Design of structural steel elements with the Continuous Strength Method

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PhD Thesis
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Abstract

The current practice of ultimate limit state design for steel structures involves an elastic–perfectly plastic material model and the classification of cross-sections into discrete behavioural classes. This leads to a design philosophy which is simple, but generally over-conservative. The Continuous Strength Method is a strain based design approach which allows for the beneficial influence of strain hardening. At the core of the method is a base curve which relates the deformation capacity of a cross-section to its cross-section slenderness. Deformation capacity is defined through a strain ratio, which is the ratio of the maximum strain that a cross-section can endure to its yield strain. The formulation for the base curve was derived by means of stub column and bending tests collected from the literature. Knowing the limiting strain and assuming plane sections remain plane, the resistance of cross-sections to combinations of axial load and bending moments can be calculated by integrating the stresses arising from a suitable strain hardening material model over the area of the cross-section. Analytical and design expressions have been developed, and the resistance predictions for open and closed cross-section shapes have been compared.
with existing collated test data, and shown to give additional capacity over current design approaches, with a reduction in scatter and a more consistent method. Beyond the analysis of the cross-section, the method has been extended to the global instability of pin-ended columns by utilising moment–curvature–thrust curves. The curves were paired with an assumed buckled displacement shape to find applicable equilibrium configurations, and to extract the peak axial loads for producing buckling curves. The column buckling curves showed two distinct regions based on the global slenderness of the column. Firstly a region of global-dominated failure, where the columns failed by a loss of overall flexural rigidity, and secondly a local-dominated failure region, where the mid-height cross-sections failed by local buckling. The local cross-section failure mode allowed for axial loads greater than the cross-section yield loads. The column buckling curves were found to be dependent on the initial out-of-straightness, the cross-section geometry and the material yield stress. An experimental program provided insight into the cross-section resistance of hot-rolled rectangular hollow sections (RHS). The experiments included 32 material tensile coupon tests, eight stub column tests and four simply supported beam tests, and exhibited little strain hardening. Overall, a series of developments to a strain based approach for steel structures has been presented, and areas for future developments have also been highlighted.
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Chapter 1

Introduction

1.1 Background

Design rules for structural steel elements often include simplifications that allow swift and conservative estimates of capacity to be obtained, such as the ability to withstand combinations of axial loads, shear forces and bending moments. Some of these simplifications are at the material level, where structural steel is usually treated as an elastic–plastic or rigid–plastic material, some are based on equations limited to elastic conditions or small deflections, whilst other approximations involve the grouping of similar behaviour, as in the case of cross-section classification for the treatment of local buckling. This leads to the ultimate design strength of a cross-section or member being limited by its elastic or plastic capacity under various combinations of external loading. These approximations are widely used in modern design codes, such as the European design standard EN 1993-1-1 (2005) (Eurocode 3). The consequence can be a design philosophy that makes compromises on the true structural response and produces designs that do not fully utilise the potential of the material.
1.2 The Continuous Strength Method

Solutions to many of the highlighted problems involve extending the Continuous Strength Method (CSM), a method which aims to harmonise different aspects of structural steel design under one deformation based approach, and to offer more consistent and continuous resistance functions. It provides a strong link between topics such as local buckling, global instability, material modelling, indeterminate structures and cross-section analysis. The method originates from the latest work by Gardner (2008) and Wang (2011), where the basic foundations for the approach were described, building on the previous work from Gardner (2002) and Gardner and Nethercot (2004). At the core of the CSM is a base curve, which establishes a relationship between the deformation capacity of a cross-section and its cross-section slenderness. From the base curve, a maximum allowable strain is determined and the strain distribution throughout the cross-section is defined. Once the strain distribution is known, the cross-section stresses follow via a chosen material model, where an elastic, linear hardening stress–strain relationship has been selected for structural steel. The cross-section resistance to axial loads and bending moments is then obtained by integrating the stresses throughout the cross-section.

The working philosophy is a ground up approach, as the base of the method starts from the analysis of the cross-section and continues to the analysis of the member. Within the method, higher level analysis such as member buckling, relies deeply on the lower level analysis of the cross-section, thus creating a unified method with components that are inseparable from one another. As it is the strains that drive the method and not the material model or cross-section stresses, there is great potential in extending the method fully to other materials. Adoption of the CSM in engineering design will allow for benefits to be seen in construction, as structural steel members may be designed to resist axial loads and bending moments up to 10% greater than current European design code allowances. More efficient utilisation and prediction of member capacity allows economic benefits through
material weight savings, and the ability to select the most appropriate cross-section for a particular loading demand. Unifying the separate design codes for various construction materials (structural steel, stainless steel and aluminium) would provide a single universal design methodology.

1.3 Research structure

In this thesis, cross-section design expressions are developed for combinations of axial load and bending moment, and flexural buckling design formulae, derived through strain based considerations, for simple pin-ended columns are established. The cross-section shapes considered are I-sections and square, rectangular, circular and elliptical hollow sections with a limited study of equal and unequal angle cross-sections in bending. An experimental program in which hot-rolled rectangular hollow sections were tested, provided insight into the cross-section behaviour of such members. Having developed a range of design provisions utilising the strain based methods, reference is made to published laboratory test data, together with that generated herein, to confirm the applicability of the CSM predictions and to reinforce the established design philosophy.

1.4 Thesis layout

This thesis is laid out as detailed in this subsection. Chapter 2 contains a literature review, which summarises the Eurocode 3 and element interaction methods of determining the slenderness of a cross-section, introduces various strain based analysis methods from published literature, highlights models for determining intermediate elastic–plastic moment capacities, and describes the current Eurocode 3 rules for calculating the resistance of a cross-section to axial load and bending moments. Material modelling and design methods that utilise strain hardening to predict bending moment resistances greater than the plastic
moment capacity are also described. Stub column and simple beam test data from published literature are summarised for a variety of cross-section shapes. Traditional methods of describing the global flexural buckling behaviour of a pin-ended column are provided, as well as the Eurocode 3 design buckling curves.

Chapter 3 introduces local buckling of cross-sections, compares quantitatively the two different definitions of cross-section slenderness, examines commonly used material models, and then presents and justifies the chosen Eurocode 3 compliant design bi-linear material model. The deformation capacity of a cross-section is defined through a CSM strain ratio and curvature ratio, both being established for hot-rolled and cold-formed structural steel, along with the necessary corrections for corner strength enhancement and the occurrence of local buckling prior to yielding. A summary of the gathered stub column test data is presented and the deformation capacities plotted to form base curves, which are continuous relationships between the cross-section deformation capacity and the cross-section slenderness.

The axial and bending resistances of cross-sections in the strain hardening range are considered in Chapter 4. The axial capacity of a cross-section is based on uniform strains distributed throughout the cross-section, in conjunction with the design bi-linear material model. The resulting design axial load equation is then compared to stub column test data and yield loads. Uni-axial bending is rigorously investigated both analytically and numerically for six cross-section shapes, bending about either the major or minor axis and for any ratio of failure strain to yield strain (strain ratio). The analytical expressions are simplified into design expressions and compared to the ultimate loads from the gathered bending test data. Angle cross-sections bent about an arbitrary bending axis are also investigated. Example design calculations are provided for two cross-section shapes, a plastic strain ratio is defined to indicate when the plastic moment has been reached, and finally the effective flexural rigidity of a cross-section is introduced both analytically and approximately.
Combined axial load and bending resistances are examined in Chapter 5. Axial loading and uni-axial bending are treated with a uniform and linearly varying strain driven numerical model, the results of which are compared with a simple rigid–plastic model and with the Eurocode 3 design provisions. Interaction curves are presented in both plastic and CSM normalised form, and used to create moment–curvature–thrust curves. A bi-axial bending interaction model is formed from a cross-section planar strain distribution limited by the CSM limiting strain, to give interaction curves and design equations. The bi-axial model is extended further by adding uniform axial strains, with the resulting interaction surfaces then sliced to create contours of equal axial load (bi-axial interaction curves for a given axial load). General combined loading design equations are developed, with statistical comparisons made to the numerical model via a residual. The cross-section slenderness calculation procedure is defined for combined loading, so that it is possible to calculate appropriate elastic buckling coefficients for any applied loading combination.

Chapter 6 provides full details of an experimental study of hot-rolled rectangular hollow sections including tests on tensile coupons, stub columns and beams in three and four-point bending. The axial and bending capacities of the cross-sections are compared with their yield loads and plastic moments, and shortfalls in obtaining these values are attributed to the extended yield plateau of the material.

The geometrically and materially non-linear column buckling problem is investigated in Chapter 7, for an imperfect pin-ended column. Traditional elastic, first yield and CSM ultimate resistance solutions form bounds to the true response. This true column response is found by solving numerically the governing beam–deflection differential equation using moment–curvature–thrust curves, or approximately by simplifying the deflected shape of the column to an assumed half sine wave. The importance of the mid-height cross-section is emphasised, load–lateral deflection curves are plotted, and the peak loads are used to give column buckling curves for various initial imperfections and global slenderness values.
A design fit is derived that distinguishes between the local-dominated failure of the mid-height cross-section and the global-dominated failure of the whole column. The design fits are compared to the numerical results and the range of Eurocode 3 column buckling curves.

The key findings of the thesis and overall conclusions, together with recommendations for future work, are reported in Chapter 8.
Chapter 2

Literature review

This chapter provides a review of literature that is relevant to this thesis, with an emphasis on inelastic local buckling of cross-sections and global buckling of columns. Further literature is introduced in the individual thesis chapters.

2.1 Local buckling and cross-section slenderness

To determine the susceptibility of a cross-section to local buckling, a slenderness definition is needed which considers the contributing factors of geometry, material properties and stress distribution. This section details the two main methods, the critical plate element method and the element interaction method.


In Eurocode 3, the treatment of local buckling and the definition of cross-section slenderness for plated sections (e.g. box and I-sections) is governed by the most slender plate element in the cross-section, based upon the flat plate width method from Table 5.2 of EN 1993-1-1 (2005). For a rectangular plate element, the elastic buckling coefficients $k_\sigma$ from Tables 4.1
and 4.2 of EN 1993-1-5 (2006) are calculated for each internal and out-stand element. For short column members in axial compression, the stresses at each plate end are equal, and so out-stand elements have $k_σ = 0.43$ and internal elements have $k_σ = 4.0$. For flexure, the beneficial effects of tensile stresses causes internal elements such as webs in bending to use $k_σ = 23.9$ typically. Plate slenderness $\bar{\lambda}_p$ may be defined in the non-dimensional form of Eqn (2.1)

$$\bar{\lambda}_p = \sqrt{\frac{f_y}{f_{cr}}} = \sqrt{\frac{12(1 - \nu^2)235}{\pi^2 E k_σ} \left( \frac{c}{t c} \right)},$$

where $f_y$ is the material yield stress, $f_{cr}$ is the elastic buckling stress, $c$ is the flat plate width or height, $t$ is the plate thickness, $\epsilon = \sqrt{235/f_y}$, $E$ is the Young’s modulus and $\nu$ is the Poisson’s ratio, generally taken as $\nu = 0.3$ for steel. The plate slenderness of all the elements that make up the cross-section are evaluated, with the critical and governing element determined from the highest value of $\bar{\lambda}_p$. Since this definition of cross-section slenderness is based only upon the most slender element, the effect of the interaction between connected plates is not accounted for. Consider the rectangular and square hollow sections (RHS and SHS) in Figure 2.1 subjected to uniform compressive stresses. The web element of flat height $d$ will define the slenderness of the whole cross-section for both shapes, and therefore both cross-section shapes will have the same slenderness. If the two local buckling deformed shapes are compared, the flanges of the rectangular hollow section will provide more restraint to the web than those of the square hollow section due to the relatively higher flexural rigidity. For the webs of the rectangular hollow sections, the buckling coefficient should be greater than $k_σ = 4.0$, that is greater than the hinged end supports assumption. The axial load capacity of both cross-sections cannot be effectively estimated using the same buckling coefficients, and this prompts the need for a cross-section slenderness definition that considers the inter-connectivity between the plate elements.
For the treatment of circular hollow sections with depth $D$ and thickness $t$, the cross-section slenderness is given, with $f_{cr}$ defined from Timoshenko and Gere (1961), by

$$\bar{\lambda}_p = \sqrt{\frac{f_y}{f_{cr}}} = \sqrt{\frac{235}{E} \frac{\sqrt{3(1-\nu^2)} \left( \frac{D}{te^2} \right)}{2(1-\nu^2)E}}. \quad (2.2)$$

The circular hollow section $D/te^2$ class limits in EN 1993-1-1 (2005) are identical for both compression and bending, with no distinction made between the two loading cases. For elliptical hollow sections, the effective diameter $D_e = 2b^2/a$ from Chan and Gardner (2008a) is used in place of $D$ in order to account for the aspect ratio of the cross-section $b/a$. When $b/a = 1$, the cross-section is a circular hollow section and $D_e = D$, and when $b/a > 1$, $D_e$ and hence $\bar{\lambda}_p$ increase, as the sides of the ellipse get flatter, causing a reduction in the elastic buckling stress $f_{cr}$ of the cross-section.

### 2.1.2 Element interaction

Seif and Schafer (2010) examined the problem of representing the cross-section slenderness by the most slender plate element, as the issue is also present in the American National Specification AISC 360-10 (2010). The authors formulated equations that included the
effects of element interaction for cross-sections (previous analytical and numerical solutions were also reported by Bulson (1970)). They began by rearranging the elastic critical buckling stress $f_{cr}$, for the buckling coefficient $k_\sigma$ with

$$k_\sigma = \frac{f_{cr} 12(1 - \nu^2)}{\pi^2 E} \left(\frac{c}{t}\right)^2.$$  \hspace{1cm} (2.3)

The elastic critical buckling stress of a series of cross-sections, including element interaction, was then calculated numerically for compression and bending stress distributions by the finite strip analysis program CUFSM as described by Li and Schafer (2010). This method used the cross-section centreline geometry for various cross-section shapes, and ignored the effects of welds and root radii. Design fits were found for the results of the finite strip analysis and were summarised in tabular form.

In the CSM, $\tilde{\lambda}_p$ may be calculated using the buckling coefficients that consider element interaction from Seif and Schafer (2010), or by conservatively ignoring element interaction as in EN 1993-1-5 (2006). In general, it is recommended to use the best description of cross-section slenderness available; however, the most slender plate element method may be favoured in some instances since: 1) the EN 1993-1-5 (2006) method gives conservative results 2) the Seif and Schafer (2010) method is not currently available for combinations of axial force and bending moments and 3) the element interaction procedure is not presently recognised in EN 1993-1-1 (2005).

### 2.2 Strain based methods

To assess the capability of a cross-section to resist local buckling, relationships are needed that link the buckling resistance to the cross-section slenderness. In the European design code EN 1993-1-1 (2005), the local buckling resistance is determined by assigning the cross-section a class number from 1 to 4, with class 1 being the most resistant to local buckling.
This classification system considers the stress distribution throughout the cross-section, the material yield strength and the width-to-thickness ratios of the component plate elements.

Shifferaw and Schafer (2012) presented a relationship for bending between the strain demand $C_{y/b}$, which is the ratio of the ultimate strain (i.e. failure strain) and yield strain $\epsilon_u/\epsilon_y$ as calculated from finite element models, and the cross-section slenderness $\lambda = \sqrt{M_{el}/M_{cr}}$, where $M_{el}$ is the elastic moment and $M_{cr}$ is the elastic critical local buckling moment. This method was used for the calculation of the inelastic moment capacity of cold-formed members considering the influence of local buckling, and is given in Eqn (2.4) by

$$C_{y/b} = \frac{\epsilon_u}{\epsilon_y}. \quad (2.4)$$

Juhás (2010) provided a relationship between the ultimate strain $\epsilon_u$ and a web slenderness $\beta$, for the prediction of the elastic–plastic moment capacity of steel members, which is the intermediate moment between the elastic ($M_{el}$) and plastic ($M_{pl}$) moments. The relationship is defined by Eqn (2.5), which gives the ultimate strain for a web element in bending

$$\epsilon_u = 18.9 + 0.004 \left( \frac{f_y}{235} \right) \left( \frac{130\sqrt{235/f_y - \beta}}{\beta^2} \right)^2. \quad (2.5)$$

BS 5950-1 (2000) gives various interpolations between the elastic and plastic section moduli of the cross-section, to give an effective plastic section modulus for I-sections, rectangular hollow sections and circular hollow sections. The effective width approach of Knobloch and Fontana (2006) is a strain based method for the local buckling of steel cross-sections subjected to fire.
2.3 Strain hardening modulus

In Section 3.3 a bi-linear material model is used within the CSM to represent the stress-strain behaviour of structural steel by using a strain hardening modulus $E_{sh}$ less than the Young’s modulus $E$. Wang (2011) gave the following $E_{sh}/E$ values for use in such a material model, based on the material ultimate stress $f_u$ and the yield stress $f_y$:

Hot-rolled I-sections: \[ \frac{E_{sh}}{E} = 0.015 \frac{f_u}{f_y} - 1.0 - \frac{0.7}{0.7} \]

but \[ \frac{E_{sh}}{E} \leq 0.015 \quad (2.6) \]

Cold-formed hollow sections: \[ \frac{E_{sh}}{E} = 0.01 \frac{f_u}{f_y} - 1.0 - \frac{0.25}{0.25} \]

but \[ \frac{E_{sh}}{E} \leq 0.015 \quad (2.7) \]

Hot-rolled box sections:

\[ \frac{E_{sh}}{E} = \begin{cases} 
0.003 \frac{f_u}{f_y} - 1.0 - \frac{0.3}{0.3} & \frac{f_u}{f_y} \leq 1.3 \\
0.007 \frac{f_u}{f_y} - 1.3 - \frac{0.3}{0.3} + 0.003 & \frac{f_u}{f_y} > 1.3 \end{cases} \]

but \[ \frac{E_{sh}}{E} \leq 0.01 \quad (2.8) \]

Kemp et al. (2002) suggested a value of $E_{sh}/E = 0.0133$ for a bi-linear moment–curvature model, for use in predicting the strain hardening potential of hot-rolled I-sections. From 50 mill tests, Byfield et al. (2005) found $E_{sh}/E = 0.0129$ initiating at a strain of $6\varepsilon_y$, with little variation between grade S275 and S355 steel, and between cross-section sizes. With reference to Annex C of EN 1993-1-5 (2006), a value of $E_{sh}/E = 0.01$ is recommended, to be used in material models for finite element methods of analysis. This is paired with the requirement from EN 1993-1-1 (2005) which states that structural steel should also satisfy $f_u/f_y \geq 1.1$. Combining these criteria gives the suggested value for $E_{sh}$ as
\[
\frac{E_{sh}}{E} = \frac{f_u}{f_y} - \frac{1}{10} \quad \text{with} \quad \frac{E_{sh}}{E} \leq 0.01.
\] (2.9)

The inability of constant $E_{sh}$ models, such as Kemp et al. (2002) and EN 1993-1-5 (2006), to incorporate $f_u/f_y$ means that there is no allowance for materials that may have a restricted potential to strain harden. The limiting case where $f_u/f_y = 1$, should, by definition, be a perfectly plastic material with $E_{sh} = 0$. The proposed CSM model addresses this issue by giving lower values of $E_{sh}/E$ for low values of $f_u/f_y$.

### 2.4 Flexural strain distribution

Byfield and Nethercot (1998) confirmed from strain gauges attached to I-section beams, that a strain distribution varying linearly with depth, up to and beyond the plastic moment capacity is a valid assumption. This assumption is also noted by Chen and Atsuta (1976) based on observations from experiments, as cross-section dimensions are generally considerably smaller than beam lengths, permitting the neglect of shear deformations.

### 2.5 Test data

An extensive experimental database was assembled for this research, based upon published experimental results from stub column tests and stocky beams bending about the major axis. The data are used in Chapter 3 to construct relationships between the deformation capacity and the slenderness of cross-sections, and in Chapter 4 to compare the CSM design cross-section capacities to those of tested members. The collected test data are summarised in the subsections below, where the following abbreviations are used: UB (universal beam), UC (universal column), SHS (square hollow section), RHS (rectangular hollow section), CHS (circular hollow section), EHS (elliptical hollow section), I (I-section), + (cruciform section), CF (cold-formed), HR (hot-rolled) and W (welded).
2.5.1 Stub column data

Stub column test data were gathered from published literature that gave information about structural steel specimens tested beyond their peak loads, and had generally reached axial loads and strains above the yield values. The interest was in stub columns that had a low global slenderness ($\bar{\lambda} \leq 0.1$, see Chapter 7). The data sources are described in more detail in Section 3.6, where any assumed values for cross-section slenderness calculations or for material properties are also stated. The total number of tests gathered is 206, which are summarised in Table 2.1.

2.5.2 Bending data

Experimental data were collected on simply supported beams that were not influenced by lateral torsional buckling; thus they were either of short length or laterally supported to prevent any tendency to buckle globally. Hollow sections and I-sections bending about either the major or minor axis were sought, although only major axis bending data were found, with information recorded on the cross-section geometry and the material properties. The data were used to extract the maximum moments $M_u$ the test specimens had reached relative to their plastic moments $M_{pl}$, which could then be plotted against the cross-section slenderness. Where available, measurements of the curvatures or strains at the peak moments were also obtained, and the ultimate curvatures were used to represent the cross-section deformation capacity. The total number of bending tests is 72, as summarised in Table 2.2, with 4pt and 3pt indicating four-point and three-point bending respectively.
Table 2.1: Summary of the gathered stub column test data.

<table>
<thead>
<tr>
<th>Source</th>
<th>Shapes</th>
<th>No.</th>
<th>Type</th>
<th>$f_y$ [N/mm$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Akiyama et al. (1996)</td>
<td>SHS</td>
<td>45</td>
<td>CF, W</td>
<td>250-500</td>
</tr>
<tr>
<td>Gardner et al. (2010)</td>
<td>SHS, RHS</td>
<td>20</td>
<td>CF, HR</td>
<td>360-500</td>
</tr>
<tr>
<td>Zhao and Hancock (1991)</td>
<td>SHS, RHS</td>
<td>10</td>
<td>CF</td>
<td>430-490</td>
</tr>
<tr>
<td>Hu et al. (2011)</td>
<td>SHS, RHS</td>
<td>6</td>
<td>CF</td>
<td>360-380</td>
</tr>
<tr>
<td>Gao et al. (2009)</td>
<td>SHS</td>
<td>4</td>
<td>CF</td>
<td>790</td>
</tr>
<tr>
<td>Rasmussen and Hancock (1992)</td>
<td>SHS, +, I</td>
<td>18</td>
<td>W</td>
<td>740-750</td>
</tr>
<tr>
<td>Elchalakani et al. (2002)</td>
<td>RHS, CHS</td>
<td>11</td>
<td>CF</td>
<td>350-490</td>
</tr>
<tr>
<td>O’Shea and Bridge (1997)</td>
<td>CHS</td>
<td>7</td>
<td>CF, HR</td>
<td>185-360</td>
</tr>
<tr>
<td>Liew and Xiong (2010)</td>
<td>CHS</td>
<td>2</td>
<td>HR</td>
<td>380</td>
</tr>
<tr>
<td>Greiner et al. (2008)</td>
<td>I, RHS</td>
<td>10</td>
<td>HR, W</td>
<td>310-400</td>
</tr>
<tr>
<td>Giakoumelis and Lam (2004)</td>
<td>CHS</td>
<td>2</td>
<td>HR</td>
<td>340-370</td>
</tr>
<tr>
<td>Jiao and Zhao (2003)</td>
<td>CHS</td>
<td>2</td>
<td>CF</td>
<td>430</td>
</tr>
<tr>
<td>Zhao (2000)</td>
<td>CHS</td>
<td>12</td>
<td>CF</td>
<td>1350</td>
</tr>
<tr>
<td>Sakino et al. (2004)</td>
<td>CHS, SHS</td>
<td>30</td>
<td>CF</td>
<td>260-840</td>
</tr>
<tr>
<td>Chan and Gardner (2008a)</td>
<td>EHS</td>
<td>25</td>
<td>HR</td>
<td>360-430</td>
</tr>
<tr>
<td>Tutuncu and O’Rourke (2006)</td>
<td>CHS</td>
<td>2</td>
<td>CF</td>
<td>330</td>
</tr>
</tbody>
</table>

Table 2.2: Summary of the gathered bending test data.

<table>
<thead>
<tr>
<th>Source</th>
<th>Shapes</th>
<th>No.</th>
<th>Type</th>
<th>$f_y$ [N/mm$^2$]</th>
<th>Load</th>
<th>Span [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zhao and Hancock (1991)</td>
<td>RHS, SHS</td>
<td>10</td>
<td>CF</td>
<td>430-490</td>
<td>4pt</td>
<td>1000</td>
</tr>
<tr>
<td>Wilkinson and Hancock (1998)</td>
<td>RHS</td>
<td>44</td>
<td>CF</td>
<td>430-530</td>
<td>4pt</td>
<td>-</td>
</tr>
<tr>
<td>Gardner et al. (2010)</td>
<td>RHS, SHS</td>
<td>6</td>
<td>CF, HR</td>
<td>400-500</td>
<td>3pt</td>
<td>1100</td>
</tr>
</tbody>
</table>
2.6 Cross-section capacity

2.6.1 Uni-axial bending

At the ultimate limit state, a cross-section subjected to flexure is typically designed on the basis of its elastic \( M_{el} = W_{el} f_y \) or plastic moment capacity \( M_{pl} = W_{pl} f_y \), where \( W_{el} \) and \( W_{pl} \) are the elastic and plastic section moduli, and \( f_y \) is the material yield stress. The choice between the two bending capacities is based on the susceptibility of the cross-section to local buckling, which is assessed by considering the width-to-thickness ratios of the elements that make up the cross-section through a process known as cross-section classification. For slender cross-sections, where local buckling occurs prior to the initiation of yielding, reduced moment capacities are assigned. In EN 1993-1-1 (2005) stocky cross-sections are assigned to class 1 or class 2, and \( M_{pl} \) is taken as the design resistance \( M_{c,Rd} \).

For class 3 cross-sections, the elastic moment capacity \( M_{el} \) is used, and for slender cross-sections which fall into class 4, an effective area (or modulus) approach is used. For cross-sections that can reach at least the elastic moment capacity (class 3 or better), the EN 1993-1-1 (2005) resistances are given by Eqn (2.10), with \( \gamma_{M0} = 1.0 \) as a partial factor,

\[
M_{c,Rd} = \frac{W_{pl} f_y}{\gamma_{M0}} \quad \text{for class 1 or 2,} \quad M_{c,Rd} = \frac{W_{el} f_y}{\gamma_{M0}} \quad \text{for class 3.} \quad (2.10)
\]

This approach results in a step from \( M_{el} \) to \( M_{pl} \) at a particular cross-section slenderness limit, and in addition, as there is no distinction between the most stocky and slender cross-sections in a class group, they are treated incorrectly as behaving the same. This has led to the proposal of various elastic–plastic moment transitions to eliminate the discontinuity. The simplest model by Greiner et al. (2008) is a linear transition between \( M_{el} \) and \( M_{pl} \) for box sections and I-sections. This gave the intermediate elastic–plastic moment \( M_{ep} \) as shown in Figure 2.2.
Juhás (2010) presented $M_{ep}$ as the plastic moment minus an elastic–plastic bending moment of the web $M_{el,w}$, which was related to the ultimate strain $\epsilon_u$ attainable by the cross-section, divided by the yield strain $\epsilon_y$, with $\epsilon_u$ as a function of the web slenderness,

$$M_{ep} = M_{pl} - M_{el,w} \left( \frac{\epsilon_u}{\epsilon_y} \right)^2.$$ \hfill (2.11)

Another parabolic transition between the elastic and plastic moment capacities was presented by Shifferaw and Schafer (2012), and was also a function of the ultimate strain to yield strain ratio $\epsilon_u/\epsilon_y$, which for symmetric cross-sections took values between 1 and 3. This design curve was given by Eqn (2.12)

$$M_{ep} = M_{el} + (M_{pl} - M_{el}) \left[ 1 - \left( \frac{\epsilon_u}{\epsilon_y} \right)^2 \right].$$ \hfill (2.12)

The British design code BS 5950-1 (2000) gives various interpolations between the elastic and plastic section moduli of a cross-section, to give an effective plastic section modulus for I-sections, rectangular hollow sections and circular hollow sections.
For a bi-linear material model, with linear strain hardening, Wang (2011) suggested two methods for estimating the major axis bending capacity of an I-section; the first was an explicit calculation involving the cross-section dimensions, and the second was a simplified formulation using the elastic and plastic section moduli. For hot-rolled and cold-formed cross-sections, the first method assumed that the maximum stress was the local buckling stress $f_{lb}$, and that this was constant within the flanges, producing a stress step at the top of the web to a value of $f_{lb,web} = f_{lb}[h/(h + T)]$.

Partitioning the stress distribution into triangular and rectangular components gave the bending capacity $M_{c,Rd,y}$ as

$$M_{c,Rd,y} = f_{lb}BT(h + T) + \frac{2f_ytY^2}{3} + \frac{tf_y}{4}(h^2 - 4Y^2) + \frac{t}{6}(f_{lb,web} - f_y)(h^2 - Yh - 2Y^2)$$  \hspace{1cm} (2.13)

with $Y = 0.5D(\epsilon_{lb}/\epsilon_y)^{-1}$ as the distance either side of the neutral axis to first yield. This equation can be normalised with respect to the major axis plastic moment to give

$$\frac{M_{c,Rd,y}}{M_{pl,y}} = \frac{f_{lb}BT(h + T)}{f_yW_{pl,y}} + \frac{2t}{3W_{pl,y}} \frac{D^2}{4} \left( \frac{\epsilon_{lb}}{\epsilon_y} \right)^{-2} + \frac{t}{4W_{pl,y}} \left[ h^2 - D^2 \left( \frac{\epsilon_{lb}}{\epsilon_y} \right)^{-2} \right]$$
\[ + \frac{t}{6W_{pl,y}} \left( \frac{f_{lb}h}{f_y(h + T)} - 1 \right) \left[ h^2 - \frac{Dh}{2} \left( \frac{\epsilon_{lb}}{\epsilon_y} \right)^{-1} - \frac{\epsilon_{lb}}{2} \left( \frac{\epsilon_{lb}}{\epsilon_y} \right)^{-2} \right]. \]  \quad (2.14)

Substituting in \( f_{lb} = f_y + E_{sh}(\epsilon_{lb} - \epsilon_y) \) from a bi-linear material model gives

\[ \frac{M_{c,Rd,y}}{M_{pl,y}} = \left[ 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{lb}}{\epsilon_y} - 1 \right) \right] \frac{BT(h + T)}{W_{pl,y}} + \frac{tD^2}{6W_{pl,y}^2} \left( \frac{\epsilon_{lb}}{\epsilon_y} \right)^{-2} + \frac{t}{4W_{pl,y}} \left[ h^2 - \frac{Dh}{2} \left( \frac{\epsilon_{lb}}{\epsilon_y} \right)^{-1} - \frac{\epsilon_{lb}}{2} \left( \frac{\epsilon_{lb}}{\epsilon_y} \right)^{-2} \right] \]  

\[ + \frac{t}{6W_{pl,y}} \left( \left[ 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{lb}}{\epsilon_y} - 1 \right) \right] \frac{h}{(h + T)} - 1 \right) \left[ h^2 - \frac{Dh}{2} \left( \frac{\epsilon_{lb}}{\epsilon_y} \right)^{-1} - \frac{\epsilon_{lb}}{2} \left( \frac{\epsilon_{lb}}{\epsilon_y} \right)^{-2} \right]. \]  \quad (2.15)

The method was derived for I-sections but can also be applied to box sections by doubling the web thickness term. The results are not however applicable to minor axis bending or other cross-section shapes. This first method involves a significant amount of computation due to its explicit inclusion of the cross-section geometry, and so a second simpler method was formulated, which used a quadratic transition from \( M_{cl,y} \) to \( M_{pl,y} \) at \( \epsilon_{lb}/\epsilon_y = 3 \). The assumed stress distribution for \( 1 \leq \epsilon_{lb}/\epsilon_y \leq 3 \) is shown in Figure 2.4a, and gives the moment capacity as

\[ \frac{M_{c,Rd,y}}{M_{pl,y}} = 1 - \left( 1 - \frac{W_{cl,y}}{W_{pl,y}} \right) \left[ \left( \frac{\epsilon_{lb}}{\epsilon_y} \right)^{-2} - \left( \frac{\epsilon_{lb}}{\epsilon_y} - 1 \right) / 18 \right]. \]  \quad (2.16)

For stockier cross-sections, strain hardening was introduced by adding a linearly varying stress distribution to a fully plastic stress distribution. This strain hardening stress distribution started from zero at the neutral axis and increased to \( f_{lb} - f_y \) at the compressive outer-fibres. For \( 3 < \epsilon_{lb}/\epsilon_y \leq 15 \) the second method uses Figure 2.4b to give

\[ \frac{M_{c,Rd,y}}{M_{pl,y}} = 1 + \frac{W_{cl,y}}{W_{pl,y}} \frac{E_{sh}}{E} \left( \frac{\epsilon_{lb}}{\epsilon_y} - 3 \right). \]  \quad (2.17)
Kemp et al. (2002) proposed a bi-linear moment–curvature relationship based on bending tests on hot-rolled UB and UC sections. This model is plotted in Figure 2.5 and provided a simple calculation to obtain moment capacities above $M_{pl,y}$, that allowed for strain hardening. The major axis plastic normalised moment $M_y/M_{pl,y}$ is plotted against the curvature ratio $\kappa/\kappa_e$, which is the curvature $\kappa$ divided by the curvature at the elastic limit. The initial elastic slope $EI_y$ was used to describe moments up to the elastic moment $M_{el,y}$, set equal to $0.9M_{pl,y}$ at $\kappa_e$, and then an inelastic slope of $E_{sh}I_y$ described moments above the elastic moment up to a maximum of $1.08M_{pl,y}$ at $\kappa = 16\kappa_e$. The strain hardening modulus was taken as $E_{sh} = E/75$. The advantage of the method is its simplicity, but some disadvantages are that there is no distinction between different cross-section shapes, the assumption that $M_{el,y} = 0.9M_{pl,y}$ is only valid for shape factors of 1.11 (reasonable for a UB or UC), and a relatively high curvature ratio is required in order to attain $M_{pl,y}$.
Byfield and Nethercot (1998) proposed two methods for incorporating strain hardening into the bending resistance of I-sections. The first and simplest method involved replacing the yield stress in a rectangular stress block assumption with \( f_{1.5} \) (the stress at a strain of 1.5\%), and gave the design moment resistance as \( M_u = W_{pl,y} f_{1.5} \). The second method utilised the \( f_{1.5} \) stress at the outer-fibres only, and provided the continuous expression of Eqn (2.18) for the stress distribution \( f \), at position \( y \) throughout the cross-section depth,

\[
f = y^\frac{1}{2} f_{1.5} \left( \frac{D}{2} \right)^{-\frac{3}{2}}. \tag{2.18}
\]

Integration of the stresses \( f \) throughout an I-section of depth \( D \), width \( B \), flange thickness \( T \), web height \( h \) and web thickness \( t \), gave the moment resistance function as

\[
M_u^* = \frac{5}{22} f_{1.5} BD^2 - \frac{10}{11} f_{1.5} (B - t) \left( \frac{D}{2} - T \right)^{\frac{11}{11}} \left( \frac{D}{2} \right)^{-\frac{1}{2}}. \tag{2.19}
\]
This explicit equation is only applicable to I-sections (and extended to box sections) bending about the major axis. Separate equations would need to be developed for minor axis bending or for other cross-section shapes. In addition, since the resistance function is independent of the cross-section slenderness, it is assumed that the cross-section is stocky enough to attain the outer-fibre strain $\epsilon_{1.5}$, that is $\epsilon_{1.5}/\epsilon_y = 1.5E/100f_y$. This corresponds to 11.5 times the yield strain for S275 steel and 8.9 times for S355 steel, and so the $M^*_u$ equation is defined for a group of stocky cross-sections that can reach these strains.

2.6.2 Axial load and uni-axial bending

For axial loading and bending about one axis, design codes provide interaction equations. The EN 1993-1-1 (2005) provisions for the major axis bending of a class 1 or class 2 I-section, are bi-linear interaction curves that are functions of the cross-section web area to gross area. For the major axis, the reduced moment capacity in the presence of axial load $M_{N,y,Rd}$ is presented as a reduction to the plastic capacity $M_{pl,y}$ as

$$M_{N,y,Rd} = M_{pl,y} \left( \frac{1 - n}{1 - 0.5a} \right) \quad \text{but} \quad M_{N,y,Rd} \leq M_{pl,y}. \quad (2.20)$$

This equation uses the following relationships between the applied axial load $N_{Ed}$, the yield load $N_y$, and $A, B$ and $T$, which are the cross-section area, flange width and flange thickness respectively,

$$n = \frac{N_{Ed}}{N_y} \quad \text{and} \quad a = \frac{A - 2BT}{A} \quad \text{but} \quad a \leq 0.5. \quad (2.21)$$

For axial load and minor axis bending of an I-section, a linear-parabolic interaction curve is given by the following equations
\[ M_{N,z,Rd} = M_{pl,z} \quad \text{for} \quad n \leq a \quad (2.22) \]

\[ M_{N,z,Rd} = M_{pl,z} \left[ 1 - \left( \frac{n - a}{1 - a} \right)^2 \right] \quad \text{for} \quad n > a. \quad (2.23) \]

For box sections with wall thickness \( T \), a bi-linear model is used for both bending axes. For the major axis, this is

\[ M_{N,y,Rd} = M_{pl,y} \left( \frac{1 - n}{1 - 0.5a_w} \right) \quad \text{but} \quad M_{N,y,Rd} \leq M_{pl,y} \quad (2.24) \]

\[ a_w = \frac{A - 2BT}{A} \quad \text{but} \quad a_w \leq 0.5, \quad (2.25) \]

and similarly for the minor axis

\[ M_{N,z,Rd} = M_{pl,z} \left( \frac{1 - n}{1 - 0.5a_f} \right) \quad \text{but} \quad M_{N,z,Rd} \leq M_{pl,z} \quad (2.26) \]

\[ a_f = \frac{A - 2hT}{A} \quad \text{but} \quad a_f \leq 0.5. \quad (2.27) \]

For circular hollow sections, the following simple equation is given by EN 1993-1-1 (2005),

\[ M_{N,y,Rd} = M_{N,z,Rd} = M_{pl}(1 - n^{1.7}). \quad (2.28) \]

There are currently no codified provisions for elliptical hollow sections.
2.6.3 Bi-axial bending

The EN 1993-1-1 (2005) design model for the bi-axial bending of a cross-section when there is no applied axial load \( N_{Ed} = 0 \), takes the plastic moment normalised power form of Eqn (2.29)

\[
\left( \frac{M_{Ed,y}}{M_{pl,y}} \right)^\alpha + \left( \frac{M_{Ed,z}}{M_{pl,z}} \right)^\beta \leq 1
\]

(2.29)

where \( M_{Ed,y} \) and \( M_{Ed,z} \) are the design major and minor axis bending moments, \( M_{pl,y} \) and \( M_{pl,z} \) are the major and minor axis plastic moment resistances, and the two exponents \( \alpha \) and \( \beta \) take the values from Table 2.3. These exponents are both 2 for circular hollow sections and take constant values for box sections and I-sections. Eqn (2.29) does not allow for strain hardening and so the EN 1993-1-1 (2005) method is limited to providing moments that do not exceed the plastic moments \( M_{pl,y} \) and \( M_{pl,z} \).

Table 2.3: EN 1993-1-1 (2005) \( \alpha \) and \( \beta \) exponents for bi-axial bending.

<table>
<thead>
<tr>
<th></th>
<th>UB, UC</th>
<th>RHS, SHS</th>
<th>CHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>2</td>
<td>1.66</td>
<td>2</td>
</tr>
<tr>
<td>( \beta )</td>
<td>1</td>
<td>1.66</td>
<td>2</td>
</tr>
</tbody>
</table>

2.6.4 Axial load and bi-axial bending

EN 1993-1-1 (2005) extends Eqn (2.29) to the general case of combined loading to give Eqn (2.30), which is the same bi-axial power relationship, but now depends on the axial load \( n = N_{Ed}/N_y \),

\[
\left( \frac{M_{Ed,y}}{M_{N,y,Rd}} \right)^\alpha + \left( \frac{M_{Ed,z}}{M_{N,z,Rd}} \right)^\beta \leq 1.
\]

(2.30)

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where $M_{Ed,y}$ and $M_{Ed,z}$ are again the applied major axis and minor axis bending moments, $M_{N,y,Rd}$ and $M_{N,z,Rd}$ are the reduced plastic moments in the presence of axial loads, and the exponents $\alpha$ and $\beta$ now take the values given in Table 2.4.

Table 2.4: EN 1993-1-1 (2005) $\alpha$ and $\beta$ exponents for axial load and bi-axial bending.

<table>
<thead>
<tr>
<th></th>
<th>UB, UC</th>
<th>RHS, SHS</th>
<th>CHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>2</td>
<td>min[6, 1.66/(1 − 1.13$n^2$)]</td>
<td>2</td>
</tr>
<tr>
<td>$\beta$</td>
<td>max[1, 5$n$]</td>
<td>min[6, 1.66/(1 − 1.13$n^2$)]</td>
<td>2</td>
</tr>
</tbody>
</table>

Notice that for bi-axial bending, with or without an axial load, the cross-section shapes are grouped; UB and UC as well as SHS and RHS, are treated in the same manner and neither of the exponents relate to cross-section geometry. The CHS exponents are geometry and axial load level independent.

2.7 Flexural buckling

Inelastic column buckling is addressed in Chapter 7 of this thesis and builds upon the concepts described in this subsection. For a simple pin-ended column subjected to a concentric axial load $N$, and with $v_0$ and $v$ as the initial deflections and additional lateral deflections under load respectively, the differential equation that describes the equilibrium between the imposed bending moment $M = N(v_0 + v)$ and the internal resisting moment for a cross-section at distance $x$ along the member length is

$$\frac{d^2v}{dx^2} E'I = -N(v + v_0). \quad (2.31)$$

In Eqn (2.31), $E'I$ is the effective flexural rigidity at location $x$, measured from the end of the column of length $L$. This differential equation uses a linearised curvature assumption
from a small slope approximation, and various elastic solutions to Eqn (2.31) are available. When the initial deflections are zero \(v_0 = 0\), the solution gives the Euler (1744) elastic buckling load \(N_{cr} = \pi^2 EI/L^2\). A second order elastic analysis can be performed when \(v_0 \neq 0\) by limiting the material stresses to the yield stress, which will be governing at the most compressed outer-fibres of the mid-height cross-section. This occurs when the axial compressive stresses from the axial load \(N\), combine with the compressive bending stresses from moment \(M\) on the concave side of the laterally deflecting column. The axial load that corresponds to first yield is the Ayrton–Perry–Robertson load from Ayrton and Perry (1886) and Robertson (1925), which can be presented in the yield load normalised form of Eqn (2.32). This is the solution for the case when the initial bent shape is of a half sine wave shape \(v_0 = d_0 \sin(\pi x/L)\) with amplitude \(d_0\).

\[
\frac{N}{N_y} = \frac{1}{2} \left[ 1 + \frac{N_{cr}}{N_y} (1 + \eta_e) \right] - \frac{1}{2} \sqrt{\left[ 1 + \frac{N_{cr}}{N_y} (1 + \eta_e) \right]^2 - 4 \frac{N_{cr}}{N_y} \eta_e} \quad \eta_e = \frac{Ad_0}{W_{el}}. \quad (2.32)
\]

When stresses in any cross-section exceed the yield stress, material non-linearity affects the effective flexural rigidity term of the left-hand side of equation Eqn (2.31). Prior to yielding, the effective flexural rigidity \(E' I\) is equal to the elastic flexural rigidity \(EI\), otherwise the flexural rigidity is a function of the axial load and the curvature \(\kappa\). It will be seen in Section 5.1.9 that moment-curvature-thrust curves provide the necessary information to solve the general differential equation directly.

### 2.7.1 Inelastic column buckling method of Horne (1956)

The method of Horne (1956) involved the identification of three zones along the length of a column, depending upon whether yielding had occurred in the tensile and compressive outer-fibres. A simplified stress-strain curve was used, which gave the associated stress distributions for the plastic zones of a column, with most yielding located at mid-height.
This analytical method was derived for pin-ended columns of solid rectangular cross-section shape, subjected to a point load at an eccentricity, and for fixed-ended columns with ends at a given slope and with no load eccentricity. Such analytical methods are only suitable for cross-sections of simple geometry and for idealised stress–strain relationships.

2.7.2 Tangent and double modulus methods

For a column that is carrying an axial load greater than the yield load \( N > N_y \), there is an average cross-section stress \( f_m = N/A \) at a uniform strain \( \epsilon_m \). If the axial load is increased by a small amount, the change in the strain distribution will consist of a further uniform strain component and a linearly varying component (relating to an increase in curvature). The zero strain neutral axis will shift from the centroidal axis, and the area of the cross-section above this neutral axis will become more compressed, and the area below will experience greater tension. The corresponding stress distribution will depend upon the material stress–strain curve. The increase in compressive strains will be associated with the tangent modulus \( E_t \), which is the gradient of the material stress–strain curve, while decreasing strains will lead to strain reversal, for which \( E \) applies (Figure 2.6). It was shown by Engesser (1895) that this leads to the cross-section reduced modulus

\[
E_r = \frac{E_t I_1 + EI_2}{I},
\]

(2.33)

where \( I \) is the second moment of area about the centroidal axis, and \( I_1 \) and \( I_2 \) are the second moments of area corresponding to the areas above and below the instantaneous neutral axis. Whether or not strain reversal takes place, depends upon the relative increases in the uniform and linear strains for an increment of loading. Uniform compressive strain increases will be insufficient to overcome strain reversal in a moment dominated loading case. However, if the uniform strain increases are dominant, then no strain reversal can
occur and $E_r$ becomes equal to $E_t$, which leads to a lower flexural rigidity and more conservative model. Later, Shanley (1947) examined the relationship between $E_t$ and $E_r$ using a simplified rigid and hinged bars model, on a geometric basis. The value of $E_r$ is often used as a replacement for $E$ in the standard Euler buckling load $N_{cr}$, and so such a model does not consider the effects of initial out-of-straightness as the initial configuration is perfectly straight with $v_0 = 0$. This disregards the moments that are induced from the onset of loading and their amplification afterwards, which may greatly reduce the axial load obtained.

![Figure 2.6: Loading and unloading on material stress–strain curve.](image)

The assumption that a column member is perfectly straight until a critical axial load is reached is not representative of a typical column which may have an initial out-of-straightness, and so modern design codes such as EN 1993-1-1 (2005), utilise equations based on the Ayrton–Perry–Robertson formula of Eqn (2.32) to account for local and global imperfections.
2.7.3 European design standard EN 1993-1-1 (2005)

The combination of the Ayrton–Perry–Robertson formula and the research by Maquoi and Rondal (1978), forms the foundation of the EN 1993-1-1 (2005) buckling curves for columns. There are five buckling curves labelled a₀, a, b, c and d, which are plotted in Figure 2.7. For class 1, 2 and 3 cross-sections, Eqn (2.34) is used to calculate the design buckling load \( N_{b,Rd} \), presented as a reduction \( \chi \) to the yield load \( N_y \). The reduction factor \( \chi \) is a function of the global slenderness \( \bar{\lambda} = \sqrt{N_y/N_{cr}} \),

\[
\frac{N_{b,Rd}}{N_y} = \chi = \frac{1}{\phi + \sqrt{\phi^2 - \lambda^2}} \quad \text{with} \quad \phi = 0.5 \left[ 1 + \alpha (\bar{\lambda} - 0.2) + \bar{\lambda}^2 \right]. \tag{2.34}
\]

Some characteristics of the EN 1993-1-1 (2005) approach are: 1) a maximum reduction factor of \( \chi = 1.0 \), which limits the buckling load to the yield load 2) a plateau for low global slenderness values \( \bar{\lambda} \leq 0.2 \), for which the axial load is constant at the yield load and 3) an imperfection factor \( \alpha \), which accounts for variations in cross-section geometry and in steel grade.

![Figure 2.7: EN 1993-1-1 (2005) design buckling curves for columns.](image-url)
This chapter summarised two methods of determining the slenderness of a cross-section, described various strain based analysis methods and models for determining intermediate elastic–plastic moment capacities, and described the current European design rules for calculating cross-section resistance. Methods that utilise strain hardening to give bending moment resistances greater than the plastic moment capacity were also given. Test data, which will be used later in this thesis were summarised, and methods for analysing the flexural buckling behaviour of pin-ended columns were given, including the latest Eurocode 3 allowance.

This research looks at progressing the current work on the CSM with respect to local buckling, refinements to the base curves, advancing uni-axial bending, and extending the method further into new territory. This extension includes incorporating various new cross-section shapes (angle, circular hollow and elliptical hollow sections), defining more general analytical and design flexural equations, developing numerical methods, creating a combined loading model with design equations, creating moment–curvature–thrust curves, and introducing a CSM flexural buckling model. The methods to achieve this are analytical, with statistical methods for residuals, curve fitting and comparisons with test data, and with the development of numerical methods for generating approximate solutions.
Chapter 3

Local buckling and strain limits

3.1 Introduction

The Continuous Strength Method (CSM) has two key components. The first component is a base curve, which defines the maximum strain that a cross-section can endure $\varepsilon_{\text{cs}}$, as a function of the cross-section slenderness. Development of the base curve, utilising both compression and bending test data, is described in the following subsections. The second component is a material model that allows for the influence of strain hardening; this is described in Section 3.3.

If a structural member is resistant or insensitive to global buckling (e.g. fully braced columns or beams with closely spaced lateral restraints), then its peak capacity will depend upon the local strength of the most heavily loaded cross-section. Local plate buckling may
initiate before or after the onset of material yielding, with the key determining geometric factor being the relative width $B$ (or height/diameter $D$) to thickness ratio, of the plate elements that make up the cross-section. Elements that are proportioned with low width-to-thickness ratios are more resistant to local buckling. Cross-sections subjected to pure bending or axial load plus bending, experience a stress distribution which is more favourable than the axial case alone. This is often represented in design codes by higher elastic buckling coefficients and leads to reduced cross-section slenderness values. The following subsection describes two ways of defining such a cross-section slenderness.

### 3.2 Cross-section slenderness

Cross-section slenderness defines the susceptibility of a cross-section to local buckling. For the cross-section geometry shown in Figure 3.1, the elastic buckling stresses $f_{cr}$ and the buckling coefficients $k_w$, $k_b$ and $k_h$, are presented for the Seif and Schafer (2010) method of determining the complete cross-section slenderness, referred to herein as $\bar{\lambda}_{p,DSM} = \sqrt{f_y / f_{cr}}$.

![Figure 3.1: Cross-section geometry for use in Eqn (3.1) to Eqn (3.6).](image-url)
These expressions for the total cross-section slenderness are given in Eqn (3.1) to Eqn (3.6) for the loading conditions of axial compression, major axis bending and minor axis bending. For I-sections, the notation used is as follows: \( B \) is the flange width, \( T \) is the flange thickness, \( t \) is the web thickness and \( h \) is the clear distance between the flanges. For box sections, \( b = B - T \) is the centreline width between the web elements, and \( T = t/2 \) is the wall thickness, which is taken as constant around the cross-section. The Young’s modulus and Poisson’s ratio are \( E \) and \( \nu \) respectively. For I-sections

Axial compression
\[
\frac{1}{k_w} = \frac{1.5}{\left(\frac{(h+T)}{t}\right)^2} + 0.18 \quad f_{cr} = \frac{k_w \pi^2 E}{12(1 - \nu^2)} \left( \frac{t}{h+T} \right)^2 \tag{3.1}
\]

Major axis bending
\[
\frac{1}{k_h} = \frac{1.5}{\left(\frac{(h+T)}{t}\right)^2} + 0.015 \quad f_{cr} = \frac{k_h \pi^2 E}{12(1 - \nu^2)} \left( \frac{t}{h+T} \right)^2 \tag{3.2}
\]

Minor axis bending
\[
\frac{1}{k_h} = \frac{1.5}{\left(\frac{(h+T)}{t}\right)^2} + 0.008 \quad f_{cr} = \frac{k_h \pi^2 E}{12(1 - \nu^2)} \left( \frac{t}{h+T} \right)^2 \tag{3.3}
\]

and for box sections

Axial compression
\[
k_b = \frac{4.0}{\left(\frac{(h+T)}{b}\right)^{1.7}} \quad f_{cr} = \frac{k_b \pi^2 E}{12(1 - \nu^2)} \left( \frac{T}{b} \right)^2 \tag{3.4}
\]

Major axis bending
\[
\frac{1}{k_h} = \frac{0.19}{\left(\frac{(h+T)}{b}\right)^3} + 0.03 \quad f_{cr} = \frac{k_h \pi^2 E}{12(1 - \nu^2)} \left( \frac{T}{h+T} \right)^2 \tag{3.5}
\]

Minor axis bending
\[
k_b = \frac{5.5}{\left(\frac{(h+T)}{b}\right)^2} \quad f_{cr} = \frac{k_b \pi^2 E}{12(1 - \nu^2)} \left( \frac{T}{b} \right)^2. \tag{3.6}
\]
Figure 3.2 shows the ratio of the cross-section slenderness values calculated according to the Seif and Schafer (2010) and Eurocode 3 methods for axial compression, major axis bending and minor axis bending, of standard hot-rolled steel cross-sections from SCI P363 (2009). The cross-section slenderness based on the elastic buckling stress of the most slender plate element is denoted $\bar{\lambda}_{p,EC3}$. (h + T)/2T/tB

$$\frac{\bar{\lambda}_{p,DSM}}{\bar{\lambda}_{p,EC3}}$$

(a) I-sections

Figure 3.2: EN 1993-1-5 (2006) and Seif and Schafer (2010) cross-section slenderness comparisons.

Note that for a like-for-like comparison between the two procedures (element interaction and the most slender plate element calculations), the flat plate width $c$ is used to determine the $c/T$ ratio instead of the centreline geometry. The $\bar{\lambda}_{p,DSM}/\bar{\lambda}_{p,EC3}$ ratio is given by Eqn (3.7), where max is used for the most slender element according to the EN 1993-1-1 (2005) provisions, $k_\sigma$ is the plate buckling coefficient given in EN 1993-1-5 (2006) and $k_{DSM}$ is the appropriate value of $k_w$, $k_h$ or $k_b$. 62
\[
\frac{\bar{\lambda}_{p,\text{DSM}}}{\bar{\lambda}_{p,\text{EC3}}} = \frac{(\frac{c}{T})_{\text{DSM}} \frac{1}{\sqrt{k_{\text{DSM}}}}}{\max\left((\frac{c}{T})_{\text{EC3}} \frac{1}{\sqrt{k_{p}}}ight)}
\] (3.7)

In Figure 3.2a, I-sections under axial compression are found to be slightly more slender (higher $\bar{\lambda}_{p}$) with the Seif and Schafer (2010) method when $(h + T)2T/tB \approx 3$, but are stockier for all other values, giving $\bar{\lambda}_{p,\text{DSM}}/\bar{\lambda}_{p,\text{EC3}}$ between 0.85-1.0. Note that $\bar{\lambda}_{p,\text{DSM}}/\bar{\lambda}_{p,\text{EC3}}$ values greater than unity are due to approximations made in the Seif and Schafer (2010) method, rather than the true elastic buckling behaviour of the cross-sections. For major and minor axis bending, the ratio of cross-section slenderness $\bar{\lambda}_{p,\text{DSM}}/\bar{\lambda}_{p,\text{EC3}}$ falls between 0.70 and 1.0. This shows that the consideration of the total cross-section element interaction for local buckling would, in general, result in a stockier cross-section.

The conservatism in the EN 1993-1-1 (2005) methodology is also demonstrated in Figure 3.2b, which shows the same comparisons for hollow box sections. In Figure 3.2b, $\bar{\lambda}_{p,\text{DSM}}/\bar{\lambda}_{p,\text{EC3}}$ generally ranges between 0.70-1.0, except for major axis bending when $(h + T)/b \approx 2.2$, which is the approximate transition point between the flange or web being the most slender plate element.

### 3.3 Material model

The stress–strain response of structural steel can differ depending on the material grade and how the material has been manufactured, subsequently mechanically worked, and ultimately tested. Hot-rolling or cold-forming can affect the material behaviour by altering the distinctiveness of the yield point, the length of the yield plateau, and the magnitude of the strain hardening slope. Variation in material properties around structural cross-sections is also possible, such as in the case of cold-formed cross-sections, where higher strength but lower ductility is typically found in the corner regions.
3.3.1 Existing material models

Given that the stress–strain response of steel can vary significantly, it is important to utilise a material model that can represent adequately the range of characteristic material curves. Traditionally a bi-linear, elastic–perfectly plastic material model (Figure 3.3a) is used to model structural steel, with the key advantage of being very simple to analyse, but with the potential disadvantage of being overly conservative since no post-yield strain hardening is accounted for. The traditional bi-linear model can be augmented with a third linear portion to produce a tri-linear model, as shown in Figure 3.3b, which is particularly suited to some hot-rolled members that exhibit a yield plateau, followed by strain hardening. In addition to providing $E$ and $E_{sh}$, the length of the yield plateau must also be given to complete the model description. If the stress–strain ($\sigma-\epsilon$) curve is more rounded, it is common to use the Ramberg and Osgood (1943) relationship shown in Figure 3.3c and given in Eqn (3.8) by

$$\epsilon = \frac{\sigma}{E} + \alpha \frac{f_0}{E} \left( \frac{\sigma}{f_0} \right)^n$$  \hspace{1cm} (3.8)

where $\alpha \epsilon_0 = \alpha f_0 / E$ is the yield offset, commonly 0.2%, and both $\alpha$ and $n$ are determined from experimental data. Note that with this model, strains are expressed as a function of stresses, rendering the Ramberg–Osgood model difficult to apply. It will be seen later that the CSM applies conversely by defining strains first, and then determining the corresponding stresses. It will also be seen that due to the deformation and numerical based approach of the CSM, any material model can be used in conjunction with the CSM, with the prime requirement being that the cross-section can be assumed to be deforming with plane sections remaining plane.
Figure 3.3: Material models used to represent structural steel.

3.3.2 Design bi-linear material model

The idealised stress–strain relationship for hot-rolled material is given in Figure 3.4a, and exhibits a constant Young’s modulus up to a well-defined yield stress and yield strain, and then progresses onto a distinctive yield plateau.

Figure 3.4: Typical hot-rolled (HR) and cold-formed (CF) material stress–strain curves and the CSM design material model.
At the end of the plateau, strain hardening ensues, where stresses increase with strains with a varying gradient $E_{sh}$, up until an ultimate peak stress $f_u$ and peak strain $\epsilon_u$. Cold-formed steel and other non-linear materials such as stainless steel and aluminium, tend to have a more rounded stress–strain curve and so a definite yield point can be difficult to ascertain. Usually the 0.2% proof stress $f_{0.2}$ is used which is the stress at a strain of $\epsilon_{0.2} = 0.002 + \epsilon_y$, where $\epsilon_y = f_{0.2}/E$. For the proposed material model to be used in the CSM, a compromise between the hot-rolled and cold-formed curves is the simplified bi-linear model shown in Figure 3.4c. This model is selected due to its compliance with EN 1993-1-5 (2006), and allows the derivation of design expressions which are based upon an accepted (and as will be seen later, relatively conservative) material model. This model is an elastic, linear hardening relationship which consists of an initial linear region with gradient $E$, which defines stresses up to the yield stress $f_y$ (taken as $f_{0.2}$ for cold-formed cross-sections or in the case of non-linear materials), followed by a strain hardening region, described by an appropriate constant modulus $E_{sh}$. A maximum limiting strain is also set at 15 times the yield strain ($\epsilon_{csm} = 15\epsilon_y$), a value which corresponds to the material ductility requirements given in Clause 3.2.2(1) of EN 1993-1-1 (2005). This material model gives a stress $f$ at strain $\epsilon$ from the piece-wise function

$$f = \begin{cases} E\epsilon & \epsilon \leq \epsilon_y \\ f_y + E_{sh}(\epsilon - \epsilon_y) & \epsilon_y < \epsilon \leq \epsilon_{csm}. \end{cases}$$ \hspace{1cm} (3.9)

### 3.4 Strain ratio

The CSM is a deformation based design approach, founded upon a derived relationship between the failure strain of a cross-section and its local slenderness. The results of both stub column and in-plane bending tests can be used in the derivation of this relationship. A stub column is defined herein as a column with a global non-dimensional slenderness.
\( \bar{\lambda} \leq 0.1 \), where \( \bar{\lambda} = \sqrt{N_y/N_{cr}} \), \( N_y \) being the yield load of the cross-section and \( N_{cr} \) the elastic buckling load of the member. While meeting the above requirement of \( \bar{\lambda} \leq 0.1 \) to avoid any significant influence from global buckling, the test lengths \( L \) of stub columns should ideally be at least three times the larger cross-section dimension, as suggested by Ziemian (2010), in order to contain a representative distribution of geometric imperfections and residual stresses and to allow local failure modes to form without a strong influence from end effects. A typical load versus end-shortening \((N-\delta)\) curve from a stub column test is shown in Figure 3.5a, where loads above the yield load will be reached if the cross-section slenderness is sufficiently low to allow material stresses to enter the strain hardening regime.

![Figure 3.5: Load–deformation response of a stub column and the flat and corner areas of a box section.](image)

The end–shortening \( \delta \) at the ultimate load \( N_{lb} \) (generally the peak load achieved in the stub column test \( N_u \) with a local buckling failure), is divided by the length of the specimen to obtain the average failure strain of the cross-section \( \epsilon_{lb} \). The deformation capacity of the stub column is then defined as \( \epsilon_{cam} \), which is taken directly as \( \epsilon_{lb} \) for materials that exhibit
a distinct yield point and as $\epsilon_{lb} - 0.002$ for materials with a rounded stress–strain curve. The subtraction of 0.2% strain in the case of rounded stress–strain curves is to ensure compatibility with the chosen material model of Figure 3.4c and to avoid over-predictions of capacity. The CSM strain is normalised by the yield strain in Eqn (3.10), and is referred to as the strain ratio. For a cross-section that fails exactly at the yield load $N_y$, the strain ratio $\epsilon_{csm}/\epsilon_y$ is equal to unity.

$$\frac{\epsilon_{csm}}{\epsilon_y} = \begin{cases} \frac{\epsilon_{lb}}{\epsilon_y} & \text{for hot-rolled materials exhibiting a sharply defined yield point} \\ \frac{\epsilon_{lb} - 0.002}{\epsilon_y} & \text{for cold-formed and non-linear materials} \end{cases}$$

When $\epsilon_{csm}/\epsilon_y < 1$, the cross-section is of sufficiently slender proportions to invoke failure by local buckling before the yield load is reached; these cross-sections are referred to as class 4 in Eurocode 3. Cross-sections that fail at strain ratios greater than unity, can exceed beyond their yield loads and mobilise some of their strain hardening potential.

Around a cross-section, there can be variations in the material stress–strain response. This can be because of variations in residual stresses from non-uniform cooling, material non-homogeneity, and due to the manufacturing processes. For cold-formed members, the corner material can exhibit higher strength than the flat material. To account for this in the assessment of strain ratios ($\epsilon_{csm}/\epsilon_y$), the yield strain $\epsilon_y$ is determined in Eqn (3.11) on the basis of an area weighted average of the yield stresses for the flat and corner material $f_{y,f}$ and $f_{y,c}$. This calculation uses $\phi = A_f/A$ as the proportion of the area of the flat regions of the cross-section $A_f$ minus the wall thickness $T$ extending from the corners since the corner strength enhancements extend beyond the curved portions, (see Karren (1967) and Cruise and Gardner (2008)) to the gross cross-section area $A$. 

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\[
\epsilon_y = \frac{f_{u,f} \phi + f_{y,c} (1 - \phi)}{E}.
\] (3.11)

For box sections with a constant wall thickness \( T \), internal root radius \( r \) and flat lengths \( b \) and \( d \), \( \phi \) is calculated as

\[
\phi = \frac{2b + 2d - 8T}{2b + 2d + 4\frac{\pi}{2} (r + \frac{T}{2})} = \frac{b + d - 4T}{b + d + \pi (r + \frac{T}{2})}.
\] (3.12)

For cold-formed cross-sections where corner material properties were not measured, their properties were estimated using the predictive model of Gardner et al. (2010), by using the recorded flat material properties in Eqn (3.13), where \( f_{u,f} \) is the ultimate tensile stress of the flat material,

\[
\frac{f_{y,c}}{f_{y,f}} = \frac{B_c}{(r/T)^m}
\] (3.13)

with

\[
B_c = 2.9 \frac{f_{u,f}}{f_{y,f}} - 0.752 \left( \frac{f_{u,f}}{f_{y,f}} \right)^2 - 1.09
\] (3.14)

and

\[
m = 0.23 \frac{f_{u,f}}{f_{y,f}} - 0.041.
\] (3.15)

When reading the peak load and corresponding peak displacement or peak strain from a load versus end-shortening curve, it is important to appreciate that there can be an error caused by user interpretation in the value selected, which will lead to variability in the calculated strain ratio \( \epsilon_{cm}/\epsilon_y \). Consider the two curves in Figure 3.6; in Figure 3.6a, there are no obstacles in identifying the peak load at point \( A \); however, in Figure 3.6b, the local peak near the elastic limit at point \( A \) is of similar magnitude to that of point \( B \). Using \( A \) as the representative peak load is overly conservative as a significant amount of extra
deformation capacity is still available, and so point B should be used if the axial loads are similar. An additional problem arises in Figure 3.6b from points such as B that are on a very flat part of the curve, as this leads to a wide range of end-shortening values for approximately the same load. When this occurs, the final point prior to a clear drop-off in capacity should be used.

![Diagram](image)

(a) Clear peak load
(b) Less distinct peak load

Figure 3.6: Peak load ambiguity for stub column tests.

For slender cross-sections, elastic local buckling is followed by stable post-buckling, which results in increased capacities but with reduced axial stiffness. The consequence of this is that a slender cross-section can have a high deformation capacity (i.e. strain at failure), greater than $\epsilon_y$, but a peak load still below the yield load. This would result in an over-prediction of capacity when using the CSM. To avoid this, the deformation capacity of slender cross-sections is defined in Eqn (3.16) by

$$\frac{\epsilon_{csm}}{\epsilon_y} = \frac{N_u}{N_y} \quad \text{for} \quad \frac{N_u}{N_y} < 1.$$  (3.16)
3.5 Curvature ratio

For cross-sections in bending, the strain distribution is assumed to be linearly-varying through the cross-section depth, and a relationship can be made between curvature and outer-fibre strains. With the height and width of a cross-section denoted \( D \) and \( B \), \( W_{el.y} = 2I_y/D \) and \( W_{el.z} = 2I_z/B \), where \( I_y \) and \( I_z \) are the major and minor axis second moments of area, and the yield curvatures \( \kappa_{y,y} \) and \( \kappa_{y,z} \) for the major and minor axes are defined by Eqn (3.17). These are the curvatures at which a cross-section will reach its major and minor axis elastic moments \( M_{el,y} \) and \( M_{el,z} \), given as

\[
\kappa_{y,y} = \frac{M_{el,y}}{EI_y} = \frac{W_{el.y}E\varepsilon_y}{EI_y} = \frac{2\varepsilon_y}{D} \quad \text{and similarly} \quad \kappa_{y,z} = \frac{2\varepsilon_y}{B}. \tag{3.17}
\]

Assuming plane sections remain plane during bending, there is a proportional relationship between the strains at \( \pm D/2 \) and \( \pm B/2 \) and curvature. This gives the equivalence of the strain ratio \( \varepsilon_{csm}/\varepsilon_y \) to the curvature ratio \( \kappa_{csm}/\kappa_y \), where \( \kappa_{csm} \) is the failure curvature. This occurs when the strain \( \varepsilon_{csm} \) is reached in the compressive outer-fibre of the cross-section, and \( \kappa_{lb} \) is the curvature at maximum moment \( M_{lb} \) (or \( M_u \)). Hence \( \kappa_{csm} \) is related to \( \kappa_{lb} \) in a similar manner to the relationship between \( \varepsilon_{csm} \) and \( \varepsilon_{lb} \), as defined by Eqn (3.18) for the case of the major axis as

\[
\left\{
\begin{array}{c}
\frac{\kappa_{lb}}{\kappa_y} \\
\frac{\kappa_{lb} - 0.002D/2}{\kappa_y}
\end{array}
\right\}
\quad \text{for hot-rolled materials exhibiting a sharply defined yield point}
\]

\[
\kappa_{csm} \kappa_y \kappa_y
\quad \text{for cold-formed and non-linear materials.}
\tag{3.18}
\]
A similar modification to $\kappa_{csm}$ to that used in axial compression is used to distinguish between hot-rolled and cold-formed sections, by subtracting the offset strain as a curvature of $0.002/0.5D$. This allows experimental bending data to be plotted on a common deformation capacity–slenderness curve with axial test data; this is shown in Section 3.7. For slender cross-sections in bending, the definition of curvature ratio, given by Eqn (3.18) has been modified for similar reasons to those previously explained for compression with

$$\frac{\kappa_{csm}}{\kappa_y} = \frac{M_u}{M_{el}} \quad \text{for} \quad \frac{M_u}{M_{el}} < 1.$$ (3.19)

### 3.6 Stub column test data

The stub column test data summarised in Table 2.1 was gathered from previously published literature that gave information on structural steel specimens that were tested beyond their peak loads, had generally reached axial loads and strains above the yield values $N_y$ and $\epsilon_y$, and that had global slenderness values $\bar{\lambda}$ below 0.1. The collected data were used for establishing a link between the cross-section slenderness and the strain ratio as discussed previously, and also later in Section 4.1 with respect to axial strength. The following briefly summarises the data found and, where necessary, states the assumed values for the geometry or material properties that were not recorded in the original publications, but are necessary for the determination of the cross-section slenderness and strain ratio. Where the Young’s modulus was not reported, a value of $E = 210000 \text{N/mm}^2$ was assumed for structural steel. Local buckling was generally observed throughout the testing of all of the stub columns. For rectangular and square hollow sections, an internal root radius assumption of $r = T$ was used for cross-section slenderness calculations when $r$ was not recorded. When corner coupon tests were not performed, the corner enhancement predictive equation from Gardner et al. (2010) was used.
3.6.1 Closed cross-sections

Experimental data on closed cross-sections were obtained from the following studies: Akiyama et al. (1996) tested square hollow sections manufactured by roll-forming, welding and press-forming, and it was found that the roll-formed cross-sections, which had the greater extent of cold working, attained the highest yield normalised axial loads. Gardner et al. (2010) performed SHS and RHS tests, with a split between hot-rolled and cold-formed manufacturing processes, where an identifiable yield stress was observed for the hot-rolled sections, and a more rounded response from the cold-formed sections. Zhao and Hancock (1991) tested cold-formed SHS and RHS sections, for which the material showed gradual yielding and with the yield stress in the corner region about 30% higher than the flats and with significantly less elongation to failure. Hu et al. (2011) tested thick-walled cold-formed rectangular and square hollow sections, while Gao et al. (2009) investigated high strength, thin-walled box section stub columns, fabricated from two cold-formed channels welded together. Data were also gathered from concrete filled stub column test series, in which control tests were performed on the unfilled steel sections. Sakino et al. (2004) tested both circular hollow and square hollow sections, where the circular steel tubes were cold-formed from flat plate and seam welded, and the square tubes were fabricated by welding together two cold-formed channel sections. Similarly, Elchalakani et al. (2002) performed stub column tests on cold-formed square and circular hollow sections. O’Shea and Bridge (1997) tested circular hollow section stub columns, with the tubes created from rolled sheet with welded longitudinal seams. Liew and Xiong (2010) conducted an experimental investigation into axially loaded, ultra-high strength concrete filled, single- and double-skinned tubular columns, with two of the samples as bare steel S355 circular hollow sections. Giakoumelis and Lam (2004) also tested concrete filled CHS, and included two bare steel tests. Chan and Gardner (2008a) conducted an experimental programme on elliptical hollow sections that had a cross-section aspect ratio of 2. It was observed for moderately
stocky cross-sections that the cross-sections reached the yield load with a plastic plateau, and then failed via inelastic local buckling, while stockier cross-sections showed considerably more strain hardening potential before local buckling failure occurred. Tutuncu and O’Rourke (2006) tested circular hollow cross-sections with imperfections by indenting the test specimens. Zhao (2000) tested the compressive capacities of high strength circular hollow sections.

3.6.2 Open cross-sections

Rasmussen and Hancock (1992) performed stub column tests on square hollow sections, cruciform sections and I-sections, with three cross-section slenderness values for each shape. The cross-sections were fabricated from 5 mm and 6 mm thick plates with 0.2% proof stresses of 750 N/mm² and 740 N/mm² respectively. Greiner et al. (2008) investigated the local and member buckling resistance of hot-rolled and welded I-sections, as well as rectangular hollow sections with welded seams.

3.7 Base curve

The CSM base curve relates normalised cross-section failure strain $\epsilon_{csm}/\epsilon_y$ to cross-section slenderness $\bar{\lambda}_p$. Now that the strain ratio and the cross-section slenderness have been defined in Section 3.2, Section 3.4 and Section 3.5, it is possible to seek a relationship between the two from the test data that was gathered in Section 3.6. Figure 3.7a shows a graph of strain ratio versus cross-section slenderness, including all tested cross-section shapes. The test data shows a clear trend of increasing deformation capacity with reducing cross-section slenderness (i.e. lower $\bar{\lambda}_p$), with the strain at peak load sometimes exceeding $25\epsilon_y$. For the more slender cross-sections, the strain ratio drops below the elastic value of $\epsilon_{csm}/\epsilon_y = 1$. Box sections and I-sections exhibit the highest deformation capacity, followed by ellipti-
cal and circular hollow sections, based on the following cross-section slenderness: the Seif and Schafer (2010) element interaction equations for box and I-sections and Eqn (2.2) for circular and elliptical hollow sections. This would agree with the elastic post-bifurcation behaviour of these cross-section shapes as described by Allen and Bulson (1980). The data may be divided by cross-section shape and whether the test specimen was axially loaded or loaded in bending. This has been performed in Figure 3.7b, Figure 3.8a and Figure 3.8b for box and I-sections, circular hollow sections and elliptical hollow sections, respectively. A non-linear least squares fit to the collected data set in Figure 3.7b, excluding cross-sections where $\bar{\lambda}_p > 0.68$ and $\epsilon_{csm}/\epsilon_y > 15$, is given by Eqn (3.20). A similar relationship was observed from the numerical results of Torabian and Schafer (2014). The base curve is

$$\frac{\epsilon_{csm}}{\epsilon_y} = \frac{0.25}{\bar{\lambda}_p^{3.6}} \quad \text{with} \quad \frac{\epsilon_{csm}}{\epsilon_y} \leq 15 \quad \text{and} \quad \bar{\lambda}_p \leq 0.68. \quad (3.20)$$

![Figure 3.7](image)

**Figure 3.7:** Relationship between strain ratio and cross-section slenderness, with CSM base curve for box and I-sections.
The value of $\bar{\lambda}_p = 0.68$ has been found by Afshan and Gardner (2013) to represent the transition point for which plated cross-sections behave as either slender (achieving peak loads below the yield values $N_y$ and $M_{el}$), or as non-slender (achieving peak loads above the yield values). An upper bound of 15 is applied to the strain ratio to avoid excessive strains and to remain within the fracture ductility limits set out in EN 1993-1-1 (2005). Eqn (3.20) is referred to as the CSM base curve for box and I-sections, and may be used to predict the CSM failure strain $\varepsilon_{csm}$ from the cross-section slenderness $\bar{\lambda}_p$. For circular hollow sections, the CSM base curve shown in Figure 3.8a is given by Eqn (3.21), and passes through $\bar{\lambda}_p = 0.304$ at $\varepsilon_{csm}/\varepsilon_y = 1$ to give

$$\frac{\varepsilon_{csm}}{\varepsilon_y} = \frac{0.386}{1000\bar{\lambda}_p^{6.6}}$$

with $\frac{\varepsilon_{csm}}{\varepsilon_y} \leq 15$ and $\bar{\lambda}_p \leq 0.304$. (3.21)

![Figure 3.8](image1)

(a) Circular hollow sections

![Figure 3.8](image2)

(b) Elliptical hollow sections

Figure 3.8: Relationships between strain ratio and cross-section slenderness, with CSM base curves for circular and elliptical hollow sections.
For elliptical hollow sections the unity strain ratio for the curve is now at a cross-section slenderness of $\bar{\lambda}_p = 0.4$, and the base curve shown in Figure 3.8b is given in Eqn (3.22) by

$$\frac{\epsilon_{csm}}{\epsilon_y} = \frac{0.001}{\bar{\lambda}_p^{0.5}} \text{ with } \frac{\epsilon_{csm}}{\epsilon_y} \leq 15 \text{ and } \bar{\lambda}_p \leq 0.4. \quad (3.22)$$

For design, the cross-section slenderness for box sections and I-sections can be calculated using either the buckling coefficients and equations that consider element interaction, or conservatively the standard $k_\sigma$ values from EN 1993-1-5 (2006) (see Section 3.2). Eqn (2.2) is used for the cross-section slenderness calculation for circular and elliptical hollow sections, and using the effective diameter $D_e$ for the latter.

As the CSM base curves are constructed from test data for a specific range of non-slender cross-sections ($\bar{\lambda}_p \leq 0.68$ for plated cross-sections, $\bar{\lambda}_p \leq 0.304$ for circular hollow sections and $\bar{\lambda}_p \leq 0.4$ for elliptical hollow sections), the CSM is not valid for larger cross-section slenderness values (slender class 4 cross-sections). The CSM is applicable to cross-sections with a strain ratio greater than unity, limited to a maximum value of $\epsilon_{csm}/\epsilon_y = 15$. The base curve prediction of the cross-section strain ratio, based on both axial and bending test data, with $\bar{\lambda}_p$ calculated from all loading conditions, allows for the determination of the cross-section strain distribution for the calculation of the cross-section resistance; this process is described in the following chapters.

The CSM base curves are fitted through the available test data and show low scatter either side of these functions. Cross-section strength, as defined in the following chapters, is not sensitive to large differences in deformation capacity (strain ratio), as inelastic deformations are associated with a shallow strain hardening slope from the material model. Based on the material model from a strain ratio of 1 to a strain ratio of 15 (1500% increase), the variation in stresses is merely 14%.
3.8 Summary

The introduced design material model rationalised the stress–strain relationships for hot-rolled and cold-formed structural steel onto one common bi-linear model, which allowed for the effects of strain hardening, and terminated at a strain of 15 times the yield strain. The need for an appropriate and code compliant strain hardening modulus led to the chosen value of $E_{sh}/E = 0.01$.

It was found that the more accurate element interaction method would tend to produce lower cross-section slenderness values, by up to 30% for cross-sections of common proportions. A strain ratio was defined based on the strain achieved at the peak axial load in stub column tests, normalised by the yield strain. Higher values of the strain ratio indicated a greater deformation capacity and hence increased resistance to local buckling. The strain ratio can be used for general loading as the plotted data is based on axial and bending test data, with $\bar{\lambda}_p$ calculated from all loading conditions.
Chapter 4

Axial and bending resistances

The capacity of a cross-section to withstand axial loads and bending moments in isolation is examined in this chapter. For axial loading, the strain distribution is assumed to be uniform, while for bending it is taken as linearly varying; in both cases, the maximum strain that a cross-section can endure is limited to $\varepsilon_{\text{cs}}$. Based on the described strain distributions and strain limit, analytical and design expressions are derived for the CSM resistance of a cross-section to an axial load $N_{\text{cs}}$ or bending moment $M_{\text{cs}}$.

4.1 Axial load

For a column that is unaffected by global flexural buckling (such as a stub column) and is resisting an axial load only, the strains throughout the cross-section are assumed to be uniform at $\varepsilon_A$, as in Figure 4.1. When the uniform strain is less than the material yield strain, $\varepsilon_A < \varepsilon_y$, the cross-section is fully within its elastic material limit. However, when $\varepsilon_A \geq \varepsilon_y$ the cross-section is deforming inelastically and, following the CSM strain hardening material model described in Section 3.3, will reach the CSM limiting stress $f_{\text{cs}}$. Therefore, for a strain ratio that is greater than unity ($\varepsilon_{\text{cs}}/\varepsilon_y > 1$), the CSM axial resistance $N_{\text{cs}}$ will be greater than the yield load $N_y$. 

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4.1.1 Design expression

From the uniform strain model in Figure 4.1, the axial load $N_{csm}$ that a cross-section can resist is given by Eqn (4.1). Since this resistance is based on the strain ratio and slenderness of the cross-section under compressive loading, the tensile resistance can be conservatively taken as the same value by ignoring the immunity to local buckling in tension; in reality the tensile capacity will be higher. The CSM axial load is

$$N_{csm} = Af_{csm} \quad \text{or} \quad \frac{N_{csm}}{N_y} = \frac{f_{csm}}{f_y}. \quad (4.1)$$

In Eqn (4.1) the yield load $N_y = Af_y$ is the product of the cross-section area and the material yield stress, and $f_{csm}/f_y$ is obtained from the design bi-linear material model given in Eqn (4.2) by

$$\frac{f_{csm}}{f_y} = 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right). \quad (4.2)$$

The ratio of the strain hardening modulus to the Young’s modulus is $E_{sh}/E$, and the strain ratio $\epsilon_{csm}/\epsilon_y$ is determined from the appropriate base curve via the cross-section slenderness.
4.1.2 Test data

The peak loads $N_u$ obtained from all of the gathered tests on structural steel stub columns from Section 3.6 are normalised in Figure 4.2a by their corresponding yield loads and plotted against the cross-section slenderness. This plot shows that limiting cross-section axial strength to the yield load is very conservative for many cross-sections, and that additional capacity is attained, due principally to strain hardening. For a given cross-section slenderness $\bar{\lambda}_p$, I-sections and box sections give the highest normalised axial loads, followed by elliptical hollow sections and then circular hollow sections.

![Figure 4.2: Stub column test ultimate axial loads normalised by the yield load and plotted against the cross-section slenderness.](image)

The ultimate test loads $N_u$ are normalised by the yield load for I and box, CHS and EHS shapes in Figure 4.2b, Figure 4.3a and Figure 4.3b. Plotted also is the CSM predictive equation Eqn (4.1) with $E_{sh}/E = 1/100$, and with the maximum strain ratio of 15. For box sections and I-sections, the CSM equation predicts the yield load at a cross-section
slenderness of $\bar{\lambda}_p = 0.68$, and then $N_{csm}$ increases alongside the data with decreasing cross-section slenderness, giving generally conservative predictions. The value of $\bar{\lambda}_p = 0.68$ was found by Afshan and Gardner (2013) to represent the transition point for which cross-sections behave as either slender (achieving peak loads below the yield values $N_y$ and $M_{el}$), or as non-slender (achieving peak loads above the yield values).

The same behaviour follows with circular and elliptical hollow sections: as the cross-section slenderness decreases, the normalised axial resistance increases. The cross-section slenderness values for which the circular and elliptical hollow sections reach the yield loads are $\bar{\lambda}_p = 0.304$ and $\bar{\lambda}_p = 0.40$ respectively. These are the $\bar{\lambda}_p$ values for which the data appears to pass through $N_u/N_y = 1$. For circular hollow sections the equivalent EN 1993-1-1 (2005) cross-section slenderness value is based on $D/\epsilon^2 = 90$; the value of $\bar{\lambda}_p = 0.304$ equates to $D/\epsilon^2 = 100$.

![Graph](image.png)

Figure 4.3: Stub column test ultimate axial loads normalised by the yield load (CHS and EHS) and plotted against the cross-section slenderness.
Figure 4.4 to Figure 4.6 includes cross-sections, the slenderness of which, lie below the yield slenderness ($\bar{\lambda}_p = 0.68$, $\bar{\lambda}_p = 0.304$ and $\bar{\lambda}_p = 0.40$ respectively), and show the ratio of ultimate test load to CSM resistance $N_u/N_{csm}$, and the ratio of ultimate test load to yield load $N_u/N_y$. These figures are for the I and box, circular hollow and elliptical hollow sections and allow for comparisons between the CSM and EN 1993-1-1 (2005) design models. Statistical comparisons for the three cross-section shape groups are made in Table 4.1, which gives the mean, standard deviation and coefficient of variation (COV) for the CSM and yield load predictions. Elliptical hollow sections are not considered within EN 1993-1-1 (2005) but the yield load is still relevant for comparison. The CSM gives improvements to the mean and COV of the I-section and box section data when compared to EN 1993-1-1 (2005), as the mean reduces from 1.1532 to 1.0877 and the COV from 0.1029 to 0.0731. Stocky cross-sections that were significantly under-predicted by the yield load for $\bar{\lambda}_p < 0.4$, have been estimated with greater accuracy by the CSM design equation.

![Graphs](image1.png)

Figure 4.4: Test loads $N_u$ compared to $N_{csm}$ and $N_y$ resistances (box sections and I-sections).
Figure 4.5: Test loads $N_u$ compared to $N_{csm}$ and $N_y$ resistances (circular hollow sections).

Figure 4.6: Test loads $N_u$ compared to $N_{csm}$ and $N_y$ resistances (elliptical hollow sections).
Similar improvements are found for the circular hollow sections with a reduction in the mean from 1.0825 to 1.0209 and in the COV from 0.0965 to 0.0711. The statistics would seem to indicate better CSM predictions for elliptical hollow sections, but a closer observation of Figure 4.6a reveals that a number of stub columns that were previously estimated conservatively with the yield load, are predicted unsafely by the CSM; further research is needed here. The statistics provide a validation of the CSM model, and are based on 79 tests for box and I-sections, 20 tests for CHS and 22 tests for EHS.

Table 4.1: Statistics for the ultimate axial load to CSM and yield load resistances.

<table>
<thead>
<tr>
<th>Shape</th>
<th>I/box</th>
<th>CHS</th>
<th>EHS</th>
<th>I/box</th>
<th>CHS</th>
<th>EHS</th>
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</thead>
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<tr>
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<td>1.0209</td>
<td>1.0060</td>
<td>1.1532</td>
<td>1.0825</td>
<td>1.0868</td>
</tr>
<tr>
<td>COV</td>
<td>0.0731</td>
<td>0.0711</td>
<td>0.0855</td>
<td>0.1029</td>
<td>0.0965</td>
<td>0.0881</td>
</tr>
</tbody>
</table>

4.1.3 Summary

The CSM limiting strain from the base curve was paired with the design bi-linear material model, to allow stresses greater than the yield stress and axial resistances greater than the yield load. Experimental data showed that non-slender cross-sections can reach peak loads greater than the yield load by as much as 50%. The CSM predictive equation allowed greater axial capacities for decreasing cross-section slenderness, up until a maximum value for a strain ratio of 15. The yield slenderness limits were cross-section slenderness values which coincided with strain ratios of unity at the yield loads. By taking the mean and coefficient of variation of the ultimate test loads to predicted resistances, it was observed for I-sections, box sections and circular hollow sections that the CSM gave better predictions for the axial capacities. This was based on the lower mean test to prediction ratios and the reduction in scatter.
4.2 Uni-axial bending

At the ultimate limit state, a stocky cross-section subject to flexure is traditionally designed with the strain and stress distributions displayed in Figure 4.7a. For a material with no strain hardening potential, such that the maximum material stress is equal to the yield stress, this infinite strain model predicts a maximum moment called the plastic moment resistance $M_{pl} = W_{pl}f_y$, which is the product of the plastic section modulus $W_{pl}$ and the yield stress $f_y$.

![Traditional and proposed strain and stress distributions in bending.](image_url)

Figure 4.7: Traditional and proposed strain and stress distributions in bending.

Although simple to interpret and analyse (particularly as $W_{pl}$ is well tabulated), the traditional plastic model is an idealisation that does not always represent well actual behaviour. This is because any strain hardening capacity of the material is not used, and there is a stress discontinuity that causes a jump from $-f_y$ to $f_y$ at the neutral axis. This model implies infinite curvature, which cannot occur in reality as the material would fracture or the cross-section would locally buckle before high deformations could be reached. Shown in Figure 4.7b is the linear strain, bi-linear stress model (bi-linear either side of the neutral axis) used in the Continuous Strength Method (Section 3.3.2). The shortcomings of the simplified plastic moment model are recognised and accounted for, as strain hardening is
incorporated by the use of the bi-linear material model, and local buckling is included by limiting the strains in the cross-section to the limiting strain $\epsilon_{csm}$. For the given material model, the CSM allows more realistic stress and strain distributions throughout the cross-section, leading to a more suitable moment capacity.

4.2.1 Numerical model

A numerical model was developed to find accurate bending capacities of cross-sections. Standard structural sections from SCI P363 (2009) were chosen to represent cross-sections that are used in design. The investigated cross-sections were 80 universal beams (UB), 31 universal columns (UC), 79 rectangular hollow sections (RHS), 84 square hollow sections (SHS), 62 circular hollow sections (CHS) and 26 elliptical hollow sections (EHS). The numerical methods were computed in MATLAB (2012) for each cross-section, for an elastic, linear hardening material model and a range of outer-fibre strain limits.

4.2.1.1 I and box sections

Figure 4.8a shows half of an I-section in flexure, in which the outer-fibre strain at $y = D/2$ is limited to $\epsilon_{csm}$ and at a stress of $f_{csm}$. For box sections and I-sections, the moment capacity $M_{csm}$ can be calculated via the integration of the bi-linear stress distribution, by discretising the cross-section into $n$ number of thin strips. For major axis bending

$$
\frac{M_{csm,y}}{M_{pl,y}} = \int_A \frac{f y}{f_y W_{pl,y}} dA = \frac{t_i}{W_{pl,y}} \sum_i \frac{f_i}{f_y} y_i B_i \quad \text{with} \quad t_i = \frac{D}{n} \quad (4.3)
$$

where $W_{pl,y}$ and $M_{pl,y}$ are the major axis plastic section modulus and plastic moment, $y_i$ is the distance of the element centroid from the neutral axis of the cross-section, $f_i$ is the stress at $y_i$ and $f_y$ is the yield stress.
The elemental area $A_i$ is the product of the strip width $B_i$ and strip thickness $t_i$, the latter calculated by dividing the depth $D$ by $n$. As the stress profile is antisymmetric about the neutral axis, it allows the capacity of half of the cross-section to be doubled for the total capacity. For minor axis bending

$$\frac{M_{csm,z}}{M_{pl,z}} = \int_A \frac{f_z}{f_y W_{pl,z}} dA = \frac{t_i}{W_{pl,z}} \sum_i \frac{f_i z_i B_i}{f_y} \quad \text{with} \quad t_i = \frac{B}{n}.$$  \hfill (4.4)

This procedure is equally applicable to minor axis bending by using the appropriate plastic section modulus, distance $z_i$ from the neutral axis in the orthogonal direction, and changing dimensions for the minor axis. For the major axis (minor axis similar) where the strain at position $y_i$ is $\epsilon_i = 2y_i \epsilon_{csm}/D$, the stress $f_i$ can be determined as

$$\frac{f_i}{f_y} = \frac{E \epsilon_i}{f_y} = \frac{E y_i \epsilon_{csm}}{f_y D} = \frac{\epsilon_{csm}}{\epsilon_y} \frac{2y_i}{D} \quad \text{for} \quad \epsilon_i \leq \epsilon_y.$$  \hfill (4.5)
\[
\frac{f_i}{f_y} = 1 + E_{sh} \left( \frac{\epsilon_i - \epsilon_y}{f_y} \right) = 1 + E_{sh} \left( \frac{2y_i \epsilon_{csn}}{D f_y} - \frac{1}{E} \right)
\]

\[
= 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csn} y_i}{\epsilon_y D} - 1 \right) \quad \text{for} \quad \epsilon_i > \epsilon_y. \quad (4.6)
\]

### 4.2.1.2 Circular and elliptical hollow sections

For I-sections and box sections the definition of the elemental area \( A_i \) is straightforward, as the cross-section can be conveniently discretised into thin rectangles. For the elliptical hollow section in Figure 4.9a, which has inner and outer dimensions \( a_1, a_2 \) and \( b_1, b_2 \) in the \( z \) and \( y \) directions respectively, an elemental area is not of constant shape as the inner and outer radii \( r_1 \) and \( r_2 \) are functions of the angle \( \theta \). The area of such an element \( A_i \), shown in Figure 4.9b contained by the rays at \( \theta - \theta_i / 2 \) and \( \theta + \theta_i / 2 \) either side of \( r \) (which is at an angle \( \theta \)), and by the inner and outer radii \( r_a \) and \( r_b \), is determined.

![Figure 4.9: The geometry of an elliptical hollow section and an elemental area.](image)
The equation of an ellipse in polar co-ordinate form is that of Eqn (4.7),

\[ r = \frac{ab}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}. \]  

(4.7)

The general area of a sector bounded by rays at angles \( \theta_1 \) and \( \theta_2 \) with curve \( r \) is

\[ \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta = \frac{1}{2} \int_{\theta_1}^{\theta_2} \frac{a^2 b^2}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} \, d\theta = \frac{1}{2} \left[ ab \arctan \left( \frac{a}{b} \tan \theta \right) \right]_{\theta_1}^{\theta_2}. \]  

(4.8)

This integral can be checked by

\[ \frac{d}{d\theta} \left( ab \arctan \left( \frac{a}{b} \tan \theta \right) \right) = \left( \frac{ab}{1 + \frac{a^2}{b^2} \tan^2 \theta} \right) \frac{a}{b} \sec^2 \theta \]

\[ = \frac{ab^3}{b^2 + a^2 \sin^2 \theta} \frac{a}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} = \frac{a^2 b^2}{b^2 \cos^2 \theta + a^2 \sin^2 \theta}. \]  

(4.9)

Introducing the function \( F \) simplifies the subsequent formulae

\[ F = f(a, b, \theta) = \frac{ab}{2} \arctan \left( \frac{a}{b} \tan \theta \right). \]  

(4.10)

By defining the elemental angle \( \theta_i = \frac{2\pi}{m} \), then \( \theta_1 = \theta - \theta_i/2 \) and \( \theta_2 = \theta + \theta_i/2 \). For the elemental length \( r_i \) in the radial direction, it is important to note that \( r_i = r_b - r_a \) is not constant across \( \theta_i \) as \( r = f(\theta, a, b) \). With the thickness of the cross-section \( t = a_2 - a_1 = b_2 - b_1 \) split into \( n \) strips of length \( r' = t/n \), the area \( A_i \) may be found by subtracting the area of the sector for \( r_a = r - r_i/2 \) from the area formed by \( r_b = r + r_i/2 \) to give
\[ A_i = F \left( a + \frac{r'}{2}, b + \frac{r'}{2}, \theta + \frac{\theta_i}{2} \right) - F \left( a + \frac{r'}{2}, b + \frac{r'}{2}, \theta - \frac{\theta_i}{2} \right) - F \left( a - \frac{r'}{2}, b - \frac{r'}{2}, \theta + \frac{\theta_i}{2} \right) + F \left( a - \frac{r'}{2}, b - \frac{r'}{2}, \theta - \frac{\theta_i}{2} \right). \]  

(4.11)

The general elliptical hollow section result for \( A_i \) may be preserved in this form. However for the case of a circular hollow section with \( a = b = r \) it may be simplified further. Firstly the function \( F \) is simplified to give \( F_c \) as

\[ F_c = \frac{r^2}{2} \arctan(\tan \theta) = \frac{r^2}{2} \theta. \]  

(4.12)

For a circular hollow section, expanding and simplifying the general result for \( A_i \) gives

\[ A_i = \frac{1}{2} \left[ \left( \frac{r + r'}{2} \right)^2 \left( \theta + \frac{\theta_i}{2} \right) - \left( \frac{r + r'}{2} \right)^2 \left( \theta - \frac{\theta_i}{2} \right) - \left( \frac{r - r'}{2} \right)^2 \left( \theta + \frac{\theta_i}{2} \right) + \left( \frac{r - r'}{2} \right)^2 \left( \theta - \frac{\theta_i}{2} \right) \right] \]

\[ = \frac{1}{2} \left[ \left( r^2 + rr' + \frac{r'^2}{4} \right) \left( \theta + \frac{\theta_i}{2} \right) - \left( r^2 + rr' + \frac{r'^2}{4} \right) \left( \theta - \frac{\theta_i}{2} \right) - \left( r^2 - rr' + \frac{r'^2}{4} \right) \left( \theta + \frac{\theta_i}{2} \right) \right] \]

\[ + \frac{1}{2} \left( r^2 - rr' + \frac{r'^2}{4} \right) \left( \theta - \frac{\theta_i}{2} \right) \right] = \frac{1}{2} \left[ 2rr' \left( \theta + \frac{\theta_i}{2} \right) - 2rr' \left( \theta - \frac{\theta_i}{2} \right) \right] = rr' \theta_i. \]  

(4.13)

Having determined \( A_i \), the next step is to integrate the moments produced by the elements for one quarter of the cross-section and then to quadruple the result. It is convenient to continue to use polar co-ordinates and evaluate the integral from the inner to outer radii and through a 90 degree angle. For a circular hollow section with \( dA = r \, d\theta \, dr \) and \( y = r \sin \theta \), the moment integral can be converted to a numerical approximation using \( i \)
as the index of the elements for one quadrant of the cross-section, and \( r_1, r_2 \) and \( r \) as the inner, outer and element radii respectively,

\[
\frac{M_{csm}}{M_{pl}} = \int_A \frac{f_y}{f_y W_{pl}} \, dA = \frac{4}{W_{pl}} \int_{r_1}^{r_2} \frac{f}{f_y} r^2 \sin \theta \, d\theta \, dr = \frac{4}{W_{pl}} \sum_i \frac{f_i}{f_y} r_i^2 \sin \theta_i r_i'.
\] (4.14)

For an elliptical hollow section \( r \) is no longer constant (as in the circular hollow section case), and so the full \( A_i \) expression must be used. For the major axis moment

\[
\frac{M_{csm,y}}{M_{pl,y}} = \int_A \frac{f_y}{f_y W_{pl,y}} \, dA = \frac{4}{W_{pl,y}} \sum_i \frac{f_i}{f_y} r_i \sin \theta A_i = \frac{4}{W_{pl,y}} \sum_i \frac{f_i}{f_y} \frac{ab \sin \theta}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} A_i.
\] (4.15)

For the minor axis moment

\[
\frac{M_{csm,z}}{M_{pl,z}} = \int_A \frac{f_z}{f_y W_{pl,z}} \, dA = \frac{4}{W_{pl,z}} \sum_i \frac{f_i}{f_y} r_i \cos \theta A_i = \frac{4}{W_{pl,z}} \sum_i \frac{f_i}{f_y} \frac{ab \cos \theta}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} A_i.
\] (4.16)

For bending about the major and minor axes of an elliptical hollow section, the strain at element \( i \) becomes \( \varepsilon_i = r \sin \theta \varepsilon_{csm}/b_2 \) and \( \varepsilon_i = r \cos \theta \varepsilon_{csm}/a_2 \), and for a circular hollow section the elemental strain is \( \varepsilon_i = 2r \sin \theta \varepsilon_{csm}/D \). The elemental stress \( f_i/f_y \) is then calculated as follows.

For elliptical hollow sections bending about the major axis

\[
\frac{f_i}{f_y} = \begin{cases} 
\varepsilon_{csm} \frac{r \sin \theta}{\varepsilon_y} / b_2 & \varepsilon_i \leq \varepsilon_y \\
1 + \frac{E_{sh}}{E} \left( \frac{\varepsilon_{csm} r \sin \theta}{\varepsilon_y b_2} - 1 \right) & \varepsilon_i > \varepsilon_y.
\end{cases}
\] (4.17)
For elliptical hollow sections bending about the minor axis

\[
\frac{f_i}{f_y} = \begin{cases} 
\frac{\epsilon_{csm} r \cos \theta}{\epsilon_y a_2} & \epsilon_i \leq \epsilon_y \\
1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm} r \cos \theta}{\epsilon_y a_2} - 1 \right) & \epsilon_i > \epsilon_y.
\end{cases}
\]  

(4.18)

For circular hollow sections in bending

\[
\frac{f_i}{f_y} = \begin{cases} 
\frac{\epsilon_{csm} 2r \sin \theta}{\epsilon_y D} & \epsilon_i \leq \epsilon_y \\
1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm} 2r \sin \theta}{\epsilon_y D} - 1 \right) & \epsilon_i > \epsilon_y.
\end{cases}
\]  

(4.19)

To demonstrate how this numerical model can be used, a comparison between the Wang (2011) explicit model and the developed numerical model is given in Figure 4.10a, which shows the moments predicted for a UC with increasing strain ratio and with $E_{sh} = E/100$. Figure 4.10b shows the discrepancies for all standard I-sections and box sections in major axis bending, which is found by dividing the predicted moments $M_{\text{pred}}$ from Wang (2011) by the numerical values $M_{n,y}$. In general, the discrepancies are small after $\epsilon_{csm}/\epsilon_y = 2$ (less than 3%), but the Wang (2011) model can predict inaccurate results in the region of the elastic moment $M_{el,y}$ at $\epsilon_{csm}/\epsilon_y = 1$, due to the constant stress assumption in the flanges.

The simplified Wang (2011) method 2 is shown in Figure 4.11a, and the piecewise nature is evident as the two parts join at $\epsilon_{csm}/\epsilon_y = 3$. Figure 4.11b shows that for $E_{sh} = E/100$, the error is between -6% and +2% when the strain ratio is less than 3, and remains constant at approximately -3% thereafter.
Figure 4.10: Wang (2011) method 1 accuracy when compared to the numerical model.

Figure 4.11: Wang (2011) method 2 accuracy when compared to the numerical model.
4.2.2 Analytical solution

4.2.2.1 Governing equation

Assuming plane sections remain plane and normal to the neutral axis in bending, the corresponding linear strain and bi-linear stress distributions for one half of a symmetric cross-section are shown in Figure 4.12. When the strain ratio $\varepsilon_{csm}/\varepsilon_y \geq 1$, the limiting outer-fibre stress $f_{csm}$ is equal to or greater than the yield stress $f_y$, and the cross-section bending resistance $M_{csm}$ will equal or exceed $M_{el}$.

![Figure 4.12: Strain and stress distributions for a symmetric cross-section.](image)

With reference to Figure 4.12, the moment capacity of a cross-section can be expressed by Eqn (4.20) and Eqn (4.21) for major and minor axis bending respectively, in terms of the elastic $W_{el,y}, W_{el,z}$ and plastic $W_{pl,y}, W_{pl,z}$ section moduli, and moduli $W_{w,y}$ and $W_{w,z}$,

$$M_{csm,y} = W_{pl,y}f_{csm} - (W_{pl,y} - W_{el,y})f_1 - W_{w,y}f_2$$ (4.20)

$$M_{csm,z} = W_{pl,z}f_{csm} - (W_{pl,z} - W_{el,z})f_1 - W_{w,z}f_2.$$ (4.21)
Eqn (4.20) and Eqn (4.21) are the governing CSM bending equations for the major and minor axes respectively. The first yield distances $Y$ and $Z$ from NA for major and minor axis bending are

$$Y = \frac{D}{2\varepsilon_{\text{csm}}/\varepsilon_y} \quad \text{and} \quad Z = \frac{B}{2\varepsilon_{\text{csm}}/\varepsilon_y}. \quad (4.22)$$

Recalling the CSM limiting stress allows the stresses $f_1$ and $f_2$ to be found.

$$\frac{f_{\text{csm}}}{f_y} = 1 + \frac{E_{sh}}{E} \left( \frac{\varepsilon_{\text{csm}}}{\varepsilon_y} - 1 \right) \quad (4.23)$$

Stress $f_1$ is determined from the stress distribution geometry in Figure 4.12,

$$\frac{f_1 - (f_{\text{csm}} - f_y)}{Y} = \frac{f_{\text{csm}} - f_y}{\frac{D}{2} - Y} \quad \text{giving} \quad f_1 - (f_{\text{csm}} - f_y) = \frac{f_{\text{csm}} - f_y}{\varepsilon_{\text{csm}}/\varepsilon_y - 1}. \quad (4.24)$$

Re-arranging and normalising to the yield stress gives

$$\frac{f_1}{f_y} = \left( \frac{f_{\text{csm}}}{f_y} - 1 \right) \left( 1 + \frac{1}{\frac{\varepsilon_{\text{csm}}}{\varepsilon_y} - 1} \right) = \left( \frac{f_{\text{csm}}}{f_y} - 1 \right) \frac{\varepsilon_{\text{csm}}}{\varepsilon_y} - 1$$

$$= \frac{E_{sh}}{E} \left( \frac{\varepsilon_{\text{csm}}}{\varepsilon_y} - 1 \right) \frac{\varepsilon_{\text{csm}}}{\varepsilon_y} - 1 = \frac{E_{sh}}{E} \frac{\varepsilon_{\text{csm}}}{\varepsilon_y}. \quad (4.25)$$

Stress $f_2$ is then given by

$$\frac{f_2}{f_y} = \frac{f_{\text{csm}}}{f_y} - \frac{f_1}{f_y} = 1 + \frac{E_{sh}}{E} \left( \frac{\varepsilon_{\text{csm}}}{\varepsilon_y} - 1 \right) \frac{E_{sh}}{E} \frac{\varepsilon_{\text{csm}}}{\varepsilon_y} = 1 - \frac{E_{sh}}{E}. \quad (4.26)$$

Normalising Eqn (4.20), which is for major axis bending, by the plastic moment capacity $M_{pl,y} = W_{pl,y} f_y$ gives
\[
\frac{M_{csm,y}}{M_{pl,y}} = \frac{f_{csm}}{f_y} - \left(1 - \frac{W_{el,y}}{W_{pl,y}}\right) \frac{f_1}{f_y} - \frac{W_{w,y} f_2}{W_{pl,y} f_y},
\]

(4.27)

and substituting in the expressions for \(f_{csm}, f_1\) and \(f_2\) gives

\[
\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh}}{E} \left(\frac{\epsilon_{csm}}{\epsilon_y} - 1\right) - \left(1 - \frac{W_{el,y}}{W_{pl,y}}\right) \frac{E_{sh} \epsilon_{csm}}{E} \frac{W_{w,y}}{W_{pl,y}} \left(1 - \frac{E_{sh}}{E}\right)
\]

\[
= 1 + \frac{E_{sh}}{E} \left(\frac{\epsilon_{csm}}{\epsilon_y} \frac{W_{el,y}}{W_{pl,y}} - 1\right) - \frac{W_{w,y}}{W_{pl,y}} \left(1 - \frac{E_{sh}}{E}\right).
\]

(4.28)

Eqn (4.28) can also be re-arranged for normalisation by the elastic moment capacity \(M_{el,y}\),

\[
\frac{M_{csm,y}}{M_{el,y}} = \frac{W_{pl,y}}{W_{el,y}} + \frac{E_{sh}}{E} \left(\frac{\epsilon_{csm}}{\epsilon_y} - \frac{W_{pl,y}}{W_{el,y}}\right) - \frac{W_{w,y}}{W_{el,y}} \left(1 - \frac{E_{sh}}{E}\right).
\]

(4.29)

Similar equations are obtained for minor axis bending as

\[
\frac{M_{csm,z}}{M_{pl,z}} = \frac{W_{pl,z}}{W_{el,z}} + \frac{E_{sh}}{E} \left(\frac{\epsilon_{csm}}{\epsilon_y} - \frac{W_{pl,z}}{W_{el,z}}\right) - \frac{W_{w,z}}{W_{el,z}} \left(1 - \frac{E_{sh}}{E}\right)
\]

(4.30)

\[
\frac{M_{csm,z}}{M_{el,z}} = \frac{W_{pl,z}}{W_{el,z}} + \frac{E_{sh}}{E} \left(\frac{\epsilon_{csm}}{\epsilon_y} - \frac{W_{pl,z}}{W_{el,z}}\right) - \frac{W_{w,z}}{W_{el,z}} \left(1 - \frac{E_{sh}}{E}\right).
\]

(4.31)

4.2.2.2 Moduli \(W_{w,y}\) and \(W_{w,z}\) for I-sections and box sections

The term \(W_{w,y} f_2\) in Eqn (4.20) and Eqn (4.21) represents a moment \(M_{f2}\) caused by the triangular shaped stress block associated with stress \(f_2\) for \(|y| \leq Y\) (major axis)

\[
M_{f2} = W_{w} f_2 = \int_{A_Y} f_y \, dA_Y = \int_{A_Y} f_2 g(y) \, dA_Y \quad \text{giving} \quad W_w = \int_{A_y} g(y) \, dA_Y
\]

(4.32)
where the function \( g(y) \) represents the triangular stress distribution normalised by \( f_2 \). The integral is only evaluated to the first yield point \( |y| = Y \), and the associated area to integrate over is \( A_Y \). Since the stress distribution and cross-section geometry are relatively simple, the integration may be performed in a straightforward manner by discretising the cross-section into rectangles and triangles, and then computing the sum analytically. For I-sections and box sections bending about the major axis, \( W_{w,y} \) is primarily associated with the web (Figure 4.13). Note that box sections may be treated as I-sections by setting the web thickness \( t \) equal to twice the wall thickness \( T \).

From Figure 4.13a, \( W_{w,y} \) is calculated as

\[
W_{w,y} = 2t \frac{Y Y}{2 \frac{3}{3}} = \frac{t}{3} \left( \frac{D}{2} \right)^2 = \frac{tD^2}{12} \left( \frac{\epsilon_{cam}}{\epsilon_y} \right)^{-2}.
\]  

(4.33)
Eqn (4.33) is valid while the first yield point lies within the web ($Y \leq h/2$), alternatively defined as

$$\frac{D}{2} \leq \frac{h}{2} \frac{\varepsilon_{csm}}{\varepsilon_y} \quad \text{or} \quad 1 + \frac{2T}{h} \leq \frac{\varepsilon_{csm}}{\varepsilon_y}. \quad (4.34)$$

When $Y$ lies outside of the web, as in Figure 4.13b, the modulus $W_{w,y}$ has contributions from a trapezium with sides $f_2$ and $kf_2$ acting within the web, and triangular parts within the flanges. For the factor $k$ at the web and flange junction

$$\frac{k}{Y - \frac{h}{2}} = \frac{k}{D} \left( \frac{\varepsilon_{csm}}{\varepsilon_y} \right)^{-1} - \frac{h}{2} = \frac{1}{D} \left( \frac{\varepsilon_{csm}}{\varepsilon_y} \right)^{-1} \quad \text{giving} \quad k = 1 - \frac{h}{D} \frac{\varepsilon_{csm}}{\varepsilon_y}. \quad (4.35)$$

The modulus $W_{w,y}$ is then

$$W_{w,y} = \frac{BD^2}{12} \left( \frac{\varepsilon_{csm}}{\varepsilon_y} \right)^{-2} - (B - t)h \left[ k \frac{h}{2} + \frac{1}{2} (1 - k) \frac{h}{3} \right]$$

$$= \frac{BD^2}{12} \left( \frac{\varepsilon_{csm}}{\varepsilon_y} \right)^{-2} - \frac{(B - t)}{12} h^2 \left( 3 - \frac{2h}{D} \varepsilon_{csm} \frac{\varepsilon_y}{\varepsilon_y} \right) \quad (4.36)$$

which applies when the first yield point or strain ratio are bounded as

$$\frac{h}{2} < Y \leq \frac{D}{2} \quad \text{or} \quad 1 + \frac{2T}{h} \geq \frac{\varepsilon_{csm}}{\varepsilon_y} \geq 1. \quad (4.37)$$

For box sections, the general shape is the same for both bending axes, and so the $W_{w,z}$ modulus can take the same form as the $W_{w,y}$ term,

$$W_{w,z} = \frac{tB^2}{12} \left( \frac{\varepsilon_{csm}}{\varepsilon_y} \right)^{-2} \quad \text{for} \quad \frac{1}{1 - \frac{t}{B}} \leq \frac{\varepsilon_{csm}}{\varepsilon_y}. \quad (4.38)$$
When $\frac{1}{1 - \frac{t}{B}} > \frac{\epsilon_{csm}}{\epsilon_y} \geq 1$, the modulus $W_{w,z}$ becomes

$$W_{w,z} = \frac{DB^2}{12} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} - \frac{(D - t)}{12}(B - t)^2 \left[ 3 - 2 \left( 1 - \frac{t}{B} \right) \frac{\epsilon_{csm}}{\epsilon_y} \right].$$  \hspace{1cm} (4.39)

The behaviour of I-sections differs significantly for bending about the major and minor axes. Their minor axis can be treated as behaving like three rectangular plates, two with dimensions $B$ and $T$ representing the flanges and one with dimensions $h$ and $t$ as the web.

![Diagram](a) $Z$ within web  
(b) $Z$ within flanges

Figure 4.14: Modulus $W_{w,z}$ for I-sections.

With reference to Figure 4.14a, for $Z \leq t/2$, which arises when $\epsilon_{csm}/\epsilon_y \geq B/t$, $W_{w,z}$ can be expressed as

$$W_{w,z} = \frac{DB^2}{12} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} \text{ for } \frac{\epsilon_{csm}}{\epsilon_y} \geq \frac{B}{t}. \hspace{1cm} (4.40)$$

Figure 4.14b shows the case where $Z > t/2$ and the first yield point lies outside of the web. In this situation there is a triangular stress block with peak stress $jf_2$ at $z = t/2$ reducing to zero at $z = Z$, as well as a trapezoidal stress block with sides $f_2$ and $jf_2$ acting within the web.
To calculate \( j \),

\[
\frac{1}{Z} = \frac{j}{Z - \frac{t}{2}}, \quad Z - \frac{t}{2} = jZ \quad \text{giving} \quad j = 1 - \frac{t}{2Z}.
\] (4.41)

The calculation of the modulus \( W_{w,z} \) is then

\[
W_{w,z} = 2 \left[ \frac{t}{2} h \frac{j}{4} + \frac{t}{2} h \frac{(1 - j) t}{6} + \frac{TB^2}{12} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} \right] = \frac{t^3 h}{4} \left( 1 - \frac{t}{2Z} \right) + \frac{t^2 h t}{12} 2Z + \frac{TB^2}{6} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2}
\]

\[
= \frac{t^2 h}{4} + \frac{t^3 h}{8Z} \left( \frac{1}{3} - 1 \right) + \frac{TB^2}{6} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} = \frac{t^2 h}{4} + \frac{t^3 h \epsilon_{csm}}{4B} \left( \frac{2}{3} \right) + \frac{TB^2}{6} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2}
\]

\[
= \frac{t^2 h}{2} \left( \frac{1}{2} - \frac{t \epsilon_{csm}}{3B \epsilon_y} \right) + \frac{TB^2}{6} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2}. \quad (4.42)
\]

In summary, for the cross-section geometry in Figure 4.15, the exact analytical equations have been determined for the uni-axial bending of I-sections and box sections about the major or minor axis, with an elastic, linear hardening material model. These may be summarised as follows, for major axis bending

\[
\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm}}{\epsilon_y} W_{pl,y} - 1 \right) - \frac{W_{w,y}}{W_{pl,y}} \left( 1 - \frac{E_{sh}}{E} \right) \quad (4.43)
\]

and with the \( W_{w,y} \) term is defined for I-sections and box sections as

\[
W_{w,y} = \begin{cases} 
\frac{tD^2}{12} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} & 1 + \frac{2T}{h} \leq \frac{\epsilon_{csm}}{\epsilon_y} \\
\frac{BD^2}{12} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} - \frac{(B - t)}{12} h^2 \left( 3 - \frac{2h \epsilon_{csm}}{D \epsilon_y} \right) & 1 + \frac{2T}{h} > \frac{\epsilon_{csm}}{\epsilon_y} \geq 1.
\end{cases}
\] (4.44)
For minor axis bending

\[
\frac{M_{csm,z}}{M_{pl,z}} = 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm}}{\epsilon_y} \frac{W_{el,z}}{W_{pl,z}} - 1 \right) - \frac{W_{w,z}}{W_{pl,z}} \left( 1 - \frac{E_{sh}}{E} \right)
\]  

(4.45)

and the \(W_{w,z}\) term for I-sections is given by

\[
W_{w,z} = \begin{cases} 
\frac{DB^2}{12} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} & \frac{\epsilon_{csm}}{\epsilon_y} \geq \frac{B}{t} \\
\frac{t^2h}{2} \left( \frac{1}{2} - \frac{t}{3B} \frac{\epsilon_{csm}}{\epsilon_y} \right) + \frac{TB^2}{6} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} & \frac{\epsilon_{csm}}{\epsilon_y} < \frac{B}{t}
\end{cases}
\]

(4.46)

and for box sections by

\[
W_{w,z} = \begin{cases} 
\frac{tB^2}{12} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} & \frac{1}{1 - \frac{t}{B}} \leq \frac{\epsilon_{csm}}{\epsilon_y} \\
\frac{DB^2}{12} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} - \frac{(D-t)}{12} (B-t)^2 \left[ 3 - 2 \left( 1 - \frac{t}{B} \right) \frac{\epsilon_{csm}}{\epsilon_y} \right] & \frac{1}{1 - \frac{t}{B}} > \frac{\epsilon_{csm}}{\epsilon_y} \geq 1
\end{cases}
\]

(4.47)

Figure 4.15: Cross-section geometry for analytical bending equations.
4.2.2.3 Modulus $W_w$ for circular hollow sections (CHS)

The $W_w$ modulus for circular hollow sections is derived in this subsection. For the effect of the triangular $f_2$ stress block on circular hollow sections, the moment integral is evaluated for the outer radius $r_2$ up to $Y$, and then the integral for the inner radius $r_1$ up to $Y$ is subtracted, where $Y = r_2 (\epsilon_{csm}/\epsilon_y)^{-1}$. If one quarter of the cross-section is integrated (as in Figure 4.16), then the result may be multiplied by four for the entire cross-section, and will give the modulus $W_w$ corresponding to the area $A_Y$, which is the cross-section area within $|y| \leq Y$. Formally this is

$$\frac{M_{f_2}}{f_2} = W_w = \int_{A_Y} \frac{f}{f_2} y \, dA = \int_{A_Y} \left(1 - \frac{y}{Y}\right) y \, dA. \quad (4.48)$$

![Figure 4.16: Notation used for the derivation of the modulus $W_w$ for circular and elliptical hollow sections.](image)

The calculation begins with the following integration for the outer radius $r_2$, with both $y = r \sin \theta$ and $dy = r \cos \theta \, d\theta$,
\[ W_{w,r_2} = 4 \int_0^Y y \left(1 - \frac{y}{Y}\right) z \, dy = 4 \int_0^Y r_2 \sin \theta \left(1 - \frac{r_2 \sin \theta}{Y}\right) r_2^2 \cos^2 \theta \, d\theta \]

\[ = 4 \int_0^Y r_2^3 \sin \theta \cos^2 \theta \left(1 - \frac{r_2 \sin \theta}{Y}\right) \, d\theta = 4r_2^3 \int_0^Y \sin \theta \cos^2 \theta \, d\theta - \frac{4r_2^4}{Y} \int_0^Y \sin^2 \theta \cos^2 \theta \, d\theta. \]

Using the standard result for the integral of the product of sine and cosines to exponents \( n \) and \( m \) respectively

\[
\int \sin^n \theta \cos^m \theta \, d\theta = \frac{\sin^{n+1} \theta \cos^{m-1} \theta}{n + m} + \frac{m - 1}{n + m} \int \sin^n \theta \cos^{m-2} \theta \, d\theta,
\]

allows the simplification of the following two integrals

\[
4r_2^3 \int_0^Y \sin \theta \cos^2 \theta \, d\theta = 4r_2^3 \left[ \frac{\sin^2 \theta \cos \theta}{3} \right]^Y_0 + 4r_2^3 \left[ \frac{\theta - \sin \theta \cos \theta}{3} \right]^Y_0 \tag{4.51}
\]

\[
\frac{4r_2^4}{Y} \int_0^Y \sin^2 \theta \cos^2 \theta \, d\theta = \frac{4r_2^4}{Y} \left[ \frac{\sin^3 \theta \cos \theta}{4} \right]^Y_0 + \frac{r_2^4}{Y} \left[ \frac{\theta - \sin \theta \cos \theta}{2} \right]^Y_0. \tag{4.52}
\]

This gives the outer radius modulus as

\[
W_{w,r_2} = \frac{4}{3} r_2^3 \left[ \sin^2 \theta \cos \theta - \cos \theta \right]^Y_0 - \frac{r_2^4}{Y} \left[ \sin^3 \theta \cos \theta + \frac{\theta}{2} - \frac{\sin \theta \cos \theta}{2} \right]^Y_0. \tag{4.53}
\]

Eqn (4.53) is also valid for the inner radius modulus \( W_{w,r_1} \), which subtracts the cross-section hole, if \( r_2 \) is changed to \( r_1 \) and it is recognised that the integral is invalid when \( Y > r_1 \).
\[ W_{w,r1} = \frac{4}{3} r_1^3 \left( \sin^2 \theta \cos \theta - \cos \theta \right)_0^Y - \frac{r_1^4}{Y} \left( \sin^3 \theta \cos \theta + \frac{\theta}{2} - \frac{\sin \theta \cos \theta}{2} \right)_0^Y. \] (4.54)

Since these equations are in terms of \( r \) and \( \theta \), the limits \( y = 0 \) and \( y = Y \) must be changed from Cartesian co-ordinates to polar co-ordinates using \( y = r \sin \theta \) as \( \theta = \arcsin(y/r) \). For \( y = Y \) and \( r = r_1 \) and \( r = r_2 \) this gives

\[ \alpha = \arcsin \left( \frac{Y}{r_2} \right) = \arcsin \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-1} \] (4.55)

\[ \beta = \arcsin \left( \frac{Y}{r_1} \right) = \arcsin \left[ \frac{r_2}{r_1} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-1} \right], \] (4.56)

where \( \alpha \) represents the outer radius \( r_2 \) and \( \beta \) the inner radius \( r_1 \). The equations can now be simplified by evaluating at these new limits

\[ W_{w,r2} = \frac{4}{3} r_2^3 \left( 1 - \cos^3 \alpha \right) - \frac{r_2^3}{\sin \alpha} \left( \sin^3 \alpha \cos \alpha + \frac{\alpha}{2} - \frac{\sin \alpha \cos \alpha}{2} \right) \] (4.57)

\[ W_{w,r1} = \frac{4}{3} r_1^3 \left( 1 - \cos^3 \beta \right) - \frac{r_1^3}{\sin \beta} \left( \sin^3 \beta \cos \beta + \frac{\beta}{2} - \frac{\sin \beta \cos \beta}{2} \right). \] (4.58)

The inner radius modulus \( W_{w,r1} \) must be subtracted from the outer radius modulus \( W_{w,r2} \) to give the final \( W_w \), that is

\[ W_w = W_{w,r2} - W_{w,r1}. \] (4.59)

This analytical formulation is valid for \( \frac{r_2}{r_1} < \frac{\epsilon_{csm}}{\epsilon_y} \). \( W_w \) can now be used in the governing equation for the CSM moment \( M_{csm} \)

\[ \frac{M_{csm}}{M_{pl}} = 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm} W_{el}}{\epsilon_y W_{pl}} - 1 \right) - \frac{W_w}{W_{pl}} \left( 1 - \frac{E_{sh}}{E} \right). \] (4.60)
4.2.2.4 Modulus $W_w$ for elliptical hollow sections (EHS)

The equation of an ellipse in Cartesian form with $z$ and $y$ co-ordinates and minor and major radii $a$ and $b$ (with inner and outer radii $a_1, b_1$ and $a_2, b_2$) is

$$\frac{z^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{with positive branch} \quad z = a\sqrt{1 - \frac{y^2}{b^2}}.$$  \hspace{1cm} (4.61)

Following the same approach to the derivation of $W_w$ for circular hollow sections (Figure 4.16), the initial integral to solve is

$$I = 4\int_0^Y y \left(1 - \frac{y}{Y}\right) z \, dy = 4\int_0^Y y \left(1 - \frac{y}{Y}\right) a\sqrt{1 - \frac{y^2}{b^2}} \, dy$$

$$= 4a \int_0^Y y\sqrt{1 - \frac{y^2}{b^2}} \, dy - \frac{4a}{Y} \int_0^Y y^2\sqrt{1 - \frac{y^2}{b^2}} \, dy \hspace{1cm} (4.62)$$

where $Y = b_2/(\epsilon_{csm}/\epsilon_y)$ is the value of $y$ at first yield, which is constant for a given strain ratio. Introducing the following substitution $u$ for the solution of the first integral in $I$

$$u = 1 - \frac{y^2}{b^2} \quad \text{which gives} \quad -du = \frac{2y}{b^2} \, dy \hspace{1cm} (4.63)$$

$$4a \int_0^Y y\sqrt{1 - \frac{y^2}{b^2}} \, dy = -2ab^2 \int_0^Y \sqrt{u} \, du = 4ab^2 \left[\frac{u^{3/2}}{3}\right]_0^Y = -\frac{4ab^2}{3} \left[\left(1 - \frac{y^2}{b^2}\right)^{3/2}\right]_0^Y. \hspace{1cm} (4.64)$$

For the second integral in $I$, using a trigonometric substitution $y = b \sin u$ gives

$$dy = b \cos u \, du \quad \text{and} \quad \cos u = \frac{\sqrt{b^2 - y^2}}{b} = \sqrt{1 - \frac{y^2}{b^2}} \hspace{1cm} (4.65)$$

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\[- \frac{4a}{Y} \int_{0}^{Y} y^2 \sqrt{1 - \frac{y^2}{b^2}} \, dy = \frac{4a}{Y} \int_{0}^{Y} b^3 \sin^2 u \cos u \cos u \, du = \frac{4ab^3}{Y} \int_{0}^{Y} \sin^2 u \cos^2 u \, du \]

\[= - \frac{4ab^3}{Y} \left[ \frac{\sin^3 u \cos u}{4} + \frac{1}{4} \left( \frac{u}{2} - \frac{\sin u \cos u}{2} \right) \right]_{0}^{Y} \]

\[= - \frac{4ab^3}{Y} \left[ \frac{y^3}{4b^3} \sqrt{1 - \frac{y^2}{b^2}} + \frac{1}{8} \arcsin \left( \frac{y}{b} \right) - \frac{y}{8b} \sqrt{1 - \frac{y^2}{b^2}} \right]_{0}^{Y} \]

\[= - \frac{4a}{Y} \left[ \frac{y}{8} (2y^2 - b^2) \sqrt{1 - \frac{y^2}{b^2}} + \frac{b^3}{8} \arcsin \left( \frac{y}{b} \right) \right]_{0}^{Y}. \quad (4.66) \]

Bringing together for the complete integral $I$ leads to

\[I = - \frac{4ab^2}{3} \left[ \left( 1 - \frac{y^2}{b^2} \right)^{\frac{3}{2}} \right]_{0}^{Y} - \frac{4a}{Y} \left[ \frac{Y}{8} (2Y^2 - b^2) \sqrt{1 - \frac{Y^2}{b^2}} + \frac{b^3}{8} \arcsin \left( \frac{Y}{b} \right) \right]_{0}^{Y} \]

\[= - \frac{4ab^2}{3} \left[ \left( 1 - \frac{Y^2}{b^2} \right)^{\frac{3}{2}} - 1 \right] - \frac{4a}{Y} \left[ \frac{Y}{2} (2Y^2 - b^2) \sqrt{1 - \frac{Y^2}{b^2}} + \frac{b^3}{2} \arcsin \left( \frac{Y}{b} \right) \right] \]

\[= \frac{4ab^2}{3} \left[ 1 - \left( 1 - \frac{Y^2}{b^2} \right)^{\frac{3}{2}} \right] - \frac{a}{Y} \left[ \frac{Y}{2} (2Y^2 - b^2) \sqrt{1 - \frac{Y^2}{b^2}} + \frac{b^3}{2} \arcsin \left( \frac{Y}{b} \right) \right]. \quad (4.67) \]

This integral must be evaluated for the outer radius $r_2 = f(a_2, b_2)$, with the value from the inner radius $r_1 = f(a_1, b_1)$ subtracted, in order to give $W_w = W_{w,r2} - W_{w,r1}$. For $W_{w,r2}$ and $W_{w,r1}$ respectively, substitutions similar to the circular hollow section case are used.
\[ \alpha = \arcsin \left( \frac{Y}{b_2} \right) = \arcsin \left( \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-1} \right) \]  
\[ (4.68) \]

\[ \beta = \arcsin \left( \frac{Y}{b_1} \right) = \arcsin \left[ \frac{b_2}{b_1} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-1} \right]. \]  
\[ (4.69) \]

The limits are evaluated for the outer radius \((\alpha, a_2, b_2)\) and for the inner radius \((\beta, a_1, b_1)\)

\[ W_{r,r_2} = \frac{4a_2b_2^2}{3} \left[ 1 - \left(1 - \sin^2 \alpha \right)^{\frac{3}{2}} \right] - \frac{a_2}{b_2 \sin \alpha} \left[ \frac{b_2 \sin \alpha}{2} \left( 2b_2^2 \sin^2 \alpha - b_2^2 \right) \sqrt{1 - \sin^2 \alpha} + \frac{b_2^3}{2} \alpha \right] \]

\[ = \frac{4a_2b_2^2}{3} \left[ 1 - \cos^3 \alpha \right] - \frac{a_2}{b_2 \sin \alpha} \left[ \left( b_2^3 \sin^3 \alpha - \frac{b_2^3}{2} \sin \alpha \right) \cos \alpha + \frac{b_2^3}{2} \alpha \right] \]

\[ = \frac{4a_2b_2^2}{3} \left[ 1 - \cos^3 \alpha \right] - \frac{a_2b_2^2}{\sin \alpha} \left[ \sin^3 \alpha \cos \alpha + \frac{\alpha}{2} - \frac{\sin \alpha \cos \alpha}{2} \right] \]

\[ W_{r,r_1} = \frac{4a_1b_1^2}{3} \left[ 1 - \cos^3 \beta \right] - \frac{a_1b_1^2}{\sin \beta} \left[ \sin^3 \beta \cos \beta + \frac{\beta}{2} - \frac{\sin \beta \cos \beta}{2} \right]. \]  
\[ (4.70) \]

Notice that when \(a_2 = b_2 = r_2\) and \(a_1 = b_1 = r_1\), \(W_{w,r_1}\) and \(W_{w,r_2}\) collapse to the previously derived circular hollow section expressions. The analytical formula is valid for \(b_2/b_1 < \epsilon_{csm}/\epsilon_y\).

### 4.2.3 Design expression

The analytical equations in Section 4.2.2 are exact for the elastic, linear hardening material model, but are rather lengthy for practical design use due to the \(W_w\) term, which requires detailed information of the geometry of the cross-section and involves a significant amount of computation. Although the analytical equations could be programmed for frequent use,
a simple design equation is sought. Recall the governing major axis CSM moment equation normalised by the elastic moment \( M_{el,y} \), applicable to any cross-section shape symmetric about the bending axis \( y - y \),

\[
\frac{M_{csm,y}}{M_{el,y}} = \frac{W_{pl,y}}{W_{el,y}} + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm}}{\epsilon_y} - \frac{W_{pl,y}}{W_{el,y}} \right) - \frac{W_{w,y}}{W_{el,y}} \left( 1 - \frac{E_{sh}}{E} \right). \tag{4.71}
\]

To illustrate the variation of \( W_{w,y} \) with the strain ratio \( \frac{\epsilon_{csm}}{\epsilon_y} \), its value for \( M_{csm,y}/M_{el,y} = 1 \) and \( \epsilon_{csm}/\epsilon_y = 1 \) is first determined as

\[
W_{w,y} = W_{el,y} \left( 1 - \frac{W_{pl,y}}{W_{el,y}} \right) - \left( 1 - \frac{E_{sh}}{E} \right) \left( W_{el,y} - W_{pl,y} \right) \left( 1 - \frac{E_{sh}}{E} \right) = W_{pl,y} - W_{el,y} \tag{4.72}
\]

with a similar minor axis equivalent of \( W_{w,z} = W_{pl,z} - W_{el,z} \). Figure 4.17 shows the decay of \( W_{w,y} \) and \( W_{w,z} \) with respect to the strain ratio. For the major axis and with all considered cross-section shapes, \( W_{w,y} \) reduces quickly, and the values for almost all cross-sections fall under the curve

\[
W_{w,y} = (W_{pl,y} - W_{el,y}) \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2}. \tag{4.73}
\]

This is a conservative fit as the \( W_w \) terms are subtractive to \( M_{csm} \) in the governing equations. This inverse square decay is equally valid for the hollow cross-sections (RHS, SHS, CHS and EHS) bending about the minor axis in Figure 4.17b, as these cross-sections are of similar shape about both bending axes. However, for I-sections bending about the minor axis the decay is more gradual, and can be approximated in Eqn (4.74) by

\[
W_{w,z} = (W_{pl,z} - W_{el,z}) \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-1.2}. \tag{4.74}
\]

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Figure 4.17: The decay of moduli $W_{w,y}$ and $W_{w,z}$ with increasing strain ratio.

From the governing equation for major axis bending, substituting in the new simplified form of $W_{w,y}$ results in

$$\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm}}{\epsilon_y} \frac{W_{el,y}}{W_{pl,y}} - 1 \right) - \left( 1 - \frac{W_{el,y}}{W_{pl,y}} \right) \left( 1 - \frac{E_{sh}}{E} \right) \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^2. \quad (4.75)$$

By noting that in general $E_{sh}/E << 1$, the $(1 - E_{sh}/E)$ term can conservatively be taken as unity as the final term is subtractive, and the rest of the equation forced through $M_{el,y}$ at a strain ratio of 1, giving the design equation

$$\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh}}{E} \frac{W_{el,y}}{W_{pl,y}} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right) - \left( 1 - \frac{W_{el,y}}{W_{pl,y}} \right) \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^2. \quad (4.76)$$
For cross-sections bending about the minor axis this translates to a design equation of

\[
\frac{M_{csm,z}}{M_{pl,z}} = 1 + \frac{E_{sh}}{E} \frac{W_{el,z}}{W_{pl,z}} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right) - \left( 1 - \frac{W_{el,z}}{W_{pl,z}} \right) \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-\alpha}
\]

where \(\alpha = 2\) for the considered hollow cross-sections and \(\alpha = 1.2\) for I-sections. These equations only require the assignment of three ratios: the ratio of strain hardening modulus to Young’s modulus \(E_{sh}/E\), the shape factor reciprocal \(W_{el}/W_{pl}\), and the strain ratio \(\epsilon_{csm}/\epsilon_y\).

Figure 4.18a shows the predicted moment–strain ratio curve for a typical I-section bending about its major axis, and Figure 4.18b gives the error in the design equation Eqn (4.76), by dividing by the numerically exact solution from Section 4.2.1. For all considered cross-sections bending about the major axis, the design equation predicts the moment capacity accurately, with conservative results for low strain ratios and slightly higher predictions for higher strain ratios. The error is most prevalent for lower strain ratios, which is where the \(W_{w,y}\) approximation leads to a safe underestimate of 4-6% when compared to the numerical results, while for higher strain ratios the error reduces to less than 0.5%.

The moment–strain ratio curve in Figure 4.19a is for an I-section bending about its minor axis, and shows a more rounded and approximated shape compared to the numerical model curve. From Figure 4.19b the error for hollow sections (in light grey) is similar to that for major axis bending, but for the I-sections (dark grey) the error is greater, and for a wider range of strain ratios. It may be seen that the \(W_{w,z}\) term has a stronger influence on minor axis bending, and the approximation in this term for the design equation is more apparent.
Figure 4.18: Design equation for major axis bending, hollow cross-sections and I-sections.

Figure 4.19: Design equation for minor axis bending, hollow cross-sections and I-sections.
4.2.4 Asymmetric cross-sections

The analytical equations for the prediction of the bending capacity of a symmetric cross-section (Section 4.2.2) utilised the plastic and elastic section moduli to simplify the calculation. These moduli are cross-section properties that are calculated about the elastic neutral axis (ENA) and the plastic neutral axis (PNA), which are at fixed locations parallel to the chosen bending axis. For cross-sections that are symmetric about the bending axis, the ENA, PNA and instantaneous zero strain neutral axis (NA), all align at the mid-depth of the cross-section at \( y = D/2 \) for the major axis and at \( z = B/2 \) for the minor axis. This is not the case however for cross-sections bending about an axis that is not a line of cross-section symmetry, as in Figure 4.20a, which shows indicatively the locations of the PNA, ENA and NA of an unequal angle, bending about the \( y-y' \) axis, which is parallel to the smaller leg. The \( z-z' \) axis is perpendicular to the smaller leg of the angle cross-section.

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(a) The various neutral axes of an unequal angle and general rotated co-ordinate axes
(b) Mohr circle for the second moments of area of an asymmetric cross-section

Figure 4.20: Bending of angle cross-sections.
4.2.4.1 Elastic and plastic section properties

The distance $P_N$ of the plastic neutral axis from the top face of the unequal angle shown in Figure 4.20a, is such that the cross-section areas above and below the plastic neutral axis are equal. This will produce a net axial force of zero if the compressive stresses above the PNA are at the material yield stress $f_y$, and at the tensile stress $-f_y$ when below the PNA. If the PNA is located within the top flange

$$W_{pl,y} = \frac{B P_N^2}{2} + \frac{B(T - P_N)^2}{2} + h t \left( \frac{h}{2} + T - P_N \right)$$

(4.78)

while if the plastic neutral axis is located within the web

$$W_{pl,y} = \frac{t(D - P_N)^2}{2} + \frac{t(P_N - T)^2}{2} + B T \left( P_N - \frac{T}{2} \right).$$

(4.79)

The elastic neutral axis in the $y$ direction passes through the geometric centroid of the angle cross-section, and can be calculated by taking the first moment of area about the top flange face, to give the distance $E_N$

$$E_N = \frac{BT^2}{2} + \frac{ht(T + h/2)}{A}.$$ 

(4.80)

The second moment of area in the $y$ direction is then calculated about the elastic neutral axis as

$$I_y = \frac{BT^3}{12} + BT \left( E_N - \frac{T}{2} \right)^2 + \frac{th^3}{12} + h t \left( T + \frac{h}{2} - E_N \right)^2.$$ 

(4.81)

The elastic section modulus is based on whether the top outer-fibres at $y = E_N$ or the bottom outer-fibres at $y = D - E_N$ yield first, which will result in the smaller of $W_{el,y} = I_y/E_N$ and $W_{el,y} = I_y/(D - E_N)$. Similar calculations can be performed for the cross-section properties of angles bending about the $z$–$z$ axis.
4.2.4.2 Rotated second moment of areas

Figure 4.20a shows the axes about which the second moments of area $I_y$ and $I_z$ are taken. These correspond to the $y$ and $z$ directions respectively and are those directions orientated parallel to the edges of the cross-section. However bending may occur for any co-ordinate axes $y'$ and $z'$ that are orientated at a rotation $\theta$ from the unrotated $y$ and $z$ co-ordinate axes, which will give second moments of area $I_{y'}$, $I_{z'}$ and the cross moment of area $I_{yz'}$. When the $z$–$y$ axes in Figure 4.20a are rotated by an angle $\theta$ (anticlockwise positive) to become $z'$–$y'$, the co-ordinate transformation is that of

$$
\begin{bmatrix}
  z' \\
  y'
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  z \\
  y
\end{bmatrix}.
$$

The derivation of $I_{y'}$, $I_{z'}$ and $I_{yz'}$ proceeds as follows,

$$
I_{y'} = \int y'^2 \, dA = \int z'^2 \sin^2 \theta + y'^2 \cos^2 \theta + 2 y' z' \sin \theta \cos \theta \, dA
$$

$$
= \sin^2 \theta \int z^2 \, dA + \cos^2 \theta \int y^2 \, dA + 2 \sin \theta \cos \theta \int y z \, dA
$$

$$
= \sin^2 \theta I_z + \cos^2 \theta I_y + 2 \sin \theta \cos \theta I_{yz} = \frac{I_y + I_z}{2} + \frac{I_y - I_z}{2} \cos 2\theta + \sin 2\theta I_{yz}. \tag{4.83}
$$

Similarly for $I_{z'}$ with the rotated co-ordinate $z'$,

$$
I_{z'} = \int z'^2 \, dA = \int z'^2 \cos^2 \theta + y'^2 \sin^2 \theta - 2 y' z' \sin \theta \cos \theta \, dA
$$

$$
= \cos^2 \theta \int z^2 \, dA + \sin^2 \theta \int y^2 \, dA - 2 \sin \theta \cos \theta \int y z \, dA
$$

$$
= \cos^2 \theta I_z + \sin^2 \theta I_y - 2 \sin \theta \cos \theta I_{yz} = \frac{I_y + I_z}{2} - \frac{I_y - I_z}{2} \cos 2\theta - \sin 2\theta I_{yz}. \tag{4.84}
$$
For the cross moment of area $I_{y'y'}$,

$$I_{y'y'} = \int y'z' \, dA = \int z^2 \cos \theta \sin \theta + yz \cos^2 \theta - yz \sin^2 \theta - y^2 \cos \theta \sin \theta \, dA$$

$$= \cos \theta \sin \theta \int z^2 \, dA + \cos^2 \theta \int yz \, dA - \sin^2 \theta \int yz \, dA - \cos \theta \sin \theta \int y^2 \, dA$$

$$= \cos \theta \sin \theta I_z + \cos^2 \theta I_{yz} - \sin^2 \theta I_{yz} - \cos \theta \sin \theta I_y = \frac{I_z - I_y}{2} \sin 2\theta + \cos 2\theta I_{yz}.$$

(4.85)

The second moments of area $I_y$, $I_z$ and also $I_{yz}$, represent points on the Mohr circle plotted in Figure 4.20b, for the unrotated bending axes $z-z$ and $y-y$. Also plotted are the $I_{y'y'}$, $I_{z'}$, and $I_{y'z'}$ values which represent the co-ordinate axes rotated by $\theta$. Figure 4.20b shows that there exists principal second moments of area when $I_{y'y'} = 0$, these are the points $I_{11}$ and $I_{22}$, which are the intersections of the Mohr circle with the horizontal axis. These are the maximum and minimum values respectively, with the maximum principal value occurring when the rotation $\theta$ is clockwise and equal to $\alpha$. From the geometry of the Mohr circle this rotation $\alpha$ for the maximum principal value is given by

$$\tan 2\alpha = \frac{2I_{yz}}{I_y - I_z}.$$

(4.86)

With $R$ as the radius of the Mohr circle, the maximum principal second moment of area $I_{11}$ is calculated as

$$I_{11} = \frac{I_z + I_y}{2} + R = \frac{I_z + I_y}{2} + \sqrt{\left[\frac{I_y - I_z}{2}\right]^2 + I_{yz}^2}.$$

(4.87)

The elastic moment $M_{el,y'}$ at rotation $\theta$ is defined as the moment resistance (calculated from the integration of the elastic stresses for the $y'$ direction) when the cross-section has
first yielded, with yielding occurring at a distance \( y' = Y \). For a strain distribution varying linearly with depth, the distance \( Y \) will be the greater of the distances above and below the zero strain neutral axis to the extreme fibres of the cross-section in the \( y' \) direction. The rotated elastic moment is calculated from the rotated second moment of area \( I_{y'} \), and is normalised by the unrotated plastic moment \( M_{pl,y} \) to give

\[
\frac{M_{el,y'}}{M_{pl,y}} = \frac{I_{y'} f_y}{YW_{pl,y} f_y} = \frac{I_{y'}}{YW_{pl,y}}.
\]  

(4.88)

This normalised moment is plotted in Figure 4.21 with the ratio of the rotated and unrotated second moments of area \( I_{y'}/I_y \). These curves indicate that the relative gains in the second moments of area by varying \( \theta \), are greater than the gains in the elastic moment capacity. For unequal angles the maxima and minima do not need to coincide at the same orientation \( \theta \).

\[\begin{align*}
\text{(a) Equal angle} \\
\text{(b) Unequal angle}
\end{align*}\]

Figure 4.21: Variation of \( I_{y'}/I_y \) and \( M_{el,y'}/M_{pl,y} \) with the co-ordinate axes orientation \( \theta \).
4.2.4.3 Numerical model

A numerical model (Figure 4.22) was developed to calculate the neutral axis location and bending resistance of an angle cross-section at any orientation. The model is similar to that described in Section 4.2.1, except that all calculations are based on the rotated coordinate axes $y'$ and $z'$ at orientation $\theta$ (anticlockwise taken as positive). The elemental areas $A_i$ are formed by dividing the flange and web areas in both the $y$ and $z$ directions.

Each element centroid is located at $y_i$ and $z_i$, and the new effective cross-section depth in the $y'$ direction is $D'$. For the location of the correct neutral axis, zero strain neutral axes NA are assumed, that are located throughout the effective depth $D'$. The limiting CSM strain in compression is designated $\epsilon_{csm}$, and the stress distribution formed is based on the $y'$ distances from the assumed neutral axis location. In Figure 4.22, yielding is indicated to occur first at the bottom outer-fibres as the neutral axis is located at a distance less than $D'/2$ from the top. The strain ratio $\epsilon_{csm}/\epsilon_y$, as acquired from the CSM base curve, is based on tests in compression. The cross-section strains in tension, as depicted in Figure 4.22, may exceed in magnitude $\epsilon_{csm}$ so long as $\epsilon_i \leq 15\epsilon_y$ is satisfied (Eurocode 3 ductility limits). The strain at element $i$ is then based on

$$\frac{y_i}{\max[D' - NA, NA]}$$  \hspace{1cm} (4.89)

The yield normalised elemental stresses $f_i/f_y$ are calculated from the elemental strains using the bi-linear material model, as performed previously for symmetric cross-sections. The cross-section axial loads are then calculated for each of the assumed neutral axes with

$$\frac{N}{N_y} = \frac{1}{A} \sum_i A_i f_i f_y.$$  \hspace{1cm} (4.90)
Each axial load is then paired with its assumed neutral axis location, and the zero point that represents $N = 0$ is determined, indicating the correct neutral axis for which there is no net axial load. Moments are then taken about this zero strain neutral axis and normalised by the plastic moment in the unrotated orientation ($\theta = 0$),

$\frac{M_{y'}}{M_{pl,y}} = \frac{1}{W_{pl,y}} \sum_i A_i y_i f_i f_y$. \hspace{1cm} (4.91)

The numerical model has been used to analyse 66 equal angles and 47 unequal angles of standard proportions from SCI P363 (2009), by varying the strain ratio from between 1 and 15, and the axis rotation $\theta$ from $-90^\circ$ to $90^\circ$. The numerical model calculates the neutral axis position for any strain ratio by ensuring a net axial load of zero. Evolution of the neutral axis position with strain ratio is plotted in Figure 4.23 for $E_{sh} = 0$ (no strain hardening) and in Figure 4.24 for $E_{sh} = E/100$ (linear strain hardening).
Figure 4.23: Neutral axis location relative to the elastic and plastic neutral axes, $E_{sh} = 0$.

Figure 4.24: Neutral axis location relative to the elastic and plastic neutral axes, $E_{sh} = E/100$. 
These plots are for all of the considered angle cross-sections and strain ratios. The location of the neutral axis NA, always lies between the elastic neutral axis $E_N$ at $(E_N - NA)/(E_N - P_N) = 0$, and the plastic neutral axis $P_N$ at $(E_N - NA)/(E_N - P_N) = 1$. For the equal angles in Figure 4.23a and Figure 4.24a, a strain ratio of 15 is sufficient for the neutral axis to shift from the elastic neutral axis to 90% of the way to the plastic neutral axis, with minimal influence from $E_{sh}$. For the unequal angles in Figure 4.23b and Figure 4.24b, when $E_{sh} = 0$ the NA tends asymptotically to the plastic neutral axis, though when $E_{sh} = E/100$, the motion of the zero strain neutral axis is initially towards the plastic neutral axis, then changes direction and moves back towards the ENA.

4.2.4.4 Design equation for the unrotated bending axes $\theta = 0$

The design equation developed in Section 4.2.3 for the bending of symmetric cross-sections is given in Eqn (4.92) with $\alpha = 1.5$. The negative exponent of 1.5 on the final strain ratio term was found to provide a good approximation to the bending behaviour of the standard equal and unequal angles considered, for bending about the geometric $y-y$ and $z-z$ axes with the elastic and plastic section moduli. It should be noted that these section moduli $W_{pl}$ and $W_{el}$ are based on the plastic and elastic neutral axes and not the instantaneous zero strain neutral axis, and are readily available in section property tables.

\[
\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh}}{E} \frac{W_{el,y}}{W_{pl,y}} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right) - \left( 1 - \frac{W_{el,y}}{W_{pl,y}} \right) \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-1.5}
\]

(4.92)

The design equation Eqn (4.92) is plotted in Figure 4.25a and Figure 4.25b for an equal and unequal angle respectively, with the corresponding numerical model curves. For lower strain ratios the design equation over-predicts and under-predicts the true response for the equal and unequal angles respectively, by $\approx 3\%$ of $M_{pl,y}$, and then matches closely the numerical curves when $\epsilon_{csm}/\epsilon_y > 8$. 

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Figure 4.25: CSM design equation with $\alpha = 1.5$ and numerical model moment–strain ratio curve for an equal and unequal angle.

Figure 4.26: CSM design equation with $\alpha = 1.5$ normalised by the numerical model, varying with strain ratio and for all considered equal and unequal angle cross-sections.
In Figure 4.26, the CSM moment resistance $M_{csm,y}$ from Eqn (4.92) is divided by the numerical model moment resistance $M_{n,y}$ and is plotted for all considered angle cross-sections and for strain ratios in the range $1 \leq \epsilon_{csm}/\epsilon_y \leq 15$. These curves give an indication of the error in the design equation predictions; when $M_{csm,y}/M_{n,y} > 1$ the design equation has over-predicted the moment resistance, and when $M_{csm,y}/M_{n,y} < 1$ the predictions are on the conservative side. Figure 4.26a shows that equal angles are calculated in error by $\leq 4\%$ for low strain ratios, and conservatively to within $1\%$ thereafter. For the unequal angles in Figure 4.26b, the moments are also initially over-estimated by $\leq 4\%$, but then they are conservatively calculated for the majority of cross-sections over the range of strain ratios.

4.2.4.5 Moment about general bending axes at an orientation $\theta$

Since the developed numerical model is capable of calculating the moment resistance $M_y$ at any axes orientation $\theta$, the variation of the plastic normalised moment capacities $M_y/M_{pl,y}$ with $\epsilon_{csm}/\epsilon_y = 15$, for all considered equal and unequal angles have been calculated and plotted in Figure 4.27. For the equal angles presented in Figure 4.27a all cross-sections behave similarly, displaying maximum and minimum moment capacities at $\theta = +45^\circ$ and $\theta = -45^\circ$ respectively, whilst for the unequal angles plotted in Figure 4.27b, the maxima and minima locations are spread further apart. Large increases of up to 50\% over the unrotated plastic moment $M_{pl,y}$ are seen for the equal angles at $\theta = 45^\circ$, and more modest increases of up to 25\% are found for the unequal angles. Figure 4.28a shows the maximum moment capacities $M_m$ from each of the curves in Figure 4.27 divided by the unrotated ($\theta = 0$) moment capacities $M_{csm,y}$, plotted against the width-to-depth ratios $B/D$. 

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Figure 4.27: Moment resistance variation with bending axis orientation $\theta$ for all considered equal and unequal angle cross-sections.

Figure 4.28: Maximum moment resistance $M_m$ and associated co-ordinate axes orientation $\theta_m$, varying with cross-section aspect ratio $B/D$. 

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The maximum moment ratios are highest for strain ratios below 3 (SR < 3), and then they converge to the same values as the strain ratio increases. The moment gains are not as great as the maximum principal second moments of area gains, i.e. the values of $I_{11}/I_y$ are larger than $M_m/M_{\text{csm},y}$, especially for the equal angles where $B/D = 1$. The co-ordinate axes orientation $\theta_m$ at which $M_m$ occur are plotted in Figure 4.28b, which shows that the orientation of maximum moment decreases from the elastic values as the strain ratio increases, tending towards values similar to $\alpha$ (the rotation that represents the maximum principal second moment of area $I_{11}$). The optimum axes rotation for an equal angle is always $\theta_m = 45^\circ$ (this is an orientation of cross-section symmetry where $I_{y',z'} = 0$), while the maximum moments for unequal angles occur between $15^\circ$ and $30^\circ$ for $\epsilon_{\text{csm}}/\epsilon_y > 3$. The plotted fits used to represent the maximum moments $M_m$ and their corresponding bending axis rotations $\theta_m$ are

$$\frac{M_m}{M_{\text{csm},y}} = 0.45 \left(\frac{B}{D}\right)^{3.5} + 1 \quad \text{and} \quad \theta_m = 60 \frac{B}{D} - 15. \quad (4.93)$$

For an equal angle with $B = D$, $M_m$ is 1.45 times greater than the unrotated moment $M_{\text{csm},y}$. The CSM design equation may also be used with $\alpha = 1.5$ if the plastic and elastic section moduli are calculated based on the rotated co-ordinate axes. Such moduli are easily evaluated by the numerical model, which when used with the CSM design equation, will give moment $M_{\text{csm},y'}$. This moment has been divided by the moment $M_{n,y'}$ obtained from the numerical model and plotted in Figure 4.29 for all of the considered angle cross-sections. The co-ordinate axes here have been rotated in the range from $\theta = 0$ to $\theta = \theta_m$. Outside of this co-ordinate axes rotation range, Figure 4.30 shows that the error changes to between -8% to 2% with equal angles and to -8% to 10% with unequal angles. These curves in Figure 4.30 are for the CSM design equation with the exponent $\alpha = 1.2$, identical to the value used for the minor axis bending of I-sections. The errors are below 2% for both angle shapes for the maximum strain ratio of 15.
Figure 4.29: Rotated CSM design moment $M_{csm,y}$ normalised by the numerical moment, with co-ordinate axes rotated between $\theta = 0$ and $\theta = \theta_m$ degrees.

Figure 4.30: Rotated CSM design moment $M_{csm,y}$ normalised by the numerical moment, with co-ordinate axes rotated between $-90 \leq \theta < 0$ and $\theta_m < \theta \leq 90$ degrees.
4.2.5 Example calculations

In this subsection, worked examples are presented to demonstrate the application of the proposed design methods.

1) Calculate the major axis bending capacity and the axial capacity of the square hollow section in Figure 4.31.

![Figure 4.31: Square hollow section calculation example.](image)

**Bending capacity**

**Cross-section slenderness and strain ratio**

Using the most slender plate element method for determining the cross-section slenderness, the compressive top flange will be governing,

\[
\bar{\lambda}_p = \sqrt{\frac{f_y}{f_{cr}}} = \sqrt{\frac{12(1 - \nu^2)235}{\pi^2 E k_{\sigma}}} \left( \frac{c}{t\epsilon} \right) = \sqrt{\frac{12(1 - 0.3^2)235}{\pi^2 (210000)(4)}} \left( \frac{270}{15(0.814)} \right) = 0.3891. \quad (4.94)
\]
This cross-section slenderness is less than $\bar{\lambda}_p \leq 0.68$, giving a strain ratio (using the base curve) of

\[ \frac{\epsilon_{csm}}{\epsilon_y} = \frac{0.25}{\lambda^{3.6}} = \frac{0.25}{0.3891^{3.6}} = 7.477 \quad \text{where} \quad \frac{\epsilon_{csm}}{\epsilon_y} \leq 15. \quad (4.95) \]

**Design bending prediction**

\[ \frac{f_u}{f_y} > 1.1 \quad \text{and so} \quad E_{sh} = E/100; \quad \frac{W_{el,y}}{W_{pl,y}} = \frac{1548}{1829} = 0.8464 \quad (4.96) \]

The design CSM moment normalised by the plastic moment is

\[ \frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh} W_{el,y}}{E W_{pl,y}} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right) - \left( 1 - \frac{W_{el,y}}{W_{pl,y}} \right) \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} \]

\[ = 1 + 0.01(0.8464)(7.477 - 1) - (1 - 0.8464)(7.477)^{-2} = 1.052 \quad (4.97) \]

\[ = 1.052W_{pl,y}f_y = 1.052(1829000)(355)/10^6 = 683 \text{kNm}. \quad (4.98) \]

**Analytical bending prediction**

The cross-section geometry from Figure 4.31 is: $B = 300 \text{mm}$, $D = 300 \text{mm}$, $h = 270 \text{mm}$, $T = 15 \text{mm}$ and $t = 2T = 30 \text{mm}$. As $1 + 2T/h = 1 + 2(15)/270 = 1.11$ is less than the strain ratio, the modulus $W_{w,y}$ is calculated for this box section as

\[ W_{w,y} = \frac{tD^2}{12} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} = \frac{30(300)^2}{12} (7.477)^{-2} = 4025 \text{mm}^3. \quad (4.99) \]
This gives the analytical CSM moment $M_{csm,y}$ normalised by the plastic moment $M_{pl,y}$ as

$$
\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm} W_{el,y}}{\epsilon_y W_{pl,y}} - 1 \right) - \frac{W_{w,y}}{W_{pl,y}} \left( 1 - \frac{E_{sh}}{E} \right)
$$

$$
= 1 + 0.01[7.477(0.8464) - 1] - \frac{4.025}{1829}(1 - 0.01) = 1.051 \quad (4.100)
$$

$$
M_{csm,y} = 1.051W_{pl,y}f_y = 1.051(1829000)(355)/10^6 = 682 \text{ kNm}. \quad (4.101)
$$

**Axial capacity**

The strain ratio of $\frac{\epsilon_{csm}}{\epsilon_y} = 7.477$ is applicable to both bending and axial compression, as the top flange of the cross-section is governing for both loading cases ($k_\sigma = 4.0$),

$$
\frac{f_{csm}}{f_y} = 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right) = 1 + 0.01 (7.477 - 1) = 1.065 \quad (4.102)
$$

$$
N_{csm} = Af_{csm} = (300^2 - 270^2)(1.065)(355) = 6464 \text{ kN}. \quad (4.103)
$$

2) Calculate the major axis bending capacity of the elliptical hollow section in Figure 4.32.

**Bending capacity**

**Cross-section slenderness and strain ratio**

Calculating the cross-section slenderness in bending as equal to the cross-section slenderness in compression,

$$
\overline{\lambda}_p = \sqrt{\frac{f_y}{f_{cr}}} = \sqrt{\frac{235\sqrt{3(1 - \nu^2)}}{2E} \left( \frac{D_e}{te^2} \right)} = \sqrt{\frac{235\sqrt{3(1 - 0.3^2)}2(250)^2/125}{2(210000)16(0.66)}} = 0.2959. \quad (4.104)
$$
This is less than $\bar{\lambda}_p \leq 0.4$, and so the strain ratio is

$$\frac{\epsilon_{csm}}{\epsilon_y} = \frac{0.001}{\bar{\lambda}_p^{0.5}} = \frac{0.001}{0.2959^{0.5}} = 9.256$$

where $\frac{\epsilon_{csm}}{\epsilon_y} \leq 15$. (4.105)

**Design bending prediction**

Using the CSM major axis design equation, the CSM moment $M_{csm,y}$ is calculated by

$$\frac{f_u}{f_y} > 1.1 \quad \text{and so} \quad E_{sh} = E/100; \quad \frac{W_{el,y}}{W_{pl,y}} = \frac{1748}{2459} = 0.7109 \quad (4.106)$$

$$\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh} W_{el,y}}{E W_{pl,y}} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right) - \left( 1 - \frac{W_{el,y}}{W_{pl,y}} \right) \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2}$$

$$= 1 + 0.01(0.7109)(9.256 - 1) - (1 - 0.7109)(9.256)^{-2} = 1.055$$

$$M_{csm,y} = 1.055 W_{pl,y} f_y = 1.055(2459000)(355)/10^6 = 921 \text{kNm}.$$ (4.107)
Analytical bending prediction

The cross-section geometry of the elliptical hollow section for bending about the major axis is $a_2 = 125 \text{ mm}$, $a_1 = 125 - 16 = 109 \text{ mm}$, $b_2 = 250 \text{ mm}$ and $b_1 = 250 - 16 = 234 \text{ mm}$. Both $\alpha$ and $\beta$ are calculated by

$$\alpha = \arcsin \left[ \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-1} \right] = \arcsin \left[ \left( 9.256 \right)^{-1} \right] = 0.1082 \quad (4.108)$$

$$\beta = \arcsin \left[ \frac{b_2}{b_1} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-1} \right] = \arcsin \left[ \frac{250}{234} \left( 9.256 \right)^{-1} \right] = 0.1157. \quad (4.109)$$

The outer radius modulus $W_{r,r_2}$ is calculated as

$$W_{r,r_2} = \frac{4a_2b_2^2}{3} \left[ 1 - \cos^3 \alpha \right] - \frac{a_2b_2^2}{\sin \alpha} \left[ \sin^3 \alpha \cos \alpha + \frac{\alpha}{2} - \frac{\sin \alpha \cos \alpha}{2} \right]$$

$$= \frac{4(125)(250)^2}{3} \left[ 1 - 0.9826 \right] - \frac{125(250)^2}{0.1080} \left[ 0.1080^3(0.9942) + \frac{0.1082}{2} - \frac{0.1080(0.9942)}{2} \right]$$

$$= 60.76(10^3) \text{ mm}^3. \quad (4.110)$$

The inner radius modulus $W_{r,r_1}$ is calculated as

$$W_{r,r_1} = \frac{4a_1b_1^2}{3} \left[ 1 - \cos^3 \beta \right] - \frac{a_1b_1^2}{\sin \beta} \left[ \sin^3 \beta \cos \beta + \frac{\beta}{2} - \frac{\sin \beta \cos \beta}{2} \right]$$

$$= \frac{4(109)(234)^2}{3} \left[ 1 - 0.9801 \right] - \frac{109(234)^2}{0.1154} \left[ 0.1154^3(0.9933) + \frac{0.1157}{2} - \frac{0.1154(0.9933)}{2} \right]$$

$$= 51.66(10^3) \text{ mm}^3. \quad (4.111)$$
Giving an overall modulus $W_{w,r}$ of

$$W_{w,r} = W_{w,r2} - W_{w,r1} = 9.100 \times 10^3 \text{ mm}^3. \quad (4.112)$$

The analytical bending resistance is therefore

$$\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm}}{\epsilon_y} \frac{W_{el,y}}{W_{pl,y}} - 1 \right) - \frac{W_{w,y}}{W_{pl,y}} \left( 1 - \frac{E_{sh}}{E} \right)$$

$$= 1 + 0.01 [9.256 (0.7109) - 1] - \frac{9.100}{2459} (1 - 0.01) = 1.052 \quad (4.113)$$

$$M_{csm,y} = 1.052 W_{pl,y} f_y = 1.052 (2459000)(355)/10^6 = 918 \text{ kNm}. \quad (4.114)$$
4.2.6 Plastic strain ratio

The majority of cross-sections available for engineering design are classified as class 1 or class 2, that is they are geometrically proportioned so that designers are able to use the plastic moment capacity of the cross-section. For $M_{csm,y} / M_{pl,y} = 1$, Eqn (4.28), which is valid for any cross-section shape, can be re-arranged to find the major axis plastic strain ratio $\epsilon_{pl,y}/\epsilon_y$, this is the strain ratio for which the CSM moment is equal to the plastic moment.

$$\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{pl,y} W_{el,y}}{\epsilon_y W_{pl,y}} - 1 \right) - \frac{W_{w,y}}{W_{pl,y}} \left( 1 - \frac{E_{sh}}{E} \right) = 1$$

$$\frac{\epsilon_{pl,y}}{\epsilon_y} = \frac{W_{w,y}}{W_{pl,y}} \left( 1 - \frac{E_{sh}}{E} \right) \frac{E}{E_{sh}} + 1 = \frac{W_{pl,y}}{W_{el,y}} \left[ 1 + \frac{W_{w,y}}{W_{pl,y}} \left( \frac{E}{E_{sh}} - 1 \right) \right]. \tag{4.115}$$

Likewise for the minor axis, the minor axis plastic strain ratio $\epsilon_{pl,z}/\epsilon_y$ is

$$\frac{\epsilon_{pl,z}}{\epsilon_y} = \frac{W_{pl,z}}{W_{el,z}} \left[ 1 + \frac{W_{w,z}}{W_{pl,z}} \left( \frac{E}{E_{sh}} - 1 \right) \right]. \tag{4.116}$$

These plastic strain ratios are plotted in Figure 4.33, and indicate that an approximately linear relationship exists with each cross-section shape factor (for the bi-linear stress–strain model, $E_{sh}/E = 1/100$ and for the cross-sections analysed). I-sections reach their theoretical $M_{pl,y}$ at major axis plastic strain ratios between 2.1 and 3.0, and require higher deformations to reach $M_{pl,z}$, where $\epsilon_{pl,z}/\epsilon_y$ is between 4.7 and 9.0. For box sections, which have a similar shape factor for both their major and minor axes, the plastic strain ratios $\epsilon_{pl,y}/\epsilon_y$ and $\epsilon_{pl,z}/\epsilon_y$ are between 2.8-3.7 and 2.3-3.1 respectively. The circular and elliptical hollow sections exhibit major axis plastic strain ratios that are generally between 3.5-4.0.
4.2.7 Test data

Experimental data were collected for simply supported beams that were not influenced by lateral torsional buckling; thus, they were either of short length or laterally restrained to offset any tendency to buckle globally. Hollow sections and I-sections bending about either the major or minor axis were sought, although only major axis bending data were found with information recorded on the cross-section geometry and material properties. The data were used to extract the maximum bending moments $M_u$ that the test specimens reached relative to their plastic moment capacity $M_{pl}$, so that the data could be plotted against the cross-section slenderness $\lambda_p$. Zhao and Hancock (1991) conducted four-point bending tests on cold-formed box sections, and Wilkinson and Hancock (1998) investigated cold-formed rectangular hollow sections subjected to four-point bending and using various methods to transfer the load from a spreader beam to the specimens. Byfield and Nethercot (1998) tested hot-rolled I-sections with closely spaced lateral restraints and subjected to
four-point bending. For these tests, as the cross-section geometry was measured at several locations, averaged values have been taken and an area weighted average of the yield and ultimate stresses for the web and flanges has been used. Gardner et al. (2010) tested square and rectangular hollow sections in three-point bending; half of the specimens were cold-formed and half hot-rolled. Coupon tests on flat and corner material showed a defined yield stress for the hot-rolled cross-sections, and a more rounded response from the cold-formed cross-sections. Gresnigt and Foeken (2001) tested 20 inch diameter circular pipes under four-point bending and recorded the maximum moment capacities and strains.

For box sections and I-sections, Figure 4.34a shows the ratio of the ultimate test moment to plastic moment capacity $M_{u,y}/M_{p,y}$, plotted against the cross-section slenderness. The results for the circular hollow sections in Figure 4.34b are too few to draw definitive conclusions, and so more data points are required. From the plotted I and box section bending test data in Figure 4.34a, it can be observed that any design model that limits the cross-section bending capacity to the plastic moment will be conservative, as stocky cross-sections can resist additional moments up to 30% beyond the plastic moment. Almost all of the box sections and I-sections attained moment capacities greater than the plastic moment, even those that are classified as class 3, where only attainment of the elastic moment is designated by EN 1993-1-1 (2005). The bi-linear moment–curvature model from Kemp et al. (2002) gives improved moment predictions for stocky cross-sections by offering a higher bending capacity than EN 1993-1-1 (2005), but the model is conservative for higher cross-section slenderness values where the transition from 0.9 to 1.0 of the plastic moment takes place. The Continuous Strength Method design equation (Eqn (4.76) from Section 4.2.3) provides additional moment carrying capacity with decreasing cross-section slenderness, which follows well the trend of the bending test data in Figure 4.34a. The plotted CSM curve is for a strain hardening modulus of $E_{sh} = E/100$ and for a shape-factor of 1.25, which is representative of the data set. The maximum strain ratio of $\epsilon_{csm}/\epsilon_y = 15$
gives a cap to the CSM moment of $\approx 1.11M_{pl,y}$ when the cross-section slenderness is below $\tilde{\lambda}_p < 0.32$. The shape of the proposed design curve is similar to the second simplified method from Wang (2011), but offers increased capacity. However both curves provide generally a safe estimate of the data.

Figure 4.34: Normalised ultimate test moments $M_{u,y}$ for structural steel beams, compared to resistance functions.

For the gathered test data on box sections and I-sections, Figure 4.35 shows the test moment capacity $M_{u,y}$ divided by the CSM predicted moment $M_{csm,y}$ and the EN 1993-1-1 (2005) capacity $M_{EC3,y}$. The mean and coefficient of variation (COV) statistics are summarised in Table 4.2, and are based on the ratio of ultimate test moment to predicted moment, such that a mean ratio less than unity is on average unsafe. In Table 4.2, $M_{W,y}$ and $M_{K,y}$ represent the predictions from Wang (2011) and Kemp et al. (2002) respectively. Comparing all design models, the CSM provides the mean closest to unity and with the lowest coefficient of variation. The comparisons highlight an improved prediction of capacity
with the proposed CSM design method for elements in bending over EN 1993-1-1 (2005), as the design equation conservatively predicts the ultimate moment and allows utilisation of the extra reserve attributed to strain hardening. These increases in cross-section capacity equate directly to material weight savings for beam elements in construction, and provide additional reserve for design situations where partial safety factors have been eroded by unforeseen circumstances (e.g. increases in external loading). The statistics provide a validation of the CSM model for box and I-sections, and are based on 67 tests.

![Graph](image)

Figure 4.35: Ultimate test moments $M_{u,y}$ compared to CSM and Eurocode 3 bending resistances ($M_{csm,y}$ and $M_{EC3,y}$) for box sections and I-sections.

Table 4.2: Statistics comparing ultimate test moments to the various design models.

<table>
<thead>
<tr>
<th>Shape</th>
<th>$M_{u,y}$/$M_{W,y}$</th>
<th>$M_{u,y}$/$M_{K,y}$</th>
<th>$M_{u,y}$/$M_{csm,y}$</th>
<th>$M_{u,y}$/$M_{EC3,y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.1198</td>
<td>1.1767</td>
<td>1.1123</td>
<td>1.1593</td>
</tr>
<tr>
<td>COV</td>
<td>0.0708</td>
<td>0.0725</td>
<td>0.0700</td>
<td>0.0812</td>
</tr>
</tbody>
</table>

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4.2.8 Flexural rigidity

The gradient of a cross-section moment–curvature curve gives the flexural rigidity; this is the first derivative of the moment–curvature function with respect to curvature. The flexural rigidity provides information on the ability of the cross-section to take additional bending based on its current elastic or inelastic state, and is a useful cross-section property, as will be shown in Chapter 7. Given the established relationships between the moment capacity of a cross-section and its strain ratio, and the equivalence of the strain ratio to the curvature ratio when there is no axial load, the flexural rigidity may be calculated from the analytical and design equations by using $E'$ as an effective Young’s modulus. The effective flexural rigidity $E' I$ is the product of $E'$ and the second moment of area $I$. It is expected from the shape of the moment–strain ratio curves that there will be a transition from the initial elastic stiffness where $E' I = EI$, to a reduced value as the gradient of the curves decreases following the initiation and spread of material yielding. Then $E' I$ will tend towards $E_{sh} I$ as more of the material yields and curvatures $\kappa$ increase; such $E' I – \kappa$ relationships provide the link between cross-section capacity and the deformed shape of a member, as curvatures are related to the second derivative of displacements.

4.2.8.1 Analytical expression

Recall the previous governing analytical bending equation for the major axis in the plastic moment normalised form of

$$\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh}}{E} \left( \frac{W_{el,y}}{W_{pl,y}} \epsilon_{csm} - 1 \right) - \frac{W_{w,y}}{W_{pl,y}} \left( 1 - \frac{E_{sh}}{E} \right). \tag{4.117}$$

For box sections and I-sections the $W_{w,y}$ term was found to consist of two parts, depending on whether the first yield distance $Y$ from the neutral axis was within the web or the flange,
With the equivalence of \(\frac{\epsilon_{\text{csm}}}{\epsilon_y} = \kappa_{\text{csm}}/\kappa_{y,y}\) for no axial load, one can state the relationship between moment and curvature as

\[
M_{\text{csm},y} = W_{\text{pl},y} f_y + \frac{E_{sh}}{E} f_y \left( W_{\text{el},y} \frac{\kappa_{\text{csm}}}{\kappa_{y,y}} - W_{\text{pl},y} \right) - W_{w,y} \left( 1 - \frac{E_{sh}}{E} \right) f_y.
\]

(4.119)

The effective flexural rigidity \(E' I_y\) is calculated by differentiating the CSM moment resistance function with respect to CSM curvature \(\kappa_{\text{csm}}\),

\[
\frac{dM_{\text{csm},y}}{d\kappa_{\text{csm}}} = \frac{E_{sh}}{E} f_y \frac{W_{\text{el},y}}{\kappa_{y,y}} - \frac{dW_{w,y}}{d\kappa_{\text{csm}}} \left( 1 - \frac{E_{sh}}{E} \right) f_y = E_{sh} f_y - \frac{dW_{w,y}}{d\kappa_{\text{csm}}} \left( 1 - \frac{E_{sh}}{E} \right) f_y.
\]

(4.120)

For \(1 + 2T/h \leq \kappa_{\text{csm}}/\kappa_{y,y}\) and recalling that \(\kappa_{y,y} = 2\epsilon_y/D\), the first \(W_{w,y}\) term in Eqn (4.44) (corresponding to fully yielded flanges) is differentiated to give

\[
\frac{d}{d\kappa_{\text{csm}}} \left[ \frac{tD^2}{12} \left( \frac{\kappa_{\text{csm}}}{\kappa_{y,y}} \right)^{-2} \right] = - \frac{tD^2}{6\kappa_{y,y}} \left( \frac{\kappa_{\text{csm}}}{\kappa_{y,y}} \right)^{-3} = - \frac{tD^3}{12} f_y \left( \frac{\kappa_{\text{csm}}}{\kappa_{y,y}} \right)^{-3}.
\]

(4.121)

Giving the effective flexural rigidity as

\[
E' I_y = E_{sh} I_y + (E - E_{sh}) \frac{tD^3}{12} \left( \frac{\kappa_{\text{csm}}}{\kappa_{y,y}} \right)^{-3}.
\]

(4.122)

This indicates that when the flanges have fully yielded, the effective flexural rigidity consists of \(E_{sh} I_y\) for the whole cross-section, plus an elastic portion remaining in the web, which is decaying rapidly with respect to the curvature ratio.
Differentiation of the second term in Eqn (4.44) for \(1 + 2T/h > \kappa_{csm}/\kappa_{y,y} \geq 1\) (for flanges partially yielded) gives

\[
\frac{d}{d\kappa_{csm}} \left[ \frac{BD^2}{12} \left( \frac{\kappa_{csm}}{\kappa_{y,y}} \right)^{-2} - \frac{(B-t)}{12} \frac{h^2}{D} \left( 3 - \frac{2h \kappa_{csm}}{D \kappa_{y,y}} \right) \right] = -\frac{BD^3}{12} \frac{E}{f_y} \left( \frac{\kappa_{csm}}{\kappa_{y,y}} \right)^{-3}
\]

\[
+ \frac{(B-t)}{12} \frac{h^2}{D} \frac{2h}{D \kappa_{y,y}} - \frac{BD^3}{12} \frac{E}{f_y} \left( \frac{\kappa_{csm}}{\kappa_{y,y}} \right)^{-3} + \frac{(B-t)h^3}{12} \frac{E}{f_y}.
\]

(4.123)

The effective flexural rigidity when the flanges have partially yielded and the web remains elastic is then

\[
E' I_y = E_{sh} I_y + \frac{(E - E_{sh})}{12} \left[ BD^3 \left( \frac{\kappa_{csm}}{\kappa_{y,y}} \right)^{-3} - (B-t)h^3 \right].
\]

(4.124)

For the minor axis, \(E' I_z\) is easily defined from the result of Eqn (4.120),

\[
E' I_z = E_{sh} I_z - \frac{dW_{w,z}}{d\kappa_{csm}} \left( 1 - \frac{E_{sh}}{E} \right) f_y.
\]

(4.125)

This is then paired with the \(W_{w,z}\) equations for I-sections, formulated previously as

\[
W_{w,z} = \begin{cases} 
\frac{DB^2}{12} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} & \epsilon_{csm} \geq \frac{B}{t} \\
\frac{t^2h}{2} \left( \frac{1}{2} - \frac{t}{3B} \frac{\epsilon_{csm}}{\epsilon_y} \right) + \frac{TB^2}{6} \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-2} & \epsilon_{csm} \geq \frac{B}{t}.
\end{cases}
\]

(4.126)

With the equivalence of \(\epsilon_{csm}/\epsilon_y = \kappa_{csm}/\kappa_{y,z}\), using \(\kappa_{y,z} = 2\epsilon_y/B\), and for \(\kappa_{csm}/\kappa_{y,z} \geq B/t\)

\[
\frac{d}{d\kappa_{csm}} \left[ \frac{DB^2}{12} \left( \frac{\kappa_{csm}}{\kappa_{y,z}} \right)^{-2} \right] = -\frac{BD^3}{12} \frac{E}{f_y} \left( \frac{\kappa_{csm}}{\kappa_{y,z}} \right)^{-3}
\]

(4.127)
which gives an effective flexural rigidity that corresponds to a cross-section that has fully yielded except for a portion of the web,

$$E'I_z = E_{sh}I_z + (E - E_{sh}) \frac{DB^3}{12} \left( \frac{\kappa_{csm}}{\kappa_{y,z}} \right)^{-3}.$$  \hspace{1cm} (4.128)

When $\kappa_{csm}/\kappa_{y,z} < B/t$, the minor axis yield distance $Z$ is outside of the web and

$$\frac{d}{d\kappa_{csm}} \left[ \frac{t^2 h}{2} \left( \frac{1}{2} - \frac{t}{3B} \frac{\kappa_{csm}}{\kappa_{y,z}} \right) + \frac{TB^2}{6} \left( \frac{\kappa_{csm}}{\kappa_{y,z}} \right)^2 \right] = -\frac{ht^3}{6B\kappa_{y,z}} - 2\frac{TB^3}{6\kappa_{y,z}} \left( \frac{\kappa_{csm}}{\kappa_{y,z}} \right)^{-3}$$

$$= -\frac{E}{f_y} \frac{ht^3}{12} - 2\frac{E}{f_y} \frac{TB^3}{12} \left( \frac{\kappa_{csm}}{\kappa_{y,z}} \right)^{-3}.$$  \hspace{1cm} (4.129)

$E'I_z$ then represents an elastic cross-section with yielding of the flange tips

$$E'I_z = E_{sh}I_z + (E - E_{sh}) \left[ \frac{ht^3}{12} + 2\frac{TB^3}{12} \left( \frac{\kappa_{csm}}{\kappa_{y,z}} \right)^{-3} \right].$$  \hspace{1cm} (4.130)

### 4.2.8.2 Design expression

From the design equation for major or minor axis bending, approximated $W_{w,y}$ and $W_{w,z}$ were derived in Section 4.2.3. For the major axis, the design equation is of the form

$$\frac{M_{csm,y}}{M_{pl,y}} = 1 + \frac{E_{sh} W_{el,y}}{E W_{pl,y}} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right) - \left( 1 - \frac{W_{el,y}}{W_{pl,y}} \right) \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^{-\alpha}.$$  \hspace{1cm} (4.131)

With the equivalence of the strain ratio $\epsilon_{csm}/\epsilon_y$ and the curvature ratio $\kappa_{csm}/\kappa_{y,y}$ for bending without axial load, the CSM major axis bending moment as a function of curvature is given by
\[ M_{\text{cs}, y} = W_{\text{pl}, y}f_y + \frac{E_{\text{sh}}}{E}W_{\text{el}, y}f_y \left( \frac{\kappa_{\text{cs}, y}}{\kappa_{y,y}} - 1 \right) - f_y (W_{\text{pl}, y} - W_{\text{el}, y}) \left( \frac{\kappa_{\text{cs}, y}}{\kappa_{y,y}} \right)^{-\alpha}. \quad (4.132) \]

The effective flexural rigidity \( E'I_y \) is calculated by differentiating with respect to curvature

\[
E'I_y = \frac{dM_{\text{cs}, y}}{d\kappa_{\text{cs}, y}} = \frac{E_{\text{sh}}}{E} W_{\text{el}, y}f_y + f_y (W_{\text{pl}, y} - W_{\text{el}, y}) \alpha \left( \frac{\kappa_{\text{cs}, y}}{\kappa_{y,y}} \right)^{-\alpha-1} 
\]

\[
= \frac{E_{\text{sh}} I_y}{E} \epsilon_y + \frac{DE_{\text{sh}}}{2\epsilon_y} (W_{\text{pl}, y} - W_{\text{el}, y}) \alpha \left( \frac{\kappa_{y,y}}{\kappa_{\text{cs}, y}} \right)^{\alpha+1} 
\]

\[
= E_{\text{sh}} I_y + EI_y \left( \frac{W_{\text{pl}, y}}{W_{\text{el}, y}} - 1 \right) \alpha \left( \frac{\kappa_{y,y}}{\kappa_{\text{cs}, y}} \right)^{\alpha+1} \quad (4.133) 
\]

which is valid for \( \kappa_{\text{cs}, y}/\kappa_{y,y} \geq 1 \). For I-sections and hollow sections bending about the major axis, \( \alpha \) is taken as 2. For the minor axis the equation follows the same form but has a change of subscript between \( y \) and \( z \) and uses \( \alpha = 1.2 \) for I-sections; \( \alpha \) is kept as 2 for hollow sections. Comparisons between the analytical and design effective flexural rigidity predictions of an I-section with \( E_{\text{sh}} = E/100 \), are displayed in Figure 4.36 along with the corresponding plastic strain ratio of the cross-section, which is the strain ratio when \( M_{\text{cs}} = M_{\text{pl}} \). For curvatures below the yield curvatures \( \kappa_{y,y} \) and \( \kappa_{y,z} \), the effective flexural rigidity \( E'I \) remains equal to the elastic flexural rigidity \( EI \), as the cross-section is still fully elastic. Immediately after yielding there is a rapid decay in stiffness for the major axis and a more gradual loss of stiffness for the minor axis. The transition from \( EI \) towards \( E_{\text{sh}}I \) is more rapid for cross-sections with a lower shape factor; hence the more rounded shape of the minor axis curves. Theoretically a step would exist at a curvature ratio of 1 if the shape factor was unity, indicating an instant loss of all stiffness reserve above \( E_{\text{sh}}I \). The design curve shows good conformance with the analytical curve for both
bending axes, with slightly more deviation for the minor axis, highlighting the compromise in the \( W_{w,z} \) term by its representation with \( \alpha = 1.2 \), as seen previously when comparing to numerical results in Figure 4.19.

![Figure 4.36: Effective flexural rigidity of an I-section relative to its elastic value and plotted against the curvature (strain) ratio.](image)

The effective flexural rigidity may also be plotted against the area of the cross-section that has yielded \( A_y \) divided by the gross area \( A \). The yielded area is calculated by locating the elements in the numerical model that have their centroidal strains greater than the yield strain \( (\epsilon_i \geq \epsilon_y) \). For the I-section in Figure 4.37, the curves show that for the major axis, 90% of the original elastic stiffness \( EI_y \) has been lost when \( \approx 70\% \) of the gross area \( A \) has yielded, and 90% of \( EI_z \) is lost by the time \( A_y/A \approx 0.35 \). This distinction is due to the difference in the areas that are contributing the most to the second moment of area; which are the flanges for \( I_y \) and just the flange tips for \( I_z \).
4.2.9 Summary

A new model for bending has been presented which uses a linearly varying strain distribution that is controlled by a limiting strain at the extreme outer-fibres. This limiting strain is obtained from the CSM base curve as a function of the slenderness of the cross-section in flexure. A numerical method was employed to assess moment resistance equations for a variety of cross-section shapes bending about either the major or minor axis.

Analytical equations were developed based on a simple governing equation, with special attention given to a web modulus term. The additional modulus terms were approximated into a strain ratio decaying term to form a design equation. In general, the equation was conservative for major axis bending with an error term that decreased quickly as the strain ratio increased.
By comparing to the gathered test data, the Eurocode 3 methodology was shown to be conservative, particularly for cross-sections of low slenderness. The CSM design equation is recommended for use as a design alternative, as it offered improved moment predictions and with a reduction in scatter when compared with test data, thus reducing the conservative margin.

With the equivalence of the strain ratio and the curvature ratio for bending without an axial load, flexural rigidity–curvature curves were created from the CSM analytical and design moment equations.
Chapter 5

Combined axial load and bending

As the CSM base curves from Section 3.7 are based on both axial and bending test data, the cross-section strain ratio may be used for all combined loading cases. Cross-section resistance under combined loading is investigated in this chapter.

5.1 Axial load and uni-axial bending

The traditional approach for calculating the capacity of a stocky cross-section subjected to an axial load and bending about one axis, is the infinite strain, perfectly plastic model, depicted in Figure 5.1 for the major axis of an I-section. The perfectly plastic assumption means no exploitation of strain hardening and stresses limited to the yield stress $f_y$. The discontinuity of stresses at a distance $y = Y$ can only be described as the limiting case of a cross-section at infinite curvature. This model is an idealisation of the true behaviour, as in reality there will be some potential for the material to strain harden as well as a gradual change from $-f_y$ to $+f_y$ over a finite distance around $y = Y$. This traditional method of analysis is stress based, as the stress distribution is initially prescribed throughout the cross-section, and without strain appearing as an explicit variable. However, this traditional model in Figure 5.1 is simple and relatively intuitive. It will be seen in Section 5.1.1 that a strain based model provides a closer description of the structural mechanics.
When \( Y \) is located within the web as in Figure 5.1a, the equilibrium of the compressive forces \( C \), tensile force \( T_1 \) and applied axial load \( N \) is given by

\[
N = C_1 + C_2 + C_3 - T_1 = f_y \left[ BT + \left( \frac{h}{2} + Y \right) t - BT - \left( \frac{h}{2} - Y \right) t \right] = 2f_y Y t. \tag{5.1}
\]

This indicates that the axial load is taken solely by a portion of the web, and the higher the axial load, the greater the extent of web area used. The moments about \( y = 0 \) caused by \( C_1 \) and \( C_2 \) are of equal magnitude and of opposite sense and cancel each other out. The cross-section moment resistance is given by the couple produced from \( C_3 \) and \( T_1 \),

\[
M = BT (D - T) f_y + \left( \frac{h}{2} - Y \right) t \left[ 2Y + \left( \frac{h}{2} - Y \right) \right] f_y. \tag{5.2}
\]

When \( Y \) is located within the flanges, as shown in Figure 5.1b, the distance \( Y \) moves from the web and to the flanges to allow the cross-section to be in equilibrium,
\[ N = f_y \left[ BT + h t + B \left( Y - \frac{h}{2} \right) - B \left( \frac{D}{2} - Y \right) \right] = f_y [h t + B (2Y - h)]. \] (5.3)

The moment resistance is now reduced to the remaining reserve in the flanges as

\[ M = B \left( \frac{D}{2} - Y \right) \left[ 2Y + \left( \frac{D}{2} - Y \right) \right] f_y. \] (5.4)

Figure 5.2a plots the interaction curves that are formed from these equations for a typical I-section, for the axial load and major axis bending moment interaction. Figure 5.2b shows equivalent perfectly plastic expressions for the same axially loaded cross-section bending about the minor axis. Both of these curve sets are plotted alongside the results obtained from a strain based numerical model described in Section 5.1.2 with no strain hardening and for strain ratios of 1, 3 and 15, where it is seen that with increasing deformations the simplified traditional case is approached.

Figure 5.2: Traditional perfectly plastic and numerical model interaction curves for axial load and uni-axial bending (SR is the strain ratio).
The infinite strain assumption without the incorporation of strain hardening is a limiting case that can never be reached, as a cross-section will be limited by finite strains, local buckling and material fracture. It is often implicitly assumed, such as the case of a cross-section reaching its plastic moment $M_{pl}$, that strain hardening will account for the additional strength required to reach the idealised capacities, although it is seen that with a strain ratio of 15, the numerical results are very close to the simplified model.

5.1.1 Strain based model

In the same way that the CSM limits a cross-section in bending or compression to its strain ratio $\epsilon_{csm}/\epsilon_y$ (described in Chapter 4), the cross-section combination of an axial load and bending moment is also limited to the outer-fibres reaching the CSM limiting strain $\epsilon_{csm}$. Omitting the effects of global flexural buckling, the interaction of axial strains and major axis bending strains is taken as the sum of uniform strains $\epsilon_A$ and linearly varying strains with maximum magnitude $\epsilon_B$. This combination leads to distinct strain and stress profiles based on the state of strain at the lower outer-fibres, as displayed in Figure 5.3 to Figure 5.5.

For the case of simple bending, as shown in Figure 5.3a with no uniform strain $\epsilon_A = 0$, the cross-section is acting in bending only and can be described with the derived analytical or design equations for $M_{csm}$ in Section 4.2. The strain and stress profiles are antisymmetric about the zero strain neutral axis (which is located at mid-depth for symmetric sections), and the upper and lower outer-fibres reach $\pm f_{csm}$ and $\pm \epsilon_{csm}$. In Figure 5.3b, when the uniform strain $\epsilon_A$ is low, such as with a bending dominated loading case, the lower outer-fibres are beyond the tensile yield strain and develop strain hardening. The zero strain neutral axis is near the middle of the cross-section but is below the cross-section centroid.
As $\epsilon_A$ increases relative to $\epsilon_B$, tensile strains at the lower outer-fibres occur in the elastic region between $\epsilon = -\epsilon_y$ and $\epsilon = 0$ (Figure 5.4a). The neutral axis is pushed down the cross-section as it becomes dominated by compressive strains, and the region of plasticity grows on the compressive side. In Figure 5.4b, further increasing $\epsilon_A$ creates a cross-section that is completely in compression and with significant yielding and loss of stiffness. A bending moment can still be resisted and is based upon the compressive stresses and lever arm to the centroidal axis. The lower outer-fibre strains are within the compressive elastic range between $\epsilon = 0$ and $\epsilon = \epsilon_y$, and there is no longer a zero strain neutral axis within the depth of the cross-section.
In Figure 5.5a, for a low applied or induced moment and $\epsilon_{csm}/\epsilon_y$ significantly greater than unity, a state can be reached where the entire cross-section has yielded and the axial stiffness is at a minimum. All parts of the cross-section are within the material strain hardening region of the stress–strain curve where $\epsilon > \epsilon_y$. For the case of pure compression, a uniform stress of $f_{csm}$ will occur throughout the cross-section at a strain of $\epsilon_{csm}$, producing an axial resistance of $N_{csm}$ (Figure 5.5b).

![Diagram](image)

(a) Compression plastic  
(b) Simple compression

Figure 5.5: High axial states for axial load and uni-axial bending.

With the strains and stresses established, the co-existing axial load $N$ and uni-axial bending resistance $M_y$ or $M_z$ follow by integrating the stresses $f$ over the cross-section area $A$ with lever arm $y$ or $z$. The distances $y$ and $z$ are taken about the centre of curvature, which for symmetric cross-sections is about the major axis $y$–$y$ or minor axis $z$–$z$,

$$
N = \int_A f \, dA \quad M_y = \int_A f_y \, dA \quad M_z = \int_A f_z \, dA.
$$

Although strains $\epsilon_A$ and $\epsilon_B$ are linearly superimposed, the stresses are based on a bi-linear material model. Therefore $\epsilon_A$ is not solely responsible for defining the axial force and neither is $\epsilon_B$ exclusive to bending; it is the combination of $\epsilon_A$ and $\epsilon_B$ that defines the axial and bending capacity.
5.1.2 Numerical model

Although it is possible to derive exact expressions for the combined axial and uni-axial capacity of a cross-section, the results are lengthy and of limited practical use. Two stress states are shown in Figure 5.6 for an I-section, where discretising these stress profiles into nine rectangular and triangular areas, so that forces and moments can be summed, means that an explicit analytical expression will involve at least nine terms.

![Figure 5.6: Explicit discretisation of the cross-section area of an I-section.](image)

A numerical method is therefore developed to overcome the difficulties in obtaining simple analytical solutions to the interaction of an axial load and bending moment. Figure 5.7 shows the key components of the numerical model, which is based on normalising a plane strain distribution by the CSM limiting strain $\varepsilon_{cs}\text{m}$. For a cross-section under major axis bending, all strain interactions are found for which the sum of the uniform strain $\varepsilon_A$ and the maximum linear strain $\varepsilon_B$ equal $\varepsilon_{cs}\text{m}$. This then defines the failure criterion of Eqn (5.6), which is normalised by the CSM limiting strain as

$$\frac{\varepsilon_A}{\varepsilon_{cs}\text{m}} + \frac{\varepsilon_B}{\varepsilon_{cs}\text{m}} = 1. \quad (5.6)$$

The normalised uniform strain $\varepsilon_A/\varepsilon_{cs}\text{m}$ is varied from 0 to 1, indicating simple bending and simple axial loading respectively, and leaving $\varepsilon_B/\varepsilon_{cs}\text{m} = 1 - \varepsilon_A/\varepsilon_{cs}\text{m}$. 

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For the major axis with cross-section depth $D$, the CSM normalised strain at element $i$ is

$$\frac{\epsilon_i}{\epsilon_{csm}} = \frac{\epsilon_A}{\epsilon_{csm}} + \left(1 - \frac{\epsilon_A}{\epsilon_{csm}}\right) \frac{2y_i}{D}. \quad (5.7)$$

This result is easily extended to axial load and minor axis bending as

$$\frac{\epsilon_i}{\epsilon_{csm}} = \frac{\epsilon_A}{\epsilon_{csm}} + \left(1 - \frac{\epsilon_A}{\epsilon_{csm}}\right) \frac{2z_i}{B}. \quad (5.8)$$

For both bending axes the element centroid is at a distance $y_i$ or $z_i$ from the centroidal axes of the cross-section, which correspond to the lines of symmetry for symmetric cross-sections. For the bi-linear material model, the normalised element stress $f_i$ is

$$\frac{f_i}{f_y} = \frac{E\epsilon_i}{E\epsilon_y} = \frac{\epsilon_i}{\epsilon_y} \frac{\epsilon_{csm}}{\epsilon_y} \quad \epsilon_i \leq \epsilon_y \quad (5.9)$$

$$\frac{f_i}{f_y} = \frac{f_y + (\epsilon_i - \epsilon_y)E_{sh}}{f_y} = 1 + \frac{E_{sh}}{E} \left(\frac{\epsilon_i}{\epsilon_{csm}} - 1\right) \quad \epsilon_i > \epsilon_y \quad (5.10)$$
5.1.2.1 I-sections and box sections

With the elemental strains and stresses defined, the cross-section is divided into $n$ thin strips of area $A_i = B_i t_i$ (for the major axis) and keeping the element thickness $t_i = D/n$ as constant. The cross-section bending resistances are then calculated from summing the elemental area $\times$ lever arm $\times$ centroidal stress. Hence, the normalised axial load, major axis bending and minor axis bending resistances are defined as

$$\text{Axial load} \quad \frac{N}{N_y} = \int_A \frac{f}{f_y A} \, dA = \frac{t_i}{A} \sum_i \frac{f_i}{f_y} B_i \quad (5.11)$$

$$\text{Major axis bending} \quad \frac{M_y}{M_{pl,y}} = \int_A \frac{f_y}{f_y W_{pl,y}} \, dA = \frac{t_i}{W_{pl,y}} \sum_i \frac{f_i}{f_y} y_i B_i \quad (5.12)$$

$$\text{Minor axis bending} \quad \frac{M_z}{M_{pl,z}} = \int_A \frac{f_z}{f_y W_{pl,z}} \, dA = \frac{t_i}{W_{pl,z}} \sum_i \frac{f_i}{f_y} z_i D_i \quad (5.13)$$

5.1.2.2 Circular and elliptical hollow sections

The stress and strain profiles remain the same for circular and elliptical hollow sections; the only adjustments needed are altered forms of calculating the cross-section resistances by discretisation in both the radial $r$ and angular $\theta$ directions. Using the notation from the simple uni-axial bending numerical model (Section 4.2.1), the normalised axial and bending resistances for a circular hollow section are

$$\frac{N}{N_y} = \int_{r_1}^{r_2} \int_0^{2\pi} \frac{f}{f_y A} r \, d\theta \, dr = \frac{1}{A} \sum_i \frac{f_i}{f_y} r_i r' \quad (5.14)$$
\[
\frac{M}{M_{pl}} = \int_{r_1}^{r_2} \int_{0}^{2\pi} \frac{f r^2 \sin \theta}{f_y W_{pl}} \, d\theta \, dr = \frac{1}{W_{pl}} \sum_{i} f_{i} r^2 \sin \theta_{i} \, r' . \tag{5.15}
\]

For an elliptical hollow section the radius \( r \) is not constant, and the general elemental area \( A_i \) from Section 4.2.1 must be used,

\[
\frac{N}{N_y} = \int_{r_1}^{r_2} \int_{0}^{2\pi} \frac{f}{f_y A} \, dA = \frac{1}{A} \sum_{i} f_{i} A_i \tag{5.16}
\]

\[
\frac{M_y}{M_{pl, y}} = \int_{r_1}^{r_2} \int_{0}^{2\pi} \frac{f r \sin \theta}{f_y W_{pl, y}} \, dA = \frac{1}{W_{pl, y}} \sum_{i} f_{i} \frac{ab \sin \theta}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} A_i \tag{5.17}
\]

\[
\frac{M_z}{M_{pl, z}} = \int_{r_1}^{r_2} \int_{0}^{2\pi} \frac{f r \cos \theta}{f_y W_{pl, z}} \, dA = \frac{1}{W_{pl, z}} \sum_{i} f_{i} \frac{a b \cos \theta}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} A_i . \tag{5.18}
\]

The CSM normalised axial and bending resistances are found by converting the yield and plastic normalised results to form

\[
\frac{N}{N_{csm}} = \frac{N}{N_y} \frac{N_{csm}}{N_y} \quad \frac{M_y}{M_{csm, y}} = \frac{M_y}{M_{pl, y}} \frac{M_{csm, y}}{M_{pl, y}} \quad \frac{M_z}{M_{csm, z}} = \frac{M_z}{M_{pl, z}} \frac{M_{csm, z}}{M_{pl, z}} . \tag{5.19}
\]

The numerical methods were implemented in MATLAB (2012) for all of the considered SCI P363 (2009) cross-sections, with each cross-section divided into thousands of elements and subjected to the complete range of axial and bending combinations.
5.1.3 Interaction curves

The numerical model developed in Section 5.1.2 can be used to plot every interaction of axial load and bending moment for any strain ratio, from the bending only state to the pure axial state. The input data required to run each numerical procedure are the cross-section geometry, strain ratio and stress–strain curve, for which the design bi-linear material model from Section 3.3.2 is used.

5.1.3.1 Axial load and uni-axial bending (plastic normalised)

The plastic moment and yield load normalised interaction curves plotted in Figure 5.8 to Figure 5.12, are for representative cross-sections of each shape at a strain ratio of \( \epsilon_{csm}/\epsilon_y = 15 \). These figures show the increased loading that a cross-section can withstand by introducing moderate strain hardening. With no strain hardening \( (E_{sh} = 0) \), the yield load \( N_y \) and the plastic moment \( M_{pl,y} \) or \( M_{pl,z} \) are the maximum cross-section resistances, where with \( E_{sh} = E/100 \), an additional 10% in resistance is seen from the more expanded interaction curves. Also plotted on these figures are the bi-linear and parabolic–linear EN 1993-1-1 (2005) interaction curves, except for the elliptical hollow section in Figure 5.10b where no design guidance is provided. For the I-sections and box sections in Figure 5.8 and Figure 5.9 which are bending about the major axis, the EN 1993-1-1 (2005) equations generally compare well to the numerically derived curves with \( E_{sh} = 0 \). The assumption that an overestimate of cross-section capacity for high moments can be taken by strain hardening is justified, as the design curves lie below the \( E_{sh} = E/100 \) curves.

For the major axis, the curves are more rounded for box sections than for I-sections, with the UC cross-section displaying an almost linear interaction and the RHS cross-section curves expanding furthest from the origin. The circular and elliptical hollow sections behave similarly in Figure 5.10, with the continuous EN 1993-1-1 (2005) design fit providing an excellent match to the circular hollow section interaction curve when \( E_{sh} = 0 \).
Figure 5.8: Axial load and major axis bending interaction curves for I-sections, normalised by the plastic cross-section capacities.

Figure 5.9: Axial load and major axis bending interaction curves for box sections, normalised by the plastic cross-section capacities.
Figure 5.10: Axial load and major axis bending interaction curves circular and elliptical hollow sections, normalised by the plastic cross-section capacities.

Figure 5.11: I-section axial load and minor axis bending interaction curves normalised by the plastic cross-section capacities.
The axial load and minor axis bending interaction curves for the same UB, UC, RHS and EHS cross-sections are plotted in Figure 5.11 and Figure 5.12. Increases in the cross-section resistance of 10% can be seen in the interaction curves when strain hardening is included. The I-sections in Figure 5.11 perform notably different when bending about the minor axis due to the increased cross-section shape factor and ability to withstand axial loads, this produces a more expanded interaction curve that is dominated by the tension plastic state (corresponding to high moments). The enhanced ability of axial loads to be taken by the web is evident, as the web produces little flexural benefit for the minor axis bending of I-sections. The EN 1993-1-1 (2005) design predictions are accurate, but they appear to slightly over-predict if $E_{sh} = 0$; this is most apparent for the UB where it is assumed that the full plastic moment can be resisted for a range of axial loads. The rectangular hollow sections behave similarly about the minor axis as to the major axis.
### 5.1.3.2 Axial load and uni-axial bending (CSM normalised)

Instead of plotting the axial load and bending moment interactions for every strain ratio, as a suite of interaction curves normalised by the yield load \( N_y \) and the plastic moment \( M_{pl,y} \) or \( M_{pl,z} \), each curve can be anchored to the CSM axial and bending resistances \( N_{csm} \) and \( M_{csm,y} \) or \( M_{csm,z} \). This has been performed for the major axis in Figure 5.13 to Figure 5.15 and for the minor axis in Figure 5.16 and Figure 5.17, which show the linear interaction at a strain ratio of 1, intermediate interaction curves with strain ratios of 3 and 5, and one curve with the maximum strain ratio of 15 (these are labelled respectively as SR=1, SR=3, SR=5 and SR=15). The interaction curves are plotted for a strain hardening modulus of \( E_{sh} = E/100 \) and for the same cross-sections plotted in the \( N_y, M_{pl,y} \) and \( M_{pl,z} \) normalised curves. For all of these interaction curves the greatest change in the curve shape occurs for low strain ratios from SR=1 to SR=3 or SR=1 to SR=5. Beyond this and up to SR=15 the interaction curves become generally more rounded and less variable in shape.

![Interaction Curves](image)

(a) UB axial load and major axis bending  
(b) UC axial load and major axis bending

Figure 5.13: Axial load and major axis bending interaction curves for I-sections, normalised by the CSM cross-section capacities.
Figure 5.14: Axial load and major axis bending interaction curves for box sections, normalised by the CSM cross-section capacities.

Figure 5.15: Axial load and major axis bending interaction curves for circular and elliptical hollow sections, normalised by the CSM cross-section capacities.
Figure 5.16: Axial load and minor axis bending interaction curves for I-sections, normalised by the CSM cross-section capacities.

(a) UB axial load and minor axis bending  
(b) UC axial load and minor axis bending

Figure 5.17: Axial load and minor axis bending interaction curves for RHS and EHS sections, normalised by the CSM cross-section capacities.

(a) RHS axial load and minor axis bending  
(b) EHS axial load and minor axis bending
5.1.4 Cross-section slenderness

Along the axial load and uni-axial bending interaction curves, there are a set of strain and stress distributions as distinguished by the tension elastic, tension plastic, compression elastic and compression plastic states (Section 5.1.1). To determine the relevant cross-section slenderness $\bar{\lambda}_p = \sqrt{f_y/f_{cr}}$ to use, the stress distribution selected in the determination of the elastic critical buckling stress $f_{cr}$ should be representative of the applied design axial and flexural loading $N_{Ed}$ and $M_{Ed}$. With the correct cross-section stress profile, the buckling coefficients $k_\sigma$ can be found using the methodology in EN 1993-1-5 (2006). The $k_\sigma$ values will thus take account of both tensile stresses and compressive stresses.

For the projection from the origin to the design loading state represented by the point with co-ordinates $(M_{Ed}/M_{el}, N_{Ed}/N_y)$, the straight line plotted in Figure 5.18a follows

$$\frac{N}{N_y} = \left( \frac{N_{Ed}}{N_y} \frac{M_{el}}{M_{Ed}} \right) \frac{M}{M_{el}}.$$  (5.20)

This line intersects the linear elastic interaction at the point $(M'/M_{el}, N'/N_y)$, which can be expressed as

$$\left( \frac{N_{Ed}}{N_y} \frac{M_{el}}{M_{Ed}} \right) \frac{M'}{M_{el}} + \frac{M'}{M_{el}} = 1$$  (5.21)

giving intersection co-ordinates of

$$\frac{M'}{M_{el}} = \frac{1}{1 + \frac{N_{Ed}}{N_y} \frac{M_{el}}{M_{Ed}}} \quad \text{and} \quad \frac{N'}{N_y} = 1 - \frac{M'}{M_{el}}.$$  (5.22)

The corresponding stress distribution at this elastic intersection state is the addition of uniform axial stresses and linearly varying elastic bending stresses. Using $y$ as the distance from the geometric axis for major axis bending, the stress at $y$ is
\[ f = \frac{N'}{A} + \frac{M'y}{I_y} \]  

(5.23)

where \( A \) is the cross-section area and \( I_y \) is the major axis second moment of area. Using the substitution \( M_{el,y} = 2I_yf_y/D \), will give the yield normalised stress distribution Eqn (5.24).

This can be adjusted with \( M_{el,z} = 2I_zf_y/B \) for the minor axis,

\[ \frac{f}{f_y} = \frac{N'}{N_y} + \frac{M'}{M_{el,y}} \left( \frac{2y}{D} - 1 \right) = 1 + \left( \frac{2y}{D} - 1 \right) \frac{M'}{M_{el,y}} = 1 + \frac{2y}{D} - 1 + \frac{N_{Ed}M_{el,y}}{N_yM_{Ed}}. \]  

(5.24)

At the compressive outer-fibres located at \( y = D/2 \) (\( z = B/2 \) for the minor axis) the stresses will be \( f = f_1 = f_y \), which corresponds to \( \psi = 1 \) in the EN 1993-1-5 (2006) method of determining buckling coefficients \( k_x \). For the variation in \( y \) or \( z \) for internal
elements, the stresses $f_2$ at the outer-fibres located at $y = -D/2$ (or $z = -B/2$) will depend on the relative sizes of the axial load and bending moment,

$$\frac{f_2}{f_y} = \frac{f_2}{f_1} = 1 - \frac{2}{1 + \frac{N_{Ed}}{N_y} \frac{M_{el}}{M_{Ed}}}.$$  \hfill (5.25)

For the internal web elements of I-sections and box sections bending about the major axis, the equation above for $f/f_y$ can be used for $\psi$ in EN 1993-1-5 (2006), and for I-sections bending about the minor axis, the stress at the centroidal axis ($z = 0$) will be half of $1 + f/f_y$ (i.e. midway between the two flange tips). This single equation allows a designer to designate the elastic stress distribution of the entire cross-section, thus allowing the cross-section slenderness to be calculated for any general design loading case. Two elastic stress distribution examples are displayed in Figure 5.18b for an I-section orientated about its major axis with $N_{Ed}/N_y = 0.4$ and $M_{Ed}/M_{el,y} = 0.9$, and also about its minor axis with $N_{Ed}/N_y = 0.8$ and $M_{Ed}/M_{el,z} = 0.3$. The corresponding bottom outer-fibre stresses are

$$\frac{f_2}{f_1} = 1 - \frac{2}{1 + \frac{0.4}{0.9}} = -0.385 \quad \text{and} \quad \frac{f_2}{f_1} = 1 - \frac{2}{1 + \frac{0.8}{0.3}} = 0.455.$$  \hfill (5.26)

### 5.1.5 Design expression

When investigating a suitable design expression for the interaction curves, it is important to retain as much of the curve shape as possible whilst still providing a simple and continuous expression. The EN 1993-1-1 (2005) axial load and moment interaction equations are either piecewise linear–linear or linear–parabolic (Section 2.6), and gave over-estimates and constant moment values for different axial loads when compared to the numerical model in Section 5.1.3. The following equation form in Eqn (5.27) and Eqn (5.28) is a relationship that is conveniently bound to points representing an axial load of $N = N_{csn}$ and a bending moment $M = M_{csn}$. For the major axis, the design expression is
and the axial load and minor axis bending expression is

\[
\left( \frac{N}{N_{csm}} \right)^{a_y} + \left( \frac{M_y}{M_{csm,y}} \right)^{b_y} \leq 1
\]

(5.27)

and the axial load and minor axis bending expression is

\[
\left( \frac{N}{N_{csm}} \right)^{a_z} + \left( \frac{M_z}{M_{csm,z}} \right)^{b_z} \leq 1.
\]

(5.28)

The values of the exponents \(a_y, a_z\) and \(b_y, b_z\) can be varied, resulting in both rounded and straight curves, and produces interaction curves that are smooth and continuous between the two CSM anchor points. Figure 5.19 shows for the major axis, how altering one of these exponents whilst the other exponent is kept at unity, provides a high degree of control in the curve shape. With a function form found, the non-linear least squares fits that pass through the numerical cross-section interaction curves can be found in order to determine the optimum exponents.

![Figure 5.19: Variation of the \(a_y\) and \(b_y\) exponents in the design expression.](image)

(a) Varying \(a_y\) with \(b_y = 1\)

(b) Varying \(b_y\) with \(a_y = 1\)
5.1.6 Exponents $a_y$, $a_z$, $b_y$ and $b_z$

The optimum $a_y$, $a_z$, $b_y$ and $b_z$ exponents for the proposed CSM design equation form have been plotted for each cross-section and for bending about each axis. These values derive from a curve fitting routine coded in MATLAB (2012) for any strain ratio. Plotted in Figure 5.20 are the optimum $a_y$ and $b_y$ exponents for box sections bending about either axis ($a_z$ and $b_z$ for the minor axis) and I-sections bending about the major axis, for strain ratios between 1 and 15 (SR=1 and SR=15) and with $E_{sh} = E/100$.

![Figure 5.20: I-sections (major axis) and box sections (major and minor axes) optimum $a_y$ and $b_y$ exponents.](image)

Also plotted in Figure 5.20a is Eqn (5.29), a linear relationship between $a_y$ and the ratio of web area to gross area $A_w/A = ht/A$,

$$a_y = \frac{A_w}{A} + 1.2. \quad (5.29)$$
For box sections bending about the minor axis the web height \( h = B - t \), with the web thickness \( t \) taken as twice the wall thickness \( T \). Cross-sections with greater \( A_w/A \) ratios can absorb higher axial loads in the web, leading to more expanded interaction curves. The optimum \( b_y \) exponent is common throughout, taking values between 0.75 and 0.90 and with an average of approximately \( b_y = 0.82 \), which is less than 1.0 and so causes an inwards bend for the interaction curves towards the origin. The grouping of similar interaction behaviour for strain ratios of 3 and above allows simpler expressions to be used.

Figure 5.21 shows that for circular hollow sections both \( a_y \) and \( b_y \) vary slightly with \( D/t \), taking values of approximately 1.8 and 0.82 for the strain ratio range \( 3 \leq \epsilon_{csm}/\epsilon_y \leq 15 \). Figure 5.22 shows similar behaviour for elliptical hollow sections but with slightly higher \( a_y \) values, with \( a_y = 1.95 \) selected with \( b_y = 0.82 \). This pairing of \( a_y \) and \( b_y \) exponents corresponds only to elliptical hollow sections with an aspect ratio of \( D_b/D_a = 2 \).

Figure 5.21: Circular hollow section optimum \( a_y \) and \( b_y \) exponents.
Figure 5.22: Elliptical hollow section (major axis) optimum $a_y$ and $b_y$ exponents.

Figure 5.23: I-section (minor axis) optimum $a_z$ and $b_z$ exponents.
For the I-sections in Figure 5.23 under combined axial load and minor axis bending, a higher strain ratio of 5 is required before the convergence of the $a_z$ exponents, and there is a steeper relationship between $a_z$ and $A_w/A$. The $a_z$ exponents increase rapidly up to 6 from the linear interaction value of 1, whilst the $b_z$ exponents first increase up to 1.8 and then decrease to as low as 0.4. The straight line approximations through the optimum numerical $a_z$ and $b_z$ exponents are given in Eqn (5.30) by

$$
a_z = 8 \frac{A_w}{A} + 1.2 \quad \text{and} \quad b_z = 0.8 - 0.5 \frac{A_w}{A}.
$$

(5.30)

In Figure 5.24 constant exponent values again appear for elliptical hollow sections for bending about the minor axis, the exponents are $a_z = 1.65$ and $b_z = 0.82$. For all of the figures from Figure 5.20 to Figure 5.24, a rapid transition is seen from the $a_y, a_z, b_y$ and $b_z$ exponents all being equal to unity at the elastic strain ratio of $\epsilon_{csm}/\epsilon_y = 1$ (SR=1), to higher values at strain ratios greater than 3 or 5.

![Figure 5.24: Elliptical hollow section (minor axis) optimum $a_z$ and $b_z$ exponents.](image)
The exponents to be used in the design interaction equations are summarised in Table 5.1 and Table 5.2 for the major and minor axes respectively.

Table 5.1: Design $a_y$ and $b_y$ exponents for axial load and major axis bending.

<table>
<thead>
<tr>
<th>All</th>
<th>Box and I-sections</th>
<th>Circular hollow sections</th>
<th>Elliptical hollow sections</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \leq \frac{\epsilon_{csm}}{\epsilon_y} &lt; 3$</td>
<td>$3 \leq \frac{\epsilon_{csm}}{\epsilon_y} \leq 15$</td>
<td>$3 \leq \frac{\epsilon_{csm}}{\epsilon_y} \leq 15$</td>
<td>$3 \leq \frac{\epsilon_{csm}}{\epsilon_y} \leq 15$</td>
</tr>
<tr>
<td>$a_y$</td>
<td>1</td>
<td>$\frac{A_w}{A} + 1.2$</td>
<td>1.8</td>
</tr>
<tr>
<td>$b_y$</td>
<td>1</td>
<td>0.82</td>
<td>0.82</td>
</tr>
</tbody>
</table>

Table 5.2: Design $a_z$ and $b_z$ exponents for axial load and minor axis bending.

<table>
<thead>
<tr>
<th>All</th>
<th>Box sections</th>
<th>I-sections</th>
<th>Elliptical hollow sections</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \leq \frac{\epsilon_{csm}}{\epsilon_y} &lt; 3$</td>
<td>$3 \leq \frac{\epsilon_{csm}}{\epsilon_y} \leq 15$</td>
<td>$3 \leq \frac{\epsilon_{csm}}{\epsilon_y} &lt; 5$</td>
<td>$5 \leq \frac{\epsilon_{csm}}{\epsilon_y} \leq 15$</td>
</tr>
<tr>
<td>$a_z$</td>
<td>1</td>
<td>$\frac{A_w}{A} + 1.2$</td>
<td>2</td>
</tr>
<tr>
<td>$b_z$</td>
<td>1</td>
<td>0.82</td>
<td>1</td>
</tr>
</tbody>
</table>

5.1.7 Envelopes

The following plots in Figure 5.25 to Figure 5.29 show the CSM design predictions and the associated numerical model envelopes. These comparisons have been performed for each cross-section shape with strain ratios between 3 and 15 (5 and 15 for I-sections bending about the minor axis). These plots are used to determine how accurately the design interaction curves match the numerical model. The axial load and uni-axial bending interaction curves very tightly align with the numerical model envelopes, showing that the rationalising performed with the strain ratio grouping and with the linear fits for the exponents in Section 5.1.6, has not affected accurate predictions.
Figure 5.25: I-section axial load and major axis bending interaction envelopes.

Figure 5.26: Box section axial load and major axis bending interaction envelopes.
Figure 5.27: Circular and elliptical hollow section axial load and major axis bending interaction envelopes.

Figure 5.28: I-section axial load and minor axis bending interaction envelopes.
Figure 5.29: Rectangular and elliptical hollow section axial load and minor axis bending interaction envelopes.

5.1.8 Comparisons with test data

For combined loading limited test data were found, as the discovered literature were either for cross-sections that were slender, or long members that could not be considered to be influenced by only local buckling. Test data from Little (1978) are plotted in Figure 5.30, alongside the corresponding EN 1993-1-1 and CSM design interaction curves. The test data comprised six square hollow sections with strain ratios greater than 1, subjected to axial compression plus uni-axial bending. The data have been divided into two groups: the Data(a) set contains cross-sections with \(1 \leq \epsilon_{csm}/\epsilon_y < 3\) and the Data(b) set covers those with \(3 \leq \epsilon_{csm}/\epsilon_y \leq 15\). The CSM interaction curves and EN 1993-1-1 interaction curves are labelled with (a) or (b) to indicate the data set to which they apply. For Data(a), which are class 2, the linear CSM and bi-linear EN 1993-1-1 curves apply, giving conservative and unconservative predictions respectively. For Data(b), the cross-sections are class 1,
leading to accurate CSM predictions and slightly unsafe EN 1993-1-1 estimates. Table 5.3 summarises the means and coefficients of variation of the EN 1993-1-1 and CSM design model predictions. The statistics are based on the projection $R_u$ from the origin to the test data point, and $R_{csm}$ or $R_{EC3}$ to the associated intersection with the CSM or EN 1993-1-1 interaction curves. For combined loading, the CSM provides safe-side predictions whilst EN 1993-1-1 yields a reduced scatter but unsafe mean predictions.

Table 5.3: Overall comparison of ultimate combined loading capacities from test data with those predicted by the CSM and EN 1993-1-1.

<table>
<thead>
<tr>
<th></th>
<th>$R_u$/$R_{csm}$</th>
<th>$R_u$/$R_{EC3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.090</td>
<td>0.949</td>
</tr>
<tr>
<td>COV</td>
<td>0.0632</td>
<td>0.0231</td>
</tr>
<tr>
<td>No. of tests</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.30: Comparison of test data for cross-sections under combined loading with EN 1993-1-1 (EC3) and CSM predictions.
5.1.9 Moment–curvature–thrust relationships

Expressions for the moment–curvature–thrust ($M–\kappa–N$) relationships of solid rectangular sections with an elastic–perfectly plastic material model can be found in Chen and Atsuta (1976), which also describes other approximations for various design shapes. Finding accurate curves for cross-section shapes used in design and with more realistic material stress–strain curves is significantly more challenging as a continuous function is needed in the entire $M–\kappa–N$ domain, that starts initially straight in the elastic region and then transitions through to the strain hardening regime. Therefore attention turns to the developed numerical model for the interaction between axial load and uni-axial bending, as it can be used to find all possible strain distributions that give the same cross-section axial load.

In Figure 5.31, for an I-section with a maximum strain ratio of 15, the horizontal line for a fixed axial load value of $N/N_y = 0.4$, will give a set of moment resistances at the intersections with each interaction curve. Note that interaction curves with strain ratios (SR) below unity (i.e. elastic) are also important, and that not all interaction curves will be intersected, as when the yield normalised axial load $N/N_y$ increases, fewer moment possibilities occur. The intersection moment values can be plotted against the curvature, this curvature is related to the linearly varying strain components $\epsilon_B$ or $\epsilon_C$ in the numerical model. This procedure creates $M–\kappa$ curves for a fixed axial load $N/N_y$ as plotted in Figure 5.32 to Figure 5.36 for a maximum curvature ratio $\kappa/\kappa_y$ of 15. The numbers next to each curve indicate the axial load level $N/N_y$, from $N/N_y = 0$ to $N/N_y = 1.0$. There is a clear distinction between the strain ratio and the curvature ratio when there is an axial load present, as the curvature ratio will always be less than the strain ratio when there is a uniform strain $\epsilon_A$ (recall $\epsilon_A+\epsilon_B=\epsilon_{csm}$ or $\epsilon_A+\epsilon_C=\epsilon_{csm}$ at failure).
Figure 5.31: Fixed level $N/N_y = 0.4$ in axial load and uni-axial bending interaction space for an I-section.

Figure 5.32: Moment–curvature–thrust plots for I-sections (major axis) $N \geq 0$. 
Figure 5.33: Moment–curvature–thrust plots for box sections (major axis) $N \geq 0$.

Figure 5.34: Moment–curvature–thrust plots for circular and elliptical hollow sections (major axis) $N \geq 0$. 
Figure 5.35: Moment–curvature–thrust plots for I-sections (minor axis) $N \geq 0$.

Figure 5.36: Moment–curvature–thrust plots for rectangular and elliptical hollow sections (minor axis) $N \geq 0$. 
For both bending axes the gradient of the curves (which corresponds to the effective flexural rigidity $E'I$) reduces from $EI$ and tends towards $E_{sh}I$ for higher curvatures. This transition occurs more slowly for the cross-sections with higher shape factors due to the rate of stress redistribution. All cross-section shapes behave similarly for axial load and major axis bending. For $N/N_y \leq 0.2$, the effect on the moment–curvature curves is modest, with a greater influence and straightening of the curves occurring for increased axial loads ratios $N/N_y$. For I-sections bending about the minor axis the moment–curvature curves are affected only slightly for $N/N_y \leq 0.4$, but thereafter, increasing the axial load rapidly reduces the moment carrying capacity, as the axial load is no longer easily carried by the web. The rectangular and elliptical hollow sections plotted in Figure 5.36 behave similarly in minor axis bending to major axis bending.

5.2 Bi-axial bending

The planar sections remain plane assumption in Section 4.2 for cross-sections bending about one axis is visualised in Figure 5.37a and Figure 5.37b for the major and minor axes respectively, and is shown at the point where the limiting strain $\epsilon_{csm}$ is reached. The strain distribution can then be converted to a stress distribution via the bi-linear material model, and then integrated for the cross-section bending resistance. Figure 5.37a and Figure 5.37b give stress and strain profiles which are identical across the cross-section, as there is no variation in the $z$ direction for major axis bending and no variation in the $y$ direction for minor axis bending. When there is a combination of both major and minor axis bending (without an axial load) the strain distribution is more complex, as shown in Figure 5.37c and Figure 5.38. The strain distribution is still planar, but instead of the whole width of outer-fibres at $y = \pm D/2$ or $z = \pm B/2$ at strains of $\pm \epsilon_{csm}$, only the very furthest corner fibres (for box and I-sections) are at the limiting CSM strains and stresses.
Figure 5.37: Plane strain assumption for uni-axial and bi-axial bending (compression positive and NA is the zero strain neutral axis).

The zero strain neutral axis of a doubly symmetric cross-section, which is in the plane \( \epsilon = 0 \), now rotates around a fulcrum at the geometric centroid by an angle \( \theta \), giving a series of parallel equal strain lines (Figure 5.38). This leads to variations in the stress and strain profiles across the cross-section width \( B \) for the major axis bending component, and across the cross-section depth \( D \) for the minor axis bending component. As the cross-section areas above and below the neutral axis are equal, and the stresses either side of the neutral axis are also anti-symmetric, the integration process to determine the cross-section axial load leads to zero net force. Due to the infinite number of lines of symmetry for the circular hollow section in Figure 5.38b, with \( M_{csm} \) as the moment about the rotated neutral axis, \( M_y = M_{csm} \cos \theta \) and \( M_z = M_{csm} \sin \theta \),

\[
\cos^2 \theta + \sin^2 \theta = \left( \frac{M_y}{M_{csm}} \right)^2 + \left( \frac{M_z}{M_{csm}} \right)^2 = 1.
\]  

This equation form is identical to that in EN 1993-1-1 (2005).
5.2.1 Numerical model

The numerical model described in Section 5.1 for axial load and uni-axial bending interactions, is developed for bi-axial bending in this subsection.

Figure 5.38: Cross-section strain distributions for bi-axial bending.

Figure 5.39: Numerical model for bi-axial bending.
5.2.1.1 Box sections and I-sections

Figure 5.39a shows for an I-section (box sections similar), the parallel lines of the key strains and stresses for bending about both axes. These are the zero strain neutral axis NA, the CSM limiting strains and stresses \( \pm \epsilon_{csm} \) and \( \pm f_{csm} \) at the outer-fibres, and the yield strains and stresses \( \pm \epsilon_y \) and \( \pm f_y \). The numerical model divides the cross-section into many elements \( i \), each of which has an elemental area \( A_i \) and an element centroid at distances \( y_i \) and \( z_i \) from the bending axes \( y-y \) and \( z-z \).

For each box section and I-section, the task is to find all orientations of a planar strain surface which are limited by the CSM strain \( \epsilon_{csm} \), and to calculate the resulting cross-section resistance from the stress distribution. The first step is to normalise the failure criterion \( \epsilon_B + \epsilon_C = \epsilon_{csm} \) by the limiting strain \( \epsilon_{csm} \),

\[
\frac{\epsilon_B}{\epsilon_{csm}} + \frac{\epsilon_C}{\epsilon_{csm}} = 1 \tag{5.32}
\]

where \( \epsilon_B \) and \( \epsilon_C \) are the maximum strain values associated with the major and minor axis linearly varying strain profiles, such that \( \epsilon_B/\epsilon_{csm} \) can be varied from 0 for minor axis bending only, to 1 for major axis bending only, which leaves \( \epsilon_C/\epsilon_{csm} = 1 - \epsilon_B/\epsilon_{csm} \). With \( D \) as the total cross-section depth and \( B \) as the total cross-section width, the CSM normalised strain in the cross-section at any point \( (y_i, z_i) \) is given in Eqn (5.33) by

\[
\frac{\epsilon_i}{\epsilon_{csm}} = \frac{\epsilon_B}{\epsilon_{csm}} \frac{2y_i}{D} + \left(1 - \frac{\epsilon_B}{\epsilon_{csm}}\right) \frac{2z_i}{B}. \tag{5.33}
\]

With the strain at each element centroid defined, the elemental stress \( f_i \), normalised by the yield stress from the design bi-linear material model is
The normalised major and minor axis cross-section bending components are then

\[
\frac{f_i}{f_y} = \begin{cases} 
  \frac{\epsilon_i}{\epsilon_{csm}} \frac{\epsilon_{csm}}{\epsilon_y} & \epsilon_i \leq \epsilon_y \\
  1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_i}{\epsilon_{csm}} - 1 \right) & \epsilon_i > \epsilon_y.
\end{cases}
\]  

(5.34)

The normalised major and minor axis cross-section bending components are then

\[
\frac{M_y}{M_{pl,y}} = \frac{1}{W_{pl,y}} \sum_i \frac{f_i}{f_y} y_i A_i \quad \text{and} \quad \frac{M_y}{M_{csm,y}} = \frac{M_y}{M_{pl,y}} \frac{M_{csm,y}}{M_{pl,y}} 
\]  

(5.35)

\[
\frac{M_z}{M_{pl,z}} = \frac{1}{W_{pl,z}} \sum_i \frac{f_i}{f_y} z_i A_i \quad \text{and} \quad \frac{M_z}{M_{csm,z}} = \frac{M_z}{M_{pl,z}} \frac{M_{csm,z}}{M_{pl,z}} 
\]  

(5.36)

where \( M_{csm,y}/M_{pl,y} \) and \( M_{csm,z}/M_{pl,z} \) are calculated from the exact analytical expressions defined in Section 4.2.2.

5.2.1.2 Circular and elliptical hollow sections

For circular and elliptical hollow sections it is not appropriate to use Eqn (5.33) for the location of the limiting strain, since no point exists on the outer radius \( r_2 \) of the cross-sections with co-ordinates \( y = b_2 \) and \( z = a_2 \) (i.e. equivalent to the compressive corner fibre of box sections and I-sections). Therefore a new strain failure criterion that is different from \( \epsilon_B + \epsilon_C = \epsilon_{csm} \) must be defined for the fibres on the outer radius of the cross-section.

Note that the maximum linearly varying strains \( \epsilon_B \) and \( \epsilon_C \) must still not exceed \( \epsilon_{csm} \), otherwise the limiting strain will be exceeded at either \( y = b_2 \) or \( z = a_2 \). The strain \( \epsilon \) at any location \( y, z \) (or \( r, \theta \)) on the cross-section is

\[
\epsilon = \epsilon_B \frac{2y}{D_b} + \epsilon_C \frac{2z}{D_a} = \epsilon_B \frac{2r \sin \theta}{D_b} + \epsilon_C \frac{2r \cos \theta}{D_a}.
\]  

(5.37)
With the radius as $r$, and using the substitution $u = b^2 \cos^2 \theta + a^2 \sin^2 \theta$,

$$r = \frac{ab}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} = \frac{ab}{\sqrt{u}} \quad (5.38)$$

the rate of change of $r$ with respect to $\theta$ is

$$\frac{dr}{d\theta} = -\frac{ab}{2} \left( b^2 \cos^2 \theta + a^2 \sin^2 \theta \right)^{-\frac{3}{2}} \left( -b^2 \cos \theta \sin \theta + a^2 \sin \theta \cos \theta \right)$$

$$= \frac{ab (b^2 - a^2) \cos \theta \sin \theta}{u^\frac{3}{2}}. \quad (5.39)$$

With $b = D_b / 2$ and $a = D_a / 2$, the maximum strain around the outer radius is found by

$$\frac{d\epsilon}{d\theta} = \frac{\epsilon_B}{b} \frac{d}{d\theta} r \sin \theta + \frac{\epsilon_C}{a} \frac{d}{d\theta} r \cos \theta = \frac{\epsilon_B}{b} \left( r \cos \theta + \sin \theta \frac{dr}{d\theta} \right) + \frac{\epsilon_C}{a} \left( -r \sin \theta + \cos \theta \frac{dr}{d\theta} \right) = 0$$

$$\epsilon_B a \left[ \frac{ab \cos \theta}{\sqrt{u}} + \frac{ab (b^2 - a^2) \cos \theta \sin^2 \theta}{u^\frac{3}{2}} \right] + \epsilon_C b \left[ \frac{ab (b^2 - a^2) \cos^2 \theta \sin \theta}{u^\frac{3}{2}} - \frac{ab \sin \theta}{\sqrt{u}} \right] = 0$$

$$\epsilon_B a^2 b \cos \theta \left[ \frac{1}{\sqrt{u}} + \frac{(b^2 - a^2) \sin^2 \theta}{u^\frac{3}{2}} \right] + \epsilon_C b^2 a \sin \theta \left[ -\frac{1}{\sqrt{u}} + \frac{(b^2 - a^2) \cos^2 \theta}{u^\frac{3}{2}} \right] = 0$$

$$\epsilon_B a \cos \theta \left[ 1 + \frac{(b^2 - a^2) \sin^2 \theta}{u} \right] + \epsilon_C b \sin \theta \left[ -1 + \frac{(b^2 - a^2) \cos^2 \theta}{u} \right] = 0$$

$$\epsilon_B ab^2 \cos \theta - \epsilon_C a^2 b \sin \theta = b \epsilon_B \cos \theta - a \epsilon_C \sin \theta = 0 \quad \text{giving} \quad \tan \theta = \frac{b \epsilon_B}{a \epsilon_C}. \quad (5.40)$$
Therefore when $\epsilon_B/\epsilon_C \rightarrow 0$ then $\theta \rightarrow 0$ and the maximum strain occurs at $z = a_2$ (minor axis bending), and when $\epsilon_B/\epsilon_C \rightarrow \infty$ then $\theta \rightarrow \pi/2$ and the maximum strain occurs at $y = b_2$ (major axis bending). For all combinations between these two limits, the strain on the outer radius reaches the CSM limiting strain when

$$
\epsilon_{csm} = \epsilon_B \frac{r \sin \theta}{b} + \epsilon_C \frac{r \cos \theta}{a} = \epsilon_B \frac{rb \epsilon_B}{b \sqrt{b^2 \epsilon_B^2 + a^2 \epsilon_C^2}} + \epsilon_C \frac{ra \epsilon_C}{a \sqrt{b^2 \epsilon_B^2 + a^2 \epsilon_C^2}} = \frac{r (\epsilon_B^2 + \epsilon_C^2)}{\sqrt{b^2 \epsilon_B^2 + a^2 \epsilon_C^2}}
$$

which gives the failure criterion as $\epsilon_{csm}^2 = \epsilon_B^2 + \epsilon_C^2$. For a given maximum major axis bending strain $\epsilon_B$, a point in the compressive quadrant (where major and minor axis compressive stresses combine) at angle $\theta = \arctan(b_2 \epsilon_B/a_2 \epsilon_C)$ on the outer radius of a circular or elliptical hollow section, will be at $\epsilon_{csm}$ if

$$
\frac{\epsilon_C}{\epsilon_{csm}} = \sqrt{1 - \left(\frac{\epsilon_B}{\epsilon_{csm}}\right)^2}.
$$

(5.41)

5.2.2 Interaction curves

5.2.2.1 Bi-axial bending (plastic normalised)

The output from the numerical model developed in Section 5.2.1 are interaction curves for the bi-axial bending of a given cross-section and strain ratio. Examples of these interaction curves normalised by the plastic moments $M_{pl,y}$ and $M_{pl,z}$ are plotted in Figure 5.40 to Figure 5.42 for a strain ratio of 15, with and without strain hardening. For the I-sections, box sections and circular hollow sections, the numerical models with $E_{sh} = E/100$ and $E_{sh} = 0$ are compared to the EN 1993-1-1 (2005) predictions (elliptical hollow sections have no design code comparison).
Figure 5.40: Bi-axial bending interaction curves for I-sections, normalised by the plastic moments.

Figure 5.41: Bi-axial bending interaction curves for box sections, normalised by the plastic moments.
The inclusion of strain hardening with $E_{sh} = E/100$ gives a resistance increase to the interaction curves of about 10% of the plastic moments when compared to $E_{sh} = 0$. For the I-sections in Figure 5.40, the interaction curves are smooth and more expanded for lower major axis moments. The EN 1993-1-1 (2005) equation with constant exponents (Section 2.6) appears appropriate for the represented UC when $E_{sh} = 0$, as the design curve follows the numerical model closely, but it fails to recognise the different interaction curve shape produced by the UB and gives an overly conservative prediction. It is seen that a strain ratio of 15 is not sufficient for the I-section interaction curves to pass through $M_{pl,z}$ at $M_y = 0$, making the EN 1993-1-1 (2005) curve slightly unconservative for $E_{sh} = 0$ at some minor axis dominated moment combinations. For the representative box sections in Figure 5.41, the numerical model interaction curves with no strain hardening overlap the EN 1993-1-1 (2005) predictions. The strain ratio of 15 is sufficient to approach the plastic moments $M_{pl,y}$ and $M_{pl,z}$ with $E_{sh} = 0$, and so the interaction curves almost intercept
these points at the $M_y$ and $M_z$ axes. The curve shapes are similar for the rectangular and square hollow sections, and a slight necking between the two numerical curves ($E_{sh} = 0$ and $E_{sh} = E/100$) is seen mid-length around the curves. For the circular and elliptical hollow sections in Figure 5.42, the curve shapes are similar and display an extra 10% capacity with $E_{sh} = E/100$.

### 5.2.2.2 Bi-axial bending (CSM normalised)

In Section 5.1 it was found that when the axial load and uni-axial bending interaction curves were normalised by the CSM axial load $N_{csm}$ and bending moment $M_{csm,y}$ or $M_{csm,z}$, the curves began to overlap for particular ranges of strain ratio. Similar behaviour is observed in Figure 5.43 to Figure 5.45 for the CSM normalised interaction curves for bi-axial bending. For the I-sections in Figure 5.43 the curve overlapping occurs for a strain ratio of 5 (SR=5), which is the same value as the axial load and minor axis bending curves.

![Diagrams](image)

Figure 5.43: I-section bi-axial bending, CSM normalised interaction curves.
Figure 5.44: Box section bi-axial bending, CSM normalised interaction curves.

Figure 5.45: Circular and elliptical hollow section bi-axial bending, CSM normalised interaction curves.
The box section curves begin to overlap at a strain ratio of 3 (SR=3), which is also identical to the axial load and uni-axial bending interaction curves about either bending axis. The circular and elliptical hollow sections show little to no dependency on the strain ratio, and so maintain the same curve shape throughout. The interaction curves also show high minor axis moment $M_z/M_{csm,z}$ retention for an extended range of major axis moments $M_y/M_{csm,y}$. For the hollow cross-sections in Figure 5.44 and Figure 5.45 the interaction curves are smooth, have no observable kinks, and are symmetric about the line $M_y/M_{csm,y} = M_z/M_{csm,z}$ for the square and circular hollow sections, and are approximately symmetric for the rectangular and elliptical hollow sections. Owing to the symmetry characteristics of a circular hollow section, any strain ratio produces the same interaction curve, which is the path of a quarter circle between the CSM normalised anchor points at the plot axes. Elliptical hollow sections follow closely the quarter circle shape, but the interaction curves contract very slightly for higher strain ratios.

### 5.2.3 Cross-section slenderness

Between the CSM limiting major axis bending and minor axis bending moments $M_{csm,y}$ and $M_{csm,z}$, there are a set of moment combinations and stress distributions on the bi-axial bending interaction curves. To determine the relevant cross-section slenderness $\bar{\lambda}_p = \sqrt{f_y/f_{cr}}$, the stress distribution used in the determination of the elastic critical buckling stress $f_{cr}$ should be representative of the point in the $M_y-M_z$ space, and based upon the applied major and minor axis bending moments $M_{Ed,y}$ and $M_{Ed,z}$. With the correct stress distribution, the buckling coefficients $k_\sigma$ can be found by the methodology of EN 1993-1-5 (2006). For the applied loading projection from the origin to the applied load state represented by the point $(M_{Ed,y}/M_{el,y}, M_{Ed,z}/M_{el,z})$, the line plotted in Figure 5.46 is

$$\frac{M_z}{M_{el,z}} = \left(\frac{M_{Ed,z}}{M_{el,z}} \frac{M_{el,y}}{M_{Ed,y}}\right) \frac{M_y}{M_{el,y}}.$$  

(5.42)
This projected line will intersect the linear elastic interaction boundary at the point
\((M_{y\text{el,y}}/M_{\text{el,y}}, M_{z\text{el,z}}/M_{\text{el,z}})\), which can be expressed as

\[
\left( \frac{M'_{\text{Ed,z}}}{M_{\text{el,z}}} \frac{M_{\text{el,y}}}{M'_{\text{Ed,y}}} \right) \frac{M'_{y}}{M_{\text{el,y}}} + \frac{M'_{z}}{M_{\text{el,z}}} = 1
\]

\[
\frac{M'_{y}}{M_{\text{el,y}}} = \frac{1}{1 + \frac{M'_{\text{Ed,z}} M_{\text{el,y}}}{M_{\text{el,z}} M'_{\text{Ed,y}}}} \quad \text{and} \quad \frac{M'_{z}}{M_{\text{el,z}}} = 1 - \frac{M'_{y}}{M_{\text{el,y}}}. \tag{5.43}
\]

The corresponding stress distribution (compression positive) at this intersection state is
given by the addition of the linearly varying major axis and minor axis elastic bending stresses as

\[
f = \frac{M'_{y}}{I_{y}} + \frac{M'_{z}}{I_{z}}. \tag{5.44}
\]
For the major axis with $W_{el,y} = 2I_y/D$ and the minor axis with $W_{el,z} = 2I_z/B$, the yield normalised stress is

$$\frac{f}{f_y} = \frac{M_y}{M_{el,y}} \frac{2y}{D} + \frac{M_z}{M_{el,z}} \frac{2z}{B} = \left( 1 + \frac{1}{M_{Ed,z} M_{el,y}} \right) \frac{2y}{D} + \left( 1 - \frac{1}{1 + \frac{M_{Ed,z} M_{el,y}}{M_{el,z} M_{Ed,y}}} \right) \frac{2z}{B}. \quad (5.45)$$

For a box section or I-section at the corner fibre $z = B/2, y = D/2$ this equation gives $f/f_y = 1$. Similarly for $z = -B/2, y = -D/2$ the stress will be $f/f_y = -1$, and then for $z = -B/2, y = D/2$ the corner stress is

$$\frac{f_c}{f_y} = \frac{2}{1 + \frac{M_{Ed,z} M_{el,y}}{M_{el,z} M_{Ed,y}}} - 1. \quad (5.46)$$

When $M_{Ed,z} = 0$ this gives $f_c/f_y = 1$, which is the loading state of major axis bending only, and as $M_{Ed,y} \rightarrow 0$ then $f_c/f_y \rightarrow -1$, giving only a minor axis stress distribution.

![Elastic stress distributions for $M_y$ and $M_z$ interactions.](image)

Figure 5.47: Elastic stress distributions for $M_y$ and $M_z$ interactions.
For all combinations in-between, the stress distributions will be either major axis or minor axis bending dominated depending on the orientation of the zero strain neutral axis NA (Figure 5.47). To define fully the elastic stress distribution for the I-section in Figure 5.47b, a designer also needs the stress \( f_w/f_y \) at the end of the web at \( z = 0, y = D/2 \),

\[
\frac{f_w}{f_y} = \frac{1}{1 + \frac{M_{Ed,z}}{M_{el,x}} \frac{M_{el,y}}{M_{Ed,y}}}. \tag{5.47}
\]

5.2.4 Design expression

The same CSM normalised power equation form that was used for axial load and major or minor axis bending interactions in Section 5.1.5 can be used for bi-axial bending. This equation form was found to be simple and effective at describing a wide variety of interaction curve shapes, as well as being continuous between the CSM anchor points \( (N_{csm}, M_{csm,y} \text{ and } M_{csm,z}) \) on both plot axes. For consistent and clear use of notation, \( a_y \) and \( a_z \) were previously used to represent the exponents on the normalised axial terms, and \( b_y \) and \( b_z \) for the exponents on the normalised major axis and minor axis bending terms, here \( \beta \) and \( \gamma \) are used to give Eqn (5.48),

\[
\left( \frac{M_y}{M_{csm,y}} \right)^\beta + \left( \frac{M_z}{M_{csm,z}} \right)^\gamma \leq 1. \tag{5.48}
\]

5.2.5 Exponents \( \beta \) and \( \gamma \)

The numerical model from Section 5.2.1 is extended to find the optimum \( \beta \) and \( \gamma \) exponents via a curve fitting method, for each cross-section and for every strain ratio. Figure 5.48 shows for I-sections how these exponents vary with the ratio of major axis to minor axis plastic section moduli \( W_{pl,y}/W_{pl,z} \). The greatest variations occur between \( 1 \leq \epsilon_{csm}/\epsilon_y < 5 \) (SR=1 to SR=4), and then beyond this the \( \beta \) values settle to between 2.0 and 4.0, and the
The \( \gamma \) exponent drops from 1.0 to between 0.6 and 0.9. The linear fits in Eqn (5.49) are also plotted, showing that cross-sections with higher \( W_{pl,y}/W_{pl,z} \) ratios have larger and smaller values of \( \beta \) and \( \gamma \) respectively. These fits are

\[
\beta = 2 + 0.15 \frac{W_{pl,y}}{W_{pl,z}} \quad \text{and} \quad \gamma = 0.8 - 0.015 \frac{W_{pl,y}}{W_{pl,z}}.
\]  

Figure 5.49 shows results of the same \( \beta \) and \( \gamma \) optimisation process but applied to hollow box sections (RHS and SHS). For square hollow sections, where the double symmetry of the cross-sections gives \( W_{pl,y}/W_{pl,z} = 1 \), both the \( \beta \) and \( \gamma \) exponents take the value of 1.6, which is similar to the EN 1993-1-1 (2005) designation of 1.66. For the rectangular hollow sections, the values of \( \beta \) and \( \gamma \) decrease and increase respectively by large and small amounts as the ratio of the plastic section moduli increases. The linear fits are
\[ \beta = 1.6 - 0.15 \left( \frac{W_{pl,y}}{W_{pl,z}} - 1 \right) \quad \text{and} \quad \gamma = 1.6. \quad (5.50) \]

Figure 5.49: Box section bi-axial bending \( \beta \) and \( \gamma \) exponents.

For circular hollow sections a power optimisation process is not required, as the special symmetry properties of the cross-section and its strain distribution leads to an independence from the strain ratio. With \( W_{pl,y}/W_{pl,z} = 1 \) the \( \beta \) and \( \gamma \) exponents for circular hollow sections are both 2, which forms a quadrant of a circle in the major axis–minor axis interaction space. For the elliptical hollow sections presented in Figure 5.50, the exponents lay close enough to the circular values to be approximated with \( \beta \) and \( \gamma \) equal to 2. This allows some rationalising by the grouping of similar behaviour for these two tubular shapes. As was the case for axial load and uni-axial bending interactions with strain ratios that were less than 3, the linear interaction \( \beta = \gamma = 1 \) may be used for bi-axial bending for \( \epsilon_{csm}/\epsilon_y < 3 \).

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Figure 5.50: Elliptical hollow section bi-axial bending $\beta$ and $\gamma$ exponents.

Table 5.4 summarises the optimum exponents for the investigated cross-section shapes. Also included for I-sections are the intermediate exponent values of $\beta = 2$ and $\gamma = 1$, to ease the transition from a strain ratio of 3 to a strain ratio of 5, rather than use $\beta = \gamma = 1$ for $\epsilon_{csm}/\epsilon_y < 5$ and being excessively conservative.

$$\left( \frac{M_y}{M_{csm,y}} \right)^\beta + \left( \frac{M_z}{M_{csm,z}} \right)^\gamma \leq 1. \quad (5.51)$$

Table 5.4: Design $\beta$ and $\gamma$ exponents for bi-axial bending ($W_r = W_{pl,y}/W_{pl,z}$).
5.2.6 Envelopes

Figure 5.51 to Figure 5.53 show for each cross-section shape, the design expression from Section 5.2.4 with the linear and constant fits for the $\beta$ and $\gamma$ exponents, plotted with the numerical model envelope for the strain ratios where the bi-axial interaction curves overlap. For all of the represented cross-sections, the numerical model envelopes are in tight proximity to the simple design curves, with the interaction curves either passing through the middle of the numerical model envelopes or deviating only slightly from them. The CSM normalised power equation continues to show excellent ability in describing the continuous interaction curves, as was previously discovered with its use for axial load and uni-axial bending in Section 5.1.

![Design equation vs Numerical envelope](image-url)

Figure 5.51: I-section bi-axial bending interaction curves and numerical model envelopes.
Figure 5.52: Box section bi-axial bending interaction curves and numerical model envelopes.

Figure 5.53: Circular and elliptical hollow section bi-axial bending interaction curves and numerical model envelopes.
5.3 Axial load and bi-axial bending

The CSM strain distribution for bi-axial bending, as investigated in Section 5.2, utilised a strain plane rotating about the centroid of a doubly symmetric cross-section, and was constructed from the addition of linearly varying strains with maximum values of \( \varepsilon_B \) and \( \varepsilon_C \). This is presented again in Figure 5.54a, where two corner points at \( y = \pm D/2, z = \pm B/2 \) have reached the limiting strains \( \pm \varepsilon_{csm} \), and the zero strain neutral axis has rotated to be at an angle to the major and minor axes. For the extension of the bi-axial bending model to include axial loads, the uniform compressive strain \( \varepsilon_A \) is reintroduced, which has the effect of preventing the tensile corner strain at \( y = -D/2, z = -B/2 \) from reaching \( -\varepsilon_{csm} \) and accelerating the compressive corner strain at \( y = D/2, z = B/2 \) reaching \( \varepsilon_{csm} \). This is shown in Figure 5.54b and Figure 5.54c for both relatively low and high uniform strains, corresponding to low and high axial loads respectively. The strain distribution can be visualised as a rotated plane shifted in the compressive strain direction towards \( \varepsilon_{csm} \); the higher the value of \( \varepsilon_A \) the lesser the extent to which the plane can rotate, and hence the lower the flexural capacity of the cross-section.

![Strain distributions in cross-sections under combined axial load and bi-axial bending](image)

(a) No compressive strain \( \varepsilon_A \)  
(b) Low compressive strain \( \varepsilon_A \)  
(c) High compressive strain \( \varepsilon_A \)

Figure 5.54: Strain distributions in cross-sections under combined axial load and bi-axial bending; compression is positive and NA is the zero strain neutral axis.
For bending about both axes and with no uniform strain, the area and distribution of stresses in compression equal those in tension and no net axial force is produced. However in Figure 5.54b and Figure 5.54c the areas in tension reduce as $\epsilon_A$ increases. This will lead to different stress states across the cross-section, as seen in the case of axial load and uni-axial bending.

The interaction curves for axial load and uni-axial bending, as well as for bi-axial bending, can be plotted in three-variable space ($N$, $M_y$, $M_z$) with respect to the elastic cross-section capacities $M_{el,y}$, $M_{el,z}$ and $N_y$ as in Figure 5.55a, or to the CSM values $M_{csym,y}$, $M_{csym,z}$ and $N_{csym}$ as in Figure 5.55b. These plots show that the investigations so far have considered the interaction between two of the three variables $N$, $M_y$, $M_z$ (or $\epsilon_A$, $\epsilon_B$, $\epsilon_C$), with the third taken as zero. These two variable interaction curves define the edges or boundaries of an interaction surface projecting out into $M_y$, $M_z$, $N$ space.

Figure 5.55: Two variable interaction curves forming the boundary of interaction surface.
The linear interaction where strains and stresses are limited to the yield values corresponds to a strain ratio of unity (SR=1) and produces the triangular interaction surfaces in Figure 5.55, while an expanded surface is created for cross-sections that can exceed the elastic capacities.

5.3.1 Numerical model

The numerical model employed is a simple extension of that used for bi-axial bending in Section 5.2.1; the only modification that is required is the inclusion of a uniform strain. For each cross-section, orientations of a planar strain surface shifted in the compressive strain direction are created that are limited by the CSM strain $\epsilon_{csm}$. The strain distribution drives a stress distribution, which is then integrated for the cross-section capacities.

5.3.1.1 I-sections and box sections

This model continues to divide the cross-section geometry into $i$ elements each of area $A_i = D_iB_i$ (where $D_i = D/n_y$, $B_i = B/n_z$), with $D$ and $B$ as the cross-section depth and width and with element centroids at distances $y_i$ and $z_i$ from the $y$–$y$ and $z$–$z$ axes. The new failure criterion is normalised by the limiting strain $\epsilon_{csm}$ and given as

$$\frac{\epsilon_A}{\epsilon_{csm}} + \frac{\epsilon_B}{\epsilon_{csm}} + \frac{\epsilon_C}{\epsilon_{csm}} = 1$$

(5.52)

where $\epsilon_A$, $\epsilon_B$ and $\epsilon_C$ have been described previously in Section 5.1 and Section 5.2. The numerical analysis procedure is initiated with a given value of $\epsilon_A/\epsilon_{csm}$ from between 0 and 1, and then the parameter $\epsilon_B/\epsilon_{csm}$ can be varied from 0 for no major axis bending, to $1 - \epsilon_A/\epsilon_{csm}$ for axial load plus major axis bending. This limits the maximum value of the minor axis bending parameter $\epsilon_C/\epsilon_{csm} = 1 - \epsilon_A/\epsilon_{csm} - \epsilon_B/\epsilon_{csm}$. The strain $\epsilon_i$ at element $i$ is then based on the addition of $\epsilon_A$, $\epsilon_B$ and $\epsilon_C$, all normalised by $\epsilon_{csm}$.
\[
\frac{\epsilon_i}{\epsilon_{csm}} = \frac{\epsilon_A}{\epsilon_{csm}} + \frac{\epsilon_B}{\epsilon_{csm}} \frac{2y_i}{D} + \left( 1 - \frac{\epsilon_A}{\epsilon_{csm}} - \frac{\epsilon_B}{\epsilon_{csm}} \right) \frac{2z_i}{B}.
\] (5.53)

### 5.3.1.2 Circular and elliptical hollow sections

For the bi-axial bending numerical model described in Section 5.2.1, it was shown that as a point \( y = D_b/2, z = D_a/2 \) did not exist on the outer radius of a circular or elliptical hollow section, a modified strain failure criterion was required. The same is true when a uniform strain is added to two linearly varying strains. The strain at \( y, z \) or \( r, \theta \) on the cross-section is

\[
\epsilon = \epsilon_A + \epsilon_B \frac{2y}{D_b} + \epsilon_C \frac{2z}{D_a} = \epsilon_A + \epsilon_B \frac{2r \sin \theta}{D_b} + \epsilon_C \frac{2r \cos \theta}{D_a}
\] (5.54)

where the radius \( r \) of an ellipse at angle \( \theta \) is defined with minor and major diameters \( 2a = D_a \) and \( 2b = D_b \) as

\[
r = \frac{ab}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}.
\] (5.55)

As the uniform strain \( \epsilon_A \) is constant across the cross-section and so not a function of \( \theta \), the previous result of \( d\epsilon/d\theta \) from the bi-axial bending model may be used for indicating the angle at which the limiting strain is reached \( \epsilon = \epsilon_{csm}, \)

\[
\tan \theta = \frac{b \epsilon_B}{a \epsilon_C}.
\] (5.56)

This leads to a result similar to the bi-axial bending case but with an additional \( \epsilon_A \) term

\[
\epsilon_{csm} = \epsilon_A + \epsilon_B \frac{r \sin \theta}{b} + \epsilon_C \frac{r \cos \theta}{a} = \epsilon_A + \frac{\epsilon_B^2 + \epsilon_C^2}{\sqrt{\epsilon_C^2 + \epsilon_B^2}} = \epsilon_A + \sqrt{\epsilon_B^2 + \epsilon_C^2}.
\] (5.57)
For a given $\epsilon_A$ and $\epsilon_B$ pair, which combined do not exceed $\epsilon_{csm}$, the reserve in $\epsilon_C$, before the failure criterion is triggered is

$$\epsilon_C^2 = (\epsilon_{csm} - \epsilon_A)^2 - \epsilon_B^2$$  \hfill (5.58)

or expressed in CSM normalised form as

$$\frac{\epsilon_C}{\epsilon_{csm}} = \sqrt{\left(1 - \frac{\epsilon_A}{\epsilon_{csm}}\right)^2 - \left(\frac{\epsilon_B}{\epsilon_{csm}}\right)^2}.$$  \hfill (5.59)

For whichever cross-section shape is analysed, the associated stresses normalised by the yield stress are the same as for bi-axial bending, as too are the cross-section capacities but now with the inclusion of the yield load or CSM load normalised axial resistance. These normalised axial loads are

$$\frac{N}{N_y} = \frac{1}{A} \sum_i f_i A_i \quad \text{and} \quad \frac{N}{N_{csm}} = \frac{N}{N_y} / \frac{N_{csm}}{N_y}.$$  \hfill (5.60)

The output from the numerical model is a suite of interaction curves between $M_y$ and $M_z$ for constant values of $\epsilon_A/\epsilon_{csm}$, as plotted in Figure 5.56a for a UB and Figure 5.56b for a RHS. It is important to recognise that these interaction surfaces must first be sliced with a series of planes parallel to the $M_y$–$M_z$ plane to show the contour lines of constant axial load $N/N_{csm}$. This has been performed in Figure 5.57 for the same UB and RHS cross-sections in Figure 5.56 for axial load levels at fixed spacing.
Figure 5.56: Numerical model interaction surfaces with curves of equal $\epsilon_A/\epsilon_{csm}$.

Figure 5.57: Numerical model interaction surfaces with contours of equal $N/N_{csm}$. 

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5.3.2 Interaction curves

5.3.2.1 Axial load and bi-axial bending (plastic normalised)

From the interaction surfaces that are produced by the numerical model, slices are cut of constant $N/N_y$ to determine the bi-axial bending interaction curves for fixed axial load levels i.e. to give contours of $n = N/N_y$ in the $M_y-M_z$ plane. This is demonstrated for I-sections in Figure 5.58, rectangular and square hollow sections in Figure 5.59 and for circular and elliptical hollow sections in Figure 5.60, with a strain ratio of 15, and also with and without strain hardening ($E_{sh} = E/100$ and $E_{sh} = 0$). For comparison, these figures also include the EN 1993-1-1 (2005) design curves for combined loading as described in Section 2.6. For no axial load (where $n = 0$) the interaction curves are identical to the bi-axial bending interaction curves displayed previously in Section 5.2.2, where the 10% benefits beyond the EN 1993-1-1 (2005) plastic capacities are seen.

![I-section bi-axial bending interaction curves at fixed axial load levels $n = N/N_y$ (plastic normalised).](image_url)

Figure 5.58: I-section bi-axial bending interaction curves at fixed axial load levels $n = N/N_y$ (plastic normalised).
Figure 5.59: Box section bi-axial bending interaction curves at fixed axial load levels $n = N/N_y$ (plastic normalised).

As $N/N_y$ increases, the curves from the numerical model and the EN 1993-1-1 (2005) design model become more rectangular in shape and contract towards the origin (the $N$ axis). The I-sections give surface shapes that are asymmetric about $M_y/M_{csm,y} = M_z/M_{csm,z}$, while the box sections hold more surface symmetry, showing that there is a bias for I-sections towards minor axis bending, and a balance between both bending axes for box sections. The EN 1993-1-1 (2005) design model shows very good correspondence with the interaction curves produced from the numerical model when $E_{sh} = 0$, but offer slightly more conservative predictions. The cross-section resistance gains from enabling strain hardening increase to around 15% of the plastic moments for high axial loads when $N/N_y \geq 0.8$, and then the interaction curves contract to an apex located at $N = N_{csm}$. The circular hollow section curves in Figure 5.60a are perfectly symmetric about $M_y/M_{pl,y} = M_z/M_{pl,z}$, as quadrants of a circle are produced for each $N/N_y$. The elliptical hollow sections in Figure 5.60b produce similar curves to the circular hollow section curves.
5.3.2.2 Axial load and bi-axial bending (CSM normalised)

By transferring the numerical model interaction surfaces into the CSM normalised domain \((N_{csm}, M_{csm,y}, M_{csm,z})\), the surfaces are bound to a unit box and are restricted from progressively expanding outwards away from the origin as the strain ratio increases. Doing so also allows compatibility with the previous CSM normalised design equations for axial load and uni-axial bending, and bi-axial bending.

The following figures (Figure 5.61 to Figure 5.63) for the six cross-section shapes show the contours for five axial load levels \((N/N_{csm})\) and for two overlapping strain ratios, these strain ratios are SR=3 or SR=5 for box sections and I-sections respectively, and the maximum strain ratio of SR=15 for both.

Figure 5.60: Circular and elliptical hollow section bi-axial bending interaction curves at fixed axial load levels \(n = N/N_y\) (plastic normalised).
For lower axial load levels when $N/N_{csm} \leq 0.4$ for I-sections and $N/N_{csm} \leq 0.6$ for tubular sections, the higher strain ratio surface is slightly on the outside of the lower strain ratio surface, and for higher axial loads the reverse is true. This behaviour was previously observed as a folding back of the axial load and uni-axial bending curves for low moments, and is also visible in Figure 5.61 for the I-sections at low minor axis moments for low axial loads. In general, the difference between the interaction surfaces for the two displayed strain ratios is most prominent for higher axial loads. Care must be taken however as this is partly because the surfaces for lower axial loads are more vertically orientated (i.e. perpendicular to the $M_y-M_z$ plane. As a consequence, because the interaction curves are on a surface slice parallel to the $M_y-M_z$ plane, the offset between the curves for higher axial loads is more pronounced. The contours should then be treated with caution when considering the goodness of fit of any proposed design interaction surface; a more appropriate residual method is developed in Section 5.3.7.

Figure 5.61: I-section bi-axial bending interaction curves at fixed axial load levels $N/N_{csm}$ (CSM normalised).
Figure 5.62: Box section bi-axial bending interaction curves at fixed axial load levels $N/N_{cs}$ (CSM normalised).

Figure 5.63: Circular and elliptical hollow section bi-axial bending interaction curves at fixed axial load levels $N/N_{cs}$ (CSM normalised).
5.3.3 Cross-section slenderness

To determine the relevant cross-section slenderness $\bar{\lambda}_p = \sqrt{f_y/f_{cr}}$, the selected stress distribution in the determination of the elastic critical buckling stress $f_{cr}$ should be representative of the applied axial load and bending moments. With the correct stress distribution the buckling coefficients $k_\sigma$ can be found by the methodology of EN 1993-1-5 (2006), or alternatively by the Direct Strength Method (Section 2.1.2) for the total cross-section slenderness. For the combination of axial load and bending moments about both axes, the straight line from the origin to the applied loading state $N_{Ed}/N_y$, $M_{Ed,y}/M_{el,y}$, $M_{Ed,z}/M_{el,z}$ (shown in Figure 5.64) will intersect the linear interaction surface at

$$\frac{N'}{N_y} + \frac{M_y'}{M_{el,y}} + \frac{M_z'}{M_{el,z}} = 1. \quad (5.61)$$

Figure 5.64: Projection of the applied loading onto the elastic interaction surface.

It will be seen in Section 5.3.7 for the derivation of a residual, such an intersection is simply a scalar $c$ multiplied by the design loading vector with components.
\[
\frac{N'}{N_y} = c \frac{N_{Ed}}{N_y}, \quad \frac{M'_y}{M_{el,y}} = c \frac{M_{Ed,y}}{M_{el,y}}, \quad \frac{M'_z}{M_{el,z}} = c \frac{M_{Ed,z}}{M_{el,z}}.
\] (5.62)

Combining the presented equations gives the scalar \( c \) as

\[
c = \left( \frac{N_{Ed}}{N_y} + \frac{M_{Ed,y}}{M_{el,y}} \right)^{-1}.
\] (5.63)

The corresponding stress distribution (compression positive) at this elastic intersection state is given by the addition of uniform compressive stresses and linearly varying major and minor axis elastic bending stresses,

\[
f = \frac{N'}{A} + \frac{M'_y}{I_y} \frac{2y}{D} + \frac{M'_z}{I_z} \frac{2z}{B}.
\] (5.64)

For the major axis with \( W_{el,y} = 2I_y/D \) and the minor axis with \( W_{el,z} = 2I_z/B \), the yield normalised stress at any point \( y, z \) is

\[
f_y = \frac{N'}{N_y} + \frac{M'_y}{M_{el,y}} \frac{2y}{D} + \frac{M'_z}{M_{el,z}} \frac{2z}{B}.
\] (5.65)

For a box section or I-section at the corner fibre \( z = B/2, y = D/2 \) this gives \( f/f_y = 1 \).

At \( z = -B/2, y = -D/2 \), the most tensile stress (or least compressive stress) will be

\[
f_c = \frac{N'}{N_y} - \frac{M'_y}{M_{el,y}} - \frac{M'_z}{M_{el,z}} = 2 \frac{N'}{N_y} - 1.
\] (5.66)

For the corner at \( z = -B/2, y = D/2 \) the stress is

\[
f_c = \frac{N'}{N_y} + \frac{M'_y}{M_{el,y}} - \frac{M'_z}{M_{el,z}} = 1 - 2 \frac{M'_z}{M_{el,z}}
\] (5.67)
and for the opposite corner at \(z = B/2, y = -D/2\)

\[
\frac{f_c}{f_y} = \frac{N'}{N_y} - \frac{M_y'}{M_{el,y}} + \frac{M_z'}{M_{el,z}} = 1 - 2\frac{M_y'}{M_{el,y}}. \tag{5.68}
\]

For an I-section, the stresses \(f_w\) at the web ends at \(z = 0, y = D/2\) and \(z = 0, y = -D/2\) are needed to fully define the elastic stress distribution of the cross-section,

\[
\frac{f_w}{f_y} = \frac{N'}{N_y} + \frac{M_y'}{M_{el,y}} \quad \text{and} \quad \frac{f_w}{f_y} = \frac{N'}{N_y} - \frac{M_y'}{M_{el,y}} \tag{5.69}
\]

both of which can be alternatively calculated by taking the middle value between the two adjacent corners.

### 5.3.4 Design expression

In this subsection, design interaction expressions are developed on the basis of the numerical results. The boundary of the interaction surface consists of the three CSM normalised interaction curves on the axis planes, that is two axial load and uni-axial bending curves and one bi-axial bending curve. For compatibility, any proposed design surface should match with these interaction curves by including the previously developed equations in Section 5.1.5 and Section 5.2.4, and fit new information in-between with a smooth surface.

The proposed design model in Figure 5.65 consists of bi-axial interaction curves on planes that are parallel to the \(M_y-M_z\) plane, and that are anchored not to the CSM moments, but to reduced moments \(M_{R,y}\) and \(M_{R,z}\) that take account of the axial load. The equation form of Eqn (5.70) is used due to its proven ability in modelling interaction curves, but the denominators change as it is inappropriate to normalise to CSM moments for \(N > 0\), as \(M_{csm,y}\) and \(M_{csm,z}\) can only be reached when \(N = 0\). The equation contains reduced moment normalised terms, raised to exponents \(\beta\) and \(\gamma\).
\[ \left( \frac{M_y}{M_{R,y}} \right)^\beta + \left( \frac{M_z}{M_{R,z}} \right)^\gamma \leq 1 \] (5.70)

The moments \( M_{R,y} \) and \( M_{R,z} \) are axial load reduced moments, found by re-arranging the axial load and uni-axial bending expressions from Section 5.1.5 into the forms of Eqn (5.71) and Eqn (5.72) by using \( \psi = N/N_{csm} \). The ratio of web area to gross area \( A_w/A \) is \( A_r \), taken as \( A_r = 0.6 \) for circular hollow sections and \( A_r = 0.75 \) or \( A_r = 0.45 \) for the major or minor axis of an elliptical hollow section. Carried forwards from the axial load and uni-axial bending equations is the use of \( a_y = b_y = a_z = b_z = 1 \) for strain ratios below 3, which continues to form the conservative linear interaction for more slender cross-sections. The previous \( b_y \) and \( b_z \) values of 0.82 have been changed to 0.8 with little consequence.

\[ M_{R,y} = M_{csm,y} \left( 1 - \psi^{a_y} \right)^{\frac{1}{b_y}} \] (5.71)

\[ M_{R,z} = M_{csm,z} \left( 1 - \psi^{a_z} \right)^{\frac{1}{b_z}} \] (5.72)
Table 5.5: Exponents $a_y, a_z$ and $b_y, b_z$.

<table>
<thead>
<tr>
<th>$3 \leq \frac{\epsilon_{csm}}{\epsilon_y} &lt; 5$</th>
<th>$5 \leq \frac{\epsilon_{csm}}{\epsilon_y} \leq 15$</th>
<th>$3 \leq \frac{\epsilon_{csm}}{\epsilon_y} \leq 15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UB, UC</td>
<td>RHS, SHS, CHS, EHS</td>
<td></td>
</tr>
<tr>
<td>$a_y$</td>
<td>$A_r + 1.2$</td>
<td>$A_r + 1.2$</td>
</tr>
<tr>
<td>$a_z$</td>
<td>2</td>
<td>$8A_r + 1.2$</td>
</tr>
<tr>
<td>$b_y$</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>$b_z$</td>
<td>1</td>
<td>$0.8 - 0.5A_r$</td>
</tr>
</tbody>
</table>

The design equations simplify, when the appropriate loading terms are taken as zero. For example, when there is no axial load, the reduced moments become the CSM moments and convert Eqn (5.70) into the design bi-axial bending equation, and when either $M_y$ or $M_z$ are zero, the equations collapse into the simple axial load and uni-axial bending forms of $M_z \leq M_{R,z}$ and $M_y \leq M_{R,y}$ respectively. When both the axial and minor axis bending components are zero, Eqn (5.70) gives $M_y = M_{csm,y}$.

5.3.5 Exponents $\beta$ and $\gamma$

The $\beta$ and $\gamma$ exponents are known for $N/N_{csm} = 0$ from Section 5.2.5; now the task is to represent these exponents with new fits that include $N/N_{csm}$ as a variable. Exponents $\beta$ and $\gamma$ are then functions of three variables, shape ($W_{pl,y}/W_{pl,z}$), strain ratio ($\epsilon_{csm}/\epsilon_y$) and axial load ($N/N_{csm}$). The aim is to form simple linear or quadratic fits for use in design, and to eliminate the dependence on the strain ratio when the interaction surfaces overlap. For the two I-sections in Figure 5.66 and Figure 5.67, the $\beta$ exponent decreases smoothly from the initial $N = 0$ starting value to around $\beta = 1$ when $N/N_{csm} < 0.5$, from where it then stays relatively constant before peaking again at high axial loads.
Figure 5.66: UB $\beta$ and $\gamma$ exponents for combined axial load and bi-axial bending.

A $\beta$ fit is plotted that reduces with the axial load and then plateaus, and averages the curves for the 5, 10 and 15 strain ratio interaction surfaces. For the exponent $\gamma$ the variation is much more pronounced, displaying a rapid increase with axial load from $\gamma = 0.8$ up to 10. A non-linear fit is used based on the ratio $W_{pl,y}/W_{pl,z}$ and with a maximum on the exponent $\gamma$ of $\gamma = 8$ for high axial loads. The rapid decline of $\gamma$ after $N/N_{csm} = 0.9$ is waived, as it represents the small kinked portions of the interaction surfaces where the reducing moments $M_{R,y}$ and $M_{R,z}$ are low. The $\beta$ and $\gamma$ fits plotted in Figure 5.66 and Figure 5.67 are

\[
\beta = 2 + 0.15 \frac{W_{pl,y}}{W_{pl,z}} - 5 \left( \frac{N}{N_{csm}} \right)^{1.5} \quad \text{with} \quad \beta \geq 1.3 \quad (5.73)
\]

\[
\gamma = 0.8 + \left( 15 - \frac{W_{pl,y}}{W_{pl,z}} \right) \left( \frac{N}{N_{csm}} \right)^{2.2} \quad \text{with} \quad \gamma \leq 8. \quad (5.74)
\]
Both the UB and UC I-sections share the same $\beta$ and $\gamma$ fits, with the differences between the two cross-section shapes distinguished by the ratio of plastic section moduli $W_{pl,y}/W_{pl,z}$. For the rectangular hollow section in Figure 5.68, the values of $\beta$ are larger than those for $\gamma$ as the cross-section shape is not identical about both bending axes, but the variability in the two exponents is lower than it is for the I-sections. For low axial load levels where $N/N_{csm} < 0.25$, the $\beta$ exponent does not deviate significantly from the starting value at $N = 0$; this is also the case with the exponent $\gamma$ when $N/N_{csm} < 0.5$. After these initial flat regions there is a rapid increase in the exponent $\beta$ up to 4 and a modest increase for the exponent $\gamma$ to 2.5, before a sudden drop with both of the exponents at high axial loads where the interaction surfaces fold back to a triangular shape near $N = N_{csm}$. The plotted fits for $\beta$ and $\gamma$, averaged across the overlapping strain ratio surfaces (SR=3 to SR=15), are given by the following equations.
\[ \beta = 1.75 + \frac{W_{pl,y}}{W_{pl,z}} \left[ 2 \left( \frac{N}{N_{csm}} \right)^2 - 0.15 \right] \quad \text{with} \quad \beta \leq 1.7 + \frac{W_{pl,y}}{W_{pl,z}} \]  

(5.75)

\[ \gamma = 1.6 + \left( 3.5 - 1.5 \frac{W_{pl,y}}{W_{pl,z}} \right) \left( \frac{N}{N_{csm}} \right)^2 \quad \text{with} \quad \gamma \leq 3.7 - \frac{W_{pl,y}}{W_{pl,z}} \]  

(5.76)

Figure 5.68: RHS \( \beta \) and \( \gamma \) exponents for combined axial load and bi-axial bending.

For the SHS shown in Figure 5.69, the \( \beta \) and \( \gamma \) exponents are identical due to \( \frac{W_{pl,y}}{W_{pl,z}} = 1 \). The behaviour with respect to \( N/N_{csm} \) is similar to that of the rectangular hollow section, by possessing initial flat regions for low axial loads and then with increases up to \( \beta = \gamma = 3 \). The rectangular hollow section fit equations collapse to simpler square hollow section forms by substituting \( \frac{W_{pl,y}}{W_{pl,z}} = 1 \) giving

\[ \beta = \gamma = 1.6 + 2 \left( \frac{N}{N_{csm}} \right)^2 \quad \text{with} \quad \beta, \gamma \leq 2.7. \]  

(5.77)
Figure 5.69: SHS $\beta$ and $\gamma$ exponents for combined axial load and bi-axial bending.

Figure 5.70: CHS $\beta$ and $\gamma$ exponents for combined axial load and bi-axial bending.
Figure 5.71: EHS $\beta$ and $\gamma$ exponents for combined axial load and bi-axial bending.

For the circular and elliptical hollow sections shown in Figure 5.70 and Figure 5.71 respectively, exponent values of $\beta = \gamma = 2$ may be used.
5.3.6 Envelopes

Figure 5.72 shows the design equation fits for I-sections with a typical UB and UC, and the numerical model envelopes produced by the interaction surfaces with overlapping strain ratios of 5 and 15. The numbers shown on the figures (Figure 5.72, Figure 5.73 and Figure 5.74) are the axial load levels $N/N_{cs}$ of each contour. For increasing normalised axial load $N/N_{cs}$, the numerical model envelopes widen, and both the contour curves and the design fit curves become rectangular, with the steep, almost straight sides a consequence of the large $\gamma$ exponent values. The design fits approximate well the numerical model by providing either an averaged passage through the middle of the envelopes or by providing a conservative fit.

By contrast, the hollow sections (rectangular, square, circular and elliptical hollow sections) in Figure 5.73 and Figure 5.74 have narrower numerical model envelopes, and spread over the larger overlapping strain ratio range between 3 and 15. The strain ratio averaged design fits again show very good agreement with the numerical model for all normalised axial loads. The rectangular, square, circular and elliptical hollow sections all have very similar interaction surfaces, the main difference being the behaviour of the box-sections with increasing axial loads, by showing more rectangular interaction curves for the higher $N/N_{cs}$ values. The circular and elliptical hollow sections use constant exponent values of $\beta = \gamma = 2$ across all axial loads, which produces interaction curves close to the curve shapes created from the numerical model. For both the hollow cross-sections and the I-sections, the reduced moment anchors $M_{R,y}$ and $M_{R,z}$, which represent the axial load and uni-axial bending curves, provide good moments to normalise the interaction curves to, as the design fits approximate well with the numerical envelopes at the $M_y$ and $M_z$ plot axes.
Figure 5.72: I-section numerical model envelopes and CSM design fits for axial load and bi-axial bending.

Figure 5.73: Box section numerical model envelopes and CSM design fits for axial load and bi-axial bending.
In summary, the CSM design equations for the prediction of cross-section axial load and bi-axial bending capacity are

\[
\left( \frac{M_y}{M_{R,y}} \right)^\beta + \left( \frac{M_z}{M_{R,z}} \right)^\gamma \leq 1 \tag{5.78}
\]

\[
M_{R,y} = M_{csm,y} (1 - \psi)^{\frac{1}{\beta}} \quad \text{and} \quad M_{R,z} = M_{csm,z} (1 - \psi)^{\frac{1}{\gamma}} \tag{5.79}
\]

with the exponents as summarised in Table 5.6 and using \( A_r = A_w/A \) as the ratio of the web area to the gross area, \( A_r = 0.6 \) for circular hollow sections, \( A_r = 0.75 \) and \( A_r = 0.45 \) for the major and minor axes of an elliptical hollow section. The normalised axial load is \( \psi = N/N_{csm} \) and the ratio of major to minor axis plastic section moduli is \( W_r = W_{pl,y}/W_{pl,z} \). For \( 1 \leq \epsilon_{csm}/\epsilon_{y} < 3 \) all the stated exponents are unity.
Table 5.6: CSM design exponents $a_y, a_z, b_y, b_z, \beta$ and $\gamma$ for combined loading.

<table>
<thead>
<tr>
<th></th>
<th>$3 \leq \frac{\epsilon_{csm}}{\epsilon_y} &lt; 5$</th>
<th>$5 \leq \frac{\epsilon_{csm}}{\epsilon_y} \leq 15$</th>
<th>$3 \leq \frac{\epsilon_{csm}}{\epsilon_y} \leq 15$</th>
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<td>UB, UC</td>
<td>RHS, SHS</td>
<td>CHS, EHS</td>
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<td>$a_y$</td>
<td>$A_r + 1.2$</td>
<td>$A_r + 1.2$</td>
<td></td>
</tr>
<tr>
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<td>$2$</td>
<td>$8A_r + 1.2$</td>
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</tr>
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<td>$b_z$</td>
<td>$1$</td>
<td>$0.8 - 0.5A_r$</td>
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</tr>
<tr>
<td>$\beta$</td>
<td>$2 - 1.5\psi \geq 1$</td>
<td>$2 + 0.15W_r - 5\psi^{1.5} \geq 1.3$</td>
<td>$1.75 + W_r (2\psi^2 - 0.15) \leq 1.7 + W_r$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$0.8 + 5\psi^{2.2} \leq 4$</td>
<td>$0.8 + (15 - W_r) \psi^{2.2} \leq 8$</td>
<td>$1.6 + (3.5 - 1.5W_r) \psi^2 \leq 3.7 - W_r$</td>
</tr>
</tbody>
</table>

5.3.7 Residual

In order to assess the conformity between the design interaction surfaces and the numerical results, a residual is defined. The chosen residual definition shown in Figure 5.75a, is the distance between the trial surface represented by the small element containing point $S$, and the associated point $R$ produced by the numerical model. Note that defining the residual with a normal distance from a point on either the numerical model surface or trial surface is inappropriate, as instances can occur where the selected point has no associated point on the other surface, unless the surface resides outside the unit box in the $N_{csm}$, $M_{csm,y}$ and $M_{csm,z}$ domain.

Figure 5.75b shows some residuals between an I-section interaction surface from the numerical model (circle markers) and an example trial surface fit (cross markers), with the longer lengths between the markers signifying higher residual magnitude. As the residuals can start on either the inside or the outside of the numerical model surface, the absolute distances or squared distances need to be taken to determine their magnitude.
Figure 5.75: Residual of a surface fit to the numerical model interaction surface.

The line from the origin to the known numerical point $R$, which has a $(M_y, M_z, N)$ coordinate of $(M'_y, M'_z, N')$, will intersect the proposed surface at point $S$. The distance from the origin to the surface intersection is $|S| = a|R|$, using the following definitions for $|R|$ and the two angles $\omega$ and $\phi$

$$|R| = \sqrt{M'^2_y + M'^2_z + N'^2}$$  \hspace{1cm} (5.80)

$$\tan \phi = \frac{M'_z}{M'_y} \hspace{1cm} \tan \omega = \frac{N'}{\sqrt{M'^2_y + M'^2_z}}$$  \hspace{1cm} (5.81)

where the angle $\phi$ is measured from the $M_y$ axis and the angle $\omega$ from the $M_y-M_z$ plane. The co-ordinate of point $S$ is given by $(M'_{y,s}, M'_{z,s}, N_{s})$, where the following equations show that this surface intersection point is simply a scaled position vector of the point $R$. 

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\[ N_s = |S| \sin \omega = \frac{|S|N'}{|R|} = a\frac{|R|N'}{|R|} = aN' \]  

(5.82)

\[ M_{y,s} = |S| \cos \omega \cos \phi = \frac{|S|\sqrt{M_y'^2 + M_z'^2}}{|R|}\frac{M_y'}{\sqrt{M_y'^2 + M_z'^2}} = a\frac{|R|M_y'}{|R|} = aM_y' \]  

(5.83)

\[ M_{z,s} = |S| \cos \omega \sin \phi = \frac{|S|\sqrt{M_y'^2 + M_z'^2}}{|R|}\frac{M_z'}{\sqrt{M_y'^2 + M_z'^2}} = a\frac{|R|M_z'}{|R|} = aM_z'. \]  

(5.84)

For a trial design surface, the scalar \( a \) must be found for every data point of a numerical model interaction surface, for a given cross-section and strain ratio. This is performed by solving implicitly for the single root of a function \( f \) with the following format

\[ f(aN', aM_y', aM_z') \left( \frac{aN'}{N_{csm}}, \frac{aM_y'}{M_{csm,y}}, \frac{aM_z'}{M_{csm,z}} \right) = 0. \]  

(5.85)

For the proposed CSM design fit in Section 5.3.4, the function \( f \) to find the root \( a \) of is

\[ f = \left( \frac{aM_y'}{M_{csm,y}} \left[ 1 - \left( \frac{aN'}{N_{csm}} \right)^{\frac{1}{a_y}} \right]^{\frac{1}{a_y}} \right) + \left( \frac{aM_z'}{M_{csm,z}} \left[ 1 - \left( \frac{aN'}{N_{csm}} \right)^{\frac{1}{a_z}} \right]^{\frac{1}{a_z}} \right) - 1. \]  

(5.86)

The residuals are then presented as \((1 - a)100\), where a positive value of the residual corresponds to the numerical point lying outside the design interaction surface and so a conservative prediction of capacity (for a negative value, the opposite is true).

### 5.3.7.1 Residual plots

Figure 5.76 to Figure 5.78 display plots of the residuals as percentages on \( \epsilon_{csm}/\epsilon_y = 15 \) interaction surfaces, for a typical cross-section of each of the six shapes. For these residual plots, each greyscale map is calibrated to be 50% grey at a residual of 0%, white at the
maximum magnitude and black at the minimum residual magnitude. For the I-sections in Figure 5.76 the magnitude of the residuals are less than 3.5%, with the selected UB on the safe side across the interaction surface and the UC slightly non-conservative across a band on the surface at low axial loads. For both I-sections, regions of more negative residuals appear towards the centre of the surfaces, with two other concentrated negative areas at the $M_{csm,y}$ and $N_{csm}$ end points.

![Image](a) UB  
![Image](b) UC

Figure 5.76: Residuals between the CSM design interaction surfaces and the numerical results, for I-sections with a strain ratio of 15.

The residual plots are more uniform for the box sections in Figure 5.77, showing no significant high or low regions and displaying residuals of lower magnitudes, with the highest residuals around 2.5%. The surface centres show areas of residuals with somewhat higher positive values. The residuals are lower for the circular and elliptical hollow sections in Figure 5.78, which show a band of more positive values at $N/N_{csm} = 0.5$ and more negative values either side.
Figure 5.77: Residuals between the CSM design interaction surfaces and the numerical results, for box sections with a strain ratio of 15.

Figure 5.78: Residuals between the CSM design interaction surfaces and the numerical results, for circular and elliptical hollow sections with a strain ratio of 15.
5.3.7.2 Mean and standard deviation

The standard deviation of the residuals for a given cross-section and strain ratio, gives a measure of the goodness of fit between the proposed design surfaces and the numerical model surfaces. The mean of the residuals will show if the design surface fits have generally a conservative tendency with a positive mean, or an unconservative tendency with a negative mean. The mean and standard deviation of the residuals for each interaction surface and using the design fits are plotted in Figure 5.79 to Figure 5.81, with a strain ratio range from 3 to 15.

The I-sections in Figure 5.79 have a positive (conservative) mean except for the UB at strain ratios above 10, while the UC are predicted more safely by an average of 1-2%. The two parts of the I-section design fit for strain ratios above and below 5, give significant differences in the mean and standard deviation values, with the simpler $\epsilon_{csm}/\epsilon_y < 5$ portion being more conservative. For $\epsilon_{csm}/\epsilon_y \geq 5$ the standard deviation is approximately constant at 1% for the UB, and is slightly lower for the UC at 0.75%. For the box sections in Figure 5.80, which have continuous combined loading design fits for the strain ratio range $3 \leq \epsilon_{csm}/\epsilon_y \leq 15$, there exists a slight mean over-prediction for a selection of low strain ratios and a conservative mean of up to 2.5% for all other strain ratios.

Square hollow sections are predicted less conservatively but more accurately, as they have a lower standard deviation due to the increased variability in the cross-section geometry of the rectangular hollow sections (two cross-sections in particular are distinct from the RHS group). Both box section shapes however have low standard deviations, mostly below 1% for the entire strain ratio range. The same plots for the simple circular and elliptical hollow section fits ($\beta = \gamma = 2$) are plotted in Figure 5.81, which displays means and standard deviations similar in behaviour to the square hollow section case. No observable distinction between the two cross-section shapes is found in the mean, but less residual variation is found with circular hollow sections from a reduced standard deviation.
Figure 5.79: Mean and standard deviation of the residuals for I-sections.

Figure 5.80: Mean and standard deviation of the residuals for box sections.
Figure 5.81: Mean and standard deviation of the residuals for circular and elliptical hollow sections.
5.3.8 Example calculation

Calculate whether the square hollow section in Figure 5.82a can withstand the design combined loading $N_{Ed}/N_y = 0.2$, $M_{Ed,y}/M_{el,y} = 0.1$ and $M_{Ed,z}/M_{el,z} = 0.3$.

![Cross-section geometry and material properties](image)

(a) Cross-section geometry and material properties

![Elastic stress distribution](image)

(b) Elastic stress distribution

Figure 5.82: Square hollow section example for combined loading.
Combined loading cross-section slenderness and strain ratio

The scalar $a$ representing the intersection of the applied loading with the elastic linear interaction surface is

$$a = \left( \frac{N_{Ed}}{N_y} + \frac{M_{Ed,y}}{M_{el,y}} + \frac{M_{Ed,z}}{M_{el,z}} \right)^{-1} = (0.2 + 0.1 + 0.3)^{-1} = 1.667. \quad (5.87)$$

At $z = -B/2, y = -D/2$ the least compressive stress (in this case tensile) is

$$\frac{f_c}{f_y} = 2a \frac{N_{Ed}}{N_y} = 1 - 2(1.667)(0.2) = -0.3332. \quad (5.88)$$

For the corner at $z = -B/2, y = D/2$ the stress is

$$\frac{f_c}{f_y} = 1 - 2a \frac{M_{Ed,z}}{M_{el,z}} = 1 - 2(1.667)(0.3) \approx 0 \quad (5.89)$$

and for the opposite corner at $z = B/2, y = -D/2$

$$\frac{f_c}{f_y} = 1 - 2a \frac{M_{Ed,y}}{M_{el,y}} = 1 - 2(1.667)(0.1) = 0.6666. \quad (5.90)$$

This gives the elastic cross-section stress distribution shown in Figure 5.82b. From EN 1993-1-5 (2006), this elastic stress distribution gives a critical element with $\sigma_2/\sigma_1 = 0.6666$, and so $k_\sigma = 8.2/(1.05 + 0.6666) = 4.78$. The cross-section slenderness and strain ratio are

$$\bar{\lambda}_p = \sqrt{\frac{12(1 - \nu^2)235}{\pi^2 E k_\sigma}} \left( \frac{c}{te} \right) = \sqrt{\frac{12(1 - 0.3^2)235}{\pi^2(210000)(4.78)}} \left( \frac{270}{15(0.814)} \right) = 0.3559 \quad (5.91)$$

$$\frac{\epsilon_{csm}}{\epsilon_y} = 0.25 \frac{\lambda_p^{3.6}}{0.3559^{3.6}} = 10.31 \quad \text{where} \quad \frac{\epsilon_{csm}}{\epsilon_y} \leq 15. \quad (5.92)$$
Design axial and bending capacities

\[
\frac{f_u}{f_y} > 1.1 \quad \text{and so} \quad E_{sh} = E/100; \quad \frac{W_{el,y}}{W_{pl,y}} = \frac{1548}{1829} = 0.8464 \quad (5.93)
\]

\[
\frac{M_{csm,y}}{M_{pl,y}} = \frac{M_{csm,z}}{M_{pl,z}} = 1 + \frac{E_{sh}}{E} \frac{W_{el,y}}{W_{pl,y}} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right) - \left( 1 - \frac{W_{el,y}}{W_{pl,y}} \right) \left( \frac{\epsilon_{csm}}{\epsilon_y} \right)^2
\]

\[
= 1 + 0.01(0.8464)(10.31 - 1) - (1 - 0.8464)(10.31)^{-2} = 1.077 \quad (5.94)
\]

\[
M_{csm,y} = 1.077W_{pl,y}f_y = 1.077(1829000)(355)/10^6 = 699 \text{ kNm} \quad (5.95)
\]

\[
\frac{N_{csm}}{N_y} = 1 + \frac{E_{sh}}{E} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right) = 1 + 0.01 (10.31 - 1) = 1.093 \quad (5.96)
\]

Combined loading

With \( A_r = A_w/A = 2(270)(15)/(300^2 - 270^2) = 0.4737 \), \( \psi = 0.2/1.093 = 0.1830 \) and \( W_r = W_{pl,y}/W_{pl,z} = 1 \), the \( a_y, a_z, b_y, b_z, \beta \) and \( \gamma \) exponents are calculated as

\[
a_y = a_z = A_r + 1.2 = 0.4737 + 1.2 = 1.674 \quad \text{and} \quad b_y = b_z = 0.8 \quad (5.97)
\]

\[
\beta = \gamma = 1.75 + W_r(2\psi^2 - 0.15) = 1.75 + [2(0.183)^2 - 0.15] = 1.667. \quad (5.98)
\]

\[
M_{R,y} = M_{R,z} = M_{csm,y} \left( 1 - \psi^{a_y} \right)^{\frac{1}{b_y}} = 699 \left( 1 - 0.183^{1.674} \right)^{\frac{1}{0.8}} = 648.5 \text{ kNm}. \quad (5.99)
\]

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The applied moments are $M_y = M_{Ed,y} = 0.1 M_{el,y} = 0.1[1.548(355)] = 54.95 \text{kNm}$ and $M_z = M_{Ed,z} = 0.3 M_{el,z} = 0.3[1.548(355)] = 164.9 \text{kNm}$, giving

$$\left( \frac{M_y}{M_{R,y}} \right)^{\beta} + \left( \frac{M_z}{M_{R,z}} \right)^{\gamma} = \left( \frac{54.95}{648.5} \right)^{1.667} + \left( \frac{164.9}{648.5} \right)^{1.667} = 0.118 \quad (5.100)$$

which is less than 1 and so the combined loading is handled safely. By inspection this could have been predicted as the scalar $a$ was greater than 1, indicating that the design loading point lies beneath the elastic interaction surface.

### 5.4 Summary

In this chapter, the CSM strain limited concepts for axial and bending resistances alone have been extended to cover cross-sections under combined axial load and uni-axial or bi-axial bending. The numerical model used a planar sections remain plane assumption to combine uniform strains with linearly varying strains to find a set of strain distributions and then formulate interaction curves and surfaces.

A curve fitting method was employed which optimised the proposed design expression form through each interaction curve. The design equation produced curves on planes that were parallel to the $M_y-M_z$ plane and that were anchored to the axial load and uni-axial bending curves. The accuracy of the design equation was very good, deviating little from the numerical model envelopes.

A residual was defined as the distance between the design surface and the numerical model surface with a line projected from the origin. The mean and standard deviation of the residuals for each cross-section and strain ratio, showed that the means were generally less than 3% and on the conservative side, while the standard deviation was usually found to be below 1.5%.
The axial load and uni-axial bending interaction curves were transformed into moment–
curvature–thrust curves by considering the strain states that gave the same axial load. This produced curves of varying flexural rigidity, as functions of curvature and axial load.

As any combination of axial load and bending moments may need designing for, expressions were formed which designated the equivalent elastic stress distribution in the cross-section for use in cross-section slenderness calculations.
Chapter 6
Laboratory testing

6.1 Introduction

Rectangular hollow section members of grade S355 steel were tested by the author in a range of structural configurations at the University of Applied Sciences of Western Switzerland in Fribourg. The tests were conducted to provide information on hot-rolled hollow sections via axial and bending tests, as the majority of gathered test data has been based on cold-formed cross-sections and hot-rolled I-sections. The nominal cross-section depth and width were $D = 120$ mm and $B = 80$ mm, with two thickness values, $T = 4$ mm and $T = 5$ mm. Specimens were cut for flat face tensile coupons, stub columns and simply supported beams in four and three-point bending. The members started at lengths of $L = 6.0$ m and were labelled with the identification numbers 41-44 and 51-54, where the first digit indicates the plate thickness and the second digit represents the beam number. Tensile coupon tests utilised the final 630 mm end-lengths of the members, with the identifier TF for the flanges and TW for the webs to indicate which face the coupon was cut from.
A total of 32 tensile coupons were tested, four for each member (Section 6.2). Eight stub columns of nominal length $L = 370 \text{ mm}$ were labelled 1ST5–4ST5 for $T = 5 \text{ mm}$ and 1ST4–4ST4 for $T = 4 \text{ mm}$. The remaining 5.0 m lengths of each member were used for simple four- and three-point bending tests (Section 6.4) all with spans of 2.3 m. A general view of the laboratory is shown in Figure 6.1a.

### 6.2 Material testing

Tensile coupon tests were performed on the material extracted from the flat faces of the members to give a total of 32 specimens. Each coupon was cut from the ends of the members and away from the weld location. The identification convention consists first of the member number, followed by the web or flange identifier TW or TF, and finally by the letter R for a repeat; for example, coupon 42TF is from member number 2, of 4 mm thickness and taken from the flange. The tensile coupons were 300 mm long, with nominal
cross-section dimensions of 5 mm × 10 mm or 4 mm × 10 mm, and the material had a nominal yield stress of 355 N/mm² and was hot-rolled structural steel. After cutting the coupons and edge cleaning, the cross-section dimensions \( B \) and \( T \) were recorded by a micrometer at three locations along the middle of the coupons; they were then tested in the rig shown in Figure 6.2. Once a tensile coupon was gripped in place, a 20 mm clip gauge was attached, and a constant rate of strain (0.045%/s) was applied. Of the 32 coupon tests only 41TWR and 54TFR had measurement issues, related to the ultimate and fracture strains, and coupon 54TWR was not tested due to cutting complications. The tested coupons are shown in Figure 6.3. Two representative stress–strain curves are plotted in Figure 6.4 and Figure 6.5, which show that the 4 mm and 5 mm thickness coupons behaved similarly, but with the former possessing both higher yield and ultimate stresses. After the initial elastic region a yield plateau is observed, before strain hardening initiates up until the ultimate tensile stress.

![Testing rig and specimen](image)

(a) Testing rig          (b) Specimen and attached clip gauge

Figure 6.2: Tensile coupon test rig, and a gripped specimen.
Table 6.1: Measured tensile coupon geometry and material properties.

<table>
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<tr>
<th>ID</th>
<th>B [mm]</th>
<th>T [mm]</th>
<th>E [N/mm²]</th>
<th>$f_y$ [N/mm²]</th>
<th>$f_u$ [N/mm²]</th>
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Figure 6.3: Tensile coupons after testing.

The recorded yield stress $f_y$ was taken as the average of the yield plateau between the strains of 0.5% and 3%, while the Young’s modulus was taken as the gradient between 20% and 80% of $f_y$.

Figure 6.4: Stress–strain curve of 4 mm coupon 43TF.
All coupon data are summarised in Table 6.1, in which $f_u$ is the ultimate tensile stress, $\varepsilon_u$ is the strain at the ultimate tensile stress and $\varepsilon_f$ is the fracture strain over a 20 mm gauge length ($\bar{E}$, $\bar{f}_y$ and $\bar{f}_u$ are the coupon averaged values). The recorded average values are: Young’s modulus $E = 203,900$ N/mm$^2$, yield and ultimate stresses $f_y = 425$ N/mm$^2$, $f_u = 525$ N/mm$^2$ (4 mm) and $f_y = 398$ N/mm$^2$, $f_u = 506$ N/mm$^2$ (5 mm), and the ultimate and fracture strains for the 4 mm and 5 mm thickness as $\varepsilon_u = 0.171$, $\varepsilon_f = 0.267$ and $\varepsilon_u = 0.186$, $\varepsilon_f = 0.286$.

### 6.3 Stub columns

Stub columns of length $L = 370$ mm were cut from the ends of each 6.0 m long member, providing specimens that were three times longer than the cross-section depth of $D = 120$ mm. The identifiers are such that 2ST5 represents the stub column from member 2 of 5 mm wall thickness. The eight tests were partitioned into two groups: 1) of 4 mm thickness
and with identifiers 1ST4 to 4ST4 (four tests) and 2) of 5 mm thickness and with identifiers 1ST5 to 4ST5 (four tests). Cross-section dimensions were recorded by micrometer at both ends of the stub columns, and are summarised in Table 6.2, for which the subscripts 1 to 4 represent repeat measurements. The internal root radii were found to be approximately equal to the wall thickness \( T \). The stub columns were weighed before testing, and the resulting mass \( m \), cross-section dimensions and assumed density \( \rho = 7850 \text{ kg/m}^3 \) were used to calculate the cross-section area \( A \). An array of four 50 mm induction displacement transducers were positioned at the corners of the bottom end-plate to measure the overall end-shortening. Four strain gauges were attached to the column mid-faces to measure the true local strains. The strain gauges were HBM 120\( \Omega \) resistance and 10 mm in length. The testing machine was a 3000 kN capacity Walter+Bai hydraulic rig, shown in Figure 6.6a; a stub column, with the end-plates, strain gauges and displacement transducers, is shown in Figure 6.6b. The stub columns were loaded under displacement control beyond their peak loads, with the rate of displacement kept constant at 0.025 mm/s until the peak load, and then the rate increased soon after. The locally deformed shapes consisted of alternate inward and outward buckles. For five of the stub columns these buckles formed near the column bases (1ST4, 2ST4, 3ST4, 2ST5 and 3ST5), for two specimens the local buckling occurred at mid-height (4ST4 and 4ST5) and for 1ST5 they formed near the top. Photos of the stub columns after testing are shown in Figure 6.7 and Figure 6.8.

The raw data for the end–shortening measurements from the LVDTs (Linear variable displacement transducers) and the strains from the strain gauges are shown for each column in Figure 6.9 to Figure 6.16. In these figures the axial load \( N \) is normalised by the yield load \( N_y \) (calculated as the product of the cross-section area \( A \) and the tensile coupon yield stress \( f_y \)), the end–shortening \( \delta \) is divided by the initial length \( L \), and the measured strains \( \varepsilon \) are divided by the yield strain \( \varepsilon_y \).
Table 6.2: Measured geometric properties, normalised ultimate loads, and strains $\epsilon_{ub}$ at ultimate load from the stub column tests.

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Figure 6.6: Stub column test arrangement (pre-test).

(a) Specimens 1ST4 and 2ST4  
(b) Specimens 3ST4 and 4ST4

Figure 6.7: 4 mm post-test deformed stub columns.
Figure 6.8: 5 mm post-test deformed stub columns.

Figure 6.9: Raw load versus end-shortening and strain gauge measurements for stub column 1ST4.
Figure 6.10: Raw load versus end–shortening and strain gauge measurements for stub column 2ST4.

Figure 6.11: Raw load versus end–shortening and strain gauge measurements for stub column 3ST4.
Figure 6.12: Raw load versus end–shortening and strain gauge measurements for stub column 4ST4.

Figure 6.13: Raw load versus end–shortening and strain gauge measurements for stub column 1ST5.
Figure 6.14: Raw load versus end–shortening and strain gauge measurements for stub column 2ST5.

Figure 6.15: Raw load versus end–shortening and strain gauge measurements for stub column 3ST5.
All specimens reached loads greater than the yield load. For the 4 mm thickness stub columns, the peak loads were reached soon after yielding at LVDT end–shortenings of $\delta/L \approx 0.005$, and then post-peak unloading occurred quite suddenly with the axial loads reducing to below 80% of $N_y$ by $\delta/L \approx 0.01$. The strain gauges show that after the cross-sections reached the yield load, local buckling occurred and the strain gauge measurements became distorted, as the strains escalated rapidly or reversed sign entirely. In general, two strain gauges stayed in compression and two reversed into tension, based on their location on either the tensile or compressive faces of a buckled shape. Since there were differences between the end–shortening measurements from the displacement transducers and those calculated from the strain gauges, a correction was required that combined both sets of measurements. The strain gauges provided the correct initial Young’s modulus as they were directly in contact with the column faces, but they gave less useful information when influenced by local buckling. The displacement transducers provided good post-yield
information but included deformation of the end-plates, leading to a Young’s modulus that was found to be in error by up to 50%. The correcting method described by the Centre for Advanced Structural Engineering (1990) is used, which assumes that the set-up effects are proportional to the applied stress. This method uses the factor $k$ from Eqn (6.1)

$$k = \frac{L}{2} \left( \frac{1}{E_{\text{LVDT}}} - \frac{1}{E_{\text{SG}}} \right) \quad (6.1)$$

where $E_{\text{LVDT}}$ is the Young’s modulus calculated from the LVDT readings and $E_{\text{SG}}$ is the equivalent calculation of $E$ from the strain gauges. The corrected end displacement $\delta_c$ is then the difference between the LVDT displacements and the set-up displacements as $\delta_c = \delta_{\text{LVDT}} - 2kf$, where the stress $f = N/A$.

The corrected curves are given in Figure 6.17 to Figure 6.20, where the mean values from the four LVDTs and the four strain gauges were used. The peak loads $N_u$ and local buckling strains $\epsilon_{lb}$ are also plotted; these are tabulated in Table 6.2. The average yield normalised peak load $N_u/N_y$ for the four 4 mm thickness stub columns is 1.062, and the average peak strain for these specimens is 0.256%. The average normalised peak load $N_u/N_y$ for the 5 mm thickness stub columns (excluding 2ST5) is 1.044, and the average peak strain for these specimens is also 0.256%. As the peak loads and peak strains are similar for the two stub column groups, no differences in the post-peak behaviours will be registered if the peak strains are used. The corrected curves also show the strain $\epsilon_{lb}$, which marks where the stub columns first unload sharply. In these stub column tests, the load decreases beyond the initial peak load (located near the yield strain) and then the columns unload differently for the 4 mm and 5 mm plate thickness, with the latter exhibiting a shallower unloading gradient.
Figure 6.17: Corrected load versus end–shortening data for stub columns 1ST4 and 2ST4.

Figure 6.18: Corrected load versus end–shortening data for stub columns 3ST4 and 4ST4.
Figure 6.19: Corrected load versus end–shortening data for stub columns 1ST5 and 2ST5.

Figure 6.20: Corrected load end–shortening data for stub columns 3ST5 and 4ST5.
6.4 Bending tests

6.4.1 Four-point bending

As part of the experimental programme, two simply supported beams of total lengths $L_T = 2500$ mm were tested under four-point bending. The identifiers for these two members are SS5.4P and SS4.4P for the nominal $T = 5$ mm and $T = 4$ mm thickness respectively. The experimental set-up is shown schematically in Figure 6.21a and the post-test deformed state of beam SS5.4P is given in Figure 6.21b. The cross-section geometries in Table 6.3 were measured at both beam ends and all readings were taken by a micrometer with repeat measurements (subscripts 1 through 4). The nominal cross-section depth and width for both members were $D = 120$ mm and $B = 80$ mm. The final 100 mm of the beams overhung the supports, giving spans of $L = 2300$ mm. The elastic and plastic section moduli were calculated using the EN 10210-2 (2006) method for box sections. This method takes the internal corner radius as $T$, and the external corner radius as $1.5T$; these assumptions were found to closely match the actual cross-section geometry.

Figure 6.21: Simply supported four-point bending tests.
Table 6.3: Measured geometry and normalised ultimate moments for the simply supported four-point bending tests.

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The following parameters are used

\[ A_g = \left(1 - \frac{\pi}{4}\right) \left(1.5T\right)^2 \quad A_k = \left(1 - \frac{\pi}{4}\right) T^2 \] (6.2)

\[ h_g = \frac{D}{2} - \frac{(10 - 3\pi)1.5T}{12 - 3\pi} \quad h_k = \frac{D - 2T}{2} - \frac{(10 - 3\pi)T}{12 - 3\pi} \] (6.3)

\[ I_g = \left[\frac{1}{3} - \frac{\pi}{16} - \frac{1}{3(12 - 3\pi)}\right] \left(1.5T\right)^4 \quad I_k = \left[\frac{1}{3} - \frac{\pi}{16} - \frac{1}{3(12 - 3\pi)}\right] T^4 \] (6.4)

to calculate the second moment of areas \( I_y \) and \( I_z \), the elastic section moduli \( W_{el,y} \) and \( W_{el,z} \), and the plastic section moduli \( W_{pl,y} \) and \( W_{pl,z} \),

\[ I_y = \frac{1}{12} \left[BD^3 - (B - 2T)(D - 2T)^3\right] - 4(I_g + A_g h_g^2) + 4(I_k + A_k h_k^2) \] (6.5)

\[ I_z = \frac{1}{12} \left[DB^3 - (D - 2T)(B - 2T)^3\right] - 4(I_g + A_g h_g^2) + 4(I_k + A_k h_k^2) \] (6.6)

\[ W_{el,y} = \frac{2I_y}{D} \quad W_{el,z} = \frac{2I_z}{B} \] (6.7)
\[ W_{pl,y} = \frac{BD^2}{4} - \frac{(B - 2T)(D - 2T)^2}{4} - 4A_g h_g + 4A_k h_k \]  
(6.8)

\[ W_{pl,z} = \frac{DB^2}{4} - \frac{(D - 2T)(B - 2T)^2}{4} - 4A_g h_g + 4A_k h_k. \]  
(6.9)

The two beams were loaded under displacement control via a loading bridge that consisted of the following features:

- An I-section floor beam that was connected to a strong floor by two high-strength 32 mm diameter bars, which were threaded through two thick plates mounted on the top flange of the floor beam (see Figure 6.22a).

- Connected into the top flange of the floor beam were two 250 kN hydraulic jacks, each supported by four threaded bars (Figure 6.22b). Load cells LC3 and LC4, which were located underneath the jacks, measured the jack forces, and a single displacement transducer LVDT0 fed the vertical displacements to the hydraulic control panel.

- A rectangular hollow section cross-beam was used to link the two jacks, and was supported by threaded bars that passed through the centre of the jacks. The cross-beam was then bolted into a double webbed I-section spreader beam, which had half-round loading points screwed into the beam underside.

- The jacks were pre-loaded with 5.0 kN of force by zeroing the load cells and then tightening the end nuts during live measurement readings. The masses of the cross-beam and spreader beam were recorded, and the weights added to the pre-load force.

Roller supports were arranged at each beam end, with the right support layered with a film of grease between the beam underside and the supporting plate. Two load cells LC1 and LC2 were placed underneath the left support and a thick plate of equal height inserted underneath the right support to make up the height difference. The left support
is shown in Figure 6.23 and is similar to the right configuration. The plates supporting the beams were 80 mm wide and 30 mm thick, rested upon 30 mm diameter rollers, and extended further than the width of the cross-sections. The plate and roller pairs were positioned between angles that were stiffened by welded plates, with gaps left to allow for the activation of the load cells. The support angles were bolted into I-section columns that were anchored into the strong floor by threaded bars. Instrument readings were monitored at a frequency of 2 Hz through HBM CatmanEasy software and using HBM Spider8 data acquisition hardware. Several low load elastic cycles were initially performed to achieve load symmetry between the jacks and for verticality of the loading bridge. This was performed by adjusting components within the set-up using a spirit level and by monitoring LVDT and load cell readings. After the initial jack pre-loading, the set-up was zeroed and loaded up to and beyond the peak load. The slowest loading rate of the hydraulic machine was maintained for a period of time after the system peak, from which the rig loading rate was increased until the test ended.

(a) Floor beam, threaded bars and plates

(b) Hydraulic jacks and rectangular hollow section cross-beam

Figure 6.22: Floor beam to strong floor connection and loading bridge.
Figure 6.23: Left roller support.

Figure 6.24: Deflected shapes of the four-point bending tests during testing.

An array of seven, 100 mm displacement transducers (labelled LVDT1 to LVDT7 in Figure 6.21a) measured the vertical displacements underneath the beams at fixed horizontal
distances from the centre-line of the left support; these distances were \( x = [400, 730, 950, 1150, 1350, 1570, 1900] \) mm. Figure 6.24 shows smoothed spline fits through the LVDT readings, with the vertical displacements \( v \) plotted as percentages of the span length \( L \). Loads up to the peak load gave deflected beam shapes that were approximately symmetric, but for post-peak unloading the beam deflections favoured the right and left loading points for the SS5.4P and SS4.4P beams respectively. The maximum deflections at the peak loads were around 27 mm, which is equivalent to 1.2\% of \( L \).

For normalising the beam end rotations it is necessary to calculate a yield rotation \( \theta_y \), which for the four-point bending configuration is the rotation at the beam ends when the cross-sections between the point loads first reach the elastic moment. This idea is similar to the yield curvature \( \kappa_y \), which gives the curvature at first yield. For elastic loading and assuming small deflections, the curvature and bending moment distributions are related by the following equation

\[
\kappa = \frac{d^2v}{dx^2} = -\frac{M}{EI} = -\frac{1}{EI} (P[x] - P[x-a] - P[x-b]) (6.10)
\]

for which \( P \) is the vertical reaction at each support (half the total load \( P_T \)), \( EI \) is the flexural rigidity, and \( a \) and \( b \) are the distances of the left and right point loads from the left support. The slope distribution is found by integrating the curvature distribution,

\[
\theta = \frac{dv}{dx} = \int \frac{d^2v}{dx^2} dx = -\frac{P}{EI} \left( \frac{[x]^2}{2} - \frac{[x-a]^2}{2} - \frac{[x-b]^2}{2} \right) + K_1 (6.11)
\]

and then the displacements \( v \) are found by integrating once more, with \( K_1 \) and \( K_2 \) as the constants of integration

\[
v = \int \frac{dv}{dx} dx = -\frac{P}{EI} \left( \frac{[x]^3}{6} - \frac{[x-a]^3}{6} - \frac{[x-b]^3}{6} \right) + K_1x + K_2. (6.12)
\]
Applying the boundary condition \( x = 0, v = 0 \) gives \( K_2 = 0 \), and the boundary condition at the right end \( x = L, v = 0 \) gives

\[
0 = -\frac{P}{6EI} \left( L^3 - [L - a]^3 - [L - b]^3 \right) + K_1 L. \tag{6.13}
\]

By using \( a = 730 \text{ mm}, b = 1570 \text{ mm}, L = 2300 \text{ mm} \) and \( M_{el} = Pa = EI\kappa_y [\text{Nmm}] \), constant \( K_1 \) is calculated to be

\[
K_1 = 5.73 \times 10^5 \frac{P}{EI} = 785\kappa_y \tag{6.14}
\]

which is the first yield end rotation \( \theta_y \) at \( x = 0 \). Inclinometers INCL1 and INCL2 that were accurate in a ±10 degree range were fixed at both beam ends to record the end rotations. The left end rotation \( \theta \) is normalised by \( \theta_y \) and plotted in Figure 6.25 with the moment under the point loads \( M_y = Pa \), which is divided by the plastic moment \( M_{pl,y} \) as calculated from the measured cross-section properties.

Figure 6.25: Normalised moment–rotation curves for the two four-point bending beams.
The initial low load seating effects were corrected to match the elastic gradient between 0.6 and 0.7 of $M_{pl,y}$. The $M_y$-$\theta$ graphs in Figure 6.25 show the increased post-peak strength of the SS5.4P beam compared to the SS4.4P beam. The peak moments were reached at a rotation ratio of about $\theta/\theta_y = 1.3$.

Figure 6.26 plots the displacement readings $v$ from instruments LVDT2, LVDT4 and LVDT6, which were the displacement transducers located underneath the beam loading points and at mid-span. These measured displacements $v$ are normalised by the theoretical elastic displacements $v_e$, which are calculated from the measured cross-section geometry and material properties.

![Figure 6.26: LVDT2, LVDT4 and LVDT6 displacement readings for 4-point bending tests.](image)

The method of Chan and Gardner (2008b) is used for calculating the curvature $\kappa$ of the central region of a simply supported beam in four-point bending. This takes the central region as the distance between the two loading points and under constant bending moment as the segment of a circular arc with radius of curvature $R$,..
\[
\kappa = \frac{1}{R} = \frac{8(D_M - D_L)}{4(D_M - D_L)^2 + l^2}
\] (6.15)

where \(D_L\) is the average of the displacement readings underneath the point loads from LVDT2 and LVDT6, \(D_M\) is the LVDT4 displacement reading at mid-span, and \(l\) is the distance between the point loads, approximately one third of the beam length \(l = 840\, \text{mm}\).

With the curvatures \(\kappa\) known, the moment–curvature \((M_y - \kappa)\) curves are plotted in Figure 6.27, with moments normalised by the plastic moment and the curvatures \(\kappa\) normalised by the major axis yield curvature \(\kappa_{y,y}\). The curve shapes are similar to the \(M_y - \theta\) curves but have sharper unloading portions, and the peak moments are found at curvature ratios \(\kappa/\kappa_{y,y}\) between 1.15 and 1.25. Similar to the stub column tests in Section 6.3, the local buckling curvature \(\kappa_{lb}\) has been located at the point of sharp unloading.

![Figure 6.27: Normalised moment–curvature curves for the four-point bending tests.](image)
6.4.2 Three-point bending

Two beams labelled SS4.3P and SS5.3P of 4 mm and 5 mm thickness respectively, were tested under three-point bending in the configuration of Figure 6.28. These tests had a total beam length of $L_T = 2500$ mm and had a clear span of $L = 2300$ mm due to 100 mm overhangs at each end. The members were of the same nominal cross-section dimensions as the four-point bending tests and are summarised in Table 6.4. Measurements were by micrometer with repeat readings taken, and the elastic and plastic section moduli were calculated by the EN 10210-2 (2006) method described previously. The loading bridge configuration and support arrangements were identical to those described for four-point bending, with the exception that a 60 mm wide plate was used as a loading point. The local failure mode underneath the single loading point is displayed in Figure 6.29a, and shows a downward buckle of the top flange paired with outward buckling at the compressive regions of the webs. Slight local buckling was observed near the peak load, and then large local deformations were seen during the unloading phase of the tests. A single LVDT was placed in contact with the underside of the beam at mid-span, and two pairs of LVDTs separated by a distance of 220 mm were positioned horizontally at the beam ends. This arrangement as seen in Figure 6.29b, was to check that the measurements from the inclinometers were in agreement with rotations calculated at the beam ends.

![Figure 6.28: Three-point bending of simply supported beams SS4.3P and SS5.3P.](image-url)
Table 6.4: Measured cross-section geometry for the three-point bending tests.

<table>
<thead>
<tr>
<th>ID</th>
<th>$D_1$ [mm]</th>
<th>$D_2$ [mm]</th>
<th>$B_1$ [mm]</th>
<th>$B_2$ [mm]</th>
<th>$T_1$ [mm]</th>
<th>$T_2$ [mm]</th>
<th>$T_3$ [mm]</th>
<th>$T_4$ [mm]</th>
<th>$L_T$ [mm]</th>
<th>$M_{u,y}/M_{pl,y}$</th>
<th>$\theta_u/\theta_y$</th>
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<td>SS4.3P</td>
<td>119.87</td>
<td>119.66</td>
<td>80.02</td>
<td>80.35</td>
<td>3.90</td>
<td>4.08</td>
<td>3.89</td>
<td>3.87</td>
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<td>0.876</td>
<td>1.279</td>
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<td>119.77</td>
<td>80.08</td>
<td>80.26</td>
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<td>3.92</td>
<td>3.84</td>
<td>3.87</td>
<td></td>
<td></td>
<td></td>
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<tr>
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<td>119.60</td>
<td>119.78</td>
<td>80.12</td>
<td>80.26</td>
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<td>4.70</td>
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<td>2499</td>
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<td>4.68</td>
<td>4.73</td>
<td>4.82</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Failure mode underneath loading points  
(b) Inclinometer and LVDTs at left beam end (right similar)

Figure 6.29: Local buckling mode and beam end instrumentation detail.

The calculation of the yield rotation $\theta_y$ for the three-point bending set-up, starts with the curvature distribution along the beam

$$\kappa = \frac{d^2v}{dx^2} = -\frac{M}{EI} = -\frac{1}{EI} \left( P[x] - 2P[x - L/2] \right)$$

(6.16)
for which $P$ is the reaction at each support, equal to half of the total load $P_T$. The slope distribution is found by integrating the curvature distribution to give

$$\theta = \frac{dv}{dx} = \int \frac{d^2v}{dx^2} \, dx = -\frac{P}{EI} \left( \frac{x^2}{2} - \frac{2[x - L/2]^2}{2} \right) + K_1 \tag{6.17}$$

and the displacements $v$ are found by integrating once more

$$v = \int \frac{dv}{dx} \, dx = -\frac{P}{EI} \left( \frac{x^3}{6} - \frac{2[x - L/2]^3}{6} \right) + K_1x + K_2. \tag{6.18}$$

Applying the left end boundary condition $x = 0, v = 0$ gives $K_2 = 0$, and invoking the right end boundary condition $x = L, v = 0$ gives

$$0 = -\frac{P}{6EI} \left( L^3 - 2[L/2]^3 \right) + K_1L \tag{6.19}$$

using $M_{el} = PL/2 = EI\kappa_y$ [Nmm] results in

$$K_1 = \frac{P}{6EIL} \left( L^3 - \frac{2L^3}{8} \right) = \frac{2EI\kappa_y}{6EIL^2} \frac{3L^3}{4} = \frac{L\kappa_y}{4} \tag{6.20}$$

which is equal to the yield rotation $\theta_y$ at the end $x = 0$, as $\theta = K_1$ at $x = 0$.

The presence of pre-loading in the loading bridge, caused by the tightening of the nuts above the loading beam, was not recognised in these two tests, and so the plotted moment–rotation curves in Figure 6.30 are a lower bound. Pre-loading forces were subsequently recorded in the four-point bending tests, and were taken into account by adding to the applied load from the jacks. The moment–rotation curves in Figure 6.30 for the three-point bending tests, show that the ultimate moment capacities are just below 90% of the plastic moments. The post-peak load resistance is substantially higher for the 5 mm thickness beam SS5.3P than for the 4 mm beam SS4.3P.
Taking the strain hardening modulus as zero

As strain hardening was not developed in the stub column tests, the $E_{sh}/E = 0.01$ value will not be appropriate for use in the design axial load $N_{csm}$, and should instead be taken as zero. The CSM design equation for axial loading with $E_{sh}/E = 0$ gives

$$\frac{N_{csm}}{N_y} = 1 + \frac{0}{E} \left( \frac{\epsilon_{csm}}{\epsilon_y} - 1 \right) = 1.$$  \hspace{1cm} (6.21)

This implies that the maximum design axial load for a stub column that exhibits no strain hardening is $N_{csm} = N_y$. Similarly for the described bending tests, the cross-sections did not show evidence of strain hardening and did not reach the plastic moment. For this type of behaviour the strain hardening modulus also needs to be taken as zero when using the CSM design equations in Section 4.2.3. Therefore in flexure, cross-sections with strain...
ratios greater than 1 will return moments in-between the elastic and plastic moments ($M_{el} \leq M_{csm} \leq M_{pl}$). With $E_{sh} = 0$ the design equation Eqn (4.76) collapses to

$$\frac{M_{csm,y}}{M_{pl,y}} = 1 - \left(1 - \frac{W_{el,y}}{W_{pl,y}}\right) \left(\frac{\epsilon_{csm}}{\epsilon_y}\right)^{-2}.$$  \hspace{1cm} (6.22)

Using this equation with the strain ratios at the peak moments, which were 1.15 and 1.25 for the 4 mm and 5 mm thickness respectively, gives moments of $0.870 M_{pl,y}$ and $0.890 M_{pl,y}$, which are conservative estimates to the actual values of $0.925 M_{pl,y}$ and $0.944 M_{pl,y}$. Further investigation is required to determine under which situations $E_{sh}$ should be taken as zero and not $E_{sh}/E = 0.01$, at present, observations suggest that adjustments may be needed for hot-rolled box sections, as these cross-sections were of a material that may have made attaining stresses associated with strain hardening difficult.

6.6 Summary

An experimental investigation of grade S355 structural steel rectangular hollow sections was performed, including tests on stub columns and simply supported beams. Two wall thickness values of 5 mm and 4 mm were tested, giving class 1 and class 2 cross-sections respectively according to EN 1993-1-1 (2005).

Tensile coupon tests revealed material yield stresses that were 10-20% higher than the nominal values. Both coupon thickness values displayed a yield plateau that extended up to a strain of approximately 3.5%, after which strain hardening was observed.

The stub column test specimens failed by local buckling at peak loads close to the yield load and at strains close to the yield strain, with the thicker specimens displaying a more gradual post-peak unloading behaviour. The stub columns did not exhibit ultimate axial loads influenced by strain hardening and strains remained within the material yield plateau.
The simply supported beam tests failed by local buckling and did not reach moments greater than the plastic moments. All of the test members had cross-sections that were of plastic design proportions (class 1 and 2), and so permit the use of the plastic moment capacity in design, though the cross-sections did not generally reach their plastic moment capacities. The extended material yield plateau encouraged peak system loads to occur shortly after first yielding and with no benefit from strain hardening.
Chapter 7

Flexural buckling

7.1 Introduction

This chapter explores a CSM approach to predicting column buckling behaviour, describing methods for determining the peak load and full loading–unloading deflection curve of an axially loaded pin-ended column. This work builds on the cross-section moment–curvature–thrust curves developed in Section 5.1.9.

A pin-ended prismatic column of length $L$ is depicted in Figure 7.1a, which in its unloaded state forms the deflected shape $v_0(x)$ with a maximum mid-height displacement $d_0$; this represents the initial out-of-straightness in the member. After the application of a concentric axial load $N$, a new total deflected shape $v_t = v + v_0$ (initial $v_0$ plus additional displacements $v$) which is in equilibrium with the applied axial load will have a maximum displacement at $x = L/2$ of $d_t = d + d_0$. 

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Figure 7.1: Pin-ended column withstanding a concentric axial load.

For the perfectly straight configuration when $d_0 = v_0 = 0$, Euler (1744) gave the lowest elastic critical buckling load $N_{cr}$, here related independently to the major and minor axis effective lengths $L_y$ and $L_z$, corresponding to a half sine wave shape $v = d \sin(\pi x/L)$,

$$ N_{cr,y} = \frac{\pi^2 EI_y}{L_y^2} \quad \text{and} \quad N_{cr,z} = \frac{\pi^2 EI_z}{L_z^2}. \quad (7.1) $$

By assuming that the initial deflected shape takes the form of $v_0 = d_0 \sin(\pi x/L)$, Timoshenko and Gere (1961) described a link between the axial load and the mid-height lateral deflection, which can be written in the form of Eqn (7.2). These equations describe the elastic load–maximum lateral deflection relationships, which are asymptotic to the Euler buckling load for $d_0 > 0$. This is plotted on Figure 7.1b as curve Elastic for a given initial imperfection $d_0/L$ at $x = L/2$. 

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\[
\frac{N}{N_{cr,y}} = 1 - \frac{d_{0,y}}{d_{t,y}} \quad \text{and} \quad \frac{N}{N_{cr,z}} = 1 - \frac{d_{0,z}}{d_{t,z}}
\] (7.2)

Eqn (7.2) considers independently buckling about the major and minor axes, utilising the corresponding suffixes \(y\) and \(z\). The Elastic curve in Figure 7.1a highlights a fundamental characteristic of the imperfect column system, which is that of amplification of lateral deflections due to geometric effects.

On the application of the concentric axial load \(N\), a bending moment distribution \(M(x) = Nv_t(x)\) is formed throughout the column height. This creates further lateral deflections, which in turn creates additional bending moments, and the process cycles until equilibrium is achieved. These geometric effects mean that the flexural rigidity, which provides the stiffness to oppose the lateral deflections, becomes an important variable, as the deformed geometry provides a feedback into the system. Further complicating the response of the column is that the elastically derived equations are only valid while the material stresses and strains are below the yield values \((f \leq f_y\) and \(\epsilon \leq \epsilon_y\)), while in practice the inelastic behaviour of the material, such as its strain hardening potential and failure strains are also important. A yield limited approach which offers a lower bound estimate of the axial load, assuming that the cross-section strain ratio is at least unity, is found by tracing along the elastic deflections of Eqn (7.2) until \(f = f_y\) is reached at the most stressed outer-fibres, which occurs when axial compressive stresses combine with the bending compressive stresses on the concave side of the column. This is commonly represented by the Perry–Robertson formula from Eqn (2.32), which is plotted in Figure 7.1b as the Yield limit curve, for a suite of \(d_0\) (initial imperfection) values.

An upper bound \(N-d_t\) curve can be found by adjusting the axial load and uni-axial bending interaction curves from Section 5.1 with \(d_t = M/N\), by dividing each moment on the \(N-M\) interaction curves by the axial loads. For a given strain ratio this represents the ultimate cross-section capacity for which local buckling is the failure mode; loads above this curve
are not permitted. This upper bound curve is plotted on Figure 7.1b as *Ultimate limit*, and can be represented well by the modified CSM design equation Eqn (7.3), or be calculated directly and more accurately by the interaction curves from the numerical model described in Section 5.1.2,

\[
\left(\frac{N}{N_{csm}}\right)^{a_y} + \left(\frac{Nd_{t,y}}{M_{csm,y}}\right)^{b_y} = 1 \quad \text{and} \quad \left(\frac{N}{N_{csm}}\right)^{a_z} + \left(\frac{Nd_{t,z}}{M_{csm,z}}\right)^{b_z} = 1. \tag{7.3}
\]

The actual response, which is plotted on Figure 7.1b as the curve *Actual*, begins at \( N = 0 \) and \( d_t = d_0 \), and initially follows the *Elastic* curve until the *Yield limit* curve is reached, indicating first yielding at the mid-height cross-section. After the first yield point, the flexural rigidity will quickly decline at \( x = L/2 \) as the material plastically deforms, but some additional load up until the peak load \( N_u \) can still be carried. After this inelastic peak load, which is plotted in Figure 7.1b as *Peak load*, the column can no longer be in equilibrium for an increase in loading, and so an unloading path is followed and paired with further lateral deflections. This unloading continues until the *Ultimate limit* curve is reached, which is when the strains at the critical mid-height cross-section reach the limiting strain \( \epsilon_{csm} \). The difficulty in finding the load of interest \( N_u \), is that past the first yield point a combination of geometric effects and material non-linearity combine to deteriorate the global stiffness of the column.

### 7.2 Differential equation solution

The differential equation Eqn (7.4) describes the equilibrium between an external applied moment \( M \), and the internal resisting moment \( E'I\kappa \)

\[
E'I\kappa = E'I\frac{d^2v}{dx^2} \frac{1}{\left[1 + \left(\frac{dv}{dx}\right)^2\right]^{\frac{3}{2}}} = -M. \tag{7.4}
\]
In Eqn (7.4) the complete large deflection curvature definition is used for $\kappa$. For a pin-ended column of length $L$ the boundary conditions are $v(0) = 0$ and $v(L) = 0$. With a given axial load and initial mid-height deflection $d_0$, Eqn (7.4) is a boundary valued, non-linear, second order, ordinary differential equation that can be reformatted into two first order, ordinary differential equations with $M = Nv_t = N(v_0 + v)$

$$v_1 = v \quad \text{and} \quad v_2 = \frac{dv_1}{dx} = \frac{dv}{dx}$$

(7.5)

$$\frac{dv_2}{dx} = \frac{d^2v}{dx^2} = -\frac{N(v_0 + v_1)}{E'I} \left(1 + v_2^2\right)^{\frac{3}{2}}.$$  

(7.6)

The key complication is that $E'I$ is itself a function of curvature, which includes the highest derivative $d^2v/dx^2$ and also varies with $x$. It is required that the flexural rigidity be a function of the total deflection such that $E'I = f(v_t)$. Recall the moment–curvature–thrust curves derived from the work reported in Section 5.1 on cross-sections under axial load and uni-axial bending. These curves corresponded to various moment–curvature relationships for a given axial load level $N/N_y$. Two examples of these $M–\kappa–N$ curves are shown in Figure 7.2 for bending about the major and minor axes of an I-section. For every $(M, \kappa, N)$ point on the moment–curvature–thrust curves, the gradient $E'I$ can be calculated and plotted against $M/N = v_t$. This produces the effective flexural rigidity–total deflection curves of Figure 7.3. The effective flexural rigidity starts at the elastic value $E'I = EI$ and stays constant while the cross-section is elastic, before dropping sharply as the cross-section yields and more material switches from a stiffness of $E' = E$ to $E' = E_{sh}$. For the minor axis, where the shape-factor of the I-section is higher than for the major axis, the spread of yielding and hence the stiffness reduction is more gradual. As the normalised axial load increases, the loss of stiffness occurs sooner with respect to $v_t/L$ and the drop towards $E_{sh}I$ is steeper.
Figure 7.2: Moment–curvature–thrust curves for a typical I-section (SR=strain ratio).

Figure 7.3: Effective flexural rigidity–total deflection curves of a typical I-section.
These $E'I-v_t$ plots allow $E'I$ in Eqn (7.6) to be replaced with the function $f(v_t)$, which gives Eqn (7.7). The function $f$ is formed by linearly interpolating between the discrete data points from Figure 7.3.

$$\frac{dv_2}{dx} = -\frac{N [d_0 \sin(\pi x/L) + v_1]}{f(d_0 \sin(\pi x/L) + v_1)} (1 + v_2^2)^{\frac{3}{2}}$$  

(7.7)

The differential equations are solved in MATLAB (2012) with the boundary valued ODE solver BVP4C.m, with the resulting solutions accurate to a relative error of 0.1%. The results are for a UC cross-section buckling about the major axis, with $d_0/L = 1/250$, $\bar{\lambda} = \sqrt{N_y/N_{cr}} = 1$ and a peak load (marked with a grey line in the following figures) of $N_u = 0.557N_y$. The additional displacements $v(x)$ in Figure 7.4a are normalised by $v_m = \max(v)$ in Figure 7.4b, which shows that the displaced shape of the column up until the peak load is very close to a half sine wave.

![Displacement plots](image.png)

(a) Length normalised additional displacements  
(b) Maximum deflection normalised additional displacements

Figure 7.4: Displacements from the BVP4C solution of a UC column under increasing axial load (grey curve $N = N_u$).
The deformed shape then becomes more pointed during the post-peak unloading phase as the mid-height cross-section loses its flexural rigidity due to material yielding. The slope and curvature profiles in Figure 7.5a and Figure 7.5b highlight the presence of a yielding zone between $x/L = 0.4$ and $x/L = 0.6$. Note that at the peak load the curvature ratio $\kappa/\kappa_{y,y}$ is relatively low, but that it increases rapidly as yielding spreads.

![Figure 7.5: Rotations and curvatures from the BVP4C solution of a UC column under increasing axial load (grey curve $N = N_u$).](image)

The distribution of yielding at the peak load and during the unloading phase is shown in Figure 7.6 to Figure 7.9, which illustrate the stress magnitude distribution throughout the column (areas of lower stress are black and areas that are white have yielded). The peak load stress distribution in Figure 7.6 shows no observable yielding regions, and is similar to an elastic stress distribution. The axial stresses and bending stresses in Figure 7.7 combine to produce an area of intense yielding on the compressive side ($y = -D/2$) with a smaller concentration at the more tensile side ($y = D/2$). Yielding expands to cover a region...
between $x/L = 0.4$ and $x/L = 0.6$ in Figure 7.8. The distribution of yielding in Figure 7.9 becomes closer to symmetric about $y = 0$, as flexure becomes dominant, caused by larger $M = Nv_t$ moments:

Figure 7.6: Stress distribution for a UC column at $N = N_u = 0.557N_y$ (peak load).

Figure 7.7: Stress distribution for a UC column at $N = 0.50N_y$ (unloading).

Figure 7.8: Stress distribution for a UC column at $N = 0.45N_y$ (unloading).

Figure 7.9: Stress distribution for a UC column at $0.40N_y$ (unloading).
7.3 Solution using an approximated half sine wave deflected shape

The moment–curvature–thrust curves in Figure 7.2 can be converted to curvature–total deflection–thrust curves by the following steps: 1) Take all of the points along a moment–curvature–thrust curve for a fixed axial load level $N/N_y$. 2) Divide each of the moments $M$ by the axial load $N$ to get $v_t$ (recall that the moment $M$, axial load $N$ and total lateral deflection $v_t$, are linked by $M = Nv_t$). 3) Plot these calculated total lateral deflection points $v_t$ against the curvature $\kappa$, noting that for a deflected shape where $v_t$ is defined as positive, the curvature $\kappa$ will be negative.

![Curvature-deflection-thrust plots of a typical I-section](image)

Figure 7.10: Curvature–deflection–thrust plots of a typical I-section.

The result of performing this transformation is the suite of curves in Figure 7.10, which are plotted for the major and minor axes of a typical I-section. For a given load level $N/N_y$, each curve gives the direct relationship between the internal curvature $\kappa$ and the total lateral deflection $v_t$. 

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\[ \kappa = \frac{\frac{d^2 v}{dx^2}}{1 + \left(\frac{dv}{dx}\right)^2} = f(v_t). \]  

(7.8)

The minor axis of an I-section can take higher axial loads for a given normalised moment \( v_{t,z} N_y / M_{pl,z} \), leading to \( \kappa - v_t - N \) curves which extend further in the curvature ratio direction.

By assuming that the slope \( dv/dx \) is significantly smaller than unity, this can be further simplified to \( \kappa = \frac{d^2 v}{dx^2} = f(v_t) \). The initial curvature \( \kappa_0 \) caused by \( v_0 \) is assumed not to contribute to internal stresses, but the applied moment \( M = Nv_t \) depends on the total deflections (initial deflections \( v_0 \) and the additional deflections \( v \)). Note the difference between this transformation and the transformation performed in the direct approach in Section 7.2. Here, \( M - \kappa - N \) curves are converted into \( \kappa - v_t - N \) curves, where as previously the \( M - \kappa - N \) curves were converted into \( E' I - v_t - N \) curves by taking the gradient at each point.

This formulation is similar to the method of von Kármán and Chwalla (von Kármán (1908); von Kármán (1910) and Chwalla (1928)) as described by Bleich (1952). Their method considered an initially straight column of solid rectangular cross-section, a plane sections remain plane assumption, and a point load \( N \) applied at an eccentricity to the cross-section centroid. The stress distribution was the sum of an average axial component and a flexural component. By considering the equilibrium of the cross-section, a set of stress distributions was found. By defining a limiting strain similar in function to \( \epsilon_{asm} \), an eccentricity, an average stress and bending stress distribution, a curvature \( \kappa \) was calculated which then allowed the determination of the deflection \( v \). The pairing of the deflection and curvature points resulted in curvature being expressed as a function of \( v \), thus arriving at similar plot types to those shown in Figure 7.10.
7.3.1 Use of a half sine wave as the approximate deflected shape

It is known from the elastic solutions of the pin-ended column system that up until first yielding the deflected shape is a half sine wave; it is also seen in Figure 7.4b that at the peak axial load the deflected shape is almost indistinguishable from a half sine wave. Figure 7.5b also shows that the curvatures at the mid-height of the column accelerate rapidly once yielding occurs, and encourage the column to deflect into a more pointed shape. For large lateral deflections, the curvature distribution is elastic except for around \( x = L/2 \), which is where the hinge zone formation pushes curvatures towards high negative values. The actual deflected column shape at the peak load will be somewhere in-between the elastic and hinge-like cases, but closer to the elastic half sine wave shape provided the normalised mid-height lateral deflections are not excessive.

For a given mid-height displacement \( d_t \), any assumed deflected shape will have a curvature that is greater or equal in magnitude to the elastic curvature at \( x = L/2 \). This increased curvature will correspond to a greater axial load on the curvature–displacement–thrust plots, as each curve lies underneath other curves of lower axial loads. The conclusion can be made that for the pin-ended column with concentric concentrated load, using a half sine wave as the assumed total deflection shape will produce a conservative estimate of the peak load. This conclusion was also described by Westergaard and Osgood (1928), who considered the cases of a load applied eccentrically to a straight column and to a column with an initially deflected half sine shape. From this standpoint one can now allow the curvature \( \kappa \) to be defined as a function of \( v_t \) and \( d_0 \) by

\[
v = d \sin \left( \frac{\pi x}{L} \right) \quad \frac{dv}{dx} = \left( \frac{\pi}{L} \right) d \cos \left( \frac{\pi x}{L} \right) \quad \frac{d^2v}{dx^2} = -\left( \frac{\pi}{L} \right)^2 d \sin \left( \frac{\pi x}{L} \right). \quad (7.9)
\]
From the full definition of curvature and with \( d_t = d_0 + d \),

\[
\kappa = \frac{\frac{d^2v}{dx^2}}{\left[1 + \left(\frac{dv}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{-(\frac{\pi}{L})^2 (d_t - d_0) \sin \left(\frac{\pi x}{L}\right)}{\left[1 + \left(\frac{\pi}{L}\right)^2 (d_t - d_0)^2 \cos^2 \left(\frac{\pi x}{L}\right)\right]^{\frac{3}{2}}}.
\] (7.10)

For the mid-height of the column where \( x = L/2 \)

\[
\kappa = -\left(\frac{\pi}{L}\right)^2 (d_t - d_0).
\] (7.11)

Therefore as \( v_t = d_t \) at mid-height, the intersections between the straight line Eqn (7.11) and the curvature–total deflection–thrust curves represent the assumed half sine wave states at the \( x = L/2 \) cross-section. Examples are given in Figure 7.11 for the selected I-section buckling about each bending axis and with a maximum strain ratio of 15.

Figure 7.11: Intersections between the half sine wave function \( \kappa = f(d_t) \) at \( x = L/2 \), and the cross-section \( \kappa = f(v_t) \) curves.
Each curve can be paired with: 1) Zero intersections if the axial load level is above the peak load of the member. 2) Two intersections if sufficient deformation capacity is present or one intersection if not. 3) A single intersection when the straight line Eqn (7.11) just touches the curve, indicating the peak load of the member. The single end-point for where an intersection coincides with the very end of a curvature–deflection–thrust curve, indicates that the ultimate local buckling capacity has been reached and the limiting strain \( \varepsilon_{csm} \) has been met.

The following process is executed in MATLAB (2012) for each of the six shapes and for all cross-sections:

- Define an axial load, generally starting at \( N/N_y = 0.001 \).
- Extract the moment–curvature–thrust (\( M–\kappa–N \)) curve of the cross-section based on a large number of \( \varepsilon_{csm}/\varepsilon_y \) values between 0 and 15.
- Transform the \( M–\kappa–N \) curve into a curvature–total deflection–thrust curve (\( \kappa–v_t–N \)).
- Find the one or two half sine wave shape intersections that indicate a valid axial load, and store the \( v_t \) value(s).
- Incrementally increase the axial load (step size of 0.001\( N_y \) used) and repeat the previous steps while one or two intersections are present.
- Stop the process when no intersections are found as the peak load has been exceeded.
- Plot the loading and unloading displacements (smallest and largest respectively) at the mid-height of the column with the associated axial loads.
7.4 Load–displacement curves

A numerical procedure was described in Section 7.3.1 that began with a cross-section and a fixed axial load, and created a curvature–lateral deflection curve from a moment–curvature curve. Then the intersections were found between the curvature–lateral deflection curve and a $\kappa$–$v_t$ line, where the latter was based on a half sine wave deflected shape with initial out-of-straightness magnitude $d_0$. The intersections gave loading and unloading $v_t$ values for the selected axial load, and can now be used to create a $N$–$d_t$ curve. These load–deflection curves trace the axial load from zero up until the peak load, and are presented in the following figures for each considered cross-section shape (UB, UC, RHS, SHS, CHS and EHS), with a yield stress of $f_y = 355 \text{ N/mm}^2$ and a strain ratio of 15. These curves all follow their elastic paths until first yielding, continue to take additional loading to reach a peak load, and subsequently unload down paths that tend towards the ultimate limit curves. Eventually the curves touch the ultimate limit curves at high lateral deflections; these ultimate limit curves indicate the state when the cross-section strain ratio is reached.

7.4.1 Load–displacement curves for when $\bar{\lambda} = 1$ and $d_0/L$ is variable

For the following figures (Figure 7.12 to Figure 7.16) the initial out-of-straightness values are $d_0/L = [1/100, 1/250, 1/500, 1/1000]$ and the member lengths are calculated such that $\bar{\lambda} = \sqrt{N_y/N_{cr}} = 1$. The consequences of increasing $d_0/L$ are to reduce the peak load by issuing a greater influence from geometric effects, and to create a more gradual unloading response. For the I-sections buckling about the major axis in Figure 7.12, the peak loads are reached soon after first yielding and with little additional load carrying capacity. This may be explained by the limited potential to carry extra bending moments once the flanges have first yielded, as I-sections possess a low major axis shape factor $W_{pl,y}/W_{el,y}$. 

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Figure 7.12: $N - d_{y,t}$ curves for I-sections buckling about major axis with varying $d_{0,y}/L$.

Figure 7.13: $N - d_{y,t}$ curves for box sections buckling about major axis with varying $d_{0,y}/L$. 
Figure 7.14: $N$–$d_{t,y}$ curves for circular and elliptical hollow sections buckling about major axis with varying $d_{0,y}/L$.

Figure 7.15: $N$–$d_{t,z}$ curves for I-sections buckling about minor axis with varying $d_{0,z}/L$. 
Figure 7.16: $N-d_{t,z}$ curves for rectangular and elliptical hollow sections buckling about minor axis with varying $d_{0,z}/L$.

Figure 7.17: Effects of varying the strain ratio on the load–lateral displacement curves, for an elliptical hollow section buckling about the major axis.
Figure 7.18: Effects of varying the strain ratio on the load–lateral displacement curves, for an elliptical hollow section buckling about the major axis.

Figure 7.17 and Figure 7.18 show load–lateral displacement curves for an elliptical hollow section, where the strain ratio has been varied. The strain ratio must be reduced to below 2.0, in order to affect the peak load. This failure, which occurs when the normalised load–deflection response meets the ultimate limit curves, indicates that local buckling is influencing the peak load. For higher strain ratios, local buckling occurs beyond the peak load.
7.4.2 Load–displacement curves for when $\bar{\lambda}$ is variable and $d_0/L = 1/500$

The numerical model may also keep $d_0/L$ fixed (here it is kept at 1/500) to investigate the effects of varying the global slenderness $\bar{\lambda}$ between 0.1 and 2.0. The results are presented in Figure 7.19 to Figure 7.23, and show that for all six shapes the general form of the load–displacement curves for a given $\bar{\lambda}$ are similar. All of these curves are plotted for a material yield stress of $f_y = 355$ N/mm$^2$ and a strain ratio of 15.

When $\bar{\lambda} = 2.0$ the peak loads are approximately 20% of the yield load and occur at relatively high lateral displacements of $\approx 8d_0$, although this is small when compared to the column length $L$. Columns with a global slenderness of $\bar{\lambda} = 2.0$ also exhibit broad and rounded curves with a gradual loss of stiffness up to the peak loads, and with shallow unloading stages. As the global slenderness reduces towards $\bar{\lambda} = 0.25$, the mid-height lateral displacements at which the peak loads are reached also reduce, and approach the initial magnitude $d_0$. The loading portions of the curves are where the columns have positive stiffness, as the gradients to the curves are positive, becoming steeper as $\bar{\lambda}$ reduces. As $\bar{\lambda}$ reduces the peaks of the curves also become more pointed and significantly higher loads are obtained, with the peak loads almost reaching $N_y$ for when $\bar{\lambda} = 0.25$.

Within the global slenderness range of $0.1 < \bar{\lambda} < 0.25$ there exist $\bar{\lambda}$ values for which there are no longer unloading stages to the curves, since the limiting CSM strain is achieved prior to the occurrence of the peak load. The form of the load–deflection curves in the region of the peak load are more rounded for the circular and elliptical hollow sections shown in Figure 7.21, than for the I-sections and box sections shown in Figure 7.19 and Figure 7.20. For all cross-section shapes the peak loads reduce as $\bar{\lambda}$ increases and occur at higher $d_t/L$ ratios.
Figure 7.19: $N-d_{t,y}$ curves for I-sections buckling about the major axis with varying $\bar{\lambda}$.

Figure 7.20: $N-d_{t,y}$ curves for box sections buckling about the major axis with varying $\bar{\lambda}$. 
Figure 7.21: $N-d_{t,y}$ curves for circular and elliptical hollow sections buckling about the major axis with varying $\bar{\lambda}$.

Figure 7.22: $N-d_{t,z}$ curves for I-sections buckling about the minor axis with varying $\bar{\lambda}$.
Figure 7.23: \( N - d_{t,z} \) curves for rectangular and elliptical hollow sections buckling about the minor axis with varying \( \bar{\lambda} \).

7.5 Buckling curves of axial load versus global slenderness

7.5.1 Initial mid-height imperfection \( d_0/L \) as variable

The values of most interest from the load–lateral deflection curves are the peak axial loads, as they represent the maximum axial loads the columns can sustain before becoming unstable. The global slenderness \( \bar{\lambda} \) and the initial global geometric imperfection \( d_0/L \) can be varied, and the numerical model used to extract the peak loads only. The results are the buckling curves presented in Figure 7.24 to Figure 7.28 for a representative cross-section of each shape, with a maximum strain ratio of 15, a material yield stress \( f_y = 355 \text{ N/mm}^2 \) and for \( d_0/L = [1/250, 1/350, 1/500, 1/750, 1/1000, 1/1500] \). When the global slenderness is large, all buckling curves converge towards the Euler buckling load, which has a value close to \( N/N_y = 0.2 \) for \( \bar{\lambda} = 2 \). Significant separation between the buckling
curves occurs for intermediate global slenderness values, highlighting the importance of the \(d_0/L\) ratio. Higher magnitudes of the initial out-of-straightness produce straighter curves, and conversely columns that are initially straighter produce more rounded buckling curves that flatten towards the yield load.

All curves abruptly change direction between \(0.1 \leq \lambda \leq 0.2\), which is where the columns begin to behave like stub columns and obtain peak loads greater than the yield load. This supports the use of a global slenderness limit to denote when \(N_y\) has been achieved; however the presented curves would suggest that a fixed point of \(N = N_y\) at \(\lambda = 0.2\), as used in EN 1993-1-1 (2005) may not be suitable, as the yield loads are often reached only at lower \(\lambda\) values. As the global slenderness tends to zero, the axial loads tend to \(N_{csm}\) and so the curves become coincident at that point. All columns buckling about the major axis (Figure 7.24 to Figure 7.26) and the rectangular and elliptical hollow sections buckling about the minor axis (Figure 7.28) show similar buckling curve shapes.

![Figure 7.24: \(N_y\) normalised buckling curves for I-sections buckling about the major axis with varying \(d_{0,y}/L\).](image)

\[N_{csm}\]
Figure 7.25: $N_y$ normalised buckling curves for box sections buckling about the major axis with varying $d_{0,y}/L$.

Figure 7.26: $N_y$ normalised buckling curves for circular and elliptical hollow sections buckling about the major axis with varying $d_{0,y}/L$. 

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Figure 7.27: $N_y$ normalised buckling curves for I-sections buckling about the minor axis with varying $d_{0,z}/L$.  

Figure 7.28: $N_y$ normalised buckling curves for rectangular and elliptical hollow sections buckling about the minor axis with varying $d_{0,z}/L$.  

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7.5.2 Effect of varying the strain ratio on the buckling response

The effects of varying the strain ratio whilst keeping the initial out-of-straightness $d_0/L$ fixed are presented in Figure 7.29 to Figure 7.34. These plots give the yield load $N_y$ and CSM load $N_{csm}$ normalised buckling curves for the minor axis buckling of an I-section and rectangular hollow section column, with $f_y = 355 \text{ N/mm}^2$ and for a variety of strain ratios, to examine the effect of varying the strain ratio on the buckling response.

The elastic limit buckling curves (i.e. for a strain ratio, SR=1) represent the first yield condition, and are identical to the Ayrton–Perry–Robertson predictions, giving buckling curves that intersect $N = N_y$ at $\bar{\lambda} = 0$ and display no curve kink at low $\bar{\lambda}$ values. When the strains are allowed to increase to strain ratios of 3 or 5 (SR=3 and SR=5), the buckling curves for the rectangular hollow section shift slightly upwards, with most of the load increases occurring towards the low global slenderness values. The same is true for the I-section but with much greater observable increases in the load carrying capacity. This behaviour was also observed for I-sections in the load–lateral deflection ($N$–$d_{t,z}$) curves, with the attainment of higher peak loads relative to the first yield loads. The axial load increases are more subtle when the initial out-of-straightness is lower, as seen for the $d_0/L = 1/1500$ curves, where the differences are the smallest.

The buckling curves show a low sensitivity to the strain ratio for global slenderness values beyond $\bar{\lambda} \approx 0.17$, due to the reduced importance of the cross-section slenderness, as after this slenderness the global stiffness is lost before the mid-height cross-section reaches its ultimate capacity. For the buckling curves with the maximum strain ratio of 15 (SR=15) and for when $\bar{\lambda}$ is low, the departure from the other three curves is abrupt and a kink forms. This kink distinguishes the benefits from strain hardening, and leads to increases in the axial load of about 10% of the yield load.
Figure 7.29: \( N_y \) normalised buckling curves for buckling about the minor axis with \( d_{0,z}/L = 1/250 \) and the strain ratio variable.

Figure 7.30: \( N_y \) normalised buckling curves for buckling about the minor axis with \( d_{0,z}/L = 1/750 \) and the strain ratio variable.
Figure 7.31: $N_y$ normalised buckling curves for buckling about the minor axis with $d_{0,z}/L = 1/1500$ and the strain ratio variable.

Figure 7.32: $N_{csm}$ normalised buckling curves for buckling about the minor axis with $d_{0,z}/L = 1/250$ and the strain ratio variable.
Figure 7.33: $N_{csm}$ normalised buckling curves for buckling about the minor axis with $d_{0,z}/L = 1/750$ and the strain ratio variable.

Figure 7.34: $N_{csm}$ normalised buckling curves for buckling about the minor axis with $d_{0,z}/L = 1/1500$ and the strain ratio variable.
When normalised by the CSM axial load $N_{csm}$ as in Figure 7.32 to Figure 7.34, the buckling curves are divided by different axial loads for each strain ratio, which gives greater separation between the curves. Hence, to assess the buckling curves relative to one another, the $N_y$ normalised curves are used. The effect of normalising by the CSM load is most apparent for the SR=15 buckling curves, as these have the highest $N_{csm}$ loads and so become the lowest curves. For the UB cross-section, the SR=1 curves are not the highest curves as a sharp change in the curve shape occurs up to SR=3; this is more noticeable for the higher initial imperfection to length ratios $d_{0,z}/L$.

7.6 Development of CSM design buckling curves

A design equation is now sought that approximates the developed CSM numerical buckling curves. This equation form should consist of two distinct parts, a smooth curve to represent the global-dominated failure region and a linear fit for the local-dominated failure region. The $N_y$ normalised form will be used due to the strain ratio insensitivity in the global failure region.

7.6.1 Buckling curves for the global-dominated failure region

Recall the Ayrton–Perry–Robertson first yield formula in the yield load normalised form

$$
\frac{N}{N_y} = \frac{1}{2} \left[ 1 + \frac{N_{cr}}{N_y} (1 + \eta_e) \right] - \frac{1}{2} \sqrt{\left[ 1 + \frac{N_{cr}}{N_y} (1 + \eta_e) \right]^2 - 4 \frac{N_{cr}}{N_y}} \eta_e = \frac{A d_0}{W_{el}}, \quad \text{(7.12)}
$$

The global slenderness $\bar{\lambda}$ is defined as

$$
\bar{\lambda}^2 = \frac{N_y}{N_{cr}} = \frac{N_y L^2}{\pi^2 EI} \quad \text{which gives} \quad L = \bar{\lambda} \sqrt{\frac{\pi^2 EI}{N_y}}. \quad \text{(7.13)}
$$
The initial mid-height displacement may be written implicitly as a function of $\bar{\lambda}$ by

$$d_0 = \frac{d_0}{L} L = \frac{d_0}{L} \sqrt{\frac{\pi^2 EI}{N_y}} \bar{\lambda}. \quad (7.14)$$

For a selected cross-section and material, both the elastic flexural rigidity $EI$ and yield load $N_y$ are constant, and so for a given $d_0/L$, the elastic $\eta_e$ can be replaced with $\eta = \omega \bar{\lambda}$ where $\omega$ is a function of $d_0/L$. This gives

$$\frac{N}{N_y} = \frac{1}{2} \left[ 1 + \frac{1 + \omega \bar{\lambda}}{\lambda^2} \right] - \frac{1}{2} \sqrt{\left[ 1 + \frac{1 + \omega \bar{\lambda}}{\lambda^2} \right]^2 - \frac{4}{\lambda^2}}. \quad (7.15)$$

Further control of the buckling curve shape may be provided to Eqn (7.15) by allowing for the scaling of the curve shape in the $\bar{\lambda}$ direction by using $\delta \bar{\lambda}$, where the $\delta$ parameter is greater than or equal to unity,

$$\frac{N}{N_y} = \frac{1}{2} \left[ 1 + \frac{1 + \omega \delta \bar{\lambda}}{(\delta \lambda)^2} \right] - \frac{1}{2} \sqrt{\left[ 1 + \frac{1 + \omega \delta \bar{\lambda}}{(\delta \lambda)^2} \right]^2 - \frac{4}{(\delta \lambda)^2}}. \quad (7.16)$$

And finally, to allow for the translation of the buckling curves in the global slenderness direction, $\bar{\lambda}$ is replaced by $\bar{\lambda} - c$, which changes the slenderness at which the yield load is attained, from $\bar{\lambda} = 0$ to $\bar{\lambda} = c$. With this adjustment, and some further formatting, Eqn (7.16) becomes

$$\frac{N}{N_y} = \Phi - \frac{1}{2} \sqrt{\Phi^2 - \frac{4}{\rho^2}} \quad \text{with} \quad \Phi = 1 + \frac{1 + \omega \rho}{\rho^2} \quad \text{and} \quad \rho = \delta (\bar{\lambda} - c). \quad (7.17)$$

Figure 7.35 shows how the $\omega$ and $\delta$ parameters adjust the buckling curve shape, when one parameter is varied and the other is held constant ($c = 0$ for all curves). When $\omega = 0$ and $\delta = 1$, the resulting curve represents the piecewise buckling curve of a perfectly straight column, and as $\omega$ increases the initial imperfections increase. When $\delta$ increases
the curves scale towards the $N/N_y$ axis, with larger scaling occurring for more globally slender columns (when $\bar{\lambda}$ is larger). As all buckling curves are normalised by the yield load $N_y$ at $\bar{\lambda} = c$, this design fit is suitable for the global-dominated failure region, as the peak axial load is always less than the yield load of the column.

![Graphs](image)

(a) $\omega$ variation, $\delta = 1$

(b) $\delta$ variation, $\omega = 0.4$

Figure 7.35: Influence of the buckling curve parameters $\omega$ and $\delta$ on the shape of the CSM design equation.

The curve fitting task is to find the parameters $\omega$, $\delta$ and $c$ as constants or as simple functions of the web area to gross area $A_r = A_w/A$, as used in Section 5.1 for axial load and uni-axial bending. In addition, the buckling curves, and hence $\omega$, $\delta$ and $c$, also depend on the out-of-straightness $d_0/L$, the material yield stress $f_y$ and to some degree the strain ratio. For the major axis buckling of all the considered cross-sections and for the minor axis buckling of the hollow sections (RHS, SHS, CHS and EHS), $\omega$ is sought as a linear function of $A_r$, where the gradient and intercept are functions of $d_0/L$ (I-sections buckling about the minor axis will be seen to behave differently),
\[ \omega = f \left( \frac{d_0}{L} \right) + g \left( \frac{d_0}{L} \right) = \left( m_1 \frac{d_0}{L} + c_1 \right) A_r + \left( m_2 \frac{d_0}{L} + c_2 \right). \]  

(7.18)

Functions \( f(d_0/L) \) and \( g(d_0/L) \) are plotted in Figure 7.36 with a yield stress of \( f_y = 275 \text{ N/mm}^2 \) and with \( m_1 = 35, c_1 = 0, m_2 = 115 \) and \( c_2 = 0 \).

![Figure 7.36: Linear functions \( f \) and \( g \) for parameter \( \omega \).](image)

The linear fitting process is repeated for material yield stresses of \( f_y = 355 \text{ N/mm}^2 \) and \( f_y = 450 \text{ N/mm}^2 \), where it is found that the best fit for the \( \omega \) parameter is

\[ \omega = (115 + 35 A_r) \frac{d_0}{L} \sqrt{\frac{275}{f_y}}. \]  

(7.19)

Eqn (7.19) gives a good approximation to the optimum values calculated from a non-linear least squares regression routine in MATLAB (2012), as displayed in Figure 7.37, Figure 7.38 and Figure 7.39 for the different material yield stresses.
Figure 7.37: Numerical and approximated $\omega$ and $\delta$ parameters, $f_y = 275$ N/mm$^2$.

Figure 7.38: Numerical and approximated $\omega$ and $\delta$ parameters, $f_y = 355$ N/mm$^2$. 

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Also plotted in these figures are the optimum $\delta$ values that correspond to each value of $\omega$ when $c = 0.1$, and the constant conservative value of $\delta = 1.06$, a higher $\delta$ returns a lower curve, as shown in Figure 7.35b. The $\omega$ fit from Eqn (7.19) is selected such that it errs on the conservative side by over-predicting $\omega$ when $d_0/L = 1/250$ at high $A_r$ values. Eqn (7.19) is valid for strain ratios between 2 and 15, where it was found that the buckling curves were largely independent of the strain ratio. It is observed that increasing the material yield stress has the beneficial effect of reducing the $\omega$ parameter, with most reductions occurring for higher initial imperfections $d_0/L$. In general $\omega$ varies significantly with $d_0/L$ and moderately with $f_y$ and $A_w/A$.

The process is repeated separately for I-sections buckling about the minor axis, and the results for functions $f(d_0/L)$ and $g(d_0/L)$ are shown in Figure 7.40 for $\delta = 1.15$, $c = 0.15$ and for strain ratios between 3 and 15; now the intercepts $c_1$ and $c_2$ are non-zero.
(a) Function $f \left( \frac{d_0}{L} \right)$ for the $\omega$ gradient

(b) Function $g \left( \frac{d_0}{L} \right)$ for the $\omega$ intercept

Figure 7.40: Linear functions $f$ and $g$ for I-sections buckling about the minor axis.

(a) $f_y = 275 \text{ N/mm}^2$

(b) $f_y = 355 \text{ N/mm}^2$

Figure 7.41: Numerical and approximated $\omega$, $f_y = 275 \text{ N/mm}^2$ and $f_y = 355 \text{ N/mm}^2$ for I-sections buckling about the minor axis.
Linear fits described by Eqn (7.20) can now be used to represent the optimum \( \omega \) values plotted in Figure 7.41 and Figure 7.42,

\[
\omega = \left[ \left( 50 \frac{d_0}{L} + 0.04 \right) A_r + 120 \frac{d_0}{L} + 0.02 \right] \sqrt{\frac{275}{f_y}}. \tag{7.20}
\]

### 7.6.2 Buckling curves for the local-dominated failure region

The interface between the global-dominated failure region described in Section 7.6.1 and the local-dominated failure region for globally stocky columns, is summarised in Figure 7.43. The global failure equation Eqn (7.17) is valid until a transition global slenderness \( \bar{\lambda}_t \), where \( \bar{\lambda}_t = 0.20 \) for I-sections buckling about the minor axis and \( \bar{\lambda}_t = 0.15 \) otherwise. The yield load \( N_y \) would be reached at \( \bar{\lambda} = c \), if the global slenderness and Eqn (7.17) were allowed to continue below \( \bar{\lambda}_t \).
Figure 7.43: Transition between local and global failure modes.

For the local failure region $\bar{\lambda} \leq \bar{\lambda}_t$ a straight line is drawn from $N_t$ to $N_{csm}$, where $N_t$ is the transition axial load at the transition global slenderness, found by inserting $\bar{\lambda} = \bar{\lambda}_t$ into Eqn (7.17). This linear, local failure model is defined in Eqn (7.21) by

$$\frac{N}{N_y} = \frac{N_{csm}}{N_y} - \left( \frac{N_{csm} - N_t}{N_y} \right) \frac{\bar{\lambda}}{\bar{\lambda}_t} \quad \text{for} \quad \bar{\lambda} \leq \bar{\lambda}_t.$$  \hspace{1cm} (7.21)

Bringing both the local and global failure equations together for $\epsilon_{csm}/\epsilon_y \geq 3$ and for $275 \leq f_y \leq 450 \text{[N/mm}^2]\text{],}

$$\frac{N}{N_y} = \begin{cases} 
\frac{N_{csm}}{N_y} - \left( \frac{N_{csm} - N_t}{N_y} \right) \frac{\bar{\lambda}}{\bar{\lambda}_t} & \text{for} \quad \bar{\lambda} \leq \bar{\lambda}_t \\
\frac{\Phi}{2} - \frac{1}{2} \sqrt{\Phi^2 - \frac{4}{\rho^2}} & \text{for} \quad \bar{\lambda} > \bar{\lambda}_t
\end{cases} \quad \text{for} \quad \bar{\lambda} \geq \bar{\lambda}_t.$$

where

$$\Phi = 1 + \frac{1 + \omega \rho}{\rho^2} \quad \text{and} \quad \rho = \delta(\bar{\lambda} - c).$$  \hspace{1cm} (7.24)

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where $N_t/N_y$ is calculated by evaluating Eqn (7.23) at $\bar{\lambda} = \bar{\lambda}_t$, and $\bar{\lambda} = \sqrt{N_y/N_{cr}}$. For I-sections buckling about the minor axis, $c = 0.15$, $\delta = 1.15$, $\bar{\lambda}_t = 0.20$ and

$$\omega = \left[ \left( 50 \frac{d_0}{L} + 0.04 \right) A_r + 120 \frac{d_0}{L} + 0.02 \right] \sqrt{\frac{275}{f_y}}. \tag{7.25}$$

Otherwise, $c = 0.10$, $\delta = 1.06$, $\bar{\lambda}_t = 0.15$ and

$$\omega = (115 + 35A_r) \frac{d_0}{L} \sqrt{\frac{275}{f_y}}. \tag{7.26}$$

For $\epsilon_{csm}/\epsilon_y < 3$, Eqn (7.23) is used with $\delta = 1$, $c = 0$ and $\omega = \eta_e/\bar{\lambda}$ using $\eta_e = Ad_0/W_{el}$, which will collapse precisely to the Ayrton–Perry–Robertson formula,

$$\Phi = 1 + \frac{1 + \omega \rho}{\rho^2} = 1 + \frac{1 + \eta_e}{\lambda^2} \tag{7.27}$$

$$\frac{N}{N_y} = \frac{\Phi}{2} - \frac{1}{2} \sqrt{\Phi^2 - \frac{4}{\rho^2}} = \frac{\Phi - \sqrt{\Phi^2 - 4/\rho^2}}{2} \times \Phi + \sqrt{\Phi^2 - 4/\rho^2} = \frac{\Phi^2 - (\Phi^2 - 4/\rho^2)}{2(\Phi + \sqrt{\Phi^2 - 4/\rho^2})} \tag{7.28}$$

Similarly, as the EN 1993-1-1 (2005) buckling curve equation is also based on the Ayrton–Perry–Robertson formula, it can be equated to the more general CSM design form. From the CSM global failure equation

$$\frac{N}{N_y} = \frac{\Phi}{2} - \frac{1}{2} \sqrt{\Phi^2 - \frac{4}{\rho^2}} = \frac{\Phi - \sqrt{\Phi^2 - 4/\rho^2}}{2} \times \Phi + \sqrt{\Phi^2 - 4/\rho^2} = \frac{\Phi^2 - (\Phi^2 - 4/\rho^2)}{2(\Phi + \sqrt{\Phi^2 - 4/\rho^2})} \tag{7.29}$$
This result may be equated to the EN 1993-1-1 (2005) form

\[
\frac{N}{N_y} = \chi = \frac{1}{\Phi \rho^2 / 2 + \sqrt{(\Phi \rho^2 / 2)^2 - \rho^2}} = \frac{1}{\phi + \sqrt{\phi^2 - \lambda^2}}.
\] (7.30)

From which it is necessary for \(2\phi = \Phi \rho^2\) and \(\rho = \delta(\bar{\lambda} - c) = \bar{\lambda}\), the latter entails that if \(c = 0\) and \(\delta = 1\) so that \(\rho = \bar{\lambda}\), the equation is satisfied if

\[
2\phi = 1 + \alpha (\bar{\lambda} - 0.2) + \bar{\lambda}^2 = \Phi \rho^2 = 1 + \omega \rho + \rho^2 = 1 + \omega \bar{\lambda} + \bar{\lambda}^2
\] (7.31)

Which binds the \(\omega\) design fit parameter to

\[
\alpha(\bar{\lambda} - 0.2) = \omega \bar{\lambda} \quad \text{or} \quad \omega = \frac{\alpha(\bar{\lambda} - 0.2)}{\bar{\lambda}}.
\] (7.32)

7.7 Comparisons between numerical results and proposed buckling curves

Comparisons between the buckling curves produced from the numerical solution of Section 7.3 (labelled Num) and the CSM design fits of Eqn (7.22) and Eqn (7.23) (labelled CSM) using the parameters \(\omega\), \(\delta\) and \(c\), are displayed in Figure 7.44 to Figure 7.48 for the major and minor axis buckling of the six cross-section shapes. For these plots a material yield stress of \(f_y = 355\text{ N/mm}^2\) is used and the curves are plotted for three values of initial mid-height deflection \(d_0/L\). The major axis buckling curves for the I-sections in Figure 7.44 and the box sections in Figure 7.45, closely match the numerical model throughout the local- and global-dominated failure regions, with the most deviation occurring for the initial mid-height imperfection of \(d_0/L = 1/250\).
Figure 7.44: I-section major axis buckling curve fits compared to the numerical model, $f_y = 355$ N/mm$^2$.

Figure 7.45: Box section major axis buckling curve fits compared to the numerical model, $f_y = 355$ N/mm$^2$. 
Figure 7.46: Circular and elliptical hollow section major axis buckling curve fits compared to the numerical model, $f_y = 355$ N/mm$^2$.

Figure 7.47: I-section minor axis buckling curve fits compared to the numerical model, $f_y = 355$ N/mm$^2$. 

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Figure 7.48: Rectangular and elliptical hollow section minor axis buckling curve fits compared to the numerical model, $f_y = 355 \text{N/mm}^2$.

The design equations also predict well the buckling response of the circular and elliptical hollow sections for the major axis as shown in Figure 7.46, and the minor axis buckling response of the rectangular and elliptical hollow sections shown in Figure 7.48. The design fits are most accurate for lower $A_r$ values, where the largest $A_r$ values arise in the case of the I-sections buckling about the minor axis in Figure 7.47. Here, the design equations deviate slightly from the numerical results and over-predict for high global slenderness values at low imperfections, and then provide conservative predictions for global slenderness values below $\bar{\lambda} = 1$. 

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7.8 Example calculation

Calculate the minor axis buckling resistance of a pin-ended column, subjected to an applied concentric axial load, when the cross-section and member geometry are $L = 2800\text{ mm}$, $d_{0,z} = 5\text{ mm}$, $A = 27500\text{ mm}^2$, $A_w = 15000\text{ mm}^2$, $I_z = 230(10^6)\text{ mm}^4$ and with material properties of $f_y = 355\text{ N/mm}^2$ and $E = 210000\text{ N/mm}^2$. The yield load and elastic critical buckling load are

\begin{align*}
N_y &= Af_y = 27500(355) = 9763\text{ kN} \quad (7.33) \\
N_{cr} &= \frac{\pi^2EI_z}{L^2} = \frac{\pi^2(210000)(230)(10^6)}{2800^2} = 60800\text{ kN} \quad (7.34)
\end{align*}

giving a global column slenderness of

\begin{equation}
\bar{\lambda} = \sqrt{\frac{N_y}{N_{cr}}} = \sqrt{\frac{9763}{60800}} = 0.4007. \quad (7.35)
\end{equation}

For an I-section column buckling about the minor axis $\bar{\lambda}_t = 0.20$, and so as $\bar{\lambda} > \bar{\lambda}_t$, the column failure is by global-dominated buckling. With the web area to gross area ratio as $A_r = 150/275 = 0.545$ then

\begin{align*}
\omega &= \left[ \left( 50 \frac{d_{0,z}}{L} + 0.04 \right) A_r + 120 \frac{d_{0,z}}{L} + 0.02 \right] \sqrt{\frac{275}{f_y}} \\
&= \left[ \left( 50 \frac{5}{2800} + 0.04 \right) 0.545 + 120 \frac{5}{2800} + 0.02 \right] \sqrt{\frac{275}{355}} \\
&= (0.07046 + 0.2343)0.8801 = 0.2682. \quad (7.36)
\end{align*}
Using $\delta = 1.15$ and $c = 0.15$ for the minor axis buckling of I-section columns

$$\rho = \delta (\bar{\lambda} - c) = 1.15 (0.4007 - 0.15) = 0.2883$$

(7.37)

$$\Phi = 1 + \frac{1 + \omega \rho}{\rho^2} = 1 + \frac{1 + 0.2682 (0.2883)}{0.2883^2} = 13.96.$$  

(7.38)

This allows for the calculation of the yield load normalised buckling load

$$\frac{N}{N_y} = \frac{\Phi}{2} - \frac{1}{2} \sqrt{\Phi^2 - \frac{4}{\rho^2}} = \frac{13.96}{2} - \frac{1}{2} \sqrt{13.96^2 - \frac{4}{0.2883^2}} = 0.9228$$

(7.39)

giving a design buckling load of $N = 0.9228 (9763) = 9010 \text{kN}$.

### 7.9 Influence of imperfection ratio $d_0/L$

The measure of the out-of-straightness of a column is $d_0/L$, which is the mid-height initial imperfection amplitude $d_0$ divided by the total column height $L$. For an initially straight column $d_0/L = 0$, and as the initial crookedness increases the second order bending effects become more important. Usually a designer will not have accurate information on the out-of-straightness, but a maximum value (e.g. $d_0/L = 1/500$) would be given by codified tolerances such as EN 1090-2 (2008), or from experimental tests as described by Galambos (1998).

Often, flexural buckling design equations will group together the effect of global geometric imperfections and residual stresses, to create one unified and driving imperfection parameter, such as the imperfection factor $\alpha$ used in EN 1993-1-1 (2005). The CSM model that has been developed does not explicitly include residual stresses, but their effect on local buckling is implicitly included in the strain ratio. By selecting different values of $d_0/L$,
the CSM design curves can be approximately matched to the five flexural buckling curves of EN 1993-1-1 (2005), as shown in Figure 7.49 to Figure 7.51. Plotted are the buckling curves \( a_0, a, b, c \) and \( d \) from EN 1993-1-1 (2005), and the CSM design equations Eqn (7.22) and Eqn (7.23) for an I-section with equivalent \( d_0/L \) values. For the buckling curves \( a_0 \) and \( a \) plotted in Figure 7.49, the two models are overlapping except for the local failure region below \( \bar{\lambda} = 0.15 \), where the CSM allows axial resistances above \( N_y \). The equivalent \( d_0/L \) values for curves \( a_0 \) and \( a \) are 1/1000 and 1/650 respectively. For columns with larger imperfections, curves \( b, c \) and \( d \) are equivalent to \( d_0/L \) values of 1/400, 1/275 and 1/175. For these three buckling curves, the two models overlap for global slenderness values above \( \bar{\lambda} = 0.5 \), and differ for stockier columns. EN 1993-1-1 (2005) forces its curves through the point \( \bar{\lambda} = 0.2, N = N_y \), while the CSM design equations channel the curves (for the major axis) through \( \bar{\lambda} = 0.15, N = N_t \) where \( N_t \neq N_y \). Therefore the benefits of the CSM approach are increased loads for stocky columns and more conservative loads at \( \bar{\lambda} \approx 0.2 \).

Figure 7.49: Buckling curves \( a_0 \) and \( a \) from EN 1993-1-1 (2005) and equivalent CSM buckling curves.
Figure 7.50: Buckling curves \( b \) and \( c \) from EN 1993-1-1 (2005) and equivalent CSM buckling curves.

Figure 7.51: Buckling curve \( d \) from EN 1993-1-1 (2005) and equivalent CSM buckling curve.
As the CSM global-dominated failure region may be likened to the Eurocode 3 buckling curves, a combination of the two procedures could be suggested by utilising the CSM local-dominated failure region, by linking $N_{csm}$ at $\bar{\lambda} = 0$ to $N_y$ at $\bar{\lambda} = 0.2$, and continue the use of the EN 1993-1-1 (2005) buckling curves for $\bar{\lambda} > 0.2$.

### 7.10 Summary

A numerical model was developed to solve the flexural buckling problem of a pin-ended column with an elastic, linear-hardening material response subjected to a concentric axial load. The solutions of the differential equations were the displacement and slope distributions, and at the peak load the displacement profile was close to a half sine wave.

An assumed half sine wave displacement shape was introduced to conservatively represent the column up to and beyond the peak load. When permitted axial loads were selected, the mid-height cross-section was in equilibrium with its applied moment and axial load. By utilising a numerical model, loading and unloading displacements were calculated to produce a load–displacement curve. The peak axial loads from these curves were paired with the global slenderness of the members to produce buckling curves.

A design equation was constructed that consisted of a straight line between the CSM axial load and a transitional axial load for local-dominated failure, and a modified Ayrton–Perry–Robertson equation for global-dominated failure. The three parameters used in defining the global failure equation were either constants or simple linear functions.

The CSM flexural buckling model offers strain based compatibility between cross-section and member resistance, complete loading–unloading $N$–$d_t$ curves, explicit inclusion of second order effects induced by column out-of-straightness, an adaptable and concise design resistance function to describe global-dominated failure, and allows axial loads greater than the yield load for stocky columns.
Chapter 8

Conclusions

In this research, The Continuous Strength Method was extended to combined loading, applied to various new cross-section shapes and was utilised to predict the buckling load of an imperfect pin-ended column. Analytical and numerical methods have been used to investigate the structural behaviour, and then simplified with accurate design equations that are suitable for use in engineering design.

Local buckling and strain limits

Two methods of determining the slenderness of a cross-section were compared, the critical plate element method and the element interaction method, the latter of which returned lower and more precise slenderness values.

A design bi-linear material model was described for hot-rolled and cold-formed structural steel that allowed for the benefits of strain hardening. A maximum strain of 15 times the yield strain was allowed based upon ductility requirements, and a value of $E/100$ was selected to represent the strain hardening modulus.
A strain ratio and curvature ratio were introduced by taking the strains or curvatures at the peak axial loads or peak bending moments, and then normalising by the respective yield values. Higher values of the strain ratio indicated an increased resistance to local buckling. The relationships between the strain ratios and the cross-section slenderness were displayed on base curves.

Further work

- Adapt the material modelling function in the developed numerical models to cater for any stress–strain curve, and then make comparisons between the results of different structural steel models.
- Examine whether cold-formed and hot-rolled material should be treated with separate material models, for example with tri-linear and non-linear models respectively.
- Incorporate rolled corner strength enhancements for cold-formed cross-sections, by altering the yield and ultimate stresses and strains at the designated corner areas.
- Re-examine the base curves for the circular and elliptical hollow sections, and consider relaxing the strain ratio of 15 limit for the stockiest cross-sections.
- Add rectangular flat plate experimental results into the base curve data set, and determine whether the predictions from the CSM axial design equation are suitable.
- Gather more stub column and bending test data, particularly for circular and elliptical hollow sections.

Cross-section capacity

Cross-sections under uni-axial bending were thoroughly investigated through both analytical and numerical means. The analytical expressions were based on a governing equation that included subtractive moduli terms; these solutions were simplified further into design expressions and were compared to the analytical and numerical solutions.
Stub column test data were found, which showed that non-slender cross-sections can reach peak loads greater than the yield load by as much as 50%. Bending test data were also found for specimens bending about the major axis. The CSM predictive equations allowed greater axial and bending resistances, and offered both an improved statistical mean and a reduction in the scatter when compared to the current European design guidance.

A numerical method was formulated that created interaction surfaces, by computing all permutations of uniform and linearly varying strains which, when combined, did not exceed the CSM limiting strain in compression. The design equation parameters were expressed as simple linear and power expressions based on the key properties of the cross-sections. The accuracy of the fits were very good, with a residual analysis showing deviations of less than 3.5%.

Further work

- Perform a reliability analysis on the design resistance functions using the gathered test data, and determine partial safety factors.

- Extend the numerical models to include the root radius geometry of I-sections and box sections.

- Investigate the rotated bending of mono-symmetric and other asymmetric cross-sections such as channel sections, zed sections and unequal I-sections.

- Incorporate the effects of residual stresses into the numerical results.

- Extend the combined loading numerical model to allow for the analysis of asymmetric cross-sections bending about any co-ordinate axis.

- Find test data for cross-sections subject to general loading, compare with the predictions from the combined loading design equations and gather bending test data for beams bending about the minor axis.

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• Generate experimentally calibrated FEA models, and compare the results to the CSM numerical model interaction surfaces.

• Consider integration of the CSM cross-section analysis with the Overall Interaction Concept by Boissonnade et al. (2013) and the Direct Strength Method.

Laboratory testing

An experimental test program on rectangular hollow sections of grade S355 structural steel was performed for stub columns and simply supported beams. The tensile coupon tests gave yield stresses that were 10-20% higher than the nominal values. Both coupon thickness values displayed yield plateaus that extended up until a strain of approximately 3.5%, after which strain hardening was observed.

The tested stub columns failed by local buckling at peak loads close to the yield load and at end-shortenings close to the yield strain. The tested simply supported beams failed by local buckling and did not reach the plastic moment capacity. The cross-section slenderness of the tested beams and the extended yield plateau of the structural steel, did not permit benefits from strain hardening. Further work

• Determine the situations for taking the strain hardening modulus as zero.

Flexural buckling

The flexural buckling of a pin-ended column with an elastic, linear-hardening material response and loaded by a concentric axial load was investigated. A direct mathematical solution was given, that solved the second order beam–deflection differential equation for the pinned boundary conditions.

By assuming that the total displacement shape was of a half sine wave form, the analysis required the intersections between the displacement–curvature function of this assumed shape and those derived from moment–curvature–thrust curves.
The analysis gave loading and unloading displacements for plotting load–displacement curves. The peak axial loads were extracted from these curves and plotted against the global slenderness to give a suite of buckling curves. The buckling curves gave a transition point between global-dominated instability and local-dominated failure, the latter of which returned axial resistances above the yield load.

A design equation was created from the first yield Ayrton–Perry–Robertson equation for the global-dominated failure region, where axial loads were below the yield load. A linear function was then used for the local-dominated failure region to connect the CSM axial load and the transition axial load.

**Further work**

- Include the effects of shear deformations in the design model, to cater for their influence on local-failure dominated stocky columns.
- Find experimental column buckling data and compare the peak loads to the CSM design predictions.
- Incorporate residual stresses into the numerical models.
- Re-investigate the most suitable design equation form (or confirm the current) for the column buckling curves given any further changes made to the core numerical model.
- Confirm the suitability of simplified column displacement functions when the CSM model is extended to columns with different boundary conditions and loading combinations.
- Further expand the flexural buckling model to extended parameter ranges, other cross-section shapes and different material models.
Miscellaneous

Further work

- Harmonise the CSM across various metallic materials, such as aluminium, high strength steel and stainless steel, and then consider its usage with other non-metallic materials.

- Develop a CSM approach to tackle lateral torsional buckling. It will be necessary to consider: 1) torsional and warping stiffness, and their interaction with the flexural rigidity 2) the external loading application point 3) location and stiffness of lateral and torsional restraints and 4) when global instability reduces the ability of the beam to utilise strain hardening.

- Extend the individual element based methods to the assemblage of elements such as frames.

- Formulate the treatment of shear stresses and shear strain hardening, for pure shear and for the interaction with axial stresses.
References


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