is the Minkowski metric (1.1). We also introduce a spin connection $\omega_{(1)}^{ab} = -\omega_{(1)}^{ba}$. This obeys the Cartan structure equation

$$de^a_{(1)} + \omega_{(1)}^{a}_{\ b} \wedge e^b_{(1)} = T_{(2)} ,$$

(1.12)

where $T_{(2)}$ is some 2-form called the torsion. In the absence of fermions we have $T_{(2)} = 0$, and the structure equation can be taken as the definition of the spin connection in terms of the vielbeins. We then write $\omega_{(1)} = \omega_{(1)}(e)$.

In the case of non-zero torsion$^6$ there are various different treatments of the spin connection: we work in the 1.5-order formalism, in which $\omega_{(1)}$ is fixed in terms of the

$^4$Here we introduce the Hodge star $\star$, which acts on $p$-forms $A_{(p)}$ as

$$(*A_{(p)})_{M_1 \ldots M_{D-p}} = \frac{1}{p!} \sqrt{-g} \varepsilon^{M_1 \ldots M_{D-p} N_1 \ldots N_p} A_{N_1 \ldots N_p} .$$

(1.9)

$^6$The Levi-Civita connection is torsionless, and so is no longer the appropriate choice for $\Gamma$ in the presence of a non-zero $T_{(2)}$. 

14
metric and fermions but we can ignore it when varying these fields. It is then given by the usual torsionless term plus an explicit fermionic contribution. For example, in the $D = 11$ supergravity action (1.15) we have

$$\omega_{Mab} \equiv \omega_{Mab}(e) - \frac{1}{4} \left( \Psi_M \Gamma_b \Psi_a + \Psi_b \Gamma_a \Psi_M - \Psi_a \Gamma_M \Psi_b \right).$$

(1.13)

The spin connection allows us to covariantly couple fermionic kinetic terms to the Einstein-Hilbert action (1.10). Given an action $S$ describing fermions $\epsilon$ in Minkowski space, we simply replace partial derivatives $\partial_M \epsilon$ with the spin-covariant derivative

$$D_M \epsilon = \partial_M \epsilon + \frac{1}{4} \omega^{ab}_{M} \Gamma_{ab} \epsilon.$$  

(1.14)

### 1.2.1 Maximal Supergravity in $D = 10$ and 11

In dimensions $D > 11$, the minimal spinor representation has more than 32 components [67]. It follows that massless representations of the supersymmetry algebra contain fields with spins greater than 2, which we reject on physical grounds. Supergravities thus exist in up to 11 dimensions. In $D = 11$ we can have $\mathcal{N} = 1$ supergravity, and the supergravity multiplet is the only possible multiplet. The action was derived in [68]; in our conventions it is

$$S = \frac{1}{\mathcal{G}_{11}^{(11)}} \int d^{11}x \left\{ R \ast 1 - \frac{1}{2} \ast F_{(4)} \wedge F_{(4)} + \frac{1}{6} F_{(4)} \wedge F_{(4)} \wedge A_{(3)} ight.$$  

$$- \bar{\Psi}_M \Gamma^{MNP} D_N \left( \frac{\omega + \tilde{\omega}}{2} \right) \Psi_P \ast 1$$  

$$+ \frac{1}{192} \left( \bar{\Psi}_M \Gamma^{MNPQRS} \Psi_N + 12 \bar{\Psi}^P \Gamma^{QR} \Psi^S \right) \left( F_{PQRS} + \mathcal{F}_{PQRS} \right) \ast 1 \right\},$$

(1.15)

where

$$F_{(4)} = dA_{(3)}. \quad (1.16)$$

Here $\mathcal{F}_{(4)}$ and $\tilde{\omega}$ are supercovariant fields, meaning their supersymmetry transformations contain no derivatives of the supersymmetry parameter $\epsilon$. They are given by

$$\mathcal{F}_{MNPQ} \equiv F_{MNPQ} - 3 \bar{\Psi}_M \Gamma_{NP} \Psi_Q,$$

$$\tilde{\omega}_{Mab} \equiv \omega_{Mab} - \frac{1}{4} \Psi_N \Gamma_{Mab}^{NP} \Psi_P,$$

(1.17)

(1.18)
and so implicitly introduce gravitino self-interaction terms into the action.

The $D = 11$ supersymmetry algebra admits central charges $Z_{MN}$ and $Z_{MNPQR}$ [69], which carry the charges of 1/2-BPS solutions called $M2$-branes and $M5$-branes respectively. Brane charges are discussed in more detail in section 1.2.3.

In $D = 10$, the minimal spinor representation is both Majorana and Weyl, and thus has 16 components: one may have either $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supergravity. Uniquely\(^7\) for dimensions $D > 2$, there are actually two distinct maximal supergravity multiplets in $D = 10$ - one non-chiral and one chiral [67]; these are called type IIA and type IIB respectively. Both these theories are free from anomalies. The bosonic sector of the type IIA action is [72]

$$S = \frac{1}{G_N^{(10)}} \int d^{10}x \left\{ R * 1 - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{\frac{3\phi}{2}} * F_{(2)} \wedge F_{(2)} - \frac{1}{2} e^{-\phi} * H_{(3)} \wedge H_{(3)} - \frac{1}{2} e^{\frac{3\phi}{2}} * F_{(4)} \wedge F_{(4)} + \frac{1}{2} dA_{(3)} \wedge dA_{(3)} \wedge B_{(2)} \right\}, \quad (1.19)$$

where

$$\mathcal{F}_{(2)} = dA_{(1)}, \quad H_{(3)} = dB_{(2)}, \quad F_{(4)} = dA_{(3)} - H_{(3)} \wedge A_{(1)}. \quad (1.20)$$

The scalar $\phi$ is called the dilaton. The equations of motion are

$$R_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + \frac{1}{2} e^{\frac{3\phi}{2}} \left( \mathcal{F}_{MP} \mathcal{F}_N^P - \frac{1}{16} g_{MN} \mathcal{F}_{PQ} \mathcal{F}^{PQ} \right)$$

$$+ \frac{1}{2} \cdot 2! e^{-\phi} \left( H_{MPQ} H_N^{PQ} - \frac{1}{12} g_{MN} H_{PQR} H^{PQR} \right)$$

$$+ \frac{1}{2} \cdot 3! e^{\frac{3\phi}{2}} \left( F_{MPQR} F_N^{PQR} - \frac{3}{32} g_{MN} F_{PQRS} F^{PQRS} \right), \quad (1.21)$$

$$d * d\phi = - \frac{3}{4} e^{\frac{3\phi}{2}} * \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} + \frac{1}{2} e^{-\phi} * H_{(3)} \wedge H_{(3)}$$

$$- \frac{1}{4} e^{\frac{3\phi}{2}} * F_{(4)} \wedge F_{(4)}; \quad (1.22)$$

$$d \left( e^{\frac{3\phi}{2}} * \mathcal{F}_{(2)} \right) = - e^{\frac{3\phi}{2}} * F_{(4)} \wedge H_{(3)}; \quad (1.23)$$

$$d \left( e^{-\phi} * H_{(3)} \right) = \frac{1}{2} F_{(4)} \wedge F_{(4)} - e^{\frac{3\phi}{2}} * F_{(4)} \wedge \mathcal{F}_{(2)}; \quad (1.24)$$

$$d \left( e^{\frac{3\phi}{2}} * F_{(4)} \right) = F_{(4)} \wedge H_{(3)}. \quad (1.25)$$

\(^7\)We can have more than one maximal supergravity multiplet in other dimensions if we are willing to represent the graviton as something other than a metric $g_{MN}$ [70, 71].
The fermionic sector consists of a gravitino $\Psi_M$ and a spin-1/2 field $\lambda$ called the gaugino. We will not list the accompanying supersymmetry transformations here; we will present them in section 3.4.1 after switching to the string frame (defined in section 1.2.2), where they take a more compact form. The Einstein frame variations may be found in [72].

The $\mathcal{N} = 1$ supergravity multiplet can be obtained via a consistent truncation\(^8\) of either the type IIA or type IIB $\mathcal{N} = 2$ multiplet. Starting from the type IIA action, one sets

$$F_{(2)} = 0, \quad F_{(4)} = 0, \quad \quad (1.26)$$

and imposes the projection conditions

$$\left(1 - \Gamma^{11}\right) \Psi_m = 0, \quad \left(1 + \Gamma^{11}\right) \lambda = 0 \quad (1.27)$$

where

$$\Gamma^{11} = i\Gamma^1 \cdots \Gamma^{10}. \quad (1.28)$$

Additionally there exists an $\mathcal{N} = 1$ vector multiplet, which may be coupled to the supergravity multiplet. The $\mathcal{N} = 1$ supergravities are anomalous unless there are 496 vector multiples, coupled so that the gauge group is $SO(32)$ or $E_8 \times E_8$ [73]. Until recently it was believed that gauge groups $E_8 \times U(1)^{248}$ and $U(1)^{496}$ were also possibilities, but this is now disputed [74].

\(1.2.2 \quad D = 10\) Supergravity as a Limit of String Theory

As one goes below $D = 10$, the space of anomaly-free supergravities grows rapidly. Without some theoretical or phenomenological guidance, there is little reason to favour any of these theories over another. The uniqueness of the $D = 11$ action was an attractive trait to researchers interested in describing the observed matter content of the universe via dimensional reduction (see section 1.3), but this has proven difficult. For a time it was believed that $\mathcal{N} = 8$ supergravity in $D = 4$ could be a renormalisable theory\(^9\) of quantum gravity; it is now generally held that this is not the case [77], and in the perturbative approach the theory is expected to display divergences at 7-loops [78,79].

In modern times, string theory presents a clear and compelling motivation to investi-

\(^8\)A consistent truncation is a truncation consistent with the equations of motion, i.e. one where fields set to zero cannot be sourced by the remaining degrees of freedom.

\(^9\)Incidentally this theory can be obtained from $D = 11$ supergravity via a Kaluza-Klein reduction on the torus $T^7$ [75,76].
gate supergravity in $D=10$ and $D=11$. String theory is perhaps the most successful example of a renormalisable theory of quantum gravity [80, 81] (notable alternatives include loop quantum gravity [82] and asymptotic safety [83]). At its heart is the string action, describing $D$ scalar fields $X^M(\sigma)$ and $D$ fermions $\psi(\sigma)$ propagating on a 2-dimensional manifold, called the worldsheet. We will just give the bosonic part of the action:

$$S_{str} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma} \left( g_{MN}(X) \gamma^{ij} \partial_i X^M \partial_j X^N + B_{MN}(X) \epsilon^{ij} \partial_i X^M \partial_j X^N \right. + \left. \alpha' \phi(X) R_2 \right),$$

(1.29)

Here $\gamma_{ij}$ is the worldsheet metric and $R_2$ the corresponding Ricci scalar. The constant $\alpha'$ has units of length squared. The string action can be interpreted as a non-linear sigma model: it describes a string moving in a $D$-dimensional target spacetime, with coordinates $X^M$, metric $g_{MN}$, and a background 2-form $B_{MN}$ and dilaton $\phi$. Classically this action has a Weyl symmetry $^{10}$

$$\gamma_{ij} \longrightarrow \Omega^2(\sigma) \gamma_{ij},$$

(1.30)

with $\Omega(\sigma)$ an arbitrary, smooth, non-vanishing function. At the quantum level, this symmetry is anomalous - and the theory inconsistent - unless the $\beta$-functions for the couplings vanish. The contributions to the $\beta$-functions arising from the bosonic action (1.29) are [84]

$$\beta(g_{MN}) = \alpha' R_{MN} + 2\alpha' \nabla_M \nabla_N \phi - \frac{\alpha'}{4} H_{MPQ} H^P_N + O(\alpha'^2),$$

$$\beta(B_{MN}) = - \frac{\alpha'}{2} \nabla^P H_{MNP} + \alpha' H_{MNP} \nabla^P \phi + O(\alpha'^2),$$

$$\beta(\phi) = \frac{D-10}{6} - \frac{\alpha'}{2} \nabla^2 \phi + \alpha' \nabla_M \phi \nabla^M \phi - \frac{\alpha'}{24} H_{MNP} H^M_{NP} + O(\alpha'^2),$$

(1.31)

with $H_{(3)} = dB_{(2)}$. The leading order term in the dilaton running fixes fixes $D=10$ - the so-called critical dimension of string theory. Working to order $O(\alpha')$, after a

$^{10}$Naively the dilaton coupling appears to break this symmetry. In fact this term is best thought of as a 1-loop contribution to the action, which is cancelled by Weyl anomalies arising from the other terms [84]. In the classical action, this manifests itself as a mismatch in the powers of the loop parameter $\alpha'$ between the dilaton and the other couplings.
rescaling\textsuperscript{11}
\[ g_{MN} \rightarrow e^{-\frac{\phi}{2}} g_{MN}, \quad (1.32) \]

one easily recognises these anomaly conditions as a consistent truncation of the equations (1.21-1.24). Specifically they are the equations of motion of the $\mathcal{N} = 1$ supergravity multiplet in $D = 10$. There are five distinct choices for the omitted fermionic terms in equation (1.29). These lead, via the beta functions, to the full type IIA or type IIB supergravities, or either the $SO(32)$ or $E_8 \times E_8$ gauged $\mathcal{N} = 1$ supergravities\textsuperscript{12}.

Thus the consistent $D = 10$ supergravities arise as a limit of string theory in which all length scales, \textit{i.e.} the characteristic radii of curvature of the various bundles (tangent bundle, \textit{etc.}), are large compared to the string length $\sqrt{\alpha'}$. Of course, the string action (1.29) must be quantised. In the path integral, one sums over worldsheet topologies [85]:

\[
Z \equiv \frac{1}{\text{Vol}} \int \mathcal{D} \gamma \mathcal{D} X \mathcal{D} \psi e^{-S_{str}},
\]

\[
= \frac{1}{\text{Vol}} \int DX \sum_{\text{genus}} e^{-2\phi \chi} \int \mathcal{D} \gamma \mathcal{D} \psi \exp \left[ - \frac{1}{4\pi \alpha'} \int d^2 \sigma \sqrt{-\gamma} (g_{MN}(X)\gamma^{ij} \partial_i X^M \partial_j X^N + iB_{MN}(X)\epsilon^{ij} \partial_i X^M \partial_j X^N) \right],
\quad (1.33)
\]

with $\chi$ the Euler number of the worldsheet, and so the appropriate loop parameter is $e^\phi$. Since our supergravity action is classical, we must also require that $e^\phi \ll 1$ everywhere.

Transformations of the worldsheet fields have been discovered which exchange perturbative and non-perturbative states [86–89]. These are known as dualities. When translated into transformations on the effective supergravity description, these dualities map the type IIA theory onto type IIB, and vice versa. Similarly they swap the $\mathcal{N} = 1$ supergravities amongst themselves. Coupled with the observation [90–92] that the $D = 10$ theories can all be obtained via dimensional reduction from the unique $D = 11$ (1.15), this is taken as evidence for some underlying 11-dimensional parent theory of quantum gravity, commonly referred to as M-theory.

\textsuperscript{11}The metric scaling here is often called the string frame, and the scaling in equation (1.19) the Einstein frame.

\textsuperscript{12}Two string theories, called type I and $SO(32)$-gauged heterotic, both lead to the same $SO(32)$ gauged supergravity via this procedure.
1.2.3 Extended Objects

The string action (1.29) can be coupled to other extended objects, called branes. Though the development of the brane actions came relatively recently, they are still considered a fundamental part of the theory, with worldsheet boundaries necessitating their inclusion [93,94]. In the appropriate limits, they too must have a supergravity description: we shall discuss this shortly. Firstly, though, we shall briefly sketch the theory of their origin.

For simplicity we will just consider a string propagating in a flat target space, i.e. we will set
\[ g_{MN} = \eta_{MN}, \quad B_{MN} = 0, \quad \phi = \text{constant}. \]  
(1.34)

Though this will be good enough for our purposes, this is not quite right: a brane will inevitably backreact on the target space; it will act as a source of stress-energy for the metric. It will also source either \( B_{(2)} \) or one of the other p-form fluxes in \( D = 10 \), which appear as couplings of the fermionic terms omitted in (1.29). We will return to this momentarily.

Choose worldsheet coordinates \( \sigma^i = (\sigma, \tau) \), with\(^{13} \) \( \sigma \in (0, 2\pi) \) and \( \tau \in (-\infty, \infty) \) spacelike and timelike directions respectively. At the boundaries of the chart, one often imposes periodic boundary conditions\(^{14} \) \( X_M(0, \tau) = X_M(2\pi, \tau) \); we say the string is a closed string. In this case the boundary is a fictitious one. Varying the scalar field in (1.29), we obtain an equation of motion

\[ \frac{1}{\sqrt{-\gamma}} \partial_i \left( \eta_{MN} \gamma^{ij} \partial_j X^N \right) = 0. \]  
(1.35)

If we now relax our periodic boundary conditions, this same variation will pick up certain boundary terms, which must separately vanish:

\[ \left[ \int d\tau \sqrt{-\gamma} \eta_{MN} \gamma^{ij} \partial_j X^N \delta X^M \right]_{\sigma=0}^{2\pi} = 0, \]  
(1.36)

\[ i.e. \] we must impose some alternative boundary conditions. Taking \( \sigma_* \) to be either 0

\(^{13}\)In this section we are assuming that the worldsheet has the topology of either a strip \( I \times \mathbb{R} \) or a cylinder \( S^1 \times \mathbb{R} \). More general worldsheets may be constructed by sewing together such segments: we think of the seams as string-string interactions. Even more generally one may include punctures [80].

\(^{14}\)We impose similar boundary conditions for the fermions \( \psi(\sigma, \tau) \), with the additional option of a twisting \( \psi(0, \tau) = \pm \psi(2\pi, \tau) \).
or $2\pi$, we have the usual choices of Dirichlet:

$$\delta X^M|_{\sigma=\sigma_*} = 0, \quad (1.37)$$

or Neumann boundary conditions:

$$\partial^\nu X^M|_{\sigma=\sigma_*} = 0. \quad (1.38)$$

A worldsheet boundary with $p + 1$ Neumann\textsuperscript{15} and $9 - p$ Dirichlet boundary conditions is constrained to lie on a $(p + 1)$-dimensional surfaces of the $D = 10$ target space. We call these strings open strings, and these surfaces $Dp$-branes.

These boundary conditions can be enforced by adding Lagrange multipliers to (1.29). The effective $Dp$-brane action can be then derived from the beta functions, just as in section 1.2.2. On the brane we choose coordinates $\xi^r$, with $r = 1, \ldots, p+1$, and consider a target space embedding $Y^M(\xi)$. Then, in the string frame, the leading order part of the action is $[94,97]$

$$S_{Dp} = - \int d^{p+1}\xi \left\{ e^{-\phi} \sqrt{-\det [g_{rs} + \mathcal{F}_{rs}]} + e^{\mathcal{F}_{(2)}} \wedge \bigoplus_n A_{(n)} \right\}, \quad (1.39)$$

where

$$\mathcal{F}_{rs} = 2\pi \alpha' F_{rs} + B_{rs}. \quad (1.40)$$

Here $F_{(2)} = dA_{(1)}$ is a worldvolume abelian field, and $g_{rs}$ and $B_{rs}$ are the pullbacks of the bulk metric and 2-form gauge potentials:

$$g_{rs} = \partial_r Y^M \partial_s Y^N g_{MN}, \quad B_{rs} = \partial_r Y^M \partial_s Y^N B_{MN}. \quad (1.41)$$

The $A_{(n)}$ are the pullbacks of the other form fields, so e.g. $A_{(1)}$ and $A_{(3)}$ for type IIA, and their dual potentials $A_{(5)}$ and $A_{(7)}$. In this so-called Wess-Zumino term, it is understood that we select the $(p + 1)$-forms from the sum.

We expect the $D = 10$ supergravity background corresponding to a single $Dp$-brane to have the reduced symmetry $SO(1, p) \times SO(9 - p)$. Since the supersymmetry algebra (1.4) generates translations, there must be some supersymmetry breaking associated with the brane. In fact the $Dp$-branes are the $1/2$-BPS objects carrying the central

\textsuperscript{15}Here we are assuming that the timelike target space coordinate $X^1$ has Neumann boundary conditions. Otherwise we would have a $D$-instanton $[95,96]$.\n
charges $Z(p)$. As an example, we reproduce the type IIA background corresponding to a D2-brane [98], and analyse its supersymmetry. We choose coordinates $x^\mu$, with $\mu = 1, \ldots, 3$, in the brane directions. The background geometry is

$$ds^2 = H^{-\frac{3}{2}}(\rho)\eta_{rs}dx^r dx^s + H^{\frac{3}{2}}(\rho) \left( d\rho^2 + \rho^2 d\Omega^2_6 \right),$$

with $d\Omega^2_6$ the volume element on the unit sphere $S^6$ and

$$H(\rho) = 1 + \frac{Q}{\rho^5}.$$  \hfill (1.43)

This is supported by a dilaton profile

$$\phi = \frac{1}{4} \log [H(\rho)],$$

and flux $A_{(3)}$

$$A_{\mu_1\mu_2\mu_3} = \frac{1}{16\pi} \varepsilon_{\mu_1\mu_2\mu_3}.$$ 

This satisfies the type IIA Killing spinor equations in the bulk, provided one imposes the additional projection condition

$$(1 - \Gamma_{123}) \epsilon = 0,$$

which breaks half the supersymmetry. Now if we integrate the dual flux $*F_{(4)}$ over the boundary of the 7-dimensional transverse space, \textit{i.e.} over a transverse sphere $S^6$ surrounding the D2-brane, we find a violation of Gauss’s law, \textit{i.e.} the brane carries an (electric) $A_{(3)}$ charge:

$$\frac{1}{\text{Vol}} \int_{S^6} *F_{(4)} \neq 0.$$ \hfill (1.47)

This conserved charge is one of the central charges admissible in (1.4). Rather than give a full derivation of this result, we will just note that a general embedding of a timelike D2-brane in $D = 10$ may be described by a 2-form, which can then be contracted with $\Gamma^{MN}C$ to form a central charge $Z_{(2)}$.

For each field $A_{(n)}$ in a supergravity there is a corresponding $Dp$-brane source. Anomaly considerations reveal that the brane sourcing the $B_{(2)}$ field is rather unique, in that its dynamics are not governed by equation (1.39) but rather by a considerably more complex action [99]. We call this the NS5-brane, and will discuss it again in
So far we have just discussed the bosonic action (1.39); this is sufficient for our purposes, as we will never consider solutions with non-vanishing fermions. Nevertheless the full $Dp$-brane action is supersymmetric. One way to see this is via a superspace formalism, i.e. one extends the target space coordinates $X^M$ by writing $Z^M = (X^M, \theta^\alpha)$, with $\theta^\alpha$ Grassmann numbers. On-shell, one finds that there would be a mismatch between the bosonic and fermionic worldvolume degrees of freedom were it not for an additional ‘kappa’ symmetry [100]

$$\delta z^M = 0, \quad \delta z^\alpha = (1 + \Gamma_\kappa)^\alpha_\beta \cdot \kappa^\beta (\xi),$$  \hspace{1cm} (1.48)

with $\kappa^\beta$ some Grassmann parameter. For the $Dp$-branes, the appropriate kappa-symmetry projection operator $\Gamma_\kappa = \gamma^2_\kappa$ is defined via [101,102]

$$d^{p+1}\xi \Gamma_\kappa = -\frac{1}{\sqrt{-\det [g_{rs} + \mathcal{F}_{rs}]}^2} e^{\mathcal{F}(2)} \wedge \bigoplus \frac{1}{(2n + 1)!} \Gamma_{i_1 \cdots i_{n+1}} d\xi^{i_1} \wedge \cdots \wedge d\xi^{i_{n+1}} \cdot (\Gamma^{11})^{n+1},$$  \hspace{1cm} (1.49)

where the matrices $\Gamma_{i_1 \cdots i_r}$ are the pullbacks of the bulk gamma matrices:

$$\Gamma_{i_1 \cdots i_r} = \partial_{i_1} X^{M_1} \cdots \partial_{i_r} X^{M_r} \Gamma_{M_1 \cdots M_r}.$$  \hspace{1cm} (1.50)

Remarkably, kappa-invariance of the action (1.39) implies the $D = 10$ equations of motion [103]. This is reminiscent of our discussion in section (1.2.2), where we obtained the equations of motion by demanding Weyl invariance of the string action (1.29) at the quantum level.

### 1.3 Extra Dimensions

There exist good theoretical reasons and overwhelming observational evidence that, at low energies, our universe can be described in terms of a 4-dimensional manifold [104]. Higher dimensional supergravity is interesting in its own right, but if one is to model our own universe with such a theory then one requires a mechanism to explain the dimensional surplus. As stressed in section 1.2.2, we are particularly interested in the cases of six or even seven extra dimensions.

Such a mechanism was proposed well before the advent of string theory or even
supergravity \[3, 4\]. In this Kaluża-Klein theory, the coordinate dependence of the fields is artificially restricted so that they lie in the bundles of a lower dimensional submanifold. This procedure may be mathematically convenient, physically justified by some geometrical properties of spacetime, or both.

As we will explain below, in this procedure the higher dimensional degrees of freedom are either frozen out at low energies or packaged into new, lower dimensional fields. In this fashion it is hoped that many interesting physical systems may be embedded as low energy effective actions of \(D = 10\) or \(D = 11\) supergravity, and in turn of string or \(M\)-theory.

In this section we give an overview of the Kaluża-Klein procedure, and its applications in the literature, illustrating it with the well-known example of the \(S^1\) reduction of \(D = 11\) supergravity \[72\] - an example we shall make use of in chapter 2. A more focused discussion of the class of dimensional reductions relevant to this thesis - the Pauli reductions - will be deferred to section 1.5.

### 1.3.1 Kaluża-Klein Theory

Kaluża-Klein theory is not so much a theory but rather a general approach to understanding the low energy limit of a system with extra dimensions. As such, it is best illustrated with simple and representative examples; indeed we will begin with the simplest of all field theories: a massless scalar field \(\Phi\) in \(D + 1\)-dimensional flat space, \(i.e.\) an action

\[
S_5 = \int d^{D+1}x \left( -\frac{1}{2} * d\Phi \wedge d\Phi \right).
\]

The corresponding equation of motion is the \((D + 1)\)-dimensional Klein-Gordon equation

\[
\square_{(D+1)} \Phi = 0.
\]

The first step in the dimensional reduction process is to take coordinates \((x^\mu, z)\), with \(\mu = 1, \ldots, D\), and \(z\) our extra (typically spatial) dimension. Suppose now that we identify

\[
z \sim z + 2\pi R,
\]

where \(R\) is either some finite radius \(R \in (0, \infty)\) or \(R\) is infinite. If \(R\) is finite then we say the extra dimension has been compactified, and spacetime has the topology \(\mathbb{R}^{1,D-1} \times S^1\). If \(R\) is infinite then spacetime has topology \(\mathbb{R}^{1,D}\), and really we have made no coordinate identification at all. We can then expand \(\Phi(x^\mu, z)\) as a Fourier
series\textsuperscript{16} on $S^1_z$:
\[
\Phi(x^\mu, z) = \sum_{n \geq 0} \phi_n(x^\mu) e^{inz}.
\]  
(1.54)

Substituting this expansion into equation 1.52, we obtain 2nd order differential equations for each of the Fourier modes (in this context called Kaluža-Klein modes) $\phi_n$:
\[
\Box_{(D)} \phi_0 = 0,
\]
\[
\Box_{(D)} \phi_1 = \left( \frac{1}{R} \right)^2 \phi_1,
\]
\[
\Box_{(D)} \phi_2 = \left( \frac{2}{R} \right)^2 \phi_2,
\]
\[
\vdots
\]  
(1.55)

where we stress that $\Box_{(D)}$ is the d’Alembertian on the $D$-dimensional submanifold.

The second step in the Kaluža-Klein procedure is to truncate the theory, by setting $\phi_n = 0$ for $n \neq 0$ in these equations of motion. The resulting system can be obtained by varying the $D$-dimensional action
\[
S_4 = \int d^Dx \left( \frac{-1}{2} * d\phi_0 \wedge d\phi_0 \right),
\]  
(1.56)

and thus the theory has been dimensionally reduced. We say it has been reduced on $S^1$.

For any action, we are always free to expand fields and truncate modes however we see fit; in Kaluža-Klein theory we must seek to justify ourselves. There are two possible rationales for this truncation:

- Each of the $\phi_n$ obeys the $D$-dimensional Klein-Gordon equation for a scalar with mass $m = n/R$. If one is concerned only with the dynamics of action (1.51) at low energies $E \ll R^{-2}$ then discarding the higher modes is warranted. Another way to say this is that the modes exhibit a mass gap.

- The truncation $\phi_n = 0$ for $n \neq 0$ is a consistent truncation of the equations of motion (1.55), and so any solution of the dimensionally reduced action (1.56) will also be an exact solution of the full action (1.51).

\textsuperscript{16}In the case $R = \infty$ this sum becomes an integral and the Fourier series a Fourier transform.
These two lines of reasoning are often conflated in the literature, but they are mathematically and physically distinct. If we are thinking of the dimensionally reduced action as dynamical, *i.e.* using it to calculate scattering amplitudes, then a mass gap is essential and consistency optional\(^\text{17}\). Conversely, if we are searching for solutions of the full action then a consistent dimensional reduction is a powerful tool - and a mass gap an irrelevance.

Good examples of the latter case are reductions of general relativity - or more general theories - on the timelike direction of spacetime [105]. Solutions of the reduced action are the stationary black holes, or stationary black branes, of the full theory. Since a timelike circle necessarily has infinite radius, there is no mass gap. This merely indicates that a stationary solution will never be exactly realised in spacetime: objects will always experience small, time-dependent perturbations. This doesn’t make the stationary solutions any less worthy of study.

The \(S^1\) reduction here is unusual, in that it is both a consistent truncation and it exhibits a Kaluža-Klein mass gap. In fact, dimensional reductions on \(S^1\) are always consistent: they are a particularly straightforward example of a class of dimensional reductions called group manifold reductions, which will be discussed in section (1.5.1).

A well-known dimensional reduction that we shall employ later in this thesis is the reduction of the \(D = 11\) supergravity action (1.15) on \(S^1\) [72]; we detail it here. As before, we split our coordinates \((x^\mu, z)\). We shall use circumflexes to distinguish \(D = 11\) fields from their \(D = 10\) descendants. For the metric, we make the ansatz

\[
\hat{g}_{MN} dx^M dx^N \equiv e^{-\frac{\phi}{6}} g_{\mu \nu} dx^\mu dx^\nu + e^{\frac{4 \phi}{3}} (dz + A_\mu dx^\mu)^2.
\]

(1.57)

We have implicitly truncated to the lowest Kaluža-Klein mode by taking \(g_{\mu \nu}, A_\mu, \) and \(\phi\) to be functions of \(x^\mu\) only. A simple calculation yields the relations

\[
\sqrt{-\hat{g}} = e^{-\frac{\phi}{6}} \sqrt{-g},
\]

(1.58)

and

\[
\hat{R} = e^{\frac{\phi}{6}} \left( R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{7}{12} \Delta_{(11)} \phi - \frac{1}{4} e^{\frac{2 \phi}{3}} F_{\mu \nu} F^{\mu \nu} \right),
\]

(1.59)

with \(F_{(2)}\) is as in equation (1.20) and \(\Delta_{(11)}\) is the Laplace-Beltrami operator\(^\text{18}\) in

\(^{17}\) Optional in the sense that an inconsistent truncation will only lead to parametrically small errors in our low energy effective action.

\(^{18}\) The definition of this operator may be found in section 3.1, where its properties are discussed at
\( D = 11 \). Similarly for the 3-form potential \( \hat{A}_{(3)} \) we write

\[
\hat{A}_{(3)} \equiv A_{(3)} + B_{(2)} \wedge dz ,
\]

and so

\[
\hat{F}_{(4)} = F_{(4)} + H_{(3)} \wedge (dz + A_{(1)})
\]

with \( H_{(3)} \) and \( F_{(4)} \) as in equation (1.20). We complete the procedure by substituting these ansätze into the action (1.15). One immediately finds that the reduced action is precisely the type IIA action (1.19). The consequences of this were discussed in section 1.2.2. The \( D = 11 \) and type IIA Newton couplings are related by

\[
G_N^{(10)} = \frac{1}{2\pi R} G_N^{(11)}.
\]

We note that the ansatz (1.57) is invariant under a local symmetry \( z \mapsto z - \lambda(x) \), provided we make the compensating transformation

\[
A_{(1)} \mapsto A_{(1)} + d\lambda,
\]

From the type IIA perspective, \( A_{(1)} \) is a \( U(1) \) gauge field; the global symmetry of the metric has descended to become a gauge symmetry of the reduced action. This is a typical feature of extra dimensions, one we will encounter again in section 1.5.

In this second example we have taken a shortcut, substituting our ansätze directly into the \( D = 11 \) supergravity action instead of the equations of motion. This is considerably easier, and here yields the correct action (1.19), but this is not always a valid procedure [106]. Indeed we will be compelled to work with the equations of motion when performing the Pauli reduction on \( \mathcal{H}^{2,2} \) - one of the central topics of this thesis.

We can obtain a 4-dimensional theory by successive reductions on \( S^1 \), but this is just one of many possible geometries for the extra dimensions. One is especially interested in dimensional reductions that preserve some or all of the higher dimensional supersymmetry. There are many reasons for this: supersymmetric actions have interesting lower dimensional phenomenology; their vacuum solutions are stable against small perturbations of \( e.g. \) the metric; they are often the easiest reductions we can perform analytically. Starting from \( D = 10 \), it is known that in the absence of vacuum some length.
expectation values for the higher dimensional fluxes the most general geometry respecting some supersymmetry is a Calabi-Yau 3-fold \[107\], \textit{i.e.} a compact, 6-dimensional, Kähler manifold with special holonomy. Dimensional reductions on Calabi-Yau manifolds are not believed to be consistent truncations. As a rule of thumb, the Kaluža-Klein spectrum of a dimensional reduction will have a mass gap inversely proportional to the radius of the extra dimensions. There are some notable exceptions: braneworld models can have a mass gap despite a non-compact transverse manifold; the volume of a compact hyperbolic surface may be taken to be arbitrarily large without altering the Kaluža-Klein scale \[108\].

1.4 Holography

We can extend the Einstein-Hilbert action \(1.10\) to include a cosmological constant \(\Lambda\):

\[
S = \frac{1}{G_N^{(D)}} \int d^D x \ (R \ast 1 - \Lambda \ast 1),
\]

When \(\Lambda < 0\), one finds that the vacuum of \((1.64)\) is a maximally symmetric spacetime with constant negative curvature, called \(AdS_D\). In \(D\)-dimensions, one conventionally writes

\[
\Lambda = -D (D - 1) \frac{l^2}{2},
\]

where \(l\) has units of length, and is called the \(AdS\) length scale. For an appropriate choice of coordinates, we can write the \(AdS\) metric in the planar form

\[
ds^2 = r^2 \left[ -dt^2 + \sum_{i=1}^{D-2} (dx^i)^2 \right] + l^2 \frac{dr^2}{r^2}.
\]

An important property of \(AdS\) spacetime is that timelike geodesics can reach \(r = \infty\) in finite time. There is thus a horizon there, which has a simple \(d\)-dimensional Minkowski geometry, with \(d = D - 1\).

The holographic principle is a conjectured property of quantum gravity theory, which proposes that certain gravitational field theories with spacetimes \(M\) are equivalent to certain non-gravitational field theories living on the boundaries \(\partial M\) of said spacetimes. There is strong evidence that this duality is realised in string theory via the \(AdS/CFT\) correspondence \([5–7]\). In its best-known incarnation, the correspondence states that string theory on \(AdS_5 \times S^5\) is equivalent to \(\mathcal{N} = 4\) Yang-Mills theory in \(D = 4\), in
the sense that their partition functions are equal. More precisely, suppose that we
couple some field $\varphi$ to (1.64). Then for a small value of the field $\delta\varphi$, there exists a dual
operator $O$ in the field theory such that

$$Z_{\text{gravity}}[\delta\varphi] = \langle e^{i\int d^d x \sqrt{-g_0} \delta\varphi_0 O} \rangle_{\text{CFT}},$$

with $\delta\varphi_0$ the asymptotic value of $\varphi$ as $r \to \infty$. Every field on the gravitational side
has a dual operator in the field theory, with the metric being dual to the stress-energy
tensor. The power of this correspondence is that we identify

$$\frac{l_4^4}{\alpha'^2} = 4\pi \lambda, \quad \langle e^{\phi} \rangle = \frac{\lambda}{N},$$

with $N$ the rank of the Yang-Mills gauge group $SU(N)$ and $\lambda \equiv g_{YM}^2 N$ its so-called
t'Hooft coupling. Here $\langle e^{\phi} \rangle$ is the loop parameter of string theory, c.f. equation (1.33).
When $\lambda \gg 1$ and $N \gg \lambda$ the field theory is strongly coupled, and very difficult
to analyse using field theory techniques. We see from equation (1.68) that the dual
 gravitational theory is weakly coupled and has a radius of curvature that is large
compared to the string scale. As discussed in section 1.2.2, this is exactly the regime in
which we can deploy a supergravity description of string theory. This is an example of
a strong/weak duality. In this large $N$ limit, only the planar diagrams contribute to the
field theory path integral, and the equivalence between the two theories in this sector
is well established via integrability techniques, see e.g. [109]. Beyond the planar limit,
the holographic correspondence is more speculative. We have similar equivalences for
other geometries with lower dimensional $AdS$ components.

In the supergravity limit, the asymptotics of fields $\varphi$ are determined by their scaling
dimension $\Delta$. For example, a scalar field will have

$$\delta\varphi \sim \left(\frac{r}{l_4}\right)^{\Delta-d} \delta\varphi_0 + \left(\frac{r}{l_4}\right)^{-\Delta} \delta\varphi_1 + \ldots$$

where $\delta\varphi_0$ and $\delta\varphi_1$ are free parameters and we have omitted subleading terms which
are consistently determined by the equations of motion. We have already discussed the
dual interpretation of $\delta\varphi_0$; the $\delta\varphi_1$ gives an expectation value of $O \propto \varphi_1$.

In the Wilsonian picture, a field theory is defined at some energy scale with certain
values for the couplings. It is convenient to think of a phase space $C$ of possible
couplings. As we consider successively lower energies, we integrate the higher energy
modes out of the partition function, which results in a renormalisation of the couplings.
and hence a flow along some trajectory in $C$ [110]. The general structure of these trajectories is governed by fixed points, i.e. sets of values for the couplings which are not renormalised. Consequently these fixed point theories must have a scaling symmetry. We say that a trajectory flows away from an ultraviolet (UV) fixed point and towards an infrared (IR) fixed point.

We can consider small deformations about a fixed point by perturbing the couplings of different operators. An operator whose coupling becomes parametrically larger as we flow towards the IR is said to be relevant, whilst those with shrinking couplings are called irrelevant\(^{19}\). A central theme of modern field theory is that the low energy physics of many different systems can be described by just a few IR fixed points plus their irrelevant and marginal deformations, of which there are usually a finite number. Systems which flow to the same fixed point are said to be in the same universality class.

The $AdS/CFT$ correspondence can be thought of as a geometrisation of this renormalisation group flow, with the UV corresponding to the $AdS$ horizon and the IR to the interior. Of course we are not often interested in exactly conformal field theories; we must consider some relevant deformations, which will correspond to spacetimes that are only asymptotically $AdS$, have some more complicated behaviour in the interior, before finally displaying some emergent scaling symmetry in the IR limit $r/l \ll 1$.

A rich source of strongly coupled field theory problems is condensed matter theory (CMT). We will discuss such an example in depth in chapter 4. For now we just give some of the basics of the $AdS/CMT$ dictionary. Firstly we often require that a condensed matter system have a non-zero temperature. It is believed that the dual supergravity description is the presence of a horizon in the IR [111,112] - a belief supported by the discovery that black holes obey their own version of the laws of thermodynamics [113]. Secondly, we usually consider systems with a non-zero density of particles, i.e. non-zero chemical potential. A conserved particle number is associated with a global $U(1)$ symmetry, which is dual to an abelian gauge field on the gravity side.

\(^{19}\)There also exist marginal operators, whose couplings remain unchanged under renormalisation. A fixed point with a marginal deformation is really a line of fixed points. We see from equation (1.69) that a real scalar field is relevant when $\triangle < d$. 


30
1.5 Pauli Reductions on Spheres and Hyperboloids

In a dimensional reduction, one almost always truncates to the lowest Kaluza-Klein modes. In the case of the $S^1$ reduction of a massless scalar field theory, we showed explicitly in section 1.3 that there was a single zero mode in the Kaluza-Klein expansion (1.54), and that we could consistently truncate to this mode. From this example one might conclude that these were typical features of dimensional reduction, but this is not the case. For higher spin fields, the Kaluza-Klein spectrum is usually degenerate; there can be multiple zero modes. It may be that we can consistently retain only some subset\(^{20}\) of these, or that there is no consistent truncation at all.

We also saw in section 1.3 how an isometry group $G$ of the extra dimensions could descend to become a gauge symmetry of the reduced system. A Pauli reduction is a dimensional reduction where we retain all the zero modes of these gauge fields. Consistent Pauli reductions are extremely rare; our $\mathcal{H}^{2,2}$ reduction will be a particularly interesting example.

In this section we shall briefly review some of the group theoretic arguments concerning the consistency of reductions on group and coset group manifolds. We then summarise the structure of reductions on spheres, and present the well-known consistent Pauli reduction on $S^4$, which shall be the starting point for all the results of chapter 2. Finally we motivate the introduction of the inhomogeneous hyperboloids, and explain their relation to the spheres.

1.5.1 Group Manifolds

In section 1.3 we remarked that the dimensional reduction of $S^1$ of any system lead to a consistent truncation to the zero modes. This is an extraordinary claim: there is no combination of fields, with any couplings and with any potential, such that the higher Kaluza-Klein modes are sourced by the zero modes. There is a very simple proof however. Under a global $U(1)$ symmetry $z \rightarrow z - \lambda R$, with $\lambda$ constant, we have

$$\phi_n(x) \rightarrow e^{i \lambda} \phi_n,$$

\(^{20}\)We can imagine scenarios in which we consistently truncate to some finite subset containing higher Kaluza-Klein modes along with the zero modes. In reality it is exceptionally difficult to identify consistent truncations without appealing to the kind of group theoretic properties of the zero modes we discuss in section 1.5.1.
and similarly for any other modes in the theory. It follows that \( \phi_n \) has charge \( n \) under the global symmetry. Now, an inconsistency could only arise if one of the higher Kaluza-Klein modes - we shall use \( \phi_n \) with \( n \neq 0 \) as an example - had an equation of motion of the form

\[
\Box (D) \phi_n = f(\phi_0, \ldots) + \ldots,
\]

where \( f \) is some function of the zero modes, and the omitted terms vanish when we set the higher modes to zero. But this is impossible: the right hand side must have \( U(1) \) charge \( n \), but any function of the zero modes is necessarily a singlet under this symmetry. We can immediately generalise this argument to case of the dimensional reduction on the torus \( T^k \), viewing it as \( k \) successive consistent truncations on copies of \( S^1 \).

There is a more sophisticated argument in the case of reductions on non-abelian group manifolds (Lie groups) \( G \) \cite{114,115}. Recall a group manifold is a continuous group \( G \) that is also a manifold, on which the group multiplication action \( g : G \times G \rightarrow G \) is a diffeomorphism. In the non-abelian case we can distinguish between the left action \( g_1 : g_2 \mapsto g_1 g_2 \) and the right action \( g_1 : g_2 \mapsto g_2 g_1 \). Associated with these actions are the left and right invariant Maurer-Cartan forms \( g^{-1} \cdot dg \) and \( dg \cdot g^{-1} \) respectively. These take values in the Lie algebra \( g \):

\[
g^{-1} \cdot dg = Y^a t^a dz^i,
g^i a dz^i,
\]

where \( z^i \) are coordinates on the manifold \( G \), \( t^a \) are the generators of \( g \), and \( Y \) and \( K \) are 1-forms dual to the Killing vectors of the left and right actions respectively. Now, the non-abelian generalisation of equation (1.57) is \cite{114}

\[
\hat{g}_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + g_{ij} (dz^i + K^i_j A^a_{\mu} dx^\mu) (dz^j + K^j_i A^b_{\mu} dx^\mu),
\]

We truncate to the zero mode by taking \( A_{\mu} = A_{\mu}(x) \), etc., just as before. The higher modes are charged under the left action, but the zero modes are not. Just as with the \( U(1) \) case, it follows that the consistency of the truncation is guaranteed\footnote{If we additionally have that \( G \) is unimodular then we can consistently carry out the reduction procedure at the level of the action \cite{115}, i.e. without going via the equations of motion.}. The action (1.73) is invariant under the right action of \( G \), provided we make a compensating transformation of \( A_{(1)} \). This turns out to be precisely the transformation of a gauge
potential with gauge group \( G \).

More generally we can consider dimensional reductions on coset spaces \( G/H \). One advantage of the coset reduction is that we can obtain a lower dimensional gauge group \( G \) from just \( \dim(G) - \dim(H) \) extra dimensions. To do this we need a right action of \( G \) on the \( k \in G/H \), but this requires compensating \( H \) transformations:

\[
g : G/H \rightarrow G/H \quad \text{with}, \quad g : k \mapsto h(g) \cdot k \cdot g.
\]  

This precludes any non-trivial left action, and our group theoretic proof of consistency can no longer be applied.

### 1.5.2 The \( S^4 \) Reduction

The spheres \( S^n \) are coset manifolds \( SO(n+1)/SO(n) \). Pauli reductions on the spheres will thus lead to \( SO(n+1) \)-gauged lower dimensional actions. We generically expect these reductions to be inconsistent truncations of their parent theories [117], c.f. our discussion in section 1.5.1. Remarkably there exist several examples of sphere reductions that are believed to be consistent. Starting from the maximally supersymmetric \( D = 10 \) and \( D = 11 \) actions of section 1.2.1, these are the reductions of \( D = 11 \) supergravity on \( S^4 \) [118,119] and \( S^7 \) [120], and the reduction of type IIB supergravity on \( S^5 \) [117].

We shall focus on the \( S^4 \) reduction. The resulting \( SO(5) \)-gauged action was first constructed in [121] from a purely \( D = 7 \) perspective, and was shown to be a maximally supersymmetric\(^{23} \) theory. A gauging consistent with supersymmetry requires the introduction of a separate, composite \( SO(5) \) symmetry - that is, a local symmetry without corresponding gauge bosons. The composite symmetry is an echo of the \( SL(5)/SO(5) \) U-duality group of the \( T^4 \) reduction, which we must necessarily recover as we take the radius of the sphere to infinity.

We write \( SO(5)_g \) for the gauge symmetry, and \( A = 1, \ldots 5 \) for the gauge indices. Similarly we write \( SO(5)_c \) for the composite symmetry, and \( i = 1, \ldots 5 \) for the composite indices. These groups have the usual Cartan-Killing metrics \( \eta_{AB} = \text{diag}(1,1,1,1,1) \)

---

\(^{22}\)There is an ambiguity here, in that a group \( G \) can be thought of as a coset \( G \times \tilde{G}/\tilde{G} \). We define a Pauli reduction on \( G/H \) to be a dimensional reduction in which we retain all the gauge modes of \( G \). A Pauli reduction on \( G \times \tilde{G}/\tilde{G} \) is thus distinct to a group manifold reduction \( G \) [116], and is not in general a consistent truncation.

\(^{23}\)The use of the \( 'N' \) notation is not quite universal across the \( D = 7 \) literature. In this thesis an \( N = 1 \) theory in \( D = 7 \) is one with 16 real supercharges.
and $\delta_{ij} = \text{diag}(1,1,1,1,1)$. It follows that there is no difference between covariant and contravariant indices, but in anticipation of our discussion in section 1.5.3 we shall insert explicit factors of $\eta_{AB}$ whenever we raise or lower a gauge index.

The bosonic field content consists of the metric, an $SO(5)_g$ gauge potential $A^{AB}_{(1)} = -A^{BA}_{(1)}$ transforming in the adjoint representation of $SO(5)_g$, scalars $\Pi_A^i$ in the bifundamental representation of $SO(5)_g \times SO(5)_c$, and 3-forms $S_{(3)A}$ in the fundamental representation of $SO(5)_g$. In the fermionic sector we have gravitinos $\psi_{iA}$ transforming in the spinor representation of $SO(5)_c$, and gauginos $\lambda_i$ in the spinor and fundamental representations of $SO(5)_c$. The gauginos satisfy the constraint

$$
\gamma_i \lambda_i = 0,
$$

where the $\gamma_i$ are the gamma matrices $\{\gamma_i, \gamma_j\} = \delta_{ij}$ of the $SO(5)_c$ Clifford algebra. These fields are governed by the Lagrangian\textsuperscript{24}

$$
\mathcal{L} = R \ast 1 - \ast P_{(1)ij} \wedge P_{(1)ij} - \frac{1}{2} \Pi_A^i \Pi_B^j \Pi_C^i \Pi_D^j \ast F^{AB} \wedge F^{CD} \\
- \frac{1}{2} \Pi^{-1} \epsilon_i^A \Pi^{-1} B \ast S_{(3)A} \wedge S_{(3)B} + \frac{1}{2g} \eta^{AB} S_{(3)A} \wedge D S_{(3)B} \\
- \frac{1}{4g} \epsilon_{ABCDE} \eta^{AF} S_{(3)F} \wedge F_{i(2)}^{BC} \wedge F_{i(2)}^{DE} + \frac{1}{g} \Omega_{(7)} - V \ast 1 \\
+ \left\{ -\bar{\psi}_i \gamma^\nu \rho \nabla_\nu \psi_\rho - \bar{\lambda}_i \gamma^\mu \nabla_\mu \lambda_i + \frac{1}{4} g \Pi^{-1} \epsilon_i^A \Pi^{-1} B \eta_{AB} \bar{\psi}_i \gamma^\mu \psi_\mu \\
- \frac{1}{4} g \Pi^{-1} \epsilon_i^A \Pi^{-1} B \eta_{AB} (8 \delta_{ij} \delta_{kl} - \delta_{ij} \delta_{kl}) \bar{\lambda}_i \lambda_j + g \Pi^{-1} \epsilon_i^A \Pi^{-1} B \eta_{AB} \bar{\lambda}_i \gamma^\nu \gamma^\mu \psi_\mu \\
+ \frac{1}{2\sqrt{2}} \bar{\psi}_i \gamma^\nu \gamma^\mu \gamma_i \lambda_j \Pi B_{ij} F_{vp} + \frac{1}{8\sqrt{2}} \bar{\lambda}_i \gamma_i \gamma_{kl} \gamma_i \gamma^\nu \gamma^\mu \lambda_j \Pi B_{ij} F_{vp} \\
+ \frac{1}{2\sqrt{2}} \bar{\psi}_i \gamma^\nu \gamma^\mu \gamma_i \lambda_j \Pi B_{ij} F_{vp} + \frac{1}{16\sqrt{2}} \bar{\lambda}_i \gamma_i \gamma_{kl} \gamma_i \gamma^\nu \gamma^\mu \lambda_j \Pi B_{ij} F_{vp} \\
- \frac{1}{12} \bar{\psi}_i (\gamma^\nu \gamma^\mu \gamma^\sigma \gamma^\rho) \gamma_i \psi_\tau \Pi^{-1} A \gamma_\tau S_{\nu\rho A} \\
+ \frac{1}{12} \bar{\psi}_i (\gamma^\nu \gamma^\mu \gamma^\sigma \gamma^\rho) \lambda_i \Pi^{-1} A \gamma_\tau S_{\nu\rho A} + \frac{1}{24} \bar{\lambda}_i \gamma^\nu \gamma^\mu \gamma^\sigma \gamma^\rho \lambda_i \Pi^{-1} A \gamma_\tau S_{\nu\rho A} \right\} \ast 1,
$$

\textsuperscript{24}Terms that are quartic in the fermions become increasingly complicated as one reduces the dimension: we will omit them from here onwards. We thus also omit terms from the supersymmetry transformations that are cubic in the fermions. As a consequence, we make no distinction between the ordinary and supercovariant incarnations of a field.
where we have introduced a scalar 'inverse' satisfying
\[ \Pi^{-1} A \Pi_A^j = \delta_{ij}, \quad \Pi_A^i \Pi^{-1} B = \delta_A^B, \] (1.77)
along with the \( SO(5) \)-covariant derivatives
\[ F^{AB}_{(2)} = dA^{AB}_{(1)} + gA^{AC}_{(1)} A^{DB}_{(1)} \eta_{CD}, \]
\[ DS_{(3)A} = dS_{(3)} + gA^{A}_{(1)B} \wedge S_{(3)B}, \] (1.78)
and the scalar field strength
\[ P_{(1)ij} \equiv \Pi^{-1} A \left( \delta_B^A d + gA^{A}_{(1)B} \right) \Pi_B^k \delta_{j}^k. \] (1.79)
These scalars have a potential \( V \), which can be written as
\[ V \equiv \frac{1}{2} g^2 \Pi^{-1} A \Pi^{-1} B \Pi^{-1} C \Pi^{-1} D \left( 2\eta_{AB} \eta_{CD} - \eta_{AC} \eta_{BD} \right). \] (1.80)
The form \( \Omega_{(7)} \) in the Lagrangian (1.76) is a pure Yang-Mills Chern-Simons term required for supersymmetry. The full expression for it may be found in [121]; we shall only need its variation, which takes the neat form
\[ \delta \Omega_{(7)} = \frac{1}{g} F^{AB}_{(2)} \wedge F^{CD}_{(2)} \wedge F^{EF}_{(2)} \wedge \delta A^{GH}_{(1)} \eta_{BC} \eta_{FG} \left( 2\eta_{AH} \eta_{DE} - \eta_{AD} \eta_{EH} \right). \] (1.81)
This action (1.76) is maximally supersymmetric on shell. The supersymmetry transformations for the fermions are
\[ \delta \psi_{\mu} = D_{\mu} \epsilon - \frac{1}{20} g \Pi^{-1} A \Pi^{-1} B \eta_{AB} \Gamma_{\mu} \epsilon - \frac{1}{40 \sqrt{2}} \left( \Gamma_{\mu}^{\nu \rho} - 8 \delta_{\mu}^{\nu} \Gamma^{\rho} \right) \gamma_{ij} \epsilon \Pi_A^i \Pi_B^j F^{AB}_{\nu \rho} \]
\[ - \frac{1}{60} \left( \Gamma_{\mu}^{\nu \rho} - 9 \delta_{\mu}^{\nu} \Gamma^{\rho} \right) \gamma^j \Pi^{-1} i S_{\nu \rho \sigma A}, \] (1.82)
\[ \delta \lambda_i = \frac{1}{16 \sqrt{2}} \Gamma^{\mu \nu} (\gamma_{kl} \gamma_{ij} - \frac{1}{5} \gamma_{ij} \gamma_{kl}) \epsilon \Pi_A^k \Pi_B^l F^{AB}_{\mu \nu} - \frac{1}{120} \Gamma^{\mu \nu \rho} (\gamma_{ij} - 4 \delta_{ij}) \epsilon \Pi^{-1} j S_{\mu \nu \rho A} \]
\[ + \frac{1}{2} g \Pi^{-1} k A \Pi^{-1} l B \eta_{AB} (\delta_{ik} \delta_{jl} - \frac{1}{5} \delta_{ij} \delta_{kl}) \gamma_{j} \epsilon + \frac{1}{2} \Gamma^\mu \gamma_j \epsilon P_{\mu j}. \] (1.83)
The \( SO(5) \)-covariant derivative \( D \) acts on the supersymmetry parameter \( \epsilon \) as
\[ D \epsilon \equiv d \epsilon + \frac{1}{4} \omega_{ab} \Gamma^{ab} \epsilon + \frac{1}{4} Q_{ij} \gamma_{ij} \epsilon, \] (1.84)
where $Q_{ij}$ is the $SO(5)_c$ connection
\[ Q_{ij} \equiv \Pi^{-1} i^A (\delta^B_k d + gA_{(1)A}^B) \Pi_B^k \delta_{jk}. \] (1.85)

In the bosonic sector the supersymmetry transformations are
\[
\delta e^a_\mu = \frac{1}{2} \bar{\epsilon} \Gamma^a \psi_\mu, \quad (1.86)
\]
\[
\delta A_{\mu}^{AB} = -\frac{1}{4\sqrt{2}} \bar{\epsilon} (2\bar{\epsilon} \gamma_{ijk} \psi_\mu + \Gamma_\mu [\nu \gamma_l \gamma_{ijk} \lambda_l] i^{A} \Pi^{-1} i^{B}), \quad (1.87)
\]
\[
\delta \Pi_A^i = \frac{1}{4} \bar{\epsilon} (\gamma_{i} \lambda_j + \gamma_{j} \lambda_i) \Pi_A^j, \quad (1.88)
\]
\[
\delta S_{\mu\nu\rho A} = -\frac{3}{4\sqrt{2}} i^{A}(2\bar{\epsilon} \gamma_{ijk} \psi_\mu + \bar{\epsilon} \Gamma_\mu [\nu \gamma_l \gamma_{ijk} \lambda_l] i^{B} \Pi^C^{i} F^{AC}_{\nu\rho})
- \frac{3}{2} \eta_{AB} \Pi^{-1} i^{B} D_\mu (2\bar{\epsilon} \gamma_{i} \psi_\rho + \bar{\epsilon} \Gamma_\nu \gamma_{i} \lambda_i)
+ \frac{1}{2} \epsilon \eta_{AB} \Pi^{-1} i^{B} (3\epsilon \Gamma_{[\mu\nu} \gamma_{i} \psi_\rho] - \epsilon \Gamma_{\mu\nu\rho} \lambda_i). \quad (1.89)
\]

The action (1.76) arises from a Pauli reduction of $D = 11$ supergravity on $S^4$. In [118,119] an ansatz was derived for this reduction, which we shall give presently. The authors argued that a derivation of the $D = 7$ supersymmetry transformations from the $D = 11$ supersymmetry transformations would amount to a proof of consistency, and successfully carried this out, though the usual omission of the terms of third or higher order in the fermions means this is not a complete proof.

It is the symmetry of the sphere that makes the reduction tractable. We can best exploit this symmetry by means of an embedding formalism. Take coordinates $Y^A$ on $\mathbb{R}^5$, so that the sphere embedding is
\[
(Y^1)^2 + (Y^2)^2 + (Y^3)^2 + (Y^4)^2 + (Y^5)^2 = 1. \quad (1.90)
\]

The advantage of these new coordinates it that they are treated democratically by $SO(5)$ transformations, so we can avoid complicated angular dependencies in the ansätze. Taking the flat Euclidean metric on $\mathbb{R}^5$, the induced metric is just the usual homogeneous sphere metric. In analogy with equation (1.73), we parametrise a general $SO(5)$ invariant metric by
\[
ds^2 \sim g_{AB} D^A Y^B, \quad (1.91)
\]
where \( g_{AB} \) is some metric, to be determined, and

\[
DY^A = dY^A + gA_{(1)}{}^A{}_B Y^B .
\]  
(1.92)

We now present the Pauli reduction ansatz. For the metric we write

\[
ds_{11}^2 = \Omega^2 \left\{ ds_i^2 + \frac{1}{g^2 \Omega} T_{AB}^{-1} DY^A DY^B \right\},
\]  
(1.93)

where the warp factor is

\[
\Omega = \Pi^{-1}{}_i Y^C Y^D \eta_{AC} \eta_{BD},
\]  
(1.94)

and we are now introducing matrices\(^{25}\)

\[
T^{AB} = \Pi^{-1}{}_i Y^B, \quad T_{AB}^{-1} = \Pi_A^i \Pi_B^j .
\]  
(1.95)

Similarly for the 4-form field strength \( F_{(4)} \) we write

\[
F_{(4)} = -\frac{1}{24g^3} U \Omega^{-2} \varepsilon_{ABCDEF} Y^A DY^B \wedge DY^C \wedge DY^D \wedge DY^E \\
+ \frac{1}{6g^2} \Omega^{-2} \varepsilon_{ABCDEF} T^{AF} DT^{BG} \wedge DY^C \wedge DY^D \wedge DY^E Y^I \eta_{FH} \eta_{GI} \\
+ \frac{1}{2\sqrt{2}g^2} \Omega^{-1} \varepsilon_{ABCDEF} ^{(2)} \wedge DY^C \wedge DY^D T^{EF} \eta_{FG} Y^G \\
- T^{AB} \ast S_{(3)A} \eta_{BC} Y^C + \frac{1}{g} S_{(3)A} \wedge DY^A ,
\]  
(1.96)

where \( U \) is the gauge and composite invariant

\[
U = \Pi^{-1}{}_i Y^B \Pi^{-1}{}_j Y^C \Pi^{-1}{}_j Y^D Y^F (2 \eta_{BC} \eta_{FD} - \eta_{BF} \eta_{CD}) \eta_{AE} .
\]  
(1.97)

### 1.5.3 Inhomogeneous Hyperboloids

In many fundamental respects the negatively curved analogue of the n-sphere \( S^n \) is the hyperbolic n-space \( H^n \), i.e. the simply connected Riemannian manifold of constant negative curvature. There are various different constructions of these spaces; taking our cue from equation (1.90), we realise both via a simple embedding in the vector...
space $\mathbb{R}^{n+1}$:
\[
\eta_{AB}Y^A Y^B \equiv (Y^1)^2 \pm (Y^2)^2 \pm \cdots \pm (Y^{n+1})^2 = 1, \tag{1.98}
\]
with embedding matrix $\eta_{AB} = \text{diag}(1,1,\ldots,1)$ for $S^n$ and $\eta_{AB} = \text{diag}(1,-1,\ldots,-1)$ for $H^n$. We write the embedding as $s : \mathcal{M} \rightarrow \mathbb{R}^{n+1}$ with $\mathcal{M} \cong S^n$ or $\mathcal{M} \cong H^n$ as appropriate. Writing $g^0_{AB}$ for the metric on $\mathbb{R}^{n+1}$, the metric $\gamma_{ab}$ on $\mathcal{M}$ is the induced metric $\gamma \equiv s^* g^0$. To complete the construction we choose
\[
g^0_{AB} = \begin{cases} 
\eta_{AB} & \text{for } \mathcal{M} \cong S^n, \\
-\eta_{AB} & \text{for } \mathcal{M} \cong H^n.
\end{cases} \tag{1.99}
\]
Both the embedding matrix and the ambient metric are invariant under the action of $SO(n+1)$ in the case of $S^n$ and $SO(1,n)$ in the case of $H^n$, so these are the isometry groups of the resulting induced metrics.

General Pauli reductions on spheres have the same structure we saw in section 1.5.2: we must introduce a composite symmetry $SO(n+1)_c$, and scalar ‘vielbeins’ transforming in the bifundamental representation of $SO(n+1)_g \times SO(n+1)_c$. Then the metric on the sphere has the form
\[
d s^2_{(n)} \sim g^0_{ij} \Pi^i A \Pi^j B \Pi^A D Y^A \Pi^B D Y^B. \tag{1.100}
\]
We interpret $g^0_{ij}$ and $\eta_{AB}$ as the Cartan-Killing metrics of the composite and gauge groups respectively. From equation (1.99) one would hope that we can obtain the Pauli reduction on $H^n$ in exactly the same: this would amount to a simple analytic continuation of $g^0_{AB}$ and $\eta_{AB}$. The result would be an $SO(1,n)_g$-gauged action in the lower dimensions. Unfortunately this non-compact gauging leads to a new problem. Consider the gauge kinetic term
\[
-\frac{1}{2} g^0_{ik} g^0_{jl} \Pi^i A \Pi^j B \Pi^k C \Pi^l D \ast F_{(2)}^{AB} \wedge F_{(2)}^{CD}. \tag{1.101}
\]
When the Cartan-Killing metric $g^0_{ij}$ has mixed signature, some of these kinetic terms will appear with the wrong sign. This leads to the oft-repeated statement that non-compact gauge symmetries cannot be linearly realised.

This does suggest an alternative dimensional reduction, where we take gauge group $SO(1,n)$ but composite group $SO(n+1)$ [29]. The resulting vacuum is the Riemannian manifold we obtain from an embedding matrix $\eta_{AB} = \text{diag}(1,-1,\ldots,-1)$ but
an ambient metric \( g^{0}_{AB} = \delta_{AB} \). We denote this space \( \mathcal{H}^{1,n} \) to differentiate it from \( H^n \). Its isometry group is the intersection of the symmetry groups of these matrices, i.e. the compact subgroup \( SO(n) \subset SO(1,n) \), and the inequivalence of these matrices implies that these hyperboloids are inhomogeneous. This reduction realises \( SO(n)_g \) linearly, and the full \( SO(1,n)_g \) non-linearly via the scalar vielbeins, hence avoiding ghost fields [29,30].

Fixing \( g^{0}_{AB} = \delta_{AB} \), there is an obvious generalisation to embedding matrix \( \eta_{AB} = \text{diag}(1, \ldots, 1, -1, \ldots, -1) \) with \( p \) positive entries and \( q \) negative entries, where \( p + q = n + 1 \). We denote the resulting family of Riemannian manifolds \( \mathcal{H}^{p,q} \). In this paper we will be primarily interested in the space \( \mathcal{H}^{2,2} \). Choosing coordinates \((\rho, \alpha, \beta)\) and taking the explicit embedding

\[
Y^1 = \cosh \rho \cos \alpha, \quad Y^2 = \cosh \rho \sin \alpha, \quad Y^3 = \sinh \rho \cos \beta, \quad Y^4 = \sinh \rho \sin \beta,
\]

(1.102)
we obtain \( \mathcal{H}^{2,2} \) metric

\[
d_{\mathcal{H}^{2,2}}^2 = \cosh(2\rho) d\rho^2 + \cosh^2 \rho \, d\alpha^2 + \sinh^2 \rho \, d\beta^2.
\]
(1.103)
This has cohomogeneity one. The Pauli reduction on \( \mathcal{H}^{2,2} \) leads to an \( SO(2,2) \)-gauged theory [25]. The intersection of \( SO(2,2) \) with the ambient isometry group \( SO(4) \) gives the vacuum isometry group \( U(1) \times U(1) \).

Because the Pauli reductions on \( \mathcal{H}^{p,q} \) differ from the Pauli reductions on some \( S^n \) merely by an analytic continuation of \( \eta_{AB} \), the consistency of the latter will imply the consistency of the former. From section 1.5.2, for example, we immediately have consistent truncations for Pauli reductions on \( \mathcal{H}^{4,1}, \mathcal{H}^{3,2}, \mathcal{H}^{2,3} \), and \( \mathcal{H}^{1,4} \).

We conclude with a brief discussion of the \( \mathcal{H}^{p,q} \) reduction vacua. Setting the scalar vielbeins \( \Pi_A^i \) to be the identity, we have from equation (1.80)

\[
S^5 : \quad V = -\frac{15}{2} g^2,
\]
\[
\mathcal{H}^{4,1} : \quad V = \frac{1}{2} g^2,
\]
\[
\mathcal{H}^{3,2} : \quad V = \frac{9}{2} g^2,
\]
\[
\mathcal{H}^{2,3} : \quad V = \frac{9}{2} g^2,
\]
\[
\mathcal{H}^{1,4} : \quad V = \frac{1}{2} g^2.
\]
(1.104)
We find that the sphere reduction leads to a negative cosmological constant, and hence to an $AdS$ vacuum, while the hyperboloids lead to a positive cosmological constant, and hence to $dS$ vacua. This is pretty typical [26–28]. Non-compact extra dimensions are one way to circumvent a powerful no-go theorem prohibiting positive cosmological constants in dimensionally reduced actions [33], and here we see it explicitly. This is a nice result, but it comes at a price: firstly our extra dimensions have infinite volume, so it is certainly not clear that there is a Kaluža-Klein mass gap; secondly the stability of the $AdS$ vacua of the sphere reductions does not at all imply the stability of the $dS$ vacua of the $H^{p,q}$ reductions. We shall carefully analyse the Kaluža-Klein spectrum of the $H^{2,2}$ reduction in chapter 3. Stability of $dS$ vacua is in general a thorny issue, but we shall see in section 2.3.1 that the $H^{2,2}$ equivalent of (1.80) has a runaway scalar direction that breaks the lower dimensional symmetry, yielding a stable (supersymmetric) Minkowski vacuum. Stability of $H^{p,q}$ was explored in some detail in [30].
2 The Pauli Reduction on $\mathcal{H}^{2,2}$

Consistent Pauli reductions are rare even on manifolds as symmetric as the spheres. Moreover when a sphere reduction is consistent it is all but impossible to prove it. For the $S^4$ reduction of $D = 11$ supergravity, the authors of [119] argued that deriving the known $D = 7$ supersymmetry transformations (1.82) and (1.86) from the $D = 11$ transformations was equivalent to a proof of consistency, and carried this out.

Thankfully other Pauli reductions can be extracted from the $S^4$ reduction, allowing one to circumvent the more laborious aspects of an explicit dimensional reduction, e.g. expanding fields in terms of spherical harmonics and Killing spinors. We note that the $SO(5)$ gauge group of the action (1.76) has subgroup $SO(4)$, suggesting one might be able to obtain the $S^3$ reduction in an appropriate limit. Merely setting the excess group generators to zero would, by definition, not be a Pauli reduction; nor would it be consistent. A procedure for deforming a symmetry group without discarding degrees of freedom was proposed by Wigner and Inönü [122].

Given the $S^3$ reduction, the analytic continuation [29] immediately yields the Pauli reductions on $\mathcal{H}^{1,3}, \mathcal{H}^{2,2}$, and $\mathcal{H}^{1,3}$. We are especially interested in the case $\mathcal{H}^{2,2}$, and with good reason: it is one of the few hyperboloid reductions with stable vacua [30]. Better still, a further reduction on $S^1$ enables one to embed the Salam-Sezgin model [25,123] into type IIA supergravity. It also leads to a gauged $R$-symmetry in $D = 6$ [32].

In this chapter we shall obtain this reduction via the Wigner-Inönü method. This involves a rescaling of the fields, the details of which we present in section 2.1. This procedure was first applied to the $S^4$ reduction in [24,25]. We shall extend those works by deriving the full $N = 2$ supersymmetry transformations in section 2.2 and the fermionic terms in the action in section 2.3. An interesting feature of the Wigner-Inönü contraction is that the 3-form $S_{(3)0}$ in the action (1.76) becomes the field strength of a new 2-form potential $B_{(2)}$ [24]. In section 2.2.1 we explain this in terms of self-duality in odd dimensions [118,119,121], and derive the supersymmetry transformation of $B_{(2)}$. We will spend some time exploring the structure of the $SO(p,4-p)$-gauged action, discussing its vacua in section 2.3.1 and its orbifolds in section 2.3.2, before finally applying our scalings to the $S^4$ reduction ansatz in section 2.4.
2.1 The Wigner-Inönü Contraction

Consider a Lie group $G$ with a subgroup $H < G$. The corresponding Lie algebra has the structure $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{l}$ with $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. We perform a Wigner-Inönü contraction [122] by rescaling the generators $h \in \mathfrak{h}$ and $l \in \mathfrak{l}$ as

$$h \rightarrow h, \quad l \rightarrow \frac{1}{k} \tilde{l},$$

and then taking the limit $k \rightarrow 0$. In this limit the Lie algebra structure becomes $\tilde{\mathfrak{g}} \cong \mathfrak{h} \oplus \tilde{\mathfrak{l}}$ with

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \tilde{\mathfrak{l}}] \subseteq \tilde{\mathfrak{l}}, \quad [\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}] = 0,$$

so that the group becomes a semidirect product of $H$ and an abelian group $L$:

$$G \rightarrow \tilde{G} \cong L \rtimes H.$$

Note that since we have a bijection between $\mathfrak{l}$ and $\tilde{\mathfrak{l}}$ it follows that the dimension of the group is not altered by this contraction.

Returning to our main topic, the $S^3$ reduction of type IIA supergravity has a global symmetry $SO(4)$. Using the results of section 1.3, we may lift this IIA reduction to $D = 11$ supergravity reduced on $S^1 \times S^3$. This extra $S^1$ is associated with an extra $U(1)$ symmetry, so that the total symmetry group is some semidirect product

$$G \cong U(1) \times SO(4) \subset U(1)^4 \rtimes SO(4).$$

This suggests that the $S^3$ reduction can indeed be obtained from a Wigner-Inönü contraction of a $D = 11$ ansatz with $SO(5)$ symmetry [24,124], i.e. the $S^4$ reduction.

For $k$ non-zero, all we have done is a simple field redefinition of the $SO(5)_c$ gauge potential $A^{AB}_{(1)}$. We can make compensating redefinitions of the remaining fields and couplings, so that all kinetic terms have scaling $O(k^0)$. Then in the limit $k \rightarrow 0$ we obtain our $SO(4)$-gauged theory without discarding any degrees of freedom. Supersymmetry is preserved in this limit and, once these scalings are applied to the $S^4$ reduction ansatz, the consistency of the Pauli reduction also carries over.

The $D = 7$ theory has both gauge and composite symmetries, and presumably we should isolate the $SO(4)$ parts of both. To proceed, we decompose the $SO(5)_c$ and $SO(5)_s$ indices as

$$i = (0, \alpha), \quad A = (0, \bar{A}),$$

42
with \( \alpha, A = 1, \ldots, 4 \) the \( SO(4) \) indices. To obtain the \( SO(4) \)-gauged Lagrangian from its \( SO(5) \)-gauged parent we must apply the correct scalings: the \( SO(5) \) gauge fields are an obvious place to begin. Choose

\[
g \to k^2 \tilde{g}, \quad A_{(1)}^{\bar{A} \bar{B}} \to k^{-2} A_{(1)}^{\bar{A} \bar{B}}, \quad A_{(1)}^{0 \bar{A}} \to k^3 A_{(1)}^{0 \bar{A}},
\]

so that the \( SO(4) \)-terms \( gA_{(1)}^{\bar{A} \bar{B}} \) do not scale but the remaining \( gA_{(1)}^{0 \bar{A}} \) vanish. (The gauge potentials scale the opposite way to the generators of the algebra.) The \( k^5 \) scaling here is chosen so that we might avoid fractional powers of \( k \) in the remainder of this work. Now, applying these scalings to the \( SO(5) \) gauge field strength \( F_{(2)}^{AB} \) we obtain \( SO(4) \) field strengths

\[
\tilde{F}_{(2)}^{\bar{A} \bar{B}} \equiv dA_{(1)}^{\bar{A} \bar{B}} + \tilde{g} A_{(1)}^{\bar{A} \bar{C}} \wedge A_{(1)}^{\bar{B} \bar{C}} \eta_{\bar{C} \bar{D}},
\]

\[
\tilde{F}_{(2)}^{0 \bar{A}} \equiv dA_{(1)}^{0 \bar{A}} + \tilde{g} A_{(1)}^{\bar{A} \bar{C}} \wedge A_{(1)}^{0 \bar{D}} \eta_{\bar{C} \bar{D}}.
\]

Now we must deduce the scalings for the remaining fields. Firstly we consider the scalar vielbeins \( \Pi_i^{\alpha} \). Choose a gauge such that \( \Pi_{0}^{\alpha} = 0 \). Then from the requirement that the gauge kinetic term

\[
\Pi_A^i \Pi_B^j \Pi_C^i \Pi_D^j \star F_{(2)}^{AB} \wedge F_{(2)}^{CD}
\]

does not vanish or diverge, we deduce that \( \Pi_A^0 \) and \( \Pi_A^{\alpha} \) must scale like \( k \) whilst \( \Pi_0^6 \) must scale like \( k^{-4} \). Combining this with the requirement that \( T_{AB} \) be unimodular we arrive at the following parametrisation\(^{26}\) for the scalars:

\[
T_{AB} \equiv \Pi^{-1}_A \Pi^{-1}_B \to \begin{pmatrix}
k^8 \Phi^{-1} + k^8 \Phi \frac{1}{4} M^{CD} \chi_C \chi_D & -k^3 \Phi \frac{1}{4} M^{BC} \chi_C \\
k^3 \Phi \frac{1}{4} M^{\bar{A} \bar{C}} \chi_{\bar{C}} & k^{-2} \Phi \frac{1}{4} M^{\bar{A} \bar{B}}
\end{pmatrix},
\]

\[
T_{AB}^{-1} \equiv \Pi^i_A \Pi^i_B \to \begin{pmatrix}
k^{-8} \Phi & k^{-3} \Phi \chi_B \\
k^{-3} \Phi \chi_A & k^{2} \Phi \frac{1}{4} M_{\bar{A} \bar{B}}^{-1} + k^{2} \Phi \chi_{\bar{A}} \chi_{\bar{B}}
\end{pmatrix}.
\]

The scalar vielbeins are thus

\[
\Pi_0^0 = k^{-4} \Phi \frac{1}{4}, \quad \Pi_0^{\alpha} = 0, \quad \Pi_A^0 = k \Phi \frac{1}{4} \chi_{\bar{A}}, \quad \Pi_A^{\alpha} = k \Phi^{-\frac{1}{4}} \pi_A^{\alpha},
\]

\(^{26}\) These scalings correct the scalings of [25], though since the authors there immediately perform a truncation to the \( \mathcal{N} = 1 \) theory, c.f. section 2.3.2, these incorrect entries are discarded.
\[ \Pi^{-1}_{00} = k^4 \Phi^{-1}, \quad \Pi^{-1}_{\alpha} = -k^4 \Phi^{\frac{1}{4}} \pi^{-1}_{-\alpha} \bar{\chi}_{\bar{A}}, \quad \Pi^{-1}_{\bar{A}} = 0, \]
\[ \Pi^{-1}_{\bar{A}} = k^{-1} \Phi^{\frac{1}{4}} \pi^{-1}_{-\alpha}, \]  
(2.12)

where the \( \pi_{\bar{A}}^\alpha \) are vielbeins for the unimodular matrix of scalars

\[ M^{\bar{A}\bar{B}} = \pi^{-1}_{\alpha} \bar{A}^{\alpha}_{\bar{B}}. \]  
(2.13)

The scalar field strength becomes

\[ p^{(1)}_{\alpha\beta} \equiv \pi^{-1}_{(\alpha} \left( \delta_{\bar{B}d} + \tilde{g} \bar{A}_{(1)\bar{B}} \right) \pi^{\gamma}_{\beta} \bar{\chi}_{\bar{B}}. \]  
(2.14)

Now that we have the scalings of the scalar vielbeins, we may tackle the remaining kinetic terms. The 3-form gauge fields \( S^{(3)}_{A} \) have kinetic term

\[ \Pi^{-1}_{i} \Pi^{-1}_{i} S_{(3)A} \wedge S_{(3)B}, \]  
(2.15)

and so we require

\[ S_{(3)\bar{A}} \rightarrow k S_{(3)\bar{A}}, \quad S_{(3)0} \rightarrow k^{-4} H_{(3)}. \]  
(2.16)

We find corresponding \( SO(4) \)-covariant derivatives

\[ \bar{D} S_{(3)\bar{A}} \equiv d S_{(3)\bar{A}} + \tilde{g} \bar{A}_{(1)\bar{B}} \wedge S_{(3)B}, \]
\[ \bar{D} H_{(3)} \equiv d H_{(3)}. \]  
(2.17)

Next we consider the fermions. A cursory examination of the kinetic terms of the gravitino:

\[ \bar{\psi}_\mu \Gamma^{\mu\nu} \nabla_\nu \psi_\mu, \]  
(2.18)

and the gaugino:

\[ \bar{\lambda}_i \Gamma^{\mu} \nabla_\mu \lambda_i \]  
(2.19)

reveals that the fermions \( \psi_\mu \) and \( \lambda_\alpha \) do not scale. The gauginos \( \lambda_i \) do carry an \( SO(5) \)-index, and so should split them as \( (\lambda_0, \lambda_\alpha) \) - however we can eliminate \( \lambda_0 \) using equation (1.75), \( i.e. \) by writing

\[ \lambda_0 = -\gamma_{0\alpha} \lambda_\alpha. \]  
(2.20)
Finally we remark that we have not applied any scalings to the Cartan-Killing metrics of either the gauge or composite groups.

We will wait until section 2.4 to apply these scalings to the $S^4$ reduction ansatz of section 1.5.2; for now we just check that the contracted theory really is that of the $S^3$ reduction. The metric on the internal manifold $S^4$ is of the form

$$ds^2_{(4)} = \frac{1}{g^2} T_{AB}^{-1} DY^A DY^B,$$

where we have embedded $S^4$ in $\mathbb{R}^5$ as

$$\mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 = 1.$$  

From the scalings of $g$ and $T_{AB}^{-1}$ we deduce that $\mu_0 \to k^5 \mu_0$, so the Wigner-In"on"u contraction takes $S^4$ to $\mathbb{R} \times S^3 \sim S^1 \times S^3$, as required.

### 2.2 Supersymmetry Transformations

Before we derive the $D = 7$ $SO(4)$-gauged (or equivalently $SO(2,2)$-gauged) action, we derive the associated $N = 2$ supersymmetry transformations. To do this, we simply apply the scalings of section 2.1 to the supersymmetry transformations (1.82) and (1.86), then take the limit $k \to 0$. There is only one complication: to decompose the scalar vielbeins $\Pi^A_i$ into $SO(4)$ representations, we partially picked a gauge $\Pi^A_0 = 0$.

The supersymmetry transformations do not respect this choice of gauge:

$$\delta \Pi^A_0 = \frac{1}{4} \epsilon_0^\gamma (\gamma_0 \gamma_\alpha + \delta_\alpha_\beta) \lambda_\beta \pi^0_0 \neq 0.$$  

This is not unexpected; anticommutators of supersymmetries generate gauge and composite transformations which are not compatible with a gauge fixing. We must make a compensating gauge transformation

$$\delta \Pi^A_i = \Lambda_A^B \Pi^B_i + \Pi_A^j \Lambda_j^i.$$  

We can use either the true gauge transformations $SO(5)^g$ or the composite transformations $SO(5)^c$, or indeed some combination or both. It turns out the simplest choice
is the purely composite transformation
\[ \Lambda_A^B = 0, \quad \Lambda_0^\alpha = -\frac{1}{4} \bar{c}_0 (\gamma_\alpha \gamma_\beta + \delta_{\alpha \beta}) \lambda_\beta, \quad \Lambda_\beta^\alpha = 0, \]  
(2.25)
because the only\(^{27}\) other supersymmetry transformation affected is the \(\delta \Pi^{\alpha}_A\) transformation, where it neatly cancels an existing term. It is useful to examine the effects of our scalings on the contractions of \(F_{AB}^{(2)}\) and \(S_{(3)A}\) with the scalar vielbeins before we perform the full calculation. We find that
\[
\Pi_A^i \Pi_B^j F_{(2)}^{AB} \rightarrow \begin{pmatrix}
0 & \Phi^{\frac{3}{2}} \pi^A_{\alpha} \left( \tilde{F}^{0A}_{(2)} - \chi_B \tilde{F}_{AB}^{(2)} \right) \\
-\Phi^{\frac{3}{2}} \pi^A_{\alpha} \left( \tilde{F}^{0A}_{(2)} - \chi_B \tilde{F}_{AB}^{(2)} \right) & \Phi^{\frac{3}{4}} \pi^A_{\alpha} \pi^B_{\beta} \tilde{F}_{AB}^{(2)}
\end{pmatrix},
\]  
(2.26)
and
\[
\Pi_i^{-1} S_{(3)A} \rightarrow \left( \Phi^{-\frac{1}{2}} H_{(3)}, \Phi^{\frac{3}{2}} \pi^{-1} \tilde{A} \left( \tilde{S}_{(3)A} - \chi_A H_{(3)} \right) \right). 
\]  
(2.27)
We also give the decompositions of the scalar field strength (1.79):
\[
P_{(1)ij} \rightarrow \begin{pmatrix}
\frac{1}{2} \Phi^{-1} d\Phi \\
\frac{1}{2} \Phi^{\frac{3}{2}} \pi^{-1} \tilde{A} \left( \tilde{D}_\chi \tilde{A} - \tilde{g} \tilde{A}_{(1)0\bar{A}} \right)
\end{pmatrix}, \quad p_{(1)\alpha \beta} - \frac{1}{3} \Phi^{-1} d\Phi \delta_{\alpha \beta}
\]  
(2.28)
and the \(SO(5)c\) connection (1.85):
\[
Q_{(1)ij} \rightarrow \begin{pmatrix}
0 \\
\frac{1}{2} \Phi^{\frac{3}{2}} \pi^{-1} \tilde{A} \left( \tilde{D}_\chi \tilde{A} - \tilde{g} \tilde{A}_{(1)0\bar{A}} \right)
\end{pmatrix}, \quad q_{(1)\alpha \beta}.
\]  
(2.29)
Here \(p_{(1)\alpha \beta}\) is as in equation (2.14), and \(q_{(1)\alpha \beta}\) is the \(SO(4)c\) connection
\[
q_{\alpha \beta} \equiv \pi^{-1} \tilde{A} \left( \delta^B_A d + \tilde{g} \tilde{A}_{(1)0\bar{A}} \right) \pi^B \gamma^a \delta_{\beta \gamma}.
\]  
(2.30)
Given the form of these decompositions, it is convenient \(^{24}\) to repackage some of the fields into new tensors \(G_{(1)\bar{A}}, G_{(2)}\), and \(G_{(3)\bar{A}}\), which we define as:
\[
G_{(1)\bar{A}} \equiv \tilde{D}_\chi \tilde{A} - \tilde{g} \tilde{A}_{(1)0\bar{A}}, \quad G_{(2)} \equiv \tilde{F}_{(2)} - \chi_B \tilde{F}_{AB}^{(2)},
\]  
(2.31)
\[
G_{(3)\bar{A}} \equiv \tilde{F}_{(2)}^{0\bar{A}} - \chi_B \tilde{F}_{AB}^{(2)}.
\]  
(2.32)
\(^{27}\)There is also a contribution to the gaugino variation \(\delta \lambda_\alpha\), but we neglect this as it is of third-order in the fermions.
Here this is just an algebraic convenience, but one which will have real significance in sections 2.3 and 2.3.2: these fields will diagonalise the kinetic terms for the fields \( \tilde{A}^{0 \hat{A}} \), \( \chi_{\hat{A}} \) and \( S_{(3), \hat{A}} \) in the \( SO(4) \)-gauged action; they will also be precisely the terms we truncate to obtain the \( \mathcal{N} = 1 \) theory.

Now we are ready to apply these results to the supersymmetry transformations. For the fermions, the Wigner-In"onu contraction gives

\[
\delta \psi_\mu = \tilde{D}_\mu \epsilon - \frac{1}{4} \gamma_{0 \alpha} \epsilon \Phi^{\hat{5} \pi^{-1} \alpha} \tilde{A}_{(1) \hat{A}} - \frac{1}{20} \gamma_{\pi^{-1} \alpha} \tilde{A}_{\hat{B}} \eta_{\hat{A} \hat{B}} \Phi^{\hat{1}} \Gamma_{\mu} \epsilon \\
- \frac{1}{40 \sqrt{2}} \left( \Gamma_{\mu}^{\nu \rho} - 8 \delta_{\nu \mu} \Gamma_{\rho} \right) \left( \gamma_{\alpha \beta} \epsilon \Phi^{\hat{5} \pi^{-1} \alpha} \tilde{A}_{\hat{B}} \gamma_{\frac{1}{2} A} \tilde{F}_{\nu \rho} + 2 \gamma_{0 \alpha} \epsilon \Phi^{\hat{5} \pi^{-1} \alpha} \tilde{F}_{\nu \rho} \right) \\
- \frac{1}{60} \left( \Gamma_{\mu}^{\nu \rho \sigma} - 9 \delta_{\nu \rho} \Gamma_{\mu}^{\rho \sigma} \right) \left( \gamma_{\alpha} \epsilon \Phi^{\hat{5} \pi^{-1} \alpha} \tilde{F}_{\nu \rho \sigma} + \gamma_{\alpha} \epsilon \Phi^{\hat{5} \pi^{-1} \alpha} \tilde{F}_{\nu \rho \sigma} \right),
\]

(2.34)

\[
\delta \lambda_\alpha = \frac{1}{16 \sqrt{2}} \Gamma_{\mu \nu} \left( \gamma_{\beta \gamma} \gamma_\alpha - \frac{1}{3} \gamma_{\beta \gamma} \gamma_\alpha \right) \epsilon \Phi^{\hat{5} \pi^{-1} \alpha} \tilde{A}_{\hat{B}} \gamma_{\frac{1}{2} A} \tilde{F}_{\mu \nu} + 2 \left( \gamma_{0 \beta} \gamma_\alpha - \frac{1}{3} \gamma_{0 \beta} \gamma_\alpha \right) \epsilon \Phi^{\hat{5} \pi^{-1} \alpha} \tilde{A}_{\hat{B}} \gamma_{\frac{1}{2} A} \\
- \frac{1}{120} \left( \Gamma_{\mu \nu}^{\rho} - 4 \delta_{\rho \mu} \Gamma_{\nu} \right) \left( \gamma_{\alpha \beta} \epsilon \Phi^{\hat{5} \pi^{-1} \alpha} \tilde{A}_{\hat{B}} \gamma_{\frac{1}{2} A} \tilde{F}_{\mu \nu} \right) + \frac{1}{2} \left( \tilde{G} \gamma_\beta \gamma_\alpha - \frac{1}{8} \Phi^{-1} \Gamma_{\mu \nu} \tilde{A}_{\hat{B}} \gamma_{\frac{1}{2} A} \tilde{F}_{\mu \nu} \right) + \frac{1}{2} \Gamma_{\mu \nu} \gamma_{0 \beta} \gamma_\alpha \\
+ \frac{1}{2} \Gamma_{\mu \nu} \gamma_{0 \beta} \gamma_\alpha \left( \gamma_{\lambda} \epsilon \Phi^{\hat{5} \pi^{-1} \alpha} \tilde{A}_{\hat{B}} \gamma_{\frac{1}{2} A} \tilde{F}_{\mu \nu} \right) + \frac{1}{4} \Gamma_{\mu \nu} \gamma_{0 \beta} \gamma_\alpha \left( \gamma_{\lambda} \epsilon \Phi^{\hat{5} \pi^{-1} \alpha} \tilde{A}_{\hat{B}} \gamma_{\frac{1}{2} A} \tilde{F}_{\mu \nu} \right),
\]

(2.35)

where the \( SO(4) \)-covariant derivative \( \tilde{D} \) acts on the supersymmetry parameter \( \epsilon \) as

\[
\tilde{D} \epsilon \equiv d \epsilon + \frac{1}{4} \omega_{\alpha \beta} \Gamma^{\alpha \beta} \epsilon + \frac{1}{4} \eta_{\alpha \beta} \gamma^{\alpha \beta} \epsilon.
\]

(2.36)

For the bosonic sector the Wigner-In"onu contraction is similarly straightforward. We find that:

\[
\delta \epsilon^{\alpha} = \frac{1}{2} \Gamma_{\alpha \beta} \psi_\mu, \quad \delta \tilde{A}^{0 \hat{A}}_\mu = \frac{1}{4 \sqrt{2}} \Phi^{-\hat{5} \epsilon} \left( 2 \gamma_{0 \alpha} \Psi_\mu + \Gamma_\mu \left( \gamma_{\beta \gamma} \gamma_\alpha + \gamma_{0 \alpha} \gamma_\beta \right) \lambda_\beta \right) \pi^{-\hat{1} \alpha} \hat{A} \\
+ \frac{1}{4} \Phi^{\hat{1} \epsilon} \left( 2 \gamma_{0 \beta} \Psi_\mu + \Gamma_\mu \left( \gamma_{\gamma \alpha} \gamma_\beta - \gamma_{0 \gamma} \gamma_\beta \right) \lambda_\gamma \right) \pi^{-\hat{1} \beta} \hat{B} \chi_\beta, \\
\]

(2.37)

\[
\delta \tilde{A}^{\hat{A} B}_\mu = \frac{1}{4 \sqrt{2}} \Phi^{\frac{1}{2} \epsilon} \left( 2 \gamma_{0 \alpha} \Psi_\mu + \Gamma_\mu \left( \gamma_{\gamma \alpha} \gamma_\beta - \gamma_{0 \gamma} \gamma_\beta \right) \lambda_\gamma \right) \pi^{-\hat{1} \alpha} \hat{A} \pi^{-\hat{1} \beta} \hat{B},
\]

(2.38)

\[
\delta \tilde{A}^{\hat{A} B}_\mu = \frac{1}{4 \sqrt{2}} \Phi^{\frac{1}{2} \epsilon} \left( 2 \gamma_{0 \alpha} \Psi_\mu + \Gamma_\mu \left( \gamma_{\gamma \alpha} \gamma_\beta - \gamma_{0 \gamma} \gamma_\beta \right) \lambda_\gamma \right) \pi^{-\hat{1} \alpha} \hat{A} \pi^{-\hat{1} \beta} \hat{B},
\]

(2.39)
\[
\delta \Phi = - \Phi \epsilon \gamma_\alpha \lambda_\alpha ,
\]
\[
\delta \chi_\bar{A} = \frac{1}{4} \Phi^\frac{i}{2} \epsilon \gamma_0 (\gamma_\alpha \gamma_\beta + \delta_{\alpha \beta}) \lambda_\beta \pi_A^\alpha ,
\]
\[
\delta \pi_\bar{A}^\alpha = \frac{1}{4} \epsilon \left( \gamma_\alpha \lambda_\beta + \gamma_\beta \lambda_\alpha - \frac{1}{2} \gamma_\alpha \lambda_\gamma \delta_{\alpha \beta} \right) \pi_\bar{A}^\beta ,
\]
\[
\delta H_{\mu \nu \rho} = - \frac{3}{4 \sqrt{2}} \Phi \frac{1}{4} \epsilon (2 \gamma_0 \gamma_\alpha \psi_\mu + \Gamma_\mu (\gamma_\gamma \gamma_0 \alpha \beta + \gamma_0 \alpha \beta \gamma_\gamma) \lambda_\gamma) \pi_\bar{A}^\alpha \pi_B^\beta \tilde{F}_{\nu \rho}^{\bar{A} B} ,
\]
\[
\delta S_{\mu \nu \rho \bar{A}} = - \frac{3}{4 \sqrt{2}} \Phi \frac{1}{4} \epsilon (2 \gamma_0 \gamma_\alpha \psi_\mu + \Gamma_\mu (\gamma_\gamma \gamma_0 \alpha \beta + \gamma_0 \alpha \beta \gamma_\gamma) \lambda_\gamma) \pi_\bar{A}^\alpha \pi_B^\beta \pi_C^\gamma \tilde{F}_{\nu \rho}^{\bar{A} B C}.
\]

2.2.1 \( H_{(3)} \) as a Field Strength

Varying the action (1.76), we obtain the following field equation for the 3-forms \( S_{(3)A} \):

\[
DS_{(3)A} = g_{\alpha \beta} \Pi^{-1}_{\mu} B_{\mu} \Pi^{-1}_{\nu} C_{\nu} S_{(3)C} + \frac{1}{4} \epsilon_{AC_1 ... C_4} F_{(2)}^{C_1 C_2} \wedge F_{(2)}^{C_3 C_4} + g_{\alpha \beta} \Pi^{-1}_{\mu} B_{\mu} * K_{(3)i} ,
\]

where \( K_{(3)i} \) is the fermionic contribution

\[
K_{(3)i} = \left[ \frac{1}{4} \bar{\psi}^\sigma (\Gamma_{\sigma \mu \nu \tau} + 6 g_{\sigma \mu} \Gamma_\nu g_{\rho \tau}) \gamma_i \psi^\tau - \frac{1}{2} \bar{\psi}^\sigma (\Gamma_{\sigma \mu \rho} + 3 g_{\sigma \mu} \Gamma_\nu) \lambda_i \right. \\
\left. - \frac{1}{4} \bar{\lambda}_j \Gamma_{\mu \rho \nu} \gamma_i \lambda_j \right] dx^\mu \wedge dx^\nu \wedge dx^\sigma .
\]

Now consider the fate of the 3-form \( S_{(3)0} \). Applying our scalings and taking the limit \( k \to 0 \), its equation of motion reduces to a simple Bianchi identity

\[
dH_{(3)} = \frac{1}{4} \epsilon_{C_1 ... C_4} \tilde{F}_{C_1 C_2} \wedge \tilde{F}_{C_3 C_4} .
\]
The expression on the right hand side is the exterior derivative of a Chern-Simons 3-form $\omega^{(3)}$. Stripping off the exterior derivative, Poincaré’s lemma tells us that the $SO(4)_g$ singlet $H^{(3)}$ is the sum of $\omega^{(3)}$ and the field strength of some new 2-form potential $B^{(2)}$:

$$H^{(3)} = dB^{(2)} + \omega^{(3)}, \quad (2.48)$$

where [24]

$$\omega^{(3)} = \frac{1}{4} \varepsilon^{C_1 \ldots C_4} \left( F^{C_1 C_2}_{(2)} \wedge \tilde{A}^{C_3 C_4}_{(1)} - \frac{1}{3} \tilde{g} A_{1(1)}^{C_1 C_2} \wedge \tilde{A}^{C_3 A}_{(1)} \wedge \tilde{A}^{B C_4}_{(1)} \eta_{AB} \right). \quad (2.49)$$

We can explain the origins of this metamorphosis in terms of the notion of self-duality in odd dimensions. In $D = 7$, the ungauged $\mathcal{N} = 2$ theory [123] includes 2-form potentials $B^{(2)A}$, where the index $A = 1, \ldots, 5$ now indicates that the fields transform in the fundamental representation of a global $SL(5)$ symmetry. In the ungauged theory we are free to dualise $B^{(2)A}$ to obtain 3-form potentials $S^{(3)A}$ [125]. Now, when we attempt to gauge an $SO(5)$ symmetry we are faced with a technical problem [31]: we could write down the kinetic term

$$\sim - \frac{1}{2} \Pi^{-1A} \Pi^{-1B} * DB^{(2)A} \wedge DB^{(2)B}, \quad (2.50)$$

but then the $A^{AB}_{(1)}$ terms hidden in the $SO(5)$-covariant derivatives break the symmetry

$$B^{(2)} \rightarrow B^{(2)} + d\Lambda^{(1)}, \quad (2.51)$$

which would alter the number of bosonic degrees of freedom. The resolution [118, 119, 121] of this dilemma is to first replace the $B^{(2)A}$ with their duals $S^{(3)A}$ and then gauge, writing terms in the action

$$\sim - \frac{1}{2} \Pi^{-1A} \Pi^{-1B} * S^{(3)A} \wedge S^{(3)B} + \frac{1}{2g} \eta^{AB} S^{(3)A} \wedge DS^{(3)B}. \quad (2.52)$$

This leads to the first-order ‘self-duality’ structure we see in equation (2.45). Differentiating this equation a second time reveals a (topological) mass term for the $S^{(3)A}$. On-shell, these massive fields have the same number of degrees of freedom as the ungauged two forms with the symmetry (2.51). In the Wigner-Inönü limit, $S^{(3)0}$ becomes a singlet again and the topological mass term vanishes, and consistency demands that the symmetry (2.51) be restored.
It is usual to give the supersymmetry transformation for the potentials rather than their field strengths. The transformation $\delta B_2$ can be deduced from the known transformation for $H_3$ by calculating $\delta \omega_3$. Under a general transformation it can be shown that

$$\delta \omega_3 = \frac{1}{4} \epsilon \bar{C}_1 \cdots \bar{C}_4 \left\{ d \left( \delta \tilde{A}_{(1)}^1 \bar{C}_2 \wedge \tilde{A}_{(1)}^2 \bar{C}_4 \right) + 2 \tilde{F}_{(2)}^1 \bar{C}_2 \wedge \delta \tilde{A}_{(1)}^2 \bar{C}_4 \right\}. \tag{2.53}$$

The first term in equation (2.53) is a total derivative and will presumably be a part of the $\delta B_2$ variation. We must have that the difference of the second term and the variation (2.43) be exact. Substituting in the supersymmetry transformation (2.39) for $\delta \tilde{A}_{(1)}^{AB}$, this second term becomes

$$\frac{1}{16 \sqrt{2}} \Phi^{1/2} \epsilon \bar{C}_1 \cdots \bar{C}_4 \left( 2 \bar{\epsilon} \gamma^{\alpha \beta} \psi_{[\mu} + \bar{\epsilon} \Gamma_{[\mu} (\gamma^{\gamma \alpha \beta} - \gamma^{\alpha \beta} \gamma^\gamma) \lambda_{\gamma]}) \pi^{-1}_A \bar{C}_1 \pi^{-1}_B \bar{C}_2 \tilde{F}_{\nu\rho} \bar{C}_3 \bar{C}_4 \right). \tag{2.54}$$

Because $\pi \bar{C}_A$ is unimodular, we can introduce the Levi-Civita symbol $\epsilon_{\alpha \beta \gamma \delta}$ via the relation

$$\epsilon \bar{C}_1 \bar{C}_2 \bar{C}_3 \bar{C}_4 = \pi \bar{C}_1 \alpha \pi \bar{C}_2 \beta \pi \bar{C}_3 \gamma \pi \bar{C}_4 \delta \epsilon_{\alpha \beta \gamma \delta}. \tag{2.55}$$

We also have that

$$\epsilon_{\alpha \beta \gamma \delta} = 4! \gamma_{0 \alpha \beta \gamma \delta}. \tag{2.56}$$

We can use equations (2.55) and (2.56) to write the first term in equation (2.54) as

$$-\frac{3}{4 \sqrt{2}} \Phi^{1/2} \left( 2 \bar{\epsilon} \gamma_{0 \alpha \beta} \psi_{[\mu} + \bar{\epsilon} \Gamma_{[\mu} (\gamma^{\gamma \alpha \beta} - \gamma_{0 \alpha \beta} \gamma^\gamma) \lambda_{\gamma]}) \pi_A \alpha \pi_B \beta \tilde{F}^{AB}_{\nu \rho} \right). \tag{2.57}$$

This is precisely the variation $\delta H_3$, c.f. equation (2.43). It is now apparent that $\delta H_3 - \delta \omega_3$ is indeed a total derivative, and we can immediately write down the new supersymmetry variation

$$\delta B_{\mu \nu} = \frac{3}{4 \sqrt{2}} \Phi^{1/2} \left( 2 \bar{\epsilon} \gamma_{0 \alpha \beta} \psi_{[\mu} + \bar{\epsilon} \Gamma_{[\mu} (\gamma^{\gamma \alpha \beta} - \gamma_{0 \alpha \beta} \gamma^\gamma) \lambda_{\gamma]}) \pi_A \alpha \pi_B \beta A_{\nu}^{AB} \right). \tag{2.58}$$

An unusual property of the $SO(5)$-gauged theory was that the supersymmetry variations of the fluxes $S_3 A$ included covariant derivatives of the supersymmetry parameter $\epsilon$. It was crucial for this discussion that these terms vanish under the Wigner-Inonu scalings when $A = 0$. 50
2.3 The $SO(p,4-p)$-Gauged Theory

The Pauli reduction on $\mathcal{H}^{p,4-p}$ yields an $SO(p,4-p)$-gauged, maximally supersymmetric action in $D=7$. One might assume that we could obtain this by directly applying our Wigner-Inönü scalings to the action (1.76), but in fact this leads to divergences. For example, the topological mass term for $S_{(3)0}$ has

$$
\int d^7x \frac{1}{2g} S_{(3)0} \wedge DS_{(3)0} \rightarrow k^{-10} \int d^7x \frac{1}{2g} H_{(3)} \wedge dH_{(3)},
$$

(2.59)

and there are similar $O(k^{-10})$ divergences in the other two topological terms. This apparent pathology is because of the field mutation\(^{28}\) we discussed in section 2.2.1: in the action (1.76) we vary $S_{(3)0}$, but in the new action we should be varying $B_{(2)}$.

The appropriate course of action is to apply our scalings to the $SO(5)$-gauged field equations and then write down an equivalent Lagrangian. All divergences at the level of the field equations, provided one first substitutes in equation (2.45) wherever a $\bar{D}S_{(3)0}$ term appears\(^{29}\). This procedure was carried out in [24], where the $SO(p,4-p)$-gauged Lagrangian was found to be

$$
\mathcal{L}_{(7)} = R * 1 - \frac{5}{16} \Phi^{-2} * d\Phi \wedge d\Phi - *p_{(1)\alpha\beta} \wedge p_{(1)\alpha\beta} - \frac{1}{2} \Phi^{-1} * H_{(3)} \wedge H_{(3)}
$$

$$
- \frac{1}{2} \Phi^{-1} \pi^{A}_\alpha \pi^{B}_\beta \pi^{C}_\alpha \pi^{D}_\alpha * \tilde{F}_{(2)}^{AB} \wedge \tilde{F}_{(2)}^{CD} - \frac{1}{2} \Phi^{-1} \pi^{A}_\alpha \pi^{B}_\alpha \pi^{C}_\beta \pi^{D}_\alpha * G_{(3)\tilde{A}} \wedge G_{(3)\tilde{B}}
$$

$$
- \Phi^{-1} \pi^{A}_\alpha \pi^{B}_\alpha \pi^{C}_\alpha \pi^{D}_\beta \pi^{E}_\beta \pi^{F}_\alpha * G_{(2)\tilde{G}} \wedge G_{(2)\tilde{H}} - \Phi^{-1} \pi^{A}_\alpha \pi^{B}_\alpha \pi^{C}_\beta \pi^{D}_\alpha \pi^{E}_\beta * G_{(1)\tilde{A}} \wedge G_{(1)\tilde{B}}
$$

$$
+ \frac{1}{2g} \eta^{ABC} \tilde{S}_{(3)\tilde{A}} \wedge \tilde{D} \tilde{S}_{(3)\tilde{B}} - \tilde{S}_{(3)\tilde{A}} \wedge \tilde{A}_{(1)}^{0\tilde{A}} \wedge H_{(3)} + \frac{1}{g} \varepsilon_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} \eta^{\tilde{A}\tilde{E}} \tilde{S}_{(3)\tilde{E}} \wedge \tilde{F}_{(2)}^{0\tilde{B}} \wedge \tilde{F}_{(2)}^{\tilde{C}\tilde{D}}
$$

$$
+ \frac{1}{2} \varepsilon_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} H_{(3)} \wedge \tilde{F}_{(2)}^{\tilde{A}\tilde{B}} \wedge \tilde{A}_{(1)}^{0\tilde{C}} \wedge \tilde{A}_{(1)}^{0\tilde{D}} - \frac{1}{g} \tilde{\Omega}_{(7)} - \hat{V} * 1 + \mathcal{L}_f,
$$

(2.60)

where $H_{(3)}$ obeys equation (2.48) and we treat $B_{(2)}$ as the fundamental field. Here we have a new scalar potential $\hat{V}$, given by

$$
\hat{V} = \frac{1}{2} g^2 \Phi^{\frac{1}{2}} \pi^{A}_\alpha \pi^{B}_\beta \pi^{C}_\alpha \pi^{D}_\beta (2\eta_{\tilde{A}\tilde{B}} \eta_{\tilde{C}\tilde{D}} - \eta_{\tilde{A}\tilde{C}} \eta_{\tilde{B}\tilde{D}}),
$$

(2.61)

and a new pure Yang-Mills Chern-Simons term $\tilde{\Omega}$. We extend the work [24] by pre-

\(^{28}\)The ungauged limit $g \rightarrow 0$ also leads to divergences at the level of the action - this time in all the topological terms, rather than just those involving $S_{(3)0}$ and its couplings [24]. In this case all five 3-forms $S_{(3)\tilde{A}}$ become field strengths of some potentials $B_{(2)\tilde{A}}$.

\(^{29}\)This cancellation is not obvious in form notation unless one exploits the fact that antisymmetrising over five or more $SO(4)_g$ indices always gives zero, e.g. $\varepsilon_{[\tilde{A}\tilde{B}\tilde{C}\tilde{D}\tilde{E}]} \equiv 0$. 

51
senting the fermionic part of the Lagrangian $\mathcal{L}_f$:

\[
\mathcal{L}_f = \left\{ -\bar{\psi}_\mu \Gamma^{\mu \rho} \nabla_\nu \psi_\rho - \bar{\lambda}_\alpha \Gamma^{\mu} (\delta_{\alpha \beta} + \delta_{\alpha \beta}) \nabla_\mu \lambda_\beta + \frac{1}{4} \bar{\psi}_\mu \Phi^2 \pi^{-1} \bar{\lambda}_\alpha \lambda_\beta - \frac{1}{4} \bar{\psi}_\mu \Phi^2 \pi^{-1} \bar{\lambda}_\alpha \lambda_\beta \\
+ \frac{1}{4} \bar{\psi}_\mu \Phi^2 \pi^{-1} \bar{\lambda}_\alpha \lambda_\beta + \frac{1}{4} \bar{\psi}_\mu \Phi^2 \pi^{-1} \bar{\lambda}_\alpha \lambda_\beta \right\} \ast 1
\]

with fermionic couplings

\[
\tilde{L}_{(1)00} = \bar{\psi}_\mu \Gamma_\mu \gamma_\alpha \lambda_\alpha \ dx^\mu ,
\]

\[
\tilde{L}_{(1)0\alpha} = -\frac{1}{2} \bar{\psi}_\mu \Gamma_\mu \gamma_0 (\delta_{\alpha \beta} + \gamma_\alpha \gamma_\beta) \lambda_\beta \ dx^\mu ,
\]

\[
\tilde{L}_{(1)\alpha\beta} = -\bar{\psi}_\mu \Gamma_\mu \gamma_0 \lambda_\beta \ dx^\mu ,
\]

\[
\tilde{K}_{(3)0} = \left[ \frac{1}{4} \bar{\psi}_\rho \left( \Gamma_{\sigma \mu \nu \rho} + 6 g_{\sigma \mu} \Gamma_{\nu} g_{\rho \tau} \right) \gamma_0 \psi_\tau + \frac{1}{2} \bar{\psi}_\rho \left( \Gamma_{\sigma \mu \nu} + 3 g_{\sigma \mu} \Gamma_{\nu \rho} \right) \gamma_0 \lambda_\alpha \\
+ \frac{1}{4} \bar{\lambda}_\sigma \Gamma_{\mu \rho} \gamma_0 \lambda_\beta \right] \ dx^\mu \wedge dx^\nu \wedge dx^\rho ,
\]

\[
\tilde{K}_{(3)\alpha} = \left[ \frac{1}{4} \bar{\psi}_\rho \left( \Gamma_{\sigma \mu \nu \rho} + 6 g_{\sigma \mu} \Gamma_{\nu} g_{\rho \tau} \right) \gamma_0 \psi_\tau - \frac{1}{2} \bar{\psi}_\rho \left( \Gamma_{\sigma \mu \nu} + 3 g_{\sigma \mu} \Gamma_{\nu \rho} \right) \lambda_\alpha \\
- \frac{1}{4} \bar{\lambda}_\sigma \Gamma_{\mu \rho} \gamma_0 \lambda_\beta \gamma_\alpha \gamma_\beta + \gamma_\alpha \gamma_\beta \gamma_\gamma - \delta_{\alpha \beta} \gamma_\gamma \right] \ dx^\mu \wedge dx^\nu \wedge dx^\rho ,
\]

\[
J_{(2)\alpha\beta} = -\frac{1}{\sqrt{2}} \left[ \frac{1}{2} \bar{\psi}_\rho \left( \Gamma_{\rho \mu \nu} - 2 g_{\mu \nu} g_{\rho \sigma} \right) \gamma_{\alpha \beta} \psi_\sigma + \bar{\psi}_\rho \Gamma_{\rho \mu} \psi_\sigma \gamma_0 \lambda_\beta \\
+ \frac{1}{8} \bar{\lambda}_\sigma (\gamma_\delta \gamma_\alpha \gamma_\beta + \gamma_\alpha \beta \gamma_\gamma \gamma_\delta + 2 \gamma_\alpha \beta \gamma_\gamma \delta) \Gamma_{\mu \nu} \lambda_\delta \right] \ dx^\mu \wedge dx^\nu ,
\]

\[
J_{(2)0\alpha} = -\frac{1}{\sqrt{2}} \left[ \frac{1}{2} \bar{\psi}_\rho \left( \Gamma_{\rho \mu \nu} - 2 g_{\mu \nu} g_{\rho \sigma} \right) \gamma_0 \alpha \psi_\sigma - \frac{1}{2} \bar{\psi}_\sigma \Gamma_{\rho \mu} \gamma_{0 \alpha} \lambda_\beta \right]
\]

52
\( + \frac{1}{8} \lambda_\beta (\gamma_\gamma \gamma_\alpha \gamma_\beta + 2 \gamma_\alpha \gamma_\beta \gamma_\gamma + \gamma_\beta \gamma_\alpha \gamma_\gamma) \gamma_\alpha \Gamma_{\mu \nu \lambda} \gamma_\lambda dx^\mu \wedge dx^\nu . \) \tag{2.63}

### 2.3.1 Vacua and Stability

In this section we briefly consider the vacuum structure of our action (2.60). Though it made next to no difference in the derivation of the action, the choice of Cartan-Killing metric \( \eta_{\bar{A} \bar{B}} \) will have a profound effect on the landscape of solutions. We can learn much in particular from the structure of the potential \( \tilde{V} \): setting the scalar vielbeins \( \pi_{\bar{A}}^\alpha \) to be the identity, we see that

\[
S^3: \quad \tilde{V}(\Phi) = -4\tilde{g}^2 \Phi^{\frac{1}{4}}, \\
\mathcal{H}^{3,1}: \quad \tilde{V}(\Phi) = 2\tilde{g}^2 \Phi^{\frac{1}{4}}, \\
\mathcal{H}^{2,2}: \quad \tilde{V}(\Phi) = 4\tilde{g}^2 \Phi^{\frac{1}{4}}, \\
\mathcal{H}^{1,3}: \quad \tilde{V}(\Phi) = 2\tilde{g}^2 \Phi^{\frac{1}{4}}. \tag{2.64}
\]

At fixed values of the scalar \( \Phi \) we see that the sphere reduction leads to a negative cosmological constant and the hyperboloids to positive cosmological constants, as we discussed in section 1.5.3. However the Wigner-In\'onu contraction has introduced runaway directions in this potential, corresponding to \( \Phi \to \infty \) for the sphere and \( \Phi \to 0 \) for the hyperboloids: we must switch on fluxes to stabilise \( \Phi \).

Looking at the exponents of the couplings in (2.60), it is clear we must switch on either \( H_{(3)} \) or \( F_{(2)}^{A\bar{B}} \). Both of these will break some \( D = 7 \) symmetry, so that the vacua are not \( AdS_7 \) or \( dS_7 \) but something more complicated. This is not all bad: this extra flux can exactly balance the cosmological constant to leave us with a \( D = 4 \) Minkowski space, as we will now see.

Firstly we look for an \( \mathcal{H}^{2,2} \) vacua with \( F_{(2)}^{A\bar{B}} \) flux. Varying the action (2.60), one can straightforwardly verify that the following configuration \([25]\) is an exact solution of the \( D = 7 \) theory:

\[
d s_7^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{4\tilde{g}^2} \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) + dy^2, \\
\Phi = 1, \quad \pi_{\bar{A}}^\alpha = \delta_{\bar{A}}^\alpha, \quad A_{(1)}^{12} = -A_{(1)}^{34} = -\frac{1}{2\sqrt{2}\tilde{g}} \cos \theta d\varphi , \tag{2.65}
\]

with all other fields vanishing and where we now take \( \mu = 1, \ldots, 4 \). If this solution is supersymmetric in \( D = 7 \) then, thanks to the consistency of the Pauli reduction,
it must uplift to a supersymmetric solution of type IIA supergravity. The stability of
the solution is then assured. To check this, we will need the non-vanishing vielbein
components
\[ e^a_{\mu} = 1, \quad e^\theta_{\theta} = \frac{1}{2g}, \quad e^\phi_{\theta} = \frac{1}{2g} \sin \theta, \quad e^\phi_{y} = 1, \quad (2.66) \]
and spin-connection components
\[ \omega^{\hat{\theta}\hat{\phi}} = -\omega^{\phi\hat{\theta}} = -\cos \theta \, d\varphi. \quad (2.67) \]

We now substitute this information into the Killing spinor equations, i.e. we evaluate
\[ \delta \psi_{\mu} = 0 \quad \text{and} \quad \delta \lambda_{\alpha} = 0 \]
on this background. The gaugino variations (2.35) reduce to
\[ \delta \lambda_{1} = \frac{1}{2} \tilde{g} \gamma_1 \left( 1 - \frac{1}{2} \Gamma_{\hat{\theta}\hat{\phi}} \gamma_{12} \left[ (1 + \gamma_0) + \frac{1}{5} (1 - \gamma_0) \right] \right) \epsilon = 0, \]
\[ \delta \lambda_{2} = \frac{1}{2} \tilde{g} \gamma_2 \left( 1 - \frac{1}{2} \Gamma_{\hat{\phi}\hat{\theta}} \gamma_{12} \left[ (1 + \gamma_0) + \frac{1}{5} (1 - \gamma_0) \right] \right) \epsilon = 0, \]
\[ \delta \lambda_{3} = -\frac{1}{2} \tilde{g} \gamma_3 \left( 1 - \frac{1}{2} \Gamma_{\hat{\phi}\hat{\theta}} \gamma_{12} \left[ (1 + \gamma_0) - \frac{1}{5} (1 - \gamma_0) \right] \right) \epsilon = 0, \]
\[ \delta \lambda_{4} = -\frac{1}{2} \tilde{g} \gamma_4 \left( 1 - \frac{1}{2} \Gamma_{\hat{\theta}\hat{\phi}} \gamma_{12} \left[ (1 + \gamma_0) - \frac{1}{5} (1 - \gamma_0) \right] \right) \epsilon = 0. \quad (2.68) \]
Together these equations imply, and are solved by, the projection conditions
\[ (1 - \gamma_0) \epsilon = 0, \quad (1 - \Gamma_{\hat{\theta}\hat{\phi}} \gamma_{12}) \epsilon = 0. \quad (2.69) \]

With these constraints, the gravitino transformations will also vanish provided we take
\[ \partial_{\mu} \epsilon = \partial_{\theta} \epsilon = \partial_{\phi} \epsilon = \partial_{y} \epsilon = 0, \]
so that befits a homogeneous geometry. We conclude that the Salam-Sezgin background breaks three-quarters of the total supersymmetry, i.e. that there are 8 preserved supercharges.

The solution with dilaton stabilisation based on an \( H(3) \) flux is similar, albeit with topology \( \mathbb{R}^{1,3} \times S^3 \) rather than \( \mathbb{R}^{1,3} \times S^2 \times S^1 \). It has non-vanishing fields
\[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{g^2} \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 + \sin^2 \theta \sin^2 \psi \, d\psi^2 \right), \]
\[ \Phi = 1, \quad \pi_{A}^\alpha = \delta_{A}^\alpha, \quad B_{(2)} = \frac{2 \sin^2 \theta \cos \psi}{g^2} \, d\theta \wedge d\psi. \quad (2.70) \]
It turns out, however, that this is not a supersymmetric solution; we will not reproduce
the details of the Killing spinor calculation here.
2.3.2 Orbifolds and Truncations

The action (2.60) has a very complicated structure, and consequently it is worth studying the consistent truncations of the $SO(2,2)$-gauged equations of motion. Many such truncations exist, but we are mostly concerned with the one that leaves intact $\mathcal{N} = 1$ supersymmetry. This is given by [25]

$$G_{(1)\bar{A}} = 0, \quad G_{(2)}^{\bar{A}} = 0, \quad G_{(3)\bar{A}} = 0,$$  

(2.71)

and the projection conditions

$$(1 - \gamma_0) \Psi_\mu = 0, \quad (1 + \gamma_0) \lambda_\alpha = 0.$$

(2.72)

We see that this is compatible with the supersymmetry transformations of section 2.2 provided we also impose the constraint

$$(1 - \gamma_0) \epsilon = 0,$$

(2.73)

on the supersymmetry parameter $\epsilon$. We recognise equation (2.73) as being the first of the two projection conditions (2.69) of our solution (2.65). Because we now have minimal supersymmetry in $D = 7$, no further consistent truncation consistent with supersymmetry is possible.

Given the structure of the solution (2.65), one might instead consider a second dimensional reduction to $D = 6$ - this time on $S^1$ - and then seek a consistent truncation to an $\mathcal{N} = (1, 0)$ theory. This was shown to be possible in [25], with the required projection condition equivalent to the second constraint in (2.69). The resulting action is that of the well-known Salam-Sezgin model [126], which has a unique supersymmetric vacuum $\mathbb{R}^{1,3} \times S^2$ [36]. This can be oxidised to give precisely the solution (2.65).

A primary virtue of the Salam-Sezgin model is that it yields a chiral theory in $D = 4$ [37,126]. We would like to know if we can arrive at such a theory without artificially introducing any truncations. Taking our cue from (2.65), we split our coordinates $(x^\mu, y)$ with $\mu = 1, \ldots, 6$. We then orbifold the direction $y$ by identifying

$$y \sim -y.$$

(2.74)

This introduces a $\mathbb{Z}_2$ grading of the $SO(2,2)$-gauged theory, which each field being either parity even or parity odd under the exchange $y \leftrightarrow -y$. We can imagine the
resulting singularity at $y = 0$ corresponding to some 6-dimensional domain wall on which all parity odd fields must vanish. In this way we naturally introduce a truncation. By breaking translation invariance, our domain wall must break at least half supersymmetry; explicitly we have the boundary condition

$$\bar{\epsilon}_1 \Gamma \hat{y} \epsilon_2 = 0.$$ (2.75)

To find the possible truncations, we seek all consistent supersymmetric $Z_2$ gradings of our theory, i.e. all gradings such that the Lagrangian (2.60) is parity even and such that parity is preserved under supersymmetry. The fermions are allowed different parities under the Lorentz and $SO(4)$ gamma matrices. We thus introduce projection operators

$$P_L = \frac{1}{2} (1 + \Gamma \hat{y}) \ , \quad P_R = \frac{1}{2} (1 - \Gamma \hat{y}) \ ,$$

$$P_+ = \frac{1}{2} (1 + \gamma^0) \ , \quad P_- = \frac{1}{2} (1 - \gamma^0) \ ,$$ (2.76)

and write $\Psi_{\mu L} = P_L \Psi_{\mu L}$, etc.. Our boundary condition (2.75) decomposes as

$$\bar{\epsilon}_1^+ \epsilon_2^+ - \bar{\epsilon}_1^L \epsilon_2^L - \bar{\epsilon}_1^R \epsilon_2^R = 0.$$ (2.77)

It proves useful to introduce parity parameters $p, q$, so that e.g. fields with parity $p$ transform as $\phi \rightarrow p \phi$. For notational convenience we shall here write $G_{(1)\alpha}$, etc., to denote the contraction of the form fields with the scalar vielbeins. We find the following results:

<table>
<thead>
<tr>
<th>Parity</th>
<th>Represents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>$F^{\alpha \beta}<em>{\mu y}$, $H</em>{\mu \nu \rho}$, $\Phi$, $\epsilon^a$, $\tilde{\epsilon}^\hat{y}$,</td>
</tr>
<tr>
<td>Odd</td>
<td>$F^{\alpha \beta}<em>{\mu \nu}$, $H</em>{\mu \nu \rho}$, $\epsilon^a$, $\tilde{\epsilon}^\hat{y}$,</td>
</tr>
<tr>
<td>$p$</td>
<td>$\psi^+<em>{\mu L}$, $\psi^+</em>{y R}$, $\lambda^-_{\alpha R}$,</td>
</tr>
<tr>
<td>$-p$</td>
<td>$\psi^-<em>{\mu R}$, $\psi^+</em>{y L}$, $\lambda^+_{\alpha L}$,</td>
</tr>
<tr>
<td>$q$</td>
<td>$\psi^-<em>{\mu L}$, $\psi^-</em>{y R}$, $\lambda^+_{\alpha R}$,</td>
</tr>
<tr>
<td>$-q$</td>
<td>$\psi^-<em>{\mu R}$, $\psi^-</em>{y L}$, $\lambda^-_{\alpha L}$,</td>
</tr>
<tr>
<td>$pq$</td>
<td>$G^\alpha_{\mu y}$, $G_{\mu \nu \rho \alpha}$,</td>
</tr>
<tr>
<td>$-pq$</td>
<td>$G^\alpha_{\mu \nu}$, $G_{\mu \nu \rho \alpha}$,</td>
</tr>
</tbody>
</table>

(2.78)
We have four choices for \((p, q)\), but really these are not all distinct; we can always make an additional parity transformation to take \(p = 1\), without loss of generality. If we take \(q = 1\) we find that the parity even field content defines an \(\mathcal{N} = (2, 0)\) multiplet in \(D = 6\), \textit{i.e.} a chiral theory. This corresponds to a projection \(P_R \epsilon = 0\). If we instead take \(q = -1\) we find a non-chiral \(\mathcal{N} = (1, 1)\) theory. In this case the parity even fermions are not eigenvectors of either \(\gamma_0\) or \(\Gamma^y\); things are neater if we define

\[
P_\uparrow \equiv \frac{1}{2} (1 + \Gamma^y \gamma^0) = P_+ P_L + P_- P_R, \\
P_\downarrow \equiv \frac{1}{2} (1 - \Gamma^y \gamma^0) = P_+ P_R + P_- P_L,
\]

so that our projection condition is \(P_\downarrow \epsilon = 0\). It is interesting that neither multiplet contains \(\tilde{F}^{\alpha \beta}_{\mu \nu}\), which is a key element in the reduction of the Salam-Sezgin model from \(D = 6\) to \(D = 4\). This orbifolding is closely related to the work in \cite{32}.

One quirk of this orbifold analysis is that the requirement that the action (2.60) be even under the parities (2.78) implies that the coupling constant is parity odd. Parity odd constants are not problematic in orbifold constructions \cite{127,128}, and are commonly referred to as non-zero modes\footnote{These should not be confused with the \(\lambda > 0\) eigenvalue modes in Kaluža-Klein spectra.}.

### 2.4 The Reduction Ansatz

Finally we turn to the type IIA ansätze for the \(S^3, \mathcal{H}^{2,2}, \text{etc.}\) reductions. As we sketched in section 2.1, when our Wigner-Inönu scalings are applied to the \(S^4\) reduction ansatz the internal manifold becomes \(S^1 \times S^3\). But we know that reducing \(D = 11\) supergravity on \(S^1\) just gives us type IIA supergravity.

We thus proceed as follows. We apply the scalings of section 2.1 to the \(S^4\) reduction ansatz of section 1.5.2. We then compare the results with the \(S^1\) reduction ansatz of \(D = 11\) supergravity, \textit{c.f.} section 1.3, and so extract the \(S^3\) ansatz and its analytic continuations. We will distinguish \(D = 10\) fields with tildes. Firstly though we shall need the scalings of the \(S^4\) embedding coordinates \(Y^A\). We have that

\[
Y^0 \rightarrow k^5 \tilde{Y}^0, \quad Y^{\tilde{A}} \rightarrow \tilde{Y}^{\tilde{A}},
\]

(2.80)
and we have corresponding $SO(4)$-covariant derivatives
\[
\tilde{D}\tilde{Y}^0 = d\tilde{Y}^0, \\
\tilde{D}\tilde{Y}^\hat{A} = d\tilde{Y}^\hat{A} + \tilde{g}\tilde{A}_{(1)}\tilde{Y}^B.
\] (2.81)

Now applying our scalings to the metric ansatz (1.93), we find that
\[
ds_{11}^2 \rightarrow k^{-\frac{2}{3}}\Phi^\frac{1}{2}\tilde{\Omega}^\frac{1}{2}\left\{ ds_7^2 + \frac{1}{\tilde{g}^2\Phi^\frac{1}{2}\tilde{\Omega}} \left[ \Phi^{-\frac{1}{2}}M^{-\frac{1}{2}}_{AB}\tilde{D}\tilde{Y}^A\tilde{D}\tilde{Y}^B \\
+ \Phi \left( d\tilde{Y}^0 + g\tilde{A}_{(1)0}\tilde{Y}^A + \chi_A\tilde{D}\tilde{Y}^A \right)^2 \right] \right\},
\] (2.82)

where the new warp factor is
\[
\tilde{\Omega} = \pi^{-\frac{1}{2}}A^{-\frac{1}{2}}B^{-\frac{1}{2}}C^{-\frac{1}{2}}D^{-\frac{1}{2}}. 
\] (2.83)

Comparing this with equation (1.57), we identify the M-theory circle to be $dz = \tilde{g}^{-1}d\tilde{Y}^0$, and find Kaluża-Klein vector
\[
\hat{A}_{(1)} = \tilde{A}_{(1)0}\tilde{Y}^A + \frac{1}{\tilde{g}}\chi_A\tilde{D}\tilde{Y}^A, 
\] (2.84)

and dilaton
\[
\phi = -\frac{1}{2}\log\tilde{\Omega} + \frac{5}{8}\log\Phi,
\] (2.85)

and thus we extract type IIA metric ansatz
\[
ds_{10}^2 = \Phi^\frac{1}{2}\tilde{\Omega}^\frac{1}{2}\left\{ ds_7^2 + \frac{1}{\tilde{g}^2\Phi^\frac{1}{2}\tilde{\Omega}} M^{-1}_{AB}\tilde{D}\tilde{Y}^A\tilde{D}\tilde{Y}^B \right\}.
\] (2.86)

Similarly for the fluxes we find
\[
\hat{\mathcal{F}}_{(2)} = G_{(2)}^\hat{A}\tilde{Y}^B\eta_{\hat{A}\hat{B}} + \frac{1}{\tilde{g}}G_{(1)\hat{A}}\tilde{Y}^\hat{B}, \\
\hat{\mathcal{H}}_{(3)} = -\frac{\tilde{U}}{6\tilde{g}^2\Phi^2} \varepsilon_{\hat{A}\hat{B}\hat{C}\hat{D}}\tilde{D}^\hat{A}\tilde{Y}^\hat{B}\tilde{D}^\hat{C}\tilde{Y}^\hat{D} + \frac{1}{\tilde{g}^2\Phi^\frac{1}{2}\tilde{\Omega}} \varepsilon_{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}}M^{-1}_{\hat{A}\hat{B}}\tilde{D}^\hat{C}\tilde{D}^\hat{D}\tilde{D}^\hat{E}\tilde{D}^\hat{F}\eta_{\hat{E}\hat{F}}\tilde{Y}^G\tilde{Y}^H \\
+ \frac{1}{\sqrt{2}\tilde{g}\Omega} \varepsilon_{\hat{A}\hat{B}\hat{C}\hat{D}}\tilde{F}_{(2)}\tilde{D}^\hat{A}\tilde{D}^\hat{B}\tilde{D}^\hat{C}\tilde{D}^\hat{D}\tilde{D}^\hat{E}\tilde{D}^\hat{F}\eta_{\hat{E}\hat{F}}\tilde{Y}^G + H_{(3)},
\]
\[ \tilde{F}_{(4)} = \frac{1}{g^3 \Omega^2} \varepsilon_{ABCD} M^{\tilde{A}E} M^{\tilde{F}G} G_{(1)G} \wedge \tilde{D} \tilde{Y}^{\tilde{B}} \wedge \tilde{D} \tilde{Y}^{\tilde{C}} \wedge \tilde{D} \tilde{Y}^{\tilde{D}} \eta_{EF} \eta_{FI} \tilde{Y}^{\tilde{B}} \tilde{Y}^{\tilde{I}} \]

\[ + \frac{1}{g^2 \Omega} \varepsilon_{ABCD} G^{\tilde{B}} (2) \wedge \tilde{D} \tilde{Y}^{\tilde{B}} \wedge \tilde{D} \tilde{Y}^{\tilde{C}} M^{\tilde{D}E} \eta_{EF} \tilde{Y}^{\tilde{F}} \]

\[ - \Phi \frac{1}{2} M^{\tilde{A}B} * G_{(3)A} \eta_{BC} \tilde{Y}^{\tilde{C}} + \frac{1}{g} G_{(3)\tilde{A}} \wedge \tilde{D} \tilde{Y}^{\tilde{A}}. \]  

(2.87)

Here we have introduced the gauge and composite invariant \( \tilde{U} \):

\[ \tilde{U} = \pi^{-1} \alpha^{-1} \beta^{-1} \tilde{Y}^{\tilde{A}} \eta_{BC} \tilde{Y}^{\tilde{B}} \eta_{\tilde{C}} \eta_{\tilde{D}} (2 \eta_{BC} \eta_{DF} - \eta_{BF} \eta_{CD}) \eta_{\tilde{A}E}. \]  

(2.88)

We can use this ansatz to oxidise any solution of the \( D = 7 \) action (2.60) to an exact solution of type IIA supergravity. Given the discussion in section 2.3.2, we are especially interested in the uplift of the solution (2.65). We find metric

\[ d \hat{s}_{10}^2 = \Omega^{\frac{1}{2}} (\rho) \left[ \eta_{\mu \nu} dx^\mu dx^\nu + \frac{1}{4g^2} (d \theta^2 + \sin^2 \theta d \varphi^2) + dy^2 + \frac{1}{g^2} \left( d \rho^2 \right. \right. \]

\[ + \left. \left. \frac{\cosh^2 \rho}{\cosh(2 \rho)} \left( d \alpha + \frac{1}{2} \cos \theta d \varphi \right) \right]^2 + \frac{\sinh^2 \rho}{\cosh(2 \rho)} \left( d \beta - \frac{1}{2} \cos \theta d \varphi \right) \right] \right] , \]  

(2.89)

where the warp factor is

\[ \Omega (\rho) = \cosh(2 \rho). \]  

(2.90)

The curvature of this metric is supported by the 3-form flux

\[ \hat{H}_{(3)} = \frac{\sinh(2 \rho)}{g^2 \cosh^2(2 \rho)} d \rho \wedge \left( d \alpha + \frac{1}{2} \cos \theta d \varphi \right) \wedge \left( d \beta - \frac{1}{2} \cos \theta d \varphi \right) \]

\[ + \frac{\sin \theta}{2g^2 \cosh(2 \rho)} d \theta \wedge d \varphi \wedge \left( \cosh^2 \rho d \alpha - \sinh^2 \rho d \beta \right), \]  

(2.91)

and has dilaton profile

\[ \hat{\Phi} = - \frac{1}{2} \log [\cosh(2 \rho)] . \]  

(2.92)

We henceforth refer to this as the Salam-Sezgin background. We will spend much of the next chapter discussing this solution, but for now we remark on a couple of its more interesting features. Although the \( \mathcal{H}^{2,2} \) component is present, the internal manifold is altered slightly by non-trivial fibrations of the directions \( \alpha \) and \( \beta \) over the \( S^2 \) - in fact these are Hopf fibrations. We also note the non-trivial warp factor. Unlike many metrics employed in braneworld scenarios, our warp factor is exponentially
increasing rather than decreasing as our radial coordinate $\rho$ increases. This will have some important consequences for our Kaluža-Klein spectrum.
3 Kałuża-Klein Modes of the Salam-Sezgin Background

In chapter 2 we obtained the $\mathcal{H}^{2,2}$ reduction via an analytic continuation of the $S^3$ reduction, instead of starting afresh from type IIA supergravity. We have thus sidestepped any discussion of the $\mathcal{H}^{2,2}$ Kałuża-Klein spectrum. In particular we have not justified the $SO(2,2)$-gauged theory (1.76) as some low energy effective action of type IIA supergravity. For non-compact spacetimes we generically expect a continuum of Kałuża-Klein modes, and hence no mass gap; there is no reason a priori for the $\mathcal{H}^{2,2}$ reduction to be any different. On the contrary, the observation of [25] that the $D = 7$ Newton’s constant vanishes in this theory is indicative of a fundamentally higher-dimensional theory.

In this chapter we shall study the properties of the Kałuża-Klein spectrum of the $\mathcal{H}^{2,2}$ reduction and seek a braneworld interpretation. For simplicity we shall work consider only the linearised perturbations about the Salam-Sezgin background (2.89). We shall begin in section 3.1 with a careful analysis of the Laplace-Beltrami operator on this background, relating our harmonic spectrum to the localisation of scalars in section 3.2. The results of [22,23] allow us to study graviton modes at the same time. A central result of this chapter is that there is a family of valid boundary conditions for the harmonics, parametrised by some angle $\theta$. In section 3.3 we relate this to the inclusion of a codimension-2 brane at the $\mathcal{H}^{2,2}$ origin, and discuss its backreaction. Finally in section 3.4 we motivate the addition of an NS5-brane to the system, and obtain the corresponding backreacted solution. We finish with a brief discussion.

3.1 Harmonic Functions on the Salam-Sezgin Background

Ultimately we wish to consider decompositions into Kałuża-Klein modes of linearised perturbations of the type IIA fields about the Salam-Sezgin background (2.89). In this section we restrict ourselves to a mode analysis of the harmonic functions on this geometry; we will see in section 3.2 that the harmonic spectrum will tell us much about the structure of more general modes, and about about gravitational wave perturbations in particular.

Before we begin, we note that we presented our geometry (2.89) in a coordinate chart valid for $\rho \in (0, \infty)$: we must keep in mind the need for boundary conditions at $\rho = 0$. Identifying and interpreting the appropriate boundary conditions will be
recurring theme in this chapter. Conventionally one identifies \( \beta \sim \beta + 2\pi \), so that the \((\rho, \beta)\) space is locally \( \mathbb{R}^2 \) near the origin, but in anticipation of our discussion in section 3.3 we will allow for a possible deficit angle \( \beta \sim \beta + 2\pi - \beta_0 \).

Now, we recall that a harmonic function \( F \) obeys

\[
\triangle_{(10)} F = 0, \tag{3.1}
\]

where \( \triangle_{(10)} \) is the Laplace-Beltrami operator of the background metric:

\[
\triangle_{(10)} F \equiv \frac{1}{\sqrt{-\det g_{(10)}}} \partial_M \left( \sqrt{-\det g_{(10)}} g^{MN}_{(10)} \partial_N F \right). \tag{3.2}
\]

For an unwarped geometry the Laplacian has the diagonal form \( \triangle_{(10)} = \triangle_{(4)} + \triangle_{(6)} \), and we can perform a Kaluza-Klein expansion in the eigenfunctions of the internal manifold. The warping introduces additional derivatives \( \partial_\rho \) which will obviously alter the spectrum of the Kaluza-Klein modes. Substituting in our background metric (2.89), we obtain

\[
\triangle_{(10)} = \Omega^{-\frac{1}{4}}(\rho) \left( \Box_{(4)} + \triangle_{(6)} + g^2 \Omega^{-1} \frac{d\Omega}{d\rho} \frac{\partial}{\partial \rho} \right),
\]

\[
\equiv \Omega^{-\frac{1}{4}}(\rho) \left( \Box_{(4)} + g^2 \triangle_{(\theta,\varphi,y,\alpha,\beta)} + g^2 \triangle_{KK} \right), \tag{3.3}
\]

where we have separated out the derivatives in the compact directions \((\theta, \varphi, y, \alpha, \beta)\), and the non-compact direction \( \rho \) into operators \( \triangle_{(\theta,\varphi,y,\alpha,\beta)} \) and \( \triangle_{KK} \) respectively. (Here \( \Box_{(4)} \) is the usual \( D = 4 \) d’Alembertian operator.) The compact operator \( g^2 \triangle_{(\theta,\varphi,y,\alpha,\beta)} \) has the unpleasant form:

\[
g^2 \triangle_{(\theta,\varphi,y,\alpha,\beta)} = 4g^2 \left( \frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right) + \frac{\partial^2}{\partial y^2} + \frac{g^2}{\sin^2 \theta} \left( \frac{\cosh(2\rho) - \cos^2 \theta \sinh^2 \rho}{\cosh^2 \rho} \frac{\partial^2}{\partial \alpha^2} + \frac{\cosh(2\rho) - \cos^2 \theta \cosh^2 \rho}{\sinh^2 \rho} \frac{\partial^2}{\partial \beta^2} \right. \\
+ \left. \frac{\cosh(2\rho) - \cos^2 \theta \cosh^2 \rho}{\sinh^2 \rho} \frac{\partial^2}{\partial \alpha \partial \beta} \right) + 2 \cos^2 \theta \frac{\partial^2}{\partial \alpha \partial \beta} - 4 \cos \theta \frac{\partial^2}{\partial \varphi \partial \alpha} + 4 \cos \theta \frac{\partial^2}{\partial \varphi \partial \beta} \right). \tag{3.4}
\]

Physically we can expect its modes to be discrete with separation \( \mathcal{O}(g^2) \) - that is,
the inverse square of the radius of curvature of the compact directions. We will see shortly that this of the same order of magnitude as the mass gap of \( g^2 \Delta_{KK} \), and so we will neglect these higher Kaluza-Klein modes in what follows, i.e. we will take 
\[
\partial_\theta F = \partial_x F = \partial_y F = \partial_\alpha F = \partial_\beta F = 0.
\]

The nature of the spectrum of the operator \( \Delta_{KK} \) is less apparent: since \( \rho \) is a non-compact direction, it clearly admits a continuum of non-normalisable states; it may or may not, however, admit one or more bound states. This is the central question of this section. The operator itself is simply
\[
\Delta_{KK} \equiv \frac{\partial^2}{\partial \rho^2} + \frac{2}{\tanh(2\rho)} \frac{\partial}{\partial \rho}.
\]

Using equation (3.3) we can expand in eigenfunctions of the modified Laplacian \( \Delta_{KK} \):
\[
F(x, \rho) = \sum_{\lambda_i} f_{\lambda_i}(x) \xi_{\lambda_i}(\rho) + \int_{\Lambda}^{\infty} d\lambda f_{\lambda}(x) \xi_{\lambda}(\rho),
\]
where \( \Delta_{KK} \xi_{\lambda} = -\lambda \xi_{\lambda} \), so that the \( f_{\lambda} \) are \( D = 4 \) Klein-Gordon scalar fields \( \Box(4)f_{\lambda} = \lambda g^2 f_{\lambda} \). Our notation here foreshadows our later results by allowing for a set of discrete eigenvalues \( \lambda = \lambda_i \) in addition to the expected continuum \( \lambda \in [\Lambda, \infty] \).

In this way, the problem of finding harmonic functions reduces to the spectral problem of the operator \( \Delta_{KK} \). This eigenvalue problem naturally has an associated inner product
\[
\langle \xi_{\lambda_1}, \xi_{\lambda_2} \rangle \equiv \int_{\Lambda}^{\infty} d\rho \sinh(2\rho) \xi_{\lambda_1}(\rho) \xi_{\lambda_2}(\rho),
\]
with respect to which eigenfunctions will, for appropriate Sturm-Liouville boundary conditions, be orthogonal and potentially normalisable. Using the popular technique advanced in [8], we introduce a new field \( \psi(\rho) \):
\[
\psi(\rho) \equiv \sqrt{\sinh(2\rho)} \xi(\rho).
\]

\footnote{Technically we have not specified the radius of curvature of the direction \( y \), but we are free to take it to be at least \( O(g^{-1}) \).}

\footnote{The factor \( \sinh(2\rho) \) in the inner product is the weight function \( \omega(x) \) we find when we write \( \Delta_{KK} \) in Sturm-Liouville form, c.f. equation (3.13). We shall see it arise in a more physically intuitive way in equation (3.34).}

63
Then the eigenfunction equation for $\xi(\rho)$ becomes

$$\mathcal{L}_{KK}\psi_\lambda(\rho) \equiv -\frac{d^2\psi_\lambda}{d\rho^2} + V(\rho)\psi_\lambda(\rho) = \lambda\psi_\lambda(\rho),$$  \hspace{1cm} (3.9)

with

$$V(\rho) = 2 - \frac{1}{\tanh^2(2\rho)},$$ \hspace{1cm} (3.10)

i.e. the one-dimensional Schrödinger equation for a particle in a potential $V(\rho)$. This potential is plotted in figure 1. The inner product associated with this equation is just the usual quantum mechanical inner product

$$\langle \psi_{\lambda_1}, \psi_{\lambda_2} \rangle \equiv \int d\rho \psi_{\lambda_1}(\rho)\psi_{\lambda_2}(\rho).$$  \hspace{1cm} (3.11)

Here we have essentially absorbed the spacetime measure (and Sturm-Liouville weight function) $\sinh(2\rho)$ into the field $\xi(\rho)$. If we suppose for a moment that $F$ were a physical field - and the Laplace-Beltrami equation its equation of motion - then we can interpret $\psi(\rho)$ as something more significant than merely the argument of an operator isospectral to $\triangle_{KK}$: it is the correct $\rho$-dependent part of an effective Klein-Gordan wavefunction. This steers us towards a very natural restriction to impose on the space

Figure 1: The effective potential (3.10) for the $H^{2,2}$ harmonics, shown in orange, behaves as an inverse square well near $\rho = 0$ and quickly asymptotes to a value $V_\infty = 1$. The normalised $\lambda = 0$ bound state (3.42) is shown in purple.
of admissible $\psi(\rho)$:

**Condition 1** The wavefunction $\psi(\rho)$ belongs to a (rigged) Hilbert space $H$.

At this point we pause in our analysis to introduce some important definitions and results.

### 3.1.1 Introduction to Hilbert Spaces

The appropriate formalism for Sturm-Liouville problems in general, and the Schrödinger equation in particular, is the Hilbert space - a generalisation of the vector space allowing for a formally infinite dimension. The technical distinctions between Hermitian and self-adjoint operators on Hilbert spaces are usually ignored by authors, but will have real consequences for us in section 3.1.2. Here we introduce only the minimal machinery needed to properly discuss self-adjointness, and illustrate its importance with theorem 2. For a more complete review of these ideas, see [129–131]

**Definition 1** Let $H$ be a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. Then $H$ is a Hilbert space if it is complete with respect to the metric

$$d(u,v) \equiv \sqrt{\langle u - v, u - v \rangle},$$

i.e. if every Cauchy sequence converges in $H$.

Clearly all finite dimensional vector spaces are Hilbert spaces. We will be interested in the infinite dimensional Hilbert space $L^2(0,\infty,\omega(\rho))$, i.e. the set of all functions $f(\rho) : (0,\infty) \mapsto \mathbb{R}$ such that

$$\sqrt{\langle f, f \rangle} \equiv \sqrt{\int d\rho \, \omega(\rho) f^2(\rho) < \infty},$$

(3.12)

with $\omega(\rho)$ some non-negative ‘weight’ function on $(0, \infty)$.

An important subtlety of Hilbert spaces is that their linear operators are generically defined only on some dense domain $\mathcal{D} \subset H$, e.g. the operator $-d^2/d\rho^2$ is defined on the domain of twice differentiable functions, which is dense in $L^2((0,\infty),\omega(\rho))$. For every operator/domain pair $(T, \mathcal{D})$, we can define a dual $(T^*, \mathcal{D}^*)$.

- **Definition 1** Recall that an inner product on a vector space $V$ is a map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ that is sesquilinear and positive definite.
Definition 2 Let $T : \mathcal{D} \rightarrow H$ be a linear operator on some dense domain $\mathcal{D} \subset H$. Then $v \in \mathcal{D}^* \subset H$ if $\exists w \in H$ such that
\[
\langle T(u), v \rangle = \langle u, w \rangle \quad \forall u \in \mathcal{D},
\]
and we define the adjoint $T^* : \mathcal{D}^* \rightarrow H$ to be the map $T^*(v) = w$.

In the finite dimensional case $\mathcal{D}$ and $\mathcal{D}^*$ are isomorphic. In our infinite dimensional case, however, they are almost always distinct. Note that as we increase the size of $\mathcal{D}$ the dual domain $\mathcal{D}^*$ becomes correspondingly smaller, i.e. $\mathcal{D}_1 \subseteq \mathcal{D}_2 \implies \mathcal{D}_2^* \subseteq \mathcal{D}_1^*$.

Definition 3 Let $T : \mathcal{D} \rightarrow H$ be a linear operator on some dense domain $\mathcal{D} \subset H$. Then $T$ is Hermitian if $\mathcal{D} \subseteq \mathcal{D}^*$ and $Tf = T^*f$, $\forall f \in \mathcal{D}$. $T$ is self-adjoint if additionally $\mathcal{D} = \mathcal{D}^*$.

The terms ‘Hermitian’ and ‘self-adjoint’ are used interchangeably in much of the physics literature, primarily because authors tend not to specify the domain $\mathcal{D}$ of operators. This error is usually inconsequential: the domain of a typical Hermitian operator can be implicitly and uniquely extended to render the operator self-adjoint. The distinction is important, however, as only genuinely self-adjoint operators possess the desirable spectral properties of e.g. Hamiltonians and other quantum mechanical observables, as we will now discuss.

Definition 4 Let $T : \mathcal{D} \rightarrow H$ be a linear operator on some dense domain $\mathcal{D} \subset H$. Then $\lambda \in \mathbb{C}$ is in the resolvent set $\rho(T)$ if $T - \lambda I$ is invertible and its inverse is a bounded\footnote{Recall that a linear operator $T : \mathcal{D} \rightarrow H$ is bounded if $\exists C \in \mathbb{R}$ such that $d(T(v), 0) < C \cdot d(v, 0)$, $\forall v \in \mathcal{D}$.} operator. Else $\lambda$ is in the spectrum $\sigma(T)$. We further define:

- $\lambda \in \sigma(T)$ is in the point spectrum $\sigma_p(T)$ if $\exists f \in \mathcal{D}$ such that $Tf = \lambda f$, i.e. if $\lambda$ is an eigenvalue of $T$;

- $\lambda \in \sigma(T)$ is in the continuous spectrum $\sigma_c(T)$ if $\lambda \notin \sigma_p(T)$ and the image of $(T - \lambda I)$ is dense in $H$;

- $\lambda \in \sigma(T)$ is in the residual spectrum $\sigma_r(T)$ if $\lambda \notin \sigma_p(T)$ and the image of $(T - \lambda I)$ is not dense in $H$.\footnote{Recall that a linear operator $T : \mathcal{D} \rightarrow H$ is bounded if $\exists C \in \mathbb{R}$ such that $d(T(v), 0) < C \cdot d(v, 0)$, $\forall v \in \mathcal{D}$.}
In quantum mechanics the point spectrum $\sigma_p(T)$ corresponds to bound states and the continuous spectrum $\sigma_c(T)$ to scattering states. Note that scattering states are not technically in the Hilbert space because they are not $L^2$-normalisable. One can formally include them by extending $H$ to obtain a rigged Hilbert space; we shall not discuss this, as it would bring us nothing except mathematical peace of mind.

**Theorem 2** Let $T : \mathcal{D} \to H$ be a closed Hermitian operator. If $T$ is self-adjoint then $\sigma(T) \subset \mathbb{R}$. Else $\sigma(T) \equiv \mathbb{H}^+, \mathbb{H}^-$, or $\mathbb{C}$, with $\mathbb{H}^\pm$ the half-planes

$$H^\pm \equiv \{ z \in \mathbb{C} \mid \Im(z) \gtrless 0 \}.$$ 

We see now the importance of self-adjointness. If a Hermitian operator is not self-adjoint then we must extend $\mathcal{D}$, or equivalently impose some new boundary condition on $\mathcal{D}^*$.

**Definition 5** An extension of a Hermitian operator $T : \mathcal{D} \to H$ is a linear operator $\tilde{T} : \tilde{\mathcal{D}} \to H$ such that $\mathcal{D} \subset \tilde{\mathcal{D}}$ and $\tilde{T}f = Tf, \forall f \in \mathcal{D}$.

**Definition 6** A Hermitian operator $T : \mathcal{D} \to H$ is essentially self-adjoint if it has a unique self-adjoint extension.

Most textbook discussions of the Schrödinger equation are essentially self-adjoint. This effectively means that there is no need to properly define the domain $\mathcal{D}$; there is a unique set of natural boundary conditions on the usual test functions. In section 3.1.2 we shall see that the potential (3.10) does not lead to an essentially self-adjoint problem.

We conclude with some remarks on Sturm-Liouville operators [132]. These are linear differential operators $\mathcal{L}$, acting on some dense domain $D \subset L^2((0, \infty), \omega(x))$, that can be written in the form

$$\mathcal{L}v = \frac{1}{\omega(x)} \left( -\frac{d}{dx} \left[ p(x) \frac{dv}{dx} \right] + q(x)v \right),$$

(3.13)

for some functions $p(x), q(x), \omega(x)$, where $\omega(x) > 0$. If we have boundary conditions such that $\mathcal{L}$ is self-adjoint then the eigenfunctions will form a basis, and that basis will be orthogonal with respect to the inner product.

---

35For a full proof, see [130].
3.1.2 The $\mathcal{H}^{2,2}$ Potential

Our operator $L_{KK}$ is of Sturm-Liouville form, with $\omega(\rho) \equiv p(\rho) \equiv 1$ and $q(\rho) \equiv V(\rho)$. Now let’s find the eigenvalues by imposing the usual quantum mechanical condition $\psi_\lambda \in H \equiv L^2((0, \infty), 1)$. Unfortunately we have been unable to solve this equation analytically for general eigenvalue $\lambda$. However, we note that the potential is well-behaved away from the origin, so we need only concern ourselves with the solutions near the boundaries $\rho = 0$ and $\rho = \infty$.

At $\rho = \infty$ the problem is straightforward: for $\lambda > 1$ both solutions look like a wavepacket, and so we expect scattering states; for $\lambda < 1$ one solution is exponentially growing and one is exponentially decaying, so we need to specify one boundary condition here to have $\psi \in H$. At the origin things are more complicated, thanks to the divergent potential. In fact $\rho = 0$ is a regular singular point\footnote{An nth order ordinary differential equation has a regular singular point at some $\rho_*$ if the coefficient of the ith-derivative has at most an $(n-i)$th-order pole there, for $i = 0, \ldots, n$.} of the Schrödinger equation. Solutions around regular singular points are amenable to a Frobenius method, which is a generalisation of the usual series expansion to non-integer exponents:

$$\psi_\lambda(\rho) \sim \sum_{k=0}^{\infty} c_k \rho^{k+r}. \quad (3.14)$$

Substituting such an expansion into our Schrödinger equation, we obtain an ‘indicial equation’ for the exponent $r$. In our case the indicial expansion has repeated root $1/2$, indicating that our second independent solution has a logarithmic component:

$$\psi_\lambda(\rho) \sim \sqrt{\rho} (A_\lambda + B_\lambda \log \rho). \quad (3.15)$$

Unusually both solutions are normalisable near the origin, and we need specify no boundary conditions here to have $\psi \in H$. Naively it now seems that we have a continuum of bound states for $\lambda \in (-\infty, 1)$. In fact, it is generally true that if both solutions are normalisable near the origin for any one value of $\lambda$ then they are both normalisable there for all $\lambda \in \mathbb{C}$ [130]. Since for complex eigenvalues one solution is always exponentially decaying at infinity, we naively have $\sigma(L_{KK}) \equiv \mathbb{C}$.

Comparing with theorem 2, it is clear that our Hamiltonian is not an essentially self-adjoint operator. Normalisability is not in itself a sufficient condition, and we must explicitly introduce an additional parameter $\theta$ that will play the role of a a self-
adjoint extension, $H \to (H, \theta)$. This amounts to a restriction on the dual domain $D^*$. Borrowing some results from functional analysis\textsuperscript{37}, we have that $V(\rho)$ is relatively form-bounded at the origin \cite{130}, and so the self-adjoint extensions of the full Hamiltonian are equivalent to the self-adjoint extensions of $-d^2/d\rho^2$. Given $\psi_{\lambda_1}, \psi_{\lambda_2} \in H$ we have that

$$
\int d\rho \left\{ \psi_{\lambda_1} \left( -\frac{d^2}{d\rho^2} \right) \psi_{\lambda_2} - \psi_{\lambda_2} \left( -\frac{d^2}{d\rho^2} \right) \psi_{\lambda_1} \right\} = 0
\iff (A_{\lambda_1}, B_{\lambda_1}) \propto (A_{\lambda_2}, B_{\lambda_2}).
$$

(3.18)

Our self-adjoint extension is thus:

**Condition 2** $\psi(\rho) \in (H, \theta)$ where $(H, \theta) = \{ \psi(\rho) \in H \mid \arctan \left( \frac{A}{B} \right) = \theta \}$.

We must determine numerically which bound states $\lambda < 1$ have the allowed $\theta = \arctan \left( \frac{A}{B} \right)$, where we restrict $\theta$ to the fundamental domain $\theta \in (-\pi/2, \pi/2]$. We present our results in figure 2.

Our potential (3.10) can be compared with the oft-studied inverse square potential

$$
V(\rho) = -\frac{1}{4\rho^2}.
$$

(3.19)

Since the potentials (3.10) and (3.19) coincide near the origin, it is no surprise that the inverse square well also requires a self-adjoint extension treatment \cite{133–136}. In this case one finds a spectrum of scattering states $\lambda \in (0, \infty)$ and a single bound state with a $\theta$-dependent eigenvalue $\lambda < 0$. These eigenfunctions can be solved for analytically in terms of cylindrical Bessel functions. The Schrödinger equation has a conformal symmetry

$$
\rho \rightarrow a\rho, \quad \lambda \rightarrow \frac{\lambda}{a^2},
$$

(3.20)

\textsuperscript{37}Here we are using the KLMN theorem \cite{130}, which states that if a positive self-adjoint operator $T$ and a quadratic form $S$ on a dense domain $D$ satisfy

$$
S(v, v) < a(v, Tv) + b(v, v), \quad \forall v \in D,
$$

(3.16)

for some constants $a, b \in \mathbb{R}$ with $a < 1$, then there exists a unique self-adjoint operator $\tilde{T}$ which we can interpret as $T + S$, i.e. it obeys

$$
\langle u, \tilde{T}v \rangle = \langle u, Tv \rangle + S(u, v), \quad \forall u, v \in D.
$$

(3.17)

So if $T$ were merely Hermitian, then a self-adjoint extension of $T$ would specify a self-adjoint extension of $T + S$. 

69
which is spontaneously broken by the bound state/self-adjoint extension [137]. This effect has been experimentally observed in [138]. With the $H^{2,2}$ potential (3.10), this conformal symmetry is present only in the limit $\rho \to 0$, which allows for both a mass gap and a non-negative bound state eigenvalue.

Several authors have taken a renormalisation group approach [139,140] to the inverse square potential (3.19), introducing a cutoff at some small $\rho = \varepsilon$ and demanding that the wavefunctions vanish there. As with all renormalisation calculations, this introduces an arbitrary energy scale into the system which is ultimately equivalent to $\theta$. We favour the self-adjoint extension approach; we can calculate $\theta$ directly from a codimension-2 brane source term (and will do so in section 3.3.2) but the corresponding calculation in the renormalisation picture is more opaque.

### 3.1.3 Generic $H^{p,q}$ Backgrounds

Though we shall only consider the Salam-Sezgin background (and its modifications) in the remainder of this chapter, we pause briefly here to discuss the harmonics of generic $H^{p,q}$ backgrounds, with $p, q \neq 0$. We can expect a background obtained via a consistent
Pauli reduction of $D$-dimensional supergravity on $\mathcal{H}^{p,q}$ to have the form [26,27]

$$ds^2_D = \Omega^\gamma(\rho) \left\{ g_{\mu\nu}dx^\mu dx^\nu + \frac{1}{g^2} \left( d\rho^2 + \frac{\cosh^2 \rho}{\cosh(2\rho)} d\Omega_{p-1}^2 + \frac{\sinh^2 \rho}{\sinh(2\rho)} d\Omega_{q-1}^2 \right) \right\}, \quad (3.21)$$

where $g_{\mu\nu}(x^\mu)$ is some unspecified $(D - p - q + 1)$-dimensional background metric, the warp factor $\Omega(\rho)$ is as before, and the exponent $\gamma$ is given by

$$\gamma = \frac{p + q - 2}{D - 2}. \quad (3.22)$$

One may wish\(^{38}\) to switch on fluxes that thread the non-trivial cycles of the $S^p$ and $S^q$ components of $\mathcal{H}^{p,q}$, thus altering the metrics on these spheres. Provided these fluxes do not also thread any lower dimensional non-compact directions, this will only alter the equivalent of the compact operator (3.4) and have no consequences for our spectral analysis. For this reason we consider only the simple structure (3.21). A typical $\mathcal{H}^{p,q}$ background will also have dilaton factors in the metric; we restrict ourselves to the reasonable class of solutions in which the dilaton is stabilised, and absorb the resulting constant factors into the coordinates $x^\mu$ and the constant $g$.

We now proceed as before, introducing a lower dimensional graviton perturbation and performing a separation of variables of the Laplace-Beltrami operator $\Delta_{(D)}$ to obtain an ordinary differential equation for the transverse profile $\xi(\rho)$. Once again, this can be mapped onto a one-dimensional Schrödinger equation via the introduction of a graviton wavefunction

$$\psi(\rho) = \left( \cosh \rho \right)^{\frac{p-1}{2}} \left( \sinh \rho \right)^{\frac{q-1}{2}} \xi(\rho). \quad (3.23)$$

The corresponding potential is then

$$V(\rho) = \frac{pq - 1}{2} + \frac{(p - 1)(p - 3)}{4} \tanh^2 \rho + \frac{(q - 1)(q - 3)}{4} \frac{1}{\tanh^2 \rho}, \quad (3.24)$$

in agreement with equation (3.10). We are concerned with the behaviour of this potential at large and small values of $\rho$. With the exception of the trivial case $\mathcal{H}^{1,1} \cong \mathbb{R}$,

\(^{38}\) Indeed one may be compelled to switch such terms on by the kind of instability we encountered in section 2.3.1.
the potential asymptotes to a positive constant:

\[ V_\infty \equiv \lim_{\rho \to \infty} g^2 V(\rho) = \frac{g^2 (p + q - 2)^2}{4}. \]  

(3.25)

This is a vital property of \( \Delta_{(D)} \): interpreting the eigenvalues as the masses of some physical modes, it provides a mechanism for the scattering states to decouple from the bound states below a tunable energy threshold \( \sim \mathcal{O}(g^2) \), provide these bound states exist.

For \( q = 2 \) the potential is always an inverse square well near the origin. The treatment employed in section 3.1 is again required, and we obtain self-adjoint extension relations \( \theta_p(\lambda) \). We find that the spectrum of \( \mathcal{H}^{2,2} \) is typical of the family \( \mathcal{H}^{p,2} \), i.e. for each integer \( p \) our numerical results are strongly indicative of a single admittable bound state.

For \( q \neq 2 \) this potential well is either merely absent or replaced by a potential barrier, and naively it would seem that there is no possibility of a normalisable graviton mode. We can always force a system to admit a bound state by prescribing an irregularity at the origin, but this must be supported by sources that backreacted on the geometry and alter the harmonics. We can avoid this backreaction when \( q = 2 \), and the need for a self-adjoint extension reflects this. We will return to this point in section 3.3.

3.2 Localisation of Modes

We now wish to consider the Kaluża-Klein decomposition of linearised fluctuations about the Salam-Sezgin background (2.89). From our discussion in chapter 2 we already know that there exists a certain consistent truncation of these modes, but we would like to know if this truncation can be justified in terms of a mass gap, i.e. if there exists some energy scale \( E_{KK} \) below which the action (1.76) accurately describes the (linearised) type IIA dynamics. If this is not the case then we would like to identify a different truncation - perhaps an inconsistent truncation - that does exhibit this mass gap.

For simplicity, imagine that the \( D = 10 \) action is minimally coupled to an additional massless scalar term

\[ \sim \int d^{10} x \sqrt{-g} \left( -\frac{1}{2} * d\sigma \wedge d\sigma \right). \]  

(3.26)

On our background (2.89) we take \( \sigma = 0 \). Considering now a small perturbation
\[ \sigma = \delta \sigma, \]  
we have that \( \delta \sigma(x, \rho) \) obeys the Laplace-Beltrami equation (3.1). Moreover this is the only equation we need to consider; the linearisation of the \( \sigma \)-contribution to the Einstein equations also vanishes. The results of section 3.1 then tell us that, for an appropriate choice of self-adjoint extension \( \theta \), we can truncate to a \( D = 4 \) scalar \( \delta \sigma_\kappa(x) \) with a mass \( g_\kappa(\theta) \) and an energy scale \( E_{KK} = E_{KK}(\theta, g) \).

Thanks to the non-trivial background flux (2.91) and dilaton (2.92), the linearised dynamics of the actual type IIA fields are more complicated. For example, a dilaton perturbation

\[ \phi = -\frac{1}{2} \log \left[ \cosh(2\rho) \right] - 2\delta \Phi(x, \rho), \quad (3.27) \]

requires compensating perturbations in the other fields. Otherwise the form-field and Einstein equations will not be satisfied at the linear level. One consistent set of such perturbations is

\[ \delta g_{\mu\nu} = \Omega^\frac{1}{4}(\rho) \delta \Phi(x, \rho) \eta_{\mu\nu}, \quad \delta g_{\mu\rho} = -7 \Omega^\frac{1}{4}(\rho) \delta \Phi(x, \rho) \eta_{\mu\rho}, \]

\[ \delta g_{\rho\rho} = \frac{1}{4g^2} \Omega^\frac{1}{4}(\rho) \delta \Phi(x, \rho), \quad \delta g_{\varphi\varphi} = \frac{1}{4g^2} \Omega^\frac{1}{4}(\rho) \delta \Phi(x, \rho), \]

\[ \delta g_{\alpha\alpha} = \frac{\cosh^2 \rho}{g^2 \cosh(2\rho)} \Omega^\frac{1}{4}(\rho) \delta \Phi(x, \rho), \quad \delta g_{\beta\beta} = \frac{\sinh^2 \rho}{g^2 \cosh(2\rho)} \Omega^\frac{1}{4}(\rho) \delta \Phi(x, \rho), \]

\[ \delta g_{\alpha\varphi} = \delta g_{\varphi\alpha} = \frac{\cosh^2 \rho}{2g^2 \cosh(2\rho)} \Omega^\frac{1}{4}(\rho) \delta \Phi(x, \rho) \cos \theta, \]

\[ \delta g_{\beta\varphi} = \delta g_{\varphi\beta} = -\frac{\sinh^2 \rho}{2g^2 \cosh(2\rho)} \Omega^\frac{1}{4}(\rho) \delta \Phi(x, \rho) \cos \theta, \quad (3.28) \]

and then \( \delta \Phi(x, \rho) \) again obeys the Laplace-Beltrami equation (3.1), and we can truncate to the lowest mode \( \delta \Phi_\kappa(x) \), just as with our scalar \( \sigma \).

Note that the linearisation of the Pauli reduction ansatz implies that \( \partial_\rho \delta \Phi = 0 \). Equation (3.8) tells us that this corresponds to a transverse wavefunction

\[ \psi(\rho) \sim \sqrt{\sinh(2\rho)}. \quad (3.29) \]

This is an eigenfunction of \( \mathcal{L}_{(KK)} \) with eigenvalue \( \lambda = 0 \), but is not normalisable in either the sense of a bound state or a scattering state: it is not in our Hilbert space. This is worse than simply having no mass gap: the modes retained in the Pauli reduction do not have a physical interpretation as \( D = 4 \) fields. To put it another way,
an apparently small fluctuation in the reduced action will, in $D = 10$, be arbitrarily large at large values of $\rho$.

Next we consider the Kaluza-Klein decomposition of the metric degrees of freedom. A general metric perturbation is governed by the Lichnerowicz operator $\triangle L$, and requires complicated compensating perturbations in the other fields. We shall restrict ourselves to the important case of $D = 4$ gravitational waves, i.e. perturbations of the form

$$ds^2(10) = \Omega^4(\rho) \left[ (\eta_{\mu\nu} + h_{\mu\nu}) \; dx^\mu dx^\nu + \ldots \right] ,$$  \hspace{1cm} (3.30)

where $h_{\mu\nu}(x, \rho)$ is small and transverse-traceless with respect to $\eta_{\mu\nu}$

$$\eta^{\mu\nu} h_{\mu\nu} = 0 , \quad \partial^\mu h_{\mu\nu} = 0 .$$  \hspace{1cm} (3.31)

For a given background, we expect the linearised equations of motion to depend sensitively on both the background geometry and supporting stress-energy. In a Kaluza-Klein setting, it follows that very few universal statements can be made about the spectrum of lower dimensional modes. For the spin-2 modes, we have the following surprising result [22,23]:

**Theorem 3** Let $(g_{MN}, T_{MN})$ be an exact solution to Einstein’s equations, with a $D$-dimensional warped product-manifold structure

$$g_{MN} dx^M dx^N = \Omega^4(y) \eta_{\mu\nu} dx^\mu dx^\nu + \hat{g}_{ab}(y) dy^a dy^b ,$$  \hspace{1cm} (3.32)

with general warp factor $\Omega(y)$ and general Euclidean metric $\hat{g}_{ab}$. Here $\mu, \nu = 1, \ldots, d$ and $a, b = 1, \ldots, D - d$. Consider a perturbation of the form (3.30) and (3.31). Then the linearised Einstein’s equations are equivalent to

$$\triangle_{(D)} h_{\mu\nu} = 0 ,$$  \hspace{1cm} (3.33)

where $\triangle_{(D)}$ is the $D$-dimensional Laplace-Beltrami operator (3.1).

We pause here to note the power of this theorem: the spin-2 modes are determined entirely by the harmonic functions on the background geometry. In particular, the Einstein equations do not require us to consider (at the linearised level) any compensating perturbations in the remaining fields.

The linearised equations of motion for the dilaton and form fields are not universal, and must be considered separately. Fortunately it is clear by inspection of equations
(1.22) and 1.24) that they are automatically satisfied, and also require no compensating perturbations. This is because these background fields do not thread the lower four dimensions; both $\partial_\mu \phi$ and $F_{\mu\nu}$ are zero.

These spin-2 modes mediate the gravitational force between test masses in $D = 4$, which can be described (at low energies) by an effective Newton’s law. We will return to this in section 3.2.1. Here we note only that we have thus far skirted discussion of the lower dimensional Newton’s constant $G_N^{(4)}$. This is because we were compelled to work directly with the field equations - as opposed to the action - in our construction of the Pauli reduction. In standard Kaluža-Klein reductions, c.f. section 1.3, this is simply the $D = 10$ Newton’s constant divided by the volume of the extra dimensions. In our model then, the non-compactness of $H^2$ implies a vanishing lower dimensional Newton’s constant, as was noted in [25].

We can see this more explicitly. The linearisation of the Pauli reduction ansatz (2.86) implies that we should set $\partial_\rho h_{\mu\nu}$ in equation (3.30). The effective action for these graviton modes is then [23]

$$S \sim \frac{1}{G_N^{(10)}} \int d\rho \sqrt{-g} \Omega^{-1} \int d^4x \partial_\mu h_{\nu\rho} \partial^\mu h^{\nu\rho},$$

$$\sim \frac{1}{G_N^{(10)}} \int d\rho \sinh(2\rho) \int d^4x \partial_\mu h_{\nu\rho} \partial^\mu h^{\nu\rho},$$

$$\sim \frac{1}{G_N^{(4)}} \int d^4x \partial_\mu h_{\nu\rho} \partial^\nu h^{\rho\mu},$$

and so $G_N^{(4)} = 0$. Just as with our perturbation (3.28), the effective transverse wavefunction for this mode is (3.29); a gravitational wave with the $\rho$-dependence of a bound state will contribute to the transverse integral and thus render it finite.

### 3.2.1 An Effective Newton’s Law

Thanks to theorem 3, we know that the gravitational wave spectrum is precisely the harmonic spectrum. Referencing the results of section 3.1, we conclude that a lower dimensional observer will measure a discrete graviton of mass $g\kappa$, with $\kappa(\theta) < 1$ a function of our self-adjoint extension, c.f. figure 2, plus a continuum of gravitons with masses $m \in (g, \infty)$. Given this, it is straightforward to write down an effective Newton’s law for such an observer.

More precisely, take $\rho_c$ to be some positive constant and consider an observer confined
to the $D = 4$ slice $\rho = \rho_c$. In the Newtonian limit, this observer can measure an effective potential for the force between two point masses $M_1$ and $M_2$ located on this slice with separation $r = \sqrt{\eta_{\mu\nu}\partial^{\mu}\partial^{\nu}}$. The force mediated by a gravitational wave $\tilde{h}_{\mu\nu}(x) = m^2\tilde{h}_{\mu\nu}(x)$ of mass $m$ will make a Yukawa-like contribution \[110\] to this potential:

$$U(r) \sim -\frac{M_1 M_2 e^{-mr}}{r}. \quad (3.35)$$

We need to determine the coefficient of each contribution. Firstly we introduce $G_N^{(5)}$ - the $D = 5$ Newton’s constant - obtained from the $D = 10$ Newton’s constant by integrating out the neglected compact directions:

$$\frac{1}{G_N^{(5)}} = \frac{1}{G_N^{(10)}} \int \sqrt{\det g_{(10)}} d\theta d\varphi dy d\alpha d\beta = \frac{1}{G_N^{(10)}} \cdot \frac{\pi^2 (2\pi - \beta_0)}{g^5} \cdot \text{Vol}(y), \quad (3.36)$$

where the measure to be integrated over is, in a slight abuse of notation, the part of the $D = 10$ measure with no dependence on $x^\mu, \rho$. Recall that we have not specified the compactification radius in the $y$-direction; this is reflected in the unknown but finite factor $\text{Vol}(y)$.

Now, we have gravitational waves for masses $m \in \{g\kappa\} \cup (g, \infty)$. Each Yukawa-like contribution is suppressed by the amplitude $|\psi^\lambda(\rho_c)|^2$ of the transverse wavefunction\[39\] on the $D = 4$ slice. We thus have an effective Newton’s constant for the leading order term in the potential $U(r)$:

$$G_N^{(4)} = G_N^{(5)} |\psi^\kappa(\rho_c)|^2. \quad (3.37)$$

Putting this all together, the full expression for the potential is

$$U(r) = -\frac{M_1 M_2}{r} \left( G_N^{(4)} e^{-g_{\kappa\kappa} r} + G_N^{(5)} \int_g^\infty dm e^{-mr} |\psi_{m^2/g^2}(\rho_c)|^2 \right). \quad (3.38)$$

The first term is the usual Newton’s law for (potentially massive) $D = 4$ gravity. At large values of the radius $r$ (in units of $g^{-1}$) we would like to truncate to this term, and so be able to write down an effective 4-dimensional action for a single graviton. Thanks to the mass gap, this is possible for

$$r \gg \frac{1}{g}. \quad (3.39)$$

\[39\]With our expression for $G_N^{(5)}$ the wavefunctions must be normalised to have total norm 1 for the bound state and norm 1 over a unit interval at large $\rho$ for the scattering states.
provided that $\kappa \ll 1$. This requires some tuning of the self-adjoint extension.

If the value of the self-adjoint extension does not allow a bound state, e.g. if $\theta = \pm \pi/2$, then the first term in equation (3.38) is missing, and we must evaluate the scattering state contribution to determine the effective potential. We define

$$I(s, \rho) \equiv \int_1^\infty dm \; e^{-ms|\psi_{m^2}(\rho)|^2},$$

(3.40)

so that the potential is

$$U(r) = -\frac{G_N^{(10)} M_1 M_2 g_6^6}{\pi^2 (2\pi - \beta_0) \text{Vol}(y)} \cdot \frac{I(gr, \rho_c)}{r},$$

(3.41)

and then perform the integral (3.40) numerically for different values of $\rho = \rho_c$. Our results are shown in figure 3. In our limit (3.39) we find that $I(gr, \rho_c) \sim r^\gamma$, with $\gamma$

![Figure 3](image-url)

**Figure 3:** The logarithmic behaviour of $I(s, \rho)$ as a function of $s$, for different values of $\rho = \rho_c$. Curves are plotted for $\rho_c = 0.1, 0.3, 0.6$, and $2$, in shades of purple, orange, brown, and grey respectively. For $s \gg 1$ we see a predominantly linear behaviour with negative slope, corresponding to an inverse power law term in the effective Newton’s potential (3.41).

some $\rho_c$-dependent, negative power. Rounding to the nearest integer, we have $\gamma \approx -2$ in the range of $\rho_c$ studied, so very crudely speaking we have an inverse cube law for the potential. Of course we have neglected the modes corresponding to graviton excitations with non-trivial dependence on the compact directions ($\theta, \varphi, y, \alpha, \beta$). These have
comparable masses \( m = O(g) \), and will alter the effective potential (3.41) accordingly.

### 3.2.2 Discussion

We conclude this section with some remarks. Firstly we note that our \( D = 4 \) slices are Minkowski, and so we must keep in mind the vDVZ discontinuity [141,142]. This is a phenomenon whereby a massless graviton and a graviton of arbitrary small mass lead to measurably different results for e.g. null geodesics\(^{40}\). Ideally then, we want to take \( \kappa(\theta) = 0 \). From figure 2 we see that this corresponds to a choice of self-adjoint extension \( \theta = 0 \). Happily this is one of the few values we can solve for analytically: after normalisation, the exact wavefunction is

\[
\psi_0(\rho) = \frac{2\sqrt{3}}{\pi} \sqrt{\sinh(2\rho)} \log(\tanh \rho).
\]

Whether or not \( \theta = 0 \) is a physically valid choice is something we shall address in section 3.3.

We briefly review the standard line of reasoning concerning massless graviton modes in braneworld scenarios, and contrast it with our own model. Evaluating \( \Delta_{(D)} \) on a metric of the form (3.32) and performing our usual separation of variables, we obtain the eigenvalue problem

\[
-\frac{1}{\sqrt{\bar{g}}} \partial_a \left[ \sqrt{\bar{g}} \Omega^\frac{d}{2}(y) \bar{g}^{ab} \partial_b \xi(y) \right] = \lambda \Omega^{\frac{(d-2)}{2}}(y) \xi(y),
\]

which has the associated inner product

\[
\langle \xi_{\lambda_1}, \xi_{\lambda_2} \rangle \equiv \int d^{(D-2)} \sqrt{-\bar{g}} \Omega^{\frac{(d-2)}{2}}(y) \xi_{\lambda_1}(y) \xi_{\lambda_2}(y).
\]

For the sake of clarity we specialise to the case of a transverse space with a single non-compact direction\(^{41}\) \( \rho \), writing

\[
\bar{g}_{ab} dy^a dy^b = \Omega^\frac{d}{2} d\rho^2 + \bar{g}_{\tilde{a}\tilde{b}}(\rho) d\tilde{y}^\tilde{a} d\tilde{y}^\tilde{b},
\]

and taking \( \Omega = \Omega(\rho) \). Then the partial differential equation (3.43) becomes an ordinary differential equation, and we may map the eigenvalue problem onto a 1-dimensional

---

\(^{40}\)There is some evidence [143] that the vDVZ discontinuity in braneworld models may be an artifact of linearisation, and can be absent in a fully covariant approach.

\(^{41}\)Obviously this includes the case of the Salam-Sezgin background.
Schrödinger equation by identifying a graviton wavefunction

\[ \psi(\rho) \equiv \sqrt{f(\rho)} \xi(\rho), \quad \text{with} \quad f(\rho) = \sqrt{-\tilde{g}} \Omega^{(d-1)/8}, \quad (3.46) \]

The corresponding potential is given by

\[ V(\rho) \equiv \frac{1}{4} \left[ \frac{2f''}{f} - \left( \frac{f'}{f} \right)^2 \right]. \quad (3.47) \]

Asymptotically, this is a non-zero constant - and we have a mass gap - only if \( f(\rho) \sim e^{a\rho} \), and then the potential tends to a value \( V_{\infty} \sim a^2/4 \). Whatever the background metric, equation (3.43) will always have a zero eigenvalue solution \( \xi_0(y) = 1 \), corresponding to a wavefunction \( \psi_0 = \sqrt{f} \). This is normalisable provided that

\[ \int_0^\infty d\rho \, f(\rho) < \infty. \quad (3.48) \]

As discussed at the start of section 3.2, the Pauli reduction of chapter 2 truncates to this mode even though equation (3.48) does not hold: the Salam-Sezgin background, for example, has \( f(\rho) \sim e^{2\rho} \).

When (3.48) does hold, however, it turns out that \( \rho = \infty \) is an apparent horizon \([22, 144, 145]\), and we must either extend our spacetime or specify additional boundary conditions there. This is a very general problem one encounters when trying to obtain a massless graviton mode in a model with a \( D = 4 \) Minkowski part and a non-compact transverse space.

There is a second zero eigenvalue solution, which behaves asymptotically as \( \psi_0(y) \sim e^{-a\rho} \). In a typical braneworld model one has \( a < 0 \), and this mode is of no interest, but in our case \( a = 2 \) this mode is normalisable. We propose that via this mechanism one might obtain a massless graviton without encountering an apparent horizon at infinity, \emph{i.e.} by having a transverse volume integral that not only fails to converge, \emph{c.f.} equation (3.48), but actually grows exponentially.

On the Salam-Sezgin background the measure \( f(\rho) \) vanishes at \( \rho = 0 \), and the resulting massless mode is irregular, \emph{c.f.} equation (3.42). We turn to the interpretation of this in the next section.
3.3 Irregularities at the Origin

The functional analysis techniques of section 3.1 have allowed us to identify a single gravitational wave bound state, but not to fix its mass: this is specified by some numerically determined function $\kappa(\theta)$ of the self-adjoint extension. Distinct choices of $\theta$ obviously lead to distinct physical behaviours, and we would like to have some physical rationale behind its selection. Since $\triangle_{(KK)}$ is self-adjoint on each of the Hilbert spaces $(H, \theta)$ of functions on $(0, \infty)$, we can learn nothing more from this approach. Only the behaviour of the wavefunctions at $\rho = 0$ lies outside our analysis. Indeed, since $\theta$ is defined in the $\rho \to 0$ limit, the physics at the origin is the obvious place to begin.

At $\rho = 0$ the metric can either be regular or singular. The natural assumption of regularity in the fluctuations implies that $\theta = \pi/2$, in which case there is no normalisable gravitational wave, c.f. figure 2. In this case we are free to switch to another chart which includes the origin and construct $\triangle_{(KK)}$ in these coordinates. Then the regular single point of the corresponding Schrödinger equation is seen to be just an artifact of the degeneracy of the Jacobian there in polar coordinates.

Alternatively we may allow a singularity in the metric. If we calculate curvature invariants such as the Kretschmann scalar $R_{MNPQ}R^{MNPQ}$, we do not observe a blowup in their behaviour even when we include the logarithm-type perturbations. The only compatible singularity is thus a conical singularity. This is not too unexpected given the $\mathbb{R}^2 \times S^1$ topology of $\mathcal{H}^{2,2}$.

Of course we are always free to impose irregular boundary conditions on our harmonic functions. The drawback is that any irregularity must be supported by some singular source, and that source will backreact on the geometry, rendering the original harmonic analysis moot. Conical singularities are special, in that they are supported by codimension-2 branes that do not (locally) alter the bulk geometry [17, 146–149]. Indeed, our self-adjoint extension analysis of section 3.1.2 was required for precisely this reason; it is well-known that an inverse square potential with non-trivial extension corresponds to a non-zero deficit angle $\beta_0$ [150].

3.3.1 Codimension-2 Branes

In this section we consider the coupling of a codimension-2 brane to the type IIA action (1.19), its effects on the equations of motion, and its consequences for the Salam-Sezgin background. We ultimately want to consider a 7-brane with transverse space
(\rho, \beta), since these are polar coordinates around \( \rho = 0 \). (It can be useful to consider a regularisation of the system as a codimension-1 brane wrapping the \( \beta \)-direction at some small radius \( \rho = \varepsilon \), as in [151, 152].) Firstly we discuss a general brane embedding.

We write \( \mu, \nu, \ldots \) for spacetime indices in the 8-brane directions, and \( \alpha, \beta, \ldots \) for indices in the two transverse directions. Let \( \rho \) be some radial coordinate in the transverse space. Now, we take worldvolume coordinates \( \xi_i \) and embedding coordinates \( X^M(\sigma) \), and consider a brane action of the form

\[
S_b \subset T_b \int d^8 \sigma \sqrt{-\gamma} V_b .
\]

(3.49)

where \( \gamma_{ij} \) is the induced metric

\[
\gamma_{ij} = \partial_i X^M \partial_j X^N g_{MN}(X) ,
\]

(3.50)

and \( V_b \) is some worldvolume scalar potential. When coupled to the type IIA action, this will introduce a new source term into the Einstein equations. Since the Salam-Sezgin background only involves the fluxes of the \( \mathcal{N} = 1 \) theory, we will for simplicity make the truncation (1.26). We then find

\[
R_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + \frac{1}{4} e^{-\phi} \left( H_{MPQ} H_N^{PQ} - \frac{1}{12} g_{MN} H_{PQR} H^{PQR} \right) + \frac{T_b \sqrt{-\gamma}}{2} \right) \delta^{(2)}(z) .
\]

(3.51)

Taking a canonical embedding

\[
\partial_i X^\mu = \delta_i^\mu , \quad \partial_i X^\alpha = 0 ,
\]

(3.52)

it is immediately apparent that the source term contribution will vanish in the directions \( R_{\mu
u} \) whatever the form of the metric. The same is true for \( R_{\mu\alpha} \), provided that the metric has the block-diagonal form \( g_{\mu\alpha} = 0 \). In this simple case, the brane backreacts on the metric only via a term:

\[
R_{\alpha\beta} \sim -g_{\alpha\beta} \delta^{(2)}(\rho) .
\]

(3.53)

This term induces no local backreaction of the \( \rho > 0 \) metric; its only effect\(^{42}\) is the

\(^{42}\)The deficit angle induced by a pure tension codimension-2 brane usually breaks all supersymmetry,
introduction of a conical singularity at $\rho = 0$, corresponding to a deficit angle $\beta_0$:

$$\beta_0 = \left( \frac{T_b V_b}{2 + T_b V_b} \right)^2 \pi . \quad (3.54)$$

In the case of the Salam-Sezgin background, there is a non-trivial fibration of the $H^{2,2}$ angles $(\alpha, \beta)$ over the $S^2$, and we do not have the block-diagonal form $g_{\mu \alpha} = 0$. Nevertheless one can check that (3.53) is still the only non-vanishing source term contribution.

### 3.3.2 Fixing the Self-Adjoint Extension

We now analyse the implications of the 7-brane for the Kaluza-Klein modes. As in section 3.2, we begin by considering a massless scalar $\sigma$ with a coupling (3.26) to the type IIA action. This scalar can couple to the brane action (3.49) via a potential $V_b(\sigma)$, which we write as

$$V(\sigma) = \Lambda + a_1 (\sigma - \sigma_0) + a_2 (\sigma - \sigma_0)^2 + \ldots , \quad (3.55)$$

for constants $\Lambda, a_1, a_2, \ldots$, so that the effective brane tension is $T_{\text{eff}} = T_b \Lambda$. In section 3.2 we took a background solution $\sigma = 0$; for consistency we must set $\sigma_0 = 0$ and $a_1 = 0$ in equation (3.55). Considering now a small perturbation $\sigma = \delta \sigma$ and performing the usual separation of variables, we find the modified equation for the transverse perturbation profile:

$$\left( \Delta_{(KK)} + T_b a_2 \frac{\sqrt{-\gamma}}{\sqrt{-g}} \delta^{(2)}(z) + \lambda \right) \xi_\lambda = 0 . \quad (3.56)$$

The harmonic analysis of section 3.1.2 still applies, i.e. there is a single bound state, but now the self-adjoint extension $\theta$ is no longer a free parameter. To determine its value we perform an integral [150] of equation (3.56) over a small disk $\rho = \epsilon$ in the $(\rho, \beta)$ plane. Trivially we have:

$$\int_{\mathcal{D}_\epsilon} \left( \Delta_{(KK)} + T_b a_2 \frac{\sqrt{-\gamma}}{\sqrt{-g}} \delta^{(2)}(z) + \lambda \right) \xi_\lambda , \equiv 0 . \quad (3.57)$$

though supersymmetric extensions of (3.49) have been proposed [17,153].
for $P$, any circle $\rho = \epsilon$ in the $(\rho, \beta)$ transverse space. Using our Frobenius expansion (3.15), we write
\[ \xi_\lambda(\rho) \approx \frac{1}{\sqrt{\mathcal{M}_\lambda}} (\sin \theta + \cos \theta \log \rho) \] (3.58)
with $\mathcal{M}_\lambda > 0$ some normalisation factor. Substituting this into the identity (3.57) and taking the limit $\epsilon \to 0$, we obtain the relation
\[ \theta = \arctan \left( \frac{2\pi - \beta_0}{T_b a_2} \right). \] (3.59)

We thus see that a codimension-2 brane can indeed localise a scalar mode in the Salam-Sezgin background, and moreover can do so without inducing any local backreaction - the usual downfall of braneworld models. The massless bound state $\theta = 0$ is obtained in the limit $T_b a_2 \to \infty$.

We could perform a similar analysis for the remaining modes. For the dilaton one would proceed similarly, writing down a brane potential that is compatible with the bulk solution but introduces a source term into the linearised perturbations. For the gravitational wave perturbations however, it is not obvious how one might do this: the very same cancellation in the $R_{\mu\nu}$ equation that allowed us to include a codimension-2 brane now prevents the linearised equation of motion from receiving a source term! We automatically have $\theta = \pi/2$, and an effective gravitational potential (3.41). Boundary conditions for more general brane actions are analysed in [154].

In conclusion, the addition of a 7-brane allows us to localise modes in the Salam-Sezgin background without having to worry about backreaction. But by avoiding this backreaction we exclude the one case we’re most interested in: spin-2 modes. To complete our braneworld picture, we need to go beyond the Salam-Sezgin background, and beyond the 7-brane.

### 3.4 A Backreacted 5-Brane Solution

Out of the results of section 3.3.2 comes a clear message: if we wish to localise gravity, we must introduce a more severe singularity into our background - one which will have some backreaction on the geometry and supporting stress-energy. If we must do this, then at least we can seek to retain some of the desirable features of the Salam-Sezgin solution, namely its background supersymmetry and the asymptotic structure that guarantees a Kaluža-Klein mass gap.
In this section then, we seek a type IIA supergravity that contains some form of defect at $\rho = 0$, but asymptotes to the Salam-Sezgin background (or something very similar) at large $\rho$. The naive approach would be to couple a brane source to the type IIA action, make an ansatz based on the Salam-Sezgin asymptotics, and then solve the resulting equations of motion. In practice, obtaining backreacted solutions this way is dispiritingly difficult. The challenge lies in formulating a suitable ansatz: too simple and it may not admit an exact solution; too complicated and the equations of motion may be utterly intractable.

In this naive approach, background supersymmetry is a property to be checked only after we have found a solution. We can however reverse the order of this calculation; if we fix a brane’s position and orientation, it is possible to determine the supersymmetry of the associated backreacted solution before we construct it. We can thus scan a range of brane configurations until we find one that is supersymmetric, and exploit our knowledge of its configuration to write down a better ansatz. The idea behind this approach is that, on the worldvolume, the Killing spinors of a brane background must obey the kappa symmetry projection condition

$$(1 - \Gamma_\kappa) \epsilon = 0,$$  \hspace{1cm} (3.60)$$

where $\Gamma_\kappa$ is a matrix satisfying $\Gamma_\kappa^2 = 1$, the properties of which were discussed in section 1.2.3. We can study this projection condition in the limit in which the $D = 10$ Newton’s constant vanishes, i.e. the probe limit, in which case it depends only on the brane embedding and the unadulterated Salam-Sezgin background. This strategy was successfully employed in e.g. [155,156].

We proceed as follows. Firstly in section 3.4.1 we obtain the Killing spinors $\epsilon$ of the $D = 10$ Salam-Sezgin background, identifying the necessary projection conditions. We then examine the space of plausible probe brane configurations in section 3.4.2, and use our kappa symmetry criterion to select a candidate 5-brane setup. In section 3.4.3, we split our metric based on our expected brane configuration, and in so doing we identify a transverse gravitational instanton structure. This is reminiscent of the work [157], and after a straightforward dimensional reduction to $D = 9$ we are able to apply the methods of that paper, to write down - and solve - an ansatz for the backreacted solution. Finally we revisit the problem of the Kaluza-Klein modes in section 3.4.4, to see what effect, if any, the brane has had on the graviton spectrum.
3.4.1 Killing Spinors of the Salam-Sezgin Background

We analysed the Killing spinor equations for the $D = 7$ Salam-Sezgin vacuum in section 2.3.1. We could use our consistent truncation to uplift the solutions to $D = 10$, but it is just as easy to solve the Killing spinor equations in $D = 10$ directly. For type IIA supergravity, the Killing spinor equations are most neatly expressed in the string frame, related to the Einstein frame via the conformal transformation $ds^2_{str} = e^{-2\phi}ds^2_{10}$.

As our background involves only the NS-NS sector, we will not write terms involving the R-R fields. The Killing spinor equations are then

\[
\delta \Psi_M = \left( \nabla_M - \frac{1}{8} H_{MNP} \Gamma^{NP} \Gamma^{11} \right) \epsilon = 0, \\
\delta \lambda = \left( \Gamma^M \partial_M \phi - \frac{1}{12} H_{MNP} \Gamma^{MNP} \Gamma^{11} \right) \epsilon = 0, 
\]

(3.61)

where spacetime indices are raised and lowered with the string frame metric. The string frame metric of the Salam-Sezgin background (2.89) is

\[
ds_{str}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{4g^2} \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) + dy^2 + \frac{1}{g^2} \left\{ d\rho^2 \\
+ \frac{\cosh^2 \rho}{\cosh(2\rho)} \left( d\alpha + \frac{1}{2} \cos \theta d\varphi \right)^2 + \frac{\sinh^2 \rho}{\cosh(2\rho)} \left( d\beta - \frac{1}{2} \cos \theta d\varphi \right)^2 \right\}.
\]

(3.62)

Substituting this metric into the dilatino transformation, along with our background flux (2.91) and dilaton (2.92), we find the following algebraic constraint on the supersymmetry parameter $\epsilon$:

\[
\delta \lambda = -g \left( \tanh(2\rho) \Gamma^i - \frac{1}{\cosh(2\rho)} \Gamma^{i\bar{\alpha} \bar{\beta}} \Gamma^{11} \\
+ \frac{1}{\sqrt{\cosh(2\rho)}} \left[ \cosh \rho \Gamma^{i\bar{\alpha}} + \sinh \rho \Gamma^{i\bar{\alpha}} \right] \Gamma^{11} \right) \epsilon = 0.
\]

(3.63)

The non-factorisable $\mathcal{H}^{2,2}$ coordinate dependence of this algebraic constraint tells us that the coordinate dependence of the spinor $\epsilon$ is also non-factorisable. Writing $\delta \lambda = -g M \epsilon$, we can study the structure of the algebraic constraint by examining the char-
acteristic polynomial of the matrix $M$:

$$\det(M - \lambda I) = \lambda^{16} (\lambda + 2)^8 (\lambda - 2)^8 .$$  \hfill (3.64)

This obviously has the 16 repeated roots $\lambda = 0$ needed for a $1/2$–BPS constraint. For a given basis of the $D = 10$ Dirac matrices $\hat{\Gamma}^M$, one can easily verify that there are indeed 16 linearly independent eigenvectors corresponding to these eigenvalues. In fact we observe that

$$\{M, \Gamma^{11}\} = 0 ,$$  \hfill (3.65)

and so we can simultaneously diagonalise these $\lambda = 0$ eigenvectors with respect to the operator $\Gamma^{11}$. We thus write them as $u_I^\pm$ (with $I = 1, \ldots, 8$), where

$$\Gamma^{11} u_I^\pm = \pm u_I .$$  \hfill (3.66)

Now we turn our attention to the gravitino constraints. Considering first the flat directions, it is clear we require $\partial_\mu \epsilon = \partial_\nu \epsilon = 0$, and so our candidate Killing spinor solutions are

$$\epsilon_I = f_I^\pm (\theta, \varphi, \rho, \alpha, \beta) u_I^\pm ,$$  \hfill (3.67)

for 16 unknown scalar functions $f_I^\pm$. The remaining gravitino variations thus reduce to a set of partial differential equations for each of these functions

$$\delta \Psi_\theta = \left( \frac{\partial}{\partial \theta} - \frac{1}{4 \cosh(2 \rho) \cosh(2 \rho)} \left[ \cosh \rho \Gamma^{\hat{\alpha} \hat{\alpha}} + \sinh \rho \Gamma^{\hat{\alpha} \hat{\beta}} \right] \right) \left( 1 + \Gamma^{11} \right) = 0 ,$$

$$\delta \Psi_\varphi = \left( \frac{\partial}{\partial \varphi} + \frac{1}{4} \left[ \frac{\sin \theta}{\cosh(2 \rho)} \left( \cosh \rho \Gamma^{\hat{\alpha} \hat{\alpha}} + \sinh \rho \Gamma^{\hat{\alpha} \hat{\beta}} \right) \right. \right.$$

$$\left. \left. + \frac{\cos \theta}{\cosh^2(2 \rho)} \left( \sinh \rho \Gamma^{\hat{\alpha} \hat{\alpha}} - \cosh \rho \Gamma^{\hat{\alpha} \hat{\beta}} \right) \right) \left( 1 + \Gamma^{11} \right) \right) = 0 ,$$

$$\delta \Psi_\rho = \left( \frac{\partial}{\partial \rho} + \frac{1}{2 \cosh(2 \rho) \Gamma^{\hat{\alpha} \hat{\alpha}} \Gamma^{11}} \right) \epsilon_I^\pm = 0 ,$$

$$\delta \Psi_\alpha = \left( \frac{\partial}{\partial \alpha} + \frac{\cosh^2 \rho}{2 \cosh(2 \rho)} \Gamma^{\hat{\alpha} \hat{\beta}} \left( 1 - \Gamma^{11} \right) \right.$$

$$\left. + \frac{1}{2 \cosh^2(2 \rho)} \left[ \sinh \rho \Gamma^{\hat{\alpha} \hat{\alpha}} - \cosh \rho \Gamma^{\hat{\alpha} \hat{\beta}} \Gamma^{11} \right] \right) \epsilon_I^\pm = 0 ,$$

86
\[ \delta \Psi_\beta = \left( \frac{\partial}{\partial \beta} - \frac{\sinh^2 \rho}{2 \cosh(2\rho)} \Gamma^{\hat{\phi}} (1 - \Gamma^{11}) \right) \]
\[ - \frac{1}{2 \cosh^2(2\rho)} \left[ \sinh \rho \Gamma^{\hat{\rho}} \Gamma^{11} - \cosh \rho \Gamma^{\hat{\beta}} \right] \tilde{\epsilon}_I^\perp = 0. \]

Substituting our ansatz into the variation \( \delta \Psi_\rho \), we find that there are no non-trivial solutions for the \( f_I^+ \). For the \( f_I^- \) we encounter no problems, and the remaining Killing spinor equations determine each function up to a constant factor. In particular, we find \( \partial_\beta f_I^- = \partial_\phi f_I^- = 0 \), as one would expect from lower dimensional considerations. We do find a non-trivial dependence on the \( \mathcal{H}^{2,2} \) angles: \( \partial_\alpha f_I^- \neq 0 \) and \( \partial_\rho f_I^- \neq 0 \).

We thus conclude that the Salam-Sezgin background admits 8 Killing spinors, obeying the conditions
\[ M \epsilon = 0, \quad (1 + \Gamma^{11}) \epsilon = 0. \]

This is not entirely satisfactory: although \( M \) has a non-trivial kernel, it is not a projection operator; moreover we would like to write the \( f_I^- (\rho, \alpha, \beta) \) explicitly. Taking the Killing spinor equation \( \delta \Psi_\rho = 0 \) as a guide, we write
\[ \epsilon = \exp \left\{ \frac{1}{2} (\beta - \alpha) \Gamma^{\hat{\phi}} \right\} \cdot \exp \left\{ \frac{1}{2} \arctan |\tanh \rho| \Gamma^{\hat{\rho}} \right\} \cdot \eta_0. \]

Then we find that \( \eta_0 \) is just a constant spinor, and our constraints (3.69) reduce to the true projection conditions
\[ \left( 1 - \Gamma^{\hat{\rho} \hat{\phi}} \right) \eta_0 = 0, \quad (1 + \Gamma^{11}) \eta_0 = 0. \]

### 3.4.2 Kappa Symmetry

We now wish to find a probe brane configuration that is compatible with at least some of the Killing spinor solutions of section 3.4.1, i.e. to construct via kappa symmetry considerations a projection operator that annihilates some \( \epsilon \), as defined in equations (3.70) and (3.71).

In type IIA supergravity, we have at our disposal \( Dp \)-branes, with \( p = 2, 4, 6, 8 \), and the NS5-brane, and each of these can be sensibly inserted into the Salam-Sezgin background in various ways. Before beginning any sort of systematic analysis of these

\[ \text{This counting agrees with our } D = 7 \text{ analysis in section 2.3.1. This was assured by the consistency of the Pauli reduction on } \mathcal{H}^{2,2}. \]
configurations, we look to cull the space of candidate probe brane embeddings by more
general considerations. Firstly we note that our background Killing spinors have the
same chirality with respect to $\Gamma^{11}$, and so we require that there exists at least one linear
combination $\epsilon$ of our eight background Killing spinors (3.70) such that

$$[\Gamma_\kappa, \Gamma^{11}] \epsilon = 0. \tag{3.72}$$

Looking at the form of the $Dp$-brane kappa symmetry operator (1.49) however, we see
that one always has

$$\{\Gamma_\kappa, \Gamma^{11}\} = 0, \tag{3.73}$$

and so a $Dp$-brane can never satisfy equation (3.72). We can also see this by noting
that our $\Gamma^{11}$ chirality condition is precisely the fermionic part (1.27) of the consistent
truncation of type IIA supergravity to the $\mathcal{N} = 1$ multiplet. The $\mathcal{N} = 1$ multiplet does
not contain the fluxes $F_{(2)}$ or $F_{(4)}$ which are sourced by the $Dp$-branes; we are thus left
with the NS5-brane.

Secondly we consider the symmetries of our background. To make contact with our
braneworld picture, we shall always take the brane to be localised in the $\rho$-direction
at some constant $\rho = \rho_*$, which we expect to be zero\(^{44}\). Additionally we would like to
preserve 4-dimensional Poincaré symmetry, and so the brane worldvolume must span
these lower-dimensional directions.

Now we consider the brane action. The NS5-brane is a more complicated object than
its $Dp$-brane cousin (1.39), thanks to the presence of a self-dual worldvolume 3-form
$H_{(3)}$. The usual kinetic term

$$\ast H_{(3)} \wedge H_{(3)}, \tag{3.74}$$

vanishes identically, making it impossible to write down a standard action. One solution
is to sacrifice manifest Lorentz invariance [158], splitting $H_{(3)}$ into two fields $H_{ijk}$
and $H_{ij6}$, with $i = 1, \ldots, 5$. We shall follow the equivalent but covariant approach
of [159–161], introducing an auxiliary scalar $a$ whose derivative we can identify with a
preferred direction on the worldvolume.

Some work has been done on the kappa-invariant NS5-brane action [99]. However, it
is neater to first oxidise our background (2.89) to $D = 11$ - whereupon the NS5-brane
becomes an M5-brane - and then identify a kappa symmetry operator $\Gamma_\kappa^{(11)}$. It is then

\(^{44}\)Our space of candidate embeddings includes cases in which the brane wraps the coordinate $\beta$,
which corresponds to a vanishing cycle at $\rho = 0$. Mathematically we treat this by setting $\rho_* = \iota$
and then taking the limit $\iota \to 0$. This can then be interpreted as a 4-brane distinct from the $D4$-brane.
trivial to obtain the $D = 10$ equivalent. The M5-brane action is

$$S_{PST} = -\int d^6 \xi \left\{ \sqrt{-\det \left[ g_{ij} + \tilde{\mathcal{H}}_{ij} \right]} + \frac{1}{24} \varepsilon^{ijklmn} \mathcal{H}_{ijlp} v^p v_k \mathcal{H}_{lmn} 
+ C_6 + \frac{1}{2} \mathcal{H}_{(3)} \wedge C_{(3)} \right\}, \tag{3.75}$$

where we take worldvolume coordinates $\xi^i$, with $i = 1, \ldots, 6$, and $X^M(\xi)$ to be embedding coordinates, where now $M = 1, \ldots, 11$. Here we have defined

$$\mathcal{H}_{(3)} = H_{(3)} + C_{(3)}, \tag{3.76}$$

and $v_i$ is a unit vector related to a worldvolume scalar $a(\xi)$ via

$$v_i = \frac{\partial_i a}{\sqrt{-\partial_i a \partial^i a}}, \tag{3.77}$$

and finally

$$\sqrt{-g} \tilde{\mathcal{H}}_{ij} = \frac{1}{3!} \varepsilon_{ijklmn} v^k \mathcal{H}^{lmn}. \tag{3.78}$$

As usual, $g_{ij}$ and $C_{ijk}$ are the pullbacks of the bulk metric and gauge form potential:

$$g_{ij} = \partial_i X^M \partial_j X^N g_{MN}, \quad C_{ijk} = \partial_i X^M \partial_j X^N \partial_k X^P C_{MNP}. \tag{3.79}$$

The worldvolume scalar field $a$ is an auxiliary field: its equation of motion imposes a self-duality condition on $\mathcal{H}_{(3)}$. The kappa symmetry operator $\Gamma_\kappa$ for this action is [162]:

$$\Gamma_\kappa = \frac{v^p \Gamma_p}{\sqrt{-\det \left[ g_{ij} + i \tilde{\mathcal{H}}_{ij} \right]}} \varepsilon^{ijklmn} v_i \left[ \frac{1}{5!} \Gamma_{jklmn} + \frac{1}{3! 2!} \mathcal{H}_{jkl} \Gamma_{mn} \right. \right.$$

$$\left. - \frac{1}{2 \cdot 2!} \mathcal{H}_{jkl} \mathcal{H}_{lmn} \Gamma_n \right]. \tag{3.80}$$

Since brane actions couple directly to the Kalb-Ramond potential, we will need an explicit expression for $C_{(3)}$ in the following calculations. We have

$$C_{(3)} = -\frac{1}{2 g^2 \cosh(2\rho)} \left( d\alpha + \frac{1}{2} \cos \theta d\varphi \right) \wedge \left( d\beta - \frac{1}{2} \cos \theta d\varphi \right) \wedge dz$$

$$- \frac{\cos \theta}{4 g^2} d\varphi \wedge (d\alpha - d\beta) \wedge dz + \frac{1}{2 g^2} d\alpha \wedge d\beta \wedge dz, \tag{3.81}$$
where $z$ is the coordinate on the $M$-theory circle. Here we have chosen a gauge such that only $C_{\varphi z} = -B_{\alpha z}$ are non-zero at $\rho = 0$.

Using equation (3.80), we find in $D = 10$ the candidate probe brane configuration

$$
\begin{align*}
X^1(\xi) &= \xi^1, & X^2(\xi) &= \xi^2, & X^3(\xi) &= \xi^3, & X^4(\xi) &= \xi^4, \\
X^6(\xi) &= \theta_0, & X^7(\xi) &= \varphi_0, & X^y(\xi) &= \xi^y, & X^\rho(\xi) &= \iota, \\
X^\alpha(\xi) &= -\frac{1}{2}(\xi^6 - \tau_0), & X^\beta(\xi) &= -\frac{1}{2}(\xi^6 + \tau_0),
\end{align*}
$$

(3.82)

where $\theta_0, \varphi_0$, and $\tau_0$ are constants, and we take the limit $\iota \to 0$. This corresponds to a brane with worldvolume directions $(x^1, x^2, x^3, x^4, y, \alpha + \beta)$. The resulting kappa symmetry operator is then

$$
\Gamma_\kappa = \Gamma^{123456} \left( \frac{\cosh \rho}{\cosh(2\rho)} \Gamma^\alpha - \frac{\sinh \rho}{\cosh(2\rho)} \Gamma^\beta \right).
$$

(3.83)

Applying this to our background killing spinors (3.70) and (3.71) we find that this probe brane breaks no additional supersymmetry, i.e. is compatible with all eight of our remaining supercharges.

### 3.4.3 The Backreacted Geometry

We now look to construct the backreacted solution. Since our brane configuration (3.82) wraps the direction $\alpha + \beta$, we first make a simple change of coordinates:

$$
u = \alpha + \beta, \quad v = \alpha - \beta.
$$

(3.84)

The 5-brane is localised in the directions $(\theta, \varphi, \rho, v)$, but we shall consider a smearing of the brane in the directions $(\theta, \varphi, v)$; ultimately we are only interested in the dynamics in the non-compact $\rho$-direction, and besides a smeared solution is more tractable. With this in mind, it is useful to rewrite the metric (3.62) as a fibration over this 4-dimensional transverse space:

$$
\begin{align*}
\eta_{\mu\nu} dx^\mu dx^\nu + dy^2 + \frac{1}{4g} \left[ du + \frac{1}{\cosh(2\rho)} \left( dv + \cos \theta \, d\varphi \right) \right]^2 + \Omega^{-1}(\rho) ds_4^2,
\end{align*}
$$

(3.85)
where $\Omega(\rho)$ is our usual warp factor and the transverse metric is

$$
\frac{ds_4^2}{g^2} = \frac{1}{4}\left[ \cosh(2\rho)d\rho^2 + \frac{1}{4}\cosh(2\rho) \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \\
+ \frac{\sinh^2(2\rho)}{4\cosh(2\rho)} \left( dv + \cos \theta \, d\varphi \right)^2 \right].
$$

At this stage we know enough about the shape of our eventual geometry to make an intelligent guess at an ansatz. We could then substitute the ansatz into the equations of motion, or even continue the philosophy of the previous section and substitute it into the Killing spinor equations, as in [163,164]. However there is still more we can discover about the backreacted solution. Making a change of variables

$$
r = \frac{1}{g} \sqrt{\cosh(2\rho)},
$$

we can write the transverse metric (3.86) as

$$
\frac{ds_4^2}{g} = \left( 1 - a^4 \frac{1}{r^4} \right)^{-1} dr^2 + \frac{1}{4} r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \\
+ \frac{1}{4} \left( 1 - a^4 \frac{1}{r^4} \right) r^2 \left( dv + \cos \theta \, d\varphi \right)^2,
$$

where $a = 1/g$. This is precisely the geometry of Eguchi-Hanson space - a very important example of a gravitational instanton. The exact definition of a gravitational instanton varies from author to author, but the standard properties are that it is a 4-dimensional, complete, Riemannian manifold that is both Ricci-flat and has a self-dual (or sometimes anti-self-dual) Riemann tensor.

The appearance of a gravitational instanton as the transverse part of a probe brane metric is an extremely strong indication that the methods of [157] might yield the full backreacted solution. In [157], the authors begin by writing down the usual string frame ansatz for a dilatonic magnetic brane (for some dilaton $\psi$) with a $d$-dimensional worldvolume in flat $D$-dimensional space:

$$
ds_4^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \Omega^{-1}(y) \eta_{ab} dy^a dy^b, \\
e^{\alpha\psi} \ast F_{(d)} = d\Omega \wedge d^d x, \quad e^{\beta\psi} = \Omega,
$$

where as in previous sections $x^\mu$ and $y^a$ represent brane and transverse directions.
respectively. For an appropriate choice of the dilatonic couplings $\alpha$ and $\beta$, the resulting equations of motion reduce to a single equation in the warp factor:

$$\triangle_T \left[ \Omega^{-1} \right] = 0 \quad (3.90)$$

where $\triangle_T$ is the Laplace-Beltrami operator on the flat transverse space. So far this is all very standard. But if there are additional form-fields $F(p), F(q)$ in our theory, with $p + q = d + 1$, and if these fields obey a Bianchi identity

$$dF(p) = F(p) \wedge F(q), \quad (3.91)$$

then it is possible to extend the ansatz and construct a new solution with both a magnetic $d$-brane and a non-trivial background flux $F(p)$. All we need to do is replace the flat transverse metric in the ansatz (3.89) with some non-trivial Ricci-flat metric $ds^2_{D-d}$, and set $F(p)$ equal to some non-trivial harmonic p-form on this transverse metric. Again the full equations of motion reduce to a single equation in the warp factor:

$$\triangle_T \left[ \Omega^{-1} \right] \propto F^2(p), \quad (3.92)$$

where now $\triangle_T$ is the Laplace-Beltrami operator on the new $ds^2_{D-d}$. In the reasonably common case where $p = q$ and $F(p) \equiv F(q)$, we can still find a solution of this form provided we also take $F(p)$ to be self-dual or anti-self-dual.

For the case $D-d = 4$ and $p = 2$, this extended ansatz is thus describing exactly a brane with a gravitational instanton for its transverse space. Indeed several different solutions based on the Eguchi-Hanson geometry are described in [157]. This then would seem to be an ideal way to construct our backreacted solution. It is easy to see what kind of Bianchi identity we would need to do this; both the Salam-Sezgin background and the 5-brane are associated with a non-zero type IIA flux $H(3)$, so we would like an equation relating some components of $dH(3)$ to some components of $H(3)$. Of course our type IIA 3-form field strength $H(3)$ does not obey an equation of this form, or indeed any non-trivial Bianchi identity (3.91). Fortunately we can sidestep this problem by performing a dimensional reduction of the NS-NS sector to $D = 9$ on the worldvolume coordinate $u$. We write

$$ds^2_{10} = ds^2_9 + e^{\sqrt{2}x} \left( \frac{1}{2g} du + \mathcal{A}(1) \right)^2,$$
\[
B_{(2)} = A_{(2)} + \frac{1}{2g} A_{(1)} \wedge du,
\]
\[
\phi = -\sqrt{\frac{7}{8}} \psi + \sqrt{\frac{1}{8}} \chi,
\]
(3.93)

where \( \psi, \chi, A_{(1)}, A_{(1)}, \) and \( A_{(2)} \) are \( D = 9 \) fields. We stress here that we are still in the string frame. Now, the \( D = 9 \) field strengths are

\[
F_{(2)} \equiv dA_{(1)}, \quad F_{(2)} \equiv dA_{(1)}, \quad F_{(3)} \equiv dA_{(2)} - dA_{(1)} \wedge A_{(1)}.
\]
(3.94)

We now have the Bianchi identity that we need:

\[
dF_{(3)} = - F_{(2)} \wedge F_{(2)}.
\]
(3.95)

Our path is now clear. We make the ansatz (3.89), with transverse metric the Eguchi-Hanson metric (3.86), setting \( \chi = 0, \alpha = \beta = \sqrt{7/2}, \) and \( F_{(2)} = - F_{(2)} \) equal to the self-dual harmonic form of Eguchi-Hanson space. In our coordinates this is

\[
F_{(2)} = \frac{1}{g \cosh^2(2\rho)} \left[ \sinh(2\rho) \, d\rho \wedge (dv + \cos \theta \, d\varphi) + \frac{1}{2} \cosh(2\rho) \sin \theta \, d\theta \wedge d\varphi \right].
\]
(3.96)

As described above, the \( D = 9 \) equations of motion reduce to a single equation involving the Laplace-Beltrami operator on the Eguchi-Hanson metric:

\[
\triangle_T [\Omega^{-1}(\rho)] = \frac{1}{2} F^{ab} F_{ab} = -\frac{8g^2}{\cosh^4(2\rho)}.
\]
(3.97)

We know that this equation admits a particular solution \( \Omega^{-1} = \cosh^{-1}(2\rho) \), since this ansatz must encompass the probe brane case, and the homogeneous equation is essentially the same harmonic equation we analysed in section 3.1. Not surprisingly then, the general solution is

\[
\Omega^{-1}(\rho) = \Delta + \Lambda \log[\tanh \rho] + \frac{1}{\cosh(2\rho)}.
\]
(3.98)

with \( \Delta \) and \( \Lambda \) arbitrary constants. Finally we lift our solution back to \( D = 10 \). We find that the metric is as in (3.85), albeit with our more general warp factor (3.98). For the dilaton we obtain

\[
\phi = \frac{1}{2} \log [\Omega^{-1}],
\]
(3.99)
and for the 2-form gauge potential:

\[
B_{(2)} = \frac{1}{4g^2} \left[ (1 - \Lambda) dv + \frac{1}{\cosh(2\rho)} du \right] \wedge (dv + \cos \theta \ d\varphi).
\]

(3.100)

We want our backreacted solution to have the same asymptotic structure as the Salam-Sezgin background, so we set \( \Delta = 0 \). We will relate \( \Lambda \) to the brane tension in a moment. Interestingly \( \log[tanh(\rho)] \sim -(cosh(2\rho))^{-1} \) at large \( \rho \), so the 5-brane actually results in a constant shift of some asymptotic values. One consequence of this is a \( \Lambda \)-dependent shift in the Kaluza-Klein mass gap.

It is easy to see that the gravitino variation \( \delta \Psi_\rho = 0 \) is the same for the backreacted solution as for the Salam-Sezgin background, and so the \( \rho \)-dependence of the Killing spinors is still as in (3.70). Only in the Einstein frame will there be any explicit \( \Lambda \)-dependence in the Killing spinors.

All that is left to do in this section is reconcile our new bulk metric with its NS5-brane source. Since we have found a smeared solution, we integrate the action over the \( \theta, \varphi \) directions of the transverse metric. The \( v \)-coordinate is degenerate at \( \rho = 0 \), so it doesn’t really make sense to smear over this direction; we are effectively working with a codimension-2 defect once again. The relevant part of the source action is now

\[
S_{NS5} = -\frac{T}{\text{Vol}_2} \int d^2\Omega \int d^6\xi \ e^{-\frac{\phi}{2}} \sqrt{-\det \partial_i X^M \partial_j X^N g_{MN}},
\]

(3.101)

where \( \int d^2\Omega \) implements our smearing, and \( \text{Vol}_2 \) is the value of the compensating volume integral. This source term contributes a new \( \delta(\rho) \) term to the Einstein equations. After cancelling the bulk curvature against the bulk stress-energy, we find source equation

\[
\triangle_T [\Lambda \log[tanh(\rho)] = -\frac{Tg^4}{\sqrt{g_2}} \delta^2(z),
\]

(3.102)

where \( \sqrt{g_2} \) is the smeared measure \( \int d^2\Omega \sqrt{g_4} \). Obviously \( \triangle_T [\log[tanh(\rho)] \) vanishes in the bulk - as it must - but the derivatives of the logarithmic term at the boundary \( \rho = 0 \) yield singularities. Integrating equation (3.102) over the transverse metric, we find\(^{45}\)

\[
\Lambda = -\frac{Tg^4}{\int d^4y \sqrt{g_4} \triangle_T [\log[tanh(\rho)] = -\frac{2Tg^4}{\pi}.
\]

(3.103)

\(^{45}\)For a positive tension brane we find that \( \Lambda < 0 \). We note that when \( \Lambda > 0 \) we do not have the correct metric signature at the origin anyway. In section 3.4.4 we will see that this condition is equivalent to the existence of a strictly positive weight function for the Sturm-Liouville problem.
3.4.4 Kaluža-Klein Modes and Boundary Conditions

We have already discussed in detail the connection between the Laplace-Beltrami operator and the spin-0 and spin-2 Kaluža-Klein modes of a dimensional reduction. For the backreacted metric, we perform the separation of variables just as in section (3.1), and arrive at an eigenfunction equation:

\[-\xi''(\rho) - \frac{2}{\tanh(2\rho)} \xi'(\rho) = \lambda \left( 1 + \Lambda \cosh(2\rho) \log[\tanh(\rho)] \right), \quad (3.104)\]

whose solutions determine the spectrum of $D = 4$ gravitational wave modes $\Box_{(4)} h_{\mu\nu} = \lambda g^2 h_{\mu\nu}$. This eigenfunction problem has an associated inner product

\[\langle \xi_{\lambda_1}, \xi_{\lambda_2} \rangle \equiv \int d\rho \sinh(2\rho) \left( 1 + \Lambda \cosh(2\rho) \log[\tanh(\rho)] \right) \xi_{\lambda_1} \xi_{\lambda_2}, \quad (3.105)\]

with respect to which eigenfunctions will, for appropriate Sturm-Liouville boundary conditions, be orthogonal and potentially normalisable. We have implicitly defined here a weight function $\omega(\rho)$.

The new $\rho$-dependence of the $\lambda$ term means we cannot make a field redefinition that maps this this eigenfunction equation onto a Schrödinger problem as we did before. We could however obtain a very different Schrödinger problem by introducing $\Psi(\rho) = \sqrt{\sinh(2\rho)} \xi(\rho)$, whereupon we would obtain a potential

\[V_{\lambda}(\rho) = 2 - \frac{1}{\tanh^2(2\rho)} + \lambda \left( 1 + \Lambda \cosh(2\rho) \log[\tanh(\rho)] \right). \quad (3.106)\]

Whereas before we were looking for the eigenvalues $\lambda$ in some fixed Schrödinger potential $V(\rho)$, here we would be the studying the set of zero modes across a family of effective potentials $V_{\lambda}(\rho)$.

In the limit $\rho \gg 1$ we see that we have scattering states for $\lambda > (1 - \Lambda)^{-1}$, and for $\lambda < (1 - \Lambda)^{-1}$ we need to impose one boundary condition to have a normalisable eigenfunction at infinity. For $\rho \ll 1$, the inverse square term $-1/4\rho^2$ is dominant, and we once again have an apparent continuum of bound states $\lambda \in (-\infty, (1 - \Lambda)^{-1})$ with universal small-$\rho$ behaviour:

\[\xi_{\lambda}(\rho) \sim A_{\lambda} + B_{\lambda} \log \rho. \quad (3.107)\]

Now, we can try to fix the ratio $\theta = \arctan(A_{\lambda}/B_{\lambda})$ the same way we did in section
3.3.2. Looking at the relevant part of the Einstein equations:

\[
\left( \Delta_T - \frac{T g^4 \Omega(\rho) \delta(z)}{\sqrt{g_2}} \right) \xi_\lambda = 0 ,
\]  

(3.108)

we note the presence of a factor \( \Omega(\rho) \) multiplying the source term. For the Salam-Sezgin background, we have \( \Omega(\rho) = 1 \), and then integrating over a small volume containing the origin we pick up a \( B_\lambda \) contribution from the first term and an \( A_\lambda \) contribution from the second, and thus we obtain a consistency condition relating \( \theta \) to the brane tension. In our backreacted geometry however, we have \( \Omega \sim (\Lambda \log \rho)^{-1} \) for small \( \rho \), and the integration of the delta function also yields a \( B_\lambda \) term. We then have a consistency condition relating \( B_\lambda \) to itself, which is automatically satisfied when we impose equation (3.103).

We can extract more information from the brane setup by performing a regularisation of the NS5-brane, considering a ring-source \( \rho = \epsilon \) in \((\rho, v)\) space. Immediately outside the source, \( e.f. \) for \( \epsilon < \rho \ll 1 \) we have the usual \( \xi_\lambda(\rho) = \tilde{A}_\lambda + B_\lambda \log \rho \). We then derive a consistency equation between \( A_\lambda \) and \( B_\lambda \) by our usual method:

\[
\int_{\epsilon}^{\epsilon^+} \sqrt{-g_4} \Delta_T \left[ \xi_\lambda(\rho) \right] = 2\pi T g^4 \Omega(\epsilon) \xi(\epsilon) ,
\]  

(3.109)

which is equivalent to

\[
A_\lambda = -B_\lambda \left( \frac{\pi \Lambda}{2T g^4} + 1 \right) \log \epsilon .
\]  

(3.110)

According to equation (3.103) we should then have \( \tilde{A}_\lambda = 0 \), but this equation was derived for \( \epsilon = 0 \); in our regularised setup there will be \( \epsilon \) corrections that are potentially important. Repeating our procedure:

\[
\int_{\epsilon}^{\epsilon^+} \sqrt{-g_4} \Delta_T \left[ \Lambda \log[\tanh \rho] \right] = -2\pi T g^4 ,
\]  

(3.111)

and expanding the left hand side in terms of \( \epsilon \), we find the modified \( \Lambda-T \) relation

\[
\Lambda = -\frac{2T g^4}{\pi} \left( 1 + \frac{2}{3} \epsilon^2 \right) .
\]  

(3.112)

We now have that

\[
A_\lambda \sim \frac{2B_\lambda}{3} \epsilon^2 \log \epsilon ,
\]  

(3.113)
and so $\theta \to 0$ as $\epsilon \to 0$ whenever $T \neq 0$. Just as in the Salam-Sezgin background, a value $\theta = 0$ corresponds to a single bound state, with eigenvalue $\lambda = 0$ - i.e. a massless mode. From equation (3.104) it is obvious that $\xi_0(\rho)$ is the same analytic solution \[ \log[\tanh \rho] \] we had before, though its normalisation will be slightly different.

### 3.5 Conclusions

In these two chapters, we derived in full the Pauli reduction of type IIA supergravity on the inhomogeneous hyperboloid $H^{2,2}$, including the full set of supersymmetry transformations and the fermionic terms in the action, which can be used to generate IIA backgrounds with a non-compact component. We studied the spectrum of the Laplace-Beltrami operator on one such background and discovered that there exists a single bound state with a mass gap, provided that we allow for a non-trivial self-adjoint extension at the $H^{2,2}$ origin. We discussed the physical significance of this in terms of a conical singularity, and solved for a closely related codimension-2 brane solution.

We are always free to introduce localised irregularities at the boundaries of spacetime, though the backreaction of the necessary localised sources is then an issue. In this respect, the conical nature of the singularity discussed in section 3.3 is a distinct advantage: the metric is blind to the singularity away from the origin. By including an NS5-brane we sacrificed this property, and the harmonic analysis of section 3.1 didn’t carry over exactly. There was good reason to expect a reasonable Kaluža-Klein spectrum however: the presence of a mass gap is a robust property of the asymptotic geometry, and the effect of a brane on a spacetime is usually localised. In our case we encountered a constant shift in some asymptotic values, but this was not a serious change. Here the mass gap is proportional to $g$, which is inversely proportional to the radius of curvature of $H^{2,2}$; this result holds for equivalent solutions with other $H^{p,q}$ components. There is some similarity here with the conjectured Kaluža-Klein spectrum on compact hyperbolic surfaces [108].

We have stressed throughout this paper the distinction between the Pauli reduction as a constrained subset of type IIA backgrounds and the Pauli reduction as a lower dimensional dynamical theory, and have clarified this in terms of the norm of the lowest Kaluža-Klein mode. In this light, it would be interesting to understand the lower dimensional dynamics arising from our alternative boundary conditions - that is, to truncate the Kaluža-Klein expansion to the new bound state and integrate out the $H^{2,2}$ directions in the action. Of course, we have no reason to expect this to be
a consistent truncation. In particular, it would be interesting to know how much, if anything, survives of the Salam-Sezgin action [126]. We have limited ourselves here to the study of the scalar and graviton modes; the derivation of the low energy effective action would require an understanding of the spin-1 and fermionic spectra, which are not universal but are directly influenced by the background fluxes. Such an analysis would not be too involved, but it would then be difficult to draw conclusions which are generic for $H^{2,2}$ backgrounds as we have done here.
4 A Holographic Treatment of Fractionalisation

Classical phase transitions in field theory are a result of thermal fluctuations, and consequently are inaccessible at zero temperature. There exists an entirely separate class of phase transitions at $T = 0$ driven by the ubiquitous quantum fluctuations: we call these quantum critical points (QCP) [165]. One might naively assume that the physics at absolute zero is of no interest physically, while in fact the existence of a QCP has important consequences for the finite temperature phase diagram. When the quantum fluctuations are smaller than the thermal fluctuations, the finite temperature version of the quantum critical theory remains a valid description. The result is a characteristic quantum critical wedge in phase space, broadening as we heat up the system.

Quantum critical points are typically described by strongly coupled field theory, making them extremely difficult to analyse with conventional tools; the strong/weak duality of the AdS/CFT correspondence provides a viable alternative. In this chapter we shall employ holographic methods in a study of the fractionalisation transition. Before we present our gravitational treatment of the problem, we give a brief overview of fractionalisation in condensed matter physics.

In field theory, an electron is a fundamental fermionic field with integer charge. After renormalisation a system of interacting electrons will still have fermionic fields - which by convention are still called electrons - but they are not the fundamental fields we started with; they are complicated functions of the original fields called quasiparticles. Normally these quasiparticles differ from their parents only in their mass, or their coupling constants, but sometimes the effect can be more dramatic. After fractionalisation - a quantum phase transition - the spin and charge of the electron become deconfined: seem to split into separate particles, known as spinons and holons. It is even possible for the spinons to have non-integer charge.

In a simple model of fractionalisation, one considers fields $s$ and $b$, representing the spin and charge degrees of freedom respectively, and a composite field $c = sb^\dagger$. An emergent $U(1)$ gauge field $A_{(1)}$ is introduced, under which $s$ and $c$ are both charged, but $c$ is gauge-invariant. One then posits an action for the fields $s, c, A_{(1)}$ (the slave-boson model) or for $b, c, A_{(1)}$ (the slave-fermion model). These theories are discussed in more detail in [39–42]. The strong coupling of these actions means they have yet to yield useful predictions for quantities of interest, e.g. transport coefficients.

At $T = 0$, a finite density of fermions (represented by a chemical potential $\mu \neq 0$) will
have the configuration with the lowest possible energy. Because two fermions cannot have the same state, the occupied states will define a ball in momentum space centered on zero: we call this a Fermi surface. Luttinger’s theorem asserts that the total charge \( Q \) of a system obeys

\[
Q = \sum_i a_i q_i V_i ,
\]

where the \( V_i \) are the volumes of the Fermi surface of each fermionic field\(^{46}\), and the \( q_i \) the corresponding charges. For a review, see [39, 167]. The unspecified constants \( a_i \) of proportionality are given by the degeneracy of the momentum modes. This is an intuitive result, and proven for a wide range of field theories at the quantum level.

Fermi liquid theory describes the low energy physics of a system with a Fermi surface; it is perhaps the most important condensed matter model, with many real-world theories falling into its universality class. Fractionalised phases are a rare example of a non-Fermi liquid, and its properties are poorly understood. One striking feature is a violation of the Luttinger theorem [53]:

\[
Q = Q_F + \sum_i a_i q_i V_i ,
\]

with \( Q_F \) some non-zero ‘fractionalised’ charge. This is because the volume of a Fermi surface is an inherently gauge-invariant value. In the slave-boson model it is clear that in our fractionalisation model the spinon is not gauge-invariant under the emergent \( U(1) \) symmetry.

In a gravitational background, charge can be sourced either by bulk fields or by a charged IR horizon. Depending on whether the bulk fields are fermionic\(^{47}\) or bosonic, we call a solution with bulk charge an electron star [46] or a superconductor [43–45] respectively. It was proposed in [112] that a charged horizon is equivalent to a deconfined phase in the dual field theory, and is thus a candidate description of fractionalisation [39, 167, 170]. We shall consider a system with superconductor and charged horizon solutions and study its phase diagram; this is a bosonic analogue of [47, 48].

In this chapter we will refer to various phases as fully fractionalised (solutions with

\(^{46}\)A bosonic analogue of Luttinger’s theorem relates the total charge to the transverse Magnus force on a vortex [166].

\(^{47}\)It is difficult to solve for backgrounds with non-vanishing fermions, so one ordinarily models the fermionic matter as a perfect fluid. A proper treatment of an electron star based on the Dirac equation exists in [168, 169].
a charged horizon but no bulk charge), partially fractionalised (solutions with both a 
charged horizon and bulk charge present), or cohesive\footnote{In field theory we call the non-fractionalised phase ‘mesonic’, but this would be inappropriate in a purely bosonic system.} (bulk charge only). By the fractionalisation transition we specifically mean a zero temperature phase transition between the partially fractionalised and cohesive phase. A bulk charge, \textit{i.e.} a superconducting solution, is associated with a broken $U(1)$ symmetry, so there can exist a second phase transition between partially and fully fractionalised phases.

Our aim is to construct a $D = 4$ bottom-up model of gravity that exhibits a fractionalisation transition, and to identify the $T = 0$ geometries in each of the phases. By a bottom-up model we mean an action which has not been obtained via a consistent truncation of $D = 10$ or $D = 11$ supergravity, \textit{i.e.} a model without a known string theory embedding. The zero temperature limit of a charged black hole usually has an IR geometry $\text{AdS}_2 \times \mathbb{R}^2$, which is dual to a condensed matter phase with finite entropy. If this were the true ground state then we would have a violation of the third law of thermodynamics, but such geometries are usually unstable \cite{2,49–52}. This seems to preclude the possibility of a fractionalisation transition in simple models.

The solution is to include a coupling of our model to a neutral scalar $\Phi$ that is allowed to diverge in the IR. The resulting deformation of the solutions allows us to avoid an $\text{AdS}_2 \times \mathbb{R}^2$ geometry at $T = 0$. The divergence results in a hyperscaling symmetry, described in section 4.1; field theory phases and supergravity solutions with hyperscaling symmetry have been discussed at length in \cite{53–56}. We shall also allow the Maxwell field to couple to this scalar, so that its IR behaviour can make fractionalisation more or less favourable. Tuning the asymptotic value $\Phi_1$ of $\Phi$ thus provides a neat mechanism to trigger the transition.

\section{4.1 Hyperscaling}

We described in section 1.4 how the IR region $r/l \ll 1$ of an asymptotically $\text{AdS}$ solution can be thought of as dual to some low energy effective field theory. Low energy physics is governed by a renormalisation group fixed point plus its irrelevant and marginal operators, and so we require that the geometry have some emergent scaling symmetry in the limit $r \to 0$.

The simplest possibility is that we recover the $\text{AdS}$ metric (1.66), albeit with a different length scale $l_{IR}$ than in the UV; this will be the case for the critical geometry
in section 4.2.2. The scaling symmetry is then
\[ r \rightarrow \lambda^{-1} r, \quad t \rightarrow \lambda t, \quad x^i \rightarrow \lambda x^i. \tag{4.3} \]
This scaling treats the field theory directions \((t, x^i)\) democratically, but we are often interested in condensed matter systems that break Lorentz invariance. A Lifshitz scaling \(t \rightarrow \lambda^z t\) is a typical example, but we shall consider the more general hyperscaling symmetry \([53]\):
\[ r \rightarrow \lambda^{\frac{d-2}{2}} r, \quad t \rightarrow \lambda^z t, \quad x^i \rightarrow \lambda x^i, \tag{4.4} \]
under which the line element scales as
\[ ds^2 \rightarrow \lambda^\theta ds^2. \tag{4.5} \]
We call \(z\) the dynamical critical exponent and \(\theta\) the hyperscaling parameter. In a \(d\)-dimensional field theory, one expects the entropy \(S\) to scale with temperature as
\[ S \sim T^{\frac{d-1-\theta}{z}}, \tag{4.6} \]
so that \(\theta\) represents a shift in the effective dimension of the system.

### 4.2 A Bottom-Up Model of Bosonic Fractionalisation

We wish to construct and analyse a bosonic analogue of the holographic fractionalisation transitions observed in \([47,48]\). In addition to Einstein-Hilbert gravity, the minimal set of ingredients needed to describe a cohesive phase are a \(U(1)\) gauge field \(A_{(1)}\) and a corresponding charged complex scalar \(\sigma\) whose vacuum expectation will spontaneously break this symmetry. We shall drive the transition between the cohesive and fractionalised phases by tuning the gauge coupling in the IR; we can do this dynamically via a neutral scalar \(\Phi\), which we shall refer to as a dilaton. The appropriate arena for this investigation is thus Einstein-Maxwell-Dilaton theory with a charged scalar. The most general\(^{49}\) form of such an action is
\[
S = \frac{1}{16\pi G_N} \int d^4x \left\{ R + 1 - \frac{1}{2} \ast d\Phi \wedge d\Phi - Z_{\sigma}(|\sigma|) * \tilde{D}\sigma \wedge D\sigma \right\}
\]

\(^{49}\)One might additionally consider a Chern-Simons term of the form \(F \wedge F\); they are not uncommon in consistent truncations of supergravity. For the restricted ansatz we are about to consider however, these terms will make no contribution to the equations of motion.
\[-\frac{1}{2}Z_F(\Phi) \ast F(2) \wedge F(2) - V(\Phi, |\sigma|) \ast 1\]  
\hspace{1cm} (4.7)

with gauge covariant derivative

\[D\sigma = d\sigma - iqA^{(1)}(\sigma).\]  
\hspace{1cm} (4.8)

Here we have implicitly set some coupling \(Z_\Phi(\Phi) = 1\) - that is, put the kinetic term of the dilaton \(\Phi\) in a canonical form - by a judicious field redefinition. There is an obstruction to doing the same for the charged scalar; introducing real fields \(\eta\) and \(\varphi\):

\[S = \eta e^{iq\varphi},\]  
\hspace{1cm} (4.9)

one finds that for generic \(Z_\sigma\) there exists a field redefinition yielding a canonical kinetic term for \(\eta\) but not \(\varphi\). The best we can do is write

\[-Z_\sigma(|\sigma|) \ast \tilde{D}\sigma \wedge D\sigma \longrightarrow - * d\eta \wedge d\eta - q^2 Z_\varphi(\eta) \ast D\varphi \wedge D\varphi,\]  
\hspace{1cm} (4.10)

where now

\[D\varphi = d\varphi - A^{(1)}(\varphi),\]  
\hspace{1cm} (4.11)

and \(Z_\varphi\) is some coupling which behaves as \(Z_\varphi(\eta) \sim \eta^2\) for small values of \(\eta\). The potential becomes some function \(V(\Phi, \eta)\).

Before we present the particular class of models that we shall study, we discuss the rôle of the coupling \(Z_F(\Phi)\) in phase diagram. As explained in section 4, we are interested in systems that admit hyperscaling geometries, and we expect these to be generated by a divergent dilaton \(\Phi \rightarrow \pm\infty\) in the IR. Under such a divergence, many reasonable choices for the coupling \(Z_F(\Phi)\) will also diverge; equivalently the effective gauge coupling \(e^2 = Z_F^{-1}(\Phi)\) will vanish, and we can expect a fractionalised IR geometry. When the dilaton diverges with the opposite sign, \(Z_F\) may either diverge again or, more interestingly, vanish. In the latter case the effective gauge coupling diverges in the IR, and no fractionalisation is possible. At zero temperature there lies, between these phases, a quantum critical point with finite dilaton, \(i.e.\) a fractionalisation transition.

Meanwhile, the interplay of the potential \(V(\Phi, \eta)\) and the profile of the Maxwell field \(A^{(1)}\) may or may not lead, in these limits, to a non-zero value of \(\eta\) in the bulk. We thus expect a second phase transition between a fully fractionalised and a partially fractionalised phase. This transition will define a superconducting dome in phase space.
With this motivation in mind, we select a class of potentials $V(\Phi, \eta)$ possessing a single, finite stationary point and two runaway directions for $\Phi$, and zero and non-zero stationary points for $\eta$:

$$\ell^2 V(\Phi, \eta) = -V_0^2 \cosh \left( \frac{b\Phi}{\sqrt{3}} \right) - 2\eta^2 + g_\eta^2 \eta^4.$$  \hspace{1cm} (4.12)

where, without loss of generality, $b > 0$. We require that, for small values of $\Phi$ and $\eta$, the potential has an expansion

$$\ell^2 V(\Phi, \eta) = -6 - \Phi^2 - 2\eta^2 + \ldots,$$  \hspace{1cm} (4.13)

so that the model admits a pure $AdS_4$ solution with $AdS$ length $l$, and so that the scalars $\Phi$ and $\eta$ are both\footnote{The scalars appear to have different mass terms here, but this is simply due to the differing normalisations of their kinetic terms in the action (4.7).} dual to field theory operators with scaling dimension $\Delta = 2$. Of course this fixes $V_0 = \sqrt{6}$ and $b = 1$, and all our numerical results will be for these values. We shall nevertheless present in section 4.2.2 analytic solutions for all values of these parameters, referring them as IR geometries with the implicit understanding that there do not exist domain walls connecting them with an $AdS_4$ UV except in our special case.

For the coupling $Z_F(\Phi)$ we choose a function asymmetric in $\Phi$, diverging for $\Phi \to \infty$ and vanishing for $\Phi \to -\infty$:

$$Z_F(\Phi) = Z_0^2 e^{a\Phi/\sqrt{3}},$$  \hspace{1cm} (4.14)

where, without loss of generality, $a > 0$. For the solutions in this chapter, the scalar shall have a constant phase $\varphi = \varphi_0$, which we can take to be zero; we do not need to describe a $Z_\varphi(\eta)$. Again, we shall consider solutions of action (4.7) for a general choice of parameters. All numerical results will be for the case

$$a = 1, \quad b = 1, \quad g_\eta = 1, \quad V_0 = \sqrt{6}, \quad Z_0 = 1,$$  \hspace{1cm} (4.15)

and for charge

$$q = 2.$$  \hspace{1cm} (4.16)

In both the IR and UV geometries, and ultimately in their interpolating solution, we
shall make an ansatz corresponding to a planar metric

\[ ds^2 = -f(r)e^{-\beta(r)}dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^2} \left( dx^2 + dy^2 \right), \]  

(4.17)

and a gauge potential

\[ A_{(1)} = A_t(r)dt. \]  

(4.18)

We stress once again that we are working in coordinates in which the IR is the limit \( r/l \ll 1 \) and the UV \( r/l \gg 1 \). We shall set \( l = 1 \). From the form of the resulting Maxwell equation, we see that the presence of a fractionalised charge behind a horizon \( r = r_+ \) is indicated by a non-vanishing value of

\[ Q_F = \lim_{r \to 0} \sqrt{-g} Z_F(\Phi) F_{rt}. \]  

(4.19)

### 4.2.1 UV Geometries

We will ultimately be interested in domain wall solutions of our action (4.7) that are asymptotically \( AdS_4 \). At large \( r \) we seek a series solution about \( AdS_4 \), expanding in powers of \( r^{-1} \):

\[
\begin{align*}
    f(r) &= \frac{r^2}{l^2} + \frac{2\eta_1^2 + \Phi_1^2}{4} \frac{\eta_1^2}{r} + \frac{lG_1}{r} + \ldots, \\
    \beta(r) &= \beta_0 + \frac{l^2 (2\eta_1^2 + \Phi_1^2)}{4r^2} + \ldots, \\
    A_t(r) &= l e^{-\frac{\beta_0}{2}} \left( \mu - \frac{Q}{r} + \ldots \right), \\
    \eta(r) &= \frac{l^{\eta_1}}{r} + \frac{l^2 \eta_2}{r^2} + \ldots, \\
    \Phi &= \frac{l\Phi_1}{r} + \frac{l^2 \Phi_2}{r^2} + \ldots, 
\end{align*}
\]  

(4.20)

for some free constants \( G_1, \beta_0, \eta_1, \eta_2, \Phi_1, \Phi_2, \mu, Q \) corresponding to relevant deformations of the system. The constant \( \beta_0 \) corresponds to the boundary speed of light. We are free to set it to unity via a rescaling of our time coordinate; we shall leave it as a free parameter here, preferring to use our coordinate freedom to fix an equivalent constant in the infrared.

Holographically \( \Phi_1 \) and \( \eta_1 \) both correspond to source terms for relevant operators. We shall set \( \eta_1 = 0 \), so that the charged scalar has no external source and any breaking of the \( U(1) \) symmetry must happens spontaneously. For the dilaton however, we shall
allow $\Phi_1 \neq 0$. In fact we shall consider $\Phi_1$ to be some external parameter, and study the phases of the system as we vary the dimensionless quantity $\Phi_1/\mu$.

### 4.2.2 Zero Temperature IR Geometries

As discussed in 4.2, we may classify the IR geometries according to the behaviour of the dilaton as $r \to 0$. At zero temperature, the divergence of the dilaton is not masked by a black hole horizon, and we expect a solution of the form

$$\Phi \sim \frac{\sqrt{3} \chi_{\Phi}}{a} \log r,$$

(4.21)

for some constant $\chi_{\Phi}$. With this divergence the potential is schematically $V \sim r^{\chi_{\Phi}} + r^{-\chi_{\Phi}}$. We find that there are no exact solutions of this form: the subleading term in the potential induces subleading terms in all the fields; the dilaton, gauge potential, and metric components must all be written as series solutions. We write

$$f(r) = f_0 r^{\chi_f} \left( 1 + \sum_k c_f^{(k)} r^{kn} \right),$$

$$f(r) e^{-\beta(r)} = g_0 r^{\chi_g} \left( 1 + \sum_k c_g^{(k)} r^{kn} \right),$$

$$A_t(r) = A_0 r^{\chi_A} \left( 1 + \sum_k c_A^{(k)} r^{kn} \right),$$

$$\eta(r) = \eta_0 r^{\chi_\eta} \left( 1 + \sum_k c_\eta^{(k)} r^{kn} \right),$$

$$\Phi(r) = \frac{\sqrt{3} \chi_{\Phi}}{a} \log r \left( 1 + \sum_k c_{\Phi}^{(k)} r^{kn} \right),$$

(4.22)

for some constants $\chi_f, \chi_g, \chi_A, \chi_\eta, f_0, \beta_0, A_0, \eta_0$ and coefficients $\{c^{(k)}\}$. The stepsize $n$ in these series is some integer multiple of $\chi_{\Phi}$. Since we have not fixed $\beta_0$ in equation (4.20), we are still free to rescale our time coordinate to set $g_0 = 0$. In this section we shall present the leading order solution only; the coefficients $\{c^{(k)}\}$ are then fully determined by the equations of motion. The series were truncated at a much higher order for the purpose of the numerics.
About any IR solution we expect to find leading order deformations of the form

\[ \delta f = a^{IR}_f r^{\nu_f}, \quad \delta \beta = a^{IR}_{\beta} r^{\nu_{\beta}}, \quad \delta A_t = a^{IR}_{A} r^{\nu_A}, \]
\[ \delta \eta = a^{IR}_{\eta} r^{\nu_{\eta}}, \quad \delta \Phi = a^{IR}_{\Phi} r^{\nu_{\Phi}}, \]

(4.23)

for some fixed exponents \( \{\nu\} \), and for some parameters \( \{a^{IR}\} \) which are free modulo certain linear constraints. There are then the usual subleading corrections, e.g. for \( \delta f \) one has

\[ \delta f = a^{IR}_f r^{\nu_f} \left( 1 + \sum_j d^{(j)}_f r^{jm} \right), \]

(4.24)

for some coefficients \( \{d^{(j)}\} \) which are fully determined by the \( \{a^{IR}\} \). Because in our model (4.7) we are not calculating our deformations about an exact solution, but rather the leading order part of the series solution (4.22), we require a more complicated set of subleading corrections:

\[ \delta f = a^{IR}_f r^{\nu_f} \left( 1 + \sum_j \sum_k d^{(j,k)}_f r^{jm+kn} \right). \]

(4.25)

Again, we will just give the leading order parts of the deformations, though we retained higher order terms when carrying out numerical calculations. We will only list the irrelevant and marginal\(^{51} \) deformations, as the relevant deformations are of no use to us.

**Fractionalised IR Solutions** In this case some or all of the flux is sourced by the horizon, i.e. the quantity \( Q_F \) as defined in equation (4.19) is non-vanishing. We find a fractionalised solution of the form (4.22), with leading order parts

\[ f(r)e^{-\beta(r)} = r^2 \frac{12 + a^2 - b^2}{(a+b)^2}, \]
\[ f(r) = f_0 r^{\frac{2(a-b)}{a+b}}, \]
\[ A_t(r) = A_0 r^{\frac{12 + (4a-b)(a+b)}{(a+b)^2}}, \]
\[ \Phi(r) = - \frac{4\sqrt{3}}{a + b} \log r, \]

\(^{51} \)A marginal deformation is just an undetermined coefficient in (4.22), i.e. a parameter in a family of solutions, but for consistency of style we shall write one solution from each family and separately list any marginal deformations with their irrelevant relatives. This better reflects the way these families were handled numerically.
with coefficients

\[ f_0 = \frac{(a + b)^4 V_0^2}{4(6 + a(a + b))(12 + (3a - b)(a + b))}, \]
\[ A_0^2 = \frac{4(6 - b(a + b))}{(12 + (3a - b)(a + b))Z_0^2}. \]  

The higher order corrections have a stepsize \( n = 4 \). In this solution the charged scalar \( \eta \) is identically zero; this is the IR limit of a fully fractionalised solution. Not surprisingly, there exists a marginal deformation with leading order part corresponding to a shift in the IR vacuum expectation value of \( \eta \):

\[ \delta \eta = \eta_0, \]  

which continuously connects (4.26) to a family of partially fractionalised solutions. There is a second deformation - this time an irrelevant deformation associated with the dilaton - of the form

\[ \delta \Phi = a_\Phi^{IR} r^{\nu_\Phi}, \quad \delta f = a_f^{IR} r^{\nu_f}, \quad \delta \beta = a_\beta^{IR} r^{\nu_\beta}, \quad \delta A_t = a_A^{IR} r^{\nu_A}, \]  

where the \( \nu_f \) and \( \nu_A \) are given by

\[ \nu_f = \nu_\Phi + \frac{2(a - b)}{a + b}, \]
\[ \nu_A = \nu_\Phi + \frac{12 + (3a - b)(a + b)}{(a + b)^2}. \]  

The equations of motion fix the exponent \( \nu_\Phi \), and fix the coefficients \( \{a^{IR}\} \) up to an overall constant. For \( a \neq b \) these expressions take an exceptionally unwieldy form; for the simplified case \( a = b \) that we shall study numerically, we find exponent

\[ \nu_\Phi = \frac{3 + b^2}{6b^4} \left(-3 + \sqrt{81 - 24b^2}\right), \]  

and coefficients

\[ \{a_f, a_\beta, a_A, a_\Phi\} = a^{IR} \left\{-f_0, 2, \frac{A_0 \nu_\Phi b^2}{2(3 - b^2)}, -\frac{b}{\sqrt{3}}\right\}, \]
with an overall free magnitude $a^{IR}$. This is similar to the irrelevant deformation of the fractionalised phase in [47]. This is not surprising: the form of the bulk matter is not important when all the charge is sourced by an IR horizon.

As we expected, the metric in solution (4.26) has a hyperscaling symmetry. The dynamical critical exponent is

$$z = \frac{12 + (a - 3b)(a + b)}{a^2 - b^2},$$

(4.33)

and the hyperscaling parameter is

$$\theta = \frac{4b}{b - a}.$$  

(4.34)

We note that both these parameters are formally undefined when $a = b$, which includes our primary case $a = b = 1$. This does not indicate any physical discontinuity; they only enter the entropy scaling relation (4.6) in the combination $(d - \theta)/2$, which remains finite in the limit $a \to b$. This limit is discussed further in [171].

**Critical Solution**  A key feature of the critical IR solution is that the dilaton remains finite: from the form of the potential (4.13) we expect $\Phi = 0$. We find two such solutions, both having metric the $AdS_4$ metric and a vanishing gauge field $A_{(1)}$. They are distinguished by the IR value of $\eta$: the first sits at the $\eta = 0$ (stable) maximum of the potential (4.13), *i.e.* it is the IR part of a solution (4.20); the second sits at the minimum $\eta = g_\eta^{-1}$, and has a shifted value of the $AdS$ length scale. We are interested in the latter solution:

$$f(r) e^{-\beta(r)} = r^2,$$

$$f(r) = \frac{r^2}{R_{IR}^2},$$

$$A_t(r) = 0,$$

$$\Phi(r) = 0,$$

$$\eta(r) = \pm g_\eta^{-1},$$

(4.35)

with length scale

$$R_{IR}^2 = \frac{6g_\eta^3 \ell^2}{1 + g_\eta^2 V_0^2}.$$  

(4.36)
Unlike the fractionalised and cohesive cases, this is an exact solution requiring no sub-leading corrections. This solution has no flux, and so the scalar \(\eta\) carries no charge despite its non-zero vacuum expectation value. However there exists a marginal deformation involving \(A_t(r)\), and so the full interpolating solution (which is an \(AdS_4\) to \(AdS_4\) domain wall) may have a non-zero \(Q\). This deformation has the form

\[
\delta A_t = a^{IR}_A r^{\nu_A}
\]

with exponent

\[
\nu_A = \frac{1}{2} \left( -1 + \sqrt{1 + \frac{48\ell^2 q^2}{Z_0^2 (1 + g_0^2 V_0^2)}} \right).
\]

There is a second, independent deformation - this time irrelevant - involving \(\eta\):

\[
\delta \eta = a^{IR}_n r^{\nu_n},
\]

with exponent

\[
\nu_n = \frac{3}{2} \left( -1 + \sqrt{1 + \frac{32g_0^2}{3 (1 + g_0^2 V_0^2)}} \right).
\]

**Cohesive IR Solution**  Finally we seek the cohesive solutions, by allowing \(\Phi \to -\infty\) and demanding that the quantity \(Q_F\) vanishes. Similar to our remarks in the critical case, a non-zero charge is not necessarily apparent in the IR solution: we only require that there exist deformations that allow this in the full, interpolating solution. We find a solution of the form (4.22), with leading order part

\[
f(r)e^{-\beta(r)} = r^2, \\
f(r) = f_0r^{\frac{2(3-b^2)}{3}}, \\
A_t(r) = 0, \\
\Phi(r) = \frac{2b}{\sqrt{3}} \log r,
\]

with coefficient

\[
f_0 = \frac{3V_0^2}{4(9-b^2)}.
\]
The higher order corrections have a stepsize \( n = 2/3 \). Note that we have not specified the form of the charged scalar in the solution (4.41). Just as in the fractionalised phase, \( \eta = 0 \) is a valid solution, and there is a marginal deformation corresponding to a shift \( \delta \eta = \eta_0 \) in the leading order vacuum expectation value. However there is a second marginal deformation, which depends on the value of \( \eta_0 \) in a curious fashion. Its general form is complicated, but for our important case \( a = b \) it can be written as

\[
\delta A_t = a^{IR} R^{\frac{3+b^2}{6}+\nu_A}.
\] (4.43)

where the exponent \( \nu_A \) is not a constant, but rather a function:

\[
\nu_A(\eta) = \frac{3 + b^2}{6} \left( -2 + \sqrt{1 + \frac{72q^2\eta_0^2}{Z_0 f_0(3 + b^2)^2}} \right).
\] (4.44)

As in equation (4.25), this is merely the leading order contribution to a deformation series solution; there are additional corrections in powers of \( 2\nu_A \). If we wish to switch on this deformation - and we must switch it on if we are to have non-zero charge - then we have the non-trivial consistency condition that these corrections be subleading, which translates into the constraint on the condensate:

\[
\eta_0^2 > \eta_* \equiv \frac{(3 + b^2)^2 f_0 Z_0^2}{24q^2}.
\] (4.45)

A similar structure was seen in the zero-temperature superconducting solutions of [172]. In conclusion, there exists a one-parameter connected family of IR solutions (4.41) taking all values of \( \eta \), but this is not true of the cohesive solutions interpolating between (4.41) and (4.20).

This metric also has a non-trivial hyperscaling symmetry. After an appropriate change of variables, we this time find dynamical critical exponent \( z = 1 \) and a hyperscaling parameter

\[
\theta = -\frac{2b^2}{3 - b^2}.
\] (4.46)

### 4.2.3 Finite Temperature IR Geometries

Recall that a non-zero temperature is represented holographically by a geometry with a horizon at some \( r = r_+ \). This horizon masks the dilaton divergence, and so we must
make a very different ansatz in the IR:

\[
\begin{align*}
  f(r) &= f_0(r - r_+) , \\
  \beta(r) &= \beta_0 , \\
  A_t(r) &= A_0(r - r_+) , \\
  \eta(r) &= \eta_0 , \\
  \Phi(r) &= \Phi_0 ,
\end{align*}
\]  

(4.47)

for some constants \( f_0, \beta_0, A_0, \eta_0 \) and \( \Phi_0 \). Again this is just the leading order part of a series solution: we expect subleading corrections in powers of \((r - r_+)\).

Since the fractionalisation transition is a quantum critical point, we do not expect to see any discontinuity at finite temperature between the partially fractionalised and cohesive phases, though we do expect the characteristic quantum critical wedge to be clearly visible at very low temperatures. For this reason we do not attempt to classify the possible IR geometries here.

The existence of \( U(1) \) symmetry breaking vacuum expectation value for \( \eta \) is not a purely IR statement, so we do expect a sharp transition between the partially and fully fractionalised phases, even at non-zero temperature. There is a critical temperature associated with this transition. This transition typically defines a superconducting dome [172,173]; our model is no different, with a well-defined dome apparent in figure 6.

4.2.4 Numerical Results

For each of the geometries of sections 4.2.2 and 4.2.3, one can construct domain wall solutions of action (4.7) with these IR limits and with asymptotics the \( AdS_4 \) asymptotics of section 4.2.1. In this section we shall briefly describe the resulting phase diagram of our system. Before we present our results, we briefly summarise our numerical methods.

We employ a common variation of the shooting method. Firstly we must specify some initial guess for the seven UV constants \((G_1, \beta_a, \Phi_1, \Phi_2, \eta_2, \mu, Q)\) and for the two IR constants \((a^{IR}, \eta_0)\) corresponding to the two deformations about the fractionalised and cohesive phases. We can write these values as a vector \( v_i \), with \( i = 1, \ldots, 9 \). After some manipulation, it can be shown that the equations of motion arising from our ansatz (4.17) are first order in \( f(r) \) and \( \beta(r) \) and second order in \( A_t(r), \Phi(r), \eta(r) \).
solution can therefore be specified by the value of \( u_p(r) \) at some \( r \), where \( p = 1, \ldots, 8 \) and
\[
 u(r) \equiv \{ f(r), \beta(r), A_i(r), A'_i(r), \Phi(r), \Phi'(r), \eta(r), \eta'(r) \}.
\] (4.48)

Given some choice of parameters \( v_i \), we evaluate our IR geometry at some \( r = r_{\text{min}} \ll 1 \) to form the vector \( u_p(r_{\text{min}}) \). With these boundary conditions, we numerically integrate our equations of motion from \( r = r_{\text{min}} \) to some \( r = r_{\text{mid}} \), with \( r_{\text{mid}} \) of \( \mathcal{O}(1) \). We thus find a vector \( u_p^- \equiv u_p(r_{\text{mid}}) \). Similarly we evaluate our UV geometry at some \( r = r_{\text{max}} \gg 1 \) to form the vector \( u_p^+ \equiv u_p(r_{\text{max}}) \) and numerically integrate from \( r = r_{\text{max}} \) to \( r = r_{\text{min}} \) to obtain \( u_p^+(r_{\text{mid}}) \). We have found a domain wall solution when
\[
 d(u^+, u^-) = \sqrt{\langle u^+ - u^-, u^+ - u^- \rangle} < \epsilon
\] (4.49)
for some numerical tolerance \( \epsilon \ll 1 \). If equation (4.49) is not satisfied, we apply Newton’s method, i.e. we introduce a new choice of parameters
\[
 \tilde{v}_i = v_i - J^{-1}_{ip} \left( u_p^+ - u_p^- \right),
\] (4.50)
and iterate the procedure. Here \( J^{-1} \) is the generalised inverse \( (J^T J)^{-1} J^T \) of the Jacobian
\[
 J_{pi} = \frac{\partial u_p}{\partial v_i}.
\] (4.51)

Now, our parameter space of free parameters \( n_i \) is restricted by the boundary conditions \( u_p^+ = u_p^- \), and so we expect a 1-parameter family of solutions for both the fractionalised and cohesive phases. We consider the phase diagram as a function of the dimensionless parameter\(^{52} \Phi_1/\mu \).

### Zero Temperature Domain Walls

We now apply our shooting method to the zero-temperature IR geometries of section 4.2.2. We find that the cohesive phase exists for values \( \Phi_1/\mu < g_f \approx -0.131 \). We find two distinct branches for the fractionalised phase: a fully fractionalised (\( \eta \equiv 0 \) globally) phase that exists for all values of \( \Phi_1/\mu \); and a partially fractionalised phase that exists in the range \( g_f < \Phi_1/\mu < g_{\text{SC}} \approx 0.621 \).

The unique critical solution has \( \Phi_1/\mu \approx g_f \).

A holographic measure of fractionalisation is the ratio of the charge \( Q_F \) (c.f. equation (4.19)) to the total charge \( Q \). We plot this quantity in figure 4. As required, this

\(^{52}\)In practice we first fix \( \Phi_1/\mu \) and then apply our shooting method to find the corresponding isolated solution (or solutions). Then \( J \) is a square matrix, and the generalised inverse is just the usual inverse.
Figure 4: The proportion of fractionalised charge in each of the three phases, as a function of $\Phi_1/\mu$. The fully fractionalised phase, shown in green, and the cohesive phase, shown in dark red, have $Q_F/Q$ unity and zero respectively. Between the critical points $g_f$ and $g_{SC}$, the partially fractionalised phase, shown in light red, interpolates between these two values. Supplementary ($T/\mu \simeq 10^{-3}$) data is indicated by a dashed line.

ratio is zero in the cohesive phase, unity in the fully fractionalised phase, and takes some intermediate value in the partially fractionalised phase. Moreover the partially fractionalised phase interpolates between these limits between $\Phi_1/\mu = g_f$ and $\Phi_1/\mu = g_{SC}$.

These results strongly indicate that there are two phase transitions in our system: a fractionalisation transition at $\Phi_1/\mu = g_f$ between cohesive and partially fractionalised phases; and a second transition between broken and unbroken $U(1)$ phases at $\Phi_1/\mu = g_{SC}$. To make a definitive statement we must calculate the free energy of each phase; we require that the fully fractionalised branch be thermodynamically non-preferred whenever it coexists with the cohesive or partially fractionalised branches. The free energy is given by a combination of the UV parameters [2]:

$$\Omega = -\frac{1}{3}\mu Q - \frac{1}{3}\Phi_1\Phi_2.$$  \hspace{1cm} (4.52)

For each branch, the dimensionless quantity $\Omega/\mu^3$ is given in figure 5. We indeed find that the fully fractionalised phase has a higher free energy. This plot allows us to draw some conclusions as to the nature of these transitions. At $\Phi_1/\mu = g_{SC}$ we find a kink
in the first derivative of the free energy, implying a second-order transition there. This is typical of transitions between broken (superconducting) and unbroken $U(1)$ phases. For the fractionalisation transition $\Phi_1/\mu = g_f$ we computed the first two derivatives of the free energy and found them to be continuous, so this is at least a third-order transition. We were unable to accurately compute higher derivatives.

It proved numerically unprofitable to calculate the free energies of the cohesive and partially fractionalised branches in the region immediately adjacent to the $\Phi_1/\mu = g_f$ critical point. We have thus supplemented figures 4 and 5 with low temperature ($T/\mu \simeq 10^{-3}$) data.

**Finite Temperature Domain Walls**  From our zero temperature analysis a clear picture has emerged of the structure of the phase diagram. We support our conclusions with a brief discussion of the finite temperature domain walls.$^{53}$

Just as in the zero temperature case, we find that a broken $U(1)$ phase exists only for some range $\Phi_1/\mu < f_{SC}$, where now $f_{SC}$ is some function $f_{SC}(T)$. The unbroken (fully fractionalised) phase exists for all values of $\Phi_1/\mu$, but is not thermodynamically

---

$^{53}$We thank J. Sonner and B. Withers for carrying out this part of the analysis, and for producing figures 6, 7, and 8. These results are found in [2].
preferred. The function $f_{SC}(T)$ thus describes the superconducting dome discussed in section 4.2.3. This is clearly displayed in figure 6. A good test of our $T = 0$ results is that the superconducting dome intersects $T = 0$ at our critical point $g_{SC} \approx 0.621$. We see that our results are indeed consistent.

Recall that the finite temperature IR solutions of section 4.2.3 have a finite dilaton, while the zero temperature geometries of section 4.2.2 are classified by the sign of a logarithmic dilaton divergence. In figure 7 we plot the constant leading order part of the dilaton, c.f. equation (4.47). We see that the divergent dilaton behaviour begins to emerge at very low temperatures, with the quantum critical point having the expected $\Phi_0 = 0$.

The entropy scaling relation (4.6) offers a final, non-trivial check that we have identified the correct $T = 0$ geometries. Writing $S \sim (T/\mu)^{\frac{2}{\alpha}}$, we plot the value of $\alpha$ in figure 8. From equations (4.33) and (4.34) we expect a value $\alpha = 2$ in the fractionalised phase. From equation (4.46) we expect $\alpha = 2/3$ in the cohesive phase. The finite temperature results are entirely consistent with these values. Moreover we see can clearly see the characteristic wedge shape of the quantum critical point imprinted on the data.

![Figure 6: Finite temperature solutions with a broken $U(1)$ symmetry are shown for various values of $\Phi_1/\mu$. This phase exists only for a certain subset of the parameter space $(\Phi_1/\mu, T/\mu)$, the edge of which defines a superconducting dome, indicated by the dashed line. The dome intersects $T = 0$ at the critical point $\Phi_1/\mu = g_{SC} \approx 0.621$.](image)
4.3 Conclusions

In our introduction to this chapter we touched upon the subject of entropy at zero temperature. Charged black holes at $T = 0$ usually have an IR geometry $AdS_2 \times \mathbb{R}^2$, and the $AdS_2$ component is dual to a $d = 1$ field theory. From the entropy scaling relation (4.6) we see that a $d = 1$ field theory with $\theta = 0$ has $S \sim T^0$ at low temperature. Equivalently we say that our true $d = 3$ system has $z = \infty$. The third law of thermodynamics (in the form of Nernst’s theorem) states that the ground state of a condensed matter system has vanishing entropy. One usually takes this to mean that the fractionalised phase is unstable whenever a dual field theory exists\(^5\). A good example of this can be found in [50,51], in which an action of the form (4.7) was obtained via a consistent truncation of a dimensional reduction of $D = 11$ supergravity on a Sasaki-Einstein manifold [175]. There the $z = \infty$ fractionalised phase was exactly masked by a cohesive phase with lower free energy. For similar instabilities, see e.g.

\(^5\)It is generally believed that this is the case whenever an action with asymptotically $AdS$ solutions can be embedded in $D = 10$ or $D = 11$ supergravity. Curiously the work [174] seems to offer a counterexample: a $T = 0$ black hole solution carrying a magnetic charge with stability guaranteed by an unbroken supersymmetry.
Figure 8: The entropy scaling parameter $\alpha$ as a function of $T/\mu$ and $\Phi_1/\mu$. At low temperatures, $\alpha$ approaches the predicted values $\alpha = 2/3$ in the cohesive phase and $\alpha = 2$ in the fractionalised phase. A characteristic quantum critical wedge is clearly visible, consistent with an apex $\Phi_1/\mu \approx g_f$.

[176].

The new ingredient in our zero temperature solutions is hyperscaling. In the fractionalised phase we have the expected $z = \infty$, but this is compensated for by an equally divergent parameter $\theta$. The result is an entropy scaling $S \sim T$, so that Nernst’s theorem is satisfied and the fractionalised phase is stable. The same mechanism was employed in [47,171].

Generally we have that the entropy of a black hole is proportional to the area of its horizon. It follows that the price of a zero entropy fractionalised phase is a naked singularity - this is indeed the case in our solution (4.26). In a string theory embedding we expect higher order derivative terms in the action to become important when the radius or curvature is of the order the string length $\sqrt{\alpha'}$. The divergent dilaton is also expected to invalidate the supergravity description in the deep IR by introducing quantum corrections [177,178]. Nevertheless the structure we have seen in this chapter will still describe fractionalisation transitions over an intermediate range of energy.
We remarked in section 4.1 that a non-zero hyperscaling parameter could be thought of as a shift in the effective dimension of the system. Normally one is interested in a $\theta > 0$ that lowers the effective dimension, but in our model (4.7) we have seen a negative hyperscaling in both the fractionalised and cohesive phases. Intriguingly this suggests a dimensional surplus at low energies, i.e. that our model can be embedded in some higher dimensional action with an extra dimension that decompactifies as $r \to 0$. This was seen in [51], where the apparently singular geometries with negative hyperscaling were uplifted to regular higher-dimensional solutions [2,179]. In this way one might be able to avoid stringy corrections in a model of fractionalisation.
References


