Generalized seismic wavelets

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SUMMARY
The Ricker wavelet, which is often employed in seismic analysis, has a symmetrical form. Seismic wavelets observed from field data, however, are commonly asymmetric with respect to the time variation. In order to better represent seismic signals, asymmetrical wavelets are defined systematically as fractional derivatives of a Gaussian function in which the Ricker wavelet becomes just a special case with the integer derivative of order 2. The fractional value and a reference frequency are two key parameters in the generalization. Frequency characteristics, such as the central frequency, the bandwidth, the mean frequency and the deviation, may be expressed analytically in closed forms. In practice, once the statistical properties (the mean frequency and deviation) are numerically evaluated from the discrete Fourier spectra of seismic data, these analytical expressions can be used to uniquely determine the fractional value and the reference frequency, and subsequently to derive various frequency quantities needed for the wavelet analysis. It is demonstrated that field seismic signals, recorded at various depths in a vertical borehole, can be closely approximated by generalized wavelets, defined in terms of fractional values and reference frequencies.

Key words: Time-series analysis; Numerical solutions; Computational seismology; Wave propagation.

INTRODUCTION
The Ricker wavelet is a well-known symmetrical waveform in the time domain (Ricker 1953). In order to better represent practically observed non-Ricker forms of seismic signals (Hosken 1988), the symmetric Ricker wavelet is generalized to be asymmetrical.

While the Ricker wavelet is the second derivative of a Gaussian function, generalization is achieved by modifying the derivative order from the integer ‘2’ to a fractional value. For mathematical convenience, the base function is the same Gaussian function, rather than other alternative forms. Therefore, generalized wavelets are systematically defined by fractional derivatives of a Gaussian function.

Generalized wavelets have similar Fourier spectra because they are derived from the same Gaussian function. Their spectra differ from each other only in a frequency-related factor $(i \omega)^u$, where $\omega$ is the angular frequency and $u$ is the fractional order of the time derivative. Although there are possibly different definitions for field seismic wavelets, the current paper provides a systematic definition of non-Ricker wavelets and meanwhile can use the Ricker wavelet as a benchmark. This paper will prove that the power spectrum of a generalized wavelet is close to a Gaussian distribution, and thus the mean frequency is approximately equal to the central frequency. The degree of similarity between a generalized wavelet spectrum and a Gaussian distribution depends upon the fractional value.

For a generalized wavelet with a variable fractional value $u$, the central frequency and the frequency band can be expressed analytically, using a special function, the Lambert $W$ function (Lambert 1758, 1772; Euler 1779; Corless et al. 1996; Banwell & Jayakumar 2000; Valluri et al. 2000; Packel & Yuen 2004; Shafee 2007; Wang 2015a,b). The mean frequency and its deviation can also be derived analytically in terms of the Gamma function. Thus, the analytical relationships between the theoretical properties (the central frequency and the bandwidth) and statistical parameters (the mean frequency and its deviation) may be established.

While these frequency analyses are inspired by previous studies (Wang 2015a,b) on the Ricker wavelet, which is just a special case with $u = 2$, the systematic definition of generalized wavelets and the non-trivial development of frequency relationships will certainly set up a solid foundation for field seismic signal analysis. In practice, given a discrete Fourier spectrum, the mean frequency and the standard deviation can be evaluated numerically. Once these two quantities are measured from field seismic data, the analytical expressions mentioned above can be used to derive the fractional value, the reference frequency and other frequency parameters for wavelet and spectral analysis.

WAVELETS DEFINED IN THE FREQUENCY DOMAIN
According to Ricker (1943, 1944), a wavelet function (of the displacement, velocity or acceleration type) may be expressed as a
polynomial of various derivatives of a potential function. Therefore, we set the potential function in (negative) Gaussian:

\[ g(\tau) = -\sqrt{\pi} \omega_0 \exp \left( -\frac{\alpha^2}{4}(\tau - \tau_0)^2 \right), \]

where \( \tau \) is a time variable (in seconds), \( \tau_0 \) is the time position of the symmetrical centre and \( \omega_0 \) is a reference frequency (in radians per second). This \( \omega_0 \) parameter is inversely proportional to the deviation of the Gaussian distribution and should physically reflect the viscoelastic property of the subsurface media. Setting the potential function as a negative Gaussian will produce a conventional polarity of the wavelets.

In this section, generalized wavelets are defined in the frequency domain. The Fourier transform of the Gaussian function (1) is

\[ G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tau) \exp(-i\omega \tau) \, d\tau \]

\[ = -\exp \left( -\frac{\alpha^2}{\omega_0^2} \right) \exp(-i\omega \tau_0). \]  

(2)

For any fractional or integer order derivative of the Gaussian function, with respect to the time \( \tau \), the frequency spectrum may be expressed as \( G(\omega) \) multiplied by a frequency factor \( (i\omega)^u \):

\[ (i\omega)^u G(\omega) = \omega^u \exp \left( -\frac{\omega^2}{\omega_0^2} \right) \exp \left( -i\omega \tau_0 + i\pi \left( 1 + \frac{u}{2} \right) \right), \]

where \( u \) is the order of a time-domain derivative. A normalized spectrum \( \Phi^{(u)}(\omega) \) is set, by multiplying a factor of \( \omega_0^{-u/2} \omega^{u/2} \exp(u/2) \) to spectrum (3), as

\[ \Phi^{(u)}(\omega) = \left( \frac{\omega}{2} \right)^{-u/2} \omega^u \exp \left( -\frac{\omega^2}{2 \omega_0^2} + \frac{u}{2} \right) \exp(-i\omega \tau_0 + i\pi \left( 1 + \frac{u}{2} \right)). \]  

(4)

Performing an inverse Fourier transform on \( \Phi^{(u)}(\omega) \), one can generate a time-domain wavelet \( g^{(u)}(\tau) \). When \( u \) is an integer, the inverse Fourier transform can be derived analytically. But, for any fractional value of \( u \), the inverse Fourier transform needs to be calculated numerically.

Fig. 1 displays a series of wavelets \( g^{(u)}(\tau) \) (in solid curves), defined with a sample reference frequency \( \omega_0 = 60\pi \) rad s\(^{-1} \), equivalent to the ordinary frequency of 30 Hz. The fractional value \( u \) varies between 0.4 and 2.2. These wavelets centred at time \( \tau_0 = 0.05 \) s are overlaid by approximations (the dashed curves) obtained using time-domain fractional derivatives, described in the following section.

**WAVELETS DEFINED BY FRACTIONAL DERIVATIVES**

Since wavelets are commonly presented in the time domain, this section presents generalized wavelets as time-domain fractional derivatives of a Gaussian function.

For a derivative of order \( u \), which can be either a positive fraction or an integer, of the Gaussian function \( g(\tau) \), the following definition of Caputo (1967) is used:

\[ g^{(u)}(\tau) = \frac{1}{\Gamma(m-u)} \int_{0}^{\tau} (\tau - \xi)^{m-u-1} g^{(m)}(\xi) \, d\xi, \]  

(5)

where \( g^{(m)}(\tau) \equiv d^m g(\tau)/d\tau^m \), \( m \) is an integer, \( m - 1 \leq u < m \), \( g^{(m)}(\tau) \) is the \( m \)th integer-order derivative and \( \Gamma(s) \) is the Gamma function. Eq. (5) can be understood as a Laplace convolution of two causal functions:

\[ g^{(u)}(\tau) = g^{(m)}(\tau) * h(\tau), \]  

(6)

where \( g^{(m)}(\tau) = 0 \), for \( \tau < 0 \), and

\[ h(\tau) = \begin{cases} \tau^{m-u-1} \Gamma(m-u), & \tau > 0, \\ 0, & \tau \leq 0. \end{cases} \]  

(7)

The convolution expression (6) is equivalent to a Fourier transform domain multiplication between frequency spectra of \( g^{(m)}(\tau) \) and \( h(\tau) \). The Fourier transform of \( h(\tau) \) is the factor \( (i\omega)^u \) in eq. (3).

For fractional derivatives, there are various definitions, such as conventional Riemann–Liouville integral (Podlubny 2002) and its modification (Jumarie 2006). These definitions form the integer-order derivative \( d^n d^m \) outside the integral, rather than being inside of the integral. They are theoretically equivalent, based on the Leibniz integral rule. Caputo’s definition is used here because any integer-order derivative \( g^{(m)}(\tau) \) can be easily expressed in an analytical form, and fractional derivatives can be written explicitly as follows.
For 0 \leq u < 1, where m = 1:

\[
g^{0}(\tau) = \frac{\sqrt{\pi} \omega_0^u}{2 \Gamma(1-u)} \int_0^\tau (\tau - \xi)^{-u}(\xi - \tau_0) \times \exp \left(-\frac{\omega_0^2}{4}(\xi - \tau_0)^2\right) d\xi. \tag{8}
\]

For 1 \leq u < 2, where m = 2:

\[
g^{1}(\tau) = \frac{\sqrt{\pi} \omega_0^u}{2 \Gamma(2-u)} \int_0^\tau (\tau - \xi)^{-u-1} \left(1 - \frac{\omega_0^2}{2}(\xi - \tau_0)^2\right) \times \exp \left(-\frac{\omega_0^2}{4}(\xi - \tau_0)^2\right) d\xi. \tag{9}
\]

For 2 \leq u < 3, where m = 3:

\[
g^{2}(\tau) = \frac{3\sqrt{\pi} \omega_0^u}{4 \Gamma(3-u)} \int_0^\tau (\tau - \xi)^{-u-2}(\xi - \tau_0) \times \left(\frac{\omega_0^2}{6}(\xi - \tau_0)^2 - 1\right) \exp \left(-\frac{\omega_0^2}{4}(\xi - \tau_0)^2\right) d\xi. \tag{10}
\]

Considering special cases with the integer values u = 0, 1, 2, the corresponding m values are m = 1, 2, 3, eqs (8)-(10) will be

\[
g^{0}(\tau) = g^{(-1)}(\xi), \quad g^{1}(\tau) = g^{(-1)}(\tau) - g^{(-1)}(0). \tag{11}
\]

What one expected here is g^{0}(\tau) = g^{(-1)}(\tau), for u = 0, 1, 2. This expectation is fulfilled if g^{(-1)}(0) is zero-valued. Hence, a mathematical condition for Caputo’s integral (5) is

\[
g^{(-1)}(\tau) = 0, \quad \text{for} \quad \tau \leq 0. \tag{12}
\]

This condition elicits the importance of the time-shift \tau_0 in the Gaussian function (1). The time-shift will make the above condition be satisfied by Caputo’s definition (5), and explicitly by three eqs (8–10), in which the integrals are implemented using the following Simpson’s 3/8 rule in each subinterval:

\[
\int_a^b f(\xi) d\xi \approx \frac{(b-a)}{8} \left(f(a) + 3f \left(\frac{2a+b}{3}\right) + 3f \left(\frac{a+2b}{3}\right) + f(b)\right). \tag{13}
\]

A wavelet is defined by the normalized derivative \phi^{0}(\tau) = g^{0}(\tau), where the overbar denotes the normalization. Fractional derivatives are approximations due to the integral singularity when \xi \to \tau. However, Fig. 1 indicates that these approximated wavelets (dashed curves) have some negligible errors, only when u = 0.8 and 1.8, if they are compared to the accurate waveforms (solid curves), obtained by a Fourier transform method, presented in the previous section.

**FREQUENCY CHARACTERISTICS**

For any real-valued wavelet \phi^{\omega}(\tau), the frequency spectrum \Phi^{\omega}(\omega) needs to be considered only for \omega \geq 0. The amplitude spectrum is

\[
|\Phi^{\omega}(\omega)| = \frac{u^{\nu/2}}{\omega_0^\nu} \omega_0^u \exp \left(-\frac{\omega^2}{\omega_0^2} + \frac{u}{2}\right). \tag{14}
\]

Fig. 2 displays the amplitude spectra |\Phi^{\omega}(\omega)| with different u values (0.4 \leq u \leq 2.2). The peak frequency is

\[
\omega_p = \omega_0 \sqrt{\frac{u}{2}}. \tag{15}
\]

For example, the peak frequencies are \omega_p = (\sqrt{\frac{2}{3}}\omega_0, \sqrt{\frac{5}{3}}\omega_0, \omega_0) for the first, one-and-a-half and second derivatives, respectively. This proves that only the Ricker wavelet with u = 2 has its peak frequency equal to \omega_0. When u < 2, the peak frequency is generally smaller than the reference frequency.

The peak amplitude is unity. To determine the frequency band, an equation is set up so that the amplitude spectrum equals to one half,

\[
\left(\frac{\omega}{\omega_0}\right)^u \exp \left(-\frac{\omega^2}{\omega_0^2} + \frac{u}{2}\right) = \frac{1}{2}. \tag{16}
\]

Manipulating this equation leads to the following form:

\[
-\frac{2}{u} \frac{\omega_0^2}{\omega_0^2} \exp \left(-\frac{2\omega^2}{u\omega_0^2}\right) = -\frac{1}{2^{2/u}} e^\nu. \tag{17}
\]

where e is Euler’s number, e = \exp(1). Eq. (17) is an inverse exponential equation,

\[
z \exp z = x, \tag{18}
\]
Table 1. Numerical values of the Lambert $W(x)$ function, where $x = -(2^v/e)^{-1}$, corresponding to various $u$ values.

<table>
<thead>
<tr>
<th>$u$</th>
<th>$x$</th>
<th>$W_{-1}(x)$</th>
<th>$W_0(x)$</th>
</tr>
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<tr>
<td>0.4</td>
<td>-0.011496</td>
<td>-6.307471</td>
<td>-0.011631</td>
</tr>
<tr>
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<td>-5.472285</td>
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<tr>
<td>0.7</td>
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<td>-4.480656</td>
<td>-0.053565</td>
</tr>
<tr>
<td>0.8</td>
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</tr>
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<td>0.9</td>
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<td>-0.085914</td>
</tr>
<tr>
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</tr>
<tr>
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<td>-3.518224</td>
<td>-0.117307</td>
</tr>
<tr>
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<td>-3.370224</td>
<td>-0.132260</td>
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<tr>
<td>1.3</td>
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<td>-0.146646</td>
</tr>
<tr>
<td>1.4</td>
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<tr>
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<td>2.2</td>
<td>-0.195903</td>
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<td>-0.252065</td>
</tr>
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</table>

with a solution $z = W(x)$, where $W(x)$ is the Lambert $W$ function (Corless et al. 1996; Wang 2015a). Hence the solution to this equation is

$$W\left(\frac{-1}{2^{v/e}}\right) = \frac{2}{u} \frac{\omega^2}{\omega_0}.$$  \hspace{1cm} (19)

For $x < 0$, the $W(x)$ function has two branches, $W_{-1}(x) \leq -1$ and $W_0(x) \geq -1$. Then, the frequency band $[\omega_{-1}, \omega_2]$ may be analytically defined as

$$\omega_{-1} = \omega_0 \sqrt{\frac{u}{2} W_0\left(-\frac{1}{2^{v/e}}\right)},$$

$$\omega_2 = \omega_0 \sqrt{\frac{u}{2} W_{-1}\left(-\frac{1}{2^{v/e}}\right)}. \hspace{1cm} (20)$$

Correspondingly, the central frequency, the geometric centre of the frequency band, is

$$\omega_c = \frac{\omega_0}{2} \left(\sqrt{\frac{u}{2} W_{-1}\left(-\frac{1}{2^{v/e}}\right)} + \sqrt{\frac{u}{2} W_0\left(-\frac{1}{2^{v/e}}\right)}\right). \hspace{1cm} (21)$$

and the half-bandwidth is

$$\omega_b = \frac{\omega_0}{2} \left(\sqrt{\frac{u}{2} W_{-1}\left(-\frac{1}{2^{v/e}}\right)} - \sqrt{\frac{u}{2} W_0\left(-\frac{1}{2^{v/e}}\right)}\right). \hspace{1cm} (22)$$

For practical application, Table 1 lists numeric values of the Lambert $W$ function: $W_{-1}(x)$ and $W_0(x)$, where $x = -(2^{v/e})^{-1} < 0$, corresponding to a series of $u$ values, 0.4 ≤ $u$ ≤ 2.2.

**Statistical Properties**

The statistical properties of the discrete Fourier spectrum of a field seismic signal can be described by the mean frequency and the standard deviation. This section derives analytical expressions for these two frequency parameters.

The mean frequency and its deviation may be evaluated from the power spectrum $|\Phi^{(u)}(\omega)|^2$ by using (Berkhourt 1984; Cohen & Lee 1989)

$$\omega_m = \frac{\int_0^\infty \omega |\Phi^{(u)}(\omega)|^2 d\omega}{\int_0^\infty |\Phi^{(u)}(\omega)|^2 d\omega},$$

$$\omega_o = \sqrt{\frac{\int_0^\infty (\omega - \omega_m)^2 |\Phi^{(u)}(\omega)|^2 d\omega}{\int_0^\infty |\Phi^{(u)}(\omega)|^2 d\omega}}. \hspace{1cm} (23)$$

The three definite integrals are

$$\int_0^\infty |\Phi^{(u)}(\omega)|^2 d\omega = \frac{\omega_0 e^{u}}{2\sqrt{2\pi u}} \Gamma\left(u + \frac{1}{2}\right), \hspace{1cm} (24)$$

$$\int_0^\infty \omega |\Phi^{(u)}(\omega)|^2 d\omega = \frac{\omega_0^2 e^{u}}{4\pi u^{3/2}} \Gamma(u), \hspace{1cm} (25)$$

$$\int_0^\infty (\omega - \omega_m)^2 |\Phi^{(u)}(\omega)|^2 d\omega = \frac{\omega_0^2 e^{u}}{4\pi u^{5/2}} \left\{ \Gamma\left(u + \frac{1}{2}\right) \times \left(\frac{1}{2} + u + \frac{1}{2} + \sqrt{\frac{2\omega_0^2}{\omega_0^2}}\right) \right\} - 2\omega_0^2 \Gamma(u). \hspace{1cm} (26)$$

Then, the analytical expression for the mean frequency is

$$\omega_m = \frac{\omega_0}{\sqrt{2}} \frac{u \Gamma(u)}{\Gamma(u + \frac{1}{2})}. \hspace{1cm} (27)$$

and for its deviation is

$$\omega_o = \omega_0 \sqrt{\frac{1}{2} + u - \left(\frac{u \Gamma(u)}{\Gamma(u + \frac{1}{2})}\right)^2}. \hspace{1cm} (28)$$

For $u = 2$, where $\Gamma(u) = 1$, $\Gamma(u + \frac{1}{2}) = \frac{1}{4}\sqrt{\pi}$, the statistic quantities of the Ricker spectrum are (Wang 2015b)

$$\omega_m = \frac{4}{3} \sqrt{\frac{2}{\pi}} \omega_0, \hspace{1cm} \omega_o = \omega_0 \sqrt{\frac{5}{4} \frac{32}{9\pi}}. \hspace{1cm} (29)$$

Fig. 3 displays the power spectra, with annotations of the mean frequencies and the deviations, of wavelets defined by different $u$ values. It reveals that the mean frequency $\omega_m$ is close to the central frequency $\omega_c$. This observation was found first in the case $u = 2$, when the mean frequency was evaluated from the power spectrum of a Ricker wavelet (Wang 2015b).

Using the two statistical parameters $\omega_m$ and $\omega_o$, an equivalent Gaussian function can be constructed. Fig. 3 above also shows how the wavelet power spectra (solid curves) are close to the Gaussian distribution (dashed curves).

Frequency characteristics can be presented as the reference frequency $\omega_0$ multiplied by fraction-dependent coefficients:

$$\begin{bmatrix} \omega_p \\ \omega_c \\ \omega_b \\ \omega_m \\ \omega_o \end{bmatrix} = \begin{bmatrix} \alpha_p(u) \\ \alpha_c(u) \\ \alpha_b(u) \\ \alpha_m(u) \\ \alpha_o(u) \end{bmatrix}. \hspace{1cm} (30)$$

These frequency coefficients ($\alpha_p, \alpha_c, \alpha_b, \alpha_m, \alpha_o$) can be found from eqs (15), (21), (22), (27) and (28), respectively. Fig. 4 compares
(2) the peak frequency $\omega_p$ is less than the central frequency $\omega_c$ and the mean frequency $\omega_m$;
(3) the half-bandwidth $\omega_b$ is wider than the standard deviation $\omega_\sigma$; and
(4) variations in the half-bandwidth $\omega_b$ and the deviation $\omega_\sigma$ are both relatively small, along with the fraction.

**FIELD SIGNAL ANALYSIS**

Once the mean frequency and its deviation are evaluated from field seismic spectra, the fractional value $u$ and the reference frequency $\omega_0$ can be derived.

The fractional value of $u$ can be uniquely determined by the ratio of the standard deviation to the mean frequency, using the following equation:

$$
\left( \frac{1}{2u} + 1 \right) \left( \frac{\Gamma(u + \frac{1}{2})}{\sqrt{u!} (u)} \right)^2 - 1 = \frac{\omega_\sigma}{\omega_0},
$$

in which the factor related to the ratio of the Gamma functions can be expressed as an asymptotic series (Graham et al. 1994),

$$
\frac{\Gamma(u + \frac{1}{2})}{\sqrt{u!} (u)} = 1 - \frac{1}{8u} + \frac{1}{128u^2} + \frac{5}{1024u^3} - \frac{21}{32768u^4} + \cdots.
$$

For the nonlinear eq. (31) with a single variable, a simple iterative procedure can find an optimal $u$ value.

Once the fraction $u$ is determined, the reference frequency can be determined based on the sum of $\omega_0^2$ and $\omega_\sigma^2$, using the following expression:

$$
\omega_0 = \sqrt{\frac{\omega_0^2 + \omega_\sigma^2}{1 + 2u}}.
$$

This expression suggests to use both parameters $\omega_0$ and $\omega_\sigma$, instead of only $\omega_0$ or only $\omega_\sigma$, for estimating $\omega_0$. It is an effort to minimize any potential bias error in these two values, numerically evaluated from a discrete Fourier spectrum.

Fig. 5 displays a series of field waveforms recorded at different depths in a vertical borehole. These waveforms (solid curves) are extracted from a vertical seismic profiling (VSP) data set generated by a dynamite source shot in a 15 m depth hole. They are the first arrivals, obtained by median filtering, which removes the VSP upgoing waveform, and cosine-square tapering, which suppresses the downgoing free-surface multiples behind the first arrivals (Wang 2014).

In the frequency domain, the mean frequency $\omega_0$ and the standard deviation $\omega_\sigma$ are evaluated from the power spectra. Fig. 5 clearly indicates that the mean frequency $\omega_0$ is decreasing gradually along the depth, while the standard deviation $\omega_\sigma$ is almost a constant. Based on $\omega_0$ and $\omega_\sigma$, the fractional value $u$ and the reference frequency $\omega_0$ are also derived. The reference frequency $\omega_0$ has a relatively small variation, as the deviation from the mean of $72.32\pi$ is $2.87\pi$. The fractional value $u$ is also decreasing gradually from 1.9 down to 1.2, similar to the variation of the mean frequency $\omega_0$.

The strong covariance between $u$ and $\omega_0$, $\omega_\sigma$ can be explained analytically by making an approximation to eq. (27) as

$$
\omega_0 \approx \omega_0 \sqrt{\frac{u}{2}} \left( 1 + \frac{1}{8u} \right),
$$

if assuming the reference frequency $\omega_0$ to be a constant. The correlation coefficient between $u$ and $\omega_0$ is 0.97 in this case (Fig. 6).
**CONCLUSIONS**

In order to better represent field seismic signals, asymmetrical wavelets are defined by fractional derivatives of a Gaussian function. The Ricker wavelet is just a special case with an integer derivative of order 2. Since these wavelets and the Ricker wavelet are mathematically derived from the same Gaussian function, their spectral properties are similar to each other and differ in a frequency-related factor $(i\omega)^{u}$. This factor is a frequency-domain representation of the fractional derivative with respect to time. Generalized wavelets match field seismic signals with high correlations.

For various wavelets, analytical expressions are found for the central frequency, the bandwidth, the mean frequency and the deviation. The first two frequency characteristics of a wavelet are expressed by the Lambert $W$ function. The last two statistical properties are presented in terms of the Gamma function. In practice, once the mean frequency and its deviation are numerically evaluated from discrete Fourier spectra of field seismic data, the analytical expressions mentioned above can be used to uniquely determine the fractional value of $u$ and the reference frequency $\omega_0$, and can subsequently be used by generalized wavelets, as shown in Fig. 5. The correlation coefficients between the field waveforms and the theoretical wavelets have an average value of 0.95. An accurate wavelet is necessary in seismic inversion, either for the reflectivity series or for the velocity variation.
to derive the peak frequency, the central frequency and the band-
width. These are frequency parameters that are needed practically
for wavelet and spectral analyses.

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