Optimization Based Control of Nonlinear Constrained Systems

by

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March 2014

A Thesis submitted in fulfilment of requirements for the Diploma and degree of

Doctor of Philosophy

of

Imperial College London

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PhD thesis:
Optimization Based Control of Nonlinear Constrained Systems

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Abstract

This thesis is in the field of Optimal Control. It addresses research questions concerning both the properties of optimal controls and also schemes for control system stabilization based on the solution of optimal control problems.

The first part is concerned with the derivation of necessary conditions of optimality for two classes of optimal control problems not covered by earlier theory. The first is the class of optimal control problems with a combination of mixed control-state constraints and pure state constraints in which the dynamics are described by a differential inclusion under weaker hypotheses than have previously been considered. The second is the class of optimal control problems in which the dynamics take the form of a non-smooth differential equation with delays, and where the end-time is included in the decision variables. We shall demonstrate that these new optimality conditions lead to algorithms for solution of certain optimal control problems not amenable to earlier theory.

Model Predictive Control (MPC) is an approach to control system design based on solving, at each control update time, an optimal control problem. This is the subject matter of the second part of the thesis. We derive new MPC algorithms for constrained linear and nonlinear systems which, in certain significant respect, are simpler to implement than standard schemes, and which achieve performance specifications under more general conditions than has previously been demonstrated. These include stability and feasibility.

Keywords: optimal control, nonlinear control, linear systems, predictive control, stability, state constraints, mixed constraints, feasibility, necessary conditions, time delays, maximum principle, free time problems, differential inclusions.
Declaration of Originality

I declare that the work presented in this thesis is my own, except where explicitly stated.

[Signature]

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Preface

This thesis was written while the author was a research fellow, supported by the Marie Curie ITN SADCO. The author’s primary supervisor was Prof. Richard Vinter. As required under the terms of the fellowship, the author was seconded for periods of three months each to Faculdade de Engenharia da Universidade do Porto and to Mathematischen Institut Universität Bayreuth.

While the overarching theme of the thesis is optimal control, it divides into two parts. The first, and larger part, is on the derivation of new necessary conditions of optimality in optimal control. This work was carried out under the supervision of Prof. Vinter, with advise also of academic staff at the University of Porto. The second part is on an approach to control system design based on solving a sequence of optimal control problems. This work was proposed by Prof. Grüne and carried out in collaboration with him during, and subsequent to, the Bayreuth secondment. Each of the technical chapters of the thesis is largely self-contained, and is the subject of a publication or paper submitted for publication. The first part makes extensive use of nonsmooth analysis. A summary of the main concepts is provided at the beginning for convenience of the reader. An introduction to the second part of the thesis is also provided in Chapter 7.

Acknowledgment

I wish to express my deep gratitude to Richard Vinter, for giving me the opportunity to work in such a stimulating environment that is Imperial College in the first place, for his support and guidance throughout this Ph.D., for his precious advice and encouragements
to pursue my own research interests, for all the things he taught me both as a researcher and as a person, for making this experience such an enjoyable one, for organizing lovely country walks around the UK with his wife Donna and her American students, and for countless many other helpful suggestions he gave me some of which significantly improved the presentation of this thesis.

Part of the results presented in this thesis are the consequence of fruitful discussions and interactions I have had with several people during the time this research has been carried out. For this I thank Lars Grüne, Karl Worthmann, Piernicola Bettiol, Paola Falugi and Helmut Maurer.

The financial supported by the European Union under the 7th Framework Programme FP7-PEOPLE- 2010-ITN Grant agreement number 264735-SADCO, is gratefully acknowledged. The many events organized within the training network program SADCO have been essential for my professional education. I wish to thank all the people of the network for making this experience so rewarding.

I would also like to thank all my friends and all the amazing people I have met during these three years. I thank Piermarco Cannarsa for introducing me to the world of research.

Above all I thank my family.
Contents

Abstract

Declarations

Preface

Contents

List of Figures

I Optimality Conditions

Chapter 1. Basic Concepts of Nonsmooth Analysis

1.1 Some Notation and Background

1.2 Normal Cones

1.3 Subdifferentials

1.4 Subdifferential Calculus

1.4.1 Perturbation Analysis

Chapter 2. Optimal Control

2.1 The Maximum Principle

2.2 Nonsmooth Necessary Conditions
2.3 Differential Inclusions ........................................... 40
2.4 A Free End-Time Problem .................................. 45
2.5 State Constraints ............................................. 47

Chapter 3. Optimal Control Problems with State Constraints 53
3.1 Literature Review ............................................. 54
3.2 Stratified Necessary Conditions .............................. 58
3.3 Counter Examples and Discussion ......................... 63
3.4 Proof of Theorems and Corollaries ......................... 67

Chapter 4. Mixed Constraints 79
4.1 Literature Review ............................................. 79
4.2 Necessary Conditions: Main Results ...................... 83
4.3 Examples ..................................................... 86
4.4 Special Cases ............................................... 89
4.5 Proofs of Theorems ......................................... 91

Chapter 5. Euler-Lagrange Conditions for Delayed Systems 95
5.1 A Delayed Lagrange problem ............................... 96
5.2 Differential-difference inclusion ............................ 106
5.2.1 Hypotheses reductions & Preliminaries ............... 108
5.2.2 Proof of Theorem 5.2.1 ................................. 114
5.3 Free Time Problems ........................................ 121
5.4 The Maximum Principle .................................... 131

Chapter 6. Free time optimal control problems with time delays 133
6.1 Literature Review ............................................. 134
6.2 A Free End-Time Optimal Control Problem for Retarded Systems ................................................. 136
6.3 A Maximum Principle .................................................................................................................. 137
6.4 Sensitivity Relations ................................................................................................................... 140
6.5 Computation of Minimizers ....................................................................................................... 142
6.6 Numerical Examples .................................................................................................................. 143
  6.6.1 Linear Dynamics ..................................................................................................................... 144
  6.6.2 Control of a renewable resource ............................................................................................ 146

II Model Predictive Control ............................................................................................................. 149

Chapter 7. Overview of Model Predictive Control ............................................................................. 151
  7.1 Stability of Discrete Time Systems ............................................................................................ 151
  7.2 The MPC Algorithm ................................................................................................................. 155
  7.3 The Dynamic Programming Principle ....................................................................................... 158
  7.4 Stability of MPC Schemes: Classical Theory ........................................................................... 159
  7.5 Stability of MPC Schemes: Recent Developments ................................................................... 163
  7.6 Robustness ............................................................................................................................... 164

Chapter 8. Stability and feasibility of state constrained MPC ............................................................ 167
  8.1 Relations to the Literature ......................................................................................................... 168
  8.2 Model Predictive Control ........................................................................................................... 169
  8.3 Recursive Feasibility and Asymptotic Stability ........................................................................ 171
    8.3.1 Asymptotic Stability on Level Sets ...................................................................................... 172
    8.3.2 Global stability .................................................................................................................... 175
  8.4 Linear Systems .......................................................................................................................... 177
    8.4.1 Characterization of the Viability Kernel for Linear Systems ............................................... 178
Chapter 9. Linear MPC and Continuity of the Value Function 189

9.1 Model Predictive Control ........................................ 190
9.2 Stability on Level Sets ........................................... 193
9.3 The Basin of Attraction ........................................... 195
9.4 Stationarity of Feasible Sets ..................................... 197
9.5 Continuity of $V_\infty$ .............................................. 199
9.6 An Illustrative Example ........................................... 202
9.7 Sufficient Conditions for Continuity of $V_\infty$ ................. 205
  9.7.1 Set-Valued Analysis ......................................... 205
  9.7.2 Sufficient Conditions for Continuity of $G$ from (9.5.2) .... 206

Chapter 10. Concluding Remarks 211

10.1 Methodology ..................................................... 212
10.2 Contributions ................................................... 213
10.3 Directions for future research .................................. 217

Notation 219

Bibliography 225
List of Figures

1.1 Left: A proximal normal vector $\zeta$ to $S$ at $x_0$. Right: Nonsmooth boundary and limiting normals. .................................................. 20

1.2 Illustration of Ekeland’s Theorem .................................................. 30

2.1 Unbounded Dynamic ................................................................. 44

4.1 Illustration of Example 4.3.1. The function $g(\cdot)$ on the left and the set $S(t, w) \cap [-1, +\infty)$ on the right .................................................. 87

5.1 Geometric idea of the proof of Lemma 5.2.3 .................................. 111

6.1 End-time value function and performance of algorithm based on sensitivity formulae, for various starting times: $T_0 = 0.5(\circ), T_0 = 3.5(\circ)$ .................................................. 145

6.2 End-time value function and performance of algorithm based on sensitivity formulae, for various starting times: $T_0 = 20(\circ), T_0 = 0.6(\circ)$ .................................................. 147

6.3 Example 2: optimal state variable (right) and optimal input variable (left) .................................................. 147

8.1 Illustration of the pointwise bounded value function $V_\infty(\cdot)$ for Example 8.3.6 .................................................. 177

8.2 Illustration of the finite time controllability for Example 8.4.12 .................................................. 187

9.1 Illustration of Example 9.4.2 .................................................. 198
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.2</td>
<td>(left): Representation of two trajectories (dotted curves in red) for the system with control $u = 1$ at each step, starting at $(1,0)$ and $\Gamma$. The feasible set $\mathcal{F}_1$ in white, $\mathcal{N}$ in yellow (oval shaped). (right): The constraints defining $\mathcal{F}_1$ (blue) and $\mathcal{F}_2$ (yellow) intersect in $\Omega$ (on the half space $x_2 \leq 0$). Analogously $\Gamma$ is defined as intersection of $\mathcal{F}_2$ and $\mathcal{F}_3$ (red, $\mathcal{F}<em>3 = \mathcal{F}</em>\infty$). $\Theta$ is the intersection with the line $x_1 = 1$.</td>
</tr>
<tr>
<td>9.3</td>
<td>Number of steps required to reach the origin, from the inner color (1 step) to the outer one (6 steps).</td>
</tr>
<tr>
<td>9.4</td>
<td>Representation of the set $U(x)$ as a section of $\mathcal{E}$.</td>
</tr>
<tr>
<td>9.5</td>
<td>Continuity proof in $\mathbb{R}^2$, Theorem 9.7.3(iii).</td>
</tr>
<tr>
<td>9.6</td>
<td>On the left the constraint set $\mathcal{C}$ for example 9.7.4. On the right $\mathcal{C}$ is projected onto the plane $x_2 = -1$.</td>
</tr>
</tbody>
</table>
Part I

Optimality Conditions
Chapter 1

Basic Concepts of Nonsmooth Analysis

Techniques of nonsmooth analysis are widely used through this part of the thesis. We describe here some of the key concepts. For more details see [16, 21, 25, 55].

1.1 Some Notation and Background

Euclidean Space: Let $x$ be a point in $\mathbb{R}^n$, the Euclidean norm of $x = (x_1, \ldots, x_n)$ is written $|x|:

$$|x| := \sqrt{x_1^2 + \ldots + x_n^2}.$$ 

The closed unit ball in $\mathbb{R}^n$ is denoted by $B$:

$$B := \{ x \in \mathbb{R}^n : |x| \leq 1 \}.$$ 

We write $B(x, r)$ or $x + rB$ to indicate the closed ball of radius $r > 0$, centered at $x \in \mathbb{R}^n$. Given a nonempty subset $S \subset \mathbb{R}^n$, $\bar{S}$ denotes the closure of $S$. The interior of $S$ is written $\text{int}(S)$ while $\text{bdry}(S) := \bar{S} \setminus \text{int}(S)$. The distance of a point $x \in \mathbb{R}^n$ from the set $S$ is defined by

$$d_S(x) := \inf_{y \in S} |y - x|.$$ 

Theorem 1.1.1. Given a subset $S \subset \mathbb{R}^n$, the distance function $d_S : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz
continuous with Lipschitz constant $k = 1$. Moreover we have have that a point $x \in \bar{S}$ if and only if $d_S(x) = 0$.

In some sense $d_S(.)$ gives an analytical description of points that lie in $S$. The indicator function $I_S(.)$ will also serve for this purpose:

$$I_S(x) := \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{otherwise.} \end{cases}$$

$Proj_S(x)$ denotes the projection of a point $x \in \mathbb{R}^n$ onto the closed set $S \subset \mathbb{R}^n$. It is the set

$$Proj_S(x) := \{s \in S : d_S(x) = |x - s|\}.$$  

We have that $Proj_S(x) \neq \emptyset$. It may contain more than one point.

**Space of Functions**: Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function (l.s.c.), with possibly unbounded values. We write $f \in \mathcal{F}(\mathbb{R}^n; \mathbb{R})$. The domain of a lower semicontinuous function $f$ is defined by

$$\text{dom } f := \{x \in \mathbb{R}^n : f(x) < +\infty\}.$$  

The epigraph of $f$ is given by

$$\text{epi } f := \{(x, \lambda) \in \text{dom } f \times \mathbb{R} : \lambda \geq f(x)\}.$$  

Here are some well-known elementary facts about l.s.c. functions (see, e.g. [12]):

1. $f$ is l.s.c., if and only if $\text{epi } f$ is closed in $\text{dom } f \times \mathbb{R}$.

2. $f$ is l.s.c., if and only if for every $x \in \text{dom } f$ and for every $\epsilon > 0$ there exists a neighborhood $V$ of $x$ such that

$$f(y) \geq f(x) - \epsilon, \ \forall y \in V.$$
In particular then
\[ \liminf_{y \to x} f(y) \geq f(x). \]

3. If \( f_1 \) and \( f_2 \) are l.s.c., then \( f_1 + f_2 \) is l.s.c..

4. If \( (f_i)_{i \in I} \) is a family of l.s.c. functions then their superior envelope is also l.s.c., that is, the function \( f \) defined by
   \[ f(x) := \sup_{i \in I} f_i(x) \]
   is l.s.c..

5. If \( S \subset \mathbb{R}^n \) is compact and \( f \) is l.s.c., then \( \inf_S f \) is achieved.

Observe that if \( S \subset \mathbb{R}^n \) is a closed set then \( I_S(\cdot) \) is l.s.c.

The space of absolutely continuous functions \( z : [S,T] \subset \mathbb{R} \to \mathbb{R}^n \) is written \( W^{1,1}([S,T]; \mathbb{R}^n) \). It is equipped with the norm
\[ \|z\|_{W^{1,1}} = z(S) + \int_S^T |\dot{z}(t)| \, dt. \]

For \( p \in [1, \infty) \), \( L^p([S,T]; \mathbb{R}^n) \) denotes the Banach space of measurable \( \mathbb{R}^n \)-valued functions \( z : [S,T] \subset \mathbb{R} \to \mathbb{R}^n \) such that \( \int_S^T |z(t)|^p \, dt < \infty \), with norm
\[ \|z\|_{L^p} = \left( \int_S^T |z(t)|^p \, dt \right)^{1/p}. \]

\( L^\infty([S,T]; \mathbb{R}^n) \) denotes the Banach space of measurable essentially bounded functions \( z : [S,T] \subset \mathbb{R} \to \mathbb{R}^n \), with the norm
\[ \|z\|_{L^\infty} = \inf_{\text{nullsets } I \subset [S,T]} \sup_{t \in [S,T] \setminus I} |z(t)|. \]

**Multifunctions:** Take two sets \( X \) and \( Y \), a multifunction \( F : X \rightrightarrows Y \) is a mapping from \( X \) to subsets of \( Y \). We will also refer to a multifuntion also as a multivalued function...
or a set-valued map. The graph of $F$, written $\text{Gr} F$, is the set

$$\text{Gr} F := \{(x, y) : y \in F(x)\}.$$ 

We say that $F$ is closed if $\text{Gr} F$ is a closed set for the topology of $X \times Y$. $\text{Proj}_S(\cdot)$, for example, is a closed multifunction, and is a function if and only is $S$ is convex.

**Measures:** We write $\text{NBV}^+(S, T)$ for the space of functions of bounded variation from an interval $[S, T] \subset \mathbb{R}$ to $\mathbb{R}$ that are increasing, vanishing at the point $S$ and right continuous on $(S, T)$. As is well know each function $\nu \in \text{NBV}^+[S, T]$ uniquely determines a positive Borel measure on $[S, T]$ and vice versa. We loosely refer then to elements $\nu \in \text{NBV}^+[S, T]$ as positive measures. The total variation of $\nu$ is defined by $\|\nu\|_{T.V.} := \int_{[S,T]} \nu(dt)$. To give some intuition about this fact observe that if $\mu$ is a positive measure then the function

$$f_\mu(t) := \begin{cases} \mu(S, t] & t > S \\ 0 & t = S \end{cases}$$

satisfies $f_\mu(\cdot) \in \text{NBV}^+[S, T]$. In the other direction, given a function $f(\cdot) \in \text{NBV}^+[S, T]$ its weak derivative $df$ in the sense of distributions defines a positive measure on $[S, T]$, that gives meaning to

$$\int_{[S,T]} \phi f = -\int_{[S,T]} \phi df,$$

for every continuously differentiable function $\phi(\cdot)$ vanishing at $S$ and $T$. ($\phi'$ denotes the derivative of $\phi$).

We say that a sequence $(\nu_i)_{i \in \mathbb{N}}$ of positive measure converges weak$^*$ to a measure $\nu$ if, for every continuous function $f$ on $[S, T]$, we have that

$$\lim_{i \to \infty} \int_{[S,T]} f(t) \, \nu_i(dt) = \int_{[S,T]} f(t) \, \nu(dt).$$

Given a weak$^*$ convergent sequence $\nu_i \to \nu$, there exists a countable subset $\mathcal{I} \subset (S,T)$
such that
\[
\int_{[S,t]} \nu_i(ds) \to \int_{[S,t]} \nu(ds),
\]
for all \( t \in ([S,T] \setminus I) \).

**Proposition 1.1.2.** Take a weak\(^*\) convergent sequence \( \{\mu_i\} \) in \( NBV^+[S,T] \), a sequence of Borel measurable functions \( \{\gamma_i : [S,T] \to \mathbb{R}^n\} \), and a sequence of closed, uniformly bounded multifunctions \( \{A_i : [S,T] \to \mathbb{R}^n\} \). Take also a closed and convex multifunction \( A(\cdot) \) and a positive measure \( \mu \). Assume that

\[
\limsup_{i \to \infty} GrA_i \subset GrA
\]
\[
\gamma_i(t) \in A_i(t) \text{ \( \mu_i \)-a.e., for } i = 1, 2, \ldots
\]

and

\[
\mu_i \to \mu_0 \text{ weakly}^*.
\]

Then, along a subsequence,

\[
\gamma_i(t)\mu_i(dt) \to \gamma_0(t)\mu_0(dt) \text{ weakly}^*,
\]

where \( \gamma_0(\cdot) \) is a Borel measurable function that satisfies

\[
\gamma_0(t) \in A(t) \text{ \( \mu_0 \)-a.e.}
\]

**Proof.** A proof of this result can be found in [55], Proposition 9.2.1. \qed

### 1.2 Normal Cones

Normal vectors to a certain set \( S \subset \mathbb{R}^n \) at a point \( x_0 \in bdry(S) \) may intuitively be described as ‘orthogonal’ directions along which trajectories depart from \( S \) at a maximum rate. In the case in which \( S \) is a smooth manifold defined as the level set of a continuously differentiable function \( h : \mathbb{R}^n \to \mathbb{R} \):

\[
S := \{ x \in \mathbb{R}^n : h(x) \leq 0 \},
\]
Figure 1.1: Left: A proximal normal vector $\zeta$ to $S$ at $x_0$. Right: Nonsmooth boundary and limiting normals.

the (unique) normal direction at $x_0 \in \text{bdry}(S)$ is represented by the gradient $\nabla h(x_0)$ at $x_0$ (we here assume that $\nabla h(x_0) \neq 0$). In this case we can think of the normal vector $v = \nabla h(x_0)$ as the vector that maximizes the rate of increase of $t \mapsto h(x_0 + tv)$ at $t = 0$. (Observe that $h(x_0 + tv)$, for $t$ sufficiently small, can be approximated, using Taylor expansion, by $h(x_0) + \nabla h(x_0) \cdot tv$). Generalizations to sets $S$ with nonsmooth boundaries permit there to have more than one such $v$.

**Definition 1.2.1.** Given a closed set $S \subset \mathbb{R}^n$ and a point $x \in S$, the proximal normal cone to $S$ at $x_0$, written $N_P^S(x_0)$, is the set

$$N_P^S(x_0) := \{ \zeta \in \mathbb{R}^n : \exists t > 0 \text{ so that } d_S(x_0 + t\zeta) = t|\zeta| \}.$$  

Element in $N_P^S(x_0)$ are called proximal normals. $N_P^S(x_0) = \{0\}$ whenever $x_0 \in \text{int}(S)$.

A geometric interpretation of proximal normals is provided in Figure 1.1: $\zeta \in N_P^S(x_0)$ if there exists a point $x \in \mathbb{R}^n$ such that $x_0 \in \text{Proj}_S(x)$ and $\zeta$ is a scaled version of $x - x_0$, i.e. there exists $\lambda > 0$ such that

$$\zeta = \lambda(x - x_0).$$

The normal cone provides local information about the nature of the boundary of the set $S$. From this geometric interpretation it is not surprising that proximal normals can
also be expressed as vectors satisfying the following quadratic inequality. (We recall, once again, that details for all the results presented in this introductory chapter can be found in [16, 21, 25, 55]).

**Proposition 1.2.1.** A vector \( \zeta \in N_{x_0}^P(S) \) iff there exists \( \sigma = \sigma(\zeta, x_0) \geq 0 \) such that

\[
\zeta \cdot (x - x_0) \leq \sigma |x - x_0|^2 \quad \forall x \in S
\]

In the case \( S \) is convex such inequality holds with \( \sigma = 0 \).

**Remarks.**

(a) For every \( x_0 \in S \), \( N_{x_0}^P(S) \) is a convex cone. However it needs neither be open nor close and it can be trivial, \( N_{x_0}^P(S) = \{0\} \), at points where the set \( S \) is concave and it is not \( C^2 \)-smooth. Consider for example a set \( S \) defined by

\[
S := \{(x, y) \in \mathbb{R}^2 : y \geq -|x|\}.
\]

It is easily seen that the proximal normal cone is trivial at the origin. (Indeed we cannot find a ‘ball’, as in the interpretation of Figure 1.1 that intersect \( S \) only at the origin).

(b) Assume that \( S \) admits a representation of the form

\[
S = \{x \in \mathbb{R}^n : h(x) \leq 0\},
\]

where \( h(.) \in C^2 \) is twice continuously differentiable. (We also assume \( \nabla h(x_0) \neq 0 \). Then, for \( x_0 \in \text{bdry}(S) \), \( N_{x_0}^P(S) = \{\lambda \nabla h(x_0) : \lambda \geq 0\} \).

(c) The multifunction \( x \mapsto N_{x_0}^P(S) \) is not closed (we recall that a set-valued map is said to be closed if its graph is closed, see page 18). This means that there may exist converging sequences \( (x_i, \zeta_i) \to (x_0, \zeta_0) \), with \( x_i \in S \) and \( \zeta_i \in N_{x_i}^P(S) \), such that \( \zeta_0 \notin N_{x_0}^P(S) \). Many useful relationships, involving normal cones, are derived as limit processes with respect to the basepoint \( x \). In the analysis of optimization problems, for example (that are often regarded as limit of perturb problems), the
closure of $N_P^S(\cdot)$ is a fundamental property to establish relationships involving normal directions to a set.

**Definition 1.2.2.** Given a closed set $S \subset \mathbb{R}^n$ and a point $x \in S$, the limiting normal cone to $S$ at $x_0$, written $N_S(x_0)$, is the set

$$N_S(x_0) := \{ \zeta \in \mathbb{R}^n : \text{there exists } x_i \to x_0, \zeta_i \to \zeta \text{ such that } x_i \in S, \zeta_i \in N_P^S(x_i) \}.$$

Element in $N_S(x_0)$ are called limiting normals.

Now, $N_S(\cdot)$ has closed graph, and for every point $x \in \text{bdry}(S)$ we have that $N_S(x)$ contains nonzero elements (see Figure 1.1). For $x \in \text{int}(S)$,

$$N_S(x) = N_S^P(x) = \{0\}.$$

However $N_S(x)$ may not be convex. Convexity turns out to be particularly useful when we have to deal with functions that have values in the limiting normal cone.

**Theorem 1.2.3** (Mazur’s Theorem). Let $X$ be a Banach space and let $p_i$ be a sequence in $X$ which converges weakly to a limit $p$. Then there exists a sequence made up of convex combinations of the $p_i$’s that converges strongly to $p$.

In this thesis we will make extensive use of this result. A typical situation will be the following. Take a sequence $\{p_i(\cdot)\}_{i \in \mathbb{N}} \subset L^1([S,T];\mathbb{R}^n)$ that converges weakly in $L^1$ to a certain $p(\cdot)$. Assume also, that for a sequence $x_i \to x_0$, $x_i \in S$, the relationship

$$p_i(t) \in N_S(x_i) \text{ a.e.}$$

is satisfied. In general, we cannot expect that $p(t) \in N_S(x_0)$ holds in the limit. We may use though Mazur’s Theorem to conclude that $p(t) \in \text{co } N_S(x_0)$ a.e. $t \in [S,T]$. (’co’ stands for convex hull).

**Definition 1.2.4.** Given a closed set $S \subset \mathbb{R}^n$ and a point $x \in S$, the Clarke normal cone to $S$ at $x_0$, written $N_C^S(x_0)$, is the set

$$N_C^S(x_0) := \text{co } N_S(x_0)$$
Element in $N_S(x_0)$ are called Clarke normals.

1.3 Subdifferentials

We generalize the concept of ‘gradient’ for functions that are not differentiable. Throughout this section $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function. We simply write $f \in \mathcal{F}$.

**Definition 1.3.1.** Take $f \in \mathcal{F}$ and a point $x \in \text{dom } f$.

(i) The proximal subdifferential of $f$ at $x$, written $\partial_P f(x)$, is the set

$$\partial_P f(x) := \{\zeta \in \mathbb{R}^n : (\zeta, -1) \in N_{epf}(x, f(x))\}.$$ 

Elements in $\partial_P f(x)$ are called proximal subgradients.

(ii) The limiting subdifferential of $f$ at $x$, written $\partial f(x)$, is the set

$$\partial f(x) := \{\zeta \in \mathbb{R}^n : (\zeta, -1) \in N_{epf}(x, f(x))\}.$$ 

Elements in $\partial f(x)$ are called limiting subgradients.

(iii) The Clarke subdifferential of $f$ at $x$, written $\partial_C f(x)$, is the set

$$\partial_C f(x) := \text{co } \partial f(x).$$ 

Elements in $\partial f(x)$ are called Clarke subgradients or generalized gradients.

Subdifferentials inherit properties from the respective normal cones. Therefore, for example, the multifunction $x \mapsto \partial f(x)$ is closed and

$$\partial f(x) = \{\zeta \in \mathbb{R}^n : \text{there exists } x_i \to x_0, \zeta_i \to \zeta \text{ such that } x_i \in \text{dom } f, \zeta_i \in \partial_P f(x_i)\}.$$ 

Now, however, $\partial f(x)$ may be empty at points $x$ where $N_S(x)$ contains only elements of the type $(\zeta, 0)$. We call these elements asymptotic subgradients.
Definition 1.3.2. Take \( f \in \mathcal{F} \) and a point \( x \in \text{dom } f \). The asymptotic limiting subdifferential of \( f \) at \( x \), written \( \partial^\infty f(x) \), is the set

\[
\partial^\infty f(x) := \{ \zeta \in \mathbb{R}^n : (\zeta, 0) \in N_{\text{epi}\,f}(x, f(x)) \}.
\]

Observe that

\[
N_{\text{epi}\,f}(x, f(x)) = \bigcup_{\lambda > 0} \lambda (\partial f(x), -1) \bigcup \{(\partial^\infty f(x), 0)\}.
\]

Proposition 1.2.1 implies the following representation of proximal subgradients.

Theorem 1.3.3. Let \( f \in \mathcal{F} \) and \( x \in \text{dom } f \). Then \( \zeta \in \partial P f(x) \) iff there exist \( \sigma \geq 0 \) and \( \eta > 0 \) such that

\[
f(y) \geq f(x) + \zeta \cdot (y - x) - \sigma |y - x|^2, \quad \forall y \in x + \eta \mathbb{B}.
\]

If \( f \) is convex then \( \sigma = 0 \).

Let \( \zeta \in \partial P f(x_0) \) for some \( x_0 \in \text{dom } f \). Then

\[
f(y) - f(x_0) \geq \zeta \cdot (y - x_0) - \sigma |y - x_0|^2, \quad \forall y \in x_0 + \eta \mathbb{B}. \tag{1.3.1}
\]

Assume now that \( f(.) \) is Lipschitz continuous on a neighborhood \( V_0 \) of \( x_0 \) with Lipschitz constant \( K \). As it is well known, in a smooth setting, Lipschitz behavior around a point \( x_0 \) is equivalent to having \( |\nabla f(x)| \leq K \), for some constant \( K \) and for every \( x \) in an arbitrarily ‘small’ neighborhood of \( x_0 \) at which the gradient exists. The same result holds true, in a nonsmooth setting:

Theorem 1.3.4. Take \( f \in \mathcal{F} \) and \( x_0 \in \text{dom } f \). Assume that \( f \) is Lipschitz continuous on a neighborhood \( V_0 \) of \( x_0 \) with Lipschitz constant \( K \). Then:

\( i \) \( \partial f(x_0) \) is nonempty and \( \partial f(x_0) \subset K \mathbb{B} \)

\( ii \) \( \partial^\infty f(x_0) = \{0\} \).

Viceversa, if either (i) or (ii) are satisfied, then \( f \) is Lipschitz continuous on a neighborhood of \( x_0 \).
1.4 Subdifferential Calculus

Proof. The proof of this result can be found in [55], Chapter 5. □

Let us see some examples.

Example 1.3.1. (In each of the following cases $f$ is a function from $\mathbb{R}$ to $\mathbb{R}$)

(i) $f(y) = |y|$ and $x = 0$. Then

$$\partial_P f(x) = \partial f(x) = \partial_C f(x) = [-1, 1] \quad \text{and} \quad \partial^\infty f(x) = \{0\}.$$

(ii) $f(y) = -|y|$ and $x = 0$. Then

$$\partial_P f(x) = \emptyset, \quad \partial f(x) = \{-1, 1\}, \quad \partial_C f(x) = [-1, 1] \quad \text{and} \quad \partial^\infty f(x) = \{0\}.$$

(iii) $f(y) = |y|^{1/2}$ and $x = 0$. Then

$$\partial_P f(x) = \partial f(x) = \partial_C f(x) = (-\infty, +\infty) \quad \text{and} \quad \partial^\infty f(x) = (-\infty, +\infty).$$

(iv) $f(y) = \text{sign}\{y\}|y|^{1/2}$ and $x = 0$. Then

$$\partial_P f(x) = \partial f(x) = \partial_C f(x) = \emptyset \quad \text{and} \quad \partial^\infty f(x) = [0, +\infty).$$

Unbounded or empty subdifferentials give warning that the slopes of the function around the basepoint may not be bounded. In the presence of such pathologies, asymptotic subdifferentials supply additional information about the directions of these arbitrarily large slopes. In Example 1.3.1(iv), we have seen that $\partial f(0) = \emptyset$ and $\partial^\infty f(0) = [0, +\infty)$. This tells us that the ‘infinite’ slope near the origin is positive.

1.4 Subdifferential Calculus

We recall now some of the main results about subdifferentials. See [16, 21, 25, 55] for details.

A. (Links between Normal Cones and Subdifferentials) Take a closed set $S \subset \mathbb{R}^n$ and $x \in S$. Then

$$\partial_P d_S(x) = N^P_S(x) \cap \mathcal{B} \quad \text{and} \quad \partial_P I_S(x) = N^P_S(x).$$
B. (Limits of derivatives) Take a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), a point \( x \in \mathbb{R}^n \) and any subset \( \Omega \subset \mathbb{R}^n \) having Lebesgue measure zero. Assume that \( f \) is Lipschitz continuous on a neighborhood of \( x \). Then, by Rademacher’s theorem, \( f \) is almost everywhere differentiable and

\[
\text{co} \partial f(x) = \text{co} \{ \zeta : \exists x_i \rightarrow x, x_i \notin \Omega, \nabla f(x_i) \text{ exists and } \nabla f(x_i) \rightarrow \zeta \}.
\]

C. (Sum Rule) Take \( f_i \in \mathcal{F}, \ i = 1, \ldots, m \), and a point \( x \in \cap_i \text{dom} f_i \). Define \( f = f_1 + \ldots + f_m \). Assume that all the functions, except possibly one, are Lipschitz continuous on a neighborhood of \( x \). Then

\[
\partial f(x) \subset \partial f_1(x) + \ldots + \partial f_m(x).
\]

D. (Max Rule) Consider a family \( f_i \in \mathcal{F}, \ i = 1, \ldots, m \), and a point \( x \in \cap_i \text{dom} f_i \). Assume that all the functions, except possibly one, are Lipschitz continuous on a neighborhood of \( x \). Then:

\[
\partial (\max_{i=1, \ldots, m} f_i)(x) \subset \{ \sum_{i \in I(x)} \lambda_i \partial f_i(x) : \lambda_i \geq 0 \text{ for all } i \in I(x) \text{ and } \sum_{i \in I(x)} \lambda_i = 1 \}
\]

in which

\[
I(x) := \{ i \in \{1, \ldots, m\} : f_i(x) = \max_j f_j(x) \}.
\]

E. (Chain Rule) Take a locally Lipschitz continuous function \( G : \mathbb{R}^n \rightarrow \mathbb{R}^m \), a lower semicontinuous function \( g : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\} \), and a point \( \bar{u} \in \mathbb{R}^n \) such that \( G(\bar{u}) \in \text{dom} g \). Define the lower semicontinuous function \( f(u) := g(G(u)) \). Assume that:

The only vector \( \eta \in \partial^\infty g(G(\bar{u})) \) such that \( 0 \in \partial (\eta \cdot G)(\bar{u}) \) is \( \eta = 0 \).

Then

\[
\partial f(\bar{u}) \subset \{ \xi : \text{there exists } \eta \in \partial g(G(\bar{u})) \text{ such that } \xi \in \partial (\eta \cdot G)(\bar{u}) \}.
\]
1.4. SUBDIFFERENTIAL CALCULUS

1.4.1 Perturbation Analysis

In this thesis, techniques based on perturbation analysis will be used in several contexts. The idea is to replace a given problem, that may be difficult to analyze, by a perturb version of it that is easier to investigate and that gives approximate solutions of the original problem.

Inf Convolutions

Imagine, for example, that we would like to study the properties of some function $g : \mathbb{R}^n \rightarrow \mathbb{R}$, but we are prevented from doing so because, say, $g(.)$ is not sufficiently regular. A standard approach to dealing with this difficulty is to construct a family $\{g_\epsilon(.) : \epsilon > 0\}$ of regularized functions, that are easier to analyze, such that $\lim_{\epsilon \downarrow 0} g_\epsilon(.) = g(.)$, and obtain information on $g(.)$ in the limit. One such procedure is given by convolving $g(.)$ with a smooth mollifier $\phi_\epsilon(.)$, depending on a parameter $\epsilon > 0$, that is nonnegative valued, vanishes outside $\epsilon \mathbb{B}$ and satisfies

$$\int \ldots \int \phi_\epsilon(x) \, dx_1 \ldots dx_n = 1.$$  

Thus,

$$g_\epsilon(x) := \int \ldots \int g(y)\phi_\epsilon(x - y) \, dy_1 \ldots dy_n.$$  

The function $g_\epsilon(.)$ so constructed is now continuously differentiable. However, there is not an easy link between the derivative of this ‘integral’ convolution and subgradients of the original function $g(.)$. We use instead a procedure called inf convolution where the integral operation to generate the family $\{g_\epsilon(.) : \epsilon > 0\}$ is replaced by a minimization procedure

$$g_\epsilon(x) := \inf_{y \in \mathbb{R}^n} \{g(y) + \epsilon^{-1}|y - x|^2\}. \quad (1.4.1)$$

The key properties of the quadratic inf convolutions are reported in the following proposition.

Proposition 1.4.1. Take a Lipschitz continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ with Lipschitz constant $K$. For each $\epsilon > 0$, take $g_\epsilon(.)$ to be the function obtained by the quadratic inf
Take any $x \in \mathbb{R}^n$ and $\epsilon > 0$. Let $\bar{y}$ be a vector achieving the infimum in Proposition 1.4.1 (one such vector exists), and set
\[
\eta_{\epsilon}(x) := 2\epsilon^{-1}(x - \bar{y}).
\]

Then

(a) $g_\epsilon(\cdot)$ is Lipschitz continuous with Lipschitz constant $K$.

(b) $g(x) \geq g_\epsilon(x) \geq g(x) - \frac{K^2}{4}\epsilon$.

(c) $g_\epsilon(z) - g_\epsilon(x) \leq \eta_{\epsilon}(x) \cdot (z - x) + \epsilon^{-1}|z - x|^2$ for all $z \in \mathbb{R}^n$.

(d) $\eta_{\epsilon}(x) \in \partial g(\bar{y})$.

(e) $|x - \bar{y}| \leq K\epsilon$.

Observe, in particular, that $\lim_{\epsilon \downarrow 0} g_\epsilon(x) = g(x)$ and $\lim_{\epsilon \downarrow 0} \eta_{\epsilon}(x) \in \partial g(x)$ (such a limit exists because $g(\cdot)$ is a Lipschitz function). Furthermore, this inf convolution technique allows to regularize minimization procedures. Say we would like to solve the following optimization
\[
\inf_{x \in S} f(x) + g(x),
\]
where $S$ is some closed subset of $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ is some continuously differentiable function and $g : \mathbb{R}^n \to \mathbb{R}$ is assumed to be only Lipschitz. Let us look at the ‘regularized’ problem
\[
\inf_{x \in S} f(x) + g_\epsilon(x).
\]
Because of Proposition 1.4.1, we would expect that a minimizing point $x_{\epsilon}$, for the reformulated problem, can be taken to be arbitrarily close to a solution of the original problem (see Theorem 1.4.2). By Proposition 1.4.1 then, $x_{\epsilon}$ is also a minimum for the problem
\[
\inf_{x \in S} f(x) + g_\epsilon(x_0) + \eta_{\epsilon}(x_0) \cdot (x - x_0) + \epsilon^{-1}|x - x_0|^2,
\]
where $g(\cdot)$ has been replaced in the minimization process by a continuously differentiable function. Thus, now, we have a standard ‘smooth’ optimization problem.
Perturbed Minimization

Minimization problems are among the mathematical problems that are most frequently encountered in applications. Consider the following:

Minimize $f(x)$ over $x \in S$, \hspace{1cm} (M)

where $f \in \mathcal{F}$ and $S$ is a closed subset of $\mathbb{R}^n$. In the classical framework, in which $f(.)$ is continuously differentiable, we can search for minimizers among points $x$ that satisfy

$$\nabla f(x) = 0,$$

whenever $x \in \text{int} S$ and evaluate $f(.)$ on the boundary, to check whether minimizers are located on the boundary. Here we may use the information that if $\nabla f(x) \neq 0$, then $v = -\nabla f(x)$ is a direction of decrease of the function, i.e.

$$f(x + tv) < f(x), \quad \text{for } t > 0 \text{ sufficiently small.} \hspace{1cm} (1.4.2)$$

However, it is possible to formulate the minimization problem (M) without requiring that $f(.)$ is differentiable and, in the modern theory, a framework is often adopted in which $f(.)$ is assumed to be merely lower semicontinuous. It is known that lower semicontinuity is (almost) the only requirement for a minimum of (M) to exist: if $S$ is compact or $f$ is coercive, then (M) has a solution. In the general case we may need to add a ‘small’ perturbation to $f$ in order to achieve a minimum (see Figure 1.2).

**Theorem 1.4.2** (Ekeland). Take a complete metric space $(X, d(.,.))$, a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, a point $x_0 \in \text{dom } f$, and $\varepsilon, \lambda > 0$. Assume that

$$f(x_0) \leq \inf_{x \in X} f(x) + \lambda \varepsilon.$$

Then there exists $x_\lambda \in X$ such that

(a) $f(x_\lambda) \leq f(x_0)$ and $d(x_\lambda, x_0) \leq \lambda.$

(b) $f(x_\lambda) \leq f(x) + \varepsilon d(x, x_\lambda)$ for all $x \in X$. 

Therefore the function \( x \mapsto f_\varepsilon(x) := f(x) + \varepsilon d(x, x_\lambda) \) attains a minimum at \( x_\lambda \).

The fact that the theorem allows \((X, d(\ldots))\) to be an arbitrary complete metric space adds great flexibility to the problem. Note that, even in the case in which \( f(\cdot) \) is differentiable, the perturbed function \( f_\varepsilon(\cdot) \) is not. Subdifferential calculus gives a tool to address nonsmooth optimization problems.

**Proposition 1.4.3.** Take \( f \in \mathcal{F} \) and \( x \in \text{dom } f \). Assume that \( x \) achieves the minimum value of \( f \) over a neighborhood of \( x \). Then

\[
0 \in \partial_P f(x).
\]

Conversely, if \( f \) is convex and \( 0 \in \partial_P f(x) \), then \( x \) is a global minimum of \( f \).

Furthermore:

if \( 0 \notin \partial_C f(x) \), and if \( \zeta \) is an element of \( \partial_C f(x) \) having minimal norm, then \( v := -\zeta \) satisfies [1.4.2].

**Proof.** See [25], Theorem 1.5 and Corollary 2.7.
1.4. SUBDIFFERENTIAL CALCULUS

Exact Penalization

A research theme that has been particularly productive in the solution of constrained optimization problems is to recast \((M)\) as a problem involving no constraints

\[
\inf_{x \in \mathbb{R}^n} f(x) + Kg(x). \tag{PM}
\]

Here \(g(x)\) is a penalty function that takes value zero when \(x \in S\) and is greater than zero when \(x \notin S\). This penalizes, in the minimization process, points \(x \notin S\). We would expect, then, that for large \(K\) the perturb problem would yield a point \(\bar{x}\) that approximately solve the original problem \((M)\) and approximately satisfies the constraints. The reformulation \((PM)\) is in general easier to solve and gives approximate solutions of the original problem. It is sometimes helpful to consider penalty functions that are not differentiable. In the extreme case, when \(g(x) = I_S(x)\) we have that the penalization is \textit{exact}, in the sense that

\[
\inf_{x \in S} f(x) = \inf_{x \in \mathbb{R}^n} f(x) + I_S(x).
\]

Therefore, if \(\bar{x}\) is a minimum for \((M)\) and \(f(.)\) is Lipschitz continuous, then

\[
0 \in \partial(f + I_S)(\bar{x}) \subset \partial f(\bar{x}) + N_S(\bar{x}).
\]

When \(f\) is continuously differentiable we obtain the following well known fact

\[
-\nabla f(\bar{x}) \subset N_S(\bar{x}),
\]

and thus minimizing points \(\bar{x}\) need to be sought either from \(\bar{x} \in \text{int}(S)\) where \(\nabla f(\bar{x}) = 0\), or from \(\bar{x} \in \text{bdry}(S)\) where \(\nabla f(\bar{x}) + \zeta = 0\), and \(\zeta \in N_S(\bar{x})\).

In some applications, however, we would like to use penalty functions \(g(.)\) that are more regular than \(I_S(.)\) and that are defined everywhere, i.e. \(\text{dom } g = \mathbb{R}^n\). Surprisingly perhaps, the distance function \(d_S\) provides an exact penalization as well.

\textbf{Theorem 1.4.4.} Let \((X,d(.,.))\) be a metric space. Take \(S \subset X\) and \(f : X \to \mathbb{R}\). Assume

\[
\]
that \( f \) satisfies a Lipschitz condition on \( X \) with Lipschitz constant \( K \). Let \( \bar{x} \) be a minimizer for

\[
\inf_{\bar{S}} f.
\]

Choose any \( \bar{K} \geq K \). Then \( \bar{x} \) is a minimizer also for the unconstrained minimization problem

\[
\inf_{X} f + \bar{K} d_{S}.
\]

If \( \bar{K} > K \) and \( S \) is a closed set, then the converse assertion is also true: any minimizer \( \bar{x} \) for the unconstrained problem is also a minimizer for the constrained problem and so, in particular, \( \bar{x} \in S \).

Proof. See [55], Theorem 3.2.1.
Chapter 2

Optimal Control

In this chapter we provide an overview of Optimal Control Theory, in preparation for the technical material presented in the subsequent chapters. For more details on the result presented see, e.g., [21, 55].

2.1 The Maximum Principle

Optimal control concerns the study of optimization problems of the following form

\[
(P) \begin{cases}
\text{Minimize } J(x(.), u(.)) := \ell(x(T)) + \int_0^T L(t, x(t), u(t)) \, dt \\
\text{over absolutely continuous } \mathbb{R}^n\text{-valued functions } x(.) : [0, T] \to \mathbb{R}^n \\
\text{and measurable } \mathbb{R}^m\text{-valued functions } u(.), \text{ such that:} \\
\dot{x}(t) = f(t, x(t), u(t)) \quad t \in [0, T] \text{ a.e.,} \\
u(t) \in U \quad t \in [0, T] \text{ a.e.,} \\
x(0) = x_0 \quad \text{and } x(T) \in C.
\end{cases}
\]

The data for \((P)\) comprises an interval \([0, T] \subset \mathbb{R}, T > 0\), functions \(\ell : \mathbb{R}^n \to \mathbb{R}, L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}\) and \(f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\), and sets \(C \subset \mathbb{R}^n\) and \(U \subset \mathbb{R}^m\).

We will call \((P)\) an optimal control problem. The system \((f, U)\) is composed of a dynamic function \(f\) and a control set \(U\). The variable \(x\) is referred to as the state of the system while \(u\) is referred to as the control. The functional \(J(.)\) is called cost or objective function.

**Hypothesis 1** (The classical regularity hypotheses). The function \(\ell\) is continuously dif-
CHAPTER 2. OPTIMAL CONTROL

ferentiable. The functions $f$ and $L$ are continuous, and admit derivatives $f_x(t, x, u)$ and $L_x(t, x, u)$ with respect to $x$ which are continuous in all variables $(t, x, u)$.

The time evolution $x(t)$ of the state depends on the choice of a control policy $u(.)$ and it is referred to as state trajectory. A process $(x(.), u(.))$, consisting of a control policy $u(.)$ and the corresponding state trajectory $x(.)$, is said to be admissible if the constraints in $(P)$ are satisfied. This means that $u(.)$ is a measurable function with values in the control space $U$ and $x(.)$ is an absolutely continuous solution of the differential equation

$$
\dot{x}(t) = f(t, x(t), u(t)) \quad t \in [0, T] \text{ a.e.},
$$

$$
x(0) = x_0,
$$

which satisfies $x(T) \in C$.

The aim is to find a control policy $\bar{u}(.)$, such that the process $(\bar{x}(.), \bar{u}(.))$ is admissible and minimizes the cost $J(x(.), u(.))$ over all admissible processes $(x(.), u(.))$. We call an admissible process $(\bar{x}(.), \bar{u}(.))$ a local minimizer if $J(\bar{x}(.), \bar{u}(.)) \leq J(x(.), u(.))$ for all admissible processes $(x(.), u(.))$ that satisfy $\|x - \bar{x}\| \leq \epsilon$, for some $\epsilon > 0$. It is common to refer to a local minimizer $(\bar{x}(.), \bar{u}(.))$ as an optimal (or local optimal) process or solution. In particular we are interested in finding a set of conditions that characterizes and restricts the search for a minimizer of $(P)$, i.e. conditions satisfied by any optimal process $(\bar{x}(.), \bar{u}(.))$. We call this set of conditions **Necessary Conditions**.

The norm $\| . \|$, in the definitions above, refers to the norm induced by the chosen topology of the state trajectories. Typically $\| . \|$ refers to the $L^\infty$ norm, i.e.

$$
\|x\|_{L^\infty} := \sup_{t \in [0, T]} |x(t)|.
$$

We also consider the $W^{1,1}$ norm on the space of absolutely continuous functions, written $W^{1,1}([0, T]; \mathbb{R}^n)$:

$$
\|x\|_{W^{1,1}} := |x(0)| + \int_0^T |\dot{x}(t)| \, dt.
$$

A local minimizer with respect to the $W^{1,1}$ topology is called a $W^{1,1}$ local minimizer while
2.1. THE MAXIMUM PRINCIPLE

a local minimizer with respect to the $L^\infty$ norm is called a strong local minimizer. Observe that it is preferable, if possible, to derive necessary conditions for $W^{1,1}$ local minimizers. Indeed the $W^{1,1}$ norm is stronger than the $L^\infty$ norm and therefore the class of $W^{1,1}$ local minimizers is larger than the class of $L^\infty$ minimizers. It follows that, by choosing to work with $W^{1,1}$ local minimizers, we are carrying out a sharper analysis of the local nature of necessary conditions than would be the case if we chose to derive conditions satisfied by strong local minimizers. However during this introduction, for simplicity, we will use the anonymous norm $\|\cdot\|$ to indicate the $L^\infty$ norm and we will refer to a strong local minimizer simply as a local minimizer.

The hypotheses on the data do not imply that every measurable control $u(\cdot)$ generates a state trajectory $x(\cdot)$. We could formulate a linear growth assumption on the dynamics function to guarantee this, but we will not assume this at the moment. In fact in this thesis we are only interested in necessary conditions. Therefore we will always assume the existence of at least one admissible process. Notice however that if the control set $U$ is unbounded then the integral $\int_0^T L(t, x(t), u(t)) \, dt$ may not be well defined. In this case we extend the meaning of ‘admissible’ by requiring an admissible pair for $(P)$ to satisfy the constraints of the problem and to be such that $J(\cdot)$ is well defined. Nevertheless for the moment we assume the following.

**Hypothesis 2.** The set $C \subset \mathbb{R}^n$ is closed and $U \subset \mathbb{R}^m$ is bounded.

The first general set of necessary conditions for an optimal control problem of the form $(P)$ was given in [48] by Pontryagin, Boltyanskii, Gamkrelidze and Mishchenko. This result dates from about 1960. Before recalling it let us first introduce the Hamiltonian associated to $(P)$.

**Definition 2.1.1.** The Hamiltonian function $H^\lambda : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ associated to the problem $(P)$ is defined by

$$H^\lambda(t, x, p, u) := p \cdot f(t, x, u) - \lambda L(t, x, u).$$

When $\lambda = 1$ we usually write $H$ in place of $H^1$. The maximized Hamiltonian of the
problem is the function $\mathcal{H}$ defined by

$$\mathcal{H}^\lambda(t, x, p) := \sup_{u \in \mathcal{U}} H(t, x, p, u).$$

**Theorem 2.1.2** (Pontryagin Maximum Principle). Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a local minimizer for $(P)$. Assume Hypotheses 1 and 2. Then there exist an absolutely continuous function $p(\cdot) \in W^{1,1}([0, T]; \mathbb{R}^n)$, called the co-state or adjoint arc, and $\lambda \in \{0, 1\}$, satisfying the nontriviality condition

$$(p(t), \lambda) \neq 0 \quad \forall t \in [0, T],$$

the transversality condition

$$-p(T) = \lambda \nabla \ell(\bar{x}(T)) + \eta, \text{ for some } \eta \in N_C(\bar{x}(T))$$

the adjoint equation

$$-\dot{p}(t) = H^\lambda_x(t, \bar{x}(t), p(t), \bar{u}(t)), \text{ a.e.,}$$

and the maximum principle or Weierstrass condition

$$H^\lambda_x(t, \bar{x}(t), p(t), \bar{u}(t)) = H^\lambda(t, \bar{x}(t), p(t)), \text{ a.e..}$$

If the problem is autonomous (that is if $f$ and $L$ do not depend on $t$), then one may add to the conclusions the constancy of the Hamiltonian

$$\mathcal{H}^\lambda(\bar{x}(t), p(t)) = c, \text{ a.e.}$$

for some constant $c \in \mathbb{R}$.

**Remarks.**

(a) A couple $(p, \lambda)$ satisfying the necessary conditions listed above is commonly called ‘multiplier’. Observe that if $(p, \lambda)$ is a multiplier, then every positive multiple $\alpha (p, \lambda)$, $\alpha > 0$, is also a multiplier. The multiplier $(p, \lambda)$ in Theorem 2.1.2 can

$^1\nabla$ denotes the gradient operator. $N_C$ denotes the set of orthogonal directions. It is defined in Chapter 1.
2.1. THE MAXIMUM PRINCIPLE

then be chosen to satisfy the following condition

\[ \lambda + \|p\|_{L^\infty} = 1. \]

(b) The adjoint and state equations together resemble the classical Hamiltonian system of differential equations, precisely

\[ \dot{x} = H_p^\lambda(t, x, p, u), \quad -\dot{p} = H_x^\lambda(t, x, p, u). \]

The terminology for the Hamiltonian function \( H \) was inspired by this fact.

(c) The case \( \lambda = 0 \) is called abnormal case. In this degenerate situation the necessary conditions convey no useful information, because they make no reference to the cost. This may reflect the situation in which the constraints are so restrictive that they identify an optimal solution regardless of the cost.

(d) The Weierstrass condition provides, in principle, values of the optimal control \( \bar{u} = \bar{u}(t, x, p) \) as a function of \( (t, x, p) \). Knowledge of \( \bar{u} \) can then be used to solve the adjoint equation and the state equation. Indeed, they provide a system of \( 2n \) differential equations and we have \( 2n \) boundary conditions given by the transversality condition and by the initial conditions.

(e) Theorem 2.1.2 may fail when we allow an unbounded control set \( U \). In this case we have to compensate with a further assumption.

**Hypothesis 3.** There exists \( \epsilon > 0 \) and an integrable function \( k(\cdot) \in L^1(0, T) \) such that

\[ |(f_x, L_x)(t, x, \bar{u}(t))| \leq k(t), \text{ a.e.,} \]

for all \( x \) such that \( |x - \bar{x}(t)| \leq \epsilon. \)

Note that this assumption concerns only the control \( \bar{u} \) and it is localized around \( \bar{x} \).
2.2 Nonsmooth Necessary Conditions

Hypothesis 1 is overly restrictive for some applications of interest. We have seen how we can deal with unbounded controls, but we may still need to allow for discontinuous dynamics with respect to time or yet allow functions that are not differentiable. Nonsmooth behavior of the functions involved in the optimal control problem (P) arises in a variety of contexts. Indeed, a continuous function that models some dynamical process is in general not differentiable. This assertion is made precise by the following result that is a consequence of Baire Category Theorem:

\[\text{Theorem 2.2.1. Let } S \subset \mathbb{R}^n \text{ be a compact set, and } C(S) := \{ f : S \to \mathbb{R} \mid f \text{ is continuous} \} \text{ be the space of continuous functions equipped with the } L^\infty \text{ topology. Then the set of nowhere differentiable functions over } S \text{ is dense in } C(S).}\]

Differentiability then is a severe assumption. Yet necessary conditions such as Theorem 2.1.2 are given in terms of the derivative of the functions involved in the optimization problem. An important breakthrough occurred in the 1970s, when Clarke’s theory of generalized gradients, based on generalizations of the concept of subdifferential of a convex function, allowed for a local description of “nonsmooth” functions. This provided a bridge to necessary conditions of optimality for nonsmooth variational problems. For an introductory account on nonsmooth optimal control problems see [16, 21, 25, 55]. Let us recall the optimal control problem (P)

\[
\begin{aligned}
\text{Minimize } J(x(\cdot), u(\cdot)) := \ell(x(T)) + \int_0^T L(t, x(t), u(t)) \, dt \\
\text{over absolutely continuous } \mathbb{R}^n\text{-valued functions } x(\cdot) \in W^{1,1}([0, T]; \mathbb{R}^n) \\
\text{and measurable } \mathbb{R}^m\text{-valued functions } u(\cdot), \text{ such that:} \\
\dot{x}(t) = f(t, x(t), u(t)) \quad t \in [0, T] \text{ a.e.,} \\
u(t) \in U \quad t \in [0, T] \text{ a.e.,} \\
x(0) = x_0 \text{ and } x(T) \in C,
\end{aligned}
\]

where now we consider the following assumptions.

**Hypothesis 4.** (i) \(\ell(\cdot)\) is locally Lipschitz continuous and \(C \subset \mathbb{R}^n\) is closed.
(ii) \( f(.,x,.) \) and \( L(.,x,.) \) are \( \mathcal{L} \times \mathcal{B} \) measurable for each \( x \in \mathbb{R}^n \) (\( \mathcal{L} \) and \( \mathcal{B} \) denote, respectively, the \( \sigma \)-algebras of Lebesgue subsets of \( \mathbb{R} \) and of Borel subsets of \( \mathbb{R}^m \).)

There exist \( \epsilon > 0 \) and \( k(.,.) : [0,T] \times \mathbb{R}^m \rightarrow \mathbb{R} \) such that \( t \mapsto k(t,\bar{u}(t)) \) is integrable and
\[
|f(t,x,u) - f(t,x',u)| + |L(t,x,u - L(t,x',u))| \leq k(t,u)|x - x'|
\]
for all \( x, x' \in \bar{x}(t) + \epsilon \mathcal{B} \) and \( u \in U \).

The following Theorem is due to Clarke, see [16].

**Theorem 2.2.2 (Nonsmooth Maximum Principle).** Let \((\bar{x}(.),\bar{u}(.))\) be a local minimizer for \((P)\). Assume Hypothesis 4. Then there exist an absolutely continuous function \( p(.) \in W^{1,1}([0,T];\mathbb{R}^n) \), called the co-state or adjoint arc, and \( \lambda \in \{0,1\} \), satisfying the nontriviality condition
\[
(p(t),\lambda) \neq 0 \ \forall t \in [0,T],
\]
the transversality condition
\[
-p(T) \in \lambda \partial \ell(\bar{x}(T)) + N_C(\bar{x}(T)),
\]
the adjoint inclusion (below \( \text{co} \partial_x \) indicates the convex hull of the subdifferential with respect to \( x \))
\[
-\dot{p}(t) \in \text{co} \partial_x H^\lambda(t,\bar{x}(t),p(t),\bar{u}(t)), \ a.e.,
\]
and the maximum principle or Weierstrass condition
\[
H^\lambda(t,\bar{x}(t),p(t),\bar{u}(t)) = H^\lambda(t,\bar{x}(t),p(t)), \ a.e..
\]

If the problem is autonomous (that is if \( f \) and \( L \) do not depend on \( t \)), then one may add to the conclusions the constancy of the Hamiltonian
\[
\mathcal{H}^\lambda(\bar{x}(t),p(t)) = c, \ a.e.
\]
for some constant \( c \in \mathbb{R} \).

This theorem generalizes the original Pontryagin maximum principle (Theorem 2.1.2). Nonsmooth, Lipschitz behavior is now allowed and the necessary conditions cover
a much larger range of applications. A proof of this theorem is typically derived from necessary conditions for a more general class of optimal control problems, see [16,21,25,55]. We investigate such conditions in the next section.

### 2.3 Differential Inclusions

A convenient framework for the study of optimal control problems is to consider for every point \((t, x)\) the whole set of possible directions

\[
F(t, x) := f(t, x, U) = \{f(t, x, u) : u \in U\}.
\]

Although it may seem more artificial this framework gives much more flexibility to our formulation and represents an ideal mathematical environment in which to prove various types of necessary conditions. The new optimal control problem is formulated as following

\[
(P_t) \begin{cases}
\text{Minimize } g(x(T)) \\
\text{over absolutely continuous } \mathbb{R}^n\text{-valued functions } x(\cdot) \in W^{1, 1}([0, T]; \mathbb{R}^n) \\
\text{satisfying the following constraints:} \\
\dot{x}(t) \in F(t, x(t)) \quad t \in [0, T] \text{ a.e.,} \\
x(0) = x_0 \quad \text{and} \quad x(T) \in C.
\end{cases}
\]

Differential equations are replaced in this setting by the more general concept of differential inclusions. Indeed for a.e. \(t \in [0, T]\) now the admissible velocities belong to a certain set \(F(t, x(t))\). \(F : [0, T] \times \mathbb{R}^n \rightharpoonup \mathbb{R}^n\) is a given set-valued function, that for every point \((t, x)\) associates a set \(F(t, x) \subset \mathbb{R}^n\). Set-valued maps are sometimes called multifunctions or multivalued functions. Note that standard optimal control problems such as \((P)\) are included in the formulation above. It is sufficient to consider an extra state \(z(\cdot)\) and define a new dynamic constraint set \(F(t, x) := \{(f(t, x, u), L(t, x, u)) : u \in U\}\). Problem \((P)\) is
2.3. **DIFFERENTIAL INCLUSIONS**

then equivalent to the following

\[
\begin{align*}
\text{Minimize } g(x(T)) + z(T) \\
\text{over absolutely continuous } \mathbb{R}^{n+1}\text{-valued functions } (x(\cdot), z(\cdot)) \in W^{1,1}([0, T]; \mathbb{R}^{n+1}) \\
satisfying the following constraints:
(\dot{x}(t), \dot{z}(t)) \in \{ (f(t, x, u), L(t, x, u)) : u \in U \} \quad t \in [0, T] \text{ a.e.,}
(x(0), z(0)) = (x_0, 0) \quad \text{and} \quad (x(T), z(T)) \in C \times \mathbb{R}.
\end{align*}
\]

This procedure of absorbing the integral cost in the dynamic constraints is named state augmentation.

The optimal control problem \((P_I)\) naturally models situations in which the control constraints depend on time \(U = U(t)\) or on the state \(U = U(x)\) or on both \(U = U(t, x)\). We refer to state dependent control constraints as *mixed constraints*. A typical form of mixed state-control constraints is

\[
U(t, x) = \{ u \in \mathbb{R}^m : u \in U, \phi(t, x, u) \leq 0, \psi(t, x, u) = 0 \},
\]

where \(\phi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\phi\) and \(\psi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\psi\) are given functions. Nevertheless we focus at the moment on optimal control problems \((P_I)\) described by a general multivalued function \(F\). In so doing we can forget about controls and mixed constraints, and use properties of set-valued maps to derive necessary optimality conditions. (see [2] for details on set-valued analysis).

Absolutely continuous function \(x(\cdot)\) that satisfy the differential inclusion \(\dot{x}(t) \in F(t, x(t))\) a.e. are called *\(F\)-trajectories*. If additionally they satisfy all the other constraints in \((P_I)\), they are referred to as *admissible \(F\)-trajectories*. The concept of local minimizer is easily generalized in this setting. An admissible \(F\)-trajectory \(\bar{x}(\cdot)\) is said to be a local minimizer if, for some \(\epsilon > 0\), we have \(g(\bar{x}(T)) \leq g(x(T))\) whenever \(x(\cdot)\) is an admissible \(F\)-trajectory satisfying \(\|x - \bar{x}\| \leq \epsilon\). Define graph of \(F(t, \cdot)\) by \(\text{Gr} F(t, \cdot)\).

**Hypothesis 5.**  \(i\) The function \(g(\cdot)\) is locally Lipschitz and the set \(C\) is closed. The
multifunction \( t \mapsto \text{Gr} F(t,.) \) is measurable and, for some \( \epsilon > 0 \), the following set is closed for a.e. \( t \),
\[
\{(x,v) \in \text{Gr} F(t,.) : |x - \bar{x}(t)| \leq \epsilon \}.
\]

(ii) There exist \( \epsilon > 0 \) and \( k(\cdot) \in L^1 \) such that the following hold for a.e. \( t \) and each \( x, x' \in \bar{x}(t) + \epsilon B \),
\[
F(t,x) \subset F(t,x') + k(t)|x - x'|B.
\]

Conditions (ii) can be equivalently expressed by
\[
d^H(F(t,x), F(t,x')) \leq k(t)|x - x'|,
\]
in which \( d^H(\cdot,\cdot) \) is the Hausdorff distance metric: given two sets \( A, B \subset \mathbb{R}^n \), \( d^H \) is defined by
\[
d^H(A,B) := \max\{\sup_{x \in A} d_B(x), \sup_{x \in B} d_A(x)\}.
\]

This explains why we refer to a multifunction \( F(\cdot,\cdot) \) satisfying Condition (ii) as a Lipschitz continuous multifunction with respect to the state \( x \).

**Theorem 2.3.1** (Extended Euler-Lagrange Conditions, see [55]). Let \( \bar{x}(\cdot) \) be a local minimizer for \((P_1)\). Assume Hypothesis \( \text{[\ref{symrev}]} \). Then there exist and absolutely continuous function \( p(.) \in W^{1,1}([0,T];\mathbb{R}^n) \), called the co-state or adjoint arc, and \( \lambda \in \{0,1\} \), satisfying the nontriviality condition
\[
(p(t), \lambda) \neq 0 \ \forall t \in [0,T],
\]
the transversality condition
\[
-p(T) \in \lambda \partial g(\bar{x}(T)) + NC(\bar{x}(T)),
\]
the adjoint inclusion
\[
\dot{p}(t) \in \text{co} \{\eta : (\eta, p(t)) \in N_{\text{Gr} F(t,\cdot)}(\bar{x}(t), \dot{\bar{x}}(t))\}, \ \text{a.e.},
\]
and the maximum condition or Weierstrass condition
\[
p(t) \cdot v \leq p(t) \cdot \dot{\bar{x}}(t), \ \forall v \in F(t,\bar{x}(t)), \ t \in [0,T] \ \text{a.e.}.
\]
2.3. **DIFFERENTIAL INCLUSIONS**

**Remarks.**

(a) The (maximized) Hamiltonian associated with this problem is defined by

\[ H(t, x, p) := \sup_{v \in F(t, x)} p \cdot v. \]

Therefore we may write the maximum condition as

\[ H(t, \bar{x}(t), p(t)) = p(t) \cdot \dot{\bar{x}}(t), \text{ a.e..} \]

(b) This set of necessary conditions is referred to as generalized or extended Euler-Lagrange conditions. The Euler-Lagrange conditions, originally, refer to a set of necessary conditions for problems in the calculus of variations that are optimization problems in which the state of the system is not subject to dynamic constraints, i.e. \( F(t, x) = \mathbb{R}^n \). Indeed \((P_I)\) could be interpreted as an ‘extended’ calculus of variations problem where the cost is reformulated as following

\[
\text{Minimize } g(x(T)) + \int_0^T I_{GrF(t,.)}(x(t), \dot{x}(t)) \, dt.
\]

We used the word ‘extended’ because the cost now is allowed to take the value \( +\infty \), a feature that was not considered in the classical theory of calculus of variations.

(c) The Lipschitz condition \((ii)\) in Hypothesis 5 proves to be inadequate when we allow unbounded dynamics. Consider the example in \( \mathbb{R}^2 \) illustrated in Figure 2.1 where \( F(t, x, y) := \{(v_1, v_2) : v_1 + xv_2 \leq 0\} \). It is easily seen that

\[ d^H(F(t, x, y), F(t, x', y')) = +\infty \]

whenever \( x \neq x' \). To overcome this problem Loewen and Rockafellar proposed in [44] a generalization of Hypothesis 5 \((ii)\) called epi-Lipschitz condition. This asserts the existence of an integrable function \( k(.) \) and constants \( \beta \geq 0 \) and \( \epsilon > 0 \) such that

\[
F(t, x) \cap B(\dot{x}(t); N) \subset F(t, x') + (k(t) + \beta N)|x - x'|B
\]
holds for all \( N \in \mathbb{N}, x, x' \in \mathbb{B}(\bar{x}(t); \epsilon) \) and \( t \in [0, T] \) a.e..

(d) The nonsmooth maximum principle is easily deduced from the extended Euler-Lagrange conditions. Indeed, if \( F(t, x) = f(t, x, U) \) for some set \( U \subset \mathbb{R}^m \), then after suitable reductions (see [55, Chapter 6]), the adjoint inclusion takes the following form

\[
\dot{p}(t) \in \text{co} \ \{ \eta : (\eta, p(t)) \in N_{Grf(t, \bar{u}(t))}(\bar{x}(t), \dot{\bar{x}}(t)) \}, \text{ a.e.} \tag{2.3.1}
\]

Now we can use the fact that for a function \( h : \mathbb{R}^n \to \mathbb{R}^n \) and a point \( y \) such that \( h(\cdot) \) is Lipschitz continuous on a neighborhood of \( y \) we have

\[
(\eta, \beta) \in N_{Grh}(y, h(y)) \implies \eta \in \partial[-\beta \cdot h](y).
\]

(See [55, Proposition 5.4.2].)

We conclude from (2.3.1) that \( \dot{p}(t) = \sum_{i=0}^{n} \lambda_i \eta_i \), for some \( \lambda_i \geq 0 \) such that \( \sum_{i=0}^{n} \lambda_i = 1 \), and some \( \eta_i \) such that \( \eta_i \in \partial_x[-p(t) \cdot f(t, \bar{x}(t), \bar{u}(t))] \). Therefore

\[
-\dot{p}(t) \in \text{co} \ \partial_x H(t, \bar{x}(t), p(t), \bar{u}(t)),
\]

as desired. Similar argumentations may be exploited to give ‘explicit’ necessary conditions for optimal control problems featuring mixed state-control constraints.
(see Chapter 4). ‘Explicit’ in the sense that the necessary conditions are expressed in terms of the functions that define the mixed constraints).

### 2.4 A Free End-Time Problem

An important feature of some optimal control problems is that the time interval over which the optimization is performed is not fixed, and the final time $T$ is itself a choice variable.

\[
\begin{align*}
(P_F) \quad \text{Minimize} & \quad g(T, x(T)) \\
& \text{over absolutely continuous } \mathbb{R}^n\text{-valued functions } x(. \in W^{1,1}([0, T]; \mathbb{R}^n) \\
& \text{and interval } [0, T] \text{ satisfying the following constraints:} \\
& \dot{x}(t) \in F(t, x(t)) \quad t \in [0, T] \text{ a.e.,} \\
& x(0) = x_0 \quad \text{and} \quad (T, x(T)) \in C.
\end{align*}
\]

The cost function $g(.)$ now depend on the time horizon $T$. In this case we need to extend the concept of local minimizer. The minimization is now conducted over pairs $(T, x(.))$ composed of a final horizon time $T$ and an $F$-trajectory $x(.)$ defined over the interval $[0, T]$. To measure the distance between two pairs $(T_1, x_1(.))$ and $(T_2, x_2(.))$ we need first to extend the state trajectories on the whole interval $[0, +\infty)$ so to give a meaning to $\|x_1 - x_2\|$. We do so by defining any $F$-trajectory $(T, x(.))$ to be constant after its interval of definition, to wit $x(\tau) = x(T)$ for $\tau \geq T$. Let a pair $(\bar{T}, \bar{x}(.))$ be admissible, i.e. it satisfies the constraints in $(P_F)$. We say that it is a local minimizer if, for some $\epsilon > 0$ and all admissible $(T, x(.))$ satisfying

\[
|T - \bar{T}| + \|x - \bar{x}\| \leq \epsilon
\]

we have that $g(\bar{T}, \bar{x}(\bar{T})) \leq g(T, x(T))$.

This class of problems is generally recast as a fixed-time problem by means of a time transformation. Suppose that $(\bar{T}, \bar{x}(.))$ is a local minimizer for $(P_F)$. Fix $\alpha \in (0, 1)$ and
consider the new fixed time optimal control problem

\[
(P_F) \quad \begin{cases}
\text{Minimize } g(\tau(\bar{T}), y(\bar{T})) \\
\text{over absolutely continuous } \mathbb{R}^{n+1}\text{-valued functions } (\tau(.), y(\cdot)) \in W^{1,1}([0, T]; \mathbb{R}^{n+1}) \\text{satisfying the following constraints:} \\
(\dot{\tau}(t), \dot{y}(t)) \in \{(w, wv) : w \in [1 - \alpha, 1 + \alpha], v \in F(\tau(t), y(t))\} \quad t \in [0, T] \text{ a.e.,} \\
(\tau(0), x(0)) = (0, x_0) \quad \text{and} \quad (\tau(\bar{T}), y(\bar{T})) \in C.
\end{cases}
\]

The important feature of \((P_F)\) is that there exists a transformation \(Q\) that maps an admissible state trajectory \((\tau(.), y(\cdot))\) into an admissible state trajectory of the original problem \((P_F)\). Moreover this transformation preserves the value of the cost. To be precise \(Q\) is defined as following

\[
Q(\tau(.), y(\cdot)) = (\tau(\bar{T}), y(\tau^{-1}(\cdot))).
\]

Here \(\tau : [0, \bar{T}] \to [0, \tau(\bar{T})]\) is an increasing function defined by

\[
\tau(t) := \int_0^t w(s) \, ds
\]

for some function \(w(.)\) such that \(w(s) \in [1 - \alpha, 1 + \alpha]\) a.e.. We can define a trajectory of the original problem by \(x(t) := y(\tau^{-1}(t))\). Indeed \(x(.)\) is now defined on the interval \([0, T = \tau(\bar{T})]\) and satisfies

\[
\dot{x}(t) = \dot{y}(\tau^{-1}(t)) \cdot \frac{1}{w(\tau^{-1}(t))} \in F(t, x(t)).
\]

Since the trajectory \((\tau(t) \equiv t, \bar{x}(t))\) is admissible for \((P_F)\) and \(Q(\tau(t) \equiv t, \bar{x}(t)) = (\bar{T}, \bar{x}(\cdot))\) it follows that \((\tau(t) \equiv t, \bar{x}(t))\) is optimal for \((P_F)\). \(\text{Theorem 2.3.1}\) asserts the existence of a multiplier \((r(.), p(.), \lambda) (r(.) \text{ relative to the additional variable } \tau)\) satisfying the extended Euler-Lagrange conditions. In particular we have from the transversality condition that

\[
(\lambda \partial g(\bar{T}, \bar{x}(\bar{T})), N_C(\bar{T}, \bar{x}(\bar{T}))),
\]

(2.4.1)
and from the *maximum condition* that

\[ r(t) + p(t) \cdot \dot{x}(t) = \max \{ w(r(t) + p(t) \cdot v) : v \in F(t, \bar{x}(t)) \text{ and } w \in [1 - \alpha, 1 + \alpha] \}. \]

This last condition in particular implies, fixing \( v = \dot{x}(t) \), that

\[ -r(t) = H(t, \bar{x}(t), p(t)) \text{ a.e.,} \quad (2.4.2) \]

where \( H(t, \bar{x}(t), p(t)) = p(t) \cdot \dot{x}(t) \) is the Hamiltonian of the original problem. It follows that \( H \) is actually an absolutely continuous function and its value at \( \bar{T} \) is fixed by the transversality condition. Notice also that if the problem is autonomous, then the dynamic in \((\overline{P}_F)\) does not depend on \( \tau \). Therefore in the autonomous case \( r(t) \equiv 0 \) and we can add to the necessary conditions the constancy of the Hamiltonian, (in a stronger form for free end-time problems since the constant \( c = 0 \))

\[ H(\bar{x}(t), p(t)) = 0, \ \forall t \in [0, \bar{T}], \quad (2.4.3) \]

In summary: Let \((\bar{T}, \bar{x}(.)\)) be a local minimizer for the free end-time optimal control problem \((P_F)\). Then it satisfies the necessary conditions of Theorem 2.3.1 at the optimal time \( T = \bar{T} \). Indeed, if we freeze the end-time \( T = \bar{T} \), then \( \bar{x}(.) \) is, in particular, a local minimizer for the fixed end-time problem. Moreover the extra degree of freedom given by the fact that \( T \) is a decision variable is accommodated by extra necessary conditions such as Equation 2.4.1 and Equation 2.4.2. If the problem is autonomous, then one may add to the conclusions the constancy of the Hamiltonian Condition 2.4.3.

### 2.5 State Constraints

We may want to investigate situations where the state trajectory is not allowed to enter certain regions of the state space. In such circumstances we include a pathwise constraints in the formulation of the problem, say

\[ h(t, x(t)) \leq 0, \ \forall t \in [0, T] \]
where \( h : [0, T] \times \mathbb{R}^n \to \mathbb{R} \) is a given function.

**Hypothesis 6.** \( h \) is upper semicontinuous and locally Lipschitz with respect to \( x \) (uniformly with respect to \( t \)).

We come back in this section to studying the smooth framework of Section 2.1. This is because we want to give an intuition of the effect that the state constraint have on the necessary conditions. The optimal control problem of interest is now the following

\[
\begin{align*}
\text{(Pc)} & \quad \minimize \ell(x(T)) \\
& \text{over absolutely continuous } \mathbb{R}^n\text{-valued functions } x(\cdot) \in W^{1,1}([0,T]; \mathbb{R}^n) \\
& \quad \text{and measurable } \mathbb{R}^m\text{-valued functions } u(\cdot), \text{ such that:} \\
& \quad \dot{x}(t) = f(t, x(t), u(t)) \quad t \in [0, T] \text{ a.e.,} \\
& \quad u(t) \in U \quad t \in [0, T] \text{ a.e.,} \\
& \quad h(t, x(t)) \leq 0, \quad \forall t \in [0, T], \\
& \quad x(0) = x_0 \quad \text{and} \quad x(T) \in C.
\end{align*}
\]

We have chosen to formulate the state constraints as a functional inequality because it is a convenient starting point for the derivation of necessary conditions for other types of state constraints.

(a) Consider the following formulation of the state constraints

\[ x(t) \in K(t), \quad t \in [0, T] \]

where \( K(\cdot) \) is a given multivalued function. In this case we can choose \( h \) to be \( h(t, x) := d_{K(t)}(x) \).

(b) The case of multiple constraint inequalities

\[ (h_1, \ldots, h_k)(t, x(t)) \leq 0 \]

is dealt with defining \( h(t, x) := \max \{ h_1(t, x), \ldots, h_k(t, x) \} \). Observe here the importance of the Lipschitz hypothesis on \( h \).
(c) Consider a problem in which the constraint is only imposed on a closed subset $\mathcal{I} \subset [0, T],
\tilde{h}(t,x(t)) \leq 0, \forall t \in \mathcal{I}$

If $\tilde{h}$ is a bounded continuous function such that $\tilde{h}(t,.)$ is Lipschitz continuous, uniformly in $t$, then the constraint can be equivalently expressed as

$$h(t,x(t)) \leq 0, \forall t \in [0, T]$$

where

$$h(t,x) = \begin{cases} \tilde{h}(t,x) & \text{if } t \in \mathcal{I} \\ \inf_{(t,x)} \tilde{h}(t,x) & \text{otherwise}. \end{cases}$$

Again here notice that it is important for this reformulation to consider state constraint functionals that are merely upper semicontinuous.

What effect does the state constraint have on the necessary conditions? To see this let us assume for the moment that $h$ is continuous, and admit a continuous derivative $h_x(t,x)$ relative to $x$. We may expect the problem to be reformulated as a state constraint free optimal control problem by penalizing trajectories that do not satisfy the relation $h(t,x(t)) \leq 0$. In other words if $(\bar{x}(.),\bar{u}(.))$ is an optimal process for $(\mathcal{P}_C)$, then it is also optimal for the new state constraint free problem

$$(\mathcal{P}_C) \left\{ \begin{array}{l} \text{Minimize } \ell(x(T)) + \int_0^T h(t,x(t))m(t) \, dt \\ \dot{x} = f(t,x,u), u \in U, \text{ a.e., } x(0) = x_0 \text{ and } x(T) \in \mathcal{C}, \end{array} \right.$$ 

where $m(.)$ is a suitable chosen, non negative value function such that

$$m(t) = 0 \text{ if } t \in \{s \in [0,T] : h(s,\bar{x}(s)) < 0\}.$$
We can apply Theorem 2.3.1 to $(\bar{P}_C)$ and obtain the following conditions

\[
(q(t), \lambda) \neq 0 \quad \forall t
\]

\[-q(T) \in \lambda \nabla \ell(\bar{x}(T)) + N_C(\bar{x}(T))
\]

\[-\dot{q}(t) = q(t)f_x(t, \bar{x}(t), \bar{u}(t)) - \lambda h_x(t, \bar{x}(t))m(t), \text{ a.e.}
\]

\[q(t) \cdot f(t, \bar{x}(t), \bar{u}(t)) = \sup_{u \in U} q(t) \cdot f(t, \bar{x}(t), u)
\]

for some multiplier $(q(.), \lambda)$. These conditions capture the essence of the necessary conditions with state constraints. However, we have not been very precise about the nature of the function $m(.)$ that may possibly take unbounded values. A rigorous derivation of the necessary conditions require the introduction of a function of bounded variation $\nu : [0, T] \to \mathbb{R}$ such that

\[
\nu \text{ is constant on any subinterval of } \{t \in [0, T] : h(t, \bar{x}(t)) < 0\},
\]

\[(q, \lambda, \nu) \neq 0
\]

and, for all $t \in (0, T]$

\[-q(t) = -q(0) + \int_S^t q(s)f_x(s, \bar{x}(s), \bar{u}(s)) \, ds - \int_{[S, T]} h_x(s, \bar{x}(s))d\nu(s).
\]

Of course now $q(.)$ is a function of bounded variation itself. Since we want to express necessary conditions in terms of absolutely continuous functions it is customary to define a new co-state $p(.)$ that is obtained from $q(.)$ by subtracting the ‘irregular’ term $\int_{[S, T]} h_x(s, \bar{x}(s))d\nu(s)$:

\[
p(t) := \begin{cases} q(t) - \int_{[S, t]} h_x(s, \bar{x}(s))d\nu(s) & \text{if } t \in [S, T) \\ q(T) - \int_{[S, T]} h_x(s, \bar{x}(s))d\nu(s) & \text{if } t = T. \end{cases}
\]

Now $p(.) \in W^{1,1}([0, T]; \mathbb{R}^n)$ and together with $\lambda$ and $\nu$ satisfies the following set of necessary optimality conditions: the nontriviality condition

\[(p, \lambda, \nu) \neq 0,
\]
the transversality condition

\[-(p(T) + \int_{[S,T]} h_x \, d\nu) \in \lambda \nabla \ell(\bar{x}(T)) + N_C(\bar{x}(T))\]

the adjoint equation

\[-\dot{p}(t) = \left( p(t) + \int_{[S,t]} h_x \, d\nu \right) \cdot f_x(t, \bar{x}(t), \bar{u}(t)), \text{ a.e.,}\]

and the maximum principle or Weierstrass condition

\[\left( p(t) + \int_{[S,t]} h_x \, d\nu \right) \cdot f(t, \bar{x}(t), \bar{u}(t)) = \sup_{u \in U} \left( p(t) + \int_{[S,t]} h_x \, d\nu \right) \cdot f(t, \bar{x}(t), u), \text{ a.e.}\]

The necessary conditions may then be expressed in terms of a measure, since any function of bounded variation \( \nu \) uniquely determines a Borel measure \( \mu \) such that \( \mu(I) = \int_I d\nu(t) \) for all closed subintervals \( I \subset [0,T] \).
Chapter 3

Optimal Control Problems with State Constraints

The concept of stratified necessary conditions for optimal control problems, whose dynamic constraint is formulated as a differential inclusion, was introduced by F. H. Clarke. These are conditions satisfied by a feasible state trajectory that achieves the minimum value of the cost over state trajectories whose velocities lie in a time-varying open ball of specified radius about the velocity of the state trajectory of interest. Considering different radius functions stratifies the interpretation of ‘minimizer’. In this chapter we prove stratified necessary conditions for optimal control problems involving pathwise state constraints. As was shown by Clarke in the state constraint-free case, we find that, also in our more general setting, the stratified necessary conditions yield generalizations of earlier optimality conditions for unbounded differential inclusions as simple corollaries. Some simple examples are provided, giving insights into the nature of the hypotheses invoked for the derivation of stratified necessary conditions and into the scope for their further refinement.
3.1 Literature Review

This chapter provides necessary conditions of optimality for state constrained optimal control problems in which the dynamic constraint is modelled as a differential inclusion:

\[
\begin{aligned}
(P) \quad \left\{ \begin{array}{l}
\text{Minimize } \ell(x(S), x(T)) \\
\text{over arcs } x(.) \in W^{1,1}([S,T];\mathbb{R}^n) \text{ satisfying } \\
\dot{x}(t) \in F(t, x(t)), \text{ for a.e. } t \in [S,T], \\
(x(S), x(T)) \in E, \\
h(t, x(t)) \leq 0, \text{ for all } t \in [S,T].
\end{array} \right.
\end{aligned}
\]

The data for \((P)\) comprises an interval \([S,T] \subseteq \mathbb{R} \ (T > S)\), a set \(E \subseteq \mathbb{R}^n \times \mathbb{R}^n\), functions \(\ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) and \(h : [S,T] \times \mathbb{R}^n \to \mathbb{R}\), and a set-valued map \(F : [S,T] \times \mathbb{R}^n \rightharpoonup \mathbb{R}^n\).

Absolutely continuous arcs \(x(.)\) that satisfy the differential inclusion \(\dot{x}(t) \in F(t, x(t))\) a.e. are called \(F\)-trajectories. If additionally they satisfy all the other constraints in \((P)\), they are referred to as feasible \(F\)-trajectories. Feasible \(F\)-trajectories that minimize the cost over the set of feasible \(F\)-trajectories are called minimizers.

Suppose for the time being that the state constraint \(h(t, x(t)) \leq 0\) is absent from the above formulation. (This case is captured by setting \(h(.,.) \equiv -1\).) The earliest general necessary conditions for an \(F\)-trajectory \(\bar{x}(.)\) to be a minimizer were provided by F. H. Clarke, under hypotheses that included the requirement that the multifunction \(F(.,.)\) is convex valued and bounded and that \(F(t, .)\) is Lipschitz continuous w.r.t. the \(x\) variable, in the following sense: there exists \(\epsilon > 0\) and \(k(,) \in L^1\) such that

\[
F(t, x) \subset F(t, x') + k(t)|x - x'| \mathbb{B} \quad \text{for all } x, x' \in \bar{x}(t) + \epsilon \mathbb{B}, \text{ a.e..} \tag{3.1.1}
\]

The necessary conditions assert that there exists an absolutely continuous function \(p(.)\) called the co-state function which, together with the minimizing \(F\)-trajectory \(\bar{x}(.)\), satisfies a generalization of Hamilton’s system of equations (the ‘Hamiltonian inclusion’), and a set of boundary conditions (the ‘transversality conditions’). Subsequent work on such
conditions (see [44], [40] and, for an expository account, [55]) culminated in a set of conditions valid for unbounded, possibly non-convex valued differential inclusions, in which the generalized Hamiltonian inclusion was replaced by a combination of the Generalized Euler Lagrange inclusion

\[ \dot{p}(t) \in \text{co}\{q \mid (q, p(t)) \in N_{GrF(t,\cdot)}(\bar{x}(t), \dot{\bar{x}}(t))\} \text{ a.e.} \] (3.1.2)

and the Weierstrass condition

\[ p(t) \cdot \dot{\bar{x}}(t) \geq p(t) \cdot v \text{ for all } v \in F(t, \bar{x}(t)), \text{ a.e..} \] (3.1.3)

(The notation \( N_{GrF(t,\cdot)} \) will be recalled shortly.) Also, the Lipschitz continuity hypothesis on \( F(t,\cdot) \) [3.1.1] was replaced by a less restrictive condition, a typical example of which is: there exist \( k(.) \in L^1, \epsilon > 0 \) and \( \beta \geq 0 \) such that, for all \( N \geq 0 \),

\[ F(t,x) \cap (\dot{\bar{x}}(t) + N \mathbb{B}) \subset F(t,x') + (k(t) + \beta N)|x - x'| \mathbb{B} \] (3.1.4)

for all \( x, x' \in \bar{x}(t) + \epsilon \mathbb{B}, \text{ a.e..} \)

(Notice that the Lipschitz constant grows linearly with the size of the intersecting ball on the left side.) The motivation here is that, for unbounded differential inclusions, the earlier imposed Lipschitz continuity condition [3.1.1] is overly restrictive. If, for example, we wish to cover differential inclusions representing a two state differential inequality of the form

\[ \dot{x}_1 \leq x_1 \dot{x}_2 \]

the Lipschitz continuity is not satisfied because the Hausdorff distance between the sets \{\((e_1, e_2)|e_1 \leq x_1 e_2\}\} and \{\((e_1, e_2)|e_1 \leq x'_1 e_2\}\} is infinite, for \( x_1 \neq x'_1 \). On the other hand, condition [3.1.4], which we call the ‘pseudo-Lipschitz’ continuity condition with linear growth, is clearly satisfied. (Condition [3.1.4] is also referred in the literature as epi-Lipschitz continuity condition.)

More recently, Clarke has taken this research in a new direction, by introducing the concept
CHAPTER 3. OPTIMAL CONTROL PROBLEMS WITH STATE CONSTRAINTS

of ‘stratified’ necessary conditions, w.r.t. a given radius function \( r(t) \) or, more generally, a radius multifunction. Here, the object is to derive necessary conditions of optimality when the feasible \( F \)-trajectory \( \bar{x}(.) \) is a minimizer in comparison to all feasible \( F \)-trajectories \( x(.) \) whose velocities are restricted to satisfy the condition:

\[
\dot{x}(t) \in \dot{\bar{x}}(t) + r(t) \mathbb{B}.
\]

(The notation \( \mathbb{B} \) denotes the open unit ball in \( \mathbb{R}^n \)). Clarke showed that if the data satisfies a form of the pseudo-Lipschitz condition (3.1.4), now incorporating the radius function \( r(.) \), namely: there exist \( \epsilon > 0 \) and \( k(.) \in L^1 \) such that

\[
F(t, x) \cap (\dot{x}(t) + r(t) \mathbb{B}) \subset F(t, x') + k(t)|x - x'| \mathbb{B} \quad \text{for all } x, x' \in \bar{x}(t) + \epsilon \mathbb{B}, \text{ a.e.,}
\]

and also satisfies a technical condition, referred to as the ‘tempered growth’ condition, then the generalized Euler Lagrange inclusion (3.1.2) and a restricted form of the Weierstrass condition (3.1.3) are satisfied (for some \( p(.) \)), that is

\[
p(t) \cdot \dot{x}(t) \geq p(t) \cdot v \quad \text{for all } v \in F(t, \bar{x}(t)) \cap (\dot{x}(t) + r(t) \mathbb{B}), \text{ a.e.}
\]

The stratified necessary conditions have a number of useful consequences:

First, they yield as straightforward corollaries necessary conditions under refinements of the pseudo-Lipschitz condition with ‘linear growth’ hypothesis (3.1.4); these are obtained by considering a sequence of radius functions tending to infinity, a.e. We discuss several of these below. Some such refinements could, possibly, be obtained by techniques similar to those originally used to prove necessary conditions under hypothesis (3.1.4), but only at the price of taking apart and reworking a rather complicated analysis.

Second, they are the starting point for the derivation of state-of-the-art necessary conditions for problems with ‘mixed’ (control and state) constraints [24], [23]. This is because, if the controlled differential inclusion and the mixed constraint is recast as a
3.1. LITERATURE REVIEW

differential inclusion, the differential inclusion satisfies the hypotheses for application of
the stratified necessary conditions, under very general hypotheses on the original data.

Third they provide a framework for studying necessary conditions for a weak minimizer
\( \bar{x}(.) \), i.e., an admissible arc that, for some \( \delta > 0 \), is minimizing w.r.t. admissible arcs that satisfy
\[
|x(t) - \bar{x}(t)| + |\dot{x}(t) - \dot{\bar{x}}(t)| < \delta \quad \text{a.e.}
\]

For problems in the calculus of variations in which no dynamic constraints are imposed
on feasible trajectories, it can be shown, via straightforward application of the Pontryagin
maximum principle or direct variational analysis, that weak minimizers (in the above
sense) satisfy the Euler Lagrange, transversality conditions and a restricted form of
the Weierstrass condition. If, however, we now add a dynamic constraint in the form
of a differential inclusion \( \dot{x} \in F(t,x) \), weak minimizers may fail to satisfy the above
necessary conditions, even in seemingly benign situations, in which \( F(t,.) \) is convex valued
and satisfies the global Lipschitz continuity condition (3.1.1). The stratified theory per-
mits us to explore the ‘extra’ hypotheses required to exclude this surprising phenomenon.

The principal contribution of our approach is to show that the stratified necessary con-
ditions remain valid (with appropriate modifications), when a pathwise state constraint,
\( h(t,x(t)) \leq 0 \), is included in the problem formulation (P). The conditions obtained
reduce precisely to those in [18] and [20] when the state constraint is omitted. We allow
general (set valued) radius multifunctions \( R(.) \) in place of \( \dot{\bar{x}}(.) + r(.)B \), as in [20]. We also
examine examples aimed at providing insights into the ultimate limitations on possible
generalizations of such necessary conditions.

Finally, we comment on the methodology for proving stratified necessary conditions. It
might be thought that this was a simple matter of replacing (for each number \( N > 0 \)) the
possibly unbounded multifunction \( F(.,.) \) by the bounded multifunction \( F(t,.) \cap (\dot{x}(t) +
(r(t) \wedge N)B) \), applying known necessary conditions, and passage to the limit, as \( N \to \infty \).
CHAPTER 3. OPTIMAL CONTROL PROBLEMS WITH STATE CONSTRAINTS

But this would require $F(t,.) \cap (\dot{x}(t) + (r(t) \wedge N) \mathbb{B})$ to have Lipschitz continuity properties; unfortunately, however, the constructed multifunction can fail to be continuous (let alone Lipschitz continuous in some useful sense), even if $F(t,.)$ satisfies the (global) Lipschitz condition (3.1.1). Consider the case $F(x) = \{1 + x\} \cup \{0\}$ and $\bar{x} \equiv 0$. The dynamic $F$ is clearly Lipschitz continuous. Yet the intersection

$$F(x) \cap \mathbb{B} = \begin{cases} \{0\} & \text{if } x > 0 \\ F(x) & \text{otherwise} \end{cases}$$

turns out to be discontinuous at the origin. We follow a simple and effective proof technique proposed by Clarke (’lifting’), now in the state-constrained setting, in which we deal not directly with $F(t,.) \cap (\dot{x}(t) + (r(t) \wedge N) \mathbb{B})$ itself, but instead through a proxy, a related multifunction taking as values sets in a higher dimensional state space, which has much better regularity properties.

We refer the reader to Chapter 1 for details on the notation used in this chapter. See also the Notation table at the end of the thesis. Observe, in particular, that in this chapter $\mathbb{B}$ denotes the open unit ball in $\mathbb{R}^n$. Its closure is written $\bar{\mathbb{B}}$.

3.2 Stratified Necessary Conditions for State Constrained Optimal Control Problems

The object is to derive necessary conditions for a feasible $F$-trajectory $\bar{x}(.)$ to be a minimizer. Local minimizers both with respect to the state and the velocity are considered. Some direct implications of these conditions are then stated on the form of corollaries. Proofs will follow in later sections.

Take a feasible $F$-trajectory $\bar{x}(.)$ for $(P)$. We shall say that $R(.)$ is a radius multifunction (for $\bar{x}(.)$) if $R(.)$ is a measurable multifunction and, for each $t \in [S,T]$, $R(t)$ is an open
convex set such that
\[ \dot{x}(t) \in R(t) \quad \text{a.e..} \]

**Definition.** \( \bar{x}(.) \) is a \( W^{1,1} \) local minimizer w.r.t. a given radius multi-function \( R(.) \) for \( \bar{x}(.) \), if there exists \( \epsilon > 0 \) such that
\[ \ell(x(S), x(T)) \geq \ell(\bar{x}(S), \bar{x}(T)) \]
for all feasible \( F \)-trajectories \( x(.) \) such that \( \|x - \bar{x}\|_{W^{1,1}} \leq \epsilon \) and
\[ \dot{x}(t) \in R(t) \quad \text{a.e..} \]

We will also refer to a \( W^{1,1} \) local minimizer w.r.t. a given radius multi-function \( R(.) \) simply as a \( R(.) \)-weak minimizer.

The following hypotheses are invoked: for some \( \epsilon > 0 \),

(H1) \( \ell(.,.) \) is Lipschitz continuous on a neighborhood of \((\bar{x}(S), \bar{x}(T))\). \( E \subset \mathbb{R}^n \times \mathbb{R}^n \) is a closed set. The radius multifunction \( R(.) \) is measurable and takes values open convex sets.

(H2) \( h(.,.) \) is upper semicontinuous near \((t, \bar{x}(t))\), for all \( t \in [S, T] \), and there exists a constant \( k_h \) such that
\[ |h(t, x') - h(t, x)| \leq k_h |x' - x|, \]
for all \( t \) in \([S, T]\) and all \( x', x \in \bar{x}(t) + \epsilon \mathbb{B} \).

(H3) \( F(.,.) \) takes values non-empty subset of \( \mathbb{R}^n \). The restriction of \( F(t,.) \) to \( \bar{x}(t) + \epsilon \mathbb{B} \) has closed graph. \( F(.,.) \) is measurable w.r.t. the product \( \mathcal{L} \times \mathcal{B} \)-algebra, where \( \mathcal{L} \) and \( \mathcal{B} \) denote the \( \sigma \)-algebras of Lebesgue subsets of \([S, T]\) and of Borel subsets of \( \mathbb{R}^n \) respectively.

(H4) There exists a function \( k(.) \in L^1(S, T) \) such that
\[ F(t, x') \cap R(t) \subset F(t, x) + k(t)|x - x'| \mathbb{B}, \]
for all \( x, x' \in \bar{x}(t) + \epsilon B \), a.e. \( t \in [S, T] \).

(H5) There exist numbers \( r_0 > 0 \) and \( \gamma \in (0, 1) \), and a function \( r(.) \in L^1(S, T) \) such that
\[
r(t) \geq r_0 \text{ a.e. and }
\]
\[
i) \ \dot{x}(t) + \gamma^{-1}r(t) \ B \subset R(t)
\]
\[
ii) \ F(t, x) \cap (\dot{x}(t) + r(t) \ B) \neq \emptyset
\]
for all \( x \in \bar{x}(t) + \epsilon B \), a.e. \( t \in [S, T] \).

**Theorem 3.2.1** (Stratified necessary conditions). Let \( \bar{x}(.) \) be a \( W^{1,1} \) local minimizer for \((P)\) w.r.t. the radius multifunction \( R(.) \). Assume (H1)-(H5). Then, there exist an arc \( p(.) \in W^{1,1}([S, T]; \mathbb{R}^n) \), a nonnegative number \( \lambda \), a monotone non-decreasing function \( \mu(.) \in NBV^+[S, T] \) and a \( \mu \)-integrable function \( m(.) \), such that

\[
(i) \ \lambda + \|p\|_{L^\infty} + \|\mu\|_{T.V.} = 1,
\]
\[
(ii) \ \dot{p}(t) \in \co \left\{ \eta : (\eta, p(t) + \int_{[S,t]} m(s)\mu(ds)) \in N_{\text{Gr} F(t,.)}(\bar{x}(t), \dot{x}(t)) \right\} \text{ a.e.,}
\]
\[
(iii) \ \left( p(S), -\left[ p(T) + \int_{[S,T]} m(s)\mu(ds) \right] \right) \in \lambda \partial \ell(\bar{x}(S), \bar{x}(T)) + N_E(\bar{x}(S), \bar{x}(T)),
\]
\[
(iv) \ (p(t) + \int_{[S,t]} m(s)\mu(ds)) \cdot \dot{x}(t) \geq (p(t) + \int_{[S,t]} m(s)\mu(ds)) \cdot v
\]

for all \( v \in F(t, \bar{x}(t)) \cap R(t) \) a.e. \( t \in [S, T] \),
\[
(v) \ m(t) \in \partial^\infty_x h(t, \bar{x}(t)) \mu\text{-a.e. and supp}\{\mu\} \subset \{t : h(t, \bar{x}(t)) = 0\}.
\]

Here \( \partial^\infty_x h(t, x) \) is the set
\[
\partial^\infty_x h(t, x) := \{ \xi \mid \text{there exist } x_i \to x, t_i \to t, \xi_i \to \xi, \text{ s. t., for each } i, \}
\]
\[
\nabla_x h(t_i, x_i) \text{ exists, } \xi_i = \nabla_x h(t_i, x_i) \text{ and } h(t_i, x_i) > 0 \}.
\]

We recall that the pathwise state constraint has been formulated as a functional inequality constraint \( h(t, x(t)) \leq 0 \), in which \( h(.,.) \) is upper semicontinuous and uniformly Lipschitz continuous in the second variable. The versatility of this formulation was first noted and employed by Clarke in [16]. It covers the intrinsic formulation \( x \in A(t) \) employed elsewhere in the literature (see, e.g. [1]), in which \( A(.) \) is a closed valued
lower semicontinuous multifunction since, in this case, we may take \( h(t, x) := d_{A(t)}(x) \).

The way in which the hybrid subgradient \( \partial^*_x h(\cdot, \cdot) \) is defined, via limits of gradients at points \((t_i, x_i)\) at which \( h(t_i, x_i) > 0\), is crucial to ensuring non-triviality of the necessary conditions in this case (it implies that \( m(\cdot) \) satisfies \(|m(t)| = 1 \) \( \mu\text{-a.e.}, \) indeed for a pair \((t_i, x_i)\) such that \( x_i \notin A(t_i) \), the gradient \( \nabla d_{A(t_i)}(x_i) \) exists if and only if \( \text{proj}_{A(t_i)}(x_i) = \{a_i\} \) contains a single element, and in this case \( \nabla d_{A(t_i)}(x_i) = \frac{x_i - a_i}{|x_i - a_i|} \).

Notice that the condition \( \text{supp}\{\mu\} \subset \{t : h(t, \bar{x}(t)) = 0\} \) is inserted for emphasis; it is actually implied by \( m(t) \in \partial^*_x h(t, \bar{x}(t)) \) \( \mu\text{-a.e.}, \) since \( \partial^*_x h(t, \bar{x}(t)) \) is an empty set at those times \( t \) where \( h(t, \bar{x}(t)) < 0\). (See ([55], Chap. 9) for further discussion of this state constraint description.)

The first corollary we state is a variant on the stratified necessary conditions in which a ‘bounded slope’ condition, and an accompanying compatibility condition, replace \((H4)\) and \((H5)\). This set of necessary conditions, proved by Clarke [18] in the state constraint-free case, has been particularly useful as a starting point for deriving necessary conditions covering optimal control problems falling outside the formulation \((P)\), for example problems involving mixed state/velocity constraints [23].

**Corollary 3.2.2.** The assertions of Theorem 3.2.1 remain valid when hypotheses \((H4)\) and \((H5)\) are replaced by \((H4)^*\), \((H5)^*\):

\((H4)^*\) There exist \( \epsilon > 0 \) and \( k(\cdot) \in L^1(S, T) \) such that

\[ |x - \bar{x}(t)| < \epsilon, v \in F(t, x) \cap R(t), (\alpha, \beta) \in N_{GrF(t, \cdot)}(x, v) \Rightarrow |\alpha| \leq k(t)|\beta|. \]

\((H5)^*\) For some \( \omega > 0 \),

\[ \dot{x}(t) + \omega k(t) B \subset R(t). \]

The strict relation between Conditions \((H4)\) and \((H4)^*\) should not be surprising. Indeed, we may see this as a generalization of the classical result that links Lipschitz functions with almost everywhere differentiable functions that have uniformly bounded derivatives. Suppose \( F \) were a single-valued function, independent of \( t \). Then, for a point \( x \) such that
CHAPTER 3. OPTIMAL CONTROL PROBLEMS WITH STATE CONSTRAINTS

∇F(x) exists, we would have that (see Remark (d) of Theorem 2.3.1)

\[(\alpha, \beta) \in N_{GrF(x)}(x, F(x)) \Rightarrow \alpha = -\beta \cdot \nabla F(x).\]

The relation between Conditions (H4) and (H4)* is made more precise in the proof of Corollary 3.2.2. See also [20]. Notice, however, that Condition (H5) is a better assumption than (H5)*, because in (H5), the radius multifunction R(.) does not depend on the pseudo-Lipschitz constant k(.). Yet (H5)* is needed when we assume (H4)* in place of (H4). Justifications can be found in [20, Prop. 2]. See also [18]. We recall the results in [20] and [18] for convenience of the reader.

**Proposition 3.2.1.** Assume (H4) is satisfied. Then (H4)* is also satisfied with the same radius R(.), the same k(.), and the same \(\epsilon\).

Vice-versa, assume that (H4)* and (H5)* are satisfied. Then, for every \(\eta \in (0,1)\) and following possibly a reduction in the size of \(\epsilon\), (H4) is satisfied with the same k(.) and with the radius multifunction \((1 - \eta)R(.)\), i.e.,

\[F(t, x') \cap (1 - \eta)R(t) \subset F(t, x) + k(t)|x - x'| \mathbb{B},\]

for all \(x, x' \in x(t) + \epsilon \mathbb{B}\), a.e. \(t \in [S, T]\).

The second corollary we state provides necessary conditions under hypotheses which capture, as a special case, the ‘pseudo-Lipschitz continuity condition with linear growth’ hypothesis (3.1.4). Here we show that the rate of growth can be traded against the integrability conditions on k(.) and its scaled higher powers. This corollary was proved earlier by Clarke [18] in the state constraint-free case.

**Corollary 3.2.3.** The assertions of Theorem 3.2.1 remain valid when \(R(t) = \mathbb{R}^n\) a.e. and when hypotheses (H4) and (H5) are replaced by the hypothesis:

\[(H4)** \text{ There exist a number } \alpha > 0 \text{ and non-negative measurable functions } k(.) \text{ and } \beta(.) \text{ such that } k(.) \text{ and } t \rightarrow \beta(t)k^\alpha(t) \text{ are integrable }\]
and, for each \( N \geq 0 \),

\[
F(t, x) \cap (\dot{x}(t) + N \mathbb{B}) \subset F(t, x') + (k(t) + \beta(t)N^\alpha)|x - x'| \mathbb{B}
\]

for all \( x, x' \in \dot{x}(t) + \epsilon \mathbb{B} \) a.e.

### 3.3 Counter Examples and Discussion

Suppose that the feasible \( F \)-trajectory \( \bar{x}(.) \) is a weak minimizer for problem (P) in the sense of the discussion of the introduction, i.e. there exists \( r > 0 \) such that \( \bar{x}(.) \) is a minimizer w.r.t. admissible \( F \)-trajectories that satisfy

\[
|x(t) - \bar{x}(t)| < r \quad \text{and} \quad |\dot{x}(t) - \dot{\bar{x}}(t)| < r \quad \text{a.e.}
\]

Suppose that hypotheses (H1)-(H3) are satisfied and also the stronger (global) form (3.1.1) of the Lipschitz continuity assumption (H4) is satisfied, but not (H5). Are the generalized Euler Lagrange condition, transversality condition and some restricted form of the Weierstrass condition still valid? This question can be addressed within the framework of stratified necessary conditions, by regarding \( \bar{x}(.) \) as a \( W^{1,1} \) local minimizer w.r.t. the constant radius multifunction \( R(t) = \dot{x}(t) + r \mathbb{B} \). Clarke ([18], pp. 46-47) showed through a counter-example that, if the ‘tempered growth’ condition (H5) is dropped, this is not the case. We provide another example illustrating the critical role of the tempered growth condition: if (H5) is dropped from the hypotheses of Thm. 3.2.1 then we cannot guarantee the simultaneous satisfaction of the generalized Euler Lagrange and the transversality conditions.

**Example 1.** Consider

\[
\begin{cases}
\text{Minimize} & -x_2(1) \\
\text{over arcs} & x \in W^{1,1}([0, 1]; \mathbb{R}^2) \quad \text{satisfying} \\
\dot{x}(t) & \in F(t, x(t)) \quad \text{a.e.} \\
x_2(0) & = 0,
\end{cases}
\]

CHAPTER 3. OPTIMAL CONTROL PROBLEMS WITH STATE CONSTRAINTS

in which

\[ F(t,x) := \{(e_1,e_2) \in \mathbb{R}^2 | e_1 = 0, e_2 = k(t)x_1) \}. \]

Here, \(k(.)\) is any positive function in \(L^1(0,1)\) which is not essentially bounded.

Take an arbitrary number \(r > 0\). Then the \(F\)-trajectory \(\bar{x}(.) = (\bar{x}_1 = 0, \bar{x}_2(.) \equiv 0)\) is a minimizer w.r.t. the radius multifunction \(R(.) \equiv r\mathbb{B}\). To see this, suppose to the contrary that there exists a feasible \(F\)-trajectory \(x(.) = (x_1(.), x_2(.))\) with lower cost and such that

\[ \dot{x}(t) \in R(t) \quad \text{a.e..} \quad (3.3.1) \]

We have \(x_1 > \bar{x}_1 = 0\), whence \(\dot{x}(t) = (0, x_1 k(t))\) is not essentially bounded. This implies that, on a set of positive measure, \(\dot{x}(t) \notin R(t)\), in contradiction of \((3.3.1)\).

Hypotheses (H1)-(H4) of Thm. 3.2.1 (with \(h(.) \equiv -1\)) are satisfied for the above radius multifunction, but not (H5). This is because, for any \(\gamma \in (0,1)\) and \((x_1, x_2) \in \mathbb{R}^2\) such that \(x_1 \neq 0\), the relation

\[ F(t,x) (= \{0, x_1 k(t)\}) \notin \dot{x}(t) + \gamma^{-1}r\mathbb{B} \]

is satisfied on a set of positive measure, in violation of the tempered growth condition.

Let us now examine the generalized Euler Lagrange and transversality conditions. They assert the existence of an arc \(p(.) = (p_1(.), p_2(.))\) and \(\lambda \geq 0\), not both zero, satisfying:

\[ \dot{p}_2(t) = 0 \]
\[ \dot{p}_1(t) + p_2(t)k(t) = 0 \quad (3.3.2) \]

and

\[ p_1(0) = 0, \quad p_1(1) = 0 \]
\[ p_2(1) = \lambda. \quad (3.3.3) \]
If $\lambda = 0$ then also $p \equiv 0$, a contradiction. If, on the other hand, $\lambda > 0$ then (3.3.2) implies
\[ p_1(1) = -\lambda \int_0^1 k(t) \, dt < 0 \]
in contradiction of (3.3.3). We conclude that the generalized Euler Lagrange and transversality conditions cannot both be satisfied in this example.

Recall that the radius multifunction $R(.)$ in Thm. 3.2.1 is required to take open-set values. The second example tells us that the assertions of the theorem are no longer valid in general, if the radius function is assumed, instead, to take closed-set values.

**Example 2.** Consider the problem
\[
\begin{cases}
\text{Minimize} & -x(1) \\
\text{over arcs} & x \in W^{1,1}([0,1];\mathbb{R}^1) \text{ satisfying} \\
\dot{x}(t) & \in F(t,x) \text{ a.e.} \\
\text{and such that} & x(0) = 0,
\end{cases}
\]
in which
\[ F(t,x) = \{0\} \cup \{1 + |x|\} . \]
Consider two different choices of radius multifunctions
\[ R(t) \equiv \mathbb{B} \quad \text{and} \quad R(t) \equiv \bar{\mathbb{B}}, \]
the first of which takes values open sets and the second closed sets. Then the $F$-trajectory $\bar{x}(.) \equiv 0$ is a minimizer w.r.t. either choice of radius multifunction. To confirm this assertion suppose, contrary to the assertion, that there exists a feasible $F$-trajectory $x(.)$ w.r.t. the radius multifunction $R(.)$
\[ \dot{x}(t) \in R(t) \]
(for either choice of $R(.)$), which has lower cost. Then $x(1) > \bar{x}(1) = 0$. 

Define $\bar{t} := \min\{t' \leq 1 \mid x(t) > 0 \text{ for all } t \in [t', 1] \}$. Then, since $x(.)$ is continuous and $x(0) = 0$, we have that $\bar{t} < 1$, $x(\bar{t}) = 0$ and $x(t) > 0$ for all $t \in (\bar{t}, 1]$. We deduce that $0 < x(1) = 0 + \int_\bar{t}^1 \dot{x}(s)ds$. But then there is a set of points $t \in (\bar{t}, 1]$, having positive measure, on which $\dot{x}(t) \neq 0$ and $\dot{x}(t) = 1 + |x(t)| \notin R(t)$. This is not possible, since $x(.)$ is assumed to be feasible w.r.t. either choice of radius multifunction $R(.)$. So $\bar{x}(.)$ is indeed a minimizer w.r.t. to either radius multifunction.

There is a unique set of non zero multipliers $(p(.), \lambda)$ (modulo scaling) satisfying the generalized Euler Lagrange and transversality conditions, namely $p(.) \equiv 1, \lambda = 1$. Notice however that the Weierstrass condition

$$p(t) \cdot \dot{x}(t) = \max\{p(t) \cdot e \mid e \in F(t, \bar{x}(t)) \cap R(t)\} \text{ a.e.}$$

is satisfied for $R(.) \equiv B$ but not for $R(.) \equiv \bar{B}$.

**Example 3.** The stratified framework allows to consider examples that require less restrictive hypotheses than has previously been considered. We now construct an example in which the hypotheses of Corollary 3.2.3 are satisfied but previous set of hypotheses such as Condition (3.1.4) are not.

Consider the case of a property developer that builds two different kind of houses: “social housing” and “commercial housing”. We define

$$x_1(t) := \{\text{size of commercial housing at time } t\}$$
$$x_2(t) := \{\text{size of social housing at time } t\}$$
$$u_1(t) := \{\text{investment rate for commercial housing}\}$$
$$u_2(t) := \{\text{investment rate for social housing}\}.$$

We suppose that the estate company prefers to invest on the more expensive commercial
housing that gives a better profit. This is formalized by defining the following cost

$$\min - (x_1(T) + \beta x_2(T))$$

for some $\beta > 1$. Nevertheless investment on social housing is regulated by a governmental law. Precisely we define the constraint $u_2(t) \geq a(x_2(t))(u_1(t))^\alpha$ where

$$a(x) := M - \min\{M, x\},$$

and $M$ represent the minimum amount of social housing that should be built. It can be easily checked that the dynamic $F(t, x_1, x_2) := \{(u_1, u_2) : u_2(t) \geq a(x_2(t))(u_1(t))^\alpha\}$ satisfies Hypothesis (H4)** of Corollary 3.2.3 but it does not satisfy previous hypotheses with linear growth such as Hypothesis (3.1.4).

### 3.4 Proof of Theorems and Corollaries

The proof of Thm. 3.2.1 will be the end-result of applying known, ‘unstratified’, necessary conditions for problem $(P)$ to a sequence of optimal control problems and passage to the limit. The known necessary conditions referred to here are:

**Proposition 3.4.1.** The assertions of Thm. 3.2.1 are valid when $R(t) = \mathbb{R}^n$ for all $t$. In this case, $(H_4)$ is the Lipschitz continuity condition on $F(t, .)$: there exists $k(.) \in L^1$ and $\epsilon > 0$ such that

$$F(t, x') \subset F(t, x) + k(t)\|x - x'\| \quad \text{a.e.}$$

(3.4.1)

for all $x, x' \in \bar{x}(t) + \epsilon\mathbb{B}$. $(H_5)$ is superfluous (as it is implied by $(H_4)$) for a suitably adjusted $\epsilon$, and the qualifier ‘$v \in F(t, \bar{x}(t)) \cap R(t)$’ in the Weierstrass condition (iv) is interpreted as ‘$v \in F(t, \bar{x}(t))$’.

**Proof.** See ([55], Thm 10.3.1). Notice that, if (3.4.1) is satisfied, then hypothesis $(G2)$ of this theorem is automatically satisfied (for $\beta = 0$ and the same $k(.)$ and $\epsilon$).

We observe that a number of reductions can be performed at the outset to simplify the analysis (cf. [18]).
Lemma 3.4.2. The assertions of the Thm. 3.2.1 are valid in general if they can be verified in the special case when

(a): \( \bar{x}(.) \equiv 0 \),

(b): The interval \([S,T] = [0,1]\) and the function \( r(.) \) in hypothesis (H5) is a positive constant \( r(.) \equiv r_0 \) for some constant \( r_0 > 0 \),

(c): \( \bar{x}(.) \) is a \( \mathcal{L}^{\infty} \) local minimizer w.r.t. \( R(.) \), and not merely a \( W^{1,1} \) local minimizer, i.e. \( \bar{x}(.) \) minimizes the cost over all feasible \( F \)-trajectories \( x(.) \) satisfying \( ||x(.) - \bar{x}(.)||_{\mathcal{L}^{\infty}} \leq \epsilon \), for some fixed \( \epsilon \).

The justification for these reductions is straightforward. We need to show, in each case, that the data can be transformed so that the relevant additional hypothesis is satisfied and that the assertions of the (special case of the) theorem, applied to the transformed problem, can be interpreted as an assertion of the desired properties of the original problem. (a) is dealt with by a translation of \( \bar{x}(.) \) to the origin, following which the multifunctions \( F(t,x) \) and \( R(t) \) become \( F(t,\bar{x}(t) + x) - \{\hat{x}(t)\} \) and \( R(t) - \{\hat{x}(t)\} \). (b) is dealt with by introducing a new independent variable \( \tau \) related to the original time variable \( t \) according to \( \tau(t) = c \int_{S}^{t} r(s)ds \), in which \( c \) is a positive constant adjusted so that \( \tau(T) = 1 \). Concerning (c) we observe that, if \( \bar{x}(.) \) is merely a local \( W^{1,1} \) minimizer, it is a \( \mathcal{L}^{\infty} \) minimizer for a modified problem into which is introduced the additional constraints:

\[
|x(0) - \bar{x}(0)| \leq \epsilon \quad \text{and} \quad \int_{0}^{1} |\dot{x}(t) - \dot{\bar{x}}(t)|dt \leq \epsilon. \tag{3.4.2}
\]

The problem with additional constraints can be treated as a problem of the form \((P)\) by means of state augmentation. In view of the fact that the extra conditions (3.4.2) are not active at \( x(.) = \bar{x}(.) \), the necessary conditions for this related problem imply the desired necessary conditions for the original problem. Henceforth then we assume (a)-(c).

We next define and list salient properties of a well-known construct from convex analysis, namely the gauge function \( g(.) \) of an open convex set \( \Gamma \) containing the origin. The role of the gauge function is to quantify the extent to which a point lies in \( \Gamma \).

Lemma 3.4.3. Take an open convex set \( \Gamma \) containing the open ball of radius \( r_0 \) centered
at the origin. Define the function $g(\cdot) : \mathbb{R}^n \to [0, +\infty)$ to be

$$g(f) := \inf\{\lambda > 0 : \lambda^{-1}f \in \Gamma\}.$$ 

Then $g(\cdot)$ has the following properties:

1. $g(cx) = cg(x)$ for every constant $c > 0$.

2. $g(x) = 0$ if and only if $x = 0$.

3. $g(\cdot)$ is Lipschitz with Lipschitz constant $1/r_0$.

4. $\Gamma = \{x \in \mathbb{R}^n : g(x) < 1\}$.

In view of $(H5)$, we may choose $\eta \in (0, 0.5)$ such that, for all $x \in \epsilon \overline{B}$ and $N > \frac{r_0}{1-2\eta}$,

$$(1 - 2\eta)^{-1}r_0\mathbb{B} \subset R_N(t) \quad \text{and} \quad F(t,x) \cap r_0\mathbb{B} \neq \emptyset, \text{ a.e.} \quad \text{(3.4.3)}$$

Here

$$R_N(t) := R(t) \cap N\mathbb{B}.$$ 

Now take $g_t(\cdot)$ to be the gauge function of $R_N(t)$.

Following Clarke [18, 20] we embed information about $F(\cdot,\cdot)$ in a new multi-function $	ilde{F}(\cdot,\cdot) : [0,1] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}$ taking values subsets in $\mathbb{R}^{n+1}$: For each $(t,x) \in [0,1] \times \mathbb{R}^n$,

$$\tilde{F}(t,x) := \begin{cases} 
\{(\xi,e) | \xi \in [0,1], e \in F(t,x), g_t(e) \leq 1 - \xi \eta\} & \text{if } x \in \epsilon \overline{B} \\
\{0\} & \text{otherwise.}
\end{cases}$$

Lemma 3.4.4. $\tilde{F}(\cdot,\cdot)$ takes values non-empty sets. For a.e. $t$, $\tilde{F}(t,\cdot)$ has closed graph. Furthermore,

$$\tilde{F}(t,x) \subset \tilde{F}(t,x') + \tilde{k}(t)|x - x'|\mathbb{B}, \text{ for all } x,x' \in \epsilon \mathbb{B}, \text{ a.e.,}$$

where $\tilde{k}(t) = \left(1 + \frac{N+1}{r_0}\eta\right)k(t)$. 

Proof. The fact that \( \tilde{F}(.,.) \) takes values non-empty sets follows from (3.4.3). Showing that \( \tilde{F}(t,.) \) has closed graph is a simple exercise. To establish that \( \tilde{F}(t,.) \) is Lipschitz continuous on \( \epsilon \overline{B} \) with the stated Lipschitz constant, take \( x, x' \in \epsilon \overline{B} \) and \( (\xi e, \xi) \in \tilde{F}(t, x) \).

We have two possibilities:

A. \( \xi \leq \frac{k(t)|x-x'|}{\eta r_0} \): By (H5) there exists a velocity \( e' \in F(t, x') \) such that \( g_t(e') < 1 \). Then \( (0, 0) \in \tilde{F}(t, x') \) and

\[
|(\xi e, \xi) - (0, 0)| \leq \frac{k(t)|x-x'|}{\eta r_0}(|e| + 1) \leq \tilde{k}(t)|x-x'|.
\]

B. \( \xi > \frac{k(t)|x-x'|}{\eta r_0} \): By the pseudo Lipschitz hypothesis (H4) there exists a velocity \( e' \in F(t, x') \) such that \( |e-e'| \leq k(t)|x-x'| \). Since in this case \( k(t)|x-x'| \eta r_0 \) must be smaller than 1, \( e' \) cannot be too far from \( e \), so that \( e' \in R_N(t) \). More precisely the point \( (\xi' e', \xi') \) with \( \xi' = \xi - \frac{k(t)|x-x'|}{\eta r_0} \) is admissible. Indeed

\[
g_t(e') \leq g_t(e) + \frac{1}{r_0}|e-e'| \leq (1 - \xi \eta) + \frac{1}{r_0}k(t)|x-x'| \leq (1 - \xi' \eta).
\]

Therefore \( (\xi' e', \xi') \in \tilde{F}(t, x') \) and

\[
|(\xi e, \xi) - (\xi' e', \xi')| \leq |\xi e - \xi' e'| + |\xi - \xi'| \leq (\xi - \xi')(|e| + 1) + |e - e'| \leq \tilde{k}(t)|x-x'|.
\]

For each \( t \in [0, 1] \) define now the function \( \phi_t : \mathbb{R}^n \to \mathbb{R}_+ \) to be

\[
\phi_t(e) := \max \left\{ \frac{r_0}{2N}(g_t(e) - (1 - 2\eta)), 0 \right\}.
\]

The following properties of \( \phi_t(.) \) will be of particular significance

\[
\begin{align*}
\phi_t(e) &= 0 \quad \text{iff} \quad g_t(e) \leq 1 - 2\eta \\
|\phi_t(e) - \phi_t(e')| &\leq \frac{1}{2N}|e - e'| \\
\phi_t(e) &\geq \frac{\eta r_0}{4N} \quad \text{if} \quad g_t(e) > 1 - \frac{3}{2} \eta.
\end{align*}
\] (3.4.4)

With these definitions and constructions behind us, we are ready to start the proof of Thm. 3.2.1. This will hinge on properties of a sequence of optimization problems \((P_i)\),
involving the arbitrary sequence $\epsilon_i \downarrow 0$. Define

$$J_i(x(.), y(.)) := \max \{ \ell(x(0), x(1)) - \ell(0, 0) + \epsilon_i^2, 1 - y(1) \} + \int_0^1 \phi_t(\dot{x}(t)) dt \quad (3.4.5)$$

and consider the optimal control problem

\[
\begin{align*}
(P_i) & \quad \text{Minimize } J_i(x(.), y(.)) \\
& \quad \text{over } (x(.), y(.)) \in W^{1,1}([0,1]; \mathbb{R}^n \times \mathbb{R}) \text{ satisfying} \\
& \quad (\dot{x}(t), \dot{y}(t)) \in \tilde{F}(t, x(t)) \quad \text{for a.e. } t \in [0,1], \\
& \quad (x(0), x(1)) \in E, \ y(0) = 0 \\
& \quad h(t, x(t)) \leq 0 \quad \text{for all } t \in [0,1] .
\end{align*}
\]

Denote by $\mathcal{S}$ the set of feasible $\tilde{F}$-trajectories for $(P_i)$. Then $J_i(x(.), y(.))$ defines a continuous function on $\mathcal{S}$ with respect to the $W^{1,1}$ metric. Noting that $J_i$ is non-negative, we see that $(\tilde{x}(\cdot) \equiv 0, \tilde{y}(\cdot) \equiv t \to t)$ is an $\epsilon_i^2$ solution of $(P_i)$. It follows then from Ekeland’s Theorem that there exists $(x_i(.), y_i(.)) \in \mathcal{S}$ such that

$$\|x_i\|_{W^{1,1}([0,1]; \mathbb{R}^n)} \leq \epsilon_i \text{ and } \int_0^1 |\dot{y}_i(t) - 1| dt \leq \epsilon_i, \quad (3.4.6)$$

and $(x_i(.), y_i(.))$ is a minimizer for the perturbed problem:

\[
\begin{align*}
(\tilde{P}_i) & \quad \text{Minimize } \tilde{J}_i(x_i(.), y_i(.)) := J_i(x_i(.), y_i(.)) \\
& \quad + \epsilon_i |x(0) - x_i(0)| + \epsilon_i \int_0^1 (|\dot{x}(t) - \dot{x}_i(t)| + |\dot{y}(t) - \dot{y}_i(t)|) dt \\
& \quad \text{over } (x(.), y(.)) \in \mathcal{S} .
\end{align*}
\]

Using the usual state augmentation techniques to eliminate the integral terms in the cost, we can reformulate $(\tilde{P}_i)$ as an optimization problem to which the already known special case Prop.[3.4.1] of the theorem is applicable. Note, in particular, that the multifunction involved $\tilde{F}(t, .)$ is Lipschitz continuous. Bearing in mind that the multifunction $\tilde{F}(t, .)$ does not depend on $y$ and $y$ does not appear in the integral cost term, we observe that the costate $q$ corresponding to the $y$-variable is in fact constant. Prop.[3.4.1] tells us then that there exist, for each $i$, $p_i(.) \in W^{1,1}([0,1]; \mathbb{R}^n)$, $q_i \in \mathbb{R}$, $\lambda_i \geq 0$, $\mu_i \in NBV^+[0,1]$, $\chi_i \in [0,1]$ and a Borel measurable function $m_i : [0,1] \to \mathbb{R}^n$ such that

(A1) $q_i + \|p_i\|_{L^\infty} + \|\mu_i\|_{TV} + \lambda_i = 1$. 
(A2) \( \hat{p}_i(t) \in \text{co}\left\{ \nu : (\nu, p_i(t) + \int_{[0,t]} m(s)\mu_i(ds), q_i) \in N_{\text{Gr} \, \tilde{F}(t, \cdot)}(x_i(t), \dot{x}_i(t), \dot{y}_i(t)) \right\} \) 
\hfill + \{0\} \times \lambda_i \epsilon_i \mathbb{B} \times \lambda_i \epsilon_i \mathbb{B} + \{0\} \times \lambda_i \partial \phi_i(\dot{x}_i(t)) \times \{0\}.

\]
\( (A3) \left( p_i(0), -[p_i(1) + \int_{[0,1]} m_i(s)\mu_i(ds)] \right) \in N_E(x_i(0), x_i(1)) \) 
\hfill + \lambda_i \epsilon_i \mathbb{B} \times \{0\} + \lambda_i \chi \partial \ell(x_i(0), x_i(1)).

\( (A4) \quad q_i = \lambda_i(1 - \chi_i). \) Observe that in the case when \( \ell(x_i(0), x_i(1)) - \ell(0, 0) + \epsilon_i^2 > 1 - y_i(1) \) we have that \( \chi_i = 1. \)

\( (A5) \quad (p_i(t) + \int_{[0,t]} m_i(s)\mu_i(ds)) \cdot \dot{x}_i(t) + q_i \dot{y}_i(t) - \lambda_i \phi_i(\dot{x}_i(t)) \geq \) 
\( (p_i(t) + \int_{[0,t]} m_i(s)\mu_i(ds)) \cdot (\xi e) + q_i \xi - \lambda_i \phi_i(\xi e) - \lambda_i \epsilon_i(\xi e - \dot{x}_i(t)) + (\xi - \dot{y}_i(t)) \) 
\hfill \text{for all} \ (\xi e, \xi) \in \tilde{F}(t, x_i(t)).

\( (A6) \quad m_i(t) \in \partial \tilde{F} h(t, x_i(t)) \mu_i \text{-a.e. and} \ \text{supp}\{\mu_i\} \subset \{t : h(t, x_i(t)) = 0\}. \)

By (3.4.6) we can arrange by subsequence extraction that 
\( \dot{x}_i(t) \rightarrow 0 \ \text{a.e.,} \quad \dot{y}_i(t) \rightarrow 1 \ \text{a.e.} \)

It may be deduced from (A1) and (A2) and the Lipschitz continuity properties of \( \tilde{F}(t, \cdot) \) that the \( p_i(\cdot) \)'s are uniformly bounded and the \( \hat{p}_i(\cdot) \)'s are majorized by a common integrable function. We may conclude that, following an extraction of subsequences, \( p_i \rightarrow p \) uniformly in \( W^{1,1} \) and \( \hat{p}_i \rightarrow \hat{p} \) weakly in \( L^1 \), \( q_i \rightarrow q \), \( \lambda_i \rightarrow \lambda \), \( \mu_i \rightarrow \mu \) weakly*, \( m_i d\mu_i \rightarrow md\mu \) weakly* and \( \chi_i \rightarrow \chi \) for multipliers \( p(\cdot), q, \lambda, \mu, \chi \) such that
\[
q + \|p\|_{L^\infty} + \|\mu\|_{TV} + \lambda = 1. \quad (3.4.7)
\]

We have derived a set of conditions satisfied by \( x_i(\cdot), p_i(\cdot), q_i, \mu_i \) and \( \lambda_i \), for each \( i \). The convergence analysis of ([55], Thm. 2.5.3 and Prop. 2.6.1) permits us to deduce the following properties of their limits:

\( (B1) \quad \hat{p}(t) \in \text{co}\left\{ \nu : (\nu, p(t) + \int_{[0,t]} m(s)\mu(ds), q) \in N_{\text{Gr} \, \tilde{F}(t, \cdot)}(0, 0, 1) \right\} \ \text{a.e..} \)

\( (B2) \quad \left( p(0), -[p(1) + \int_{[0,1]} m(s)\mu(ds)] \right) \in N_E(0, 0) + \lambda \chi \partial \ell(0, 0). \)

\( (B3) \quad q \geq (p(t) + \int_{[0,t]} m(s)\mu(ds)) \cdot (\xi e) + q \xi - \lambda \phi_i(\xi e) \) a.e.
\hfill \text{for all} \ (\xi e, \xi) \in \tilde{F}(t, 0). \)
(B4) $m(t) \in \partial_x^+ h(t,0) \mu$-a.e. and $\text{supp}\{\mu\} \subset \{t : h(t,0) = 0\}$.

Note next the following lemma relating the normal cones to the graphs of $\tilde{F}(t,.)$ and $F(t,.)$.

The straightforward proof, that depends on establishing the relationship for proximal normals and then exploiting the interpretation of normal vectors as limits of proximal normals at neighbouring points, is omitted.

**Lemma 3.4.5.** Take any $x \in \varepsilon B$. If a vector $(\alpha, \beta, \gamma) \in N_{Gr \tilde{F}(t,.)}(x, \xi f, \xi)$, where $g_t(f) \leq 1 - 2\eta$, then $(\alpha, \xi \beta) \in N_{Gr F(t,.)}(x,f)$.

It follows from (B1) that

(C1) $\dot{p}(t) \in co \{ \nu : (\nu, p(t) + \int_{[0,t]} m(s)\mu(ds)) \in N_{Gr F(t,.)}(0,0) \}$ a.e..

Recall that $\phi_t(e) = 0$ if $g_t(e) \leq 1 - 2\eta$ and, in particular, that $\phi_t(0) = 0$. Also

$(1 - 2\eta)R_N(t) = \{e|g_t(e) < 1 - 2\eta\}$. It follows from (B3), upon setting $\xi = 1$, that

(C2) $(p(t) + \int_{[0,t]} m(s)\mu(ds)) \cdot e \leq 0$

for all $e \in F(t,0) \cap \left((1 - 2\eta)(R(t) \cap N\mathbb{B})\right)$.

Define $\tilde{\lambda} = \lambda \chi$. Then we see from (C1), (B2), (C2), (B4) and (3.4.7) that a restricted form of the assertions of Theorem 3.2.1, in which the Weierstrass condition holds for $v$'s in the subset $F(t,0) \cap \left((1 - 2\eta)(R(t) \cap N\mathbb{B})\right)$ of $F(t,0) \cap R(t)$, are satisfied, provided we can show that

$$\|p\|_{L^\infty} + \|\mu\|_{T.V.} + \tilde{\lambda} \neq 0.$$  

To confirm this ‘non-triviality’ relation, we make essential use of the Weierstrass condition (A5) for each $i$. Let $\mathcal{I} \subset [0,1]$ be the set of full measure comprising $t \in [0,1]$ such that $(\dot{x}_i(t), \dot{y}_i(t)) \in \tilde{F}(t, x_i(t))$, the Weierstrass condition (A5) and hypotheses (H4) and (H5) are valid for each $i$. We can always arrange by extracting a suitable subsequence that one of the following three cases applies:

**Case 1:** $\dot{y}_i(t) = 1$ a.e., in which case $y_i(1) = 1$, for all $i$.

**Case 2:** There exist $\{t_i\} \subset \mathcal{I}$ and $\alpha \in [0,1]$ such that $t_i \uparrow 1$ and $\dot{y}_i(t_i) < 1 - \alpha$ for all $i$. 

**Case 3:** 

Case 3: There exists \( \{t_i\} \subset I \) such that \( t_i \uparrow 1, \dot{y}_i(t_i) < 1 \) for each \( i \), and \( \dot{y}_i(t_i) \to 1 \).

Case 1: if \( y_i(t) \equiv 1 \) for all \( i \) then, for each \( i \), \( x_i(\cdot) \) is actually an \( F \)-trajectory. Since it satisfies the constraints of problem (P) we must have \( \ell(x_i(0), x_i(1)) - \ell(0, 0) + \epsilon_i^2 > \ell(x_i(0), x_i(1)) - \ell(0, 0) \geq 0 = 1 - y_i(1) \). It follows that the term \( (1 - y_i(1)) \) is not active in the evaluation of ‘max’ operation in the cost. Consequently \( \chi_i = 1 \) and \( q_i = 0 \). So \((q, \lambda) = (0, \tilde{\lambda})\) and it follows from (3.4.7) that

\[
\|p\|_{L^\infty} + \|\mu\|_{T.V.} + \tilde{\lambda} = 1.
\]

Case 2: Consider the Weierstrass condition (A5). According to (H5) since \((1-2\eta)^{-1}r_0B \subset R_N(t)\) and \( F(t, x_i(t)) \cap r_0B \neq \emptyset \) a.e., we can choose \( e \in F(t, x_i(t)) \) such that \( g_t(e) < 1 - 2\eta \).

Set \( \xi = 1 \). We have for each \( i \)

\[
(p_i(t_i) + \int_{[0,t_i]} m_i(s)\mu_i(ds)) \cdot \dot{x}_i(t_i) + q_i\dot{y}_i(t_i) - \lambda_i\phi_i(\dot{x}_i(t_i))
\geq (p_i(t_i) + \int_{[0,t_i]} m_i(s)\mu_i(ds)) \cdot e + q_i - \lambda_i\epsilon_i(|e - \dot{x}_i(t_i)| + |1 - \dot{y}_i(t_i)|).
\]

We obtain in the limit as \( i \to +\infty \) the relation

\[
(||p||_{L^\infty} + k_h\|\mu\|_{T.V.})N > q\alpha.
\]

If \( q \neq 0 \) then \( ||p||_{L^\infty} + k_h\|\mu\|_{T.V.} > 0 \). If on the other hand \( q = 0 \), then

\[
(q, \lambda) = (0, \tilde{\lambda})
\]

and by (3.4.7),

\[
||p||_{L^\infty} + ||\mu||_{T.V.} + \tilde{\lambda} = 1.
\]

Note that, in this case, \( \lambda = 0 \) implies \( ||p||_{L^\infty} + ||\mu||_{T.V.} = 1 \).

Case 3: By definition of \( \tilde{F}(\cdot, \cdot) \) we know that \( \dot{x}_i(t) = \dot{y}_i(t)e_i \) for some \( e_i \in F(t, x_i(t)) \) such that \( g_t(e_i) \leq 1 - \dot{y}_i(t)\eta \). By extracting subsequences we can arrange that either of the
following situations must arise:

\[ g_i(e_i) < 1 - \dot{y}_i(t_i)\eta \tag{3.4.8} \]
\[ g_i(e_i) = 1 - \dot{y}_i(t_i)\eta \tag{3.4.9} \]

for each \( i \). Suppose first (3.4.8) is true. Take any \( \xi \in (\dot{y}_i(t_i), 1] \) such that \( g_i(e_i) \leq 1 - \eta\xi \) and fix \( e = e_i \). Then, from (A5) again,

\[
(p_i(t_i) + \int_{[0,t_i]} m_i(s)\mu_i(ds)) \cdot (e_i(\dot{y}_i(t_i) - \xi)) + q_i(\dot{y}_i(t_i) - \xi) \geq \lambda_i(\phi_i(\dot{y}_i(t_i)e_i) - \phi_i(\xi e_i)) - \lambda_i e_i(\|\xi e_i - \dot{y}_i(t_i)e_i| + |\xi - \dot{y}_i(t_i)|).
\]

Since \( |e_i| \leq N \) and \( \phi_i(\cdot) \) has Lipschitz constant \( 1/2N \),

\[
|\phi_i(\dot{y}_i(t_i)e_i) - \phi_i(\xi e_i)| \leq \frac{1}{2} |\xi - \dot{y}_i(t_i)|.
\]

Estimating terms as before, dividing across the inequality by \( (\xi - \dot{y}_i(t_i)) \) and passage to the limit gives

\[
(\|p\|_{L^\infty} + k_h\|\mu\|_{T.V.})N + \frac{1}{2}\lambda \geq q.
\]

But \( \lambda = \chi\lambda + (1 - \chi)\lambda = \tilde{\lambda} + q \), so

\[
(\|p\|_{L^\infty} + k_h\|\mu\|_{T.V.})N + \frac{1}{2}\tilde{\lambda} \geq \frac{1}{2}q.
\]

It follows that, if \( q > 0 \), (\( \|p\|_{L^\infty} + k_h\|\mu\|_{T.V.})N + \frac{1}{2}\tilde{\lambda} > 0 \). If, on the other hand, \( q = 0 \), then \( \lambda = \tilde{\lambda} \) and \( \|p\|_{L^\infty} + \|\mu\|_{T.V.} + \tilde{\lambda} = 1 \), by (3.4.7).

Suppose finally that (3.4.9) is true. Take \( \xi = 1 \) and take \( e \) to be a point \( e_0 \in F(t_i, x_i(t_i)) \cap NB \) such that \( g_i(e_0) \leq 1 - 2\eta \). Then, still from (A5), we have

\[
(\|p_i\|_{L^\infty} + k_h\|\mu_i\|_{T.V.})2N + \lambda_i e_i 3N \geq q_i(1 - \dot{y}_i(t_i)) + \lambda_i \phi_i(\dot{x}_i(t_i)).
\]

Since \( g_i(\dot{x}_i(t_i)) = \dot{y}_i(t_i)g_i(e_i) = \dot{y}_i(t_i)(1 - \dot{y}_i(t_i)\eta) \rightarrow 1 - \eta \) (recall that in case 3, \( \dot{y}_i(t_i) \rightarrow 1 \)), by (3.4.4) and (3.4.9) we have that \( \phi_i(\dot{x}_i(t_i)) \geq \eta p_0 \frac{\eta}{4N} \) for all \( i \) big enough. Thus

\[
(\|p\|_{L^\infty} + k_h\|\mu\|_{T.V.})2N \geq \lambda \eta p_0 \frac{\eta}{4N} = (\tilde{\lambda} + q) \eta p_0 \frac{\eta}{4N}.
\]
If $q > 0$ then $\|p\|_{L^\infty} + k_\lambda \|\mu\|_{T.V.} > 0$. If $q = 0$, then $(q, \lambda) = (0, \tilde{\lambda})$ and $\|p\|_{L^\infty} + \|\mu\|_{T.V.} + \tilde{\lambda} = 1$ follows by (3.4.7).

The foregoing demonstrates that, in all cases the multipliers arising in the limit are non-trivial. We may scale them so that

$$\|p\|_{L^\infty} + \|\mu\|_{T.V.} + \tilde{\lambda} = 1.$$ (3.4.10)

Reviewing (C1), (B2), (C2), (B4) and (3.4.10), we see that all assertions of the theorem has been proved except that the Weierstrass condition is satisfied, a.e., only with respect to $e$’s satisfying

$$e \in F(t,0) \cap \left( (1 - 2\eta) \left( R(t) \cap N\mathbb{B} \right) \right).$$

Now take $\eta_i \downarrow 0$ and $N_i \to \infty$. (B1), (B2), (B4), (C1), (C2) and (3.4.10) are satisfied with $(p, \mu, \tilde{\lambda}, m)$ replaced by some $(p_i, \mu_i, \tilde{\lambda}_i, m_i)$. It is important to note that the Euler Lagrange inclusion (B1), whose right side is evaluated at $\bar{x} \equiv 0$, ensures that the integral bound on the $\dot{p}_i(.)'$s is independent of $i$. So we may extract subsequences that are in relevant respects convergent and yield in the limit multipliers $(p, \mu, \tilde{\lambda}, m)$ with the required properties. The Weierstrass condition is now strengthened to allow

$$e \in F(t, \bar{x}(t)) \cap R(t).$$

The fact that, for each $t \in [0,1]$, $R(t)$ is an open set is required to satisfy this analysis.

**Proof of Cor. 3.2.2**

*Proof*. We assume that $\bar{x}(\cdot) \equiv 0$. As earlier remarked, there is no loss of generality in so doing. Clarke [20, Prop. 2] shows that if a multifunction $F(\cdot, \cdot)$ satisfies the bounded slope condition (H4)* and condition (H5)* then, for any $\eta \in (0,1)$, it satisfies also the pseudo Lipschitz condition (H4) w.r.t. the radius multi-function $(1 - \eta)R(.)$ and with the same Lipschitz constant $k(\cdot)$, following, possibly, a reduction in the size $\epsilon$. Also, (H5) in this setting is a consequence of (H4) and (H5)* if, as we can always arrange, $k(t) \geq 1$ a.e and $\epsilon \leq \omega$. Application of Thm. 3.2.1 now yields the assertions of Cor. 3.2.2 except that the Weierstrass condition applies only w.r.t. the restricted radius multi-function $(1 - \eta)R(.)$. The full set of necessary conditions is provided by taking any sequence $\eta_i \downarrow 0$, obtaining the
3.4. **PROOF OF THEOREMS AND COROLLARIES**

necessary conditions for each \( i \) with reference to \((1 - \eta_i)R(.)\), and passage to the limit. \( \square \)

**Proof of Cor. 3.2.3**

**Proof.** We may assume that \( \epsilon \) has been chosen such that \( \epsilon < 1 \) and \( k(t) \geq k_0 \) for some \( k_0 > 0 \). Setting \( N = 0 \) in (H4)** yields

\[
\dot{x}(t) \in F(t, x) + 2\epsilon k(t)B \quad \text{for all} \quad x \in x(t) + \epsilon B \text{ a.e.}.
\]

Take any \( m \geq 4 \). Setting \( N = mk(t) \) also yields

\[
F(t, x) \cap (\dot{x}(t) + mk(t)B) \subset F(t, x') + (k(t) + m^\alpha \beta(t)k^\alpha(t))B,
\]

for all \( x, x' \in x(t) + \epsilon B \) a.e..

We see that (H4) and (H5) are satisfied for \( \gamma = \frac{1}{2} \), \( r(t) = 2k(t) \), the radius function \( R(t) = \dot{x}(t) + mk(t)B \) and Lipschitz constant \((k(t) + m^\alpha \beta(t)k^\alpha(t)) \in L^1 \). Apply Thm. **3.2.1** to obtain the assertions of the corollary, but in which the Weierstrass condition is valid only for \( \epsilon \in F(t, x(t)) \cap R(t) \). Now take \( m_i \uparrow \infty \). Replacing \( m \) by \( m_i \) for each \( i \), we obtain the full assertions of the corollary from these conditions, in the limit as \( i \to \infty \), as in our earlier analysis. \( \square \)
Chapter 4

On Optimal Control Problems with Mixed Constraints

Hybrid first-order necessary conditions are derived for nonsmooth, constrained optimal control problems. They imply, as a special case, both a Pontryagin type maximum principle and a generalization of the Euler-Lagrange equation of the calculus of variation. A notable feature is the presence of constraints, on the state and on the state and control jointly, within the study of stratified necessary conditions introduced in Chapter 3 for optimal control problems with pure state constraints.

4.1 Literature Review

This chapter concerns the derivation of necessary conditions of optimality for optimal control problems. Given functions $\ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\Lambda : [S,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, we want to minimize the cost $J(\cdot,\cdot)$ defined as follows

$$J(x(\cdot), u(\cdot)) := \ell(x(S), x(T)) + \int_S^T \Lambda(t, x(t), u(t)) \, dt . \tag{4.1.1}$$

The pairs $(x(\cdot), u(\cdot))$ over which the optimization task is to be performed are solutions of the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad a.e. \ t \in [S,T] . \tag{4.1.2}$$
Here $[S,T]$ is an interval of $\mathbb{R}$ and $f(.)$ is a function mapping $[S,T] \times \mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$. A solution of (4.1.2) is referred to as a process $(x(.),u(.))$, where for a given Lebesgue measurable control variable $u(.)$, the $\mathbb{R}^n$-valued arc $x(.)$ is a Carathéodory solution of the differential equation. $W^{1,1}([0,T];\mathbb{R}^n)$ (sometimes written $W^{1,1}$) denotes the set of absolutely continuous $\mathbb{R}^n$ valued functions on $[0,T]$ equipped with the norm

$$
\|x\|_{W^{1,1}} := |x(S)| + \int_S^T |\dot{x}(t)| \, dt.
$$

Elements $x(.) \in W^{1,1}([S,T];\mathbb{R}^n)$ are called arcs.

Assume for the time being that the control $u(.)$ is subject to constraints $u(t) \in U$, for some control set $U$ and that $\Lambda \equiv 0$. In particular our hypotheses will always allow for such restriction by a state augmentation technique. Earlier ‘nonsmooth’ necessary conditions for a process $(\bar{x}(.),\bar{u}(.))$ to be a minimizer were provided by Clarke, [13, 15]. Such conditions assert the existence of an arc $p(.)$ satisfying an adjoint inclusion

$$
-\dot{p}(t) \in \partial C \langle p(t), f(t,.,\bar{u}(t)) \rangle (\bar{x}(t)), \text{ a.e. } t \in [S,T] \tag{4.1.3}
$$

together with a transversality condition and the following Weierstrass condition

$$
\max_{u \in U} \langle p(t), f(t,x(t),u) \rangle = \langle p(t), f(t,\bar{x}(t),\bar{u}(t)) \rangle, \text{ a.e. } t \in [S,T]. \tag{4.1.4}
$$

Subsequent works by de Pinho and Vinter [29] replaced the adjoint inclusion (4.1.3) with one which incorporates the stationarity condition of the maximum in (4.1.4), that is, for a.e. $t \in [S,T]$

$$
(-\dot{p}(t),0) \in \partial C \langle p(t), f(t,.,\bar{u}(t)) \rangle (\bar{x}(t),\bar{u}(t)) - \{0\} \times N_C^U(\bar{u}(t)). \tag{4.1.5}
$$

The new adjoint inclusion essentially coincide with the Euler-Lagrange equation for the generalized problem of Bolza [14]. Observe that, in order to derive the adjoint inclusion (4.1.3), no regularity assumptions are required for the dynamic $f(.)$ with respect to the control $u$. In contrast (4.1.5) requires $f(.)$ to be integrably Lipschitz in $x$ and $u$ and the
set $U$ to be closed. Notice, also, that (4.1.3) and (4.1.5) are two different conditions when

$$\partial C(p, f(\cdot))(x, u) \neq \partial C(p, f(\cdot, u))(x) \times \partial C(p, f(\cdot, u))(u),$$

and each of them can supply different information about minimizers (see examples in [22] and [29]). Applicability of the above necessary conditions have been extended to more general optimization problems in which the processes involved in the minimization are subject to constraints, such as, endpoint constraints $(x(S), x(T)) \in E$ for some set $E \subseteq \mathbb{R}^n \times \mathbb{R}^n$, inequality state constraints

$$h(t, x(t)) \leq 0,$$  \hspace{1cm} (4.1.6)

where $h$ is a function $h : [S, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ and mixed state-control constraints

$$(x(t), u(t)) \in S(t),$$  \hspace{1cm} (4.1.7)

where $S(t) \subseteq \mathbb{R}^n \times \mathbb{R}^m$ for any $t \in [S, T]$. Recently Clarke and de Pinho [23] provided necessary conditions for an optimal control problem under constraints (4.1.7), when the set-valued map $S(\cdot)$ satisfies a bounded slope condition and an accompanying compatibility condition. They show that such assumptions are very general and subsume many of the assumptions typically invoked for the derivation of necessary conditions with mixed constraints.

In this chapter we derive necessary conditions for optimal control problems with both state and mixed (state and control) constraints, when the set $S(\cdot)$ satisfies a bounded slope condition. Such extension was already shown in [6] in a non stratified framework. Our analysis is significantly easier and direct. The state constraints function $h(\cdot)$ is only assumed to be upper semicontinuous with respect to the time variable $t$ in place of continuous. This is useful when we consider, for example, the constraint $h(\cdot)$ to be active only
CHAPTER 4. MIXED CONSTRAINTS

on a subinterval \( I \subset [S, T] \). \( h(.) \) can then be redefined to suit our formulation as

\[
\begin{cases}
  h(t, x) & t \in I \\
  m & \text{otherwise}
\end{cases}
\]

where \( m \) is a lower bound for \( h(.) \). Our approach is based on [5]. We show that the mixed
constraints can be absorbed into the dynamic as

\[ F(t, x) := \{(f(t, x, u), u) : (x, u) \in S(t)\} . \]

The problem is recast as a differential inclusion optimal control problem with (only) state
constraints where the results in Chapter 3 can be applied.

Along the lines of the studies on ‘stratified’ necessary conditions we consider ‘\( R(.)\)-weak’
minimizers, i.e. minimizers with respect to all feasible processes whose control actions are
constrained in a time-varying convex set \( R(t) \). (Notice that in chapter 3 we referred to this
kind of minimizers as local \( W^{1,1} \) minimizers with respect to the radius multifunction \( R(.) \).
This choice is inspired by the fact that the adjoint inclusion(s) exploit only properties of
the dynamic function \( f(.) \) around the optimal control \( \bar{u}(.) \) of interest. All the hypotheses
on the data are also restricted to such convex set \( R(t) \), to vary \( t \in [S, T] \).

In the case of differential inclusions the bounded slope condition translates into a type
of Lipschitz condition, in the Hausdorff topology, around the optimal trajectory, now
\( \dot{x}(t) = (f(t, \bar{x}(t), \bar{u}(t)), \bar{u}(t)) \). In [5] and [18] it is shown that such Lipschitz condition (or
the corresponding bounded slope condition) is not sufficient, alone, to establish necessary
conditions for ‘\( R(.)\)-weak’ minimizers (see also Section 3.3 of the previous chapter). A
compatibility condition, named the tempered growth condition, has to be assumed: there
exist numbers \( r_0 > 0 \) and \( \gamma \in (0, 1) \), and a function \( r(.) \in L^1 \) such that \( r(t) \geq r_0 \) a.e. and

i) \( \dot{x}(t) + \gamma^{-1} r(t) \mathbb{B} \subset R(t) \)

ii) \( F(t, x) \cap (\dot{x}(t) + r(t) \mathbb{B}) \neq \emptyset \).

In [18] and [5] examples are provided showing the necessity of the nonempty intersection
in (ii). We show that the requirement \( r(t) \geq r_0 > 0 \) is also needed. An other example is
provided showing that the use of a convex set-valued map $R(\cdot)$ in place of a time-varying ball, around the optimal process of interest, can give more information into the scope of finding optimal processes. A parallel between the bounded slope condition and the Lipschitz continuity of the set $S(\cdot)$ is discussed as well.

Finally we decouple the control action $u(\cdot) = (v(\cdot), w(\cdot))$. This allow, in particular, to recover as a special case both the adjoint inclusions (4.1.3) and (4.1.5).

Concerning the organization of the chapter: in the next section we state the main results whose proofs are postponed at the end of the chapter. Two examples are provided in section 4.3 which illustrate the advantages and the necessity of the hypotheses invoked. In particular by allowing a convex set-valued map $R(\cdot)$ we may enlarge the set of competing processes gaining stronger conclusions. Section 4.4 is dedicated to special cases in which the mixed constraint is described by equality and inequality functionals.

### 4.2 Necessary Conditions: Main Results

We consider the following optimal control problem

\[
\begin{align*}
\text{Minimize } & J(x(\cdot), u(\cdot)) = \ell(x(S), x(T)) + \int_S^T \Lambda(t, x(t), u(t)) \, dt, \\
\text{over processes } & (x(\cdot), u(\cdot)) \text{ such that} \\
(P) & \\
\hat{x}(t) = f(t, x(t), u(t)), & \text{a.e. } t \in [S, T], \nonumber \\
u(t) = (v(t), w(t)), & \text{a.e. } t \in [S, T], \\
(x(t), v(t)) \in S(t, w(t)), & \text{a.e. } t \in [S, T], \\
h(t, x(t)) \leq 0, & t \in [S, T], \\
(x(S), x(T)) \in E. & 
\end{align*}
\]

A process $(x(\cdot), u(\cdot))$ satisfying all the constraints in $(P)$ is called feasible.

**Definition 4.2.1.** The feasible process $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is an $R(\cdot)$-weak minimizer if it mini-
mizes the cost
\[ J(x(\cdot), u(\cdot)) \geq J(\bar{x}(\cdot), \bar{u}(\cdot)) \]
over all the feasible processes \((x(\cdot), u(\cdot))\) satisfying
\[ \|x - \bar{x}\|_{W^{1,1}} \leq \epsilon \quad \text{and} \quad u(t) \in R(t) \quad \text{a.e..} \]
The Lebesgue measurable set valued map \(R(\cdot)\) takes open and convex values and \(\bar{u}(t) \in R(t)\) a.e.. This is a generalization to the case \(|u - \bar{u}(t)| < r\) setting \(R(t) = \bar{u}(t) + r_0B\).

All the hypotheses on the data are restricted around the local minimizer \((\bar{x}(\cdot), \bar{u}(\cdot))\).

Given a set \(S \subseteq [S, T] \times \mathbb{R}^n \times \mathbb{R}^m\) we define \(S(t) := \{(x, u) : (t, x, u) \in S\}\). In the same way we can define the sets \(S(t, w), S(t, x), S(t, x, w)\) and \(R(t, w)\). For some \(\epsilon > 0:\)

(H1) \(\ell(\cdot, \cdot)\) is Lipschitz continuous on a neighborhood of \((\bar{x}(S), \bar{x}(T))\). \(W(\cdot)\) has Borel measurable graph and \(E\) is closed.

(H2) \(h(\cdot, \cdot)\) is upper semicontinuous near \((t, \bar{x}(t))\), for all \(t \in [S, T]\), and there exists a constant \(k_h\) such that
\[ |h(t, x') - h(t, x)| \leq k_h|x' - x|, \]
for all \(t \in [S, T]\) and all \(x', x \in \bar{x}(t) + \epsilon B\).

(H3) \(f(\cdot)\) and \(\Lambda(\cdot)\) are Borel measurable functions.

For a.e. \(t \in [S, T]\) and every \(w \in W(t)\) the functions \((x, v) \mapsto f(t, x, v, w)\) and \((x, v) \mapsto \Lambda(t, x, v, w)\) are Lipschitz continuous on a neighborhood of
\[ ((\bar{x}(t) + \epsilon B) \times R(t, w)) \cap S(t, w), \]
The Lipschitz constants, possibly depending on \(t\) and \(w\), are \(L \times B\) measurable functions (\(L\) and \(B\) denote, respectively, the \(\sigma\)-algebras of Lebesgue subsets of \(\mathbb{R}\) and of Borel subsets of \(\mathbb{R}^d\)). We use the notation \(k_f^L(t, w), k_f^B(t, w), k_\Lambda^L(t, w)\) and \(k_\Lambda^B(t, w)\) respectively for \(f\) and \(\Lambda\) with respect to each component. Therefore for any \((x_1, v_1)\) and \((x_2, v_2)\) on a neighbourhood of \(((\bar{x}(t) + \epsilon B) \times R(t, w)) \cap S(t, w),\)
\[ |f(t, x_1, v_1, w) - f(t, x_2, v_2, w)| \leq k_f^L(t, w)|x_1 - x_2| + k_f^B(t, w)|v_1 - v_2| \]
4.2. NECESSARY CONDITIONS: MAIN RESULTS

\[ |\Lambda(t,x_1,v_1,w) - \Lambda(t,x_2,v_2,w)| \leq k^\Lambda_x(t,w)|x_1 - x_2| + k^\Lambda_v(t,w)|v_1 - v_2|. \]

(H4) There exists a measurable function \( k_S(.) \) such that \( k_S(t) \geq k_0 > 0 \) and for every \( w \in W(t) \) and \( (x,v) \in ((\bar{x}(t) + \epsilon B) \times R(t,w)) \cap S(t,w) \) the following holds almost everywhere
\[
(\alpha, \beta) \in N^P_{S(t,w)}(x,v) \implies |\alpha| \leq k_S(t)|\beta|.
\]

(H5) There exists \( \eta > 0 \) such that \( (\bar{u}(t) + \eta k_S(t)B) \subseteq R(t) \)

(H6) The following functions are summable
\[
k^f_x(t,\bar{w}(t)), \ k^\Lambda_x(t,\bar{w}(t)), \ k_S(t)[k^f_u(t,\bar{w}(t)) + k^\Lambda_u(t,\bar{w}(t))].
\]

Hypothesis (H4), named Bounded Slope Condition, was first introduced by Clarke for differential inclusions in [18]. We assume \( S(t,w) \) to be locally closed at relevant points, so that (H4) is well defined. Observe that the Lipschitz condition (H3) for the functions \( f(t,x,.) \) and \( \Lambda(t,x,.) \), is only assumed with respect to the \( v \)-part of the control. In the unmixed case \( u \equiv w \in W(t) \) and no Lipschitz regularity is required with respect to the control. When \( u \equiv v \) and \( S(t) = \{(x,u) : u \in U\} \) we recover the adjoint inclusion, condition (4.1.5). We are now ready to state our main theorem. It provides an extension to [5], [23] and [6]. Define the hamiltonian of the system to be
\[
H_\lambda(t,x,v,w,p) := (p,f(t,x,v,w)) - \lambda \Lambda(t,x,v,w).
\]

**Theorem 4.2.2.** Let \((\bar{x}(.),\bar{u}(.))\) be \( R(.) \)-weak minimizer as in definition [4.2.1] and \( \bar{u}(.) = (\bar{v}(.),\bar{w}(.)) \). Assume (H1)-(H6). Then, there exist a multiplier \((p(.),\mu(.),\lambda) \in W^{1,1} \times NBV^+[S,T] \times \mathbb{R}_{\geq 0}\) and a \( \mu \)-integrable function \( m(.) \), such that

\[(N) \lambda + \|p\|_{L^\infty} + \|\mu\|_{T.V.} = 1.\]

\[(A) (\dot{p}(t),0) \in \partial_C H_\lambda(t,\ldots,\bar{w}(t),q(t))(\bar{x}(t),\bar{v}(t)) - N^C_{S(t,w(t))}(\bar{x}(t),\bar{v}(t)) \ a.e..\]

\[(T) (p(a),-q(b)) \in \lambda \partial \ell(\bar{x}(S),\bar{x}(T)) + N_E(\bar{x}(S),\bar{x}(T)).\]
(W) For almost every $t \in [S,T]$ and any $u \in R(t) \cap S(t, \bar{x}(t))$, $u = (v, w)$, $w \in W(t)$, we have that

$$H_\lambda(t, \bar{x}(t), \bar{u}(t), q(t)) \geq H_\lambda(t, \bar{x}(t), u, q(t)).$$

(C) $q(t) := p(t) + \int_{[a,t]} m(s) \mu(ds)$ when $t \in (a,b)$ and $q(b) := p(t) + \int_{[S,T]} m(s) \mu(ds)$. $m(t) \in \partial^> x h(t, \bar{x}(t)) \mu$-a.e. and $\text{supp} \{\mu\} \subset \{t : h(t, \bar{x}(t)) = 0\}$.

Above, the generalized gradient $\partial^> x h(t, \bar{x}) := \{\lim (t_i, x_i) \to (t, x) \nabla h(t_i, x_i) : \text{the gradient } \nabla h(t_i, x_i) \text{ exists and } h(t_i, x_i) > 0\}$. It is well known that the set of lower semicontinuous functions with uniformly bounded subdifferentials coincides with the set of Lipschitz continuous functions. In the same fashion the Bonded slope condition (H4) is strictly related to Lipschitz type regularities for the set $S$ (this is shown in [18]).

**Theorem 4.2.3.** Suppose that $k_S(t)k^+_x(t, \bar{w}(t)) \in L^1(a,b)$. Then theorem 4.2.2 is still valid when the bounded slope condition (H4) is replaced by a pseudo Lipschitz condition (H4$'$) There exists a measurable function $k_S(\cdot)$ such that for every $w \in W(t)$ and $x, x' \in \bar{x}(t) + \epsilon B$

$$S(t, w, x) \cap R(t, w) \subseteq S(t, w, x') + k_S(t) |x - x'| B, \text{ a.e.}$$

and (H5) is replaced by the following tempered growth condition (H5$'$): there exist numbers $r_0 > 0$ and $\gamma \in (0, 1)$, and a function $r(\cdot) \in L^1(a,b)$ such that $r(t) \geq r_0$ a.e. and

i) $\bar{u}(t) + \gamma^{-1} r(t) B \subseteq R(t)$

ii) $S(t, \bar{w}, x) \cap (\bar{v}(t) + r(t) B) \neq \emptyset$.

### 4.3 Examples

In engineering applications we often have situations where certain variables $u_i$, $i = 1, \ldots, N$, control respectively pressure, temperature, volume and other physical quantities. In addition, in tracking problems, every variable is allowed to deviate of a specified amount from a reference control action. For such applications it may be essential to allow convex restrictions of the type $u = (u_1, \ldots, u_N) \in \prod_{i=1}^N [u^-_i, u^+_i]$ (say $\bar{u} = 0$ is the reference under consideration). In the next example we show that such convex restrictions can give more information about a certain minimizer than the classical approach $|u - \bar{u}| < \epsilon$. Indeed
4.3. EXAMPLES

Figure 4.1: Illustration of Example 4.3.1. The function $g(.)$ on the left and the set $S(t, w) \cap [-1, +\infty)$ on the right.

necessary conditions can be used as sufficient conditions for an arc not to be a minimizer. Imagine that a certain optimal control problem is formulated as in (P). Theorem 4.2.2 is first applied to have a general idea of possible optimal candidates $(\bar{x}, \bar{u})$ with respect to control constraints $|u - \bar{u}| < \epsilon$ for some $\epsilon > 0$. The problem is then reformulated to allow for larger controls. Is $(\bar{x}, \bar{u})$ still a minimizer for the reformulated problem? In other words if the constraint $|u - \bar{u}| < \epsilon$ is now relaxed, can we still apply Theorem 4.2.2 to check whether the candidate $(\bar{x}, \bar{u})$ satisfies the necessary conditions? We illustrate this point in the next example.

**Example 4.3.1.** Minimize the following functional cost

$$-\frac{1}{3} x(1) + \int_0^1 L(x(t), u(t)) \, dt$$

over pairs $(x(.), u(.))$ such that $u = (v, w) \in \mathbb{R}^2$, $\dot{x}(t) = v$, $x(0) = 0$, $x(1) \in (-\infty, 0]$ and the following inequality mixed constraint is satisfied

$$v + |x| \leq w.$$ 

The Lagrangian $L(.)$ is defined as

$$L(x, u) := g(v) - |x| + w,$$

where $g : \mathbb{R} \to \mathbb{R}$ is the function represented in figure 4.1 and defined as, $g(v) := \min\{1, |v|, 2 - v\}$. We claim that $(\bar{x} \equiv 0, \bar{u} \equiv 0)$ is a $R(.)$-weak minimizer for the convex
map $R(t) := (-1, +\infty) \times \mathbb{R}$, for all $t \in [0, 1]$. Indeed, when $v > -1$, 

$$\frac{1}{3} x(1) + \int_0^1 L(x(t), u(t)) \, dt \geq \int_0^1 g(v) + v \geq 0.$$ 

Notice that a constraint on $v$ of the type $v \in (-1, +\infty)$ is necessary for otherwise: take $v = -\alpha$ for all $t \in [0, 1]$ and $w = -\alpha + \alpha t = v(t) + |x(t)|$. The feasible process $(x(\cdot), u(\cdot))$ has a strictly negative cost, for $\alpha > 3/2$, indeed

$$\frac{1}{3} x(1) + \int_0^1 L(x(t), u(t)) \, dt = \frac{1}{3} \alpha + \int_0^1 (1 - \alpha) \, dt = 1 - \frac{2}{3} \alpha.$$

Notice also (as illustrated in figure 4.1) that the set $S(t, w) = \{(x, v) \in \mathbb{R}^2 : v + |x| \leq w\}$ satisfies the Bounded slope condition, while $S(t, w) \cap [-1, +\infty)$ violates it at the point $(1 + w, -1)$ for which $(1, 0) \in N^P_{S(t)\cap[-1, +\infty)}(1 + w, -1)$.

As soon as we relax constraints on the admissible value for $v$, $(\bar{x} = 0, \bar{u} = 0)$ stop being an $R(\cdot)$-weak minimizer but the necessary conditions for ball-shaped $R(\cdot)$ fail to detect this behaviour. We apply necessary conditions, as described in Theorem 4.2.2, in both the cases $R(t) = (-2, +\infty) \times \mathbb{R}$ and $R(t) = 2 \mathbb{B}$, for all $t \in [0, 1]$. We know that $(\bar{x} = 0, \bar{u} = 0)$ is not a $R(\cdot)$-weak minimizer for such choices of $R(\cdot)$. Yet, in the case $R(t) = 2 \mathbb{B}$, the necessary conditions fail to eliminate it as a possible minimizer. Necessary conditions for this system are valid in normal form $\lambda = 1$. The Weierstrass condition $(W)$ guarantees that for almost every $t \in [S, T]$, $u \in R(t) \cap S(t, 0)$

$$p(t) \cdot v \leq L(0, u) = g(v) + w.$$

1. In the case $R(t) = 2 \mathbb{B}$ we have that $v \in (-2, 2)$ and the condition $u \in S(t, 0)$ implies $v \leq w$. This yields the estimation for $p$, $p(t) \in [(1/2), 1]$, for a.e. $t \in [0, 1]$. Every value for $p \in [(1/2), 1]$ is feasible for conditions $(T)$ and $(A)$ too.

2. When we allow a convex set-valued map $R(t) = (-2, +\infty) \times \mathbb{R}$ the Weierstrass condition supplies extra information. Indeed for $v > 1$ the Lagrangian $L(0, u) = 2 - v + w$ and $p(t) \leq \frac{2}{v} \to 0$ as $v \to +\infty$.

Our second example shows that condition $(H5)$ is necessary.
Example 4.3.2. Consider the following optimal control problem

\[
(P) \begin{cases}
\text{Minimize } -x(1) \\
\dot{x}(t) = u, \ a.e. \ t \in [S, T], \\
u(t) \geq |x(t)|, \ a.e. \ t \in [S, T], \\
x(0) = e^{-1}, \\
u \in R(t) := (-1, 1) \ a.e. \ t \in [S, T].
\end{cases}
\]

The admissible control \(u(t) = |x(t)|\) yields the trajectory \(x(t) = e^{t-1}\). This is the only feasible process. Indeed \(x(1) = 1\) and any other pairs \((x, u)\) would violate the condition \(u(t) \in (-1, 1)\) a.e.. Observe that the bounded slope condition is satisfied for this problem with \(k_S(t) \equiv 1\). The Weierstrass condition \((W)\) in Theorem 4.2.2 provide the estimation

\[p(t) \cdot (e^{t-1} - u) \geq 0, \ \forall u \in (e^{t-1}, 1) \ a.e..\]

This inequality implies that \(p(t) \leq 0\). On the other hand the transversality condition provides \(p(1) = \lambda \geq 0\). The contradiction is due to the fact that \((H5)\) is not satisfied for this problem. Indeed we cannot find a constant \(\eta > 0\) satisfying

\[e^{t-1} + \eta < 1\]

for a.e. \(t \in [0, 1]\). The tempered growth condition \((ii)\) in Theorem 4.2.3 is satisfied for this example but Hypothesis \((i)\) is not.

4.4 Special Cases

Suppose that the set \(S(.)\) has the special structure

\[S(t) := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : \phi(t, x, u) \in \Phi(t)\}.\]

(4.4.1)

Here \(\phi : [S, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\kappa\) and \(\Phi : [S, T] \to \mathbb{R}^\kappa\) is Lebesgue measurable and takes values closed sets. Notice that this setting covers, in particular, the case of equality and inequality constraints defining \(\Phi(t)\) respectively as \(\{0\}\) and \((-\infty, 0]\). We consider then the
CHAPTER 4. MIXED CONSTRAINTS

non-decouple problem

\[
(P_S) \begin{cases}
\text{Minimize } J(x(.), u(.)) \\
\text{over processes } (x(.), u(.)) \text{ s.t.} \\
\dot{x}(t) = f(t, x(t), u(t)), \text{ a.e.} \\
\phi(t, x(t), u(t)) \in \Phi(t), \text{ a.e.} \\
h(t, x(t)) \leq 0, \\
(x(S), x(T)) \in E.
\end{cases}
\]

Now the Lipschitz hypothesis for \( f(\cdot) \) and \( \Lambda(\cdot) \) is supposed on both the \( x \) and \( u \) variable.

\( (H3') \) For a.e. \( t \in [S, T], f(\cdot, \cdot), \Lambda(\cdot, \cdot) \) and \( \phi(\cdot, \cdot) \) are integrably Lipschitz continuous on a neighbourhood of \( (\bar{x}(t) + \epsilon B) \times R(t) \cap S(t). \)

We want to obtain explicit necessary conditions involving directly the function \( \phi(\cdot, \cdot) \) and the set valued map \( \Phi(\cdot) \) in place of the implicit \( N_{S(t)}^C \). To such end we recall a result of [23], precisely Proposition 4.1, according to which if \( (\alpha, \beta) \in N_{S(t)}(x, u) \) and the following constraint qualification holds true\footnote{Apply the chain rule (part E of Section 1.4) to the characteristic function \( I_{S} = I_{S} \circ \phi \)}

\[
\lambda \in N_{\Phi(t)}(\phi(t, x, u)), \quad 0 \in \partial(\lambda, \phi(t, \cdot, \cdot))(x, u) \implies \lambda = 0.
\]

then \( (\alpha, \beta) \in \partial(\lambda, \phi(t, \cdot, \cdot))(x, u) \) for some \( \lambda \in N_{\Phi(t)}(\phi(t, x, u)) \).

\( (CQ) \) There exists \( M : [S, T] \to \mathbb{R} \) measurable such that for a.e. \( t \in [S, T], (x, u) \in S(t), \)

\( x \in \bar{x}(t) + \epsilon B \) and \( u \in R(t) \) we have

\[
\lambda(t) \in N_{\Phi(t)}^L(\phi(t, x, u)), (\alpha, \beta) \in \partial(\lambda, \phi(t, \cdot, \cdot))(x, u) \implies |\lambda| \leq M(t)|\beta|.
\]

The latter has the advantage of implying the Bounded Slope Condition hypothesis (H4) with \( k_S(t) = M(t)k_{\phi}^S(t) \), which we will assume to be integrable, (see [23] for details).

Corollary 4.4.1. Let \( (\bar{x}(\cdot), \bar{u}(\cdot)) \) be a \( R(\cdot) \)-weak minimizer for \( (P_S) \). Assume \( (H1), (H2), \)

\( (H3'), (CQ) \) and hypotheses \( (H5) \) and \( (H6) \) where \( k_S(t) = M(t)k_{\phi}^S(t) \). Then there exists a multiplier \( (p(\cdot), \mu, m(\cdot), \lambda_0) \) which satisfies all the conclusions of Theorem 4.2.2. If in addition \( \phi \) is strictly differentiable at \( (\bar{x}(t), \bar{u}(t)) \) a.e., then there exists a measurable function
4.5. PROOFS OF THEOREMS

Let \( \lambda : [S, T] \to \mathbb{R}^n \) satisfying

\[
\lambda(t) \in N_{\phi(t)}^{C}(\phi(t, \bar{x}(t), \bar{u}(t))) \quad \text{a.e.}
\]

such that the adjoint inclusion takes the explicit multiplier form

\[
(-\dot{p}(t), 0) \in \partial C \{ (q(t), f(t, \ldots)) - \lambda_0 \Lambda(t, \ldots) - \langle \lambda(t), \phi(t, \ldots) \rangle \}(\bar{x}(t), \bar{u}(t)), \ a.e..
\]

Furthermore \( \lambda \) satisfies the inequality

\[
|\lambda(t)| \leq M(t) \{ |q(t)|k_f^L(t) + \lambda_0 k^A_u(t) \}, \ a.e.
\]

4.5 Proofs of Theorems

**Theorem 4.2.2.** We reformulate (P) as a differential inclusion optimal control problem with state constraints by including the mixed constraint in the dynamic. We use then Corollary 3.2.2 of Chapter 3 and techniques developed in [23]. Observe at the outset that, without loss of generality, the integral cost in (P) can be neglected by setting \( \Lambda(\cdot) \equiv 0 \). This reduction is standard when the Lagrangian \( \Lambda(\cdot) \) and the dynamic \( f(\cdot) \) share the same hypotheses. We also assume that

1. For every \( t \in [S, T] \), \( W(t) \) is a finite set.

2. There exist \( C > 0 \) such that, for almost all \( t \in [S, T] \) and \( w \in W(t) \):

\[
(x, v) \in (\bar{x}(t) + \epsilon B) \times R(t, w) \cap S(t, w) \implies |k_f^L(t, w) - k_f^L(t, \bar{w})| + k_S(t)|k_f^L(t, w) - k_f^L(t, \bar{w})| + |f(t, x, u) - \dot{x}(t)| \leq C.
\]

Justifications in adopting such reductions can be found in [55, Lemma 6.3.1]. For each \( t \in [S, T] \) and \( x \in \bar{x}(t) + \epsilon B \), define

\[
F(t, x) := \{(f(t, x, u), \theta(t, u)) : u = (v, w), (x, v) \in S(t, w), w \in W(t)\}. \tag{4.5.1}
\]

Here \( \theta(t, u) := c(t)(u - \bar{u}(t)) \) for some integrable function \( c(t) \) that will be set later on. The new optimization problem where the dynamic is modelled as a differential inclusion
is the following

\[
\begin{aligned}
&\text{(P)} \quad \begin{cases}
\text{Minimize} & \ell(x(S), x(T)) \\
\text{over arcs} & (x(\cdot), z(\cdot)) \in W^{1,1} \text{ satisfying}
\end{cases} \\
&\begin{cases}
(\dot{x}(t), \dot{z}(t)) \in F(t, x(t)), & \text{a.e. } t \in [S, T], \\
h(t, x(t)) \leq 0, & \text{for all } t \in [S, T], \\
(x(S), x(T)) \in E, & z(a) = 0.
\end{cases}
\end{aligned}
\]

The \( F \)-trajectory \((\bar{x}(t), \bar{z}(t) \equiv 0)\) is a \( W^{1,1} \) local minimizer with respect to the set-valued map \( \bar{R}(\cdot) \) defined as

\[
\bar{R}(t) := \mathbb{R}^n \times c(t) (R(t) - \bar{u}(t)).
\]

All the hypotheses under which Corollary 3.2.2 of Chapter 3 is valid are satisfied provided we can show appropriate regularity properties for \( F(\cdot) \). The following Lemma states exactly the requirements needed.

**Lemma 1.** Fix \( w \in W(t) \) and \( (x, v) \in \bar{x}(t) + \varepsilon \mathbb{B} \times R(t, w) \cap S(t, w) \) and take \((\alpha, \beta, \tau) \in N_{GrF(t, \cdot)}^L(x, f(t, x, u), \theta(t, u))\). An integrable function \( k(\cdot) \) can be found such that for almost every \( t \in [S, T] \),

\[
|\alpha| \leq k(t)(|\beta| + |\tau|).
\]

Furthermore

\[
(\alpha, 0) \in \partial_L \{-\langle \beta, f(t, \cdot, w) \rangle - \langle \tau, \theta(t, \cdot, w) \rangle + 2k(t)(|\beta| + |\tau|)d_S(t, w)(\cdot, \cdot)\}(x, v).
\]

The function \( k(\cdot) \) can be taken as follows

\[
k(t) := \max_{w \in W(t)} k_x^L(t, w) + k_S(t) \max_{w \in W(t)} k_u^L(t, w).
\]

Observe that thanks to our reductions \( k(\cdot) \) is an integrable function and the max operation is well defined. Lemma 1 is essentially proved in [23, Proposition 9.1] where \( c(\cdot) \) is defined by means of the function \( k(\cdot) \).

According to Corollary 2.2 of [5] there exist an absolutely continuous function \( p(\cdot) \in W^{1,1} \), a constant \( r \), a nonnegative number \( \lambda \), a monotone non-decreasing function \( \mu(\cdot) \in \text{NBV}^+[S, T] \) and a \( \mu \)-integrable function \( m(\cdot) \), such that
4.5. PROOFS OF THEOREMS

(i) \( \lambda_0 + |r| + \|p\|_{L^\infty} + \|\mu\|_{T.V.} = 0 \),

(ii) \( \dot{p}(t) \in \text{co}\left\{ \eta : (\eta, p(t) + \int_{[a,b]} m(s)\mu(ds), r) \in N^L_{GrF(t,\cdot)}(\bar{x}(t), \hat{x}(t), 0) \right\} \) a.e.,

(iii) \( \left( p(a), -\left[ p(b) + \int_{[S,T]} m(s)\mu(ds) \right] \right) \in \lambda \partial_L \ell(\bar{x}(S), \bar{x}(T)) + N_E(\bar{x}(S), \bar{x}(T)) \),

(iv) \( r = 0 \).

(v) For any \((e_1, e_2) \in F(t, \bar{x}(t)) \cap \tilde{R}(t)\), a.e. \( t \in [S, T] \),

\[ \langle p(t) + \int_{[a,t]} m(s)\mu(ds), \hat{x}(t) \rangle \geq \langle p(t) + \int_{[a,t]} m(s)\mu(ds), e_1 \rangle + \langle r, e_2 \rangle \]

(f) \( m(t) \in \partial^\nu_C h(t, \bar{x}(t)) \) \( \mu \)-a.e. and \( \text{supp}\{\mu\} \subset \{ t : h(t, \bar{x}(t)) = 0 \} \).

By (ii) and Lemma 1

\[ (\eta, 0) \in \partial_L \{-\langle q(t), f(t,.,.,\bar{w}(t)) \rangle + 2k(t)|q(t)|d_{S(t,\bar{w}(t))}(.,.)\}(\bar{x}(t), \bar{v}(t)). \]

Recall that \( q(t) := p(t) + \int_{[a,t]} m(s)\mu(ds) \) for \( t \in (a, b) \). By definition \( \partial_C = \text{co} \partial \). Also, by well know properties of generalized gradients, given two Lipschitz functions \( f \) and \( g \),
\( \partial_C(-f) = -\partial_C f \) and \( \partial_C(f + g) \subseteq \partial_C f + \partial_C g \). Then

\[ (-\dot{p}(t), 0) \in \partial_C \{-\langle q(t), f(t,.,.,\bar{w}(t)) \rangle \}(\bar{x}(t), \bar{v}(t)) \]
\[ -\partial_C \{2k(t)|q(t)|d_{S(t,\bar{w}(t))}(.,.)\}(\bar{x}(t), \bar{v}(t)), \] (4.5.2)

Remark. Condition (4.5.2) yields a stronger conclusion than the adjoint inclusion (A) stated in Theorem 4.2.2. It, indeed, gives an \( L^1 \) bound on the norm of \( \dot{p}(t) \), which is a fundamental property when considering perturbed problems and sequences.

Theorem 4.2.3. The proof technique is the same as in the proof of Theorem 4.2.2, but it will be the end-result of applying Theorem 3.2.1 of Chapter 3 instead of Corollary 3.2.2. It is sufficient, then, to show that hypotheses (H4) and (H5) of [5, Theorem 2.1] are satisfied for the dynamic defined in (4.5.1), i.e.

(L) There exists a function \( k_F(.) \in L^1(S, T) \) such that

\[ F(t, x') \cap \tilde{R}(t) \subset F(t, x) + k_F(t)|x - x'| \mathbb{B}, \]
for all $x, x' \in \bar{x}(t) + \epsilon \mathbb{B}$, a.e. $t \in [S, T]$.

(G) There exist numbers $\bar{r} > 0$ and $\nu \in (0, 1)$, and a function $r_F(.) \in L^1(a, b)$ such that $r_F(t) \geq \bar{r}$ a.e. and

i) $\dot{x}(t) + \nu^{-1}r_F(t) \mathbb{B} \subset \tilde{R}(t)$

ii) $F(t, x) \cap (\dot{x}(t) + r_F(t) \mathbb{B}) \neq \emptyset$

for all $x \in \bar{x}(t) + \epsilon \mathbb{B}$, a.e. $t \in [S, T]$.

As in [5] we make the additional reduction $r(t) \equiv r_0$ for some constant $r_0$. (L) and (G) are satisfied for

$$
\nu \in (\gamma, 1), \quad r_F(t) := \gamma^{-1} \nu c(t) r(t) \quad \text{and} \quad c(t) := \frac{k_f^f(t) + k_u^f(t)r_0}{r_0(\gamma^{-1} \nu - 1)}.
$$

It is a direct consequence of assumptions (H4$'$) and (H5$'$). We verify, for example, (G)(ii).

By (H5$'$)(ii), for every $x \in \bar{x}(t) + \epsilon \mathbb{B}$ (without loss of generality $\epsilon < 1$), there exists $v \in S(t, \bar{w}, x)$ such that $|v - \bar{v}| \leq r_0$. In particular $(f(t, x, v, \bar{w}), c(t)(u - \bar{u})) \in F(t, x)$ and

$$
|f(t, x, v, \bar{w}), c(t)(u - \bar{u}) - (f(t, x, \bar{u}), 0)| \leq k_f^f(t)\epsilon + k_u^f(t)r_0 + c(t)r_0 = r_0(\gamma^{-1} \nu - 1)c(t) + c(t)r_0 = r_F(t).
$$
Chapter 5

Euler-Lagrange Conditions for Delayed Systems

This chapter is the first of two chapters on necessary conditions for optimal control problems, in which time delays are present in the dynamic constraint. They differ according to the manner in which the dynamic constraint is formulated. Here, it takes the form of a retarded differential inclusion. In the next chapter the dynamics are described instead by a differential/delay equation. The necessary conditions for the retained differential inclusion are of independent significance. But they are also of interest because, as we shall see in the next chapter, their derivation can be used as a step to obtaining a very general Pontryagin-type Maximum Principle for differential delay-equation problems.

There is already a literature on necessary conditions for retarded differential inclusion problems. In this chapter we improve on earlier work in several respects. First, our formulation is, in some respects, more general. Most notably, it permits the end-time to be free. (In the earlier literature of retarded differential inclusion problems, the end-time is fixed, or has been treated incorrectly.) Second, we strengthen earlier statements of necessary conditions (even when we do not take account of the new conditions associated with the free end-time), replacing the Hamiltonian inclusion by the more precise (partially convexified) Euler Lagrange Inclusion.

Proof techniques based on nonsmooth analysis, perturbations and limit processes via Eke-
lands variational principle has been successfully used in the last 30 years to address
problems which does not incorporate delays, c.f. [55]. We extend applicability of such
techniques to optimization problems featuring constant delays on the state variable.

First necessary conditions are proved for a simple, delayed problem in the calculus of
variations in which the end-time is fixed. These necessary conditions will represent
the basic step for the derivation of more general set of necessary conditions valid for optimal
control problems in which the system dynamic is modeled by differential inclusions. The
final step consists in the introduction of additional freedom in the formulation of the
problem by considering free end-time problems.

5.1 A Delayed Lagrange problem

\[
\text{(FL)} \begin{aligned}
\text{Minimize } & \ell_0(x(S)) + \ell_1(x(T)) + \int_{S-h_k}^{S} \Lambda(t, x(t)) \, dt + \\
& \int_{S}^{T} L(t, x(t-h_0) \ldots, x(t-h_k), \dot{x}(t)) \, dt \\
\text{subject to } & x(t) \in E(t) \text{ a.e. } t \in [S-h_k, S), \\
& \dot{x}(t) \in R(t) \text{ a.e. } t \in [S, T], \\
& x(S) \in C_0.
\end{aligned}
\]

Here \( h_0 = 0 < h_1 < \ldots < h_k \) are given real numbers. Minimization is conducted over
absolutely continuous functions \( x(.) \in W^{1,1}([S, T]; \mathbb{R}^n) \) extended to an \( L^\infty \)
function on \([S-h_k, S]\). We refer to a state trajectory \( x(.) \) on \([S-h_k, T]\) as an arc. An arc \( x(.) \) is said
to be feasible if the following conditions are satisfied

\[ x(t) \in E(t) \text{ a.e. } t \in [S-h_k, S), \quad \dot{x}(t) \in R(t) \text{ a.e. } t \in [S, T] \quad \text{and} \quad x(S) \in C_0. \]

Let the feasible arc \( \bar{x}(.) \) be a local minimizer for \( (FL) \), i.e. for some \( \delta > 0 \) the arc \( \bar{x}(.) \) achieves the minimum of the cost over all feasible arcs \( x(.) \) satisfying

\[ \|x - \bar{x}\|_{L^\infty(S-h_k,S)} + \|x - \bar{x}\|_{W^{1,1}(S,T)} \leq \delta. \]

We write

\[ \bar{z}(t) := (\bar{x}(t-h_0), \ldots, \bar{x}(t-h_k)). \]
5.1. A DELAYED LAGRANGE PROBLEM

Hypotheses for the data are as follows:

(L1) The functions $\ell_0(\cdot)$ and $\ell_1(\cdot)$ mapping $\mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz continuous and the set $C_0 \subset \mathbb{R}^n$ is closed.

(L2) $E : [S - h_k, S) \to \mathbb{R}^n$ is a Lebesgue measurable multifunction which takes as values closed subsets of $\mathbb{R}^n$. Furthermore $E(t) \subset r\mathbb{B}$ for some $r \in \mathbb{R}$.

(L3) The Lagrangian $L(t, z, v)$ is assumed to be Lebesgue measurable with respect to $t$ and locally Lipschitz with respect to $(z, v)$, i.e. for any $N > 0$ there exist $\delta > 0$ and $k(\cdot) \in L^1$ such that, for a.e. $t \in [S, T]$,

$$|L(t, z, v) - L(t, z', v')| \leq k(t)(|z - z'| + |v - v'|),$$

for all $z, z' \in \bar{z}(t) + \delta\mathbb{B}$ and $v, v' \in (\dot{x}(t) + N\mathbb{B}) \cap (R(t) + \delta\mathbb{B})$.

(L4) $\Lambda(\cdot, x)$ is Lebesgue measurable for all $x \in \bar{x}(t) + \delta\mathbb{B}$ and $\Lambda(t, \cdot)$ is integrably Lipschitz with Lipschitz constant $k_{\Lambda}(\cdot) \in L^1(S - h_k, S)$.

(L5) $R : [S, T] \to \mathbb{R}^n$ is a Lebesgue measurable multifunction which takes as values closed subsets of $\mathbb{R}^n$.

**Theorem 5.1.1.** Assume hypotheses (L1)-(L5) and let the feasible arc $\bar{x}(\cdot)$ be a local minimizer for (FL). Then there exist arcs $q_i(\cdot) \in W^{1,1}(S - h_i, T; \mathbb{R}^n)$, $(i = 0, \ldots, k)$ satisfying

(a) A transversality condition at the end-time $t = T$

$$q_i(t) = 0, \quad t \in [T - h_i, T], \quad \forall \ i = 1, \ldots, k \quad \text{and} \quad -q_0(T) \in \partial\ell_1(\bar{x}(T))$$

and at time $t = S$

$$q_0(S) + \cdots + q_k(S) \in \partial\ell_0(\bar{x}(S)) + NC_0(\bar{x}(S)).$$

(b) An Euler-Lagrange type condition for a.e. $t \in [S - h_k, S]$ and a.e. $t \in [S, T]$, respectively

(b1) $-(\dot{q}_1(t)\chi_{[S - h_1, S]}(t) + \cdots + \dot{q}_k(t)\chi_{[S - h_k, S]}(t)) \in co\partial\Lambda(t, \bar{x}(t)) + coN_{E(t)}(\bar{x}(t))$
(b2) \((\dot{q}_0(t-h_0), \ldots, \dot{q}_k(t-h_k)) \in \text{co}\left\{ \eta : (\eta, q_0(t) + \ldots + q_k(t)) \in \partial L(t, \bar{x}(t-h_0), \ldots, \bar{x}(t-h_k), \dot{x}(t)) + \{0\} \times N_{\mathcal{R}(t)}(\dot{x}(t)) \right\}, t \in [S, T] \ a.e.

(c) The Weierstrass condition: the following hold for all \(u \in \mathcal{R}(t), \ a.e. \ t \in [S, T],\)

\[
(q_0(t) + \ldots + q_k(t)) \cdot u - L(t, \bar{z}(t), u) \leq (q_0(t) + \ldots + q_k(t)) \cdot \dot{x}(t) - L(t, \bar{z}(t), \dot{x}(t)).
\]

Proof. The quite lengthy proof of Theorem [5.1.1] will be divided in several steps. It uses arguments of nonsmooth analysis, measurable selections and limiting processes. Similar strategies has been employed in [56], [17] and [19] for optimal control problems without time delays.

Hypotheses reductions. Without loss of generality we can assume \(\bar{x}(.) \equiv 0\) and \([S, T] = [0, 1]\). Also, the functions \(\ell_0(\cdot), \ell_1(\cdot), \Lambda(t, \cdot)\) and \(L(t, \cdot, v)\) can be assumed to be globally Lipschitz. Hypothesis (L3) is then replaced by the stronger global form

\[(L3^*)\] For every \(N > 0\) there exists \(\delta_N > 0\) and \(k_N(\cdot) \in L^1(0, 1)\) such that

\[
|L(t, z, v) - L(t, z', v')| \leq k_N(t)(|z - z'| + |v - v'|),
\]

for a.e. \(t \in [0, 1]\) and all \(z, z' \in \mathbb{R}^{n \times (k+1)}\) and \(v, v' \in NB \cap (\mathcal{R}(t) + \delta_N B)\).

This is achieved by replacing the original function \(L(\cdot)\) by the composition

\[
\tilde{L}(t, z, v) := L(t, \cdot, v) \circ tr_{\delta}(z)
\]

of \(L(\cdot)\) with the Lipschitz function

\[
tr_{\delta}(z) := \begin{cases} 
z & z \in \delta_B \\
\frac{z}{|z|} \delta & \text{otherwise}.
\end{cases}
\]

Observe that \(\bar{x}(\cdot)\) is still a local minimizer for the transformed problem and the conclusions are not affected by such modifications away from the optimal state.
5.1. A DELAYED LAGRANGE PROBLEM

Cost Regularization. We make use of inf-convolutions and decoupling techniques to regularize the cost and apply a direct analysis of the delays. Define the function \( (i \geq 2) \)

\[
\ell_i(x) := \inf_{y \in \mathbb{R}^n} \{ \ell_1(y) + i|x - y|^2 \}
\]

Such function, known as inf-convolution, enjoys the following properties (below the sequence \((\alpha_i)_{i \in \mathbb{N}} \subset \mathbb{R}_+ \) is such that \( \alpha_i \downarrow 0 \) as \( i \to +\infty \))

1. \( \ell_i(.) \) is Lipschitz continuous on \( \mathbb{R}^n \).

2. \( \ell_1(x) - \alpha_i \leq \ell_i(x) \leq \ell_1(x) \), for every \( x \in \mathbb{R}^n \).

3. For all \( x \in \mathbb{R}^n \) exists \( z_i \in x + \alpha_i \mathbb{B} \) and \( \zeta_i \in \partial \ell_1(z_i) \) such that

\[
\ell_i(y) - \ell_i(x) \leq \zeta_i \cdot (y - x) + i|y - x|^2, \quad \forall y \in \mathbb{R}^n.
\]

We define the new regularized cost as follows

\[
J_i(\phi, x_0, w_0, \ldots, w_k, v) := \\
\ell_0(x_0) + \int_{-h_k}^0 \Lambda(t, \phi(t)) \, dt + \ell_i(x(1)) + \int_0^1 L(t, w_0(t), \ldots, w_k(t), v(t)) \, dt + \\
i \int_0^1 k_N(t)|w_0(t) - x(t - h_0)|^2 \, dt + \ldots + i \int_0^1 k_N(t)|w_k(t) - x(t - h_k)|^2 \, dt.
\]

The \( x \)-dependence of \( L(.) \) is ‘decoupled’ by introducing new control variables \( w_j, j = 1, \ldots, k \). The perturbed problem is now ‘regular’ with respect to \( x \) and we can use a direct analysis to seek for necessary conditions. Above we used the notation \( x(t) = \phi(t) \) when \( t < 0 \) and \( x(t) = x_0 + \int_0^t v(s) \, ds \) when \( t \geq 0 \). The function \( J_i(.) \) takes values on the space \( X \) defined as follows

\[
X := \left\{ (\phi, x_0, w_0, \ldots, w_k, v) \mid \phi \in L^\infty(-h_k, 0), \ \phi(t) \in E(t) \ \text{a.e. and} \ x_0 \in C_0, \right. \\
k_N w_j \in L^1(0, 1), \ j = 1, \ldots, k, \ v(t) \in N\mathbb{B} \cap \mathcal{R}(t) \ \text{a.e. and} \ \|\phi\|_{L^\infty} + |x_0| + \|v\|_{L^1} \leq \delta \right\}.
\]
We equipped such space with the metric

\[ \| (\phi, x_0, w_0, \ldots, w_k, v) \|_X := \int_{-h_k}^{0} |\phi(t)| \, dt + |x_0| + \| k_N w_0 \|_{L^1} + \ldots + \| k_N w_k \|_{L^1} + \| k_N v \|_{L^1}. \]

The normed space \((X, \| \cdot \|_X)\) is a complete metric space and the functional \(J_i(\cdot)\) is continuous on \((X, \| \cdot \|_X)\). We presently show the existence of a sequence \(\rho_i \downarrow 0\) as \(i \to +\infty\) such that

\[ J_i(0, \ldots, 0) \leq \inf_X J_i + \rho_i^2. \tag{5.1.1} \]

Take any element \((\phi, x_0, w_0, \ldots, w_k, v) \in X\). By the strengthened Lipschitz hypothesis, condition \((L3^*)\), we have that

\[ J_i(\phi, x_0, w_0, \ldots, w_k, v) \geq J_i(\phi, x_0, x(-h_0), \ldots, x(-h_k), v) \]
\[ - \int_0^1 k_N(t)|w_0(t) - x(t-h_0)| \, dt - \ldots - \int_0^1 k_N(t)|w_k(t) - x(t-h_k)| \, dt \]
\[ + i \int_0^1 k_N(t)|w_0(t) - x(t-h_0)|^2 \, dt + \ldots + i \int_0^1 k_N(t)|w_k(t) - x(t-h_k)|^2 \, dt. \]

Since \(\ell_i(x(1)) \geq \ell_1(x(1)) - \alpha_i\), for some \(\alpha_i \downarrow 0\), we can conclude by optimality that

\[ J_i(\phi, x_0, x(-h_0), \ldots, x(-h_k), v) \geq J_i(0, \ldots, 0) - \alpha_i. \]

Thus, defining the following quantities,

\[ a := \left( \int_0^1 k_N(t) \, dt \right)^{\frac{1}{2}}, \quad b_j := \left( \int_0^1 k_N(t)|w_j(t) - x(t-h_j)|^2 \, dt \right)^{\frac{1}{2}} \quad j = 0, \ldots, k \]

we have that

\[ J_i(\phi, x_0, w_0, \ldots, w_k, v) \geq J_i(0, \ldots, 0) - \alpha_i - ab_0 - \ldots - ab_k + i(b_0^2 + \ldots + b_k^2) = \]
\[ J_i(0, \ldots, 0) - \alpha_i + \sum_{j=0}^{k} \left( (i \frac{1}{2} b_j - \frac{a}{2i \frac{1}{2}})^2 - \frac{a^2}{4i} \right) \geq J_i(0, \ldots, 0) - \alpha_i - \frac{(k+1)a^2}{4i}. \]

Inequality \((5.1.1)\) is satisfied with \(\rho_i^2 = \alpha_i + \frac{(k+1)a^2}{4i}\).
Variational Analysis. Because of expression (5.1.1), Ekeland’s variational principle can be applied for the optimization problem \( \inf_X J_i(\cdot) \). According to Ekeland Theorem there exists a sequence \((\phi_i, \xi_i, w_{0i}, \ldots, w_{ki}, v_i)_{i \in \mathbb{N}} \subset X\) such that for each \( i \), \((\phi_i, \xi_i, w_{0i}, \ldots, w_{ki}, v_i)\) minimizes the perturbed functional \( \tilde{J}_i(\cdot) \) defined as
\[
J_i(\phi, x_0, w_0, \ldots, w_k, v) + \rho_i \| \left( \phi, x_0, w_0, \ldots, w_k, v \right) - (\phi_i, \xi_i, w_{0i}, \ldots, w_{ki}, v_i) \|_X
\] (5.1.2)
over elements \((\phi, x_0, w_0, \ldots, w_k, v) \in X\) and
\[
\| (\phi_i, \xi_i, w_{0i}, \ldots, w_{ki}, v_i) \|_X \leq \rho_i.
\] (5.1.3)

Define the feasible arc \( x_i(\cdot) \) according to
\[
x_i(t) := \phi_i(t) \text{ when } t < 0 \text{ and } x_i(t) := \xi_i + \int_0^t v_i(s) \, ds \text{ when } t \geq 0.
\]
Then, by inf-convolutions properties, \( \ell_i(\cdot) \) satisfies the following inequality
\[
\ell_i(x(1)) - \ell_i(x_i(1)) \leq \zeta_i \cdot (x(1) - x_i(1)) + i |x(1) - x_i(1)|^2,
\] (5.1.4)
for some \( \zeta_i \in \partial \ell_i(z_i) \) and some \( z_i \) close to \( x_i(1) \). By (5.1.3) the sequence \((z_i)_{i \in \mathbb{N}}\) is convergent and \( z_i \to 0 \) as \( i \to +\infty \).

**Definition 5.1.2.** Define, for each fixed \( i \in \mathbb{N}, \) the absolutely continuous functions \( q_{ji}(\cdot) \in W^{1,1}([-h_j, 1]; \mathbb{R}^n) \) \((j = 0, \ldots, k)\) according to
\[
\begin{align*}
\dot{q}_{ji}(t - h_j) &:= 2ik_N(t)(x_i(t - h_j) - w_{ji}(t)) \quad t \in [0, 1] \\
q_{ji}(t) &:= 0 \quad t \in [1 - h_j, 1], \ j = 1, \ldots, k \\
q_{0i}(1) &:= -\zeta_i.
\end{align*}
\]

Observe that for a.e. \( t \in [0, 1] \)
\[
|\dot{q}_{ji}(t - h_j)| \leq k_N(t) + \rho_i k_N(t).
\] (5.1.5)

This follows directly from the minimization of \( \tilde{J}_i(\cdot) \) in (5.1.2) by considering \( w_0 \mapsto \tilde{J}_i(\phi_i, \xi_i, w_0, w_1, \ldots, w_{ki}, v_i) \), \( w_k \mapsto \tilde{J}_i(\phi_i, \xi_i, w_0, \ldots, w_{(k-1)}i, w_k, v_i) \). Indeed, for each \( i \in \mathbb{N}, \) the functional cost
\[
\int_0^1 L(t, \ldots w, \ldots, v_i) \, dt + i \int_0^1 k_N |w - x_i(t - h_j)|^2 \, dt + \rho_i \int_0^1 k_N |w - w_{ji}| \, dt
\]
is minimized over $w \in L^1$ at $w_{ji}$. By measurable selection arguments (c.f. [25, Corollary 5.5]) it follows that the integrand is minimized, for a.e. $t \in [0,1]$, at $w_{ji}(t)$. Consequently, by well known properties of nonsmooth calculus

$$
\dot{q}_{ji}(t-h_{ij}) \in \partial w_j L(t, w_{0i}(t), \ldots, w_{ki}(t), v_i(t)) + \rho_k N(t)B,
$$

and (5.1.5) follows from the Lipschitz hypothesis on $L(.)$.

**Step 1.** Theorem 5.1.1 will be the end result of considering particular variations in the minimization process (5.1.2) (only with respect to some “directions”). Fix $(x_0, w_0, \ldots, w_k, v) = (\xi_i, w_{0i}, \ldots, w_{ki}, v_i)$ and let only $\phi(.)$ to vary in the minimization of $J_i(.)$. Then the functional

$$
i \int_0^{h_1} k_N(t)|w_{1i}(t) - \phi(t-h_1)|^2 dt + \ldots + i \int_0^{h_k} k_N(t)|w_{ki}(t) - \phi(t-h_k)|^2 dt + \int_{-h_k}^0 \Lambda(t, \phi(t)) dt + \rho_i \int_{-h_k}^0 |\phi(t) - \phi_i(t)| dt
$$

attains minimum at $\phi_i(.)$ over $\phi(.) \in L^\infty(-h_k, 0)$ satisfying $\phi(t) \in E(t)$ a.e. and $\|\phi\|_{L^\infty} + |\xi_i| + \|v_i\|_{L^1} \leq \delta$. We label such functional $\Phi_i(.)$. After a change of coordinate in the integral, $\Phi_i(.)$ can be rewritten as

$$
\int_{-h_k}^0 \rho_i|\phi(t) - \phi_i(t)| + \Lambda(t, \phi(t)) + ik_N(t + h_1)|w_{1i}(t + h_1) - \phi(t)|^2 \chi_{[-h_k,0]}(t) + \ldots + ik_N(t + h_k)|w_{0i}(t + h_k) - \phi(t)|^2 \chi_{[-h_k,0]}(t) dt
$$

Once again we invoke measurable selections’ results, according to which the integrand is minimized for $t \in \Omega_i$ (for some $\Omega_i \subset [-h_k, 0]$) at $\phi_i(t)$ over $\phi \in E(t)$. We restrict attention to $t \in \Omega_i$ where the relation $\phi_i(t) + |\xi_i| + \|v_i\|_{L^1} < \delta$ is strictly satisfied and the constraint can be neglected. Note that the sequence $(\Omega_i)_{i \in \mathbb{N}}$ converges to a set of full measure. By definition 5.1.2 the absolutely continuous functions $q_{ji}(.)$ satisfy

$$
-(\dot{q}_{1i}(t) \chi_{[-h_1,0]}(t) + \ldots + \dot{q}_{ki}(t) \chi_{[-h_k,0]}(t)) \in \partial \Lambda(t, \phi_i(t)) + N(t, \phi_i(t)) + \rho_i B, \quad (5.1.6)
$$

for all $t \in \Omega_i$. Inclusion (5.1.6) can be regarded as a forerunner of the Euler-Lagrange
condition (b1) in the assertions of Theorem 5.1.1.

**Step 2.** We now fix $\phi \equiv \phi_i$. Observe at the outset that for any $a, a_i, b, b_i \in \mathbb{R}^n$ the following inequalities hold true

$$
|a - b|^2 = |a_i - b_i|^2 + |(a - a_i) - (b_i - b)|^2 + 2\langle a_i - b_i, a - a_i \rangle - 2\langle a_i - b_i, b - b_i \rangle
$$

(5.1.7)

We use the above calculation (5.1.7) with $a = x(t - h_j)$, $a_i = x_i(t - h_j)$, $b = w_j(t)$, $b_i = w_{ji}(t)$, for any $i \in \mathbb{N}$, $j = 0, \ldots, k$ and $t \in [0, 1]$. Then by (5.1.2) and (5.1.4), for any $(\phi_i, x_0, w_0, \ldots, w_k, v) \in X$ the following functional $\Psi_i(x_0, w_0, \ldots, w_k, v)$

$$
\ell_0(x_0) + i|x(1) - x_i(1)|^2 + \int_0^1 L(t, w_0(t), \ldots, w_k(t), v(t)) dt +
2i \int_0^1 k_N(t)|w_0(t) - w_0(t)|^2 dt + \ldots + 2i \int_0^1 k_N(t)|w_k(t) - w_{ki}(t)|^2 dt +
2i \int_0^1 k_N(t)|x(t - h_0) - x_i(t - h_0)|^2 dt + \ldots + 2i \int_0^1 k_N(t)|x(t - h_k) - x_i(t - h_k)|^2 dt +
(q_0(0) + \ldots + q_{ki}(0))(x_0 - \xi_i) - \int_0^1 (q_0(t) + \ldots + q_{ki}(t))(v(t) - v_i(t)) dt
$$

(5.1.8)

attains a minimum at $(\xi_i, w_0, \ldots, w_k, v_i)$ over elements $(\phi_i, x_0, w_0, \ldots, w_k, v) \in X$. Here we replaced $\ell_i(x(1))$ with $\ell_i(x(1)) + \zeta \cdot (x(1) - x_i(1)) + i|x(1) - x_i(1)|^2$ according to (5.1.4), and made use of integration by part. In particular take any $q = q_{ji}$ and $h = h_j$

$$
\int_0^1 \dot{q}(t - h)(x(t - h) - x_i(t - h)) dt =
\int_0^{1+h} \dot{q}(t)(x(t) - x_i(t)) dt =
q(1)(x(1) - x_i(1)) - q(0)(x(0) - x_i(0)) - \int_0^1 q(t)(v(t) - v_i(t)) dt
$$

Note that the initial history of the state trajectory is fixed $x(t) = \phi_i(t)$, $t < 0$ and that,
by definition, \( q(t) \equiv 0 \) on \([1 - h, 1] \).

As a function of the initial condition \( x_0 \mapsto \Psi_i(x_0, w_{0i}, \ldots, w_{ki}, v_i) \), the optimality at \( \xi_i \) yields the following “proximal” transversality condition

\[
q_{0i}(0) + \ldots + q_{ki}(0) \in \partial \ell_0(\xi_i) + N_{C_{0i}}(\xi_i) + \rho_i B. \tag{5.1.9}
\]

**Step 3.** We proceed by fixing \( x_0 = \xi_i \) in the functional \( \Psi(\cdot) \) defined in (5.1.8). We know that \((w_{0i}(\cdot), \ldots, w_{ki}(\cdot), v_i)\) is a minimizer for

\[
\int_0^1 L(t, w_{0i}(t), \ldots, w_{ki}(t), v(t)) \, dt + (2k_i \int_0^1 k_N(t) \, dt + i) \int_0^1 \left| v(t) - v_i(t) \right|^2 \, dt + 2i \int_0^1 k_N(t) |w_{0i}(t) - w_{0i}(t)|^2 \, dt + \ldots + 2i \int_0^1 k_N(t) |w_{ki}(t) - w_{ki}(t)|^2 \, dt + \int_0^1 \dot{q}_{0i}(t - h_0)(w_{0i}(t) - w_{0i}(t)) \, dt + \ldots + \int_0^1 \dot{q}_{ki}(t - h_k)(w_{ki}(t) - w_{ki}(t)) \, dt + \rho_i \int_0^1 k_N(t) |w_{0i}(t) - w_{0i}(t)| \, dt + \ldots + \rho_i \int_0^1 k_N(t) |w_{ki}(t) - w_{ki}(t)| \, dt + \rho_i \int_0^1 k_N(t) \left| v(t) - v_i(t) \right| \, dt.
\]

Observe that the constraint \( \| \phi_i \|_{L^\infty} + |\xi_i| + \| v \|_{L^1} \leq \delta \) and \( v \in N B \) are inactive at \( v_i(t) \) for \( t \in \Omega_i \) for some set \( \Omega_i \) such that \( \text{meas} \{ \Omega_i \} \to 1 \). By now familiar measurable selections’ argument allow us to conclude that, for \( t \in \Omega_i \), the point \((w_{0i}(t), \ldots, w_{ki}(t), v_i(t))\) minimizes the integrand over \((w_{0i}, \ldots, w_k, v)\) such that \( v \in R(t) \). This provides a version of the Euler-Lagrange condition (b2), for \( t \in \Omega_i \)

\[
\left( \dot{q}_{0i}(t - h_0), \ldots, \dot{q}_{ki}(t - h_k), q_{0i}(t) + \ldots + q_{ki}(t) \right) \in \partial L(t, w_{0i}(t), \ldots, w_{ki}(t), v_i(t)) + \rho k(t) B + \{ 0 \} \times N_{R(t)}(v_i). \tag{5.1.10}
\]

Notice at this point the importance of the Lipschitz condition \((L3^*)\) on a neighborhood of \( R(t) \) in order to derive (5.1.10) and write \( \partial L \).
Step 4. It remains to derive an approximated Weierstrass condition (c). We claim that

\[
L(t, w_{0i}(t), \ldots, w_{ki}(t), v) - (q_{0i}(t) + \ldots + q_{ki}(t))(v - v_i(t)) + \rho_i k_N(t)\|v - v_i(t)\| \geq L(t, w_{0i}(t), \ldots, w_{ki}(t), v_i(t)) \quad (5.1.11)
\]

for all \( v \in N B \cap R(t) \) and \( t \in [0, 1] \) a.e..

If this were not the case, there would exist a set of positive measure \( S \) and \( \Delta > 0 \) such that

\[
L(t, w_{0i}(t), \ldots, w_{ki}(t), v) - (q_{0i}(t) + \ldots + q_{ki}(t))(v - v_i(t)) + \rho_i k_N(t)\|v - v_i(t)\| \leq L(t, w_{0i}(t), \ldots, w_{ki}(t), v_i(t)) - \Delta \quad (5.1.12)
\]

for each \( t \in S \) and for some measurable function \( v(t) \in N B \cap R(t) \) defined on \( S \). Define the arc \( x(t) := \phi_i(t) \) for \( t < 0 \) and \( x(t) := \xi_i + \int_0^t v(s)ds \), where we extended then the function \( v(.) \) to the whole set \([0, 1]\) by setting \( v(t) = v_i(t) \) when \( t \notin S \). Notice that we can reduce the size of \( S \) to have arbitrarily small measure, say \( \text{meas}(S) = \epsilon \). Therefore \( |x(t) - x_i(t)| \leq 2N\epsilon \) for all \( t \in [0, 1] \). Fix \( x_0 = \xi_i, (w_{0i}, \ldots, w_k) = (w_{0i}, \ldots, w_{ki}) \) and take \( x(.) \) and \( v(.) \) as above. From (5.1.8) we derive

\[
\int_S L(t, w_{0i}(t), \ldots, w_{ki}(t), v_i(t)) \, dt \leq (\text{by optimality})
\]

\[
i|x(1) - x_i(1)|^2 + \int_S L(t, w_{0i}(t), \ldots, w_{ki}(t), v(t)) \, dt + 2i \int_0^1 k_N(t) |x(t - h_0) - x_i(t - h_0)|^2 \, dt + \ldots + 2i \int_0^1 k_N(t) |x(t - h_k) - x_i(t - h_k)|^2 \, dt
\]

\[
- \int_S (q_{0i}(t) + \ldots + q_{ki}(t))(v(t) - v_i(t)) \, dt + \rho_i \int_S k_N(t)\|v(t) - v_i(t)\| \, dt
\]

\[
\leq (\text{by } (5.1.12))
\]

\[
\int_S (L(t, w_{0i}(t), \ldots, w_{ki}(t), v_i(t)) - \Delta) \, dt + i4N^2\epsilon^2 + k \left(4N^2\epsilon^2 2i \int_0^1 k_N(t) \, dt \right).
\]

This shows that \( \Delta \epsilon \leq \left(i4N^2 + k4N^2 2i \int_0^1 k_N(t) \, dt \right) \epsilon^2 \) which gives the desired contradiction if \( \epsilon \) is chosen sufficiently small.
Asymptotic analysis. A familiar analysis of the limit completes the proof. Observation (5.1.5) provides a common uniform integral bound for the functions $\dot{q}_0(\cdot), \ldots, \dot{q}_k(\cdot)$.

Dunford-Pettis Theorem ensures then weak convergence in $L^1$ to some arcs $\dot{q}_j(\cdot)$,

From (5.1.3) and definition 5.1.2 we conclude that $q_{ji}(\cdot) \to q_j(\cdot)$ uniformly ($j = 0, \ldots, k$),

$\xi \to 0$, and $(\phi_i, w_{0i}, \ldots, w_{ki}, v_i) \to (0, \ldots, 0)$ strongly in $L^1$.

Conditions (5.1.6), (5.1.9), (5.1.10) and (5.1.11) provide the assertions (a), (b) and (c) in the limit. Condition (5.1.11) in the limit yields actually a weaker form of (c), i.e.

$$(q_0(t) + \ldots + q_k(t)) \cdot u - L(t, \bar{z}(t), u) \leq (q_0(t) + \ldots + q_k(t)) \cdot \dot{x}(t) - L(t, \bar{z}(t), \dot{x}(t)) \quad \text{a.e.}$$

for all $v \in N \mathbb{B} \cap \mathcal{R}(t)$. Letting $N \uparrow +\infty$ we complete the proof. Note that the Euler-Lagrange inclusion (b2) ensures that the integral bound on the $q_jN$’s be independent of $N$.

5.2 Differential-difference inclusion

In this section we consider a general optimal control problem in which the dynamic is modeled as a differential inclusion with time delays.

$$\begin{aligned}
(D) \quad & \min \ell(x(S), x(T)) \\
& \text{over arcs } x(\cdot) \in W^{1,1}(S, T) \quad \text{and} \quad x(\cdot) \in L^\infty(S - h_k, S) \quad \text{s.t.} \\
& \dot{x}(t) \in F(t, x(t - h_0), \ldots, x(t - h_k)) \quad \text{a.e.} \quad t \in [S, T] \\
& x(t) \in E(t) \quad \text{a.e.} \quad t \in [S - h_k, S] \\
& (x(S), x(T)) \in C.
\end{aligned}$$

An arc $x(\cdot)$ is said to be feasible if all the constraints in $(D)$ are satisfied. A local minimizer for problem $(D)$ is intended with the same topology used in the previous section. More precisely we say that a feasible arc $\bar{x}(\cdot)$ is a local minimizer for $(D)$ if it minimizes the cost over all feasible arcs $x(\cdot)$ ‘near’ $\bar{x}(\cdot)$, i.e.,

$$\|x - \bar{x}\|_{L^\infty(S - h_k, S)} + \|x - \bar{x}\|_{W^{1,1}(S, T)} \leq \delta \quad (5.2.1)$$
for some $\delta > 0$. Take a set-valued map $R : [S,T] \rightharpoonup \mathbb{R}^n$ whose values are open and convex sets and assume that $\dot{x}(t) \in R(t)$ a.e.. Such a function will be referred to as radius multifunction (for the arc $\bar{x}(.)$). If, in addition to (5.2.1), we require the competing feasible arcs $x(.)$ to satisfy

$$\dot{x}(t) \in R(t) \text{ a.e. } t \in [S,T]$$

then we say that $\bar{x}(.)$ is a local minimizer with respect to the radius multifunction $R(.)$. This concept of local minimizer yields a set of necessary conditions called “stratified” necessary conditions. Those conditions are stronger and more general than that provided for a local minimizer (relative only to condition 5.2.1). For a detailed discussion on stratified necessary conditions we refer the reader to [5,18].

We recall the notation

$$\bar{z}(t) := (\bar{x}(t - h_0), \ldots, \bar{x}(t - h_k)).$$

Hypotheses on the data are imposed around the minimizer $\bar{x}(.)$ and restricted to the radius multifunction $R(.)$:

(G1) $\ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and the set $C \subset \mathbb{R}^n \times \mathbb{R}^n$ is closed.

(G2) Hypothesis (L2).

(G3) $R : [S,T] \rightharpoonup \mathbb{R}^n$ is a multifunction which take as values open and convex subsets of $\mathbb{R}^n$.

(G4) $F(t,z)$ is nonempty and closed for all $(t,z) \in [S,T] \times \mathbb{R}^{n \times (k+1)}$. Its graph, denoted as $GrF(t,.)$, is closed for each $t \in [S,T]$ and $F$ is $\mathcal{L} \times \mathcal{B}^{n(k+1)}$ measurable.

(G5) There exists a function $k_F(.) \in L^1(S,T)$ and $\epsilon > 0$ such that

$$F(t,z) \cap R(t) \subset F(t,z') + k_F(t)|z - z'|B,$$

for all $z,z' \in \bar{z}(t) + \epsilon B$, a.e. $t \in [S,T]$.

(G6) There exist numbers $\bar{r} > 0$ and $\beta \in (0,1)$, and a function $r(.) \in L^1(S,T)$ such that $r(t) \geq \bar{r}$ a.e. and

i) $\dot{x}(t) + \beta^{-1}r(t)B \subset R(t)$
ii) \( F(t, z) \cap (\dot{x}(t) + r(t) \mathbb{B}) \neq \emptyset \)

for all \( z \in \bar{z}(t) + \epsilon \mathbb{B} \), a.e. \( t \in [S, T] \).

**Theorem 5.2.1.** Assume (G1)-(G6) and let \( \bar{x}(\cdot) \) be a local minimizer for (D) with respect to the radius multifunction \( R(\cdot) \). Then there exist arcs \( q_i \in W^1,1([S-h_i, T]; \mathbb{R}^n) \), \( i = 0, \ldots, k \), and a nonnegative number \( \lambda \) such that

(i) \( \|q_0\|_{L^\infty} + \lambda = 1 \).

(ii) (a) \( q_0(S) + \cdots + q_k(S), -q_0(T) \in \lambda \partial \ell(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T)) \).

(b) \( q_i(t) = 0, t \in [T - h_i, T] \), for each \( i = 1 \ldots, k \).

(iii) (a) for a.e. \( t \in [S - h_k, S] \)

\[-(\dot{q}_1(t) \chi_{[S-h_1, S]}(t) + \cdots + \dot{q}_k(t) \chi_{[S-h_k, S]}(t)) \in co N_E(\bar{x}(t)) .\]

(b) for a.e. \( t \in [S, T] \)

\[(\dot{q}_0(t - h_0), \ldots, \dot{q}_k(t - h_k)) \in co\{\eta : (\eta, q_0(t) + \ldots, q_k(t)) \in N_{GrF}(\bar{z}(t), \dot{x}(t))\} .\]

(iv) For a.e. \( t \in [S, T] \) and all \( v \in F(t, \bar{x}(t-h_0), \ldots, \bar{x}(t-h_k)) \)

\[(q_0(t) + \ldots, q_k(t)) \cdot \dot{x}(t) \geq (q_0(t) + \ldots, q_k(t)) \cdot v .\]

### 5.2.1 Hypotheses reductions & Preliminaries

The strategy of the proof follows classical ideas of recasting (D) as a Lagrange Problem including cost integrand penalty terms to take account of the dynamic constraints. In particular we penalize the cost by means of a modified distance function \( \rho(\cdot) \) (measuring the distance of \( \dot{x}(t) \) from the dynamic constraint set \( F(t, x(t-h_0), \ldots, x(t-h_k)) \)), as follows

\[\ell(x(S), x(T)) + \int_S^T \rho(t, x(t-h_0), \ldots, x(t-h_k), \dot{x}(t)) dt .\]
The new modified problem is of the type studied in \(5.1\).

**Reductions.** Notice that a number of reductions can be made at the outset. The assertions of Theorem 5.2.1 are still valid if they can be verified in the special case when

\[
\ell(x(S), x(T)) = \ell(x(T)) \quad \text{and} \quad C = C_0 \times C_1, \tag{5.2.3}
\]

\[
\bar{x}(.) \equiv 0, \tag{5.2.4}
\]

\[
r(t) \equiv r_0 \quad t \in [S, T] \quad \text{and} \quad [S, T] = [0, 1]. \tag{5.2.5}
\]

Reduction (5.2.3) is dealt with introducing an extra state \(z(.)\) and defining the set \(C_0 := \{(z, x) | z = x\}\). We then rewrite (D) as follows

\[
(D) \begin{cases}
\text{Minimize } \ell(z(T), x(T)) \\
\text{over arcs } x(.) \quad \text{and} \quad z(.) \in W^{1,1}(S, T) \quad \text{and} \quad x(.) \in L^\infty(S - h_k, S) \text{ s.t.} \\
\dot{x}(t) \in F(t, x(t - h_0), \ldots, x(t - h_k)) \quad \text{a.e. } t \in [S, T] \\
\dot{z}(t) = 0 \quad \text{a.e. } t \in [S, T] \\
x(t) \in E(t), \quad \text{a.e. } t \in [S - h_k, S] \\
(z(S), x(S)) \in C_0 \quad \text{and} \quad (z(T), x(T)) \in C_1 = C.
\end{cases}
\]

Reduction (5.2.4) is achieved by translating the origin to \(\bar{x}(.)\) regarding the \(x\)-dependent data while (5.2.5) is achieved by changing the independent variable from ‘\(t\)’ to ‘\(\tau(t) = c \cdot \int_S^T r(s) \, ds\)’, arranging the constant \(c\) such that \(\tau(T) = 1\). Henceforth we assume (5.2.3)-(5.2.5). Observe that in particular hypotheses (G5) and (G6) take the form

\((G5^*)\) There exists a function \(k_F(.) \in L^1(0, 1)\) and \(\epsilon > 0\) such that

\[F(t, z) \cap R(t) \subset F(t, z') + k_F(t)|z - z'|B,\]

for all \(z, z' \in \epsilon B\), a.e. \(t \in [0, 1]\).

\((G6^*)\) There exist numbers \(r_0 > 0\) and \(\eta \in (0, 0.5)\) such that

i) \((1 - 2\eta)^{-1} r_0 \ B \subset R(t)\)

ii) \(F(t, z) \cap r_0 \ B \neq \emptyset\)

for all \(z \in \epsilon B\), a.e. \(t \in [0, 1]\).
Gauge function. Choose any $N > (1−2\eta)^{-1}r_0$, and write $R_N(t) := R(t) ∩ N\mathbb{B}$. Then the reader may verify that hypotheses $(G5^*)$ and $(G6^*)$ are still satisfied with $R_N(.)$ in place of $R(.)$. We label these further modifications to $(G5^*)$ and $(G6^*)$ with $(G5^{**})$ and $(G6^{**})$.

We describe points which lay in the convex set $R_N(t)$, for a.e. $t ∈ [0,1]$ by means of the following gauge function $g_t : \mathbb{R}^n → \mathbb{R}_+$ of $R_N(t)$, defined as

$$g_t(f) := \inf\{λ > 0 : λ^{-1}f ∈ R_N(t)\}.$$  \hspace{1cm} (5.2.6)

**Lemma 5.2.2.** The gauge function $g_t(.)$ satisfies the following properties:

1. $g_t(cf) = cg_t(f)$ for every constant $c > 0$ and every $f ∈ \mathbb{R}^n$.
2. $g_t(f) = 0$ if and only if $f = 0$.
3. $g_t(.)$ is Lipschitz with Lipschitz constant $1/r_0$.
4. For every $t ∈ [0,1]$, $R_N(t) = \{f ∈ \mathbb{R}^n : g_t(f) < 1\}$.

Distance function. Define the function $\chi : \mathbb{R}^+ → \mathbb{R}^+$ as follows

$$\chi(d) := \begin{cases} 1 & \text{if } d \leq 1 − η \\ 1 + \frac{2N}{ηr_0}(d − (1 − η)) & \text{if } d > 1 − η. \end{cases}$$

Observe that $\chi(.)$ is a Lipschitz continuous function. Now consider the modified distance, for $t ∈ [0,1]$, $z ∈ \mathbb{R}^n \times (k+1)$, and $v ∈ \mathbb{R}^n$

$$ρ(t, z, v) := \min_{e ∈ F(t, z)} |v − eχ(g_t(e))|.$$  \hspace{1cm} (5.2.7)

For any $t ∈ [0,1]$ and $z ∈ ε\mathbb{B}$, we define $\tilde{F}(t, x)$ to be the set

$$\tilde{F}(t, z) := \{eχ(g_t(e)) : e ∈ F(t, z)\},$$  \hspace{1cm} (5.2.8)

so that $ρ(t, z, v) = d_{\tilde{F}(t,x)}(v)$. Note that, in particular, the multifunction $\tilde{F}(.)$ satisfies hypothesis (G4).
Lemma 5.2.3. Assume $(G5^{**})$ and $(G6^{**})$. Then there exists $\tilde{k}(\cdot) \in L^1(0,1)$ such that the distance function $\rho(\cdot)$ defined in (5.2.7) satisfies the following properties:

(i) $\rho(t,z,\cdot)$ is Lipschitz with Lipschitz constant 1 with respect to $v$, for any $z \in \epsilon\mathbb{B}$, a.e. $t \in [0,1]$.

(ii) $\rho(t,\cdot,v)$ is locally Lipschitz with Lipschitz constant $\tilde{k}(t)$ with respect to $z$, for a.e. $t \in [0,1]$ and for all $v \in R_N(t)$.

(iii) Take any $v \in (1-\eta)\overline{R_N(t)}$. Then $\rho(t,z,v) = 0$ if and only if $v \in F(t,z)$.

Proof. Properties (i) and (iii) are a direct consequence of the definition since $\rho(t,z,v) = d_{F(t,z)}(v)$ and since $\chi(g_t(v)) = 1$ when $v \in (1-\eta)\overline{R_N(t)}$.

The idea (see figure 5.1) is that the function $\chi(\cdot)$ pushes away points $e \in F(t,z)$ which lay outside $R_N(t)$. On the other hand thanks to hypothesis $(G6^{**})$ we can always find a point $e_0 \in F(t,z) \cap r_0\mathbb{B}$ and thus strictly inside $R_N(t)$. Therefore for any $v \in R_N(t)$ the “cost” of $e \notin R_N(t)$ (for evaluating the minimum in the definition of $\rho$) is greater than that for any point $e_0 \in F(t,z) \cap r_0\mathbb{B}$. But if $\rho(t,z,v) = |v - e\chi(g_t(e))|$ for some $e \in F(t,z) \cap R_N(t)$ then we can use the pseudo Lipschitz condition $(G5^{**})$.

Analytic arguments follow. Fix any $v \in R_N(t)$, $z \in \epsilon\mathbb{B}$, and $t \in [0,1]$ such that $(G5^{**})$ and $(G6^{**})$ are satisfied. Note in particular that $|v| \leq N$. Assume that $\rho(t,z,v) = |v - e\chi(g_t(e))|$ for some $e \in F(t,z)$ for which $g_t(e) \geq 1$. Choosing $e_0 \in F(t,z) \cap r_0\mathbb{B}$ (such $e_0$ exists according to $(G6^{**})$) we have that

$$|v - e\chi(g_t(e))| \leq |v - e_0\chi(g_t(e_0))| = |v - e_0|.$$
Recall that \( \chi \) and \( g_t \) are Lipschitz continuous with Lipschitz constant respectively \( 2N/r_0 \eta \) and \( 1/r_0 \). Therefore

\[
1 + 2N/r_0 \leq g_t(e\chi(g_t(e))) \leq g_t(v) + (r_0)^{-1}|v - e_0| \leq 1 + \frac{1}{r_0}(r_0 + N) < 1 + 2N/r_0.
\]

This last estimate provides a contradiction. For every \( z \in \varepsilon B \) the minimum in the definition of \( \rho \) is attained at a point \( e \in F(t, z) \cap R_N(t) \).

Chose any \( z, z' \in \varepsilon B \). Let \( e \) be a closest point to \( v \) (with respect to the distance defined by \( \rho \)) in \( F(t, z) \). Then, since \( e \in R_N(t) \), condition \( (G5^{**}) \) guarantees the existence of a point \( e_1 \in F(t, z') \) such that \( |e - e_1| \leq k_F(t)|z - z'| \). Observe that if \( k_F(t)|z - z'| > r_0 + N \) then according to \( (G6^{**}) \) we can find \( e_2 \in F(t, z') \cap r_0 B \) such that

\[
|e - e_2| \leq N + r_0 < k_F(t)|z - z'|.
\]

Set \( e' \in F(t, z') \) as follows

\[
e' := \begin{cases}
e_1 & \text{if } k_F(t)|z - z'| \leq r_0 + N \\
e_2 & \text{if } k_F(t)|z - z'| > r_0 + N.
\end{cases}
\]

By construction \( e' \) satisfies the following estimates

\[
|e - e'| \leq \min\{N + r_0, k_F(t)|z - z'|\}.
\]  \hspace{1cm} (5.2.9)

We are ready to prove part \( (ii) \) of the Lemma. We make use of the Lipschitz regularity of the functions \( g_t(.) \) and \( \chi(.) \) (with Lipschitz constant respectively \( r_0^{-1} \) and \( 2N(r_0 \eta)^{-1} \)) and estimates \( |e| \leq N \) and \( (5.2.9) \). Observe also that \( |e'| \leq |e - e'| + |e| \leq 2N + r_0 \). The
following holds true,

\[ \rho(t, z', v) - \rho(t, z, v) \leq |v - e'\chi(g_t(e'))| - |v - e\chi(g_t(e))| \]

\[ \leq |e'\chi(g_t(e')) - e\chi(g_t(e))| \]

\[ \leq |e'\chi(g_t(e')) - e\chi(g_t(e))| + |e\chi(g_t(e)) - e\chi(g_t(e))| \]

\[ \leq |e'\chi(g_t(e')) - e\chi(g_t(e))| + \chi(g_t(e))|e - e'| \]

\[ \leq \left( |e'| \frac{2N}{r_0^2\eta} + \chi(g_t(e)) \right) |e - e'| \]

\[ \leq \left( 2N + r_0 \right) \frac{2N}{r_0^2\eta} + 1 + \frac{2N}{r_0} |e - e'| \]

\[ \leq \left( 2N + r_0 \right) \frac{2N}{r_0^2\eta} + 1 + \frac{2N}{r_0} k_F(t) |z - z'|. \]

Assertion (ii) is verified for \( \tilde{k}(t) = \left( 2N + r_0 \right) \frac{2N}{r_0^2\eta} + 1 + \frac{2N}{r_0} k_F(t). \]

We also recall a version of the Gronwall’s Lemma for delayed systems.

**Lemma 5.2.4** (Gronwall). Let \( q_i \in W^{1,1}([S - h_i, T]; \mathbb{R}^n), (T > S, i = 0, \ldots, k, h_0 = 0 < h_1 < \ldots < h_k \in \mathbb{R}), \) be such that \( q_i(t) = 0, \) when \( t \in [T - h_i, T], i = 1, \ldots, k \) and \( q_0(T) = q_0. \) Assume that there exist nonnegative integrable functions \( k_F(.) \) and \( v(.) \) such that for a.e. \( t \in [S, T] \)

\[ |(q_0(t - h_0), \ldots, \dot{q}_k(t - h_k)| \leq k_F(t) \sum_{i=0}^{k} |q_i(t)| + v(t). \]

Then for all \( t \in [S, T] \)

\[ \sup_{i=0,\ldots,k} |q_i(t)| \leq e^{(k+1) \int_t^T k_F(s) ds} \left\{ |q_0| + \int_t^T e^{-(k+1) \int_s^T k_F(\sigma) d\sigma} v(\sigma) d\sigma \right\}. \]

**Proof.** For any \( j = 0, \ldots, k \) and \( t \in [S, T - h_j] \) it follows from the fundamental theorem of calculus that \( q_j(t) = q_j(T - h_j) + \int_{T - h_j}^t \dot{q}_j(s) \, ds \) (recall that the \( q_j \)'s are absolutely
continuous). Therefore the following estimates hold true

\[ |q_j(t)| \leq |q_j(T - h_j)| + \int_t^{T - h_j} |\dot{q}_j(s)| \, ds = |q_j(T - h_j)| + \int_{t+h_j}^T |\dot{q}_j(s - h_j)| \, ds \leq |q_0(T)| + \int_{t+h_j}^T k_F(s) \sum_{i=0}^k |q_i(s)| + v(s) \, ds \leq |q_0(T)| + \int_{t+h_j}^T k_F(s) \sum_{i=0}^k |q_i(s)| + v(s) \, ds =: g(t). \]

Observe that the inequality above was obtained for any \( t \in [S, T - h_j] \). It can nevertheless be extended for \( t \in [T - h_j, T] \) since the inequality \( |q_j(t)| \leq g(t) \) is clearly satisfied for \( t \in [T - h_j, T] \), \( q_j(\cdot) \equiv 0 \) on such interval. A quick calculation shows that

\[ \frac{d}{dt} \left[ e^{-(k+1)\int_t^T k_F(s) \, ds} g(t) \right] \geq e^{-(k+1)\int_t^T k_F(s) \, ds} v(t) \]

which in turn yields the desired conclusion integrating both sides of the inequality.

### 5.2.2 Proof of Theorem 5.2.1

Recall that following our reductions Theorem 5.2.1 is proven as soon as we derive necessary conditions for the modified problem

\[
\left\{ \begin{array}{l}
\text{Minimize } \ell(x(1)) \\
\text{over arcs } x(\cdot) \in W^{1,1}(0, 1) \text{ and } x(\cdot) \in L^\infty(-h_k, 0) \text{ s.t.} \\
\dot{x}(t) \in F(t, x(t-h_0), \ldots, x(t-h_k)) \text{ a.e. } t \in [0, 1] \\
x(t) \in E(t) \text{ a.e. } t \in [-h_k, 0) \\
x(0) \in C_0 \text{ and } x(1) \in C_1,
\end{array} \right.
\]

where \( \tilde{x}(\cdot) \equiv 0 \) is a local minimizer for \((\tilde{D})\) with respect to the multifunction \( R_N(\cdot) \).

We label \( \mathcal{S}_{[0,1]} \) the set of competing feasible arcs in the minimization of \( \ell(\cdot) \) such that \( \min_{x(\cdot) \in \mathcal{S}_{[0,1]}} \ell(x(1)) = \ell(0) \), i.e.,

\[
\mathcal{S}_{[0,1]} := \left\{ x(\cdot) \mid x(\cdot) \in L^\infty(-h_k, 0) \text{ and } x(t) \in E(t) \text{ a.e. } t \in [-h_k, 0), \right. \\
x(\cdot) \in W^{1,1}(0, 1) \text{ and } \dot{x}(t) \in F(t, x(t-h_0), \ldots, x(t-h_k)) \text{ a.e.,} \\
x(0) \in C_0, x(1) \in C_1 \text{ and } \dot{x}(t) \in (1 - \eta)\overline{R_N(t)} \text{ a.e.,} \\
\left. \|x\|_{L^\infty(-h_k, 0)} + \|x\|_{W^{1,1}(0,1)} \leq \delta \right\},
\]
and we define $X$ to be the set
\[ X := \left\{ x(.) \mid x(.) \in L^\infty(-h, 0) \text{ and } x(t) \in E(t) \text{ a.e. } t \in [-h_k, 0), \right. \\
\left. x(.) \in W^{1,1}(0, 1), x(0) \in C_0 \text{ and } \dot{x}(t) \in (1 - \eta)R_N(t) \text{ a.e.,} \\
\|x\|_{L^\infty(-h, 0)} + \|x\|_{W^{1,1}(0, 1)} \leq \delta \right\}. \]

On such spaces we define the metric
\[ \|x\|_X := \|x\|_{L^1(-h, 0)} + \|x\|_{W^{1,1}(0, 1)}. \]

With this metric the spaces $X$ and $S_{[0,1]}$ are complete. Furthermore the functional $\tilde{\ell}(.)$ defined by
\[ \tilde{\ell}(x(.)) := \ell(x(1)). \]

is Lipschitz continuous on $X$ with Lipschitz constant $k_\ell$ with respect to the norm $\|.\|_X$. The optimal control problem $(\tilde{D})$ can be then rewritten as
\[ \text{Minimize } \tilde{\ell}(x(.)) \text{ over } x(.) \in S_{[0,1]} \subset X. \]

It follows by Exact Penalization Theorem ([55, Theorem 3.2.1]) that $\bar{x} \equiv 0$ is also a minimizer for
\[ \text{Minimize } \tilde{\ell}(x(.)) + k_\ell \inf \{ \|x - y\|_X : y \in S_{[0,1]} \} \text{ over arcs } x(.) \in X. \]

We use the notation $d_{S_{[0,1]}}(x(.)) := \inf \{ \|x - y\|_X : y \in S_{[0,1]} \}$ and define the function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^+$ as
\[ \phi(e) := \max \left\{ \frac{r_0}{2} (g_t(e) - (1 - 2\eta)) , 0 \right\}. \tag{5.2.10} \]

The following properties of $\phi(.)$ will be of particular significance (for any $e, e' \in \mathbb{R}^n$):
\[ \begin{align*}
\phi(e) &= 0 \text{ if } g_t(e) \leq 1 - 2\eta, \\
|\phi(e) - \phi(e')| &\leq \frac{1}{2} |e - e'| \\
\phi(e) &\geq \frac{\eta r_0}{2} \text{ if } g_t(e) > 1 - \eta
\end{align*} \tag{5.2.11} \]
Finally define the functional $J(\cdot)$ on $X$ to be

$$J(x(\cdot)) := \int_0^1 \rho(t, x(t-h_0), \ldots, x(t-h_k), \dot{x}(t))dt + \int_0^1 \phi(\dot{x}(t))dt + dC_1(x(1)).$$

Two different situations can arise (this approach was developed by Ioffe in [40]):

A. either there exist $\delta' \in (0, \delta)$ and $C > 0$, such that for any $x(\cdot) \in X$ satisfying $\|x\|_X \leq \delta'$ we have

$$dS_{[0,1]}(x(\cdot)) \leq CJ(x(\cdot)), \quad (5.2.12)$$

B. or there exists a sequence of arcs $y_i \in X$, such that $y_i \to 0$ in $X$ as $i \to \infty$ and

$$dS_{[0,1]}(y_i(\cdot)) > (2i)J(y_i(\cdot)). \quad (5.2.13)$$

We consider first case A. The arc $\bar{x} \equiv 0$ is still a local minimizer for the problem

$$\min \left\{ \tilde{\ell}(x(\cdot)) + k_1 C J(x(\cdot)) : x(\cdot) \in X \right\}.$$  

This is a Lagrange type problem considered in section 5.1 where $R(t) = (1 - \eta)R_N(t)$ and

$$L(t, z, v) = k_1C(\rho(t, z, v) + \phi(v))$$

$$\ell_1(x) = \ell(x) + k_1CdC_1(x)$$

$$\ell_0(\cdot) \equiv 0 \quad \text{and} \quad \Lambda(\cdot) \equiv 0.$$  

All the hypotheses of Theorem 5.1.1 are satisfied (recall Lemma 5.2.3) for existence of arcs $q_i(\cdot) \in W^{1,1}(-h_i, 0), \ (i = 0, \ldots, k)$ satisfying

(a) A transversality condition at the end-time $t = 1$

$$q_i(t) = 0, \ t \in [1 - h_i, 1], \ \forall \ i = 1, \ldots, k \quad \text{and} \quad -q_0(1) \in \partial \ell_1(0) + k_1C\partial dC_1(0)$$

and at time $t = 0$

$$q_0(0) + \cdots + q_k(0) \in NC_0(0).$$

(b) An Euler-Lagrange type condition for a.e. $t \in [-h_k, 0]$ and a.e. $t \in [0, 1]$, respectively

(b1) $$-(\dot{q}_1(t)\chi_{[-h_1,0]}(t) + \cdots + \dot{q}_k(t)\chi_{[-h_k,0]}(t)) \in coN\epsilon(t)(0)$$
(b2) \((q_0(t-h_0), \ldots, q_k(t-h_k)) \in \text{co}\{\eta : (\eta, q_0(t) + \ldots + q_k(t)) \in k_t C \cdot \partial \rho(t, 0, \ldots, 0) + \{0\} \times k_t C \cdot \partial \phi(0) + \{0\} \times N_{\mathcal{R}(t)}(0)\}, t \in [0, 1] \text{ a.e.}\)

(c) The Weierstrass condition: the following hold for all \(v \in \mathcal{R}(t)\), a.e. \(t \in [0, 1] \)

\[(q_0(t) + \ldots + q_k(t)) \cdot v - k_t C(\rho(t, 0, v) - \phi(v)) \leq (q_0(t) + \ldots + q_k(t)) \cdot 0 - k_t C(\rho(t, 0, 0) - \phi(0)) = 0.\]

Case A furnishes the desired necessary conditions asserted in Theorem 5.2.1 with \(\lambda = 1\) (except a restricted form of the Weierstrass conditions that will be dealt with at the end of proof). Indeed in (b2) \(N_{\mathcal{R}(t)}(0) = \{0\}\) and \(\partial \phi(0) = \{0\}\). This follows from the fact that 0 is in the interior of \(\mathcal{R}(t)\) and that \(\phi(v) = 0\) whenever \(v \in (1 - 2\eta)R_N(t)\). The latter was pointed out in (5.2.11). Moreover the function \(\rho(.)\) coincides with the distance function in a neighborhood of the origin (Lemma 5.2.3) which implies the inclusion \(\partial \rho(t, 0, \ldots, 0) \subset N_{GrF(t, .)}(0, \ldots, 0)\). Finally from (c) we obtain the following (weak form of the) Weierstrass condition

\[(q_0(t) + \ldots + q_k(t)) \cdot v \leq 0\]

if we restrict attention to points \(v \in F(t, 0, \ldots, 0) \cap (1 - 2\eta)R_N(t)\).

**Case B.** Notice first of all that the functional \(J(.)\) is continuous on \((X, \|\|_X)\). This is an easy calculation. By [5.2.13] there exists a sequence of arcs \(y_i \in X\), such that \(y_i \to 0\) in \(X\) as \(i \to \infty\) and \(d_{S_{[0,1]}}(y_i(.) > (2i)J(y_i(.))\). We conclude that \(y_i(.)\) is a “\((2i)^{-1}d_{S_{[0,1]}}(y_i(.))\)-minimizer” for the problem of minimizing \(J(.)\) over \(X\). Observe that \(d_{S_{[0,1]}}(y_i(.)) \to 0\) as \(i \to \infty\) since \(y_i \to 0\) in \(X\) and \(0 \in S_{[0,1]}\).

We are under the hypotheses where Ekeland’s theorem can be invoked: for each \(i\), there exists \(x_i(.) \in X\) such that \(x_i(.)\) is a minimizer for the perturbed problem

\[
\min J(x(.)) + i^{-1} \left( \int_{-h_k}^{0} |x(t) - x_i(t)|dt + |x(0) - x_i(0)| + \int_{0}^{1} |\dot{x}(t) - \dot{x}_i(t)|dt \right)
\]

over arcs \(x(.) \in X\),

and \(\|x_i - y_i\|_X \leq d_{S_{[0,1]}}(y_i(.))/2\). This last inequality tells us that \(x_i(.) \notin S_{[0,1]}\), for any
i ∈ \mathbb{N}. The same inequality shows, as well, that \( \lim_{i \to \infty} \|x_i\|_X = 0 \). We have obtained once again a Lagrange problem as studied in section 5.1 with \( R(t) = (1 - \eta)R_N(t) \) and

\[
L(t, z, v) = \rho(t, z, v) + \phi(v) + i^{-1}|v - \dot{x}_i(t)|,
\]

\[
\Lambda(t, x) = i^{-1}|x - x_i(t)|,
\]

\[
\ell_1(x) = d_{C_1}(x) \quad \text{and} \quad \ell_0(x) = i^{-1}|x - x_i(0)|.
\]

Particular attention has to be paid to establishing validity of hypothesis (L3). According to Theorem 5.1.1 we can find arcs \((p_i, \lambda_i, \mu_i, m_i)\) which satisfy exist arcs \(q_{ji}(\cdot) \in W^{1,1}(-h_j, 1), \quad (j = 0, \ldots, k \text{ and } i \in \mathbb{N})\) satisfying

\((a_i)\) A transversality condition at the end-time \(t = 1\)

\[
q_{ji}(t) = 0, \quad t \in [1 - h_j, 1], \quad \forall \ j = 1, \ldots, k \quad \text{and} \quad -q_{0i}(1) \in \partial d_{C_1}(x_i(1))
\]

and at time \(t = 0\)

\[
q_{0i}(0) + \cdots + q_{ki}(0) \in i^{-1}B + N_{C_0}(x_i(0)).
\]

\((b_i)\) An Euler-Lagrange type condition for a.e. \(t \in [-h_k, 0]\) and a.e. \(t \in [0, 1]\), respectively

\[(b1_i)\] \(-\dot{q}_{ki}(t)\chi_{[-h_k, 0]}(t) + \cdots + \dot{q}_{ki}(t)\chi_{[-h_k, 0]}(t) \in i^{-1}B + \co N_{E(t)}(x_i(t))
\]

\[(b2_i)\] \(\dot{q}_{0i}(t-h_0), \ldots, \dot{q}_{ki}(t-h_k) \in \co \{\eta : (\eta, q_{0i}(t) + \cdots + q_{ki}(t)) \in \partial \rho(t, x_i(t - h_0), \ldots, x_i(t - h_k), \dot{x}_i(t)) + \{0\} \times \partial \phi(\dot{x}_i(t)) + \{0\} \times N_{R(t)}(\dot{x}_i(t))\}, \ t \in [0, 1] \text{ a.e.}\)

\((c_i)\) The Weierstrass condition: the following hold for all \(v \in \mathcal{R}(t), \text{ a.e. } t \in [0, 1]\),

\[
(q_{0i}(t) + \cdots + q_{ki}(t)) \cdot v - \rho(t, x_i(t), v) - \phi(v) - i^{-1}|v - \dot{x}_i(t)| \leq (q_{0i}(t) + \cdots + q_{ki}(t)) \cdot \dot{x}_i(t) - \rho(t, x_i(t), \dot{x}_i(t)) - \phi(x_i(t)).
\]

The fact that the modified distance function is Lipschitz is essential here to invoke Theorem 5.1.1 and to derive uniformly integrably bounds for \(\dot{q}_{ji}(\cdot)'s\) (\(j = 0, \ldots, k\) and \(i \in \mathbb{N}\)). We may conclude that, following an extraction of subsequences if necessary, \(q_{ji} \to q_j\) uniformly in \(W^{1,1}\) and \(\dot{q}_{ji} \to \dot{q}_j\) weakly in \(L^1\), for some \(q_j \in W^{1,1}(-h_j, 1), \quad j = 0, \ldots, k, \) and the \(q_j\)'s satisfy
(a) A transversality condition at the end-time $t = 1$

\[ q_j(t) = 0, \quad t \in [1 - h_j, 1], \quad \forall \ j = 1, \ldots, k \quad \text{and} \quad -q_0(1) \in NC_1(0) \]

and at time $t = 0$

\[ q_0(0) + \cdots + q_k(0) \in NC_0(0). \]

(b) An Euler-Lagrange type condition for a.e. $t \in [-h_k, 0]$ and a.e. $t \in [0, 1]$, respectively

\begin{align*}
(b1) \quad & -(\dot{q}_1(t) \chi_{[-h_1,0]}(t) + \cdots + \dot{q}_k(t) \chi_{[-h_k,0]}(t)) \in coNC_E(t)(0) \\
(b2) \quad & (q_0(t - h_0), \ldots, q_k(t - h_k)) \in co\left\{ \eta : (\eta, q_0(t) + \cdots + q_k(t)) \in \partial \rho(t, 0, \ldots, 0) \right\}, \\
& t \in [0, 1] \text{ a.e.}.
\end{align*}

(c) The Weierstrass condition: the following hold for all $v \in (1 - 2\eta)RN(t) \cap F(t, 0, \ldots, 0)$, a.e. $t \in [0, 1]$,

\[ (q_0(t) + \cdots + q_k(t)) \cdot v \leq 0. \]

The proof of the Theorem (or better a restricted form of it) is completed if we can show that $q_0(t) \neq 0$ for all $t \in [0, 1]$. Indeed (a), (b), and (c) gives the desired assertions with $\lambda = 0$.

Observe that from the inequality $\|x_i - y_i\|_X \leq dS_{[0,1]}(y_i(\cdot))/2$ we can conclude that $x_i \notin S_{[0,1]}$, for any $i \in \mathbb{N}$. This mens that either $\dot{x}_i(t) \notin F(t, x_i(t - h_0), \ldots, x_i(t - h_k))$ on a set of positive measure $\Omega_i \subset [0, 1]$ or $x_i(1) \notin C_1$.

If $x_i(1) \notin C_1$ for infinitely many $i$ then from the transversality condition (a) we obtain the estimate

\[ |q_0(1)| \geq 1 \]

which ensures the desired nontriviality condition in the limit.

If on the other hand $\dot{x}_i(t) \notin F(t, x_i(t - h_0), \ldots, x_i(t - h_k))$ on a set of positive measure $\Omega_i$ and for infinitely many $i$ then we distinguish two cases:

\begin{align*}
\text{B1 : } \quad & \dot{x}_i(t) \in \text{int}((1 - \eta)RN(t)), \quad \text{a.e. } t \in \Omega_i. \\
\text{B2 : } \quad & \dot{x}_i(t) \in \text{bdry}((1 - \eta)RN(t)), \quad \text{for } t \text{ in a set of positive measure } \widetilde{\Omega}_i \subset \Omega_i.
\end{align*}
Let us examine case B1: recalling Lemma 5.2.3 and the definition of \( \rho(\cdot) \), we have that

\[
\frac{d}{dt} \tilde{F}(t, x_i(t-h_0), \ldots, x_i(t-h_k), \dot{x}_i(t)) = \rho(t, x_i(t-h_0), \ldots, x_i(t-h_k), \dot{x}_i(t)) > 0,
\]

so that every \( \beta \in \partial \rho(t, x_i(t-h_0), \ldots, x_i(t-h_k), \dot{x}_i(t)) \) satisfies \( |\beta| = 1 \), when \( t \in \Omega_i \). Therefore by \((b2)_i\) it follows that

\[
|q_0(t) + \ldots + q_{ki}(t)| = 1 + \frac{1}{2},
\]

where we use the fact that every \( \alpha \in \partial \phi(\dot{x}_i(t)) \) satisfies \( |\alpha| \leq \frac{1}{2} \) since \( \phi(\cdot) \) is Lipschitz with Lipschitz constant \( \frac{1}{2} \). Therefore \( \|q_0 + \ldots + q_k\|_{L^\infty(0,1)} = 1 + \frac{1}{2} \) and the nontriviality condition \( \|q_0\|_{L^\infty} > 0 \) is proved by means of Gronwall’s Lemma 5.2.4.

We turn to the last case B2. Take a point \( e \in F(t, x_i(t-h_0), \ldots, x_i(t-h_k)) \cap r_0B \) whose existence is ensured by \((G6^{**})\) and fix \( v = e \) in \((c_i)\). Then

\[
|q_0(t) + \ldots + q_{ki}(t)||\dot{x}_i(t) - e| \geq \phi(\dot{x}_i(t)) - i^{-1}|e - \dot{x}_i(t)|.
\]

for a.e. \( t \in \tilde{\Omega}_i \). Notice that when \( t \in \tilde{\Omega}_i \) the derivative \( \dot{x}_i(t) \in \text{bdry}((1 - \eta)R_N(t)) \) which makes \( \phi(\dot{x}_i(t)) = r_0\eta /2 \). Dividing across the inequality by \( |\dot{x}_i(t) - e| \) which satisfies \( 0 < |\dot{x}_i(t) - e| \leq 2N \), we obtain

\[
|q_0(t) + \ldots + q_{ki}(t)| \geq \frac{r_0\eta}{4N} - i^{-1}.
\]

The right part in the inequality above is greater than 0 for \( i \) sufficiently large. Consequently the estimate \( \|q_0 + \ldots + q_k\|_{L^\infty(0,1)} > 0 \) is proved once again and the desired nontriviality condition follows by Gronwall’s Lemma 5.2.4. In all cases the multipliers arising in the limit are nontrivial.

All assertions of the theorem has been proved except that the Weierstrass condition is satisfied, a.e., only with respect to \( e \)’s satisfying

\[
e \in F(t, 0) \cap \left( (1 - 2\eta)(R(t) \cap N\mathbb{B}) \right)
\]

Now take \( \eta_i \downarrow 0 \) and \( N_i \to \infty \). Conclusions \((a)-(c)\) are satisfied with \((q_0, \ldots, q_k, \lambda)\) replaced by some \((q_{0i}, \ldots, q_{ki}, \lambda_i)\). It is important to note that the Euler Lagrange inclusion \((b2)\),
whose right side is evaluated at $\bar{x} \equiv 0$, ensures that the integral bound on the $\dot{q}_j(.)'$s is independent of $i$ (by Gronwall’s Lemma). So we may extract subsequences that are in relevant respects convergent and yield in the limit a multiplier $(q_0, \ldots, q_k, \lambda)$ with the required properties. The Weierstrass condition is now strengthened to allow for

$$e \in F(t, \bar{x}(t)) \cap R(t).$$

The fact that, for each $t \in [0, 1]$, $R(t)$ is an open set is required to satisfy this analysis.

### 5.3 A Free End-Time Optimal Control Problem for Retarded Systems

We consider the following optimal control problem, the distinguishing features of which is the presence of time delays in the dynamic constraint and the fact that the end-time is included in the decision variables:

\[
\begin{align*}
(FT) \quad &\begin{cases}
\text{Minimize } g(x(S), T, x(T)) \\
\text{over intervals } [S, T] \text{ and arcs } x(.) \in W^{1,1}([S, T]; \mathbb{R}^n) \text{ satisfying } \\
\dot{x}(t) \in F(t, x(t-h_0), \ldots, x(t-h_k)), \text{ a.e. } t \in [S,T] \\
x(t) \in E(t), \text{ a.e. } t \in [S-h_k, S] \\
(x(S), T, x(T)) \in C.
\end{cases}
\end{align*}
\]

A feasible arc $(x(.), T)$ comprises a real number $T > S$ and an absolutely continuous function $x(.) : [S, T] \rightarrow \mathbb{R}^n$ satisfying

$$\dot{x}(t) \in F(t, x(t-h_0), \ldots, x(t-h_k)) \text{ a.e. } t \in [S,T] \quad \text{and} \quad (x(S), T, x(T)) \in C,$$

in which $x(s)$, for $s < S$, is in the space of $L^\infty$ functions and satisfies $x(s) \in E(s)$ a.e.. Given a feasible arc $(\bar{x}(.), T)$ we write

$$\bar{z}(t) = (\bar{x}(t-h_0), \ldots, \bar{x}(t-h_k)).$$
We say that a feasible arc \((\bar{x}(\cdot), \bar{T})\) is a local minimizer if

\[
g(x(S), T, x(T)) \geq g(\bar{x}(S), \bar{T}, \bar{x}(T))
\]

for all feasible arcs \((x(\cdot), T)\) such that, for some \(\epsilon > 0\),

\[
d((x(\cdot), T); (\bar{x}(\cdot), \bar{T})) \leq \epsilon
\]

where

\[
d((x(\cdot), T); (x'(\cdot), T')) := \|x - x'\|_{L^\infty(S-h_S, S)} + |x(S) - x'(S)| + \int_0^{T\lor T'} |\dot{x}(t) - \dot{x}'(t)| \, dt + |T - T'|.
\]

The arc \((x(\cdot), T)\) has the interpretation \(\dot{x}(t) := 0\) when \(t > T\).

Define the ‘maximized’ Hamiltonian

\[
\mathcal{H}(t, z, p) = \max_{v \in F(t, z)} p \cdot v.
\]

We invoke the following hypotheses in which \((\bar{x}(\cdot), \bar{T})\) is the arc of interest.

(F1) \(g(\cdot)\) is locally Lipschitz and \(C \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n\) is a closed set.

(F2) Hypothesis (L2). (see page 97).

(F3) \(F(t, z)\) is nonempty and closed for all \((t, z) \in \mathbb{R} \times \mathbb{R}^{n \times (k+1)}\), \(GrF(t, \cdot)\) is closed for each \(t \in \mathbb{R}\) and \(F\) is \(\mathcal{L} \times \mathcal{B}^{n(k+1)}\) measurable. There exist \(k_F(\cdot) \in L^1(S, T)\), \(\beta \geq 0\) and \(\delta > 0\) such that for a.e. \(t \in [S, T]\)

\[
F(t, z') \cap (\dot{x}(t) + N\mathcal{B}) \subset F(t, z) + (k(t) + \beta N)|x - x'|\mathcal{B}
\]

for all \(N \geq 0\) and \(z, z' \in \bar{z}(t) + \delta\mathcal{B}\).

(F4) There exist \(c_F \geq 0\) and \(\sigma > 0\) such that for a.e. \(t \in [\bar{T} - \sigma, \bar{T} + \sigma]\)

\[
\begin{cases}
F(t, z) \subset c_F\mathcal{B}, \\
F(t, z') \subset F(t, z) + c_F|z - z'|\mathcal{B}.
\end{cases}
\]
for all $z, z' \in \bar{z}(\bar{T}) + \delta B$.

**Theorem 5.3.1.** Assume (F1)-(F4) and let $(\bar{x}(\cdot), \bar{T})$ be a local minimizer for (FT) such that $\bar{T} > S$. Then there exist $q_i \in W^{1,1}([S - h_i, \bar{T}]; \mathbb{R}^n)$, $(i = 0, \ldots, k)$, $\lambda \geq 0$ and $\zeta \in \mathbb{R}$ such that

\[
(N) \quad \|q_0\|_{L^\infty} + \lambda > 0.
\]

\[
(T) \quad (T1) \quad (q_0(S) + \ldots + q_k(S), \zeta, -q_0(\bar{T})) \in \\
\lambda \partial g(\bar{x}(S), \bar{T}, \bar{x}(\bar{T})) + NC(\bar{x}(S), \bar{T}, \bar{x}(\bar{T})).
\]

\[
(T2) \quad q_i(t) = 0, \ t \in [\bar{T} - h_i, \bar{T}], \text{ for each } i = 1 \ldots, k.
\]

\[
(E) \quad (E1) \quad -(\dot{q}_1(t)\chi_{[S-h_1,S]}(t) + \ldots + \dot{q}_k(t)\chi_{[S-h_k,S]}(t)) \in \text{co}N_{E(t)}(\bar{x}(t)), \text{ a.e.}
\]

\[
t \in [S - h_k, S]
\]

\[
(E2) \quad (\dot{q}_0(t - h_0), \ldots, \dot{q}_k(t - h_k)) \in \text{co}\left\{ \eta : (\eta, q_0(t) + \ldots + q_k(t)) \in \\
N_{GrF(t, \bar{x})}(\bar{z}(t), \dot{\bar{x}}(t)) \right\}, \text{ a.e. } t \in [S, \bar{T}].
\]

\[
(W) \quad \text{For a.e. } t \in [S, \bar{T}]
\]

\[
\mathcal{H}(t, \bar{z}(t), q_0(t) + \ldots + q_k(t)) = (q_0(t) + \ldots + q_k(t)) \cdot \dot{\bar{x}}(t).
\]

\[
(A) \quad \zeta \in \text{ess}_{t \to \bar{T}} \mathcal{H}(t, \bar{z}(\bar{T}), q_0(\bar{T})).
\]

**Proof.** Observe that when we freeze the end time $T = \bar{T}$ problem (FT) reduces to a standard fixed end-time problem of the type studied in section 5.2. Conclusions (N) to (A) are satisfied except that of the existence of the multiplier $\zeta$. Assertions (T1) and (A) are proved in several stages and by direct perturbation of the end time. To this end we first consider a smooth cost and $C = C_0 \times \mathbb{R} \times \mathbb{R}^n$. The removal of this temporary hypotheses concludes the proof. We adapt techniques developed in [55], Chapter 8.

**Step 1. (Free end points and smooth cost):** Consider the following special case of
We consider variation of the optimal arc $\bar{x}$

\[
\begin{aligned}
&\text{Minimize } g_0(x(S)) + g_1(T, x(T)) + \epsilon g_2(T) \\
&\text{over arcs } (x(\cdot), T) \text{ satisfying}
\end{aligned}
\]

\[
\begin{aligned}
&\dot{x}(t) \in F(t, x(t-h_0), \ldots, x(t-h_k)), \text{ a.e. } t \in [S, T] \\
&x(t) \in E(t), \text{ a.e. } t \in [S-h_k, S), \\
&x(S) \in C_0.
\end{aligned}
\]

where $g_1$ is twice continuously differentiable and $g_0, g_2$ are Lipschitz with Lipschitz constant respectively $k_0$ and $k_2$. The hypotheses on the data are (F2)-(F4) and $C_0 \subset \mathbb{R}^n$ is a closed set. Let $(\bar{x}(\cdot), \bar{T})$ be a local minimizer for this special case. Existence of multipliers $\lambda$ and $q_0(\cdot), \ldots, q_k(\cdot)$ satisfying conditions $(N)$, $(E)$ and $(W)$ is simply implied by Theorem 5.2.1 freezing the final time $T$ at the optimal time $T = \bar{T}$. Theorem 5.2.1 also implies the formulae

\[
-g_0(\bar{T}) = \lambda \nabla_x g_1(\bar{T}, \bar{x}(\bar{T})) \quad \text{and} \quad g_0(S) + \ldots + g_k(S) \in N_{C_0}(\bar{x}(S)) + \lambda \partial g_0(\bar{x}(S)).
\]  

(5.3.1)

We consider variation of the optimal arc $\bar{x}(\cdot)$ around $\bar{T}$ backward and forward in time. Consider the arc $(\bar{x}(\cdot)|_{[S-h_k, \bar{T}-\sigma]}, \bar{T}-\sigma)$ where the notation $\bar{x}(\cdot)|_{[S-h_k, \bar{T}-\sigma]}$ represents the arc $\bar{x}(\cdot)$ restricted to the interval $[S-h_k, \bar{T}-\sigma]$. By optimality we know that $(\bar{x}(\cdot)|_{[S-h_k, \bar{T}-\sigma]}, \bar{T}-\sigma)$ must have cost not less than $(\bar{x}(\cdot), \bar{T})$, i.e.

\[
g_1(\bar{T}-\sigma, \bar{x}(\bar{T}-\sigma)) + \epsilon g_2(\bar{T}-\sigma) \geq g_1(\bar{T}, \bar{x}(\bar{T})) + \epsilon g_2(\bar{T}).
\]  

(5.3.2)

Since $g_1(\cdot, \cdot) \in C^2$ we can apply a Taylor expansion at the point $(\bar{T}, \bar{x}(\bar{T}))$. So there exists $r_1 > 0$ such that for all $\sigma$ sufficiently small

\[
g_1(\bar{T}-\sigma, \bar{x}(\bar{T}-\sigma)) =
\[
g_1(\bar{T}, \bar{x}(\bar{T})) - \nabla_T g_1(\bar{T}, \bar{x}(\bar{T}))\sigma - \nabla_x g(\bar{T}, \bar{x}(\bar{T})) \int_{\bar{T}-\sigma}^{\bar{T}} \dot{x}(t) \, dt + r_1 \sigma^2.
\]

By the optimality condition (5.3.2) and the fact that $-q_0(\bar{T}) = \lambda \nabla_x g(\bar{T}, \bar{x}(\bar{T}))$ we conclude that

\[
\frac{1}{\sigma} \int_{\bar{T}-\sigma}^{\bar{T}} q_0(\bar{T}) \cdot \dot{x}(t) \, dt \geq \lambda \nabla_T g_1(\bar{T}, \bar{x}(\bar{T})) - \lambda k_2 - \lambda r_1 \sigma.
\]
5.3. FREE TIME PROBLEMS

Now we invoke hypothesis (F4) to conclude that, for a.e. \( t \in [\bar{T} - \sigma, \bar{T}] \),

\[
\ddot{z}(t) \in F(t, \bar{z}(t)) \subset F(t, \bar{z}(ar{T})) + c_F|\bar{z}(t) - \bar{z}(ar{T})|\mathbb{B}.
\]

Thus

\[
\frac{1}{\sigma} \int_{\bar{T} - \sigma}^{\bar{T}} \mathcal{H}(t, \bar{z}(\bar{T}), q_0(\bar{T})) \, dt + c_F |q_0(\bar{T})| \frac{1}{\sigma} \int_{\bar{T} - \sigma}^{\bar{T}} |\bar{z}(t) - \bar{z}(\bar{T})| \, dt \geq \lambda \nabla_T g_1(\bar{T}, \bar{z}(\bar{T})) - \lambda \epsilon k_2 - \lambda r_1 \sigma.
\]

which, in the limit, yields

\[
\lambda \nabla_T g(\bar{T}, \bar{z}(\bar{T})) \leq \lim_{\sigma \downarrow 0} \sup_{\bar{T} - \sigma \leq t \leq \bar{T} + \sigma} \mathcal{H}(t, \bar{z}(\bar{T}), q_0(\bar{T})) + \lambda \epsilon k_2. \tag{5.3.3}
\]

Next, we investigate variations of the optimal arc forward in time. To such end we need to extend \( \bar{x}(.) \) on the interval \([\bar{T}, \bar{T} + \sigma] \). Select a measurable function \( \nu(t) : [\bar{T}, \bar{T} + \sigma] \rightarrow \mathbb{R}^n \) such that \( \nu(t) \in F(t, \bar{z}(\bar{T})) \) a.e. \( t \in [\bar{T}, \bar{T} + \sigma] \) and

\[
\mathcal{H}(t, \bar{z}(\bar{T}), q_0(\bar{T})) = q_0(\bar{T}) \cdot \nu(t). \tag{5.3.4}
\]

We take \( \sigma < h_1 \). Define the arc \( y(t) := \bar{x}(t) \) for \( t < \bar{T} \) and \( y(t) := \bar{x}(\bar{T}) + \int_{\bar{T}}^{t} \nu(s) \, ds \) for \( t \in [\bar{T}, \bar{T} + \sigma] \). By Filippov’s Theorem (c.f. [27], Theorem 1) there exists an \( F \)-trajectory \( x(.) \) on \([\bar{T}, \bar{T} + \sigma] \) such that \( x(\bar{T}) = \bar{x}(\bar{T}) \) and

\[
\int_{\bar{T}}^{\bar{T} + \sigma} |\dot{x}(t) - \nu(t)| \, dt \leq c^{(k+1)}c_F \int_{\bar{T}}^{\bar{T} + \sigma} F(t, y(t), \bar{x}(t-h_1), ..., \bar{x}(t-h_k))(\nu(t)) \, dt.
\]

Again, by hypothesis (F4), we have that

\[
\int_{\bar{T}}^{\bar{T} + \sigma} F(t, y(t), \bar{x}(t-h_1), ..., \bar{x}(t-h_k))(\nu(t)) \, dt \leq c_F \int_{\bar{T}}^{\bar{T} + \sigma} |\bar{z}(\bar{T}) - \bar{z}(t)| + |\bar{x}(\bar{T}) - y(t)| \, dt \leq c_F \int_{\bar{T}}^{\bar{T} + \sigma} |\bar{z}(\bar{T}) - \bar{z}(t)| \, dt + (c_F \sigma)^2.
\]

At this point we carry out an analysis similar to the ‘backward’ case. From the optimality
of the arc \((\bar{x}(\cdot), \bar{T})\) it follows that

\[ g_1(\bar{T} + \sigma, x(\bar{T} + \sigma)) + \epsilon g_2(\bar{T} + \sigma) \geq g_1(\bar{T}, \bar{x}(\bar{T})) + \epsilon g_2(\bar{T}). \]

and by Taylor expansion we obtain

\[
\frac{1}{\sigma} \int_{\bar{T}}^{\bar{T} + \sigma} q_0(\bar{T}) \cdot \dot{x}(t) \, dt \leq \lambda \nabla_T g_1(\bar{T}, \bar{x}(\bar{T})) + \lambda r_1 \sigma + \lambda \epsilon k_2.
\]

Because of our choice of \(\nu(.)\) (see (5.3.4)) the following holds true

\[
\int_{\bar{T} + \sigma}^{\bar{T}} q_0(\bar{T}) \cdot \dot{x}(t) \, dt \geq \int_{\bar{T} + \sigma}^{\bar{T} + \sigma} \mathcal{H}(t, \bar{z}(\bar{T}), q_0(\bar{T})) \, dt - |q_0(\bar{T})| c_{\sigma(k+1)} c_F c_F \left\{ \int_{\bar{T}}^{\bar{T} + \sigma} |\bar{z}(\bar{T}) - \bar{z}(t)| \, dt + c_F \sigma^2 \right\}.
\]

The above calculations combine together to give

\[
\lim_{\sigma \downarrow 0} \quad \text{ess inf}_{T - \sigma \leq t \leq T + \sigma} \quad \mathcal{H}(t, \bar{z}(\bar{T}), q_0(\bar{T})) \leq \lambda \nabla_T g(\bar{T}, \bar{x}(\bar{T})) + \lambda \epsilon k_2
\]

which together with (5.3.3) implies

\[
\lambda \nabla_T g(\bar{T}, \bar{x}(\bar{T})) \in \text{ess sup}_{t \rightarrow \bar{T}} \mathcal{H}(t, \bar{z}(\bar{T}), q_0(\bar{T})) + \lambda \epsilon k_2 \mathbb{B}.
\]

\[ (5.3.5) \]

**Step 2. (Lipschitz cost):** Take a Lipschitz function \(\tilde{g}(.)\) with Lipschitz constant \(k_{\tilde{g}}\) and a closed set \(\tilde{C} \subset \mathbb{R}^n \times \mathbb{R}\) and consider the optimization problem

\[
\left\{ \begin{array}{l}
\text{Minimize } \tilde{g}(x(S), T, x(T)) \\
\text{over arcs } (x(\cdot), T) \text{ satisfying}
\end{array} \right.
\]

\[
\dot{x}(t) \in F(t, x(t - h_0), \ldots, x(t - h_k)), \text{ a.e. } t \in [S, T]
\]

\[
x(t) \in E(t), \text{ a.e. } t \in [S - h_k, S),
\]

\[
(x(S), T) \in \tilde{C}.
\]

Once again hypotheses on the data are (F2)-(F4) and we assume that \((\bar{x}(\cdot), \bar{T})\) is a local
 Indeed, take any arc \((T, \tau, x, y)\) := \(\bar{g}(x(S), \tau(S), y(S)) + i \ (|T - \tau(T)|^2 + |x(T) - y(T)|^2)\),
and consider the new optimization problem

\[
\minimize J_i(T, \tau, x, y, y) \quad \text{over arcs } (x, T) \quad \text{and } (\tau, y) \in \mathbb{R} \times \mathbb{R}^n \text{ s.t.}
\]

\[
(\dot{\tau}(t), \dot{x}(t), \dot{y}(t)) \in \{0\} \times F(t, x(t - h_0), \ldots, x(t - h_k)) \times \{0\}, \quad \text{a.e.}
\]

\[
x(t) \in E(t), \text{ a.e. } t \in [S - h_k, S),
\]

\[
(x(S), \tau(S)) \in \bar{C},
\]

\[
d((x, T); (\bar{x}, \bar{T})) + |\tau - \bar{\tau}| + |y - \bar{y}| \leq \delta'.
\]

Minimization is performed over arcs \((T, \tau, x, y)\) \(\in \mathcal{W}\) where \(\mathcal{W}\) represents the feasible arcs \((T, \tau, x, y)\) satisfying the constraints in \((P)\). \(\tau(\cdot)\) and \(y(\cdot)\) are constants and we simply write \(\tau\) and \(y\). We define a metric on \(\mathcal{W}\) as follows

\[
\|(\tau, x, y, T)\|_\mathcal{W} := \|x\|_{L^1(S - h_k, S)} + |x(S)| + \int_0^{T' \vee T} |\dot{x}(t)| \ dt + |T| + |\tau| + |y|.
\]

The functional \(J_i(\cdot)\) is continuous on \((\mathcal{W}, \|\cdot\|_\mathcal{W})\) and \((\mathcal{W}, \|\cdot\|_\mathcal{W})\) is a complete metric space. We claim that the feasible arc \((\bar{T}, \bar{T}, \bar{x}(\cdot), \bar{x}(\bar{T}))\) \(\in \mathcal{W}\) is an \(\epsilon_i^2\)-minimizer for \((P)\) for some sequence \(\epsilon_i \downarrow 0\) as \(i \to +\infty\), one for which Ekeland’s Theorem can be applied, i.e.,

\[
J_i(\bar{T}, \bar{T}, \bar{x}(\cdot), \bar{x}(\bar{T})) \leq \inf_{(T, \tau, x, y) \in \mathcal{W}} J_i(T, \tau, x, y, y) + \epsilon_i^2.
\]

Indeed, take any arc \((T, \tau, x, y) \in \mathcal{W}\), then

\[
J_i(T, \tau, x, y, y) \geq \bar{g}(x(S), \tau, x(\tau)) - k_{\overline{g}} |x(\tau) - y| + i \ (|T - \tau|^2 + |x(T) - y|^2) \geq
\]

\[
\bar{g}(\bar{x}(S), \bar{T}, \bar{x}(\bar{T})) - k_{\overline{g}} |x(\tau) - x(T)| - k_{\overline{g}} |x(T) - y| + i \ (|T - \tau|^2 + |x(T) - y|^2) \geq
\]

\[
\bar{g}(\bar{x}(S), \bar{T}, \bar{x}(\bar{T})) - \frac{k_{\overline{g}}^2(1 + \epsilon_i^2)}{4i}.
\]

The claim is proven defining \(\epsilon_i^2 := k_{\overline{g}}^2(1 + \epsilon_i^2)/(4i)\). Ekeland’s Theorem furnishes a sequence \((T_i, \tau_i, x_i(\cdot), y_i)\) such that \(\tau_i \to \bar{T}, T_i \to \bar{T}, y_i \to \bar{y}(\bar{T})\) and \(x_i(\cdot) \to \bar{x}(\cdot)\) in the topology induced by the metric \(\|\cdot\|_\mathcal{W}\), that is \(W^{1,1}\) in \([S, T]\) and \(L^1\) in \([S - h_k, S]\). Observe that
**Chapter 5. Euler-Lagrange Conditions for Delayed Systems**

A priori $x_i(\cdot)$ and $\bar{x}(\cdot)$ are defined on two different intervals. The $W^{1,1}$ convergence is intended with respect to the extensions of $x_i(\cdot)$ and $\bar{x}(\cdot)$ on the whole interval $[S, T_i \lor \bar{T}]$ as $x_i(t) := x_i(T_i)$ for any $t > T_i$ and vice versa $\bar{x}(t) := \bar{x}(\bar{T})$ for any $t > \bar{T}$. Furthermore for each $i \in \mathbb{N}$, $(T_i, \tau_i, x_i(\cdot), y_i) \in \mathcal{W}$ minimizes the functional

$$
g(x(S), \tau(S), y(S)) + \epsilon_i |\tau(S) - \tau_i(S)| + \epsilon_i |y(S) - y_i(S)| + \epsilon_i |x(S) - x_i(S)| +
$$

$$
i \left( |T - \tau(T)|^2 + |x(T) - y(T)|^2 \right) + \epsilon_i \int_{S-h_k}^{S} |x(t) - x_i(t)| \, dt
$$

$$
\epsilon_i |T - T_i| + \epsilon_i \int_{T}^{T \lor T} |\dot{x}(t) - \dot{x}_i(t)| \, dt + \epsilon_i \int_{S}^{T} |\dot{x}_i(t)| \, dt.
$$

over arcs $(T, \tau, x(\cdot), y) \in \mathcal{W}$. We are in a situation already studied in step 1. The integral cost $\epsilon_i \int_{S}^{T} |\dot{x}(t) - \dot{x}_i(t)| \, dt + \epsilon_i \int_{S-h_k}^{S} |x(t) - x_i(t)| \, dt$ is eliminated by a standard state augmentation technique. In particular

$$
\int_{S-h_k}^{S} |x(t) - x_i(t)| \, dt = \int_{S}^{T} |x(t - h_k) - x_i(t - h_k)| \cdot \chi[S,S+h_k](t) \, dt.
$$

There exist $q_{ji} \in W^{1,1}(S-h_j, T_i)$, $(j = 0, \ldots, k)$, $\zeta_i \in \mathbb{R}$, $r_i \in \mathbb{R}^n$ and $\lambda_i \geq 0$ such that (recall [9.2.4] and [9.2.5])

(a) $\|q_{0i}\|_{L^\infty} + |r_i| + |\zeta_i| + \lambda_i = 1.$

(b) $q_{ji}(t) = 0$, $t \in [T_i - h_j, T_i]$, for each $j = 1 \ldots, k$.

(c) $(q_{0i}(t-h_0), \ldots, q_{ki}(t-h_k)) \in \text{co} \left\{ \eta : (\eta, q_{0i}(t) + \ldots + q_{ki}(t)) \in N_{GrF(t, \cdot)} (x_i(t-h_0), \ldots, x_i(t-h_k), \dot{x}_i(t)) + \{(0, \ldots, 0, \lambda_i \epsilon_i \chi[S,S+h_k](t))\} \times \lambda_i \epsilon_i \mathbb{B} \right\}$, a.e. $t \in [S, T_i]$.

(d) For a.e. $t \in [S, T_i]$ and all $v \in F(t, x_i(t-h_0), \ldots, x_i(t-h_k))$

$$
(q_{0i}(t) + \ldots + q_{ki}(t)) \cdot v - \lambda_i \epsilon_i |v - \dot{x}_i(t)| \leq (q_{0i}(t) + \ldots + q_{ki}(t)) \cdot \dot{x}_i(t).
$$

(e) $-q_{0i}(T_i) = 2i \lambda_i (x_i(T_i) - y_i), \quad -\zeta_i = 2i \lambda_i (\tau_i - T_i), \quad r_i = q_{0i}(T_i),$

$$
(q_{0i}(S) + \ldots + q_{ki}(S), \zeta_i, r_i) \in \lambda_i \partial \tilde{g}(x_i(S), \tau_i, y_i) + N_{\tilde{C}}(x_i(S), \tau_i) \times \{0\} + \lambda_i \epsilon_i \mathbb{B}.
$$

(f) $2i \lambda_i (T_i - \tau_i) \in \text{ess} \sup_{t \rightarrow T_i} \mathcal{H}(t, x_i(T_i-h_0), \ldots, x_i(T_i-h_k), q_{0i}(T_i)) + \lambda_i \epsilon_i (1 + c_F) \mathbb{B}$.
Condition \((a_i)\) and \((c_i)\) and Gronwall’s Lemma \([5.2.4]\) ensure a uniform bound for \(\dot{q}_0(\cdot), \ldots, \dot{q}_k(\cdot)\). Thus, there exist \(q_0(\cdot), \ldots, q_k(\cdot), \lambda \geq 0\) and \(\zeta \in \mathbb{R}\) such that \(\dot{q}_0(\cdot), \ldots, \dot{q}_k(\cdot)\) converge weakly in \(L^1\) to \(\dot{q}_0(\cdot), \ldots, \dot{q}_k(\cdot), \lambda_i \rightarrow \lambda\) and \(\zeta_i \rightarrow \zeta\) and

\[
\begin{align*}
(a)\ &\|q_0\|_{L^\infty} + \lambda = 1, \\
(b)\ &q_j(t) = 0, \ t \in [\bar{T} - h_j, \bar{T}], \ for \ each \ j = 1, \ldots, k. \\
(c)\ &\dot{q}_0(t - h_0), \ldots, \dot{q}_k(t - h_k) \in co\{\eta : (\eta, q_0(t) + \ldots + q_k(t)) \in N_{GrF(t,)}(\bar{x}(t - h_0), \ldots, \bar{x}(t - h_k), \bar{x}(t))\}, \ a.e. \ t \in [S, \bar{T}]. \\
(d)\ &For \ a.e. \ t \in [S, \bar{T}]
\end{align*}
\]

\[
\mathcal{H}(t, \bar{x}(t), q_0(t) + \ldots + q_k(t)) = (q_0(t) + \ldots + q_k(t)) \cdot \dot{\bar{x}}(t).
\]

\[
(e)\ &(q_0(S) + \ldots + q_k(S), \zeta_0(\bar{T})) \in \lambda \partial G(\bar{x}(S), \bar{T}, \bar{x}(\bar{T})) + N_\mathcal{C}(\bar{x}(S), \bar{T}) \times \{0\}. \\
(f)\ &\zeta \in ess\mathcal{H}(t, \bar{x}(\bar{T}), q_0(\bar{T})).
\]

**Step 3. (General end-point constraints):** We are ready to prove Theorem \([5.3.1]\) in its full generality. We recall the optimization problem (FT)

\[
\begin{align*}
\text{Minimize } g(x(S), T, x(T)) \\
\text{over intervals } [S, T] \text{ and arcs } x(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n) \text{ satisfying} \\
\dot{x}(t) \in F(t, x(t - h_0), \ldots, x(t - h_k)) \text{ a.e. } t \in [S, T] \\
x(t) \in E(t) \text{ a.e. } t \in [S - h_k, S] \\
(x(S), T, x(T)) \in C,
\end{align*}
\]

where the feasible arc \((\bar{x}(\cdot), \bar{T})\) is a local minimizer. Introduce a new state trajectory \(y(\cdot)\) governed by \(\dot{y}(t) = 0\) and take a sequence \(\epsilon_i \downarrow 0\) as \(i \rightarrow +\infty\) and define

\[
g_i(x(S), y(S), T, x(T), y(T)) := \max\{g(x(S), T, y(S)) - g(\bar{x}(S), \bar{T}, \bar{x}(\bar{T})) + \epsilon_i^2, |y(T) - x(T)|\}.
\]
Observe that $g_i(\bar{x}(S), \bar{x}(T)) = \epsilon_i^2$ and that for any feasible arc $(x(.), T)$ and $y$, $g_i(x(S), y(S), T, x(T), y(T)) \geq 0$. According to Ekeland’s variational principle there exists of a sequence $(y_i, x_i(.), T_i)$ $(x_i \to \bar{x}$ in $W^{1,1}$ on $[S, T]$ and in $L^1$ on $[S - h_k, S]$, $T_i \to T$, $y_i \to \bar{x}(T)$) such that for each $i \in \mathbb{N}$, $(y_i, x_i(.), T_i)$ minimizes the functional

$$g_i(x(S), y(S), T, x(T), y(T)) + \epsilon_i \int_{S-h_k}^S |x(t) - x_i(t)| \, dt +$$

$$\epsilon_i |x(S) - x_i(S)| + \epsilon_i |y(S) - y_i(S)| + \epsilon_i \int_0^{T>T_i} |\dot{x}(t) - \dot{x}_i(t)| \, dt + \epsilon_i |T - T_i|$$

over feasible arcs $(y, x(.), T)$. According to step 2 there exist $q_{ji} \in W^{1,1}(S - h_j, T_i)$, $(j = 0, \ldots, k)$, $\zeta_i \in \mathbb{R}$, $r_i \in \mathbb{R}^n$ and $\lambda_i \geq 0$ satisfying conditions (a)-(f) of step 2:

$(A_i)$ $\|q_{0i}\|_{L^\infty} + |r_i| + \lambda_i = 1$.

$(B_i)$ $q_{ji}(t) = 0$, $t \in [T_i - h_j, T_i]$, for each $j = 1, \ldots, k$.

$(C_i)$ $(\bar{q}_0(t - h_0), \ldots, \bar{q}_k(t - h_k)) \in \text{co}\{\eta : (\eta, q_{00}(t) + \ldots + q_{ki}(t)) \in N_{G_{t_i}}(x_i(t - h_0), \ldots, x_i(t - h_k), \dot{x}_i(t)) + \{(0, \ldots, 0, \lambda_i \epsilon_i \chi_{[S, S+h_k]}(t)) \times \lambda_i \epsilon_i \mathbb{B}\}$, a.e. $t \in [S, T_i]$.

$(D_i)$ For a.e. $t \in [S, T_i]$ and all $v \in F(t, x_i(t - h_0), \ldots, x_i(t - h_k))$

$$(q_{00}(t) + \ldots + q_{ki}(t)) \cdot v - \lambda_i \epsilon_i |v - \dot{x}_i(t)| \leq (q_{00}(t) + \ldots + q_{ki}(t)) \cdot \dot{x}_i(t).$$

$(E_i)$ We define $\tilde{C} := \{(x, y, T) : (x, T, y) \in C\}$. Then

$$(q_{00}(S) + \ldots + q_{ki}(S), r_i, \zeta_i, -q_{00}(T_i), -r_i) \in$$

$$\lambda_i \partial q_i(x_i(S), y_i(S), T_i, x_i(T_i), y_i(T)) + N_{\tilde{C}}(x_i(S), y_i(S), T_i) \times \{(0, 0)\}.$$

$(F_i)$ $\zeta_i \in \text{ess} H(t, x_i(T_i - h_0), \ldots, x_i(T_i - h_k), q_{00}(T_i)) + \lambda_i \epsilon_i (1 + c_F) \mathbb{B}.$

Observe that, for $i$ sufficiently large,

$$g_i(x_i(S), y_i(S), T_i, x_i(T_i), y_i(T)) > 0$$
5.4. THE MAXIMUM PRINCIPLE

This implies what follows: say that \( \bar{g}(x, y, T) := g(x, T, y) \) then

\[
\partial g_i(x_i(S), y_i(S), T_i, x_i(T_i), y_i(T_i)) \subset \gamma_i \partial \bar{g}(x_i(S), y_i(S), T_i) \times \{(0, 0)\} + (1 - \gamma_i) \{(0, 0, 0, e, -e) : |e| = 1\}
\]

for some \( \gamma_i \in [0, 1] \). So from \( (E_i) \) above \( r_i = -q_{0i}(T_i) \) and \( |r_i| = 1 - \gamma_i \). Thus

\[
(q_0(S) + \ldots + q_k(S), \zeta_i, -q_{0i}(T_i)) \in \lambda_i \gamma_i \partial g(x_i(S), T_i, y_i) + NC(x_i(S), T_i, y_i)
\]

and Theorem 5.3.1 follows in the limit.

5.4 The Maximum Principle

The results obtained lead naturally to a form of the maximum principle of Pontryagin (when the dynamic is formulated in terms of a retarded differential equation with control) by imposing temporary additional hypotheses on the data of the original optimal control problem. The assertions of the non-smooth maximum principle are deduced from the necessary conditions for the differential inclusion problem. It can be shown (c.f. [55, Chapter 6]) that the temporary additional hypotheses can be imposed without loss of generality, and can therefore be discarded. This is presented in detailed in the next chapter.
Chapter 6

Free time optimal control problems with time delays

Solutions to optimal control problems for retarded systems\footnote{The material present in this chapter is the subject matter of the publication [9]} on a fixed time interval, satisfy a form of the Maximum Principle, in which the co-state equation is an advanced differential equation. In this chapter we present an extension of this well-known necessary condition of optimality, to cover situations in which the data is non-smooth, and the final time is free. The fact that the end-time is a choice variable is accommodated by an extra transversality condition. A traditional approach to deriving this extra condition is to reduce the free end-time problem to a fixed end-time problem by a parameterized change of the time variable. This approach is problematic for time delay problems because it introduces a parameter dependent time-delay that is not readily amenable to analysis; to avoid this difficulty we instead base our analysis on direct perturbation of the end-time. Formulae are derived for the gradient of the minimum cost as a function of the end-time. It is shown how these formulae can be exploited to construct two-stage algorithms for the computation of solutions to optimal retarded control problems with free-time, in which a sequence of fixed time problems are solved by means of Guinn’s transformation, and the end-time is adjusted according to a rule based on the earlier derived gradient formulae for the minimum cost function. Numerical examples are presented.
6.1 Literature Review

This chapter concerns optimal control problems involving retarded systems. A notable feature of the problems considered is that the end-time is a decision variable. Our aim is to provide necessary conditions of optimality, sensitivity information and numerical schemes for the solution of such ‘free end-time’ problems.

Within a traditional delay-free context, optimal control problems with free end-time can be reformulated as standard optimal control problems on a fixed time interval, by means of a parameterized transformation of the time variable. Optimality conditions for free end-time problems can be obtained from those fixed end-time problems by applying the latter to the reformulated problem. Likewise, computational schemes for fixed end-time problems translate into schemes for free end-time problems, via the transformation. In this sense, the study of free end-time problems is a footnote to fixed end-time analysis.

For retarded systems, however, reduction of free end-time problems to fixed end-time problems, for purposes of deriving optimality conditions and of computing minimizers, is problematic. This is because the time transformation, designed to fix the end-time, gives rise to a non-standard optimal control problem for retarded systems with state dependent time-delays, which are difficult to analyse.

In this chapter, we describe an alternative approach to the analysis of free end-time optimal control problems, which does not depend on a transformation of the time variable. Instead we use techniques based on direct perturbation of the end-time. The end result is a modified transversality condition, providing the extra information about the optimal end-time, expressed in terms of the ‘essential value’ of the maximized Hamiltonian. These techniques were originally used to derive optimality conditions for delay-free optimal control problems with measurably time-dependent data [26]. They are used here, for retarded systems, apparently for the first time.

The implications of the transversality condition for the computation of optimal control problems are then explored. We shall see that this transversality condition leads to formulae for the sensitivity of the minimum cost (of the optimal control problem on
a fixed time interval) to perturbations of the end-time. An algorithm for the solution of free end-time problems is proposed, in which solutions to optimal control problems on a sequence of time intervals $[0, T_i]$ are computed and the end-times $T_i$ are adjusted according to a gradient descent scheme based on the sensitivity formulae. Numerical experiments demonstrate the superior performance of algorithms which make use of the sensitivity formulae, as compared with algorithms based on numerical approximation of gradients of the final time value function.

The analytic underpinnings of the chapter is a version of the Pontryagin maximum principle for free end-time optimal control problems involving retarded systems and with non-smooth data, stated in Section 6.3. In our formulation, time delays occur only in the state variables. The time delays are of finite number and of fixed length. The initial state segment is fixed. We make some comments on the relation of this optimality condition with those in the extensive literature on optimal control of retarded systems, which includes the papers [27], [28], [43], [47] and Warga’s monograph [58]. For an early survey, see [3].

Earlier non-smooth necessary conditions for time delay problems, in which the dynamic constraint has the form of a retarded differential inclusion, are due to Clarke et al. [27], [28]. The optimality conditions of these papers take the form of a Hamiltonian inclusion. This can be applied to derive a maximum principle for problems in which dynamic constraint takes the form of delay differential equation, but this has been shown only in the case that the differential equation depends smoothly on the state variable. Our necessary condition is derived, by contrast, from a generalization of the Euler Lagrange condition, and this leads to a maximum principle allowing non-smooth dependence of the differential equation on the state variable. (We mention that, in other respects, [28] addresses a more general class of problems than that considered here, in which the initial state trajectory segment is a choice variable and the delays are distributed.) Morduckhovich had earlier derived generalization to time delay problems of the Euler Lagrange condition. But Morduckhovich’s condition is not accompanied by the Weierstrass condition (as in our result), which is an important supplement to the necessary conditions, nor are the implications for problems in which the dynamic constraint takes
the form of a delay differential equation explored. Warga [58] has shown how a wide range of optimal control problems involving time delays can be fitted to a general optimization framework, allowing non-smoothness. None of this earlier ‘non-smooth’ literature treats free end-times, the key feature of the optimal control problems in this study. Necessary conditions for smooth retarded optimal control problems with free end-time have earlier been derived by Kharatishvili and Tadumadze [43], in the form of a ‘one-sided’ transversality condition relating to the free end-time. By contrast, we present a full ‘two-sided’ transversality condition. The two sided condition is an essential tool in the derivation of our sensitivity formulae and in the construction of the proposed computational scheme.

Detailed analysis, including proofs of the optimality conditions and derivation of the sensitivity relations are presented in chapter 5.

Notation: In Euclidean space, the length of a vector $x$ is denoted by $|x|$, and the closed unit ball $\{x \mid |x| \leq 1\}$ by $B$. $d_D(x)$ denotes the Euclidean distance of the point $x$ from a given closed set $D$, namely $\min\{|x-x'| \mid x' \in D\}$. $\chi_D(.)$ denotes the indicator function of $D$, which takes the value 1 on $D$ and 0 on its complement. $\text{co} F$ denotes ‘closed convex hull’ of a set $F \subset \mathbb{R}^k$. $W^{1,1}([0,T];\mathbb{R}^n)$ (sometimes written $W^{1,1}$) denotes the set of absolutely continuous $\mathbb{R}^n$ valued functions on $[0,T]$.

6.2 A Free End-Time Optimal Control Problem for Retarded Systems

We consider the following optimal control problem, the distinguishing features of which is the presence of time delays in the dynamic constraint and the fact that the end-time is
6.3. A MAXIMUM PRINCIPLE

Minimize \( g(T, x(T)) \)
over intervals \([0, T]\), measurable functions
\( u(.) : [0, T] \to \mathbb{R}^m \) and arcs \( x(.) \in W^{1,1}([0, T]; \mathbb{R}^n) \)
satisfying
\[
\dot{x}(t) = f(t, x(t - h_0), \ldots, x(t - h_k), u(t)), \quad a.e.
\]
\( u(t) \in U(t), \quad a.e. \ t \in [0, T] \)
\( x(t) = \phi(t), \quad t \in [-h_k, 0] \)
\( (T, x(T)) \in C. \)

Here \( h_0 = 0 < h_1 < \ldots < h_k \) are given real numbers, \( g(.) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \),
\( \phi(.) : [-h_k, 0] \to \mathbb{R}^n \) and \( f : \mathbb{R} \times \mathbb{R}^{n \times (k+1)} \times \mathbb{R}^m \to \mathbb{R}^n \) are given functions and \( C \subset \mathbb{R} \times \mathbb{R}^n \) a given set. \( U(.) : \mathbb{R} \prec \mathbb{R}^m \) is a given multi-function.

A process \((x(.), u(.), T)\) comprises a real number \( T > 0 \), an absolutely continuous function
\( x(.) : [0, T] \to \mathbb{R}^n \) and a measurable function \( u(.) : [0, T] \to \mathbb{R}^m \) satisfying
\[
\dot{x}(t) = f(t, x(t - h_0), \ldots, x(t - h_k), u(t)) \quad \text{and} \quad u(t) \in U(t) \text{ a.e.,}
\]
in which \( x(s) \), for \( s < 0 \), has the interpretation \( x(s) = \phi(s) \). The first and second
components of processes are called state trajectories and control functions. A process
\((x(.), u(.), T)\) is said to be feasible if \((T, x(T)) \in C. \)

A feasible process \((\bar{x}(.), \bar{u}(.), \bar{T})\) is a minimizer if it achieves the minimum of \( g(T, x(T)) \)
over all admissible processes \((x(.), u(.), T)\).

6.3 A Maximum Principle

We invoke the following hypotheses (in which \((\bar{x}(.), \bar{u}(.), \bar{T})\) is the process of interest).

Here, and below, we adhere to the following notational convention: given any state trajectory \( x(.) \) on \([0, T] \) we extend it to \([-h_k, T]\), imposing the value \( x(t) = \phi(t) \) for \( t < 0 \),
where $\phi(.)$ is the specified ‘past history’ function of the state. We write
\[
\bar{z}(t) := (\bar{x}(t - h_0), \ldots, \bar{x}(t - h_k)).
\]

For some $\delta > 0$,

(H1) $g(., .)$ is locally Lipschitz continuous, $\phi(.) \in L^\infty(-h_k, 0)$ and $C$ is closed. $U(.)$ has Borel measurable graph.

(H2) $f(., z, .)$ is $L \times B$ measurable for each $z \in \mathbb{R}^n \times (k+1)$ ($L$ and $B$ denote, respectively, the $\sigma$-algebras of Lebesgue subsets of $\mathbb{R}$ and of Borel subsets of $\mathbb{R}^m$.) There exists $k(., .): [0, \bar{T}] \times \mathbb{R}^m \to \mathbb{R}$ such that $t \mapsto k(t, \bar{u}(t))$ is integrable and
\[
|f(t, z, u) - f(t, z', u)| \leq k(t, u)|z - z'|
\]
for all $z, z' \in \bar{z}(t) + \delta B$ and $u \in U(t)$, a.e. $t \in [0, \bar{T}]$.

(H3) There exist $c_f \geq 0, k_f \geq 0$ such that
\[
|f(t, z, u)| \leq c_f \quad \text{and} \quad |f(t, z, u) - f(t, z', u)| \leq k_f|z - z'|
\]
for all $z, z' \in \bar{z}(T) + \delta B$ and $u \in U(t)$, a.e. $t \in [\bar{T} - \delta, \bar{T} + \delta]$.

Define the Hamiltonian
\[
H(t, z = (x_0, \ldots, x_k), p, u) := p \cdot f(t, x_0, \ldots, x_k, u).
\]
and write $\mathcal{H}(t, z, p) = \max_{u \in U(t)} H(t, z, p, u)$ (the ‘maximized’ Hamiltonian).

**Theorem 6.3.1.** Assume (H1)-(H3). Let $(\bar{x}(.), \bar{u}(.), \bar{T})$ be a minimizer for (D). Then there exist $p_i(.) \in W^{1,1}([-h_i, \bar{T}]; \mathbb{R}^n)$, $i = 0, \ldots, k$, and $\lambda \geq 0$ such that $(\lambda, p_0(.) \neq 0 and

(i) (a): for a.e. $t \in [0, \bar{T}]$
\[
- (\ddot{p}_0(t - h_0), \ldots, \ddot{p}_k(t - h_k)) + \mathcal{H}(t, \bar{z}(t), p_0(t) + \ldots + p_k(t), \bar{u}(t)) \in \text{co} \partial_z H(t, z(t), p_0(t) + \ldots + p_k(t), \bar{u}(t))
\]

(b): $p_i(t) = 0$, for $t \in [T - h_i, \bar{T}]$, $i = 1, \ldots, k.$
6.3. A MAXIMUM PRINCIPLE

(ii) for a.e. \( t \in [0, \bar{T}] \),

\[
H(t, \bar{z}(t), p_0(t) + \ldots + p_k(t), \bar{u}(t)) = \max_{u \in U(t)} H(t, \bar{z}(t), p_0(t) + \ldots + p_k(t), u).
\]

(iii) for some \( \zeta \in \text{ess}_{t \to \bar{T}} \mathcal{H}(t, \bar{z}(\bar{T}), p_0(\bar{T})) \)

\[
(\zeta, -p_0(\bar{T})) \in \lambda \partial g(\bar{T}, \bar{x}(\bar{T})) + N_C(\bar{T}, \bar{x}(\bar{T})�).
\]

**Sketch of Proof.** Full details of the proof of Thm. 6.3.1 are given in chapter 5. We recall a broad outline of the proof. First necessary conditions are proved for a smooth, end-point constraint free problem in the calculus of variations with time delays, over a fixed time interval. The proof hinges on Clarke’s decoupling technique, in which state trajectories are replaced by augmented control functions. Ekeland’s theorem and inf-convolutions are employed to allow for a non-smooth cost. Exact penalty techniques from [40] are then used to prove necessary conditions for problems in which the dynamic constraint takes the form of a differential inclusion, in the form of a generalized Euler Lagrange condition, Weierstrass condition and transversality condition. The necessary conditions are generalized to allow for free end-times, by means of analysis based on direct perturbation of the end-time. Imposing temporary additional hypotheses on the data of the original optimal control problem (formulated in terms of a retarded differential equation with control), we interpret this problem as a special case of the differential inclusion problem. The assertions of the non-smooth maximum principle are deduced from the necessary conditions for the differential inclusion problem. It is finally shown that the temporary additional hypotheses can be imposed without loss of generality, and can therefore be discarded.

\[\square\]

In the case when the cost function and also the functions defining the dynamic constraints are \( C^1 \) functions w.r.t. the state and ‘delayed state’ variables, the Maximum Principle takes a simpler form, expressed no longer in terms of the collection of functions \( p_i(\cdot), \)

\( i = 1, \ldots, k \), but a single costate function \( p(\cdot) \) satisfying an advanced differential equation.
$p(\cdot)$ is related to the $p_i(\cdot)$'s according to

$$p(t) = \sum_{i=0}^{k} p_i(t) \quad \text{for } t \in [0, \bar{T}].$$

**Corollary 6.3.1.** Let $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{T})$ be a minimizer for (D). Assume, in addition to (H1)-(H3), that

$(H4)$: $z \to f(t, z, \bar{u}(t))$ is $C^1$ near $\bar{z}(t)$ a.e., the multifunction $t \to f(t, \bar{z}(t), U(t))$ is continuous at $\bar{T}$ and $g(\cdot, \cdot)$ is $C^1$ near $(\bar{T}, \bar{x}(\bar{T}))$.

(a): Smooth Free-time Maximum Principle. There exist $p(\cdot) \in W^{1,1}([0, \bar{T}]; \mathbb{R}^n)$ and $\lambda \geq 0$ such that $(\lambda, p(\cdot)) \neq (0, 0)$ and

$$(i) \quad -\dot{p}(t) = \sum_{i=0}^{k} \chi_{[0,T-h_i]}(t)p(t+h_i)\nabla_x f(t+h_i, \bar{z}(t+h_i), \bar{u}(t+h_i)), \ a.e. \ t \in [0, \bar{T}].$$

$$(ii) \quad H(t, \bar{z}(t), p(t), \bar{u}(t)) = \max_{u \in U(t)} H(t, \bar{z}(t), p(t), u) \ a.e..$$

$$(iii) \quad (H(\bar{T}, \bar{z}(\bar{T}), p(\bar{T})), -p(\bar{T})) = \lambda \nabla g(\bar{T}, \bar{x}(\bar{T})) + \eta, \ for \ some \ \eta \in N_C(\bar{T}, \bar{x}(\bar{T})).$$

(b): Smooth Fixed-time Maximum Principle. If, further, $C = \{\bar{T}\} \times \bar{C}$ (for some fixed $\bar{T}$ and $\bar{C} \subset \mathbb{R}^n$), then there exist $p(\cdot) \in W^{1,1}$ and $\lambda \geq 0$, $(p(\cdot), \lambda) \neq 0$, satisfying conditions (i)-(ii) above and, in place of (iii), the condition

$$(iii') \quad -p(\bar{T}) = \lambda \nabla_x g(\bar{T}, \bar{x}(\bar{T})) + \eta, \ for \ some \ \eta \in N_C(\bar{T}, \bar{x}(\bar{T})).$$

### 6.4 Sensitivity Relations

Denote by (D$_T$) the ‘fixed end-time’ case of (D), in which

$$C = \{\bar{T}\} \times \bar{C} \ for \ some \ \bar{C} \subset \mathbb{R}^n \ and \ g(\cdot, \cdot) \ is \ C^1. \quad (6.4.1)$$

Now $T(>0)$ is interpreted as a parameter in the fixed end-time problem.

We provide formulae governing changes in minimum cost when the end-time varies. These formulae will be used later to construct algorithms for solving the free end-time problem.
via a sequence of simpler, fixed end-time problems. Define the end-time value function $V(.)$ to be the extended valued function on $(0, \infty)$:

$$V(T) := \inf \{ g(T, x(T)) | (x(\cdot), u(\cdot), T) \text{ is an admissible process for } D_T \}.$$ 

The following proposition relates Clarke subgradients $\text{co} \partial V(\bar{T})$ and multipliers in the Maximum Principle:

**Proposition 6.4.1.** Take $\bar{T} > 0$ and consider the $T$-parameterized family of fixed time problems $(D_T)$ defined by $(D)$ under assumption in (6.4.1). Assume hypotheses (H1)-(H3). Let $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{T})$ be a minimizer for $(D_{\bar{T}})$. Assume that

(R): $V(.)$ is Lipschitz continuous on a neighborhood of $\bar{T}$.

Then there exist $p_i(\cdot) \in W^{1,1}([-h_i, \bar{T}]; \mathbb{R}^n)$, $i = 0, \ldots, k$ and $\lambda \geq 0$, $(p_0(\cdot), \lambda) \neq 0$, satisfying conditions (i) and (ii) of Thm. 6.3.1 as well as the condition

(iii') (a): $-p_0(\bar{T}) = \lambda \nabla x g(\bar{T}, \bar{x}(\bar{T})) + \xi$, for some $\xi \in N_C(\bar{x}(\bar{T}))$.

(b): $\text{ess}_{t \to \bar{T}} \mathcal{H}(t, \bar{z}(\bar{T}), p_0(\bar{T})) \cap \lambda (\nabla_T g(\bar{T}, \bar{x}(\bar{T})) - \text{co} \partial V(\bar{T})) \neq \emptyset$.

**Sketch of Proof.** Consider the free-time problem, in which the cost $g(T, x(T))$ is replaced by the cost $\tilde{g}(T, x(T)) = g(T, x(T)) - V(T)$. If $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{T})$ is a minimizer for $(D_{\bar{T}})$ then

$$\tilde{g}(\bar{T}, \bar{x}(\bar{T})) = g(\bar{T}, \bar{x}(\bar{T})) - V(\bar{T}) = 0.$$ 

On the other hand, for any $T$ and any feasible process $(x(\cdot), u(\cdot), T)$ we have, by definition of $V(.)$

$$\tilde{g}(T, x(T)) = g(T, x(T)) - V(T) \geq 0.$$ 

The preceding relations imply that $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{T})$ is a minimizer for a modified, free end-time problem, in which $g(\cdot, \cdot)$ is replaced by $\tilde{g}(\cdot, \cdot)$. The sensitivity relation results from applying the Thm. 6.3.1 to the modified problem. 

Conditions (i), (ii) and (iii')(a) are simply the standard conditions on the ‘multipliers’ $(p(\cdot), \lambda)$, in the form of the adjoint inclusion, Weierstrass condition and transversality condition, for the fixed end-time problem $(D_T)$. Interest focuses on the sensitivity relation
The significance of the relation becomes clearer when we specialize to the smooth case; here we obtain an explicit formula for the gradient of $V(.)$:

**Corollary 6.4.2.** Take $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{T})$ to be a minimizer for $(D_T)$. Assume $(H1)-(H4)$. Let $p(\cdot) \in W^{1,1}$ satisfying conditions (i), (ii) and (iii') of Cor. 6.3.1 with $\lambda = 1$. We assume further that

$(R')$: $V(.)$ is $C^1$ on a neighbourhood of $\bar{T}$.

Then

$$\nabla_T V(\bar{T}) = \nabla_T g(\bar{T}, \bar{x}(\bar{T})) - H(\bar{T}, \bar{z}(\bar{T}), p(\bar{T})).$$

### 6.5 Computation of Minimizers

We propose a methodology for the computation of minimizers for the free end-time problem $(D)$. In the absence of time delays, there is a simple procedure for adapting algorithms for the solution of fixed end-time problems with parameters (i.e. constant state components) to applications in which the end-time is free. This is based on a parameterized scaling of time, which has the effect of replacing the original free time interval $[0, T]$ by a fixed time interval while introducing an extra scalar parameter into the problem specification. This approach is problematic when time delays are present, because the scaling affects the length of the time delays. The approach of this paper avoids this difficulty. It is based instead on computing the solution at each stage of a sequence of fixed end-time, delay free problems, and adjusting the end-times according to a rule based on the preceding sensitivity relations.

We assume that the delays are commensurate. This means

**(CD):** There exist $\Delta > 0$ and integers $n_0 = 0 < \ldots < n_k$ such that

$$h_k = \Delta \times n_k, \quad \text{for } i = 0, \ldots, k.$$

Fix $\bar{T} > 0$. A classical construction, attributed to Guinn [38], permits the replacement of the fixed end-time problem $D_{\bar{T}}$ of Section 6.4 by an equivalent delay free problem on the interval $[0, \Delta]$. The construction involves wrapping state trajectories and control functions on the full interval $[0, \bar{T}]$ onto the subinterval $[0, \Delta]$. The fact that the left end-
point value of a wrapped state component must coincide with the right end-point value of
the preceding wrapped state component merely contributes extra endpoint constraints of
a standard nature. The resulting state dimension is \( N \times n \), where

\[
N = \min\{ N' | N' \times \Delta \geq \bar{T} \}.
\]

By setting the dynamics for the last wrapped state component to zero over an appropriate
subinterval of \([0, \Delta]\) we can adapt the method to cover situations in which \( \bar{T} \) is not an
integer multiple of \( \Delta \). Guinn’s construction has been exploited computationally in [?].

The following procedure is based on the availability of optimal control code to compute
the solution to a fixed end-time optimal control problem and to generate adjoint variables.

**Procedure:** Select an end-time \( T_0 \). Fix a parameter \( \alpha > 0 \). Construct end-times \( T_i \),
\( i = 1, 2, \ldots \) recursively as follows. Given \( T_i \)

1. **Step 1.** Reformulate \((D_{T_i})\) as a delay free problem (Guinn’s construction).
2. **Step 2.** Compute the solution to the reformulated problem and store the optimal process
\((x^i(\cdot), u^i(\cdot), T_i)\) and terminal value of the adjoint arc \( p^i(T_i) \).
3. **Step 3.** Compute the sensitivity of the end-time value function

\[
\gamma_i = \nabla_T g(T_i, x^i(T_i)) - \max_{u \in U(T_i)} p^i(T_i) \cdot f(T_i, x^i(T_i - h_0), \ldots, x^i(T_i - h_k), u)
\]

4. **Step 4.** Set \( T_{i+1} = T_i - \alpha \gamma_i \).

### 6.6 Numerical Examples

The procedure described in Section 6.5 has been applied to two examples of free time
optimal control problems with time delay. In each case an optimal control solver was
applied to the fixed end-time optimal control problem on a sequence of intervals \([0, T_i]\), for
\( i = 1, 2, \ldots \) following, in each case, elimination of the time delays by the Guinn transfor-
mation. Calculations were performed with the nonlinear optimization code IPOPT [57]
imported into the toolbox ICLOCS [60]. The sequence of end-times \( \{T_i\} \) was generated
using two ways for estimating the gradient of the value function \( V(\cdot) \) at the final time: firstly by exploiting the sensitivity formulae as in step 3 of the procedure described in Section 6.5 and, secondly, by finite difference approximations.

### 6.6.1 Linear Dynamics

The first example is the optimization problem

\[
\begin{aligned}
\text{Minimize } & \int_0^T (x(t)^2 + u(t)^2) \, dt + 0.1T \\
\text{over } & T > 0, \ x(\cdot) \text{ and } u(\cdot) \text{ satisfying} \\
\dot{x}(t) = & \ x(t) + 0.8x(t - r) + 0.9u(t) \\
x(t) = 1, & \ t \in [-r, 0] \\
x(T) = 0
\end{aligned}
\]

where \( r = 0.15 \). \( x(\cdot) \) and \( u(\cdot) \) take scalar values. This takes the form of problem (D) following state augmentation. The step size adopted in the gradient descent method is \( \alpha = 0.21 \), while the adopted stopping criteria are the maximum number of iterations \( i_{\text{max}} = 150 \) and \( |\nabla_T V(T_i)| < \epsilon = 2.4 \cdot 10^{-6} \).

The end-time value function \( V(\cdot) \) was computed by direct evaluation of the minimum cost for fixed end-times in a grid and interpolation, in order to gain insights into the performance of the gradient descent algorithm. The value function \( V(\cdot) \) has one global minimum. Fig. 1 illustrates how, for various starting times, the steepest descent algorithm based on sensitivity calculations successfully sought out the minimum time \( T_{\text{opt}} = 1.446 \), giving the minimum cost \( V(T_{\text{opt}}) = 5.37 \) and the initial and final value of the adjoint variable \( p(0) = 9.46 \) and \( p(T) = 0.79 \) respectively.

In further experiments, numerical evaluation

\[
\nabla_T V(T_i) \approx \frac{V(T_i) - V(T_i - h)}{h}
\]

(typical value \( h = 2.4 \cdot 10^{-6} \)) provided unreliable approximation of gradients of the value function, often leading to increments of the end-time of incorrect sign and to divergent behavior.
Figure 6.1: End-time value function and performance of algorithm based on sensitivity formulae, for various starting times: $T_0 = 0.5(\circ)$, $T_0 = 3.5(\bigcirc)$. 
6.6.2 Control of a renewable resource

The second example involves optimal harvesting of a resource, when the growth of the resource is governed by a logistic model mediated by the control action. Such models were considered in [45] and used for optimization studies in [28,30] and elsewhere.

\[
\begin{align*}
&\text{Minimize } \int_0^T e^{-\beta t} (C_E x(t)^{-1} u(t)^3 - p u(t)) \, dt + 0.1 T^2 \\
&\text{over } T > 0 \text{ and } x(.), u(.) \text{satisfying} \\
&\dot{x}(t) = a x(t) \left(1 - \frac{x(t-r)}{b}\right) - u(t) \\
&x(t) = 2, \quad t \in [-r,0] \\
x(t) \geq 2, \quad t \in [0,T] \\
u(t) \geq 0, \quad t \in [0,T].
\end{align*}
\]

The state variable \(x(t)\) and the input \(u(t)\) are, respectively, the biomass population and the harvesting effort. The parameters are the harvesting cost \(C_E = 0.2\), growth rates \(a = 3\) and \(b = 5\), discount rate \(\beta = 0.05\) and market price \(p = 2\). \(r = 0.5\). Fig. 2 shows the operation of the gradient descent procedure (based on the sensitivity formulae) with parameter \(\alpha = 0.2\) and stopping criteria \(i_{max} = 100\) (maximum number of iterations) and \(|\nabla T V(T_i)| < \epsilon = 10^{-3}\). Rapid convergence to the global minimizer was observed for a variety of starting times. Further experiments demonstrated that, often, numerical evaluation of the gradients was too inaccurate for implementation of a reliable computational scheme. The minimum time, optimal cost, terminal value of the state variable and initial value of the adjoint variable were found to be \(T_{opt} = 12.24\), \(V(T_{opt}) = -26.146\), \(x(T) = 3.439\), \(p(0) = -0.814\) The optimal control and state are shown in figures [6.3]
6.6. NUMERICAL EXAMPLES

Figure 6.2: End-time value function and performance of algorithm based on sensitivity formulae, for various starting times: $T_0 = 20(\circ), T_0 = 0.6(\circ)$.

Figure 6.3: Example 2: optimal state variable (right) and optimal input variable (left)
Part II

Model Predictive Control
Chapter 7

Overview of Model Predictive Control

Model Predictive Control (MPC), also termed receding horizon control, is an approach to control system design based on solving, at each control update time, an optimal control problem. In this thesis we consider applications of MPC to stability problems, where the aim is to steer a given system towards an equilibrium point.

7.1 Stability of Discrete Time Systems

In this first chapter we introduce the key concepts that will be required to describe properties of MPC schemes. Basic tools classically used to derive stability results are also provided.

\( \mathcal{K} \), \( \mathcal{K}_\infty \) and \( \mathcal{KL} \) functions

We define the following classes of comparison functions:

\[
\mathcal{K} := \{ \rho : \mathbb{R}_+ \to \mathbb{R}_+ \mid \rho \text{ is continuous, } \rho(0) = 0 \text{ and is strictly increasing} \} \\
\mathcal{K}_\infty := \{ \rho : \mathbb{R}_+ \to \mathbb{R}_+ \mid \rho \in \mathcal{K} \text{ and is unbounded} \} \\
\mathcal{KL} := \{ \beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \forall r \geq 0 \ \lim_{t \to \infty} \beta(r, t) = 0 \text{ and } \\
\forall t \geq 0 \ \beta(\cdot, t) \in \mathcal{K}_\infty \}
\]

In the definitions above we use the notation \( \mathbb{R}_+ \) to denote the positive real numbers. Sometimes we write \( \mathbb{R}_{\geq 0} \) in place of \( \mathbb{R}_+ \). To familiarize ourselves with these classes of
functions let us give some examples: consider the function $\alpha(r) = \tan^{-1} r$. We can easily check that $\alpha \in K$. However $\alpha \notin K_\infty$ since $\lim_{r \to \infty} \alpha(r) = \pi/2 < +\infty$. Examples of $K_\infty$ functions are $\alpha(r) = r^k$, for $k > 0$ while $\beta(r, t) = r^k e^{-t}$, to vary $k > 0$ are examples of $KLC$ functions.

We shall use concepts of asymptotic stability based on these comparison functions. We also recall some classical results due to Lyapunov.

**Equilibrium Points, Asymptotic Stability, Lyapunov Functions**

Let us consider the following discrete time system

\[
\begin{align*}
x^+ &= g(x) \\
x(0) &= x_0
\end{align*}
\]

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a given function and $x_0 \in \mathbb{R}^n$ represents the initial state of the system. The notation $x^+$ denotes the successor state in the evolution of the system. Thus $x(k + 1) := g(x(k))$, $k = 0, 1, 2, \ldots$. When we want to make explicit the dependence from the initial condition we write $x(k; x_0)$, $k \geq 0$, for a solution of the system satisfying the initial condition $x(0; x_0) = x_0$.

A point $x^* \in \mathbb{R}^n$ is said to be an *equilibrium point* for the system $x^+ = g(x)$ if $x^* = g(x^*)$. As a generalization of this concept, we say that a set $Y \subset \mathbb{R}^n$ is *forward invariant* if the evolution of the system remains in $Y$ whenever it reaches $Y$, i.e. $g(x) \in Y$ is verified for all $x \in Y$. If $x^*$ is an equilibrium point then the set $\{x^*\}$ is, in particular, forward invariant.

**Definition 7.1.1** (Asymptotic Stability). *Let $x^*$ be an equilibrium for the system $x^+ = g(x)$. We say that $x^*$ is asymptotically stable on a forward invariant set $Y$, containing $x^*$, if there exists a $KLC$ function $\beta$ such that the inequality

\[
|x(k; x_0) - x^*| \leq \beta(|x_0 - x^*|, k)
\]

is satisfied for all $x_0 \in Y$ and $k \in \mathbb{N}_0$. We refer to $\beta$ as the attraction rate.*

\footnote{$\mathbb{N}$ represents the natural numbers. To avoid confusion with different notation in the literature $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.}
7.1. **STABILITY OF DISCRETE TIME SYSTEMS**

The two main ingredients expressed by Definition 7.1.1 are the following:

**Attraction:** for all \( x_0 \in Y \)

\[
\lim_{{k \to \infty}} x(k; x_0) = x^*.
\]

**Stability:** For all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\forall x_0 \in B(x^*; \delta) \cap Y \Rightarrow |x(k; x_0) - x^*| \leq \epsilon.
\]

These properties follow directly from the fact that \( \beta \in KL \). Sometimes, with an abuse of notation, we will refer to a system (in place of a point) satisfying Definition 7.1.1 as an asymptotically stable system. The definition of Lyapunov function is given as follows.

**Definition 7.1.2 (Lyapunov Function).** Consider a system \( x^+ = g(x) \), an equilibrium \( x^* \) of such system and a forward invariant set \( Y \). A function \( V : Y \to \mathbb{R}_+ \) is called a Lyapunov function for \( x^+ = g(x) \) and \( x^* \) if the following conditions are satisfied:

(i) there exist \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \) such that

\[
\alpha_1(|x - x^*|) \leq V(x) \leq \alpha_2(|x - x^*|)
\]

for all \( x \in Y \) and

(ii) there exists \( \alpha_V \in \mathcal{K} \) such that

\[
V(g(x)) \leq V(x) - \alpha_V(|x - x^*|)
\]

for all \( x \in Y \).

**Theorem 7.1.3 (Lyapunov).** If \( V \) is a Lyapunov function on \( Y \) for the system \( x^+ = g(x) \) and the equilibrium \( x^* \), then \( x^* \) is asymptotically stable on \( Y \).

**Proof.** A proof of this result can be found for example in [36, Chapter 2]. \( \square \)

**Control Systems**

In practical applications dynamical systems are not always asymptotically stable. In these situations we may wish to perturb the system through external factors so to achieve
asymptotic stability. This is done by means of control variables that are inserted in the (discrete) differential equation. The choice of such control parameters changes the behavior of the system. This leads to the following dynamical model:

\[ x^+ = f(x, u) \] (7.1.1)

where \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a map which determines the successor state \( x^+ \) in dependence of the current state \( x \in \mathbb{R}^n \) and the control input \( u \in \mathbb{R}^m \). The state trajectory emanating from initial state \( x_0 \) and generated by the control sequence \( u = (u(k))_{k \in \mathbb{N}_0} \) is denoted by \( x_u(k; x_0), k \in \mathbb{N}_0 \). Here, the trajectory \( x_u(\cdot) \) is defined iteratively by \( x_u(k+1; x_0) = f(x_u(k; x_0), u(k)) \) and \( x_u(0, x_0) = x_0 \).

Before extending the concept of asymptotic stability to this framework let us first introduce some notation. First of all the state of the system \( x \) and the control \( u \) may be subject to constraints. We formulate constraints on admissible \( x \) and \( u \) as following

\( (x, u) \in \mathcal{E} \),

(7.1.2)

for some set \( \mathcal{E} \subset \mathbb{R}^n \times \mathbb{R}^m \). This may equivalently be written as

\[ x \in X \quad \text{and} \quad u \in U(x) \]

where the space of admissible states \( X \) is defined by

\[ X := \text{proj}_{\mathbb{R}^n}(\mathcal{E}) = \{ x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \text{ s.t. } (x, u) \in \mathcal{E} \}. \]

(7.1.3)

and, for a given admissible state \( x \in X \), the control constraints are defined by

\[ U(x) := \{ u \in \mathbb{R}^m : (x, u) \in \mathcal{E} \}. \]

We will use both the notation \( (x, u) \in \mathcal{E} \) and \( x \in X, u \in U(x) \), depending on our convenience.

**Definition 7.1.4** (Admissible control sequence). A sequence of control values \( u = \)}
(u(0), u(1), ..., u(N−1)) is called admissible for x0 ∈ X and N ∈ \( \mathbb{N} \cup \{\infty\} \) if the conditions
\[
f(x_u(k; x_0), u(k)) \in X \quad \text{and} \quad u(k) \in U(x_u(k; x_0))
\]
hold for all \( k \in \{0, 1, ..., N - 1\} \). The set of all admissible control sequences of length \( N \in \mathbb{N} \cup \{\infty\} \) is denoted by \( \mathcal{U}^N(x_0) \).

**Definition 7.1.5** (Equilibrium, Asymptotic Stability, Basin of Attraction). We say that a point \( x^* \in X \) is a (controlled) equilibrium, if there exists a control \( u^* \in U(x^*) \) such that \( f(x^*, u^*) = x^* \). The equilibrium \( x^* \) is said to be asymptotically stable if there exist a feedback law \( \mu: \mathbb{R}^n \to \mathbb{R}^m \) and a set \( S \subseteq X \) such that \( x^* \) is asymptotically stable for the resulting closed loop system \( x^+ = f(x, \mu(x)) = g(x) \) on the set \( S \). This means that for any initial state \( x_0 \in S \) the closed loop trajectory \( x_\mu(k; x_0), k \in \mathbb{N}_0 \), generated by
\[
x_\mu(k + 1; x_0) = f(x_\mu(k; x_0), \mu(x_\mu(k; x_0))), \quad x_\mu(0; x_0) = x_0,
\]
remains feasible, i.e., \( (x_\mu(k; x_0), \mu(x_\mu(k; x_0))) \in \mathcal{E} \) holds for all \( k \in \mathbb{N}_0 \), and satisfies the estimate
\[
|x_\mu(k; x_0) - x^*| \leq \beta(|x_0 - x^*|, k), \quad k \in \mathbb{N}
\]
for some KL-function \( \beta \). We refer to the set \( S \) as the basin of attraction of the asymptotically stable equilibrium \( x^* \).

### 7.2 The MPC Algorithm

All the ingredients necessary to define the MPC algorithm have been introduced. MPC provides a procedure to compute the feedback values \( \mu(x) \) that will eventually stabilize the system. The idea is as follows. We first find an open loop control \( u \) that steers the system towards the controlled equilibrium \( (x^*, u^*) \). This is achieved by minimizing the ‘distance’ of the trajectories of the system from the equilibrium. In this respect then we choose the ‘best’ control \( u \) that moves the system towards the equilibrium. We then use knowledge of open loop controls to construct a feedback control. Let us clarify this procedure. Define
running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\ell (x^*, u^*) = 0$ and

$$\eta(|x - x^*|) \leq \ell^*(x) := \inf_{u \in U(x)} \ell(x, u) \leq \overline{\eta}(|x - x^*|) \quad \forall \ x \in X \quad (7.2.1)$$

for two $K_{\infty}$-functions $\eta$, $\overline{\eta}$. The corresponding cost function $J_N : \mathbb{R}^n \times (\mathbb{R}^m)^N \rightarrow \mathbb{R}_{\geq 0}$ and optimal value function $V_N : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ are given by

$$J_N(x, u) := \sum_{k=0}^{N-1} \ell(x_u(k; x), u(k)) \quad \text{and} \quad V_N(x) := \inf_{u \in U_N(x)} J(x, u)$$

for $N \in \mathbb{N} \cup \{\infty\}$, $x \in X$ and $u \in U^N(x)$ with the convention $V_N(x) = +\infty$ if $x \notin X$ or $U^N(x) = \emptyset$. Fixing a finite prediction horizon (or optimization horizon) $N$, the MPC loop is defined as follows:

**Algorithm 1.** (Basic MPC algorithm for a system $(f, \mathcal{E})$, a controlled equilibrium $(x^*, u^*) \in \mathcal{E}$, and an horizon $N \in \mathbb{N}$.) Define $x_{\mu_N}(0; x_0) := x_0$, $k := 0$,

1. Set $x = x_{\mu_N}(k; x_0)$, solve the optimal control problem

$$\begin{cases}
\text{Minimize } J_N(x, u) = \sum_{k=0}^{N-1} \ell(x_u(k; x), u(k)) \\
\text{over } u \in U^N(x) \text{ such that } \\
x_u(k+1; x) = f(x_u(k; x), u(k)) \quad k = 0, \ldots, N-1 \quad \text{and} \quad x_u(0; x) = x,
\end{cases}$$

and denote a respective minimizing control sequence$^2$ by $\bar{u}(\cdot) \in U^N(x)$.

2. Define the MPC feedback value by $\mu_N(x) := \bar{u}(0)$.

3. Compute $x_{\mu_N}(k+1; x_0)$ by (7.1.4) with $\mu = \mu_N$, set $k := k + 1$ and go to 1.

This iteration yields a closed loop trajectory for the implicitly defined MPC feedback law $\mu_N : X \rightarrow \mathbb{R}^m$.

**A Variant of the Basic MPC Algorithm**

It is common to find in the literature variations of the basic MPC algorithm that may improve, in particular cases, the performance of the scheme. A popular one is to add

$^2$Whenever $U^N(x) \neq \emptyset$, existence of a minimizer $\bar{u}(\cdot) \in U^N(x)$ satisfying $J_N(x, \bar{u}) = V_N(x)$ is assumed in order to avoid technical difficulties. Standard assumptions on the data of the system can be given to ensure this, see [36].
7.2. THE MPC ALGORITHM

terminal constraints and/or terminal costs in the optimal control problem to be solved at each control update time.

Algorithm 2. (Extended MPC algorithm for a system \((f, E)\), a controlled equilibrium \((x^*, u^*) \in E\), and an horizon \(N \in \mathbb{N}\).) Define \(x_{\mu_N}(0; x_0) := x_0\), \(k := 0\),

1. Set \(x = x_{\mu_N}(k; x_0)\), solve the optimal control problem

\[
\begin{align*}
\text{Minimize } & J_N(x, u) = \sum_{k=0}^{N-1} \ell(x_u(k; x), u(k)) + F(x_u(N; x)) \\
\text{over } & u \in U^N(x) \text{ such that } x^+ = f(x, u) \text{ and } x_u(N; x) \in X_0
\end{align*}
\]

and denote a respective minimizing control sequence by \(\bar{u}(\cdot) \in U^N(x)\).

2. Define the MPC feedback value by \(\mu_N(x) := \bar{u}(0)\).

3. Compute \(x_{\mu_N}(k + 1; x_0)\) by (7.1.4) with \(\mu = \mu_N\), set \(k := k + 1\) and go to 1.

Here \(X_0 \subset X\) is a given subset and it is referred to as terminal constraint while \(F : X \to \mathbb{R}\) is a given function and it is referred to as terminal cost.

Definition 7.2.1. Let \(X_0 \subset X\) be a terminal constraint for the MPC algorithm. The feasible set for horizon \(N \in \mathbb{N}\) is defined by

\[F_N := \{x_0 \in X : \exists u(\cdot) \in U^N(x_0) \text{ with } x_u(N; x_0) \in X_0\}.\]

The set of admissible control sequences is now defined by

\[U^N_{X_0}(x) := \{u(\cdot) \in U^N(x) : x_u(N; x) \in X_0\}.\]

Remark 7.2.2. Observe that the costs \(\ell(\cdot)\) and the horizon \(N\), as well as, the constraint set \(E\), the terminal costs \(F(\cdot)\) and the terminal constraint \(X_0\), are all design parameters that are chosen according to performance specifications.

Recursive Feasibility

Before proceeding let us also generalize the concept of forward invariance. Observe, at the outset, that when the terminal constraint is absent from the formulation, that is \(X_0 = X\),
then the feasible set for a horizon length \( N \in \mathbb{N} \cup \{\infty\} \) is simply defined by

\[
\mathcal{F}_N := \{ x \in X : \mathcal{U}^N(x) \neq \emptyset \}.
\]  

(7.2.2)

The set \( \mathcal{F}_\infty \) is also called viability kernel. \( \mathcal{F}_\infty \) is the largest set where the MPC feedback can be defined.

**Definition 7.2.3.** A set \( C \subseteq X \) is said to be (controlled) forward invariant or viable if, for each \( x \in C \), there exists \( u \in U(x) \) such that \( f(x, u) \in C \) holds.

Observe that every forward invariant set \( C \subseteq X \) satisfies the inclusion \( C \subseteq \mathcal{F}_\infty \) and that the set of admissible states \( X \) is, in general, much larger than the viability kernel \( \mathcal{F}_\infty \). Methods which can be used in order to compute invariant sets can be found in [8]. Construction of viable set is an important topic in MPC. In order to avoid that the MPC loop becomes infeasible we should always carefully choose a viable set \( C \subseteq S \) where the MPC feedback law can be defined.

**Definition 7.2.4.** A set \( C \) is said to be recursively feasible if it is forward invariant with respect to the MPC feedback law \( \mu_N \), that is \( \mu_N(x) \in U(x) \) and \( f(x, \mu_N(x)) \in C \) for all \( x \in C \).

### 7.3 The Dynamic Programming Principle

In this section we recall a standard result in optimal control theory that will be of particular relevance for our analysis. We state the result for an optimal control problem of the following form

\[
(OC) \quad \begin{cases} 
\text{Minimize } J_N(x_0, u) \\
\text{over control sequences } u \in \mathcal{U}_X^N(x_0) \text{ such that } \\
x^+ = f(x, u) \quad \text{and} \quad x(0) = x_0. 
\end{cases}
\]

Notice that we choose the notation \( \mathcal{U}_X^N \) to denote the admissible control sequences. The case of Algorithm 1 is covered by choosing \( X_0 = \mathbb{R}^n \).

**Theorem 7.3.1** (The Dynamic Programming Principle). Consider the optimal control problem \((OC)\) and assume that the initial condition is such that \( \mathcal{U}_X^N(x_0) \neq \emptyset \). Then for
7.4 STABILITY OF MPC SCHEMES: CLASSICAL THEORY

all $N \in \mathbb{N} \cup \{+\infty\}$ and $M = 1, \ldots, N$ we have that

$$V_N(x_0) = \inf_{u \in \mathcal{U}_{N-M}^M(x_0)} \left\{ \sum_{k=0}^{M-1} \ell(x_u(k); x_0, u(k)) + V_{N-M}(x_u(M); x_0) \right\}. \quad (7.3.1)$$

If, in addition, an optimal control sequence $\bar{u}(\cdot) \in \mathcal{U}_N(x_0)$ for (OC) exists, then the inf in (7.3.1) is a min and

$$V_N(x_0) = \sum_{k=0}^{M-1} \ell(x_{\bar{u}}(k); x_0, \bar{u}(k)) + V_{N-M}(x_{\bar{u}}(M); x_0).$$

For $M = 1$ we obtain the following equality

$$V_N(x_0) = \ell(x_0, \mu_N(x_0)) + V_{N-1}(f(x_0, \mu_N(x_0))).$$

Proof. See, e.g., [36].

7.4 Stability of MPC Schemes: Classical Theory

Infinite Horizon Optimal Control

Consider the basic MPC algorithm with horizon $N = +\infty$. Now in general the value function $V_\infty$ associated to the infinite horizon optimal control problem is not bounded. But let us assume, for the moment, that we could find a recursively feasible set $\mathcal{S} \subset X$, for the infinite horizon MPC feedback law $\mu_\infty$, such that the following holds

$$\alpha_1(|x - x^*|) \leq V_\infty(x) \leq \alpha_2(|x - x^*|) \forall x \in \mathcal{S} \quad (7.4.1)$$

for some $K_\infty$ functions $\alpha_1$ and $\alpha_2$. Now, since by the Dynamic programming principle we have that

$$V_\infty(x) = \ell(x, \mu_\infty(x)) + V_\infty(f(x, \mu_\infty(x))) \forall x \in \mathcal{S},$$

and since running costs $\ell(\cdot)$ have been designed to satisfy Condition 7.2.1, we have that $V_\infty$ provides a Lyapunov function for the closed loop system $x^+ = f(x, \mu_\infty(x))$, and the desired stability results follow from Theorem 7.1.3.
Relaxed Dynamic Programming

The last result shows that the infinite horizon optimal control may be used to derive a stabilizing feedback control $\mu_\infty$. Unfortunately, a direct solution of infinite horizon optimal control problems is in general impossible, both analytically and numerically. This is why we consider the optimal control problem on a finite horizon $N$, that can be solved at least numerically. Of course now we may lose stability, feasibility, and optimality, see, e.g., [51]. The challenge is to see whether we may impose particular conditions on the data in order to maintain the nice properties given by the infinite horizon problem.

For a finite $N \in \mathbb{N}$, Theorem 7.3.1 yields the equality

$$V_N(x) = \ell(x_0, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x))).$$

The problem of finding a Lyapunov function, in principle, would be solved if we could prove that $V_N(x) \leq V_{N-1}(x)$. However this is not in general verified. A technique often used in the literature (see for example [36]), to promote the $N$-horizon optimal value function $V_N$ to a Lyapunov function consists in imposing reasonable assumptions on the data that guarantee the validity of a relaxed dynamic programming principle, i.e.,

$$V_N(x_0) \geq \alpha \ell(x_0, \mu_N(x_0)) + V_N(f(x_0, \mu_N(x_0))),$$

(7.4.2)

for some $\alpha \in (0,1]$ and for all $x_0$ in a certain viable set $\mathcal{S}$.

**Theorem 7.4.1** (Suboptimality). Suppose that the value function $V_N$ for horizon $N \in \mathbb{N}$ satisfies the relaxed dynamic programming principle for a certain $\alpha \in (0,1]$ on a set $\mathcal{S} \subset X$. Then the suboptimality estimate

$$J_\infty(x, \mu_N) \leq V_N(x)/\alpha$$

holds for all $x \in \mathcal{S}$.

**Proof.** Details of the proof are given in [36].
Stability and Suboptimality using Stabilizing Constraints

A first attempt to ensure asymptotic stability and feasibility of the MPC scheme, for a certain finite horizon $N$, was to incorporate suitable terminal constraints and costs in the optimal control problem to be solved in each MPC step, as in Algorithm 2.

**Theorem 7.4.2.** Consider the MPC algorithm with terminal constraints $(X_0,F)$ and optimization horizon $N$. Assume that

(i) $X_0$ is viable, i.e.
\[ \forall x \in X_0 \exists u_x \in U(x) : f(x,u_x) \in X_0 \]

(ii) and that the same control $u_x$ (for all $x \in X_0$) satisfies
\[ F(f(x,u_x)) + \ell(x,u_x) \leq F(x). \]

Assume also that
\[ \alpha_1(|x - x^*|) \leq V_N(x) \leq \alpha_2(|x - x^*|) \quad \text{and} \quad \ell(x,u) \geq \alpha_3(|x - x^*|) \quad \forall x \in X \]
for some $\alpha_1, \alpha_2, \alpha_3 \in K_\infty$. Then the MPC closed loop system $x^+ = f(x,\mu_N(x))$ with MPC feedback law $\mu_N$ is asymptotically stable on $F_N$. In this framework the relaxed dynamic programming principle is satisfied with $\alpha = 1$.

**Idea of the proof.** For every point $x_0 \in F_{N-1}$ by definition there exists an admissible sequence $u_{N-1}(.) \in U_{X_0}^{N-1}(x_0)$ such that $x = x_{u_{N-1}}(N-1;x_0) \in X_0$. The hypotheses on the data implies that

(i) every control sequence $u_{N-1}(.) \in U_{X_0}^{N-1}(x_0)$ can be extended to a sequence $u_N(.) \in U_{X_0}^{N}(x_0)$. Precisely
\[ u_N(k) := \begin{cases} u_{N-1}(k) & k = 0, \ldots, N-2 \\ u_x & k = N-1. \end{cases} \]

(ii) Such extension reduces the cost and it can be proved that
\[ V_N(x_0) \leq V_{N-1}(x_0). \]
Therefore $V_N$ is a Lyapunov function on the recursively feasible set $F_N$.

Unfortunately, the construction of such stabilizing constraints may be challenging and their use may considerably reduce the operating range of the MPC scheme, cf. [36, Chapter 8] or [46] for a detailed discussion.

**Stability and Suboptimality without Stabilizing Constraints**

We would like then to analyze the MPC scheme without additional terminal constraints or costs. There is a vast literature concerning the study of MPC schemes without terminal constraints. We refer the reader to [33,34,36,37,41,54,59] and the reference therein. In this framework there are essentially two main approaches to ensure asymptotic stability and feasibility of the MPC algorithm. The first method assume the existence, for every point $x \in X$ of an open loop control that steers the system sufficiently fast to the equilibrium. This hypothesis is known as controllability assumption and it states the following:

**Assumption 7.4.3.** There exists $\gamma > 0$ such that the inequality

$$V_N(x) \leq \gamma \cdot \ell^*(x), \quad \forall x \in F_N,$$

holds for all $N \geq 2$. Here $\ell^*(x) := \inf_{u \in U(x)} \ell(x,u)$.

**Remark 7.4.4.** (i) The name controllability condition stems from the fact that the inequality $V_\infty(x) < C$ requires the system to be controllable to $x^*$ sufficiently fast, since otherwise (7.2.1) implies $V_\infty(x) = +\infty$. For the particular form of the bound $\gamma$ assumed above, for instance the exponential controllability assumption w.r.t. $\ell$ used in [36, Chapter 6] would be sufficient.

(ii) These inequalities could be replaced by inequalities in which $\gamma$ depends on $N$, thus allowing for less conservative estimates, cf. [59]. However, in order to keep the presentation simple, in this thesis we will work with the assumption on a single $\gamma$.

Under Assumption 7.4.3 the following result holds (see e.g. [35]).

**Theorem 7.4.5.** Let Assumption 7.4.3 and Condition (7.2.1) be satisfied. Then for each compact set $K \subset F_\infty$ there exists $N_K \in \mathbb{N}$ such that for each $N \geq N_K$, the MPC algorithm is recursively feasible on a set $S \supseteq K$ and the value function for optimization horizon $N$
satisfies
\[ V_N(f(x, \mu_N(x))) \leq V_N(x) - \alpha \ell(x, \mu_N(x)) \]
for some \( \alpha \in (0, 1] \) and all \( x \in S \). Therefore \( V_N \) is a Lyapunov function on \( S \) which in turn gives the desired asymptotic stability result.

Observe that this Theorem in combination with Theorem 7.4.1 gives also the sub-optimality estimate
\[ \alpha V_\infty(x) \leq V_N(x) \leq V_\infty(x) \]
for all \( x \in S \). There are several techniques present in the literature to estimate the sub-optimality rate \( \alpha \) and the horizon \( N_K \) necessary to stabilize the system. We refer to [35] for details.

A different approach to obtain stability, feasibility and sub-optimality of the MPC scheme, was introduced by Kerrigan in [42, Chapter 5]. This method requires the assumption that the family of feasible sets \( \{F_N\}_{N \in \mathbb{N}} \) is stationary.

**Assumption 7.4.6.** There exists \( N_0 \in \mathbb{N} \) such that \( F_{N_0} = F_{N_0+1} \).

Note that if Assumption 7.4.6 is satisfied, then \( F_{N_0} = F_\infty \) and, in particular, \( F_\infty \) is recursively feasible for the MPC feedback law \( \mu_{N_0} \). Theorem 7.4.5 is still satisfied, with appropriate modifications (\( S = F_\infty \) and \( N_K = N_0 \)), if Assumption 7.4.3 is replaced by Assumption 7.4.6.

## 7.5 Stability of MPC Schemes: Recent Developments

The classical theory is somewhat restrictive, regarding the severity of the hypotheses that need to be imposed on the terminal cost or on the growth of the infinite horizon optimal value function. Recent Research in MPC has been directed at weakening these hypotheses.

As we shall show in the next chapter, the conditions on the terminal cost or on the value of the cost, can be replaced by a weak controllability condition. This condition is expressed in terms of properties of the value function. It can be verified directly in simple cases, and the assumption that it is satisfied for a large class of control systems would appear to be a reasonable one.
Traditionally, the problems of establishing global and local stability of MPC schemes have been treated separately. It has also been common to consider first the case of linear systems, and then consider nonlinear systems. In recent work in MPC, the cases are all treated together. As we shall see in the next chapter, stability properties can be established for general nonlinear control systems under a weak controllability condition, which can be expressed both in terms of local and global stability.

7.6 Robustness

In practical applications we seldom have perfect information of the dynamic of the system. Moreover, the measurement of the state of the system at each control update time (step 1 of the MPC algorithm) may be imprecise. In such situations we would like the asymptotic stability of the MPC closed loop to be robust (See Definition 7.6.1) with respect to measurement errors, mismatches between the dynamical formulation and the real behavior of the system, and additive disturbances. Indeed, the MPC algorithm has been constructed to produce a feedback controller, and the very reason to design a feedback controller is to cope with uncertainties.

Let us consider a generic feedback law \( \mu : \mathbb{R}^n \to \mathbb{R}^m \) and an equilibrium \( x^* \) of the closed loop \( x^+ = f(x, \mu(x)) \). Let us also assume that \( x^* \) be asymptotically stable (for the feedback \( \mu \)) with basin attraction \( S \subset X \). To take account of measurement errors, mismatches of the formulation, and additive disturbances we include parameters \( e \) and \( d \) in the formulation of the problem, i.e.,

\[
x^+ = f(x, \mu(x + e)) + d.
\] (7.6.1)

Given an initial state \( x_0 \in X \), and sequences \( e = (e_0, \ldots, e_l) \) and \( d = (d_0, \ldots, d_l) \) we use the notation \( x^{ed}_\mu(k; x_0) \) to indicate a solution of (7.6.1) with respect to the sequences \( e \) and \( d \), i.e.,

\[
x^{ed}_\mu(k + 1; x_0) = f(x^{ed}_\mu(k; x_0), \mu(x^{ed}_\mu(k; x_0) + e(k))) + d(k)
\]

for \( k = 0, 1, \ldots, l - 1 \).

**Definition 7.6.1.** The equilibrium \( x^* \), of the closed loop \( x^+ = f(x, \mu(x)) \), is said to be robustly asymptotically stable on \( \text{int}\{S\} \), if there exists a class-KL function \( \beta \) such that for
each $\epsilon > 0$ and each compact set $C \subset \text{int}\{S\}$ there exists $\delta > 0$ such that for all sequences $e = (e_0, \ldots, e_{l-1})$ and $d = (d_0, \ldots, d_{l-1})$, satisfying

(i) $\max\{|e_0|, \ldots, |e_{l-1}|, |d_0|, \ldots, |d_{l-1}|\} \leq \delta$

(ii) $x_{\mu}^{ed}(k; x) \in C$ for all $k = \{0, 1, \ldots, l\}$,

we have $|x_{\mu}^{ed}(k; x)| \leq \beta(|x|, k) + \epsilon,$ for all $k = \{0, 1, \ldots, l\}$.

The following result can be found in [32].

**Proposition 7.6.2.** If there exists a Lyapunov function $V(\cdot)$, for $x^+ = f(x, \mu(x))$ and $x^*$, defined on $S$ that is continuous on $\text{int}\{S\}$, then the equilibrium $x^*$ of the closed loop $x^+ = f(x, \mu(x))$ is robustly asymptotically stable on $\text{int}\{S\}$.

In this thesis we will discuss about robust stability only marginally. In Chapter 9 for example, we will study continuity properties of the value function, which, under suitable assumptions, as we have seen in the previous section, represents a Lyapunov function for the MPC loop. Yet we will not directly consider systems of the type (7.6.1). Nevertheless, there are several important reasons to consider the idealized and less realistic disturbances-free case: first of all the satisfactory behavior of the MPC closed loop (without disturbances) is a natural necessary condition for the correct functioning of our controller. Indeed, if we cannot ensure proper functioning of the controller in the absence of disturbances we can hardly expect it to work in real applications. Second, the assumption that our model represent an exact prediction, greatly simplifies the analysis and provides us with sufficient conditions under which the MPC algorithm works (at least in the simplified case). Moreover, as Proposition 7.6.2 suggests, we can use the analysis of the simplified case to investigate the functioning of the MPC algorithm when the more realistic case (7.6.1) is considered. The study of systems governed by (7.6.1) will be the subject of future investigations.
Chapter 8

Stability and feasibility of state constrained MPC

In this chapter we investigate stability and recursive feasibility of a nonlinear receding horizon control scheme without terminal constraints and costs but imposing state and control constraints\[1\]. Under a local controllability assumption we show that every level set of the infinite horizon optimal value function is contained in the basin of attraction of the asymptotically stable equilibrium for sufficiently large optimization horizon $N$.

For stabilizable linear systems we show the same for any compact subset of the interior of the viability kernel. Moreover, estimates for the necessary horizon length $N$ are given via an analysis of the optimal value function at the boundary of the viability kernel.

We briefly recall some notation: $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^n$. Given a set $S \subset \mathbb{R}^n$, $cl\{S\}$ denotes its closure, $int\{S\}$ its interior and $\partial S := cl\{S\} \setminus int\{S\}$ its boundary. Furthermore, a continuous function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is strictly increasing and satisfies $\eta(0) = 0$. If $\eta \in \mathcal{K}$ is also unbounded, $\eta$ is called a class $\mathcal{K}_\infty$-function. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called $\mathcal{K} \mathcal{L}$-function if it is continuous, satisfies $\beta(\cdot, t) \in \mathcal{K}_\infty$, $t \in \mathbb{R}_{\geq 0}$, is strictly decreasing in its second argument for all $r > 0$, and $\lim_{t \to \infty} \beta(r, t) = 0$ holds.

\[1\]Most of the content presented in the next two chapters has been submitted for publication, see [11] and [10]. This research has been mainly carried out while the author was visiting the University of Bayreuth.
8.1 Relations to the Literature

Model predictive control (MPC) is a controller design technique relying on the iterative solution of optimal control problems. In this chapter we study stability and recursive feasibility of nonlinear MPC schemes without stabilizing terminal constraints or costs. For such schemes, it is known that stability for sufficiently large optimization horizons can be deduced from controllability assumptions or — alternatively and almost equivalently — bounds on the optimal value functions, see [33,34,36,37,41,54,59].

Here we extend this body of literature by taking into account state constraints without assuming viability of the state constraint set or boundedness of the optimal value function on this set or its viability kernel. In the first part of the chapter we consider general nonlinear systems and assume a local controllability assumption in a neighbourhood of the equilibrium to be stabilized. Under this condition, we first analyse the behaviour of the closed loop on level sets $V^{-1}_\infty[0,C]$ of the infinite horizon optimal value function. Using a technique similar to [49] we obtain recursive feasibility and an adaptation of an argument from [33] yields asymptotic stability with $V^{-1}_\infty[0,C]$ contained in the basin of attraction, provided the optimization horizon $N$ is sufficiently large. Moreover, quantitative estimates on the necessary length of $N$ are given. This result is then extended to compact sets lying in the domain of $V_\infty$ and avoiding suitable defined exceptional regions $O$. Overall, this part of the chapter can be seen as a (discrete time) extension of [41] to the state constrained case and with additional quantitative estimates for $N$.

In the second part of the chapter we specialize the results to the linear quadratic case with convex constraints. We show that in this setting any compact subset $K$ in the interior of the viability kernel is contained in the basin of attraction for sufficiently large $N$ and give an estimate of $N$ in terms of the distance of $K$ to the boundary of the viability kernel. These quantitative results rely on an estimate of the growth of the optimal value function $V_\infty$ at the boundary of the viability kernel which we obtained adapting a technique from [31]. A particularly nice case appears when $V_\infty$ is bounded on the viability kernel and we show that this property implies stationarity of the feasible sets in the sense of [42, Chapter 5].

The chapter is organized as follows. After describing the setting in Section 8.2, Section 8.3 contains the nonlinear asymptotic stability and feasibility results. The specialization to linear systems is presented in Section 8.4.
8.2 Model Predictive Control

The MPC algorithm has already been introduced in the previous introductory chapter. We repeat the main lines to make this chapter self-contained. The reader acquainted with the definition of the MPC algorithm may wish to jump this section.

We consider discrete time systems governed by the system dynamics

\[ x^+ = f(x, u) \]  

(8.2.1)

where \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a map which determines the successor state \( x^+ \) in dependence of the current state \( x \in \mathbb{R}^n \) and the control input \( u \in \mathbb{R}^m \). The state trajectory emanating from initial state \( x_0 \) and generated by the control sequence \( u = (u(k))_{k \in \mathbb{N}_0} \) is denoted by \( x_u(k; x_0) \), \( k \in \mathbb{N}_0 \). Here, the trajectory \( x_u(\cdot) = x_u(\cdot; x_0) \) is defined iteratively by \( x_u(k + 1; x_0) = f(x_u(k; x_0), u(k)) \) and \( x_u(0; x_0) = x_0 \). Constraints for the state \( x \) and the control \( u \) are modeled by a suitably chosen subset \( \mathcal{E} \subseteq \mathbb{R}^n \times \mathbb{R}^m \), i.e., we require

\[ (x, u) \in \mathcal{E}. \]  

(8.2.2)

Hence, for a given set \( \mathcal{E} \), the set of admissible states is given by the projection of the set \( \mathcal{E} \) on the state space \( \mathbb{R}^n \), i.e.

\[ X := \text{proj}_{\mathbb{R}^n}(\mathcal{E}) = \{ x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \text{ s.t. } (x, u) \in \mathcal{E} \}. \]  

(8.2.3)

Furthermore, for a given admissible state \( x \in X \), the control constraints can be represented by \( U(x) := \{ u \in \mathbb{R}^m : (x, u) \in \mathcal{E} \} \). Using these definitions the concept of an admissible control sequence can be defined as follows.

**Definition 8.2.1 (Admissible control sequence).** A sequence of control values \( u = (u(0), u(1), \ldots, u(N-1)) \) is called admissible for \( x_0 \in X \) and \( N \in \mathbb{N} \cup \{ \infty \} \) if the conditions

\[ f(x_u(k; x_0), u(k)) \in X \quad \text{and} \quad u(k) \in U(x_u(k; x_0)) \]

hold for all \( k \in \{0, 1, \ldots, N - 1\} \). The set of all admissible control sequences of length \( N \in \mathbb{N} \cup \{ \infty \} \) is denoted by \( U^N(x_0) \).
Let $x^* \in X$ be a (controlled) equilibrium, i.e. there exists $u^* \in U(x^*)$ such that $f(x^*, u^*) = x^*$ holds. Our goal is to find a static state feedback $\mu : \mathbb{R}^n \to \mathbb{R}^m$ and a basin of attraction $S \subseteq X$ such that $x^*$ is asymptotically stable for the resulting closed loop system $x^+ = f(x, \mu(x))$. This means that for any initial state $x_0 \in S$ the closed loop trajectory $x_\mu(k; x_0)$, $k \in \mathbb{N}_0$, generated by

$$x_\mu(k + 1; x_0) = f(x_\mu(k; x_0), \mu(x_\mu(k; x_0))), \quad x_\mu(0; x_0) = x_0,$$

remains feasible, i.e., $(x_\mu(k; x_0), \mu(x_\mu(k; x_0))) \in E$ holds for all $k \in \mathbb{N}_0$, and satisfies the estimate $|x_\mu(k; x_0) - x^*| \leq \beta(|x_0 - x^*|, k)$, $k \in \mathbb{N}_0$, for some $KL$-function $\beta$.

In MPC, the feedback values $\mu(x)$ are computed by solving optimal control problems. To this end, running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \geq 0$ satisfying $\ell(x^*, u^*) = 0$ and

$$\eta(|x - x^*|) \leq \ell^*(x) := \inf_{u \in U(x)} \ell(x, u) \leq \overline{\eta}(|x - x^*|) \quad \forall x \in X$$

for two $K_\infty$-functions $\underline{\eta}, \overline{\eta}$ are defined. The corresponding cost function $J_N : \mathbb{R}^n \times (\mathbb{R}^m)^N \to \mathbb{R}_{\geq 0}$ and optimal value function $V_N : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ are given by

$$J_N(x, u) := \sum_{k=0}^{N-1} \ell(x_k(k; x), u(k)) \quad \text{and} \quad V_N(x) := \inf_{u \in \mathcal{U}^N(x)} J(x, u)$$

for $N \in \mathbb{N} \cup \{\infty\}$, $x \in X$ and $u \in \mathcal{U}^N(x)$ with the convention $V_N(x) = +\infty$ if $x \notin X$ or $\mathcal{U}^N(x) = \emptyset$. In principle, the stabilization problem could be solved by solving the optimal control problem for $N = \infty$. However, solving optimal control problems on an infinite time horizon is, in general, computationally hard. This explains why we pursue a different approach: model predictive control (MPC), also termed receding horizon control. Fixing a finite prediction horizon (or optimization horizon) $N$ and setting $x_\mu(0; x_0) := x_0$, $k := 0$, the MPC loop is as follows:

1. Set $x = x_\mu(k; x_0)$, solve the optimal control problem

$$\min_{u \in \mathcal{U}^N(x)} J_N(x, u)$$

and denote a respective minimizing control sequence $^2$ by $\bar{u} \in \mathcal{U}^N(x)$.

\footnote{Whenever $\mathcal{U}^N(x) \neq \emptyset$, existence of a minimizer $\bar{u} \in \mathcal{U}^N(x)$ satisfying $J_N(x, \bar{u}) = V_N(x)$ is assumed in...}
2. Define the MPC feedback value by $\mu_N(x) := \bar{u}(0)$.

3. Compute $x_{\mu}(k + 1; x_0)$ by (8.2.4) with $\mu = \mu_N$, set $k := k + 1$ and go to 1.

This iteration yields a closed loop trajectory for the implicitly defined MPC feedback law $\mu_N : X \to \mathbb{R}^m$. However, because of the truncation of the optimization horizon, stability, feasibility, and optimality may get lost, see, e.g., [51].

8.3 Recursive Feasibility and Asymptotic Stability

In order to guarantee that the optimal control problem in Step 1 of the MPC loop is feasible, we need to ensure $U^N(x) \neq \emptyset$ for $x = x_{\mu_N}(k; x_0), \ k \in \mathbb{N}_0$. This problem can be solved by incorporating suitable terminal constraints and costs in the optimal control problem to be solved in each MPC step. However, the construction of such stabilizing constraints may be challenging and their use may considerably reduce the operating range of the MPC scheme, cf. [36, Chapter 8] or [46] for a detailed discussion. Hence, we want to analyse the scheme without additional terminal constraints or costs. In particular, feasibility of the MPC algorithm in each step and asymptotic stability of the resulting closed loop has to be ensured. To this end, the following local controllability condition is employed.

Assumption 8.3.1. There exists a neighbourhood $\mathcal{N}$ of $x^*$ and a positive constant $\gamma \in \mathbb{R}$ such that

$$V_\infty(x) \leq \gamma \cdot \ell^\mu(x), \quad \forall \ x \in \mathcal{N} \cap X.$$  

Remark 8.3.2. Note that in contrast to previous literature on the subject, such as for example [35, 36], here we only require the inequality to hold locally around $x^*$.

In order to formalize recursive feasibility, some notation is needed. The feasible set for a horizon length $N \in \mathbb{N} \cup \{\infty\}$ is defined as

$$\mathcal{F}_N := \{x \in X : U^N(x) \neq \emptyset\}. \quad \text{(8.3.1)}$$

The set $\mathcal{F}_\infty$ is also called viability kernel. Furthermore, a set $\mathcal{C} \subseteq X$ is said to be (controlled) forward invariant or viable if, for each $x \in \mathcal{C}$, there exists $u \in U(x)$ such that order to avoid technical difficulties.
Chapter 8. Stability and Feasibility of State Constrained MPC

\( f(x,u) \in \mathcal{C} \) holds. Observe that every forward invariant set \( \mathcal{C} \subseteq X \) satisfies the inclusion \( \mathcal{C} \subseteq \mathcal{F}_{\infty} \) and that the set of admissible states \( X \) is, in general, much larger than the viability kernel \( \mathcal{F}_{\infty} \). Methods which can be used in order to compute invariant sets can be found in [8]. The set \( \mathcal{C} \) is said to be \textit{recursively feasible} if it is forward invariant with respect to the feedback law \( \mu_N \), that is, \( \mu_N(x) \in U(x) \) and \( f(x,\mu_N(x)) \in \mathcal{C} \) for all \( x \in \mathcal{C} \).

8.3.1 Asymptotic Stability on Level Sets

Ideally we would like the basin of attraction \( \mathcal{S} \) to coincide with \( \mathcal{F}_{\infty} \) since the viability kernel is the maximal set on which an admissible feedback can be defined. However, this is, in general, not possible. The reason for this is that the closer we get to the boundary of \( \mathcal{F}_{\infty} \), the more costly (in the sense of our objective \( J_N \)) it may become to steer the system to \( x^* \) and if this happens then the optimization criterion may lead the MPC closed loop to stay near the boundary of \( \mathcal{F}_{\infty} \) instead of approaching \( x^* \). Hence, a central task considered in this chapter is to estimate the basin of attraction \( \mathcal{S} \subseteq \mathcal{F}_{\infty} \) and — conversely — given a set \( K \subseteq \mathcal{F}_{\infty} \) to estimate an optimization horizon \( N \) such that \( K \subseteq \mathcal{S} \) is guaranteed.

As a first step, we consider the problem to determine a recursively feasible set. To this end, for a given horizon length \( N \in \mathbb{N} \cup \{\infty\} \) and a positive constant \( C \) define the level set

\[
V^{-1}_{N}[0,C] := \{ x \in X : V_N(x) \leq C \}.
\]

Since the running costs are supposed to satisfy (8.2.5), existence of the lower bound

\[
M := \inf_{x \in X \setminus \mathcal{N}} \ell^*(x) > 0 \quad (8.3.2)
\]

is ensured. Then, for every \( x \in V^{-1}_{N}[0,C] \setminus \mathcal{N} \), the inequality

\[
V_N(x) \leq \frac{C}{M} \cdot M \leq \frac{C}{M} \cdot \ell^*(x)
\]

holds. The parameter \( C \) can be chosen sufficiently large such that the inequality

\[
V_N(x) \leq \gamma \cdot \ell^*(x) \leq \gamma \cdot \sup_{x \in \mathcal{N} \cap X} \ell^*(x) \leq C \quad (8.3.3)
\]

holds for all \( x \in \mathcal{N} \cap X \). Summarizing, a constant \( \beta = \beta(C,M,\gamma) \) only depending on
8.3. RECURSIVE FEASIBILITY AND ASYMPTOTIC STABILITY

Assumption 8.3.1, Condition (8.2.5), and a parameter $C$ can be found satisfying

$$V_N(x) \leq \beta \cdot \ell^*(x) \quad \forall x \in V_N^{-1}[0, C] \quad \text{and} \quad \mathcal{N} \cap X \subseteq V_N^{-1}[0, C]. \quad (8.3.4)$$

This in particular shows that Assumption 8.3.1 can be extended to arbitrary level sets $V_N^{-1}[0, C]$. This fact is exploited in order to prove Theorem 8.3.3.

**Theorem 8.3.3.** Let Assumption 8.3.1 and Inequality (8.2.5) be satisfied. Take any positive real number $C$ satisfying (8.3.3) and let $M$ be defined as in (8.3.2). In addition, choose $N_0 \in \mathbb{N}$ such that the inequalities

$$C \left( \frac{\beta - 1}{\beta} \right)^{N_0 - 1} < M \quad \text{and} \quad 1 - \alpha_{N_0} > 0 \quad (8.3.5)$$

are satisfied with $\beta := \max\{C/M, \gamma\}$ and $\alpha_N := \beta^2 \left( \frac{\beta - 1}{\beta} \right)^N$. Then, for every $N \geq N_0$ and every $x \in V_N^{-1}[0, C]$, we have

$$V_N(f(x, \mu_N(x))) \leq V_N(x) - (1 - \alpha_N)\ell^*(x). \quad (8.3.6)$$

In particular $V_N(\cdot)$ is a Lyapunov function on the recursively feasible set $V_N^{-1}[0, C]$ which implies recursive feasibility and asymptotic stability of the MPC closed loop.

**Proof.** The proof is an adaptation of the arguments developed in [35] to our setting. In particular, Variant II from Section 3.2 of this chapter is used, whose idea was taken from [54]. Take any $x \in V_N^{-1}[0, C]$. Then $V_N(x) \leq C$ and by hypothesis there exists an admissible control sequence $u^* \in \mathcal{U}^N(x)$ such that $V_N(x) = J_N(x, u^*)$. If we define

$$\ell_k := \ell(x_u(k); x, u^*(k)) \quad \text{for} \quad k \in \{0, 1, \ldots, N - 1\},$$

then $V_N(x)$ can be written as $V_N(x) = \sum_{k=0}^{p-1} \ell_k + V_{N-p}(x_u^*(p); x)$ for any $p = 0, 1, \ldots, N - 1$. This implies $V_{N-p}(x_u^*(p); x) \leq C$, i.e., $x_u^*(p; x) \in V_{N-p}^{-1}[0, C]$. Since $\beta$ only depends on $C, \gamma$, and $M$ (and not on the optimization horizon) from (8.3.4) we obtain the inequality $V_{N-p}(x_u^*(p; x)) \leq \beta \ell_p$. Therefore

$$V_N(x) = \sum_{k=0}^{N-1} \ell_k \leq \sum_{k=0}^{p-1} \ell_k + \beta \ell_p. \quad (8.3.7)$$
If \( f(x, \mu_N(x)) \) is feasible, i.e., if \( f(x, \mu_N(x)) \in \mathcal{F}_N \) holds or, equivalently, \( \mathcal{U}^N(f(x, \mu_N(x))) \neq \emptyset \), we obtain the inequality

\[
V_N(f(x, \mu_N(x))) \leq \sum_{k=1}^{N-2} \ell_k + V_2(x_u^*(N-1; x))
= V_N(x) - \ell_0 - \ell_{N-1} + V_2(x_u^*(N-1; x)). \tag{8.3.8}
\]

In general, however, without additional hypotheses, we cannot guarantee feasibility of \( f(x, \mu_N(x)) \). Still, by setting \( V_2(x_u^*(N-1; x)) = +\infty \) in case of infeasibility we can extend (8.3.8) to this case.

We keep this in mind and show \( x_u^*(N-1; x) \in \mathcal{N} \cap X \) and, thus, \( V_2(x_u^*(N-1; x)) < \infty \). Indeed, by (8.3.7), \( \sum_{k=p+1}^{N-1} \ell_k \leq (\beta - 1)\ell_p \) holds which implies

\[
\sum_{k=p}^{N-1} \ell_k \geq \left( \frac{\beta}{\beta - 1} \right) \sum_{k=p+1}^{N-1} \ell_k \geq \left( \frac{\beta}{\beta - 1} \right)^2 \sum_{k=p+2}^{N-1} \ell_k \geq \ldots \geq \left( \frac{\beta}{\beta - 1} \right)^{N-p-1} \ell_{N-1}
\]

for \( p \in \{0, 1, \ldots, N-1\} \). When \( p = 0 \), since \( x \in V^{-1}_N[0, C] \), we obtain

\[
C \geq V_N(x) \geq \left( \frac{\beta}{\beta - 1} \right)^{N-1} \ell_{N-1}, \tag{8.3.9}
\]

\[
\beta \cdot \ell^*(x) \geq V_N(x) \geq \left( \frac{\beta}{\beta - 1} \right)^{N-1} \ell_{N-1}. \tag{8.3.10}
\]

According to our choice of \( N \) Inequality (8.3.9) implies \( \ell^*(x_u^*(N-1; x)) \leq \ell_{N-1} < M \) and, in view of \( x_u^*(N-1; x) \in X \) and (8.3.2), \( x_u^*(N-1; x) \in \mathcal{N} \cap X \) where our local Assumption 8.3.1 can be invoked. Consequently feasibility and \( V_2(x_u^*(N-1; x)) \leq \gamma \ell_{N-1} \leq \beta \ell_{N-1} \) hold. A further appeal to (8.3.8) and (8.3.10) now gives

\[
V_N(f(x, \mu_N(x))) \leq V_N(x) - \ell_0 + (\beta - 1)\ell_{N-1} \leq V_N(x) - (1 - \alpha_N)\ell^*(x),
\]

i.e. Inequality (8.3.6) and recursive feasibility of the level set \( V^{-1}_N[0, C] \). From this and the bounds on \( V_N \) induced by (8.2.5) and (8.3.4), the Lyapunov function property of \( V_N \) and asymptotic stability follow by standard arguments, see, e.g. [34, Section 5].

**Remark 8.3.4.** The optimization horizon \( N_0 \) guaranteeing stability in Theorem 8.3.3
grows like \( 2(C/M) \ln C \) as \( C \to +\infty \). Indeed the horizon \( N \) must satisfy
\[
N > \frac{2 \ln \beta}{\ln \beta - \ln(\beta - 1)}
\]
and \( \beta \sim (C/M) \) as \( C \to \infty \). We claim that this bound can be improved under additional hypotheses. Let \( \tilde{N} \) be a neighbourhood of the origin such that \( \tilde{N} \cap X \) is controlled forward invariant. Define \( \tilde{M} := \inf_{x \in X \setminus \tilde{N}} \ell^*(x) \) and assume that the horizon \( N \) only satisfies the first inequality in (8.3.5), now with respect to the constant \( \tilde{M} \). Then feasibility of the MPC closed loop trajectory is ensured — as proved in Theorem 8.3.3 — since \( x_u(N-1; x) \in \tilde{N} \cap X \) which is forward invariant. Since feasibility is now ensured, the estimates from [37, Section 6] can be applied to get the improved value
\[
\alpha_N = \frac{(\beta - 1)^N}{\beta^{N-1} - (\beta - 1)^{N-1}}
\]
which is positive when \( N > 2 + \frac{\ln(\beta-1)}{\ln \beta - \ln(\beta - 1)} \). Hence, this bound for the optimization horizon \( N \) behaves asymptotically as \( (C/M) \ln C \) for \( C \to +\infty \).

Often more restrictive controllability conditions are assumed in order to ensure asymptotic stability, see, e.g. [54] or [35] where our local Assumption 8.3.1 was assumed on a (controlled) invariant subset of the viability kernel. We like to point out that no ‘viability’ conditions — such as forward invariance — nor regularity hypotheses on the dynamics \( f(\cdot, \cdot) \) and the control constraint set \( U(\cdot) \) are imposed on \( X \) in this section.

### 8.3.2 Global stability

Theorem 8.3.3 implies that for each compact set \( K \subseteq X \) satisfying \( C := \sup V_\infty(K) < \infty \) the MPC controller yields asymptotic stability for \( N \geq N_0 \) with a basin of attraction \( S \supseteq V_N^{-1}[0, C] \supseteq K \). In order to analyse which kind of sets \( K \) have this property, we consider the set \( V_\infty^{-1}[0, +\infty) = \{ x \in X : V_\infty(x) < +\infty \} \) and the decreasing family of sets \( V_\infty^{-1}[n, +\infty) \supseteq V_\infty^{-1}[n+1, +\infty) \) with varying \( n \in \mathbb{N} \). For these sets we consider the set

---

3 We use the notation \( f(x) \sim g(x) \) as \( x \to \infty \) to indicate that the functions \( f(\cdot) \) and \( g(\cdot) \) have asymptotically the same behaviour, i.e., that \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \) holds.

4 Such a neighborhood \( \tilde{N} \) exists under the assumptions of Theorem 8.3.3 as one may define \( \tilde{N} \) as the interior of a sublevel set of \( V_N \).
valued limit
\[ O := \lim_{n \to \infty} V^{-1}_\infty[n, +\infty) = \bigcap_{n \in \mathbb{N}} V^{-1}_\infty[n, +\infty), \]

cf. [2, Section 1.1]. In many cases of interest the set \( O \) has zero measure. If, for example, the value function is uniformly bounded or continuous on \( V^{-1}_\infty[0, +\infty) \), then the set \( V^{-1}_\infty[0, +\infty) \cap O \) is empty. Later on, conditions ensuring one of these two stipulations are investigated for linear systems, see Section 8.4.

The relevance of the set \( O \) for MPC stems from the following claim: take any compact set \( K \subset V^{-1}_\infty[0, +\infty) \). Then we claim that there exists \( C \in \mathbb{R}_{>0} \) such that \( V^{-1}_\infty[0, C] \supseteq K \).

Suppose to the contrary, it would exist a convergent sequence \( (x_n)_{n \in \mathbb{N}_0} \subset K \) such that \( x_n \to x \in K \) and \( V_\infty(x_n) > n \). Thus \( x_n \in V^{-1}_\infty[n, +\infty) \) and \( x \in O \). This is not possible since \( x \in K \).

The following theorem shows the consequences of this claim for the MPC controller.

**Theorem 8.3.5.** Let Assumption 8.3.1 and Condition (8.2.5) be satisfied and \( K \subset V^{-1}_\infty[0, +\infty) \setminus O \) be a compact set. Then there exists \( N_K \in \mathbb{N} \) such that for each \( N \geq N_K \) the MPC closed loop is recursively feasible and asymptotically stable with basin of attraction \( S \supseteq K \).

Theorem 8.3.5 provides a nonlinear extension of the linear results shown in [49]. It tells us that, for a sufficiently large horizon, the MPC algorithm provides a recursively feasible and asymptotically stable closed loop on every compact set in which the value function is finite, as long as we avoid ’small’ areas of bad behaviour (close to \( O \)). Note that Theorem 8.3.5 is also applicable if state constraints are present and thus extends [41].

The set \( O \) can comprise points \( x \) satisfying \( V_\infty(x) = +\infty \), but may also contain points \( x \) which are controllable to \( x^* \) in finite time with \( V_\infty(x) < \infty \). The latter situation is shown in the following example.

**Example 8.3.6.** Consider the one dimensional system \( x^+ = u x^2 + (1 - u)(x - \frac{3}{2}) \) with \( X = [-\frac{1}{2}, 2] \), \( U = \{1\} \cup \{0\} \) and equilibrium \( x^* = 0 \). Here, for a cost function, say \( \ell(x, u) := |x| \), the set \( V^{-1}_\infty(0, +\infty) \) is equal to \( X \). Nevertheless, the set \( O \) is nonempty.
since $\mathcal{O} = \{1\}$ holds and $V_\infty(1)$ is finite, cf. Figure 8.1

8.4 Linear Systems

This section is dedicated to linear constrained systems

$$x^+ = Ax + Bu, \quad (x,u) \in \mathcal{E}.$$  \hfill (L)

For this class of systems we will be able to provide more precise estimates for the constants involved in the general nonlinear results of the last section. Moreover, we will be able to characterize the “exceptional set” $\mathcal{O}$ in more detail and investigate the relation between the stabilizable set $\mathcal{S}$ and the viability kernel $\mathcal{F}_\infty$.

Like for nonlinear systems, we will base our analysis on the controllability Assumption 8.3.1. It should be noted that for linear systems an alternative methodology for ensuring stability and recursive feasibility of the MPC closed loop is available, cf. [49]. However, this approach requires precise knowledge on the growth of the value function and can, thus, be seldomly applied if constraints are present. In contrast to that, techniques based on Assumption 8.3.1 can be applied since this condition is significantly easier to verify. Below, we prove that Assumption 8.3.1 can always be ensured for a large class of linear constrained systems. To this end, we make the following two assumptions.
Assumption 8.4.1. The constraint set $E$ is convex, compact, and contains the origin $(0,0)$ in its interior.

Assumption 8.4.2. The linear system described by the pair $(A,B)$ is stabilizable.

8.4.1 Characterization of the Viability Kernel for Linear Systems

In the next two propositions we characterize the viability kernel $F_\infty$ in order to gain insight into the structure of the set $S$ on which the MPC feedback law $\mu_N$ stabilizes the system.

Proposition 8.4.3. Consider the nonlinear system (8.2.1) and the viability kernel $F_\infty$ defined by (8.3.1). Then, the following claims hold:

(a) If $E$ is compact and the dynamics $f(\cdot, \cdot)$ are continuous, $F_\infty$ is compact.

(b) If $E$ is convex and the dynamics $f(\cdot, \cdot)$ are linear, $F_\infty$ is a convex set.

Proof. Without loss of generality let $F_\infty \neq \emptyset$. We show that $F_\infty$ is bounded and closed in order to prove part (a) of the proposition. Since $F_\infty \subseteq X$, boundedness follows directly from the boundedness of $E$. Hence, it remains to show $F_\infty = cl\{F_\infty\}$. Take any $x \in cl\{F_\infty\}$. Then in particular $x \in X$. By definition of closure we can find points $x_i \in F_\infty$ such that $x_i \to x$ and by definition of $F_\infty$ we can find admissible controls $u_i$ such that $f(x_i, u_i) \in F_\infty$ holds for every $i \in \mathbb{N}$. Now each pair $(x_i, u_i)$ belongs to the compact set $E$ so that extracting a subsequence if necessary $(x_i, u_i) \to (x, u) \in E$. But then by continuity $f(x, u) \in cl\{F_\infty\}$. This proves that for every $x \in cl\{F_\infty\}$, there exists $u \in U(x)$ such that $f(x, u) \in cl\{F_\infty\}$, namely, $cl\{F_\infty\}$ is a forward invariant set. Therefore $cl\{F_\infty\} \subseteq F_\infty$ which completes part (a) of the proof since the reverse inclusion is obvious.

Part (b) is a straightforward application of the respective definitions. Take $x_1, x_2 \in F_\infty$ and a convex combination $\lambda x_1 + (1-\lambda)x_2$, $\lambda \in [0,1]$, of them. By definition there exist $u_1 \in U^\infty(x_1)$ and $u_2 \in U^\infty(x_2)$ such that $(x_{u_1}(k; x_1), u_1(k)) \in E$ and $(x_{u_2}(k; x_2), u_2(k)) \in E$ for every $k \in \mathbb{N}_0$. Because of the linearity of the dynamics

$$\lambda x_{u_1}(k; x_1) + (1-\lambda)x_{u_2}(k; x_2) = x_{\lambda u_1+(1-\lambda)u_2}(k; \lambda x_1 + (1-\lambda)x_2)$$

holds and the result is a consequence of the convexity assumption on $E$. \qed
Proposition 8.4.4. Consider the viability kernel $\mathcal{F}_\infty$ given by (8.3.1) and linear dynamic as in (L). Let Assumption 8.4.1 be satisfied. Then, the following assertions hold.

(a) The set $\lambda \mathcal{F}_\infty$ is forward invariant for any $\lambda \in [0, 1]$. More precisely, take any $\lambda \in [0, 1]$ and $x \in \lambda \mathcal{F}_\infty$, there exists an admissible control sequence $u = (u(k))_{k \in \mathbb{N}} \in \mathcal{U}_\infty(x)$ such that $(x_u(k; x), u(k)) \in \lambda \mathcal{E} \subseteq \mathcal{E}$ and $x_u(k; x) \in \lambda \mathcal{F}_\infty \quad \forall \, k \in \mathbb{N}_0$.

(b) If, in addition, Assumption 8.4.2 holds, the origin is contained in the interior of the viability kernel, i.e., $0 \in \text{int} \mathcal{F}_\infty$.

Proof. Fix any $\lambda \in (0, 1]$. If $\lambda = 0$ the result is obvious being 0 an equilibrium. Given any $x \in \lambda \mathcal{F}_\infty$ we have that $x/\lambda \in \mathcal{F}_\infty$ and thus there is $u_\lambda \in \mathcal{U}_\infty(x/\lambda)$ such that $(x_u(k; x/\lambda), u_\lambda(k)) \in \mathcal{E}$ and $x_u(k; x/\lambda) \in \mathcal{F}_\infty \quad \forall \, k \in \mathbb{N}_0$.

Define the control sequence $u := \lambda u_\lambda$, we claim that $u \in \mathcal{U}_\infty(x)$. By linearity $\lambda x_u(k; x/\lambda) = x_u(k; x)$ holds and part (a) follows upon multiplication by $\lambda$.

Part (b). Since the pair $(A, B)$ is stabilizable, a feedback law $F \in \mathbb{R}^{m \times n}$ exists such that $\varrho(A + BF) < 1$ holds, i.e. all eigenvalues of the closed loop given by $A + BF$ are contained in the interior of the unit circle, cf. [39]. As a consequence, constants $C \geq 1$ and $\sigma \in (0, 1)$ exist such that, for each state $x_0 \in \mathbb{R}^n$, the closed loop solution $(x_F(k; x_0))_{k \in \mathbb{N}_0}$ generated by $x_F(k+1; x_0) = (A + BF)x_F(k; x_0)$, $x_F(0; x_0) = x_0$, satisfies

$$|x_F(k; x_0)| \leq \|(A + BF)^k\| |x_0| \leq C\sigma^k|x_0| \quad \forall \, k \in \mathbb{N}_0.$$  (8.4.1)

This shows in particular that $|(x_F(k; x_0), Fx_F(k; x_0))| \leq C\sigma^k(\|F\| + 1)|x_0|$ holds. Recall that $(0, 0) \in \text{int} \mathcal{E}$ by hypothesis. Therefore existence of an $\varepsilon$-ball $\varepsilon B \subseteq \mathcal{E}$ is ensured. Hence, $(x_F(k; x_0), Fx_F(k; x_0))$, $k \in \mathbb{N}_0$, is admissible, which implies $x_0 \in \mathcal{F}_\infty$ for arbitrary $x_0 \in \delta B$ with $C(\|F\| + 1)\delta \leq \varepsilon$. This completes the proof of the proposition. $\square$

According to Propositions 8.4.3 and 8.4.4, when Assumptions 8.4.1 and 8.4.2 are in force, the viability kernel $\mathcal{F}_\infty$ is a compact and convex set containing the origin in its interior and, for any $\lambda \in [0, 1]$, the shrunk set $\lambda \mathcal{F}_\infty$ is controlled forward invariant, i.e. for
any \( x \in \lambda \mathcal{F}_\infty \) there exists a feasible state trajectory remaining in \( \lambda \mathcal{F}_\infty \) for any time. In addition, by the fact that \( \mathcal{F}_\infty \) is the maximal forward invariant set, we have information about the behaviour of feasible trajectories on \( \partial \mathcal{F}_\infty \).

**Proposition 8.4.5.** Consider the discrete time system (8.2.1) and assume that the dynamics \( f(\cdot) \) are continuous. If \( x \in \partial \mathcal{F}_\infty \), every feasible trajectory will remain on the boundary unless it touches \( \partial \mathcal{X} \).

**Proof.** The result derives from the fact that \( \mathcal{F}_\infty \) is the maximal forward invariant set. If there were a control \( u \in U(x) \) for \( x \in \partial \mathcal{F}_\infty \setminus \partial \mathcal{X} \) such that \( f(x,u) \in \text{int} \mathcal{F}_\infty \), then by continuity this would be true on a neighbourhood of \( x \) making \( \mathcal{F}_\infty \) larger. For details we refer to [50].

8.4.2 Linear Quadratic MPC

The following Proposition provides a uniform bound for \( V_\infty(\cdot) \) on the interior of the viability kernel. This is a key ingredient in order to characterize the operating range of the MPC feedback law \( \mu_N \). In particular, the set \( \mathcal{O} \) constructed in Subsection 8.3.2 if nonempty, can contain only points of the boundary of the viability kernel.

**Proposition 8.4.6.** Let Assumptions 8.4.1 and 8.4.2 be satisfied and let quadratic running costs \( \ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \) be given by

\[
\ell(x,u) := (x^T u^T) \begin{pmatrix} Q & N \\ N^T & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}
\] (8.4.2)

with symmetric matrices \( Q \in \mathbb{R}^{n \times n} \) and \( R \in \mathbb{R}^{m \times m} \). Then, for each \( \lambda \in [0,1) \) the optimal value function is uniformly bounded from above on \( \lambda \mathcal{F}_\infty \), i.e., a constant \( M = M(\lambda) \in \mathbb{R}_{\geq 0} \) exists such that \( V_\infty(x) \leq M \) holds for all \( x \in \lambda \mathcal{F}_\infty \).

**Proof.** We borrow techniques from [31, Lemma 12]. Fix any \( \lambda \in (0,1) \) and choose \( x \in \lambda \mathcal{F}_\infty \) (the result is obvious for \( \lambda = 0 \)). As we have seen in Proposition 8.4.4a) there exists \( u_I \in U^\infty(x) \) such that

\[
(x_{u_I}(k;x), u_I(k)) \in \lambda \mathcal{E} \quad \text{and} \quad x_{u_I}(k;x) \in \lambda \mathcal{F}_\infty \quad \forall \ k \in \mathbb{N}_0,
\]

i.e. a feasible state trajectory which remains in the interior of \( \mathcal{F}_\infty \). In addition, since
(A, B) is stabilizable, a feedback law \( F \in \mathbb{R}^{m \times n} \) exists such that the corresponding closed loop \( x_F^+ = (A + BF)x_F \) satisfies Inequality (8.4.1), i.e.

\[
|x_F(k; x)| \leq \|(A + BF)^k\| |x| \leq C\sigma^k|x| \quad \forall k \in \mathbb{N}_0
\]

for some \( C \geq 1 \) and \( \sigma \in (0, 1) \) which, in particular, implies \( x_F(k; x) \to 0 \) as \( k \to \infty \). However, the pair \((x_F, Fx_F)\) may not satisfy the constraints. The idea is to take a convex combination of these two trajectories and exploit linearity and convexity of the data to show that such a combination defines a feasible trajectory which converges to 0. When a sufficiently small neighbourhood of the origin is reached, the constraints can be neglected and the feedback law \( F \) is applied. This procedure yields a uniform bound for \( V_\infty(\cdot) \). Analytic arguments follow.

Using the control sequence \( u_F \) given by \( u_F(k) := Fx_F(k; x), k \in \mathbb{N}_0 \), we have

\[
|(x_F(k; x), u_F(k))| \leq C\sigma^k(\|F\| + 1)|x| \leq L\lambda\sigma^kd_{\min} \quad \forall k \in \mathbb{N}_0 \tag{8.4.3}
\]

where \( L := C(\|F\| + 1)d_{\max}d_{\min}^{-1} \) with \( d_{\min} := \inf_{x \in \partial X} |x| > 0 \) and \( d_{\max} := \sup_{x \in X} |x| < \infty \). Hence, \((x_F(k; x), u_F(k)) \in L\lambda E\) holds for all \( k \in \mathbb{N}_0 \). If \( \lambda L \leq 1 \), \( u_F \in U^\infty(x) \) so that \( x_F(k; x) \) is feasible for every \( k \in \mathbb{N}_0 \) and a uniform bound for \( V_\infty(\cdot) \) is given by \( \sup_{x \in X} J_\infty(x, u_F) \leq \alpha \) for some \( \alpha \in \mathbb{R}_{\geq 0} \).

Otherwise, for \( \lambda L > 1 \), the control sequence \( u \) is defined as \( u(k) := \mu u_I(k) + (1 - \mu)u_F(k), k \in \mathbb{N}_0 \), with \( \mu := \frac{\lambda L - 1}{\lambda L} \in (0, 1) \). Then, by linearity of the dynamics

\[
x_u(k; x) = \mu x_{u_I}(k; x) + (1 - \mu)x_{u_F}(k; x).
\]

Our choice of \( \mu \) implies \( \mu \lambda + (1 - \mu)L\lambda \leq 1 \) and, thus, \((x_u(k; x), u(k)) \in E\) for all \( k \in \mathbb{N}_0 \) which is, in turn, equivalent to admissibility of \( u \). Now since \( x_F(k; x) \to 0 \) as \( k \to \infty \), if \( k \) is taken large enough, the pair \((x_u(k; x), u(k)) \in \varepsilon \lambda E\) for some \( \varepsilon \in (\mu, 1) \). More precisely, \((x_u(k; x), u(k)) \in \varepsilon \lambda E\) holds if (recall Estimate (8.4.3))

\[
\mu \lambda + (1 - \mu)L\lambda\sigma^k \leq \varepsilon \lambda.
\]

Call \( k^* \) the first integer such that this condition is satisfied. If for example \( \varepsilon := \mu + \frac{1 - \mu}{L} \), then \( k^* \) is such that \( \sigma^{k^*} \leq \frac{1}{L^2} \) and so it is the only integer satisfying \( \log_\sigma(\frac{1}{L^2}) \leq k^* <
log_2 \left( \frac{1}{P^2} \right) + 1.

The point \( x_u(k^*; x) \in \lambda E F_\infty \) and the procedure followed so far can be iterated, say \( m \) times, until \( x_u(mk^*; x) \in \lambda E m F_\infty \) and \( \lambda E m L \leq 1 \). We keep calling \( u \) the admissible sequence that transfers the point \( x \in \lambda E F_\infty \) to the point \( x_u(mk^*; x) \in \lambda E m F_\infty \). As soon as the condition \( \lambda E m L \leq 1 \) is satisfied, we switch to the feedback \( F \) which ensures that the system feasibly converges to the origin with uniformly bounded costs for each state contained in \( \lambda E m F_\infty \). Since \( X \) and the constraint set are bounded, the prior caused costs are also uniformly bounded since the number of steps needed in order to reach this set is bounded by \( mk^* \). Note that in particular

\[
V_\infty(x) \leq J_{mk^*}(x, u) + J_\infty(x_u(mk^*; x), u_F).
\]  

(8.4.4)

This, in addition to the convergence to the origin already shown in [31, Lemma 12], provides bounds on the number of steps required in order to reach an arbitrarily small neighborhood of the origin — independent of the chosen initial state \( x \in \lambda E F_\infty \).

We are now ready to show that for linear quadratic systems, in Theorem 8.3.5, Assumption 8.3.1 and Condition (8.2.5) can be replaced by the easily checkable Assumptions 8.4.1 and 8.4.2. Moreover, the set \( V^{-1}_\infty[0, +\infty) \setminus O \), see Theorem 8.3.5, can be replaced by the interior \( \text{int} F_\infty \) of the viability kernel.

**Theorem 8.4.7.** Let Assumptions 8.4.1 and 8.4.2 hold. Furthermore, let the dynamics be given by (L) and the running costs by (8.4.2) such that the matrix comprised of \( Q, R, \) and \( N \) is positive definite. Let \( K \subseteq \text{int} F_\infty \) be a compact set. Then, a prediction horizon \( N_K \in \mathbb{N} \) exists such that, for each \( N \geq N_K \), the MPC feedback law \( \mu_N \) asymptotically stabilizes the closed loop at \( x^* = 0 \) on a recursively feasible set \( S \supseteq K \).

**Proof.** Since the running costs \( \ell \) are positive definite there exist constants \( c, \tau \) such that \( c|x| \leq \ell^*(x) \leq \tau|x| \), i.e. Condition (8.2.5) holds. Furthermore, since the origin is contained in the interior of the constraint set \( E \) and the pair \( (A, B) \) is supposed to be stabilizable, a neighborhood \( N \) of the origin exists such that an LQR can be applied neglecting the constraints. Then, the solution \( P \) of the algebraic Riccati equation fulfills \( V_\infty(x_0) = x_0^T P x_0 \leq c|x_0|^2 \leq \rho \ell^*(x_0) \) with \( \rho := c \zeta^{-1} \) on \( N \) where \( c \) is the maximal eigenvalue of \( P \), implying Assumption 8.3.1.
Moreover, since $K$ is compact and contained in $\text{int} \mathcal{F}_\infty$, we can conclude that $K \subseteq \text{int} \lambda \mathcal{F}_\infty$ for some $\lambda \in (0, 1)$. Hence, by Proposition 8.4.6, $V_\infty$ is bounded on a neighborhood of $K$ and consequently $K \subseteq V^{-1}_\infty[0, +\infty) \setminus \mathcal{O}$.

Hence, all assumptions of Theorem 8.3.5 are satisfied and the assertion follows from this theorem.

Next, under the assumptions of Theorem 8.4.7 we are going to investigate the dependence of the horizon $N_K$ on the distance of the compact set $K$ from the boundary of the viability kernel $\partial \mathcal{F}_\infty$. To this end, denoting the control $u$ in (8.4.4) by $u_x$, we obtain

$$
sup_{x \in \lambda \mathcal{F}_\infty} V_\infty(x) \leq sup_{x \in \lambda \mathcal{F}_\infty} J_{mk^*}(x, u_x) + sup_{x \in X} J_\infty(x, u_F) \leq \beta mk^* + \alpha,$$

for constants $\alpha, \beta > 0$ only depending on the data of the problem ($m = 0$ holds for $\lambda L \leq 1$).

We emphasise that $m$ depends on $\lambda$, indeed $m$ is the first integer which satisfies $\varepsilon m \leq \frac{1}{L}$.

We also observed that $k^*$ can be chosen to be the unique integer which satisfies $\log_2 \left( \frac{1}{\lambda} \right) \leq k^* < \log_2 \left( \frac{1}{T^2} \right) + 1$. Consequently defining $\tilde{\beta} := \beta k^*$ we obtain $sup_{x \in \lambda \mathcal{F}_\infty} V_\infty(x) \leq \tilde{\beta} m + \alpha$.

Following our choice $\varepsilon = \mu + \frac{1-\mu}{L} \quad \text{(where we recall $\mu = \frac{\lambda}{L(L-1)}$)}$ we have that $\varepsilon = 1 - \frac{1-\lambda}{\lambda L}$.

This yields an estimate for the growth of $m$. More precisely $m \sim \frac{4 \ln L}{1-\lambda}$ as $\lambda \to 1$.

When we are in a sufficiently small neighbourhood of the origin, say $\delta \mathbb{B}$, constraints can be neglected and $V_\infty(x) = x^TPx$, where $P$ is the solution of the algebraic Riccati equation. This in turn gives a bound for $V_\infty(\cdot)$ of the type $V_\infty(x) \leq \rho|x|^2$ for all $x \in \delta \mathbb{B}$.

Away from the origin, when constraints are present this bound is no longer satisfied. We have shown, though, that it is possible to find constants $\tilde{\beta}$ and $\tilde{\alpha} := \max\{\rho, \alpha/\delta\}$ such that

$$V_\infty(x) \leq \tilde{\beta} m_x + \tilde{\alpha}|x|^2$$

with $m_x \to +\infty$ when $x \to \partial \mathcal{F}_\infty$. Note that the upper bound diverges to $+\infty$ as $x \to \partial \mathcal{F}_\infty$.

The discussion above can be summarised in the following corollaries.

**Corollary 8.4.8.** Let the assumptions of Proposition 8.4.6 be satisfied. Then there exist constants $\tilde{\alpha} > 0$ and $\tilde{\beta} > 0$ depending only on the data of the problem such that for any $x \in \text{int} \mathcal{F}_\infty$

$$V_\infty(x) \leq \tilde{\beta} m_x + \tilde{\alpha}|x|^2$$
with \( m_x \to +\infty \) when \( x \to \partial \mathcal{F}_\infty \). Asymptotically, \( m_x \) behaves like \( m_x \sim \frac{\omega L \ln L}{\operatorname{dist}(x; \partial \mathcal{F}_\infty)} \) as \( x \to \partial \mathcal{F}_\infty \), where \( \omega \in [\inf_{x \in \partial \mathcal{F}_\infty} |x|, \sup_{x \in \mathcal{F}_\infty} |x|] \) and \( L \) is from the proof of Proposition 8.4.6. Moreover, \( m_x = 0 \) in a sufficiently small neighbourhood of the origin.

**Proof.** We only have to verify the assertion \( m_x \sim \frac{\omega L \ln L}{\operatorname{dist}(x; \partial \mathcal{F}_\infty)} \) for some \( \omega \in [f^-, f^+] \) where \( f^- := \inf_{x \in \partial \mathcal{F}_\infty} |x| \) and \( f^+ := \sup_{x \in \mathcal{F}_\infty} |x| \). To this end, we introduce a well known construct from convex analysis to compare \( \lambda \) and \( \operatorname{dist}(x; \partial \mathcal{F}_\infty) \) when \( x \in \lambda \partial \mathcal{F}_\infty \).

Let \( g(x) := \inf \{ \gamma > 0 : \gamma^{-1} x \in \mathcal{F}_\infty \} \).

This function is Lipschitz with Lipschitz constant \( 1/f^- \).

For every \( x \in \lambda \partial \mathcal{F}_\infty \), \( \lambda \in (0, 1) \), we define \( \pi(x) \) to be a projection of \( x \) onto \( \partial \mathcal{F}_\infty \). Then \( \operatorname{dist}(x; \partial \mathcal{F}_\infty) = |x - \pi(x)| \). Moreover \( g(x) = \lambda \) and \( g(\pi(x)) = 1 \). Therefore

\[
1 - \lambda = g(\pi(x)) - g(x) \leq \frac{1}{f^-} |x - \pi(x)| = \frac{1}{f^-} \operatorname{dist}(x; \partial \mathcal{F}_\infty).
\]

On the other hand

\[
\operatorname{dist}(x; \partial \mathcal{F}_\infty) \leq |x - \lambda x| \leq \frac{1 - \lambda}{\lambda}.
\]

Using this estimate we arrive at the following estimate for the optimization horizon needed in order to ensure \( K \subseteq S \).

**Corollary 8.4.9.** Given a compact set \( K \subseteq \text{int} \mathcal{F}_\infty \), there exists a constant \( D \), only depending on the data of the problem, such that

\[
\sup_{x \in K} V_\infty(x) \leq \frac{D}{\operatorname{dist}(K; \partial \mathcal{F}_\infty)}.
\] (8.4.5)

Moreover, whenever \( N \geq N_K \), where \( N_K \) is the smallest integer satisfying

\[
N_K > 2 + \frac{\ln(\beta - 1)}{\ln \beta - \ln(\beta - 1)}
\] (8.4.6)

for \( \beta = \max\{D \cdot (M \cdot \operatorname{dist}(K; \partial \mathcal{F}_\infty))^{-1}, \gamma\} \), then the MPC closed loop is asymptotically stable.

\[\text{Given any set } \Omega \subset \mathbb{R}^n \text{ and } x \in \mathbb{R}^n, \operatorname{dist}(x; \Omega) \text{ denotes the Euclidean distance of the point } x \text{ from the set } \Omega. \text{ Given a second set } K \subset \mathbb{R}^n, \operatorname{dist}(K; \Omega) := \min_{x \in K} \operatorname{dist}(x; \Omega).\]
stable with recursively feasible basin of attraction \( S \supseteq K \). Asymptotically, \( N_K \) behaves like

\[
\frac{D}{M \cdot \text{dist}(K; \partial F_\infty)} \ln \left( \frac{D}{\text{dist}(K; \partial F_\infty)} \right).
\]

**Proof.** The bound in (8.4.5) follows directly from Corollary 8.4.8 choosing a constant \( D \) sufficiently large. Theorem 8.3.3 and Remark 8.3.4 then yield the inequalities for \( N_K \). \( \square \)

The following example illustrates that the required prediction horizon grows rapidly for initial values approaching the boundary of the viability kernel.

**Example 8.4.10.** We consider the controllable and, thus, in particular stabilizable linear system given by

\[
x^+ = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

with constraints \( X := [-100, 100] \times [-1, 1] \) and

\[
U := \left\{ u \in \mathbb{R}^2 : \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}^T \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \leq \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T \right\}.
\]

The quadratic stage costs are given by

\[
\ell(x, u) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]

Then, the minimal stabilizing horizon \( \hat{N} := \min \{ N \in \mathbb{N} : x_\mu(k; x_0) \to 0 \text{ for } k \to \infty \} \) w.r.t. the origin (controlled equilibrium for \( u^* = (0 0)^T \)) in dependence of given initial values are shown in the following table\(^6\)

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<th>0.5</th>
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</thead>
<tbody>
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<tr>
<td>( \hat{N} )</td>
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<td>6</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>16</td>
</tr>
</tbody>
</table>

\(^6\)Note that the point \((0.5, 1)^T\) is not contained in the viability kernel \( F_\infty \). \( \hat{N} \) was computed with the Matlab-routine mpc ExampleBGW.m which is available for download at [http://num.math.uni-bayreuth.de/de/team/Gruene_Lars/publications/boccia_et_al_feasibility_2013](http://num.math.uni-bayreuth.de/de/team/Gruene_Lars/publications/boccia_et_al_feasibility_2013).
8.4.3 Boundedness of $V_\infty$ on the Viability Kernel $F_\infty$

In the preceding Subsection 8.4.2 we considered the stabilization task for arbitrary compact sets contained in the interior of the viability kernel $F_\infty$. However, it follows from Theorem 8.3.3 that for each sufficiently large $N$ MPC will yield asymptotic stability with the basin of attraction $\mathcal{S}$ containing the whole viability kernel $F_\infty$ if $\sup V_\infty(F_\infty)$ is finite. In this final section we show that this property implies stationarity of the feasible sets $F_N$.

We say that the feasible sets $F_N$ become stationary, if there exists $N_0 \in \mathbb{N}$ with $F_N = F_{N_0}$ for all $N \geq N_0$. In [42, Theorem 5.3] (see also [35, Section 5.1]), it was shown that stationarity of the feasible sets is sufficient for recursive feasibility of $F_\infty$ for all optimization horizons $N \geq N_0 + 1$. In the following theorem we show that it is also necessary for $V_\infty$ being bounded on the viability kernel $F_\infty$.

**Theorem 8.4.11.** Consider the linear system (L) with positive definite quadratic running costs $\ell$ and let Assumptions 8.4.1 and 8.4.2 be satisfied. Then, if $V_\infty(x) \leq c$ holds for some $c \in \mathbb{R}_{>0}$ and all $x \in F_\infty$, the feasible sets $F_N$ become stationary for some $N_0 \in \mathbb{N}$.

**Proof.** By definition $F_N \supseteq F_\infty$. An adaptation of the proof of Proposition 8.4.3(b) shows that $F_N$ is convex and it is an easy exercise to prove that $V_N$ is convex. We prove the result by showing the existence of $N_0$ with $F_{N_0} = F_\infty$, which implies stationarity. We proceed by contradiction, i.e., we assume that $F_N \supseteq F_\infty$ holds for every $N \in \mathbb{N}$. If $N \in \mathbb{N}$ is chosen sufficiently large, then for every $x_0 \in F_N \setminus F_\infty$ we have that $V_N(x_0) > c + 2$. Indeed, any trajectory originating at $x_0$ cannot reach $F_\infty$ and in particular remains outside a ball around the origin. Fix a natural number $N \in \mathbb{N}$ with such property and observe that by convexity of the set $F_N$ we may chose $x \in F_N \setminus F_\infty$ and $y \in \partial F_\infty$ such that $\lambda y + (1 - \lambda)x \in F_N \setminus F_\infty$ for all $\lambda \in (0, 1)$. This implies the inequalities $V_N(\lambda y + (1 - \lambda)x) > c + 2$ for all $\lambda \in (0, 1)$ and $V_N(y) \leq V_\infty(y) \leq c$. Then for all $\lambda \in (0, 1)$, convexity of $V_N$ yields

$$c + 2 < V_N(\lambda y + (1 - \lambda)x) \leq \lambda V_N(y) + (1 - \lambda)V_N(x) \leq \lambda c + (1 - \lambda)V_N(x).$$

For $\lambda$ sufficiently close to 1 we obtain the desired contradiction. (Note that $V_N(x)$ is bounded).

The converse is not true in general as shown in the following Example 8.4.12.
Example 8.4.12. Consider the discrete time system in \( \mathbb{R} \) given by

\[
x^{+} = 2x + u
\]

with constraint set \( \mathcal{E} := [-1,1] \times [-1,1] \).

Since every \( x \in X = [-1,1] \) is a controlled equilibrium \( u = -x \) \( \mathcal{F}_\infty = X \) and, thus, \( \mathcal{F}_N = \mathcal{F}_\infty \) actually holds for every \( N \in \mathbb{N} \). Yet, for any positive definite quadratic cost \( V_\infty \) fails to be bounded on \( \partial \mathcal{F}_\infty \) and grows unboundedly for \( x \rightarrow \partial \mathcal{F}_\infty \), as the following computation shows.

If \( x_0 = 1 \) the only admissible control sequence \( u \) is \( u \equiv -1 \) for every time instant. Indeed \( x_u(k;1) = 1 \) for every \( k \in \mathbb{N} \). Therefore as soon as we define a cost say \( \ell(x,u) = x^2 \) we have that \( V_\infty(1) = +\infty \). The point \( x_0 = -1 \) has a similar behaviour. Every other initial point \( x_0 \in (-1,1) = X \setminus \{1,-1\} \), different from 1 and -1, can be controlled to zero in finite time by

\[
u_{x_0}(k) = -\text{sign}(x_{u_{x_0}}(k;x_0)) \min\{2|x_{u_{x_0}}(k;x_0)|,1\}.
\]

However, the closer \( x_0 \) to 1 or -1, the longer it will take before an interval of the form \([-\delta,\delta]\) for \( \delta \in (0,1) \) can be reached. Hence, as \( x_0 \rightarrow 1 \) or \( x_0 \rightarrow -1 \), the value function \( V_\infty(x_0) \) tends to \( +\infty \), cf. Figure 8.2.
Chapter 9

Linear MPC and Continuity of the Value Function

In Chapter 8 we studied stability and recursive feasibility of non-linear MPC schemes without stabilizing terminal constraints or costs but imposing state and control constraints. We then applied those results to linear systems deriving stronger properties.

We continue in this chapter with the investigation of linear MPC algorithms. Particular attention is paid to furnish sufficient conditions for the infinite horizon optimal value function $V_\infty(.)$ to be continuous. The regularity of the value function is a fundamental condition to establish robustness properties of the MPC scheme, see for details [32]. A perhaps surprising example shows that the value function can be discontinuous at the boundary of the viability kernel but only if we allow the dimension of the system $n \geq 3$.

Aim of the chapter is also to highlight the difference of our formulation with respect to previous results. In [49] stability and recursive feasibility was shown for controllable linear quadratic systems with mixed linear state and control constraints on any compact subset of $I_\infty := \text{dom } V_\infty$, the domain of the infinite horizon optimal value function. This is shown to coincide with the points that can be steered to the origin in finite time. We show the same results of [49] adapted to our setting with a particular emphasis on analyzing the basin of attraction for a given prediction horizon $N$. Stabilizable linear systems are also considered in [53] but in an unconstrained framework.
9.1 Model Predictive Control

Asymptotic stability of the discrete time linear constrained system

\[ x^+ = Ax + Bu, \quad (x, u) \in \mathcal{E} \]  

with respect to the origin is investigated. The data for \( (9.1.1) \) comprises matrices \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) and a set \( \mathcal{E} \subseteq \mathbb{R}^n \times \mathbb{R}^m \). The successor state \( x^+ \) is determined by the dynamics \((A, B)\) in dependence of the current state \( x \in \mathbb{R}^n \) and the control input \( u \in \mathbb{R}^m \).

The state trajectory emanating from initial state \( x_0 \) and generated by the control sequence \( u = (u(k))_{k \in \mathbb{N}_0} \) is denoted by \( x_u(k; x_0) \), \( k \in \mathbb{N}_0 \). Here the trajectory \( x_u \) is defined iteratively by

\[ x_u(k + 1; x_0) = Ax_u(k; x_0) + Bu(k) \quad \text{and} \quad x_u(0; x_0) = x_0. \]

For a given set \( \mathcal{E} \), the set of admissible states is given by the projection of the set \( \mathcal{E} \) onto the state space \( \mathbb{R}^n \), i.e.

\[ X := \operatorname{proj}_{\mathbb{R}^n}(\mathcal{E}) = \{ x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \text{ s.t. } (x, u) \in \mathcal{E} \}. \]

Furthermore, for a given admissible state \( x \in X \), the control constraints can be represented by

\[ U(x) := \{ u \in \mathbb{R}^m : (x, u) \in \mathcal{E} \}. \]

The constraints in \( (9.1.1) \) may equivalently be written as \( x \in X \) and \( u \in U(x) \) and we refer indistinctly to either formulations depending on our convenience. Two important concepts to be considered when dealing with constraints are feasibility and admissibility.

**Definition 9.1.1** (Admissibility and Feasibility). A sequence of control values \( u = (u(0), u(1), \ldots, u(N - 1)) \) is called admissible for \( x_0 \in X \) and \( N \in \mathbb{N} \cup \{ \infty \} \), if the conditions

\[ (x_u(k; x_0), u(k)) \in \mathcal{E} \quad \text{and} \quad x_u(N; x_0) \in X \]
9.1. MODEL PREDICTIVE CONTROL

hold for all \( k \in \{0, \ldots, N - 1\} \). The set of all admissible control sequences of length \( N \) is denoted by \( \mathcal{U}^N(x_0) \). The feasible set for a horizon length \( N \in \mathbb{N} \cup \{\infty\} \) is defined as

\[
\mathcal{F}_N := \{ x \in X : \mathcal{U}^N(x) \neq \emptyset \}.
\]

(9.1.2)

The set \( \mathcal{F}_\infty \) is also called viability kernel.

Our goal is to find a static state feedback \( \mu : \mathbb{R}^n \rightarrow \mathbb{R}^m \) which asymptotically stabilizes the system (9.1.1) on a set \( S \subseteq X \) containing the origin. This means that for any initial state \( x_0 \in S \) the closed loop trajectory \( x_\mu(k; x_0), k \in \mathbb{N}_0 \), generated by \( x_\mu(0; x_0) = x_0 \) and

\[
x_\mu(k + 1; x_0) = Ax_\mu(k; x_0) + B\mu(x_\mu(k; x_0)),
\]

remains feasible, i.e., \((x_\mu(k; x_0), \mu(x_\mu(k; x_0))) \in \mathcal{E} \) holds for all \( k \in \mathbb{N}_0 \), and satisfies the estimate

\[
|x_\mu(k; x_0) - x^*| \leq \beta(|x_0 - x^*|, k) \quad \forall k \in \mathbb{N}_0
\]

for some \( KL \)-function \( \beta \). The basic assumption on the data of (9.1.1) needed to prove stability is as follows.

**Assumption 9.1.2.** The constraint set \( \mathcal{E} \) is convex, compact, and contains the origin \((0, 0)\) in its interior. Furthermore, the linear system described by the pair \((A, B)\) is stabilizable.

MPC offers an algorithmic procedure to accomplish the stabilization task where the feedback values \( \mu(x) \) are computed by solving optimal control problems. To this end, quadratic running costs \( \ell : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0} \) specified by

\[
\ell(x, u) := (x^T u^T) \begin{pmatrix} Q & N \\ NT & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}
\]

(9.1.4)

with symmetric matrices \( Q \in \mathbb{R}^{n \times n} \), \( R \in \mathbb{R}^{m \times m} \) such that

\[
\ell^*(x) := \inf_{u \in \mathbb{R}^m} \ell(x, u) \geq \eta|x|^2
\]

(9.1.5)

holds in \( X \) for some \( \eta \in \mathbb{R}_{>0} \). This property is, e.g., satisfied if \( Q > 0 \) (positive definite), \( N = 0 \), and \( R \geq 0 \). The corresponding cost function \( J_N : \mathbb{R}^n \times (\mathbb{R}^m)^N \rightarrow \mathbb{R}_{\geq 0} \) and optimal
value function $V_N : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ are given by

$$J_N(x, u) := \sum_{k=0}^{N-1} \ell(x_u(k; x), u(k)),$$

$$V_N(x) := \inf_{u \in U^N(x)} J(x, u)$$

for $N \in \mathbb{N} \cup \{\infty\}$, $x \in X$, and $u \in U^N(x)$ with the convention $V_N(x) = +\infty$ if $x \notin X$ or $U^N(x) = \emptyset$.

Fixing a finite prediction horizon (or optimization horizon) $N$ and setting $x_{\mu_N}(0; x_0) := x_0$, $k := 0$, the MPC loop is as follows:

1. Set $x = x_{\mu_N}(k; x_0)$, solve the optimal control problem

$$\min_{u \in U^N(x)} J_N(x, u)$$

and denote a respective minimizing control sequence by $\bar{u} \in U^N(x)$.

2. Define the MPC feedback value by $\mu_N(x) := \bar{u}(0)$.

3. Compute $x_{\mu_N}(k + 1; x_0)$ by (9.1.3) with $\mu = \mu_N$, set $k := k + 1$ and go to 1.

This iteration yields a closed loop trajectory for the implicitly defined MPC feedback law $\mu_N : X \to \mathbb{R}^m$. A main obstacle to applicability of the MPC scheme described above concerns the feasibility of the MPC closed loop at each time step $k$, i.e., $U^N(x) \neq \emptyset$ at stage 1. The problem could be circumvented by incorporating suitable terminal constraints and costs in the optimal control problem to be solved in each MPC step. However, the construction of such stabilizing constraints might be challenging and can reduce the operating range of the MPC scheme, cf. [36, Chapter 8] and [46] for detailed discussions.

In such cases, MPC without stabilizing constraints or costs can provide a valid alternative which is why we analyse this variant in this paper. Without stabilizing constraints, proving feasibility of the MPC algorithm in each step and asymptotic stability of the resulting closed loop poses a considerable challenge. Ideally, we would like to find the maximal set $\mathcal{S} \subseteq X$ on which the MPC feedback law $\mu_N$ asymptotically stabilizes (9.1.1) and the closed loop $x_{\mu_N}(\cdot; x)$ remains feasible. Such set $\mathcal{S}$ is called *basin of attraction*. Observe

\footnote{Whenever $U^N(x) \neq \emptyset$, existence of a minimizer $\bar{u} \in U^N(x)$ satisfying $J_N(x, \bar{u}) = V_N(x)$ is assumed in order to avoid technical difficulties.}
that it is necessarily a subset of the following set
\[ I_\infty := \{ x \in X : \exists u \in U^\infty(x) \text{ s.t. } \lim_{k \to \infty} x_u(k; x) = 0 \} \]
comprising points \( x \in X \) that can be feasibly driven (open loop) to the origin. In order to characterize \( S \) we now introduce the following concepts of invariance. A set \( C \subseteq X \) is said to be (controlled) forward invariant or viable if, for each \( x \in C \), there exists \( u \in U(x) \) such that \( x^+ \in C \). Observe that every forward invariant set \( C \subseteq X \) satisfies the inclusion \( C \subseteq F_\infty \) and that the set of admissible states \( X \) is, in general, much larger than the viability kernel \( F_\infty \). Methods which can be used in order to compute invariant sets can be found, e.g., in [8]. The set \( C \) is said to be recursively feasible if it is forward invariant with respect to the feedback law \( \mu_N \), that is, \( \mu_N(x) \in U(x) \) and \( Ax + B\mu_N(x) \in C \) for all \( x \in C \).

### 9.2 Stability on Level Sets

We now recall some results from the previous chapter. We need them to prove other results and the different exposition may aid understanding.

Under Assumption 9.1.2 a prediction horizon length can be determined such that recursive feasibility and asymptotic stability of the MPC scheme proposed in the previous section is ensured. To this end, first a local bound on the optimal value function \( V_\infty \) is deduced which is then extended to arbitrary level sets. For a given horizon length \( N \in \mathbb{N} \cup \{\infty\} \) and a positive constant \( C \) the level set is defined as
\[
V_N^{-1}[0, C] := \{ x \in X : V_N(x) \leq C \}.
\]

**Proposition 9.2.1.** Let Assumption 9.1.2 hold and consider system (9.1.1) with quadratic running costs as in (9.1.4). Then, there exists a neighbourhood \( \mathcal{N} \subseteq X \) of the origin and a constant \( \gamma \in \mathbb{R}_{>0} \) such that the following inequality holds
\[
V_\infty(x) \leq \gamma \cdot \ell^*(x) \quad \forall x \in \mathcal{N}.
\]  

**Proof.** Since the origin is contained in the interior of the constraint set \( \mathcal{E} \) and the pair
\((A,B)\) is supposed to be stabilizable, a neighborhood \(\mathcal{N}\) of the origin exists such that an LQR can be applied neglecting the constraints. Then, the solution \(P\) of the algebraic Riccati equation fulfills \(V_\infty(x_0) = x_0^T P x_0 \leq c|x_0|^2 \leq \gamma \cdot \ell^*(x_0)\) on \(\mathcal{N}\) with \(\gamma := c \eta^{-1}\) where \(c\) is the maximal eigenvalue of \(P\) and \(\eta\) is defined in (9.1.5).

Condition (9.2.1) is used in the nonlinear MPC literature as a main assumption to prove stability cf. [36, 54]. It is referred in the literature as ‘controllability’ assumption. This stems from the fact that \(V_\infty(x) < C\) is equivalent to the system being asymptotically controllable to the origin sufficiently fast, since otherwise (9.1.5) would imply \(V_\infty(x) = \infty\).

We next show that Condition (9.2.1) can be extended to hold on arbitrary level sets. This will in turn provide the desired stability and recursive feasibility properties.

**Proposition 9.2.2.** Let the assumptions of Proposition 9.2.1 be satisfied. Then for any \(N \in \mathbb{N}\) and \(C \in \mathbb{R}_{>0}\) we have that

\[
V_N(x) \leq \beta \cdot \ell^*(x) \quad \forall x \in V_N^{-1}[0,C],
\]

for some constant \(\beta = \beta(C)\) independent of \(N\). Furthermore the constant \(C\) can be chosen sufficiently large to satisfy \(V_N^{-1}[0,C] \supseteq \mathcal{N}\) for \(\mathcal{N}\) from Proposition 9.2.1

**Proof.** Since the running costs satisfy (9.1.5), existence of the positive lower bound

\[
M := \inf_{x \in \mathcal{N}\setminus \mathcal{N}} \ell^*(x) > 0 \quad (9.2.2)
\]

is ensured. Then, for every \(x \in V_N^{-1}[0,C]\setminus \mathcal{N},\) the inequality

\[
V_N(x) \leq C = \frac{C}{M} \cdot M \leq \frac{C}{\ell(x)}
\]

holds and the first part of the Proposition is proved since, when \(x \in \mathcal{N},\) \(V_\infty(x) \leq \gamma \cdot \ell^*(x)\) by Proposition 9.2.1. Observe that the constant \(\beta = \beta(C, M, \gamma)\) only depends on the constant \(C\) and on the parameters in Inequality (9.2.1) and Condition (9.1.5). Choose \(C \in \mathbb{R}_{>0}\) to satisfy

\[
\sup_{x \in \mathcal{N}} \ell^*(x) \leq C/\gamma. \quad (9.2.3)
\]

Such \(C\) exists since the costs \(\ell(\cdot)\) are quadratic. Then, since \(\mathcal{N}\) is bounded, the last
assertion follows directly from

\[ \sup_{x \in \mathcal{N}} V_N(x) \leq \gamma \cdot \sup_{x \in \mathcal{N}} \ell^*(x) \leq C. \]

\[ \square \]

We are ready to state our stability and feasibility result.

**Theorem 9.2.3.** Consider the same hypotheses and the resulting neighbourhood \( \mathcal{N} \) as in Proposition 9.2.1. Take any positive real number \( C \) satisfying (9.2.3) and let \( M \) be defined as in (9.2.2). In addition, choose \( N_0 \in \mathbb{N} \) such that the inequalities

\[ C \left( \frac{\beta - 1}{\beta} \right)^{N_0 - 1} < M \quad \text{and} \quad 1 - \alpha_{N_0} > 0 \]  

(9.2.4)

hold with \( \beta := \max\{C/M, \gamma\} \) and \( \alpha_N := \beta^2 \left( \frac{\beta - 1}{\beta} \right)^N \). Then, for every \( N \geq N_0 \) and every \( x \in V^{-1}_N[0, C] \), we have

\[ V_N(Ax + B\mu_N(x)) \leq V_N(x) - (1 - \alpha_N)\ell^*(x). \]  

(9.2.5)

In particular, \( V_N(\cdot) \) is a Lyapunov function on the recursively feasible set \( V^{-1}_N[0, C] \) which implies recursive feasibility and asymptotic stability of the MPC closed loop.

**Proof.** The proof follows from Theorem 8.3.3 of Chapter 8 which in turn is based on ideas from [54]. Note that the assumed quadratic running cost in combination with Condition (9.1.5) imply existence of \( K_\infty \)-functions \( \varphi_1, \varphi_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) satisfying \( \varphi_1(\|x\|) \leq \ell^*(x) \leq \varphi_2(\|x\|) \) — an assumption needed in Chapter 8. 

Observe that our results can be extended to general running costs if Condition (9.2.1) and \( \varphi_1(\|x\|) \leq \ell^*(x) \leq \varphi_2(\|x\|) \) are verified.

### 9.3 The Basin of Attraction

In this section we study the relations between the basin of attraction \( S \), \( I_\infty \) and the viability kernel \( F_\infty \). By their definitions it is already known that

\[ S \subseteq I_\infty \subseteq F_\infty. \]
It is interesting to understand under which conditions the reverse inclusions are also true. Without additional conditions, the answer is no, as shown for instance in Example 8.4.12.

In order to answer this question, we need the following characterization of the viability kernel $\mathcal{F}_\infty$.

**Proposition 9.3.1.** Consider the linear system (9.1.1) and let Assumption 9.1.2 be satisfied. Then the viability kernel $\mathcal{F}_\infty$, defined in (9.1.2), is a compact and convex set containing the origin in its interior. Furthermore if $x \in \partial \mathcal{F}_\infty$, every feasible trajectory will remain on the boundary $\partial \mathcal{F}_\infty$ unless it touches $\partial X$.

**Proof.** See previous chapter.

The following proposition provides a first link between the sets $\mathcal{F}_\infty$ and $I_\infty$. It provides a uniform bound for $V_\infty$ on certain subsets of the interior of the viability kernel, a key ingredient in order to characterize the operating range of the MPC feedback law.

**Proposition 9.3.2.** Let Assumption 9.1.2 be satisfied for (9.1.1). Then, for each $\lambda \in [0,1)$ the optimal value function is uniformly bounded from above on $\lambda \mathcal{F}_\infty$, i.e., a constant $M = M(\lambda) \in \mathbb{R}_{\geq 0}$ exists such that $V_\infty(x) \leq M$ holds for all $x \in \lambda \mathcal{F}_\infty$.

**Proof.** Full details of the proof can be found in Chapter 8. It makes use of techniques developed in [31, Lemma 12]. A broad outline is as follows.

For every point $x_0 \in \text{int} \mathcal{F}_\infty$ two trajectories can be generated. One uses stabilizability of the system and the other exploits viability of $\mathcal{F}_\infty$. Accordingly a feedback law $F \in \mathbb{R}^{m \times n}$ exists such that the corresponding closed loop $x_F^+ = (A + BF)x_F$ satisfies $x_F(k;x) \to 0$ as $k \to \infty$. However, the pair $(x_F,Fx_F)$ may not satisfy the constraints while the second trajectory remains in $\mathcal{F}_\infty$ for any time but may not approach the origin. The idea is to take a convex combination of these two trajectories and exploit linearity and convexity of the data to show that such a combination defines a feasible trajectory which converges to 0. When a sufficiently small neighbourhood of the origin is reached, the constraints can be neglected and the feedback law $F$ is applied. This procedure yields a uniform bound for $V_\infty$.

Note that both properties in Assumption 9.1.2 are essential here. Simple examples can be constructed in which $V_\infty$ is unbounded and discontinuous in the interior of $\mathcal{F}_\infty$ if say $\mathcal{E}$ is not convex or $(A,B)$ is not stabilizable. Note also that according to Proposition
9.3.2 \( \text{int} F_\infty \subseteq I_\infty \). Indeed \( I_\infty \) coincides with the domain of \( V_\infty \) as a straightforward adaptation of [49, Theorem 2] shows.

Another immediate consequence of Proposition 9.3.2 concerns stability and recursive feasibility on any compact set \( K \subseteq \text{int} F_\infty \). Indeed, any such \( K \) satisfies \( K \subseteq \text{int} \lambda F_\infty \) for some \( \lambda \in (0,1) \). By Proposition 9.3.2 \( V_\infty \) is bounded on a neighborhood of \( K \) and stability and recursive feasibility follows from Theorem 9.2.3. This leads to the following theorem.

**Theorem 9.3.3.** Assume the hypotheses of Proposition 9.3.2. Let \( K \subseteq \text{int} F_\infty \) be a compact set. Then, a prediction horizon \( N_K \in \mathbb{N} \) exists such that, for each \( N \geq N_K \), the MPC feedback law \( \mu_N \) asymptotically stabilizes the closed loop at the origin on a recursively feasible set \( S \supseteq K \).

**Remark 9.3.4.** Theorem 9.3.3 corrects and improves [49, Theorem 7]. In [49] the authors allow compact sets \( K \subseteq I_\infty \) which may contain points at the boundary of \( F_\infty \) and use arguments which exploit continuity of the value function on such sets \( K \). As we show in Example 9.7.4 continuity of the value function may not be satisfied at the boundary of \( F_\infty \).

Example 8.4.10 illustrates that the required prediction horizon may grow rapidly for initial values approaching the boundary of the viability kernel.

9.4 Stationarity of Feasible Sets

In the preceding section we considered the stabilization task for arbitrary compact sets contained in the interior of the viability kernel \( F_\infty \). Particularly, it follows from Theorem 9.2.3 that, for each sufficiently large \( N \), MPC will yield asymptotic stability with the basin of attraction \( S \) containing the whole viability kernel \( F_\infty \), if \( \sup V_\infty (F_\infty) \) is finite. In Chapter 8 we showed that this property implies stationarity of the feasible sets \( F_N \).

We say that the feasible sets \( F_N \) become stationary, if there exists \( N_0 \in \mathbb{N} \) with \( F_N = F_{N_0} \) for all \( N \geq N_0 \). In [42, Theorem 5.3] (see also [35, Section 5.1]), it was shown that stationarity of the feasible sets is sufficient for recursive feasibility of \( F_\infty \) for all optimization horizons \( N \geq N_0 + 1 \). The following theorem states that it is also necessary for \( V_\infty \) to be bounded on the viability kernel \( F_\infty \).
Theorem 9.4.1. Consider the linear system (9.1.1) with positive definite quadratic running costs \( \ell \) and let Assumption 9.1.2 be satisfied. Then, if \( V_\infty(x) \leq c \) holds for some \( c \in \mathbb{R}_{>0} \) and all \( x \in \mathcal{F}_\infty \), the feasible sets \( \mathcal{F}_N \) become stationary for some \( N_0 \in \mathbb{N} \).

Proof. See Theorem 8.4.11.

Example 9.4.2. The converse is not true in general. This was shown in Chapter 8 for a discrete time system. We now show that the converse of Theorem 9.4.1 is not satisfied even in a continuous time setting. Consider the following continuous time system given by

\[
\begin{pmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{pmatrix} = \begin{pmatrix}
x_2(t) \\
u(t)
\end{pmatrix}
\]

with

\( u(t) \in [-1, 1] \) and \( \begin{cases} 
-1 \leq x_1 \leq 1 \\
x_1 - 1 \leq x_2 \leq 2 
\end{cases} \)

Here, \( \mathcal{F}_\infty \) does not coincide with \( X \) and \( \partial \mathcal{F}_\infty \) comprises points \( x \) for which \( V_\infty(x) = +\infty \) and points \( x \) for which \( V_\infty(\cdot) \) is finite (see Figure 9.4.2 for an illustration of this example).
We easily see that there exists a time $T > 0$ such that $F_T = F_\infty$ and every starting point $(x_1, x_2)^T \in X$ such that $x_2 = \sqrt{2(1-x_1)}$ has infinite cost. Indeed, once the trajectory reaches the controlled equilibrium point $(1, 0)$ it has to stay there forever: as soon as $u > 0$, $x_1(t)$ increases violating the constraint $x_1 \leq 1$. At the same time, if $u < 0$ for say an interval of time $(t, t+\varepsilon)$ we would have that

$$x_1(t+\varepsilon) - 1 = \int_t^{t+\varepsilon} \int_t^s u(r) \, dr \, ds$$

$$= \int_t^{t+\varepsilon} (t + \varepsilon - r) u(r) \, dr.$$

But then $x_1(t+\varepsilon) - 1 > x_2(t+\varepsilon) = \int_t^{t+\varepsilon} u(r) \, dr$ since

$$\int_t^{t+\varepsilon} (t + \varepsilon - r - 1) u(r) \, dr > 0.$$

We conclude this section with the following remark: if the infinite horizon optimal value function were continuous on $F_\infty$, stationarity, as proven in Theorem 9.4.1, would be fulfilled as soon as the condition $I_\infty = F_\infty$ is verified. Indeed, since $F_\infty$ is a compact set, if $V_\infty$ is pointwise bounded and continuous on $F_\infty$ then it is also uniformly bounded and Theorem 9.4.1 can be invoked.

Continuity of the value function is important for other applications in MPC, such as robustness, cf. [32].

### 9.5 Continuity of $V_\infty$

For the reasons just mentioned, the goal of this section is to deduce sufficient conditions for continuity of the value function $V_\infty$. To this end, first lower semicontinuity is derived before a sufficient condition for upper semicontinuity and, thus, continuity is given.

**Proposition 9.5.1.** Consider linear systems (9.1.1) and quadratic running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$. Let Assumption 9.1.2 be satisfied. Then, the value function $V_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous on $F_\infty$ and continuous on $\text{int}\{F_\infty\}$.

In particular, $V_\infty(\cdot)$ is strictly increasing on every ray starting from the origin and the estimate $V_\infty(\lambda x) \leq \lambda V_\infty(x)$ holds for every $\lambda \in [0, 1]$ and $x \in \mathbb{R}^n$.

**Proof.** To show that $V_\infty(\cdot)$ is a convex function is an easy exercise. Proposition 9.3.2...
implies $V_\infty(x) < \infty$ for each $x \in \text{int}\{\mathcal{F}_\infty\}$. Hence, $V_\infty(\cdot)$ is continuous on the interior of its convex domain $\text{int}\{\mathcal{F}_\infty\}$. It remains to show that $V_\infty(\cdot)$ is lower semicontinuous on $\partial\mathcal{F}_\infty$, i.e., that

$$\liminf_{y \to x, y \in \mathcal{F}_\infty} V_\infty(y) \geq V_\infty(x) \quad (9.5.1)$$

holds for every $x \in \partial\mathcal{F}_\infty$. Take a sequence $(x_i)_{i \in \mathbb{N}_0} \subset \mathcal{F}_\infty$ such that $x_i \to x$ and $\liminf_{y \to x} V_\infty(y) = \lim_{i \to +\infty} V_\infty(x_i)$. If $V_\infty(x_i) \to +\infty$ the result is obvious. We assume then, without loss of generality, that control sequences $u_i \in \mathcal{U}_\infty(x_i)$, $i \in \mathbb{N}_0$, exist, satisfying $J_\infty(x_i, u_i) \leq V_\infty(x_i) + \varepsilon$, for some $\varepsilon > 0$. Let $N \in \mathbb{N}$ be given. Then, taking a subsequence if necessary, we have that $u_i \to u \in \mathcal{U}_N(x)$ for the truncated sequence $u_i \in \mathcal{U}_N(x_i)$. Compactness of the constraint set $\mathcal{E}$ (Assumption 9.1.2) was used in order to conclude this convergence — at least for a subsequence if necessary. Continuity of $J_N(\cdot, \cdot)$ implies

$$V_N(x) \leq J_N(x, u) = \lim_{i \to \infty} J_N(x_i, u_i) \leq \liminf_{i \to \infty} J_\infty(x_i, u_i) \leq \lim_{x_i \to x} V_\infty(x_i) + \varepsilon.$$

Since the right hand side of this inequality does not depend on $N$ and $\varepsilon > 0$ was chosen arbitrarily, the desired Inequality (9.5.1) holds which implies lower semicontinuity.

\[ \square \]

**Remark 9.5.2.** The assumptions of Proposition 9.5.1 can be weakened to requiring only convexity of the running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_\geq 0$.

Proposition 9.5.1 tells us that in order to prove continuity of $V_\infty$ only upper semicontinuity has to be established. Observe at the outset that in dimension $n = 1$, when $V : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, upper semicontinuity is given for free by convexity. However, convexity is no longer sufficient when the dimension increases. Consider the following instructive example, see, e.g. [52, page 83], in $\mathbb{R}^2$

$$f(x, y) := \begin{cases} 
\frac{y^2}{2x} & x > 0 \\
0 & x = y = 0 \\
+\infty & \text{elsewhere}. 
\end{cases}$$

This function is convex since it is the support function of the parabolic convex set $C =$
\[(x, y) : x + (y^2/2) \leq 0\] namely, \(f(x, y) = \sup_{(u, v) \in C} (u, v) \cdot (x, y)\). Yet it is not upper semicontinuous at the origin. Indeed every sequence \((\frac{x^2}{2\alpha}, x), \alpha > 0\), has limit \(\alpha\) as \(x \searrow 0\).

The following theorem provides a sufficient condition in order to ensure continuity of the value function \(V_\infty\) also on \(\partial\mathcal{F}_\infty\). Explanations on set-valued analysis and a discussion of this condition are given in Sections 9.7.1 and 9.7.2 respectively.

**Theorem 9.5.3.** Suppose that the set-valued map

\[x \mapsto G(x) := \{u \in U(x) : Ax + Bu \in \mathcal{F}_\infty\}\]

\(x \in \mathcal{F}_\infty\), is continuous. Then, the value function \(V_\infty\) is continuous on \(\mathcal{F}_\infty\).

**Proof.** Observe that it is sufficient to show that

\[\limsup_{y \to x, y \in \mathcal{F}_\infty} V_\infty(y) \leq V_\infty(x) \quad \forall x \in \partial\mathcal{F}_\infty.\]

Hence, pick \(x \in \partial\mathcal{F}_\infty\). Again, we notice that if \(V_\infty(x) = +\infty\) we are done. We assume henceforth that \(V_\infty(x) < +\infty\). In this case, the dynamic programming principle implies the existence of \(N_0 \in \mathbb{N}\) and \(u \in U^{N_0}(x)\) such that

\[V_\infty(x) + \varepsilon \geq \sum_{k=0}^{N_0-1} \ell(x_u(k; x), u(k)) + V_\infty(x_u(N_0; x))\]

for some \(\varepsilon > 0\), \(x_u(k; x) \in \partial\mathcal{F}_\infty\), \(k = 0, \ldots, N_0 - 1\), and \(x_u(N_0; x) \in \text{int}\{\mathcal{F}_\infty\}\).

Now, take any \(z \in \partial\mathcal{F}_\infty\) and \(y \in \mathcal{F}_\infty\). By hypothesis the map \([9.5.2]\) is continuous at \(z\), so that for every \(u_z \in U(z)\) with \(Az + Bu_z \in \mathcal{F}_\infty\), and \(y \to z\), there exists \(u_y \in G(y)\) such that \(u_y \to u_z\). Observe that in particular \(G(y) \neq \emptyset\), for every \(y \in \mathcal{F}_\infty\), by definition of \(\mathcal{F}_\infty\). In the following calculation we use this fact for \(z = x_u(k; x)\) setting \(u(k) = u_z\) for
\[ k = 0, \ldots, N_0 - 1. \]

\[
\limsup_{y \to x, y \in F_\infty} V_\infty(y) \\
\leq \limsup_{y \to x, y \in F_\infty} \{ \ell(y, u_y) + V_\infty(Ay + Bu_y) \} \\
\leq \limsup_{y \to x, y \in F_\infty} \ell(y, u_y) + \limsup_{y \to x, y \in F_\infty} V_\infty(Ay + Bu_y) \\
= \ell(x, u(0)) + \limsup_{y \to Ax + Bu(0), y \in F_\infty} V_\infty(y) \\
\leq \ldots \leq \sum_{k=0}^{N_0-1} \ell(x_u(k; x), u(k)) + \limsup_{y \to x_u(N_0; x), y \in F_\infty} V_\infty(y) \\
= \sum_{k=0}^{N_0-1} \ell(x_u(k; x), u(k)) + V_\infty(x_u(N_0; x)) \overset{\text{(9.5.3)}}{\leq} V_\infty(x) + \varepsilon.
\]

In the last equality we used continuity of the value function in the interior of \( F_\infty \) to conclude that \( \limsup_{y \to x_u(N_0; x), y \in F_\infty} V_\infty(y) = V_\infty(x_u(N_0; x)) \).

### 9.6 An Illustrative Example

In this section we illustrate several of our results by an example in which the value function \( V_\infty \) is continuous and uniformly bounded on the viability kernel \( F_\infty \). This is used in order to illustrate the assertions of Proposition \[9.3.1\] and Theorem \[9.4.1\] i.e., it is demonstrated that the trajectory leaves the boundary of \( F_\infty \) only after touching the boundary of the constraint set \( X \) and that the feasible sets \( F_N \) become stationary. Furthermore, the forward invariant neighbourhood \( N \) of the origin from the proof of Proposition \[9.2.1\] is constructed explicitly. Most of the results are illustrated graphically to give an idea into the nature of the problem.

**Example 9.6.1.** Consider the constrained linear system

\[
\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} 1 & 1.1 \\ -1.1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u
\]

with \((x_1, x_2) \in X := [-1, 1] \times [-1, 1] \) and \( u \in U := [-1, 1] \). The running costs are defined as \( \ell(x, u) := |x|^2 + |u|^2 \), i.e. the matrix \( Q \) and \( R \) are taken equal to the identity matrix and \( N = 0 \).
Assumption 9.1.2 is fulfilled for Example 9.6.1. First, \( \mathcal{N} \) is constructed. To this end, the unique symmetric and positive definite solution \( P \) of the discrete algebraic Riccati equation

\[
P = A^T PA - A^T PB (R + B^T PB)^{-1} B^T PA + Q
\]

is computed. This yields the value function \( V_\infty(x) = x^T Px \) of the unconstrained problem. The corresponding optimal feedback law is given by \( Fx := -(R + B^T PB)^{-1} B^T PAx \), see, e.g., [7, Section 10.2]. Next, the number

\[
\rho := \min \left\{ \min_{x \in \{x \colon Fx \in \partial U\}} V_\infty(x), \min_{x \in \partial \mathcal{X}} V_\infty(x) \right\}.
\]

is computed. Then, by convexity arguments, the level set \( V^{-1}[0, \rho] \) is our desired set \( \mathcal{N} \), cf. Figure 9.2 (left).

The feasible sets \( \mathcal{F}_N, N \in \mathbb{N} \), can be explicitly determined and the equality \( \mathcal{F}_3 = \mathcal{F}_\infty \) can be shown. We observe that the system is symmetric on opposite quadrants, i.e. \( A(-x) + B(-u) = -(Ax + Bu) \) and that the point \((1, 0)\) can be steered into \( \mathcal{N} \) in four steps with controls \( u(0) = \ldots = u(3) = 1 \), see also Figure 9.2 (left).

Define the points \( \Omega, \Gamma \) and \( \Theta \) as in Figure 9.2 (right). The only control that renders points on the boundary of \( \mathcal{F}_3 \) feasible is \( u = 1 \), on the half space \( x_2 \leq 0 \), and \( u = -1 \) on the half space \( x_2 \geq 0 \). Points on the segment joining \((-1, 0)\) and \( \Omega \) can be mapped into \((-1, 0)\). In particular \(( \Omega, 1)^+ = (-1, 0) \). Points on the segment \( \overline{\Omega \Gamma} \) are mapped into \(( -1, 0 )\overline{\Omega} \) and \(( \Gamma, 1)^+ = \Omega \) as illustrated by Figure 9.2 (a). Finally the segment \( \overline{\Gamma \Theta} \) is mapped into \( \overline{\Gamma \Omega} \).

The above calculations show that Proposition 9.3.1 applies to this example. A more careful computation shows that the number of steps required to reach the origin is at most six, cf. Figure 9.3. Thus \( I_\infty = \mathcal{F}_\infty \) and indeed \( \mathcal{F}_3 = \mathcal{F}_\infty \). Finally, continuity of \( V_\infty \) always holds in \( \mathbb{R}^2 \) cf. Proposition 9.7.3.

Acknowledgement

We want to thank Daniel Walter for providing Fig. 9.3.
Figure 9.2: (left): Representation of two trajectories (dotted curves in red) for the system with control $u = 1$ at each step, starting at $(1,0)$ and $\Gamma$. The feasible set $\mathcal{F}_1$ in white, $\mathcal{N}$ in yellow (oval shaped). (right): The constraints defining $\mathcal{F}_1$ (blue) and $\mathcal{F}_2$ (yellow) intersect in $\Omega$ (on the half space $x_2 \leq 0$). Analogously $\Gamma$ is defined as intersection of $\mathcal{F}_2$ and $\mathcal{F}_3$ (red, $\mathcal{F}_3 = \mathcal{F}_\infty$). $\Theta$ is the intersection with the line $x_1 = 1$.

Figure 9.3: Number of steps required to reach the origin, from the inner color (1 step) to the outer one (6 steps).
9.7 Sufficient Conditions for Continuity of $V_\infty$

We continue our investigation about the continuity of the infinite horizon optimal value function $V_\infty(.)$. Observe that all the conditions provided can be adapted to yield continuity of the optimal value function for any given finite horizon $N$.

In particular in the following sections we give sufficient conditions under which the set-valued map $[9.5.2]$ is continuous, which according to Theorem $[9.5.3]$ ensures continuity of $V_\infty$. To this end, some concepts from set-valued analysis are needed, which we define in the first section of this appendix.

9.7.1 Set-Valued Analysis

Let $Z$ and $Y$ be metric spaces. A set-valued map from $Z$ to $Y$, $F : Z \leadsto Y$, associates a set $F(z) \subseteq Y$ to each point $z \in Z$. We say that $F$ is closed if it has closed set images. Henceforth we assume that $Y$ is compact and that $F$ and $\text{Dom } F := \{ z \in Z : F(z) \neq \emptyset \}$ are closed.

\textbf{Definition 9.7.1.} A set-valued map $F : Z \leadsto Y$ is called

- upper semicontinuous at $z \in \text{Dom } F$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that
  \[ F(z') \subseteq F(z) + \epsilon B \quad \forall z' \in z + \delta B \cap \text{Dom } F. \]

- lower semicontinuous at $z \in \text{Dom } F$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that
  \[ F(z) \subseteq F(z') + \epsilon B \quad \forall z' \in z + \delta B \cap \text{Dom } F. \]

We say that $F$ is upper (lower) semicontinuous if it is upper (lower) semicontinuous at every point $z \in \text{Dom } F$.

We say that $F$ is continuous if it is upper and lower semicontinuous on $\text{Dom } F$. Furthermore, observe that $F$ is upper semicontinuous if and only if $\text{Graph } F := \{ (z, y) \in Z \times Y : y \in F(z) \}$ is closed.
Chapter 9. Linear MPC and Continuity of the Value Function

206

\[ \mathcal{U}(x) \]

Figure 9.4: Representation of the set \( \mathcal{U}(x) \) as a section of \( \mathcal{E} \).

Definition 9.7.2. The upper and lower limit of \( F : Z \rightrightarrows Y \) at \( z \in Z \) are defined as

\[
\limsup_{z' \to z} F(z') := \{ v \in Y : \liminf_{z' \in \text{Dom} F} \text{dist}(v; F(z')) = 0 \},
\]

\[
\liminf_{z' \to z} F(z') := \{ v \in Y : \lim_{z' \in \text{Dom} F} \text{dist}(v; F(z')) = 0 \}.
\]

In particular the inclusions \( \liminf_{z' \to z} F(z') \subseteq F(z) \subseteq \limsup_{z' \to z} F(z') \) hold. Equalities hold if and only if \( F \) is respectively lower and upper semicontinuous. For details of definitions and properties of set-valued maps, we refer the reader to [?].

9.7.2 Sufficient Conditions for Continuity of \( G \) from (9.5.2)

We first observe that continuity of \( x \rightrightarrows \mathcal{U}(x) \) is a direct consequence of the definitions. Indeed \( \mathcal{U}(x) \) — as shown in Figure 9.4 — is a section of the compact and convex set \( \mathcal{E} \). Compactness of \( \mathcal{E} \) also implies, at once, that the graph of \( G(\cdot) \) is closed.

By [2, Proposition 1.5.2], \( G \) is continuous at \( x \in \mathcal{F}_\infty \) if there exists \( u \in G(x) \) such that \( Ax + Bu \in \text{int}\{\mathcal{F}_\infty\} \). In particular, this implies continuity on \( \text{int}\{\mathcal{F}_\infty\} \). \( G \) is also continuous at \( x \in \mathcal{F}_\infty \) when \( G(x) = \{ u \} \). Indeed, since \( G \) is upper semicontinuous, for any sequence \( x_n \to x \), \( x_n \in \mathcal{F}_\infty = \text{Dom} G \), we have that

\[ G(x_n) \subseteq G(x) + \epsilon_n \mathcal{B} = u + \epsilon_n \mathcal{B}, \quad \text{for some } \epsilon_n \downarrow 0. \]
9.7. SUFFICIENT CONDITIONS FOR CONTINUITY OF $V_\infty$

Therefore any sequence $(u_n)_{n \in \mathbb{N}}$ with $u_n \in G(x_n) \neq \emptyset$ will converge to $u$. Continuity of the set-valued map $G(\cdot)$, then, has to be checked only at points $x \in \partial \mathcal{F}_\infty$ for which $G(x)$ is not a singleton and $Ax + BG(x) \subseteq \partial \mathcal{F}_\infty$.

**Proposition 9.7.3.** Assume that the matrix $B$ has full rank. Then, the map $G$ from (9.5.2) and thus also the value function $V_\infty$ are continuous on the whole feasible set $\mathcal{F}_\infty$ in the following cases:

(i) $BU(x)$ is strictly convex for every $x \in \partial \mathcal{F}_\infty$.

(ii) $\mathcal{F}_\infty$ is strictly convex.

(iii) The state dimension is $n = 2$ and the constraints are of the form $\mathcal{E} = X \times U$ for $X \subseteq \mathbb{R}^2$, $U \subseteq \mathbb{R}^m$.

**Proof.** The cases (i) and (ii) follow from the considerations before this proposition. Indeed, by our convexity assumptions, for any $x \in \partial \mathcal{F}_\infty$ the intersection $Ax + BU(x) \cap \mathcal{F}_\infty = Ax + BG(x)$ is either a singleton or contains points in $\text{int} \mathcal{F}_\infty$. Those are exactly the situations in which continuity is assured.

For proving (iii), fix $u \in G(x)$, $x \in \partial \mathcal{F}_\infty$ and take a sequence of points $x_n \in \mathcal{F}_\infty$, $n \in \mathbb{N}$, such that $x_n \rightarrow x$, as $n \rightarrow +\infty$. We assume that $x$ is a point for which $Ax + BU \cap \mathcal{F}_\infty \subseteq \partial \mathcal{F}_\infty$, for otherwise $G(\cdot)$ is continuous and there is nothing to prove.

For every $n \in \mathbb{N}$, $G(x_n) \neq \emptyset$, so that there exists $v_n \in G(x_n)$. If $Ax_n + Bu_n \rightarrow Ax + Bu$, as $n \rightarrow +\infty$ the proof is concluded. Assume, then, that there exists $v \in G(x)$, $v \neq u$, such that $Ax + Bu$ is a cluster point for the sequence $(Ax_n + Bu_n)_{n \in \mathbb{N}}$. Observe that the convex combination between the origin, $Ax + Bu$ and $Ax + Bu$ is contained in...
\[ \mathcal{F}_\infty. \text{ Since } Ax + BU \cap \mathcal{F}_\infty \subseteq \partial \mathcal{F}_\infty \text{ the two convex sets } Ax + BU \text{ and } \mathcal{F}_\infty \text{ can be separated (see figure 9.5), i.e. there exists } \zeta \in \mathbb{R}^2 \text{ such that} \]
\[ \zeta \cdot (Ax + Bw) \geq \zeta \cdot (Ax + Bu) \geq \zeta \cdot z, \]
for all \( w \in U \) and \( z \in \mathcal{F}_\infty \). In particular, \( \zeta \cdot (Ax + Bu) \geq \zeta \cdot (Ax_n + Bv_n) \geq \zeta \cdot (Ax_n + Bu) \).

If \( u \in G(x_n) \) we define \( u_n := u \). Otherwise assume that \( n \in \mathbb{N} \) is such that \( Ax_n + Bv_n \) is in a neighborhood of \( Ax + Bu \). The lines \( s \in [0, 1] \mapsto s(Ax_n + Bv_n) + (1-s)(Ax + Bu) \) and \( q \in [0, 1] \mapsto q(Ax + Bu) \) must intersect at \( Ax_n + B(v_n + (1-s)u) \in \mathcal{F}_\infty \). Indeed \( s(Ax_n + Bv_n) + (1-s)(Ax + Bu) = q(Ax + Bu) \) is a linear system with two equations and two unknowns. Define \( u_n := \bar{s}v_n + (1-\bar{s})u \in G(x_n) \). In this way we construct a sequence \( (u_n)_{n \in \mathbb{N}} \) such that \( u_n \in G(x_n) \) and \( u_n \rightarrow u \) as \( n \rightarrow +\infty \). Therefore \( G(.) \) is lower semicontinuous and \( (iii) \) is proved.

The following example illustrates a situation in which \( V_\infty \) fails to be continuous.

**Example 9.7.4.** Consider the set \( \mathcal{C} \) given by the cone shown in Figure 9.6, i.e., the convex hull between the point \( V = (0, 2, -1) \) and the circle \( \mathcal{B} = \{(x_1, x_2, x_3) : x_2 = -1, |x_1|^2 + |x_3|^2 \leq 1\} \). Note that \( \mathcal{C} \) contains the origin. Define the discrete linear system
9.7. SUFFICIENT CONDITIONS FOR CONTINUITY OF $V_\infty$

\[
\begin{pmatrix}
x_1^+ \\
x_2^+ \\
x_3^+
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
+ \begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix},
\]

$u \in [-1, 1]^3$ and $x \in C$. This system satisfies Assumption 9.1.2. Moreover it can be verified that $C \equiv F_\infty$. We consider running costs $\ell(x, u) = |x|^2 + |u|^2$.

We claim that the value function $V_\infty$ is discontinuous at $(0, -1, -1)$ implying that $G$ is discontinuous, too.

Indeed $V_\infty(0, -1, -1) \leq 7$ and the origin can be reached within two steps but any point $x = (x_1, x_2, x_3)$ on the semicircle $\Gamma = \{(x_1, -1, x_3) : x_1 < 0, x_3 \leq 0$ and $|x_1|^2 + |x_3|^2 = 1\}$ has infinite cost since

\[
x^+ = Ax + BU \cap F_\infty = \begin{pmatrix}
x_1 + [-2, 0] \\
x_2 + [-1, 1] \\
x_3 + [-2, 0]
\end{pmatrix} \cap C = x,
\]

and the system does not move from such position. An illustration of this fact is given in Figure 9.6. If a feasible point $P \in x^+$, $P \neq x$ exists then by construction $P = \lambda y + (1-\lambda) y$ for some $\lambda \in (0, 1)$ and $y \in B$. Using the fact that $|y_1|^2 + |y_3|^2 \leq 1$ and that $x \in \Gamma$ we conclude that $|P_1|^2 + |P_3|^2 < 1$. This is a contradiction. Indeed $(P_1)^2 + (P_3)^2 \geq 1$ since $(P_1, P_3) \in (x_1, x_3) + [-2, 0]^2$ and $x \in \Gamma$. 


Chapter 10

Concluding Remarks

In this thesis current available theory has been extended in several significant ways that are here summarized.

In the first part, dedicated to Optimality Conditions, we derived improved necessary conditions for solutions to optimal control problems along the lines of recent work by Clarke. A specific aim was to verify whether some of this recent work on necessary conditions could be extended to cover classes of optimal control problems featuring pathwise state constraints.

For free time optimal control problems with time delays we first proved necessary conditions in a very general nonsmooth setting (a nonsmooth version of the Maximum Principle is here present for the first time in the literature) and then we used those necessary conditions to derive sensitivity information and construct new algorithms for the computation of optimal trajectories.

Finally, in the second part of the thesis, dedicated to Model Predictive Control (MPC), we proved stability and recursive feasibility of MPC schemes under weaker hypotheses than have previously been considered.

Before giving a detailed summary of the results achieved, we briefly comment on the methodology used to obtain those results.
10.1 Methodology

The first part of the thesis is concerned with the derivation of first order necessary conditions of optimality for a range of optimal control problems not previously considered in the literature. The methodology used in the derivations is typically based on perturbations analysis pioneered by Rockafellar, Clarke and others (see for example [16]). The idea is as follows. Given a solution to a problem with a feature, say a state constraint, that is difficult to analyze, we seek a nearby solution to a ‘perturbed’ problem which does not have this difficult feature (through the introduction of penalty terms, for example). We derive necessary conditions for the perturbed problem, and obtain those for the original problem by passage to the limit. This perturbational approach is different from the methodology used by Pontryagin et al. [48] in the first derivations of optimality conditions such as the Maximum Principle, based on a study of reachable sets and their approximations. The perturbational approach is particularly effective for nonsmooth optimal control problems. Nonsmooth calculus is a fundamental tool for our analysis even when the data of the problem considered is smooth.

The second part of the dissertation is concerned with stability results using Model Predictive Control (MPC) techniques. The MPC scheme provides a feedback control law that is designed to stabilize the system. This feedback is obtained by solving optimal control problems. The proofs of our stability results are based on showing (by means of a “relaxed” dynamic programming principle) that the value function of the optimization problem associated is indeed a Lyapunov function, i.e. it is decreasing over the trajectories of the system. We then specialized our results to the case in which the dynamic of the system is linear and the constraint set is convex. The algebraic Riccati equation is used to explicitly reconstruct the value function of the problem. Moreover we employed several ideas from convex analysis. The ‘gauge’ function, for example, is used to describe convex sets.

Numerical simulations are performed using Matlab routines. In Chapter 6 optimization problems were solved with the nonlinear optimization code IPOPT [57] imported into the toolbox ICLOCS [60]. A code for the example of Chapter 8 is available for download at [http://num.math.uni-bayreuth.de/de/team/Gruene_Lars/publications/boccia_](http://num.math.uni-bayreuth.de/de/team/Gruene_Lars/publications/boccia_)
10.2 Contributions

Some of the content presented in the thesis has been published or submitted for publication see [4, 5, 9, 10] and [11]. Other material as in Chapter 4 and Chapter 5 will be the subject matter of future publications.

As a whole the thesis was divided in two parts. The first part dealt with the search for open loop strategies to solutions of optimal control problems. The second part was centered on the search for closed loop strategies to system stabilization.

Part I: Optimality Conditions

Part I is composed of two main subparts. Chapters 3 and 4 are concerned with the derivation of necessary conditions of optimality for constrained systems while Chapters 5 and 6 deals with delayed systems.

Constrained Systems

Chapter 3. First it is shown that the stratified necessary conditions, introduced by Clarke in [18], remain valid (with appropriate modifications), when a pathwise state constraint is included in the problem formulation. The conditions obtained reduce precisely to those in [18] and [20] when the state constraint is omitted.

We allow general (set valued) radius multifunctions $R(.)$ in place of a ball around the optimal trajectory, as in [20]. This generalization greatly simplify the analysis when dealing with mixed constraints. Moreover, it is shown that the use of convex radius multifunction $R(.)$ is particularly useful when using necessary conditions as a tool to eliminate possible candidates (see Section 4.3).

The stratified framework considered in Chapter 3 (and subsequently extended for more general classes of problems in Chapters 4 and 5) improves on previous research in at least two directions. First the hypotheses invoked for the derivation of necessary conditions are asserted to hold only locally around optimal trajectories. This is an important feature.
when considering that necessary conditions typically give only local information. Second it leads, naturally, to the study of weak minimizers.

We examine examples aimed at providing insights into the ultimate limitations on possible generalizations of such necessary conditions. Particularly in Section 3.3 we showed the necessity of the tempered growth condition (H5) or (H5)* and of the fact that the radius multifunction takes values open sets.

The methodology for proving stratified necessary conditions represent part of our contribution. In Chapter 3 we follow a simple and effective proof technique proposed by Clarke (lifting), now in the state-constrained setting, in which we deal not directly with the original dynamic, but with a related multifunction taking as values sets in a higher dimensional state space, which has much better regularity properties. An alternative proof technique is proposed in Chapter 5. Indeed the techniques of Chapter 5 can be easily adapted to the framework of Chapter 3. Here we get rid of dynamic constraints by penalizing the cost with the violation of an arc from satisfying such constraints. We proved that such perturbed problem has the required properties for invoking known results on classical problems of calculus of variations.

Chapter 4: Validity of stratified necessary conditions is extended to cover classes of optimal control problems where not only pure state constraints but also mixed state and control constraints are considered. The theory is supported by several examples that illustrate why some of the assumptions imposed are essential. It is also shown how our framework represent a convenient starting point for the derivation of necessary conditions for many important classes of optimal control problems. Our formulation is indeed very general and our results subsume many other results previously studied in the literature (this was already observed in [23] for problems without state constraints).

Delayed Systems

Chapter 5: In this chapter we considered retarded systems. State constraints are not taken into account. However, it is our belief that the methodologies used may be extended to this case in future research. Necessary conditions for a delayed and free time optimal
control problem are derived by means of a self contained analysis. The methodology adopted is new.

Standard approaches to solve optimal control problems with free end-time are based on a time transformation that is designed to fix the end-time. For retarded systems, however, reduction of free end-time problems to fixed end-time problems, for purposes of deriving optimality conditions and of computing minimizers, is problematic. This is because the time transformation gives rise to a non-standard optimal control problem for retarded systems with state dependent time-delays, which are difficult to analyse. We describe an alternative approach to the analysis of free end-time optimal control problems, which does not depend on a transformation of the time variable. Instead we used techniques based on direct perturbation of the end-time. The end result is a modified transversality condition, providing the extra information about the optimal end-time, expressed in terms of the ‘essential value’ of the maximized Hamiltonian. These techniques were originally used to derive optimality conditions for delay-free optimal control problems with measurably time-dependent data [26]. They were used for retarded systems for the first time.

Chapter 6: The implications of necessary optimality conditions and in particular of the transversality condition for the computation of optimal solutions are explored. We show that the transversality condition leads to formulae for the sensitivity of the minimum cost (of the optimal control problem on a fixed time interval) to perturbations of the end-time. An algorithm for the solution of free end-time problems is proposed, in which solutions to optimal control problems on a sequence of time intervals \([0, T_i]\) are computed and the end-times \(T_i\) are adjusted according to a gradient descent scheme based on the sensitivity formulae. Numerical experiments demonstrate the superior performance of algorithms which make use of the sensitivity formulae, as compared with algorithms based on numerical approximation of gradients of the final time value function.

Part II: Model Predictive Control

Part II is composed of two chapters. The first chapter contains a proof of the main theorem about stability of MPC schemes. This theorem leads to several results particularly when we consider systems with linear dynamics. In the second chapter we discuss
continuity of the value function.

Chapter 8. We investigate recursive feasibility and asymptotic stability for nonlinear MPC schemes with state and control constraints without imposing stabilizing terminal constraints or costs. In the literature, stability and feasibility for this types of MPC schemes is ensured assuming a global controllability assumption on the whole space where we want stability to be achieved. In contrast we assume only a local controllability condition around the equilibrium to be stabilized. This weaker condition is particularly important for at least three reasons. First and obvious reason is that it is easier to verify. Second the results obtain, invoking only local conditions, are global, in the sense that asymptotically stability is achieved for (almost) every point for which there exists at least an open loop control that steers the system to the equilibrium. A third advantage of our local hypothesis is that it provides a framework to investigate stability for a linearization of the nonlinear model, around an equilibrium point. Indeed we show that the local controllability assumption is always satisfied for stabilizable linear systems.

We provide for the first time in the literature a relationship between controllability types assumptions and stationarity of the feasible sets. Those seemingly different hypotheses have been successfully employed to prove stability and feasibility of MPC schemes.

Chapter 9. This chapter is dedicated to linear quadratic MPC schemes. Relationships between the level sets of the infinite horizon optimal value function, the basin of attraction of the equilibrium point, and the viability kernel are provided. Moreover we carry out a careful analysis that provides several significant properties of these sets.

Continuity of the optimal value function $V_\infty$ is discussed in details. A main contribution of the chapter is to provide sufficient conditions that ensure continuity of the value function. Such sufficient conditions include as a special case inward pointing conditions classically assumed to prove continuity of the optimal value function.

Many examples are provided that illustrate our results. A perhaps surprising example
shows that the value function can be discontinuous at the boundary of the viability kernel but only if we allow the dimension of the system $n \geq 3$.

10.3 Directions for future research

Optimality Conditions

- Necessary conditions of optimality for optimal control problems with unbounded dynamics are typically derived under assumptions that include a Lipschitz type condition on the dynamic set $F(.)$. As a byproduct of our analysis, in Chapters 3 and 9 we were able to prove the validity of optimality conditions under the following assumption:

(H4)** There exist a number $\alpha > 0$ and non-negative measurable functions $k(.)$ and $\beta(.)$ such that

$$k(.) \text{ and } t \to \beta(t)k^\alpha(t) \text{ are integrable}$$

and, for each $N \geq 0$,

$$F(t, x) \cap (\dot{x}(t) + N \mathbb{B}) \subset F(t, x') + (k(t) + \beta(t)N^\alpha)|x - x'| \mathbb{B}$$

for all $x, x' \in \dot{x}(t) + \epsilon \mathbb{B}$ a.e.. Condition (H4)** is a refinement of the classical epi-Lipschitz condition introduced by Loewen and Rockafellar in [44] that requires $\alpha = 1$ and $\beta(.) \equiv \beta_0 \geq 0$. An interesting question that has not been answered in the thesis is whether this hypothesis is optimal for the derivation of necessary conditions, namely whether the integrability of the function $t \to \beta(t)k^\alpha(t)$ is necessary for Theorem 3.2.3 to hold.

- In Chapters 5 and 6 we considered free time optimal control problems featuring constant delays with respect to the state variable. Many open questions remains unaddressed.

- The maximum principle for optimal control problems described by ordinary differential equations with controls, is derived from the generalized Euler-Lagrange conditions for problems described in term of differential inclusions. Our analysis therefore does not allow delays both on the state and on the control. This
is an important class of problems to investigate in future research.

– Distributed and Time dependent controls are not considered.

– An other immediate extension to our setting is the introduction of state constraints.

– For standard optimal control problems without delays the state of a system $x(t)$ at time $t$ belongs to a finite dimensional space. When a single delay $\delta$ is included in the problem formulation the state of the system at time $t$ becomes $\{x(t + \tau) : \tau \in [-\delta, 0]\}$ that belongs to an infinite dimensional space. There are many open problems in areas where it is necessary to consider infinite dimensional state spaces, for example existence of minimizers and relaxation procedures, Hamilton-Jacobi analysis, second order local sufficiency conditions, and sensitivity analysis.

Model Predictive Control

• In Chapter 8 we gave estimates for the optimization horizon $N$ required to ensure stability of the MPC algorithm. Such performance estimates are quite conservative with respect to what it is observed numerically. Nevertheless in [36] it is shown that for some pathological situations these estimates cannot be improved. Can we derive better estimates for classes of problems that do not include those pathological situations?

• In Chapter 8 we proved stability of the MPC scheme under a local controllability assumption. Such controllability assumption is then shown to be automatically satisfied for stabilizable linear quadratic systems. This suggests the following. If a certain nonlinear system can be linearized around the equilibrium and such linearization is stabilizable then our stability results apply. What can we say about equilibrium around which the system cannot be linearized?

• The local controllability assumption employed in Chapter 8 provides stability results in the sense of Lyapunov. Can we still obtain weaker stability results by relaxing our hypotheses?

• Finally a word about robustness of the MPC scheme. It is a well known fact (see [32]) that continuity of the optimal value function $V_\infty$ (discussed at the end of Chapter 9)
and robustness of the MPC scheme are strictly related. Can our sufficient conditions for continuity of the value function be exploited in this direction?
Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.e.</td>
<td>Stands for almost everywhere</td>
</tr>
<tr>
<td>( \mathbb{R}^n )</td>
<td>Space of ( n )-vectors of real numbers</td>
</tr>
<tr>
<td>( \mathbb{N} )</td>
<td>Natural numbers</td>
</tr>
<tr>
<td>( \mathbb{R}_{\geq 0} )</td>
<td>Non-negative real numbers</td>
</tr>
<tr>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>( |M| )</td>
<td>Norm of a matrix ( M \in \mathbb{R}^{n \times m} ), ( |M| := \sup_{</td>
</tr>
<tr>
<td>( \mathbb{B} )</td>
<td>The closed unit ball in ( \mathbb{R}^n ), ( { x \mid</td>
</tr>
<tr>
<td>( \overline{A} )</td>
<td>Closure of a set ( A \subset \mathbb{R}^n )</td>
</tr>
<tr>
<td>( \text{int} A )</td>
<td>Interior of a set ( A \subset \mathbb{R}^n )</td>
</tr>
<tr>
<td>( \partial A ) or ( \text{bdry}(A) )</td>
<td>Boundary of a set ( A \subset \mathbb{R}^n ), ( A := \overline{A} \setminus \text{int} A )</td>
</tr>
<tr>
<td>( \text{co} A )</td>
<td>Convex hull of a set ( A \subset \mathbb{R}^n )</td>
</tr>
<tr>
<td>( \delta A(x) )</td>
<td>The distance of a point ( x \in \mathbb{R}^n ) from ( A \subset \mathbb{R}^n ), ( \min{</td>
</tr>
<tr>
<td>( \chi_A(.) )</td>
<td>The indicator function of ( A \subset \mathbb{R}^n ), which takes the value 1 on ( A ) and 0 on its complement.</td>
</tr>
<tr>
<td>( I_A(.) )</td>
<td>Not to be confused with ( \chi_A(.) ) defined above, this is also called indicator function but takes the value 0 on ( A ) and (+\infty) on its complement.</td>
</tr>
<tr>
<td>( \mathcal{K} )</td>
<td>Class of continuous and strictly increasing functions ( \eta : \mathbb{R}<em>{\geq 0} \to \mathbb{R}</em>{\geq 0} ) such that ( \eta(0) = 0 )</td>
</tr>
<tr>
<td>( \mathcal{K}_{\infty} )</td>
<td>Subclass of unbounded functions ( \eta \in \mathcal{K} )</td>
</tr>
<tr>
<td>( \mathcal{KL} )</td>
<td>Class of continuous functions ( \beta : \mathbb{R}<em>{\geq 0} \times \mathbb{R}</em>{\geq 0} \to \mathbb{R}<em>{\geq 0} ) such that ( \beta(\cdot,t) \in \mathcal{K}</em>{\infty} ), ( t \in \mathbb{R}<em>{\geq 0} ). ( \beta ) is strictly decreasing with respect to the second argument and for all ( r &gt; 0 ), ( \lim</em>{t \to \infty} \beta(r,t) = 0 ) holds.</td>
</tr>
<tr>
<td>( C^m([S,T];\mathbb{R}^n) )</td>
<td>( m )-times continuously differentiable ( \mathbb{R}^n )-valued functions, i.e., ( x = (x_1, \ldots, x_n) \in C^m([S,T];\mathbb{R}^n) ) are continuous and the ( i )-derivative ( \frac{d^i}{dt^i} x(t) ) exists and it is continuous ( i = 1, \ldots, m ).</td>
</tr>
</tbody>
</table>
$W^{1,1}([S,T]; \mathbb{R}^n)$ (sometimes written $W^{1,1}$.1) denotes the set of absolutely continuous $\mathbb{R}^n$-valued functions on $[S,T]$ equipped with the norm
\[
\|x\|_{W^{1,1}} := |x(a)| + \int_a^b |\dot{x}(t)| \, dt.
\]
Elements $x(\cdot) \in W^{1,1}([a,b]; \mathbb{R}^n)$ are called arcs.

$\text{Gr } \Gamma(\cdot)$ Graph of a multifunction $\Gamma(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$. It is the set \[\{(x,v) \in \mathbb{R}^n \times \mathbb{R}^k \mid v \in \Gamma(x)\}.\] A multifunction is a map which associates a set $\Gamma(x) \subseteq \mathbb{R}^k$ for any $x \in \mathbb{R}^n$.

$NBV^+[S,T]$ The space of increasing, real-valued functions $\mu(\cdot)$ on $[S,T]$ of bounded variation, vanishing at the point $S$ and right continuous on $(S,T)$. The total variation of a function $\mu(\cdot) \in NBV^+[S,T]$ is written $\|\mu\|_{T.V.}$. As is well known, each function $\mu(\cdot) \in NBV^+[S,T]$ defines a Borel measure on $[S,T]$. This associated measure is also denoted $\mu$.

**Nonsmooth Analysis (c.f. [2,16,21,25,55])**

$N_D(\bar{x})$ Normal cone (sometimes limiting normal cone) of a set $D \subset \mathbb{R}^k$ at $\bar{x} \in D$
\[
N_D(\bar{x}) := \left\{ \pi \mid \exists x_i \xrightarrow{D} \bar{x}, \pi_i \rightarrow \pi \text{ s.t.} \limsup_{x_i \stackrel{D}{\rightarrow} \bar{x}} \frac{\pi_i \cdot (x - x_i)}{|x - x_i|} \leq 0 \text{ for all } i \in \mathbb{N} \right\}.
\]
Here the notation $x_i \xrightarrow{D} \bar{x}$ indicates that $x_i \in D$ along the convergent sequence $x_i \rightarrow \bar{x}$.

$N^L_D(\bar{x})$ Limiting normal cone, see $N_D(\bar{x})$

$N^P_D(\bar{x})$ Proximal normal cone of the set $D \subset \mathbb{R}^k$ at $\bar{x} \in D$
\[
N^P_D(\bar{x}) := \{ \pi \mid \exists t > 0 \text{ s.t. } d_D(\bar{x} + t\pi) = t|\pi| \}.
\]
It can be proved that $N_D(\bar{x})$ coincides with the \[\limsup_{x \rightarrow \bar{x}} N^P_D(x),\] limsup intended in the sense of Kuratowski.

$N^C_D(\bar{x})$ Generalized (Clarke) normal cone $N^C_D(x) := \text{co } N^P_B(x)$, where co stands for convex hull.
\[ \partial f(\bar{x}) \quad \text{(limiting) sub-differential of a lower semicontinuous function} \quad f : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\} \text{ at a point } \bar{x} \in \text{dom } f := \{x \in \mathbb{R}^k \mid f(x) < +\infty\}, \]

\[ \partial f(\bar{x}) = \{\xi \mid \exists \xi_i \to \xi \text{ and } x_i \to \bar{x}, f(x_i) \to f(\bar{x}) : \limsup_{x \to x_i} \frac{\xi_i \cdot (x - x_i) - f(x) + f(x_i)}{|x - x_i|} \leq 0, \forall i \in \mathbb{N}\}. \]

\[ \partial P \! f(\bar{x}) \quad \text{Proximal sub-differential of a lower semicontinuous function} \quad f : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\} \text{ at a point } \bar{x} \in \text{dom } f := \{x \in \mathbb{R}^k \mid f(x) < +\infty\}, \]

\[ \partial P \! f(\bar{x}) := \{\xi \mid \exists \sigma, \delta > 0 : f(x) - f(\bar{x}) \geq \xi \cdot (x - \bar{x}) - \sigma|x - \bar{x}|^2, \text{ for all } x \in \bar{x} + \delta \mathbb{B}\}. \]

\[ \partial C \! f(\bar{x}) \quad \text{Generalized (Clarke) sub-differential} \quad \partial C \! f(\bar{x}) := \text{co } \partial f(\bar{x}) \]

\[ \text{ess } h(\tau) \quad \text{Essential value of measurable an essentially bounded function} \quad h(\cdot) : I \to \mathbb{R} \text{ at a point } t \in I, (I \text{ is an open interval}) \]

\[ \text{ess } h(\tau) := [a^-, a^+] \]

where

\[ a^- = \lim_{\delta \downarrow 0} \text{ess inf } h(\tau) \text{ and } a^+ = \lim_{\delta \downarrow 0} \text{ess sup } h(\tau). \]

If \( h(\cdot) \) is continuous, then \( \text{ess } h(\tau) = \{h(t)\} \).
Bibliography


