Time-Frequency Analysis on the Heisenberg Group

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Declaration of Originality

This text is based on research by myself and my collaborators. All preceding results shall be acknowledged and referenced appropriately.

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Abstract

It is the main goal of this text to study certain aspects of time-frequency analysis on the $2n+1$-dimensional Heisenberg group. More specifically, we will discuss how the well-studied notions of modulation spaces and Weyl quantization can be extended from the Euclidean space $\mathbb{R}^n$ to the Heisenberg group $\mathbf{H}_n$.

For quite a long time already this group has served as a good test object to verify which concepts and results from Euclidean (thus Abelian) analysis carry over to simple instances of non-Abelian structures.

In the case of the Weyl quantization a reasonable answer for $\mathbf{H}_n$ was first proposed by A. S. Dynin almost forty years ago, although it was studied in more detail only some twenty years after that by G. B. Folland. We will review the foundations laid by Dynin and Folland and present some new results about left-invariant differential operators and the natural product of symbols, the Moyal product.

The special tool for our analysis is a 3-step nilpotent Lie group to which we will refer as the Dynin-Folland group. As the name suggests it originates in the works of the afore-mentioned authors. The group's unitary irreducible representations are in fact the key to both the Weyl quantization and modulation spaces on $\mathbf{H}_n$.

Our results on modulation space on the Heisenberg group are based on H. Feichtinger and K. Gröchenig's coorbit theory and a more recent adaption of it by I. and D. Beltiţă, which focuses on modulation spaces arising from nilpotent Lie groups. We will use a blend of both approaches and discuss the modulation spaces induced by the Dynin-Folland group, among them a type of modulation spaces on $\mathbf{H}_n$.

Contents

Introduction

This text is dedicated to the study of certain aspects of analysis, in particular time-frequency analysis, on the $2n+1$-dimensional Heisenberg group $\mathbf{H}_n$. More precisely, it evolved from the endeavour to give a satisfying answer to the following question: is it possible to define in a plausible way modulation spaces on a quite simple non-compact non-Abelian Lie group like the Heisenberg group, say?

Since their introduction by H. Feichtinger [18] over thirty years ago modulation spaces have become a widely used tool in time-frequency analysis, especially Gabor analysis, and PDE theory. Although originally introduced as a family of Banach function spaces on arbitrary locally compact Abelian groups, the bulk of applications of modulation spaces seems to focus on $\mathbb{R}^n$.

One way to think of the (now classical) modulation spaces $M^{p,q}(\mathbb{R}^n)$ is to say that they measure the global time-frequency distribution of a signal $f \in \mathscr{S}'(\mathbb{R}^n)$ in terms of mixed $L^{p,q}$-norms on phase space, i.e., $\mathbb{R}^{2n}$. The most important tool of contemporary time-frequency analysis is the so-called short-time Fourier transform, or STFT (cf. [19]), whose point values on phase space represent simultaneously "localized" portions of the time-frequency spread of such f.

The convenient abundance and availability of diverse families of Banach function spaces like L^p-spaces, Sobolev spaces, Hölder spaces, Besov spaces, etc., on $\mathbb{R}^n$, bounded subsets of it or even on manifolds has led to questions about localization and invariance under coordinate transformations of the modulation spaces $M^{p,q}(\mathbb{R}^n)$. Both operations are of fundamental importance for the introduction of function spaces on manifolds.

A brief look at the compact Abelian group $\mathbb{T}^n$ reveals that already in this very special case the corresponding modulation spaces $M^{p,q}(\mathbb{T}^n)$ reduce to the Fourier Lebesgue spaces $\mathscr{F}\ell^q(\mathbb{Z}^n)$, and similarly one obtains $\big(M^{p,q} \cap \mathscr{E}'\big)(\mathbb{R}^n) = \big(\mathscr{F}L^q \cap \mathscr{E}'\big)(\mathbb{R}^n)$. (Cf.[62], e.g.) Moreover, it has turned out that the only C^1-changes of variables on $\mathbb{R}^n$ which leave modulation spaces invariant are in fact affine transformations (cf. [49, 62]).

This is, of course, bad news for their extension to manifolds in general, yet not necessarily so bad if the manifold is non-compact and possesses a global chart like any connected simply connected stratified (nilpotent) Lie group, for example. In such case one is furthermore tempted to make use of the Fourier analytic methods at hand (group Fourier transform, sub-Laplacian, heat kernel, etc.) to define modulation spaces in analogy to the Abelian case.

This particular approach to modulation spaces employed in the case of the stratified group $\mathbf{H}_n$ met some fierce resistance in the very Abelian nature of the notion itself. The concept of frequency shifts on non-Abelian groups is a priori not quite viable even for groups like $\mathbf{H}_n$, whose algebraic dual is very well-studied and quite simple. In Subsection 2.2.2 we show in detail which obstacles in the representation theory of $\mathbf{H}_n$ rendered this approach practically futile.

Another approach, which soon seemed more promising in terms of what could be achieved formally, was based on the theory of coorbit spaces first introduced by Feichtinger and Gröchenig [21]. The techniques in this more abstract approach are heavily based on the theory of square-integrable unitary irreducible group representations. A so-called coorbit space can be a subspace or otherwise related to the representation space $\mathcal{H}_\pi$, in practice some L^2-space, on which some locally compact group G acts via some unitary group representation π.

As the authors discovered, modulation spaces were only one family of Banach function spaces out of many that could be defined in this elegant general framework. Others include Besov spaces and Triebel-Lizorkin spaces on $\mathbb{R}^n$, Bergman spaces on the upper half-plane, Bargmann-Fock spaces, Besov spaces on compact homogeneous spaces as well as on stratified Lie groups, etc. (Cf. [20, 39, 35, 36, 37, 38, 34, 31, 7].) In the specific case of $M^{p,q}(\mathbb{R}^n)$, the representation involved is the Schödinger representation of the Heisenberg group (modulo its centre), which acts on the space $L^2(\mathbb{R}^n)$ by combined time and frequency shifts.

The quest to find a locally compact group which acts on $L^2(\mathbf{H}_n)$ via the natural $\mathbf{H}_n$-group translation and Euclidean frequency shifts, admittedly a somewhat contestable compromise, led to the rediscovery of the Dynin-Folland group, a nilpotent Lie group first introduced by A. A. Dynin [13] and some twenty years later studied from a much broader perspective by G. B. Folland [29].

The group had originally been proposed as a means to define a Weyl calculus on the Heisenberg group, in analogy to how $\mathbf{H}_n$ itself is employed to define the classical Weyl quantization on $\mathbb{R}^n$. The Dynin-Folland group was thus conceived as some sort of a Heisenberg group of the Heisenberg group (cf. [28] p. 90). The approach to view it as just one of a whole class of so-called meta-Heisenberg groups was much later proposed by Folland in the elegant paper [29], which seems to have gained little attention yet. Therein the author studies the Lie algebra and generic representations of meta-Heisenberg groups of (fully non-Abelian) two-step nilpotent groups G as well as their automorphism groups, and to some extent the Weyl calculus arising from them. In the special case of $G = \mathbf{H}_n$ one recovers the quantization proposed by Dynin. Moreover, a connection between the $\mathbf{H}_n$-Weyl quantization and the Beal-Greiner calculus on Heisenberg manifolds is established.

It is worthwhile mentioning at this point that there exists a strong connection between this Weyl quantization and modulation spaces on $\mathbf{H}_n$ as this is already the case for their $\mathbb{R}^n$-counterparts. More precisely, this relation is established by the so-called ambiguity function, a close relative of the STFT.

A considerably more abstract approach to modulation spaces induced by arbitrary unitary irreducible representations (unirreps) of nilpotent Lie groups has been proposed by Ingrid and Daniel Beltiţă [5, 6]. The techniques involved are a blend of abstract coorbit theory and the use of a Weyl quantization for nilpotent Lie groups first introduced by N. V. Pedersen [51]. Originally proposed as an intentionally transparent approach to geometric quantization associated with the co-adjoint orbits of nilpotent Lie groups (cf. [50]), Pedersen's extended calculus Weyl-quantizes tempered distributions a priori defined on the co-adjoint orbits. Each class of orbits thus induces a different Weyl-calculus defined via the corresponding unirreps. In the case of the Heisenberg group, it turns out that the Weyl correspondence proposed by Dynin agrees with Pedersen's, an indicator of soundness as we understand it.

Beltiţă and Beltiţă now associate an ambiguity function to each orbit employing tools from Pedersen's calculus. The corresponding modulation spaces are then defined in terms of the mixed $L^{p,q}$-behaviour of the ambiguity function of a vector, i.e., function or distribution, f in the representation space (or more precisely its superspace, the dual of the representation's smooth vectors). The emerging calculations are carried out rather on the Lie algebra than the Lie group itself, the latter being identified exclusively via exponential coordinates. This fact undoubtedly contributes to the main strength of their approach, its elegance, but it also excludes strong features of the original coorbit theory based on groups.

Our approach to modulation spaces can be viewed a strong blend of both. The definitions essentially follow the Beltiţă-Pedersen versions, but instead of an exclusive use of the ambiguity function we also employ a Heisenberg analogue of the STFT. This way we can assure independence of the defining window function of the modulation spaces induced by the Dynin-Folland group as well as many other nilpotent Lie groups which are given as semi-direct products and which possess square-integrable unirreps.

Our approach via the Dynin-Folland group has furthermore led us to study certain properties of the Weyl calculus on $\mathbf{H}_n$ which to our knowledge had not yet been investigated. The results we obtained constitute the second main part of this essay.

The present thesis is structured as follows. Chapter 1 recalls some important notions and concepts from Fourier analysis on locally compact groups, especially Abelian groups.

The latter are then employed to give a brief review of modulation spaces on Abelian groups, particularly $\mathbb{R}^n$, in the first half of Chapter 2. The latter half of Chapter 2 provides a short discussion about the extent of applicability of two approaches we have tested in search of an adequate framework for modulation spaces on $\mathbf{H}_n$. In particular, we will point out why the Dynin-Folland group based coorbit type approach seemed promising early on.

Chapter 3 provides a detailed meta-Heisenberg type construction of the Dynin-Folland Lie algebra and group, which right from the beginning indicates what the group's generic unirreps should look like. The construction is followed by a complete classification of unirreps in terms of Kirillov's orbit method. Chapter 3 is concluded with a brief discussion of its semi-direct product structure and the Plancherel formula for the group Fourier transform.

We will finally present our results on modulation spaces on the Heisenberg group in Chapter 4. Apart from a case-by-case study based on the classification of the Dynin-Folland unirreps, we will provide some general results for modulation spaces induced by specific semi-direct product type nilpotent groups.

Chapter 5, which comprises the second half of our results, is motivated by the relation between modulation spaces and the Weyl calculus on the Heisenberg group. After motivating Dynin's Weyl quantization on $\mathbf{H}_n$, we show that it in fact coincides with Pedersen's Weyl calculus for the Dynin-Folland group and its generic representation.

We then recall a few facts already present in Folland [29], although somewhat more explicitly. Moreover, we show that the left-invariant differential calculus on the Heisenberg group is covered by the $\mathbf{H}_n$-Weyl correspondence, namely by quantizing precisely the polynomials in the frequency variable. In particular, we prove that the $\mathbf{H}_n$-Weyl quantization coincides with the left symmetrization in the sense of Helgasson [43] (cf. Chapter II Section 4).

We continue to introduce Hörmander type symbol classes which respect the homogeneous nature of $\mathbf{H}_n$. Although these quite natural and simple symbol classes were already suggested by Dynin [13] and taken up in a slightly modified version by Folland [29], the latter authors rather make use of the subclasses of polyhomogeneous symbols. In our case we employ the full classes instead and show that the corresponding pseudodifferential operators map the Schwartz space $\mathscr{S}(\mathbf{H}_n)$ continuously into itself.

Subsequently we define a Moyal product of symbols for which we provide a full asymptotic expansion. The latter, however, does not live up to the success of its Euclidean counterpart. Interestingly, this is not so much indebted to the specific symbol classes as to the occurrence of terms one misses on $\mathbb{R}^n$ due to the less intricate nature of Euclidean phase space. Nevertheless, we can recover a closed expression of the Moyal product in the shape of an oscillatory integral. This approach finally allows us to show that the product satisfies the usual mapping properties on our symbol classes, a fact which automatically follows from the asymptotic series expansion in the case of polyhomogeneous symbols.

We conclude Chapter 5 revisiting a link to the Beals-Greiner calculus on Heisenberg manifolds [3] first established by Folland [29]. Incidentally we thereby discover another connection with the theory of modulation spaces.

List of Symbols

$\mathbb{R}$ ($\mathbb{C}$)	real (complex) numbers
$\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, ...$	complex Hilbert spaces
$\| \cdot \|_{\mathcal{H}}$	norm on $\mathcal{H}$
$\langle \cdot, \cdot \rangle_{\mathcal{H}}$	natural inner product on $\mathcal{H}$
X'	the dual space of some topological vector space X
$\mathcal{B}(\mathcal{H})$	space of bounded linear operators on $\mathcal{H}$
$\mathcal{U}(\mathcal{H})$	space of unitary operators on $\mathcal{H}$
$\mathfrak{S}_2(\mathcal{H})$	Schatten-von Neumann p-class on $\mathcal{H}$
$\|T\|_{HS}$	the Hilbert-Schmidt norm of an operator T
$\mathrm{Tr}\,(T)$	the trace of an operator T
$G, G_1, G_2, ...$	locally compact groups, in particular Lie groups
H	subgroup
N	normal subgroup
$N \rtimes H$	semi-direct product
$\mathbb{T}$	compact group $\{z \in \mathbb{C} \mid \|z\| = 1\}$
$\mathbf{H}_n$	Heisenberg group
$\mathbf{H}_{2,n}$	Dynin-Folland group
$\mathfrak{g}, \mathfrak{h}, \mathfrak{n}, ...$	Lie algebras
$\mathfrak{h}_n$	Heisenberg Lie algebra
$\mathfrak{h}_{2,n}$	Dynin-Folland Lie algebra
$x, y...$	group elements of a locally compact group
$g, h...$	group elements of a Lie group
$X = (p, q, t), ...$	group elements of $\mathbf{H}_n$
$Y = (p_Y, q_Y, t_Y), ...$	specified group elements of $\mathbf{H}_n$
$X, Y, Z, ...$	elements of $\mathbb{R}^{2n+1}$
$dx, dy, ...$	Haar measure on a locally compact group
$dg, dh, ...$	Haar measure on a Lie group
π	unitary group representation of some group G
$d\pi$	infinitesimal representation of some Lie algebra $\mathfrak{g}$

ρ, ρ_λ	Schrödinger representations of $\mathbf{H}_n$
π, π_λ	Schrödinger-type representations of $\mathbf{H}_{2,n}$
$\mathcal{O}$	co-adjoint orbit of some Lie group
$\mathcal{O}$	co-adjoint orbit corresponding to some unirrep π
$\mathcal{H}_\pi$	representation space of π
$\mathcal{H}_\pi^\infty$	semi-normed vector space of smooth vectors of π
$\mathscr{S}(\mathbb{R}^n)$	Schwartz class on $\mathbb{R}^n$
$\mathscr{S}'(\mathbb{R}^n)$	tempered distributions on $\mathbb{R}^n$
$\mathscr{S}(G)$	Schwartz class on some nilpotent Lie group G
$\mathscr{S}'(G)$	tempered distributions on some nilpotent Lie group G
$\langle\,.\,,\,.\,\rangle_{\mathscr{S}'(G)}$	sesqui-linear $\mathscr{S}'(G)$-$\mathscr{S}(G)$ duality
$M^{p,q}(\mathbb{R}^n)$	classical (unweighted) modulation space on $\mathbb{R}^n$
$M^{r,s}(\pi)$	(unweighted) modulation space induced by unirrep π
$\sigma(D,X), \mathrm{Op}^\rho$	classical Weyl quantization on $\mathbb{R}^n$ of symbol $\sigma \in \mathscr{S}'(\mathbb{R}^{2n})$
$\sigma(\mathscr{D},\mathscr{X}), \mathrm{Op}^\pi$	$\mathbf{H}_n$-Weyl quantization of symbol $\sigma \in \mathscr{S}'(\mathbb{R}^{4n+2})$
Op^π_P	Weyl-Pedersen quantization induced by unirrep π
Op^π_{BG}	Beals-Greiner quantization induced by unirrep π

1 Preliminaries

This chapter provides a brief introduction to the most basic notions and tools from abstract harmonic analysis, which will appear frequently throughout this text. In Section 1.2 we recall some facts about the Haar measure and representation theory on arbitrary locally compact groups, while Section 1.3 focuses on the $2n+1$-dimensional Heisenberg group $\mathbf{H}_n$. The most important aspects of the notation and conventions we will use throughout this text are discussed in Section 1.1.

1.1 Notation and Conventions

Following Folland's convention, shared by Feichtinger and Gröchenig, of having 2π in the exponent of the Fourier integral, we our standard definition for the Fourier transform will be

$$\widehat{f}(\xi) := \int f(x) e^{-2\pi i x \xi}\, dx.$$

We use the same convention for Fourier series on the torus group $\mathbb{T}$, which we associate with the interval $[0,1]$ equipped with the group law of the circle group S^1.

1.2 Some Elementary Ingredients for Harmonic Analysis on Groups

Since the theory of classical modulation spaces uses Fourier transformation in one form or another we will briefly introduce the required notions in the most general framework, namely for locally compact groups G. A locally compact group is a locally compact space G which moreover possesses a group structure such that group multiplication is continuous from $G \times G$ to G and such that group inversion is a homeomorphism on G. A detailed account on abstract harmonic analysis can be found in the excellent monograph Folland [30].

1.2.1 Locally Compact Groups and Haar Measure

Locally compact groups posses (up to positive mulitplicative constants) uniquely determined left-invariant and right-invariant Borel measures, the so-called left and right Haar measures, often denoted by λ and ρ. For certain types of groups these two measures coincide. These groups are referred to as unimodular groups. Examples of unimodular groups are the Abelian groups, the compact groups and all nilpotent Lie groups (cf. [30], Chapter 2). The latter types of groups in fact cover most of the groups that we will deal with in practice. Whenever the left and right Haar measures differ the so-called modular function Δ_G of G, a continuous group homomorphism from G into $\mathbb{R}^+$, grants the relation $d\rho(x) = \Delta_G(x^{-1})d\lambda(x)$. In case the of unimodular groups we thus have $\Delta_G = 1$.

In general we will always adhere to the left Haar measure, frequently denoting it simply by dx instead of $d\lambda$. In the context of Abelian groups, where the left and right Haar measures coincide, $d\rho$ may occasionally denote the Haar measure on the corresponding dual group $\widehat{G}$, but we will always explicitly fix the notation before use. Note that whenever we write $L^p(G)$, unless otherwise stated, we refer to the Lebesgue space defined by the left Haar measure on G.

Remark 1.1. We should point out that our notation for group elements will vary between $x, y, \ldots \in G$ and $g, g', h, \ldots \in G$, depending on the context and the risk of confusion with other notation, thus exhibiting a slight preference for the former one in the more abstract context of arbitrary locally compact groups whereas the latter notation will be more prevalent in the context of Lie groups. This notational ambiguity carries over to the Haar measure, which is correspondingly denoted by $dx, dg, \ldots$.

1.2.2 Unitary Irreducible Group Representations

In order to dispose of a Fourier transform on a locally compact group G, we need to employ a class of functions which corresponds to the family of

$$e_\xi : \mathbb{R}^n \to \mathbb{T} : x \mapsto e^{2\pi i\, x \cdot \xi}, \tag{1.1}$$

with $\xi \in \mathbb{R}^n$, in the case of $\mathbb{R}^n$ and to

$$e_k : \mathbb{T}^m \to \mathbb{T} : x \mapsto e^{2\pi i\, x \cdot k}, \tag{1.2}$$

with $k \in \mathbb{Z}^m$, in the case of $\mathbb{T}^m$. Theses functions are the so-called irreducible unitary group representations, or unirreps, which are defined as follows.

Definition 1.2. *Let G be a locally compact group with identity element e, and let $\mathcal{H}$ be a complex Hilbert space. A strongly continuous unitary representation (or simply unitary representation) π of G on $\mathcal{H}$ is a map $\pi : G \to \mathcal{U}(\mathcal{H})$, set of all unitary operators on H, satisfying the following properties*

(i) $\pi(x_1x_2) = \pi(x_1)\pi(x_2)$ *for all* $x_1, x_2 \in G$,

(ii) $\pi(e) = I_{\mathcal{H}}$,

(iii) $x_j \xrightarrow{G} x \quad \Rightarrow \quad \pi(x_j)u \xrightarrow{\mathcal{H}} \pi(x)u$ *for all* $u \in \mathcal{H}$,

The space $\mathcal{H}$ is called the representation space of π. It is frequently denoted by $\mathcal{H}_\pi$. Such a representation is furthermore said to be (topologically) irreducible if it satisfies

(iv) π *is non-trivial, i.e.,* $\pi \neq x \mapsto I_{\mathcal{H}_\pi} : G \to \{I_{\mathcal{H}_\pi}\} \subseteq \mathcal{U}(\mathcal{H}_\pi)$, *and the only closed subspaces of* $\mathcal{H}_\pi$ *invariant under* $\pi(G)$ *are* $\{0\}$ *and* $\mathcal{H}_\pi$ *itself.*

Two unitary representations $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\rho : G \to \mathcal{U}(\mathcal{H}_\rho)$ are said to be unitarily equivalent if there exists a unitary map $U : \mathcal{H}_\rho \to \mathcal{H}_\pi$, called equivalence, such that $\pi(x) = U\rho(x)U^*$ for all x in G.

Note that unitarity implies $(\pi(x))^* = (\pi(x))^{-1} = \pi(x^{-1})$ for all $x \in G$. Conditions (i) and (ii) tell us that π is a group homomorphism from G into $\mathcal{U}(\mathcal{H}_\pi)$. Condition (iii) expresses pointwise continuity of π, i.e., continuity in the strong operator topology. Finally, Condition (iv) states that there are no proper sub-representations of π. This property is referred to as (topological) irreducibility of π.

Now, it is easily checked that $e_\xi, \xi \in \mathbb{R}^n$, and $e_k, k \in \mathbb{Z}^m$, satisfy Conditions (i)–(iii). In both cases their corresponding representation space $\mathcal{H}_\pi$ is the one-dimensional Hilbert space $\mathbb{C}$. The proof of Condition (iv) is usually more involved, but there are several general statements which classify irreducible representations for certain types of groups. In the Abelian case, e.g., all unirreps are one-dimensional (cf. [30] Chapter 3 Corollary 3.6). Since they have to satisfy the group homomorphism properties, they must be of the form (1.1) and (1.2) in the cases $G = \mathbb{R}^n$ and $G = \mathbb{T}^m$, respectively. For compact groups one can prove that all unirreps must be finite-dimensional and every unitary representation is given as a direct sum of unirreps (cf. [30] Chapter 5 Theorem 5.2). In the case of the non-Abelian, non-compact Heisenberg group $\mathbf{H}_n$ we will see that all relevant unirreps are infinite-dimensional time-frequency shift operators on $L^2(\mathbb{R}^n)$ (cf. Subsection 1.3.3).

Example 1.3. The arguably most prototypical example of a unitary representation of a locally compact group G is its left regular representation

$$L : G \to \mathcal{U}(L^2(G)),$$
$$L(x)(f) := T_x f = y \mapsto f(x^{-1}y),$$

provided we use the left Haar measure. Its unitarity follows straight away from the left-invariance of dx. If we work with the right Haar measure or a bi-invariant Haar measure, we can define the right regular representation

$$R : G \to \mathcal{U}(L^2(G)),$$
$$R(x)(f) := T_x f = y \mapsto f(yx).$$

It is usually of interest to see how L (or R) can be decomposed into unirreps of G.

We will frequently speak about the set of all unirreps of a given group G . It makes sense to define this set up to unitary equivalence.

Definition 1.4. *Given a locally compact group G, we define its unitary dual $\widehat{G}$ to be the set of all equivalence classes (in the sense of Definition 1.2 (iv)) of irreducible unitary representations of G.*

In the case of an Abelian group G, the dual set $\widehat{G}$ possesses a natural group structure which makes it into a locally compact topological group with respect to the w^*-topology inherited from $L^\infty(G)$, thus possessing a bi-invariant Haar measure (cf. [30] p. 89). We will refer to $\widehat{G}$ as the dual group of G and its members will frequently be called characters.

1.2.3 The Group Fourier Transform

We want to conclude this section with the definition of the so-called group Fourier transform (GFT). In analogy to the Fourier transform on $\mathbb{R}^n$ given by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i\, x \cdot \xi}\, dx = \int_{\mathbb{R}^n} f(x)e_\xi(-x)\, dx = \int_{\mathbb{R}^n} f(x)(e_\xi(x))^*\, dx =: \widehat{f}(e_\xi), \quad (1.3)$$

we define the GFT of a given function $f : G \to \mathbb{C}$ evaluated at a representation π as the integral of f against π^*. Since π can be operator-valued, we will define this integral pointwise as a Bochner integral (cf. [30] for a short introduction to vector-valued integration as well as [12] for a detailed account) with respect to the Haar measure:

Definition 1.5. *Let π be an irreducible unitary representation of the locally compact group G on some Hilbert space $\mathcal{H}_\pi$. For $f \in L^1(G)$ we define its group Fourier transform at π to be the map*

$$\begin{aligned} \widehat{f}(\pi) : \mathcal{H}_\pi &\to \mathcal{H}_\pi, \\ u &\mapsto \int_G f(x)\pi(x)^* u \, dx = \int_G f(x)\pi(x^{-1}) u \, dx. \end{aligned} \tag{1.4}$$

Due to linearity and certain other properties of the Bochner integral the operator $\widehat{f}(\pi)$ is linear and bounded on $\mathcal{H}_\pi$, with $\left\| \widehat{f}(\pi) \right\| \leqslant \|f\|_{L^1(G)}$. As a function on $L^1(G)$ the map $f \mapsto \widehat{f}$ is also linear and it satisfies the convolution identity

$$\widehat{(f * g)}(\pi) = \widehat{g}(\pi)\widehat{f}(\pi), \tag{1.5}$$

provided the (left) group convolution is defined by

$$(f * g)(x) := \int_G f(y) g(y^{-1}x) dy. \tag{1.6}$$

One of the most important operations in the context of harmonic analysis on Abelian groups is the modulation of a function.

Definition 1.6. *Given a locally compact Abelian group G and a character $\xi \in \widehat{G}$ we define modulation by ξ by*

$$\begin{aligned} M_\xi : L^1_{loc}(G) &\to L^1_{loc}(G), \\ (M_\xi f)(x) &:= \xi(x) f(x), \end{aligned} \tag{1.7}$$

that is as the multiplication by the character ξ.

In electrical engineering the terminology of frequency modulation refers to conveying information via a carrier signal, say $f : \mathbb{R}^3 \to \mathbb{C}$. Modulation is performed by shifting the frequency of f, that is, by translating its frequency $\widehat{f}$ to $T_\xi \widehat{f} = \widehat{f}(\,.\, - \xi)$. But this in fact translates to a modulation of the signal f by ξ since

$$\widehat{(M_\xi f)}(\eta) = \int_{\mathbb{R}^3} e^{-2\pi i x \eta} (e^{2\pi i x \xi} f) \, dx = \int_{\mathbb{R}^3} e^{-2\pi i x(\eta - \xi)} f \, dx = \widehat{f}(\eta - \xi) = T_\xi \widehat{f}. \tag{1.8}$$

More generally, this observation (involving the same one-line calculation) holds true for any locally compact Abelian groups G as we have $\overline{\xi}(x) \cdot \eta(x) = \overline{\xi - \eta}(x)$, $x \in G$.

Much more can be said about the GFT; we refer the interested reader to [30]. We will also provide further information on special cases whenever it seems necessary.

Let us also note that will occasionally make use of the symbol $\mathscr{F}f$ or $\mathscr{F}_G$ for $\widehat{f}$. Usually, this notation is reserved for the Plancherel-Fourier transform, which extends the GFT restricted to $L^1(G) \cap L^2(G)$ to a unitary transform on $L^2(G)$ (at least for all types of G we will consider more closely). Whenever we abuse this notation, however, it will be unambiguously clear from the context.

1.3 The Heisenberg Group

In this subsection we realize the Heisenberg group $\mathbf{H}_n$ following a recipe we could call "How to construct a meta-Heisenberg group of ...." In our case, the Heisenberg group is the meta-Heisenberg group of $\mathbb{R}^n$, and we explicitly mention this type of construction at this early stage as it will play an important role throughout this text. The term meta-Heisenberg group was first used in Folland [29] and employed for meta-Heisenberg groups of 2-step nilpotent Lie groups. This, of course, includes the 1-step nilpotent Lie group $\mathbb{R}^n$, and we will point out shortly precisely what the name meta-Heisenberg group refers to.

This subsection also comprises a construction of the Schrödinger representations, the natural unirrep of $\mathbf{H}_n$ obtained via the meta-construction, and some explicit formulas for the left- and right- invariant vector fields which will be needed later on. Furthermore, we will have a look at the group Fourier transform and list some of its noteworthy properties.

1.3.1 A Meta-Type Realization of $\mathbf{H}_n$

Let us define the operators Q_k and P_j, $j, k = 1, \ldots, n$, acting on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ via

$$Q_k f(x) := x_k f(x), \tag{1.9}$$

$$P_j f(x) := \frac{1}{2\pi i}\frac{\partial f}{\partial x_j}(x), \tag{1.10}$$

where $f \in \mathscr{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

One checks easily that for any $j, k = 1, \ldots, n$,

$$[P_j, P_k] = [Q_j, Q_k] = 0, \quad [P_j, Q_k] = \frac{\delta_{j,k}}{2\pi i}\,\mathrm{I}, \tag{1.11}$$

where I denotes the identity operator. Let use the convention that the Lie bracket for two essentially adjoint operators A, B acting on $\mathscr{S}(\mathbb{R}^n)$ is defined by $2\pi i$ times the standard commutator $[A,B] := AB - BA$.

Definition 1.7. *Equalities* (1.11) *(times $2\pi i$) are called the* Canonical Commutation Relations *(CCR) or Heisenberg Commutation Relations.*

Let us denote by $\langle Q_k, P_j\rangle$ the Lie algebra generated by the operators Q_k and P_j. This means that $\langle Q_k, P_j\rangle$ is the smallest real Lie algebra of operators which contains the operators Q_k and P_j, $j,k = 1,\dots,n$, the Lie bracket being $2\pi i$ times the commutator bracket. The CCR show that

$$\langle Q_k, P_j\rangle := \mathbb{R}P_1 \oplus \ldots \oplus \mathbb{R}P_n \oplus \mathbb{R}Q_1 \oplus \ldots \oplus \mathbb{R}Q_n \oplus \mathbb{R}\mathrm{I}.$$

This Lie algebra has dimension $2n+1$ and is 2-step nilpotent. Moreover, $\langle Q_k, P_j\rangle$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_n$ whose definition we now recall.

Definition 1.8. *The* Heisenberg Lie algebra $\mathfrak{h}_n$ *is the real Lie algebra with underlying vector space* $\mathbb{R}^{2n+1}$ *endowed with the Lie bracket defined via*

$$j,k = 1,\dots n, \qquad \left.\begin{aligned} [X_{p_j}, X_{p_k}] = [X_{q_j}, X_{q_k}] = [X_{p_j}, X_t] = [X_{q_j}, X_t] = 0 \\ [X_{p_j}, X_{q_k}] = \delta_{jk} X_t, \end{aligned}\right\} \tag{1.12}$$

where $(X_{p_1},\dots,X_{p_n},X_{q_1},\dots,X_{q_n},X_t)$ *denotes the standard basis of* $\mathbb{R}^{2n+1}$.

Note that the Lie algebra isomorphism between $\langle Q_k, P_j\rangle$ and $\mathfrak{h}_n$ is

$$d\rho : \mathfrak{h}_n \longrightarrow \langle Q_k, P_j\rangle \tag{1.13}$$

defined via

$$d\rho(X_{q_k}) = 2\pi i\, Q_k, \quad d\rho(X_{p_j}) = 2\pi i\, P_j, \quad j,k = 1,\dots,n, \quad \text{and } d\rho(X_t) = 2\pi i\, \mathrm{I}.$$

The Lie algebra $\mathfrak{h}_n$ is nilpotent of step 2 and its centre is $\mathbb{R}X_t$. In standard coordinates

$$(p,q,t) := (p_1,\dots,p_n,q_1,\dots,q_n,t),$$

and similarly for (p',q',t'), its Lie bracket given by (1.12) becomes

$$[(p,q,t),(p',q',t')] \;:=\; (0,0,pq'-qp') \tag{1.14}$$

if pq' abbreviates the standard inner product of p and q' on $\mathbb{R}^n$.

Definition 1.9. *The* Heisenberg group $\mathbf{H}_n$ *is the connected simply connected Lie group corresponding to the Heisenberg Lie algebra* $\mathfrak{h}_n$.

Hence $\mathbf{H}_n$ is a nilpotent Lie group of step 2 and its centre is $\exp(\mathbb{R}X_t)$. The group law of $\mathbf{H}_n$ may be given by the Baker-Campbell-Hausdorff formula, which we now recall for a general Lie group G and corresponding Lie algebras $\mathfrak{g}$ (see, e.g., [10, p.11,12]). It reads

$$\exp_G(X) \odot_G \exp_G(Y) = \exp_G(X + Y + \frac{1}{2}[X,Y]_{\mathfrak{g}} + \frac{1}{12}([X,[X,Y]_{\mathfrak{g}}]_{\mathfrak{g}} - [Y,[X,Y]_{\mathfrak{g}}]_{\mathfrak{g}}) - \frac{1}{24}[Y,[X,[X,Y]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}} + \dots). \tag{1.15}$$

This formula always holds at least on a neighbourhood of the identity of G and in fact whenever the series in the right hand side converges. If G is a connected simply connected nilpotent Lie group, the exponential mapping $\exp_G : \mathfrak{g} \to G$ is a bijection and Formula (1.15) holds on $\mathfrak{g}$ since the series on the right hand side is finite. For the case $\mathfrak{g} = \mathfrak{h}_n$ it yields

$$\exp_{\mathbf{H}_n}(X) \odot_{\mathbf{H}_n} \exp_{\mathbf{H}_n}(Y) = \exp_{\mathbf{H}_n}\left(X + Y + \frac{1}{2}[X,Y]\right) \tag{1.16}$$

for all X, Y in $\mathfrak{h}_n$.

We now realize the Heisenberg group $\mathbf{H}_n$ using exponential coordinates. This means that we identify an element of $\mathbf{H}_n$ with an element of $\mathbb{R}^{2n+1}$ via

$$(p,q,t) = \exp_{\mathbf{H}_n}\big(\sum_{j=1}^{n}(p_j X_{p_j} + q_j X_{q_j}) + tX_t\big)$$

Hence, using this identification, the centre of $\mathbf{H}_n$ is $\{(0,0,t) : t \in \mathbb{R}\}$ and the group law given in (1.16) becomes

$$(p,q,t) \odot_{\mathbf{H}_n} (p',q',t') = \big(p+p', q+q', t+t' + \frac{1}{2}(pq' - qp')\big). \tag{1.17}$$

Remark 1.10 (On the Meta-Heisenberg-Construction)**.** This term coined by Folland refers to the construction of a nilpotent Lie group $H(G)$ of one step $m+1$, given an m-step nilpotent group G, such that the nilpotent structure of $H(G)$ is essentially given by the commutation relations of the left-invariant vector fields of G and multiplication by each coordinate function.

Remark 1.11 (On the Haar measure)**.** Since the $\mathbf{H}_n$-Haar measure coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$, we can make further use of the latter coordinates and write the $\mathbf{H}_n$-Haar measure as $dp\,dq\,dt$. It hence follows that $L^r(\mathbf{H}_n) \cong L^r(\mathbb{R}^{2n+1})$ for all $r \in \mathbb{R}^+$.

It is furthermore worth mentioning that the identification $\mathbf{H}_n \cong \mathbb{R}^{2n+1}$ allows us to define $\mathscr{S}(\mathbf{H}_n) \cong \mathscr{S}(\mathbb{R}^{2n+1})$.

1.3.2 Left-invariant Vector Fields

Let us recall that the left and right regular representations L and R (defined as in Example 1.3) of a unimodular Lie group G on $L^2(G)$ are unitary and that their infinitesimal representations yield the isomorphisms between the Lie algebra of G and the Lie algebra of the smooth right- and left-invariant vector fields on G, respectively. More precisely, the left-invariant vector field $dR(X)$ corresponding to a vector $X \in \mathfrak{g}$ at a point $g \in G$ is given by

$$dR(X)f(g) = \frac{d}{d\tau}\bigg|_{\tau=0} f(g \exp_{\mathbf{H}_n}(\tau X)),$$

for any differentiable function f on G, whereas the right-invariant vector field $dL(X)$ corresponding to X is given by

$$dL(X)f(g) = \frac{d}{d\tau}\bigg|_{\tau=0} f(\exp_{\mathbf{H}_n}(-\tau X)g).$$

Short computations in the case of the Heisenberg group $\mathbf{H}_n$ yield the following expressions for the left and right-invariant vector fields corresponding to the basis vectors X_{p_j}, X_{q_k}, X_t for $j,k = 1,\dots n$. For the left-invariant vector fields we adopt the notation

$$\left.\begin{array}{rcccl} \mathscr{D}_{p_j} & := & (2\pi i)^{-1} dR(X_{p_j}) & = & (2\pi i)^{-1}\left(\frac{\partial}{\partial p_j} - \frac{1}{2} q_j \frac{\partial}{\partial t}\right), \\ \mathscr{D}_{q_k} & := & (2\pi i)^{-1} dR(X_{q_k}) & = & (2\pi i)^{-1}\left(\frac{\partial}{\partial q_k} + \frac{1}{2} p_k \frac{\partial}{\partial t}\right), \\ \mathscr{D}_t & := & (2\pi i)^{-1} dR(X_t) & = & (2\pi i)^{-1} \frac{\partial}{\partial t}, \end{array}\right\} \tag{1.18}$$

while for the right-invariant vector fields we obtain

$$-dL(X_{p_j}) = \left(\frac{\partial}{\partial p_j} + \frac{1}{2} q_j \frac{\partial}{\partial t}\right), \quad -dL(X_{q_k}) = \left(\frac{\partial}{\partial q_k} - \frac{1}{2} p_k \frac{\partial}{\partial t}\right), \quad -dL(X_t) = \frac{\partial}{\partial t}.$$

1.3.3 The Schrödinger Representation

Here we show that there is only one possible representation of the Heisenberg group $\mathbf{H}_n$ with infinitesimal representation $d\rho$ defined in (1.13). This 'natural' representation ρ will turn out to be the well known (canonical) Schrödinger representation of $\mathbf{H}_n$.

We start with the following three observations. Firstly, from the group law, we have

$$\begin{aligned}(p,q,t) &= (0,q,0)(p,0,0)(0,0,t+\frac{pq}{2})\\ &= \exp_{\mathbf{H}_n}(q_1X_{q_1})\ldots\exp_{\mathbf{H}_n}(q_nX_{q_n})\exp_{\mathbf{H}_n}(p_1X_{p_1})\ldots\exp_{\mathbf{H}_n}(p_nX_{p_n})\\ &\quad \exp_{\mathbf{H}_n}\Big((t+\frac{pq}{2})X_t\Big).\end{aligned}$$

Secondly, from the definition of an infinitesimal representation, we know that if $d\rho$ is the infinitesimal representation of ρ, then we must have for every $X \in \mathfrak{h}_n$, and $\tau \in \mathbb{R}$

$$\rho(\exp_{\mathbf{H}_n}(\tau X)) = e^{\tau d\rho(X)},$$

where the right hand side is understood as the 1-parameter group of operators with generator $d\rho(X)$. Therefore, if it can be constructed, the representation ρ will be characterised by the 1-parameter groups with generators

$$d\rho(X_{p_j}) = 2\pi i\, P_j = \frac{\partial}{\partial x_j}, \quad d\rho(X_{q_k}) = 2\pi i\, Q_k = \times(2\pi i\, x_k) \quad \text{and} \quad d\rho(X_t) = 2\pi i\, \mathrm{I}.$$

Thirdly, it is well known that the operators $2\pi i\, P_j, 2\pi i\, Q_k$ and $2\pi i\, \mathrm{I}$ are essential skew-adjoint on $\mathscr{S}(\mathbb{R}^n)$ and generate the 1-parameter unitary groups of operators on $L^2(\mathbb{R}^n)$

$$\{e^{d\rho(\tau X_{p_j})}\}_{\tau\in\mathbb{R}}, \quad \{e^{d\rho(\tau X_{q_k})}\}_{\tau\in\mathbb{R}}, \quad \{e^{d\rho(\tau X_t)}\}_{\tau\in\mathbb{R}},$$

given respectively by

$$\begin{aligned} e^{d\rho(\tau X_{p_j})}f(x) &= f(x_1,\ldots,x_j+\tau,\ldots x_n),\\ e^{d\rho(\tau X_{q_k})}f(x) &= e^{2\pi i\tau x_k}f(x),\\ e^{d\rho(\tau X_t)}f(x) &= e^{2\pi i\tau}f(x),\end{aligned}$$

for $f \in L^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$.

From the three observations above, the unique candidate ρ for a representation of $\mathbf{H}_n$

having infinitesimal representation $d\rho$ must satisfy

$$\begin{aligned}\rho\left(\exp_{\mathbf{H}_n}(p_j X_{p_j})\right) f(x) &= e^{d\rho(p_j X_{p_j})} f(x) = f(x_1, \dots, x_j + p_j, \dots x_n),\\ \rho\left(\exp_{\mathbf{H}_n}(q_k X_{q_k})\right) f(x) &= e^{d\rho(q_k X_{q_k})} f(x) = e^{2\pi i q_k x_k} f(x),\\ \rho\left(\exp_{\mathbf{H}_n}(t X_t)\right) f(x) &= e^{d\rho(t X_t)} f(x) = e^{2\pi i t} f(x),\end{aligned}$$

for $f \in \mathscr{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, and we must have

$$\begin{aligned}\rho(p,q,t)f(x) &= e^{d\rho(q_1 X_{q_1})} \dots e^{d\rho(q_n X_{q_n})} e^{d\rho(p_1 X_{p_1})} \dots e^{d\rho(p_n X_{p_n})} e^{d\rho((t+\frac{pq}{2})X_t)} f(x)\\ &= e^{2\pi i q x}\left(e^{d\rho(p_1 X_{p_1})} \dots e^{d\rho(p_n X_{p_n})} e^{d\rho((t+\frac{pq}{2})X_t)}\right) f(x)\\ &= e^{2\pi i q x}\left(e^{d\rho(0,0,t+\frac{pq}{2})}\right) f(x+p)\\ &= e^{2\pi i q x} e^{2\pi i (t+\frac{pq}{2})} f(x+p),\end{aligned}$$

that is,

$$\rho(p,q,t)f(x) = e^{2\pi i (t + qx + \frac{pq}{2})} f(x+p). \tag{1.19}$$

Conversely, one checks easily that the expression ρ defined via (1.19) gives a unitary representation of $\mathbf{H}_n$. In fact we recognize the so-called *Schrödinger representation* of $\mathbf{H}_n$.

1.3.4 The Family of Schrödinger Representations

In this subsection we describe the complete family of Schrödinger representations ρ_λ, $\lambda \in \mathbb{R}\backslash\{0\}$, of $\mathbf{H}_n$.

We prefer to define a Lie algebra or a Lie group via a concrete description (the most common realization or the most useful for a certain purpose) rather than as a class of isomorphic objects given via a representative. Indeed, we have defined the Heisenberg Lie algebra $\mathfrak{h}_n$ via the CCR on the standard basis of $\mathbb{R}^{2n+1}$ and we have considered a concrete realization of the Heisenberg group $\mathbf{H}_n$. However, it is interesting to define other isomorphisms than $d\rho$. Indeed, let us consider the linear mapping $d\rho_\lambda : \mathfrak{h}_n \to \langle Q_k, P_j \rangle$ defined via

$$d\rho_\lambda(X_{q_k}) = 2\pi i \lambda\, Q_k, \quad d\rho_\lambda(X_{p_j}) = 2\pi i\, P_j, \quad j,k = 1,\dots,n, \quad \text{and } d\rho_\lambda(X_t) = 2\pi i \lambda \mathrm{I}\,,$$

for a fixed $\lambda \in \mathbb{R}\backslash\{0\}$.

Proceeding as for ρ, the following property is easy to check:

Lemma 1.12. *For each $\lambda \in \mathbb{R}\backslash\{0\}$, the mapping $d\rho_\lambda$ is a Lie algebra isomorphism from $\mathfrak{h}_n$ onto $\langle Q_k, P_j\rangle$. It is the infinitesimal representation of the unitary representation ρ_λ of $\mathbf{H}_n$ on $L^2(\mathbb{R}^n)$ given by*

$$\rho_\lambda(p,q,t)f(x) = e^{2\pi i\,\lambda(t+qx+\frac{1}{2}pq)}f(x+p),$$

for $f \in L^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$.

Naturally $\rho = \rho_1$.

The representations ρ_λ, $\lambda \in \mathbb{R}\backslash\{0\}$, given in Lemma 1.12 are also called Schrödinger representations. A celebrated theorem of Stone and Von Neumann says that, up to unitary equivalence, these are all the irreducible unitary representations of $\mathbf{H}_n$ that are nontrivial at the centre:

Theorem 1.13 (Stone-von Neumann)**.** *For any $\lambda \in \mathbb{R}\backslash\{0\}$, the representation ρ_λ of $\mathbf{H}_n$ is unitary and irreducible. If $\lambda, \lambda' \in \mathbb{R}\backslash\{0\}$ with $\lambda \neq \lambda'$ then the representations ρ_λ, $\lambda \in \mathbb{R}\backslash\{0\}$, are inequivalent. Moreover, if π is an irreducible and unitary representation of $\mathbf{H}_n$ such that $\pi(0,0,t) = e^{2\pi i\lambda t}$ for some $\lambda \neq 0$, then π is unitarily equivalent to ρ_λ.*

For a proof, see, e.g., [28, ch 1 §5].

For example, the mapping $\tilde{\rho}_\lambda$ of $\mathbf{H}_n$ defined via

$$\tilde{\rho}_\lambda(p,q,t) := \rho(\sqrt{|\lambda|}p, \frac{\lambda}{\sqrt{|\lambda|}}q, \lambda t),$$

is a unitary representation of $\mathbf{H}_n$ on $L^2(\mathbb{R}^n)$ which is unitarily equivalent to ρ_λ. This can be chosen as another realization of the Schrödinger representation coinciding with the character $e^{2\pi i\lambda\cdot}$ at the centre of $\mathbf{H}_n$.

The Stone-von Neumann Theorem gives an almost complete classification of the $\mathbf{H}_n$-unirreps. In fact, we see that the only other unirreps which can appear are trivial at the centre. Passing the centre through the quotient, those representations are now unirreps of $\mathbb{R}^{2n}$, hence characters of $\mathbb{R}^{2n}$. We thus have:

Theorem 1.14 (Classification of $\mathbf{H}_n$-Unirreps)**.** *Every irreducible unitary representation ρ of $\mathbf{H}_n$ on a Hilbert space H is unitarily equivalent to one and only one of the following representations:*

(i) ρ_λ, $\lambda \in \mathbb{R}\backslash\{0\}$, *acting on* $L^2(\mathbb{R}^n)$,

(ii) $\sigma_{(a,b)} : (p,q,t) \mapsto e^{2\pi i(ap+bq)}$, $a, b \in \mathbb{R}^n$, *acting on* $\mathbb{C}$.

Historical Remark 1.15. One can argue that the Heisenberg group $\mathbf{H}_n$ arose in connection with the early quantum mechanics. In fact, measurements of momentum and position of a quantum particle are, up to a factor i, represented by operators on $L^2(\mathbb{R}^n)$ given by (1.9) and (1.10) and thus satisfy the (CCR) given in (1.11). The parameter λ can then be viewed as the Planck constant, modulo normalisation. (For a presentation of these ideas cf. [28] Section 1.1, e.g.)

1.3.5 Group Center and the Reduced Heisenberg Group

The group centre $Z(\mathbf{H}_n)$ and the commutator subgroup

$$\mathbf{H}_n{}^c = \left\{ghg^{-1}h^{-1} \mid g, h \in \mathbf{H}_n\right\}$$

both coincide with the set $\{(0,0,t) \mid t \in \mathbb{R}\}$.

The periodicity in $t \in \mathbb{R}$ of the Schrödinger representation ρ_λ can be very inconvenient in the context of certain applications. As a result, the map $\rho_\lambda : \mathbf{H}_n \to \mathcal{U}(L^2(\mathbb{R}^n))$ is neither faithful, i.e., injective, nor square-integrable in the sense of Definition 2.20. It is therefore useful to occasionally use the so-called reduced Heisenberg group

$$\mathbf{H}_{n,red} := \mathbf{H}_n / \{(0,0,k) \mid k \in \mathbb{Z}\}.$$

1.3.6 Group Fourier Transform and Plancherel Formula

There is indeed quite a bit that can be said about the GFT on $\mathbf{H}_n$. In order to keep this subsection short, we very briefly collect the some important facts, in particular the Plancherel theorem. The theorem will give an answer as to why the one-dimensional representations $\sigma_{(a,b)}, a, b \in \mathbb{R}^n$, are negligible for the GFT.

For the sake of convenience, let us denote by $\widehat{f}(a,b)$ the GFT $\widehat{f}(\sigma_{(a,b)})$ and by $\widehat{f}(\lambda)$ the GFT $\widehat{f}(\rho_\lambda)$ since up to unitary equivalence these are the only possible values for $\widehat{f}(\pi), \pi \in \widehat{\mathbf{H}}^n$.

In fact there is not much to say about the case of $\pi \in [\sigma_{(a,b)}] \in \widehat{\mathbf{H}}^n$ since the GFT coincides with the Euclidean Fourier transform on $\mathbb{R}^{2n}$.

The ρ_λ-case in turn is quite different from Abelian Fourier transforms; all statements are given for arbitrary $\lambda \in \mathbb{R}\backslash\{0\}$. For a function f in $L^1(\mathbf{H}_n)$ we have observed in Subsection 1.2.3 that the operator $\widehat{f}(\lambda)$ bounded, but in fact much more can be said about the GFT in the cases $f \in L^1(\mathbf{H}_n)$ and $f \in L^2(\mathbf{H}_n)$.

Proposition 1.16. *Let $f \in L^1(\mathbf{H}_n)$ and let $g \in L^2(\mathbf{H}_n)$. Then, $\widehat{g}(\lambda)$ is compact on $L^2(\mathbb{R}^n)$ and $\widehat{f}(\lambda)$ is Hilbert-Schmidt on $L^2(\mathbb{R}^n)$. Moreover, the Hilbert-Schmidt norm of*

$\widehat{f}(\lambda)$ is given by

$$\|\widehat{f}(\lambda)\|_{HS}^2 = |\lambda|^{-n} \int_{\mathbb{R}^{2n}} |\widehat{f}(h,q,p)|^2 \, dq \, dp \tag{1.20}$$

Let us recall that the set of Hilbert-Schmidt operators on a Hilbert space H, also know as the Schatten-von Neumann 2-class $\mathfrak{S}_2(H)$, can be turned into a Hilbert space of operators if we equip it with the inner product $\langle A, B\rangle_{HS} := \mathrm{Tr}\,(B^*A)$ for $A, B \in HS(H)$. Hence, the GFT of an $L^2(\mathbf{H}_n)$-function f defines a Hilbert space-valued map $\widehat{f} : \mathbb{R}\backslash\{0\} \to \mathfrak{S}_2(L^2(\mathbb{R}^n)) : \lambda \mapsto \widehat{f}(\lambda)$ with a certain decay behaviour in λ as we observe in Identity (1.20).

A closer look reveals that the function $\widehat{f}$ is actually square-integrable over $\mathbb{R}\backslash\{0\}$ in the sense of strong Bochner integrals if we choose the right measure μ. In that case its L^2-norm equals $\|f\|_{L^2(\mathbf{H}_n)}$. This gives a very strong statement for the GFT, that is, a Plancherel-type theorem:

Theorem 1.17 (Plancherel Theorem on $\mathbf{H}_n$)**.** *Let the measure μ on $\mathbb{R}\backslash\{0\}$ be defined by $d\mu(\lambda) := |\lambda|^n \, d\lambda$. Then, the group Fourier transform $f \mapsto \widehat{f}$ restricted to $L^1(\mathbf{H}_n) \cap L^2(\mathbf{H}_n)$ extends to a unitary isomorphism $\mathscr{F}$ of $L^2(\mathbf{H}_n)$ onto $L^2(\mathbb{R}\backslash\{0\}, \mathfrak{S}_2(L^2(\mathbb{R}^n)); \mu)$. In particular, we have*

$$\|f\|_{L^2(\mathbf{H}_n)}^2 = \int_{\mathbb{R}\backslash\{0\}} \|\mathscr{F}(f)(\lambda)\|_{HS}^2 \, d\mu(\lambda)$$

for all $f \in L^2(\mathbf{H}_n)$.

This in fact is a statement of existence for the Plancherel-measure on the unitary dual $\widehat{\mathbf{H}}^n$, which moreover states that the subset $\{[\sigma_{(a,b)}] \in \widehat{\mathbf{H}}^n \mid a, b \in \mathbb{R}^n\}$ is of vanishing measure. That is, the only relevant representation for the $\mathbf{H}_n$-GFT are the Schrödinger representations.

The measure μ furthermore allows for an inversion formula for GFT of nicely-behaved functions such as the Schwartz functions, e.g.

Theorem 1.18. *For all $f \in \mathscr{S}(\mathbf{H}_n)$, the inverse Fourier transform $\mathscr{F}^{-1} : \mathscr{F}(f) = \widehat{f} \mapsto f$ is given by the formula*

$$f(p,q,t) = \int_{\mathbb{R}\backslash\{0\}} \mathrm{Tr}\left(\widehat{f}(\lambda)\pi_h(p,q,t)\right) \, d\mu(\lambda). \tag{1.21}$$

2 Modulation Spaces Revisited - A Review and Incentive

2.1 A Review of the Original Concepts with a Focus on $\mathbb{R}^n$

The first part of this chapter is mainly dedicated to the review and comparison the two different methods in the Abelian case, particularly in the case of $\mathbb{R}^n$. Each of them is formulated in a specific context of representation theory of locally compact groups and the theory of function spaces arising thereof. For each review we will briefly recall the corresponding theoretical backgrounds, highlighting important notions, technicalities and certain statements that we desire to transfer to Heisenberg group.

The second part of the chapter discusses the conclusions we draw from both approaches in view of adapting them to the $\mathbf{H}_n$-setting.

2.1.1 Classical Modulation Spaces via Uniform Frequency Decompositions and Wiener Amalgam Spaces on Locally Compact Abelian Groups

In 1983 H. G. Feichtinger introduced the concept of modulation spaces based upon a notion of Banach spaces of distributions called Wiener-type spaces or more frequently nowadays Wiener amalgam spaces (cf. Feichtinger [15]). The idea was to generalize a recipe for creating (families of) function spaces in the spirit of a very specific space introduced by N. Wiener in his study of Tauberian theorems in 1932 (cf. [72, 73]).

This space, now widely known as Wiener's space, is a Banach space of locally bounded measurable functions that are globally in $l^1(\mathbb{Z}^n)$ in the following sense: if we denote by Q the unit cube in $\mathbb{R}^n$, then a function f is said to be in $W(\mathbb{R}^n)$ if

$$\sum_{k\in\mathbb{Z}^n} \operatorname*{ess\,sup}_{x\in Q} |f(x+k)| = \left\| \left(\|f\chi_{Q+k}\|_{L^\infty(\mathbb{R}^n)} \right)_k \right\|_{l^1(\mathbb{Z}^n)} = \left\| (\|fT_{-k}\chi_Q\|_{L^\infty(\mathbb{R}^n)} \right\|_{l^1(\mathbb{Z}^n)} < \infty, \tag{2.1}$$

where χ denotes the characteristic function of a subset of $\mathbb{R}^n$ and T_y denotes the left shift by $y \in \mathbb{R}^n$.

The idea behind these spaces is that local and global behaviours, respectively, of functions and distributions are measured by two different spaces. A question that arises immediately is how much amalgamated spaces depend on the way one localizes in the first place. For that reason it is important to observe that on the one hand the amalgamated norm in (2.1) can be viewed as localizing f over a cover of $\mathbb{R}^n$, globally bounded in size, consisting of compact neighborhoods around of members of a discrete subset of $\mathbb{R}^n$, in our case the lattice $\mathbb{Z}^n$.

Alternatively, we could view it as localizing f by discrete translates indexed by some set, in our case again $\mathbb{Z}^n$, of one fixed function, usually referred to as window, which in our case is given by the characteristic function of the translate of a compact neighborhood of 0 in $\mathbb{R}^n$. Now, one can go one step further and ask what happens if we exchange discrete shifts for continuous ones and replace the discrete $l^1(\mathbb{Z}^n)$-norm by the continuous $L^1(\mathbb{R}^n)$-norm. Moreover, what happens if we take L^p and L^q-norms other than L^∞ and L^1?

Such questions and others about the possibility to interpolate between the latter amalgamates seem to have inspired Feichtinger to introduce the concept of general Wiener amalgam spaces as Banach spaces of functions and distributions over locally compact Abelian groups that possess two equivalent descriptions: one where the norms involve continuously shifted windows and another one where the shifts could be replaced by a uniform decomposition of the underlying structure, i.e., the Abelian group.

Modulation spaces eventually arose essentially as the inverse Fourier image of Wiener amalgam spaces over $\widehat{G}$. The idea behind this is the following: in analogy to Besov spaces, e.g., where the description via modules of continuity could be expressed by operations on the Fourier spectrum, modulation spaces decompose functions on a group G into frequency localized pieces.

It is then checked whether these pieces, still being functions or distributions on G, belong to certain Banach spaces over G like L^p-spaces, e.g. This is the so-called local behaviour of f. The global behaviour is checked in terms of L^q or ℓ^q-summability over the whole Fourier spectrum $\widehat{G}$. That is, modulation spaces are apparently amalgamated spaces whose local components are Banach spaces over G and whose global components are Banach space over $\widehat{G}$.

If we now Fourier transform the whole space, that is, all f on G with finite modulations space norm, we obtain an amalgamated space over $\widehat{G}$ whose local component has turned into a Fourier-Lebesgue space whereas the global one is still the same.

Many properties about Wiener amalgam spaces, like dualities, interpolation properties, etc., thus translate one-to-one to the case of modulation spaces. Also, the equivalency between of continuous and discrete descriptions is given through the Wiener amalgam perspective. It was apparently for that very reason that Feichtinger coined the name modulation spaces: shifts T_ξ applied to windows on the Fourier side are given through modulations M_ξ, i.e., multiplying by characters $\xi \in \widehat{G}$, of the inverse Fourier images of the windows on G. That is, we analyze function spaces through uniform modulations.

Let us as a first step introduce Wiener amalgam spaces on an Abelian group $\widehat{G}$. Since by Pontryagin's duality there is no distinction between an abstract locally compact Abelian group G and its dual group $\widehat{G}$ we pick the latter as the more convenient choice for the subsequent definition of modulation spaces on G. The local components that we will need to employ should include spaces like $\mathscr{F}L^p(\widehat{G})$ for $1 \leqslant p \leqslant \infty$, $C_0(\widehat{G})$, etc.

It is therefore reasonable to describe a class of spaces which satisfy properties common to those spaces. The first criterium is that they be in standard situation with respect to some weighted Fourier algebra.

Definition 2.1. *Let w be a strictly positive, locally bounded and measurable function on G which satisfies $1 \leqslant w(x)$, $w(xy) \leqslant w(x)w(y)$ and $w(x^{-1}) = w(x)$ for all x, y in G. Then w is called a weight function on G. It will be called admissible if it furthermore satisfies the so-called Beurling-Domar condition (or BD-condition)*

$$\sum_{k=1}^{\infty} k^{-2} \log(w(kx)) < \infty \tag{2.2}$$

for all x in G. A function m is called w-moderate if it is strictly positive and continuous on G, satisfying $m(xy) \leqslant m(x)w(y)$.

We define the Beurling algebra $L^1_w(G)$ to be the Banach convolution algebra $(\{f \mid fw \in L^1(G)\}, *)$ equipped with the norm $\| \,.\, \|_{L^1_w(G)} := \| \,.\, w\|_{L^1(G)}$ and its corresponding Fourier algebra to be the symmetric, pointwise multiplicative Banach algebra $A_w(\widehat{G}) := \{\widehat{f} \mid f \in L^1_w(G)\}$, equipped with the norm $||\widehat{f}||_{A_w(\widehat{G})} := \|f\|_{L^1_w(G)}$.

We furthermore define $A_{w,0}(\widehat{G})$ to be the semi-normed vector space given by the set $A_w(\widehat{G}) \cap C_c(\widehat{G})$ equipped with the natural inductive limit topology induced by the semi-normed space $C_c(\widehat{G})$.

Note that throughout the text E' stands for the topological dual of a given locally convex vector space E, usually equipped with the weak topology unless otherwise stated. (In case of double duals like L^∞ the space usually carries the w^*-topology.)

Definition 2.2. *A Banach space $(B, \| \ \|_B)$ is said to be in standard situation with respect to $A_w(\widehat{G})$ if* *(i)* $A_{w,0}(\widehat{G}) \hookrightarrow B \hookrightarrow A'_{w,0}(\widehat{G})$, *where* $\hookrightarrow$ *stands for a continuous embedding,*

(ii) $(B, \| \ \|_B)$ *is a Banach module with respect to pointwise multiplication over* $A_w(\widehat{G})$, *i.e.,* $\|hf\|_B \leqslant \|h\|_{A_w(\widehat{G})} \|f\|_B$ *for all* h *in* $A_w(\widehat{G})$ *and all* f *in* B,

(iii) $(B, \| \ \|_B)$ *is a Banach module with respect to convolution on* $\widehat{G}$ *over a Beurling algebra* $L^1_{\widehat{w}}(\widehat{G})$, *where* $\widehat{w}$ *is an admissible weight on* $\widehat{G}$ *(but not to be understood as the Fourier transform of a weight* w *on* G*!).*

The space $A_{w,0}(\widehat{G})$ obviously serves as a test function space in the case and its dual as our standard distribution space. The second embedding in Condition (i) finally justifies our recurrent reference to Banach spaces of distributions. We will furthermore speak of distributions locally belonging to B: we denote by B_{loc} all f in $A'_{w,0}(\widehat{G})$ such that hf lies in B for all h in $A_{w,0}(\widehat{G})$.

Definition 2.3. *A Banach space* $(B, \| \ \|_B)$ *in standard situation is said to be*

(i) *left-invariant (or left translation-invariant) if all left translations* T_x *are bounded operators on* B, *i.e.,* $T_x : B \to B$ *with* $\|T_x\| \leqslant C_x < \infty$ *for all* x *in* $\widehat{G}$,

(ii) *right-invariant (or right translation-invariant) if the right translations* $f \mapsto T^R_x f := f(\,.\,x)$ *are bounded operators on* B,

(iii) *translation-invariant if it is both left and right-invariant.*

We say the space is homogenous if it is

(i) *isometrically translation-invariant, i.e.,* $\|T_y f\|_B = \|f\|_B$ *for all* y *in* $\widehat{G}$,

(ii) *translation acts continuously on* B, *i.e.,* $\lim_{y\to e} \|T_y f - f\|_B = 0$.

The spaces described so far fulfill the criteria usually required of local components in Wiener amalgam spaces. Some typical examples for Banach spaces in the standard situation are the spaces $L^p(\widehat{G})$ and $L^p_w(G)$ for $1 \leqslant p \leqslant \infty$, $C_0(\widehat{G})$, $A_w(\widehat{G})$ or $\mathscr{F}L^p(G)$ and $\mathscr{F}L^p_w(G)$, $1 \leqslant p \leqslant \infty$. The latter example will be our typical local component in the description of modulation spaces as inverse Fourier images of Wiener amalgam spaces. In case of $G = \mathbb{R}^n$ even Besov-Lizorkin-Triebel spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$ (for a definition cf. Triebel [69]), among them the Sobolev spaces $W^{k,p}(\mathbb{R}^n)$, satisfy all these criteria.

Thus, let us finally define Wiener amalgam spaces and provide a few statements. The first definition given here will be the continuous version of Wiener amalgam spaces.

Definition 2.4. *Let $(B, \| \ \|_B)$ be a homogenous Banach space in standard situation with respect to $A_w(\widehat{G})$ and let $1 \leqslant q \leqslant \infty$. Let moreover v be a w-moderate function on $\widehat{G}$. We then define the Wiener amalgam space $W(B, L^q_v)(\widehat{G})$ the space of all f in B_{loc} such that for any arbitrary, but fixed non-zero window h in $A_{w,0}(\widehat{G})$ their control functions*

$$F^{(h)} : \widehat{G} \to \mathbb{C} : \xi \mapsto \|T_\xi h \, f\|_B$$

lies in $L^q_v(\widehat{G})$, i.e.,

$$\|f\|_{W(B,L^q_v)(\widehat{G})} := \left\| F^{(h)} \right\|_{L^q_v(\widehat{G})} < \infty. \tag{2.3}$$

As insinuated in the formulation of Definition 2.4, the particular choice of window does not matter. Norms with two different windows h_1 and h_2 are equivalent, thus define the same space.

The equivalent discrete definition involves as mentioned above certain partitions of the underlying space, or to be more precise, nicely behaved partitions of unity. The constituents of these partitions will have to be members of $A_w(\widehat{G})$ for the definitions to prove equivalent. Their support will be uniformly bounded and "centered" around discretely distributed points ξ_j in $\widehat{G}$, which makes it plausible for a continuously shifted fixed window to induce the same action. Also, for the partition to be globally uniform, the number of supports that intersect should be locally finite and globally bounded.

Definition 2.5. *Let us call a family $\Psi := (\psi_j)_{j \in J}$ a* **bounded uniform partition of unity** *in $A_w(\widehat{G})$ of size $\widehat{Q}$ if there exists some non-empty relatively compact set $\widehat{Q} \subseteq \widehat{G}$ and a family $\Xi = (\xi_j)_{j \in J}$ such that*

(i) $\sum_{j \in J} \psi_j = 1$,

(ii) $\sup_{j \in J} \|\psi_j\|_{A_w(\widehat{G})} =: C_{\Psi,0} < \infty$,

(iii) $\mathrm{supp}(\psi_j) \subseteq \xi_j \widehat{Q}$ *for all* $j \in J$,

(iv) $\sup_{k \in J} \sharp\{j \neq k \mid \xi_j \widehat{Q} \cap \xi_k \widehat{Q} \neq \varnothing\} =: C_{\Psi,1} < \infty$.

In this case we will say the family $\Xi = (\xi_j)_{j \in J}$ is $\widehat{Q}$-dense and relatively separated referring to $\bigcup_{j \in J} \xi_j \widehat{Q} = \widehat{G}$ and Condition (vi), respectively, or simply well-spread.

Proposition 2.6. *Given a BUPU Ψ of size $\widehat{Q}$ in $A_w(\widehat{G})$, the space $W(B, L^q_v)(\widehat{G})$ is* *defined to be the set of all distributions f in B_{loc} of finite discrete amalgamated norm*

$$||f \mid W(\widehat{Q}, B, \ell^q_v)|| := \left(\sum_{j \in J} \|\psi_j f\|_B^q \, v(\xi_j)^q \right)^{1/q} = \left\| \left(\|\psi_j f\|_B \, v(\xi_j) \right)_{j \in J} \right\|_{\ell^q_v}. \tag{2.4}$$

As in the case of Definition 2.4, where the particular choice of window did not alter the space, different BUPU's give equivalent norms, hence define the same Wiener amalgam spaces. It is thus a matter of convenience, and in fact very often determined by applications, to choose between either description; the same applies to the concrete choice of window and BUPU, respectively. The proofs of all these equivalences can be found in Feichtinger [15].

Many useful and desired properties of modulation spaces are in fact inherited from Wiener amalgam spaces since modulation spaces on locally compact Abelian groups can be viewed as Fourier transforms of Wiener amalgam spaces on their dual groups. The local components B of modulation spaces have to satisfy a few more properties since they need to fulfill certain requirements of function spaces on both G and $\widehat{G}$.

Definition 2.7. *Let G be a locally compact Abelian group. A Banach space $(B, \| \, \|_B)$ is called a BF-space on G if it is continuously embedded into the semi-normed space $L^1_{loc}(G)$.*

A BF-space B is said to be

(i) solid if $g \in L^1_{loc}(G)$, $f \in B$ and $|g(x)| \leqslant |f(x)|$ locally almost everywhere (l.a.e.) implies $g \in B$ and $\|g\|_B \leqslant \|f\|_B$,

(ii) rearrangement-invariant if $|\{x \mid |g(x)| \geqslant \alpha\}| = |\{x \mid |f(x)| \geqslant \alpha\}|$ for all $\alpha > 0$ implies $\|g\|_B = \|f\|_B$.

Finally, a solid BF-space will be referred to as a Banach function space.

Note that rearrangement-invariant BF-spaces are solid and isometrically translation-invariant. If B contains $C_c(G)$ as a dense subspace, the translations are also continuous on B, which renders B homogeneous.

Our most prominent examples for B will be weighted and unweighted L^p-spaces and other weighted versions of BF-spaces such as $B_m := \{f \mid fm \in B\}$ for a given solid, translation-invariant BF-space $(B, \| \, \|_B)$ and a w-moderate m. The space B_m is naturally equipped with the norm $\| \, \|_{B_m} := \| \, . \, m\|_B$. The translation operators T_y are then bounded on B_m with $\|T_y\| \leqslant w(y)$ for all y in G. If $C_c(G)$ is dense in B, then it is also dense in

B_m and translation is a continuous operation on B_m. Moreover, it turns out that B_m is a Banach convolution module over the Beurling algebra $L^1_w(G)$, i.e.,

$$\|f * h\|_{B_m} \leqslant \|f\|_{B_m} \cdot \|h\|_{L^1_w(G)}$$

holds for all f in B_m and all h in $L^1_w(G)$. Finally, we will call a BF-space **admissible** if it is of the form B_m as just described.

We can now give the definition of what is nowadays often referred to as classical modulation spaces.

Definition 2.8. *Let G be a locally compact Abelian group G and let w and $\widehat{w}$ be admissible weight functions in the sense of Definition 2.1 on G and $\widehat{G}$, respectively. Given an admissible BF-space $(B, \| \, \|_B)$ and a $\widehat{w}$-moderate function v on $\widehat{G}$, we define for any $1 \leqslant q \leqslant \infty$ and any arbitrary, but fixed non-zero window*

$$\varphi \in \Lambda^K_w := \{f \in L^1_w(G) \mid \operatorname{supp}(\widehat{f}) \Subset \widehat{G}\} = \{f \mid \widehat{f} \in A_{w,0}(\widehat{G})\},$$

with $\psi := \mathscr{F}\varphi$, the modulation space $M(B, L^q_v)(G)$ to be the set of all distributions f in $(\Lambda^K_w)'$ such that $f * M_\xi k$ belongs to B for all ξ in $\widehat{G}$ and $\xi \mapsto \|f * M_\xi k\|_B$ belongs to $L^q_v(\widehat{G})$, i.e.,

$$\|f\|_{M(B,L^q_v)(G)} := \left(\int_{\widehat{G}} \|f * M_\xi \varphi\|^q_B \, v(\xi)^q \, d\xi\right)^{1/q} = \left\|\xi \mapsto ||\mathscr{F}^{-1}(T_\xi \psi \cdot \mathscr{F} f)||_B\right\|_{L^q_v(\widehat{G})} < \infty \tag{2.5}$$

for $1 \leqslant q < \infty$ and

$$\|f\|_{M(B,L^\infty_v)(G)} := \sup_{\xi \in \widehat{G}} \left(\|f * M_\xi \varphi\|_B \, v(\xi)\right) = \left\|\xi \mapsto ||\mathscr{F}^{-1}(T_\xi \psi \cdot \mathscr{F} f)||_B\right\|_{L^\infty_v(\widehat{G})} < \infty \tag{2.6}$$

for $q = \infty$, respectively.

The independence of the particular choice of window follows from the general fact for Wiener amalgam spaces, i.e., from $\mathscr{F}(M(B, L^q_v)(G)) = W(\mathscr{F}B, L^q_v)(\widehat{G})$. Also, the equivalent discrete description comes for free via Proposition 2.6.

Corollary 2.9. *Let $\Psi := (\psi_j)_{j \in J}$ be a bounded uniform partition of unity in $A_w(\widehat{G})$ of* *size $\widehat{Q}$. A distribution f in Λ_w^K is a member of $M(B, L_v^q)(G)$ if and only if*

$$\left\| f \mid M(\widehat{Q}, B, \ell_v^q) \right\| := \left(\sum_{j \in J} \|f * \varphi_j\|_B^q \, v(\xi_j)^q \right)^{1/q} = \left\| \left(\|\mathscr{F}^{-1}(\psi_j \cdot \mathscr{F} f)\|_B \, v(\xi_j) \right)_{j \in J} \right\|_{\ell_v^q} < \infty \tag{2.7}$$

for $1 \leqslant q < \infty$ and

$$\left\| f \mid M(\widehat{Q}, B, \ell_v^\infty) \right\| := \sup_{j \in J} \left(\|f * \varphi_j\|_B \, v(\xi_j) \right) = \left\| \left(\|\mathscr{F}^{-1}(\psi_j \cdot \mathscr{F} f)\|_B \, v(\xi_j) \right)_{j \in J} \right\|_{\ell_v^\infty} < \infty \tag{2.8}$$

for $q = \infty$, respectively, where $\varphi_j := \mathscr{F}^{-1}\psi_j$ for all j in J.

Depending on the norm we use, that is, either (2.5) or (2.7), we will occasionally distinguish nominally between $M(B, L_v^q)(G)$ and $M(\widehat{Q}, B, \ell_v^q)$, respectively.

Some important results for modulation spaces are collected in form of the following theorem.

Theorem 2.10. *For an admissible BF-space $(B, \| \, \|_B)$, $\widehat{w}$-moderate functions v, v_1, v_2 on $\widehat{G}$ and a w-moderate function m on G the following assertions hold true.*

(i) The modulation spaces $M(B, L_v^q)(G)$, $1 \leqslant q \leqslant \infty$, are Banach spaces with respect to the norms (2.5) and (2.4), satisfying

$$\Lambda_w^K(G) \hookrightarrow M(B, L_v^q)(G) \hookrightarrow (\Lambda_w^K)'(G)$$

(ii) The spaces $M(B, L_v^q)(G)$ depend neither on the particular choice of window $k \in \Lambda_w^K$ or BUPU Ψ nor on the particular choice of weights w and $\widehat{w}$. (That is, it only matters that w and $\widehat{w}$ satisfy the (BD)-condition (2.2).)

(iii) The fact that $C_c(G)$ is dense in B implies that Λ_w^K is dense in $M(B, L_v^q)(G)$ for $1 \leqslant q < \infty$ and thus continuity of translation on $M(B, L_v^q)(G)$.

(iii) For $1 \leqslant q < \infty$ the dual space of $M(B, L_v^q)(G)$ can be identified with $M(B', L_{1/v}^{q'})(G)$.

(iv) For any $\widehat{K} \Subset \widehat{G}$ the norms $\| \, \|_B$ and $\| \, \|_{M(B, L_v^q)(G)}$ are equivalent on $\{f \in B \mid \operatorname{supp}(\widehat{f}) \subseteq \widehat{K}\}$.

(v) For $1 \leqslant p_1, q_1 < \infty$, $1 \leqslant p_2, q_2 < \infty$ *and* $\theta \in (0,1)$ *we have the interpolation space identity*

$$\big(M(L^{p_1}_{m_1}, L^{q_1}_{v_1})(G), M(L^{p_2}_{m_2}, L^{q_2}_{v_2})(G)\big)_\theta = M(L^p_m, L^q_v)(G),$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad m = m_1^{1-\theta} m_2^\theta, \quad v = v_1^{1-\theta} v_2^\theta.$$

(vi) For $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$ *and* $v_1 \leqslant v_2$ *we have*

$$M(L^{p_1}_{m_1}, L^{q_1}_{v_1})(G) \hookrightarrow M(L^{p_2}_{m_2}, L^{q_2}_{v_2})(G).$$

(vii) For $m = 1_G$, *and* $v = 1_{\widehat{G}}$ *we have* $M(L^2_1, L^2_1)(G) = M(L^2, L^2)(G) = L^2(G)$.

(viii) The Segal algebra $S_0(G) := M(L^1, L^1)(G)$, *nowadays usually referred to as the Feichtinger algebra, is invariant under the GFT:*

$$\mathscr{F}(S_0(G)) = S_0(\widehat{G}).$$

In case $G = \mathbb{R}^n \cong \widehat{\mathbb{R}^n} = \widehat{G}$ the functions $w_s : x \mapsto (1+|x|^2)^{s/2} = \widehat{w}_s : \xi \mapsto (1+|\xi|^2)^{s/2}$, with $s \in \mathbb{R}$, are a particularly common choice for admissible weights and moderate functions, respectively. In this case it turns out that we can replace $\Lambda^K_w(G)$ and $A_{w,0}(\widehat{G})$, respectively, by the Schwartz space $\mathscr{S}(\mathbb{R}^n) \cong \mathscr{S}(\widehat{\mathbb{R}^n})$.

If one chooses to use exponential weights, one leaves the realm of tempered distributions and has to use ultra-distributions instead (cf. [], e.g.).

Also, in the Euclidean case the focus mostly lies on spaces of the form $M(L^p_{s_1}, L^q_{s_2})(\mathbb{R}^n)$, where L^r_s stands for $L^r_{w_s}$ or $L^r_{\widehat{w}_s}$, respectively. Thus for classical modulation spaces on $G = \mathbb{R}^n$ one usually gives the following slightly altered definition:

Definition 2.11. *Let* $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ *and let* φ *be an arbitrary, but fixed non-zero member of* $\mathscr{S}(\mathbb{R}^n)$. *Then we define the modulation space*

$$\begin{aligned} M^{p,q}_s(\mathbb{R}^n) := \{ f \in \mathscr{S}'(\mathbb{R}^n) \mid f * M_\xi \in L^p(\mathbb{R}^n) \text{ for all } \xi \in \widehat{\mathbb{R}^n}, \\ \xi \mapsto \|f * M_\xi\|_{L^p(\mathbb{R}^n)} \in L^q_s(\widehat{\mathbb{R}^n}) \} \end{aligned}$$

and its corresponding norm is defined in analogy to (2.5) *and* (2.6).

For these spaces the following theorem lists some of the most important properties.

Theorem 2.12. *(i) For any $1 \leqslant q \leqslant \infty$ the spaces $M_s^{p,q}(\mathbb{R}^n)$ does not depend on the particular choice of window and coincides with $M(L^p, L_s^q)(\mathbb{R}^n)$. Moreover, all the norms defining either space are equivalent.*

(ii) For $1 \leqslant q < \infty$ and $s \in \mathbb{R}$ the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is dense in $M_s^{p,q}(\mathbb{R}^n)$ and we have the continuous embeddings

$$\mathscr{S}(\mathbb{R}^n) \hookrightarrow M_s^{p,q}(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n).$$

The dual space of $M_s^{p,q}(\mathbb{R}^n)$ is given by $M_{-s}^{p',q'}(\mathbb{R}^n)$.

(iii) The spaces $M_s^{2,2}(\mathbb{R}^n)$ coincide with the Sobolev spaces $H_s(\mathbb{R}^n)$.

(iv) The identification $\mathbb{R}^n \cong \widehat{\mathbb{R}^n}$ yields invariance under the Fourier transform of the Feichtinger algebra $S_0(\mathbb{R}^n) = M_0^{1,1}(\mathbb{R}^n)$, i.e., $\mathscr{F}(S_0(\mathbb{R}^n)) = S_0(\mathbb{R}^n)$.

In the following we will prove two important properties explicitly for the Euclidean case: first we will prove the recurring statement that modulation spaces are independent of the particular choice of BUPU. To this end, we will construct one model BUPU that can henceforth be used as a convenient toy example and show equivalence with any other arbitrary, but fixed abstract BUPU.

Our second proof concerns the very useful embedding property in Theorem 2.10 (*vi*). For the sake of simplicity and clarity we will give both proofs for the unweighted case $M^{p,q}(\mathbb{R}^n) = M_1^{p,q}(\mathbb{R}^n)$.

Before we start, let us briefly define a class of band-limited L^p-spaces. The construction of our model BUPU essentially involves choosing a nice compactly supported function $\rho : \widehat{\mathbb{R}^n} \mapsto \mathbb{C}$ and translating it across $\widehat{\mathbb{R}^n}$, followed by normalization.

Thus let $\rho \in \mathscr{S}(\widehat{\mathbb{R}^n})$, taking values in $[0,1]$, with $\rho(\xi) = 1$ for $0 \leqslant |\xi| \leqslant \sqrt{n/2}$ and $\rho(\xi) = 0$ for $|\xi| \geqslant \sqrt{2n}$. For such ρ we set $\rho_k := T_k\rho = \rho(\,\cdot - k)$ for $k \in \mathbb{Z}^n$. This almost yields the desired partition except for normalization: for $k \in \mathbb{Z}^n$ let

$$\sigma_k := \rho_k \left(\sum_{k \in \mathbb{Z}^n} \rho_k \right)^{-1}. \tag{2.9}$$

Normalization is perfectly possible since by construction each $\operatorname{supp}(\rho_k)$ intersects only finitely many other $\operatorname{supp}(\rho_l)$, $l \in \mathbb{Z}^n$, and the number of intersections is globally constant. Denoting by $\widehat{Q}_k$ the closed unit cube with centre $k \in \mathbb{Z}^n$, we observe that the following properties hold for all $k \in \mathbb{Z}^n$:

(i) $\sum_{k \in \mathbb{Z}^n} \sigma_k = 1$.

(ii) $\mathscr{F}^{-1}\sigma_k \in L^1_\Omega(\mathbb{R}^n)$ and there exist $C_{\Sigma,0} < \infty$ such that $\|\sigma_k\|_{A(\mathbb{R}^n)} = \|\mathscr{F}^{-1}\sigma_k\|_{L^1(\mathbb{R}^n)} \leqslant C_{\Sigma,0} < \infty$ for all $k \in \mathbb{Z}^n$.

(iii) $\operatorname{supp}(\sigma_k) \subseteq \overline{B}(k, \sqrt{2n})$ for all $k \in \mathbb{Z}^n$.

(iv) $\sup_{k\in Z} \sharp\{l \in \mathbb{Z}^n \mid \operatorname{supp}(\sigma_k) \cap \operatorname{supp}(\sigma_l) \neq \varnothing =: C_{\Sigma,1} < \infty$.

Thus, by Definition 2.5 the family $\Sigma := (\sigma_k)_{k\in\mathbb{Z}^n}$ is a BUPU. For the sake of a brief notation let us define for a given BUPU $\Psi = (\psi_k)_{k\in\mathbb{Z}^n}$ with $\varphi_k = \mathscr{F}^{-1}\psi_k$ the frequency localization operator

$$\begin{aligned} \square_k^\Psi : \mathscr{S}'(\mathbb{R}^n) &\to \mathscr{S}'(\mathbb{R}^n), \\ f &\mapsto \mathscr{F}^{-1}(\psi_k \cdot \mathscr{F} f) = f * \varphi_k. \end{aligned}$$

For a given tempered distribution f, its uniform frequency decomposition with respect to Ψ is thus given by $\{\square_k^\Psi f\}_{k\in\mathbb{Z}^n}$. (This notation is inspired by [2, 63].)

Proposition 2.13. *Any BUPU's $\Psi = \{\psi_l\}_{l\in\mathbb{Z}^n}$ in $A(\mathbb{R}^n)$ a is equivalent to the BUPU $\Sigma := (\sigma_k)_{k\in\mathbb{Z}^n}$ with σ_k defined as in (2.9) in the sense that they define equivalent norms on $M^{p,q}(\mathbb{R}^n)$ for all $1 \leqslant p, q \leqslant \infty$. In particular, this implies that all BUPU's are equivalent and define the same modulation spaces.*

Proof. Given a BUPU $\Psi = \{\psi_l\}_{l\in\mathbb{Z}^n}$ let us denote by

$$\Delta(\Sigma, \Psi, k) := \{l \in \mathbb{Z}^n \mid \operatorname{supp}(\sigma_k) \cap \operatorname{supp}(\psi_{k+l}) \neq \varnothing\}.$$

Due to Condition *(ii)* in Definition 2.5 (applied applied to both Ψ and Σ) the cardinality of $\Delta(\Sigma, \Psi, k)$ is globally bounded and of finite order. Thus, let us set $\sup_k |\Delta(\Sigma, \Psi, k)| =: C_{\Sigma,\Psi}$. We note that $\sum_{l\in\Delta(\Sigma,\Psi,k)} \psi_{k+l}(\xi) = 1$ for all $\xi \in \operatorname{supp}(\sigma_k)$. Applying Young's inequality and Condition *(ii)* again, this implies that

$$\begin{aligned} \left\|\square_k^\Sigma f\right\|_{L^p(\mathbb{R}^n)} &\leqslant \sum_{l\in\Delta(\Sigma,\Psi,k)} \left\|\mathscr{F}^{-1}\sigma_k\, \psi_{k+l}\mathscr{F} f\right\|_{L^p(\mathbb{R}^n)} = \sum_{l\in\Delta(\Sigma,\Psi,k)} \left\|\square_k^\Sigma \square_{k+l}^\Psi f\right\|_{L^p(\mathbb{R}^n)} \\ &\leqslant \sum_{l\in\Delta(\Sigma,\Psi,k)} \|\sigma_k\|_{A(\mathbb{R}^n)} \left\|\square_{k+l}^\Psi f\right\|_{L^p(\mathbb{R}^n)} \leqslant C_{\Sigma,0} \sum_{l\in\Delta(\Sigma,\Psi,k)} \left\|\square_{k+l}^\Psi f\right\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{2.10}$$

Hence, summing up and estimating the maximal number of $k+l$-summands for each

k-term by $C_{\Sigma,\Psi}$, we obtain

$$\sum_{k\in\mathbb{Z}^n} \left\|\Box_k^\Sigma f\right\|_{L^p(\mathbb{R}^n)} \leqslant C_{\Sigma,\Psi}\cdot C_\Sigma \cdot C\cdot \sum_{k\in\mathbb{Z}^n} \left\|\Box_l^\Psi f\right\|_{L^p(\mathbb{R}^n)}{}^q,$$

where C is the constant from the p-norm equivalence on $\mathbb{R}^n$. Our arguments' symmetry in Σ and Ψ finally yields the result. □

The embeddings

$$M^{1,1}(\mathbb{R}^n)\subseteq \ldots \subseteq M^{p_1,q_1}(\mathbb{R}^n)\subseteq M^{p_2,q_2}(\mathbb{R}^n)\subseteq\ldots\subseteq M^{\infty,\infty}(\mathbb{R}^n).$$

for $1\leqslant p_1\leqslant p_2\leqslant\infty$ and $1\leqslant q_1\leqslant q_2\leqslant\infty$ is based upon two facts: The q_1 - q_2-embedding is due to the analogous inclusion relation for the spaces l^{q_1} and l^{q_2}, $1\leqslant q_1\leqslant q_2\leqslant\infty$. The p_1 - p_2-embedding follows from the fact that

$$L^1_\Omega(\mathbb{R}^n)\subseteq\ldots\subseteq L^{p_1}_\Omega(\mathbb{R}^n)\subseteq L^{p_2}_\Omega(\mathbb{R}^n)\subseteq\ldots\subseteq L^\infty_\Omega(\mathbb{R}^n),$$

where for a compact set $\Omega \Subset \widehat{\mathbb{R}^n}$ we define

$$L^p_\Omega(\mathbb{R}^n) := \{f\in\mathscr{S}'(\mathbb{R}^n)\mid \operatorname{supp}(\widehat{f})\subseteq\Omega,\quad \|f\|_{L^p(\mathbb{R}^n)}<\infty\}.$$

Let furthermore $\mathscr{S}_\Omega(\mathbb{R}^n) := \{f\in\mathscr{S}(\mathbb{R}^n)\mid \operatorname{supp}(\widehat{f})\subseteq\Omega\}$.

Proposition 2.14. *For all $1\leqslant p\leqslant q\leqslant\infty$ there exists a positive constant $C_{p,q}$ such that $L^p_\Omega(\mathbb{R}^n)\subseteq L^q_\Omega(\mathbb{R}^n)$ with $\|f\|_{L^q(\mathbb{R}^n)}\leqslant C_{p,q}\,\|f\|_{L^p(\mathbb{R}^n)}$.*

Proof. Let $f\in L^p_\Omega(\mathbb{R}^n)$ and $R>0$ such that $\Omega\subseteq B(\xi_0,R)$ for some $\xi_0\in\widehat{\mathbb{R}^n}$. Let furthermore $\psi\in\mathscr{S}_{\overline{B(\xi_0,2R)}}(\mathbb{R}^n)$. We then have

$$f=\mathscr{F}^{-1}_{\mathscr{S}'(\mathbb{R}^n)}(\widehat{\psi}\cdot\widehat{f}) = f*\psi$$

in the distributional sense, which coincides with standard convolution for $f\in L^p(\mathbb{R}^n)$. By Young's inequality for $r=\infty$, we furthermore have

$$\|f\|_{L^\infty(\mathbb{R}^n)}\leqslant\|\psi\|_{L^{p'}(\mathbb{R}^n)}\cdot\|f\|_{L^p(\mathbb{R}^n)} := C_{p,\infty}\,\|f\|_{L^p(\mathbb{R}^n)}\,. \tag{2.11}$$

Hence, we have proved the statement for $q=\infty$. Now, let $1\leqslant p\leqslant q<\infty$. Supposing, without loss of generality, that $f \in L^\infty_\Omega(\mathbb{R}^n)$, it follows that

$$\|f\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \cdot |f(x)|^{q-p}\, dx\right)^{1/q} \leqslant \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|^{1-p/q} \cdot \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx\right)^{1/q}$$
$$= \|f\|^{1-p/q}_{L^\infty(\mathbb{R}^n)} \cdot \|f\|^{p/q}_{L^p(\mathbb{R}^n)} \leqslant C^{1-p/q}_{p,\infty} \|f\|^{1-p/q}_{L^p(\mathbb{R}^n)} \cdot \|f\|^{p/q}_{L^p(\mathbb{R}^n)} := C_{p,q} \|f\|_{L^p(\mathbb{R}^n)} .$$

Here we have used (2.11) for the second inequality. This completes the proof. □

We conclude this subsubsection with an application of modulation spaces techniques to a problem from PDE theory; the following argument is due to [2]: by replacing $L^p(\mathbb{R}^n)$-spaces by the modulation spaces $M^{p,q}(\mathbb{R}^n)$, one significantly improves certain a priori estimates for the Schrödinger semi-group $t \mapsto S(t) := e^{it\Delta}$ on $\mathbb{R}^n \times [0,\infty)$. The basic L^p-$L^{p'}$-estimates

$$\|S(t)f\|_{L^p(\mathbb{R}^n)} \lesssim |t|^{-n(1/2-1/p)} \|f\|_{L^{p'}(\mathbb{R}^n)}, \tag{2.12}$$

with $2 \leqslant p \leqslant \infty$, are important to solve Cauchy problems for the non-linear Scrhödinger equation. In order to control the singularity in $t = 0$ on the right-hand side of (2.12), one generally has to impose the restrictive condition $n(1/p' - 1/p) \leqslant 1$. Yet, one can readily remove the singularity if one combines Estimate 2.12 with the following one: let $\Sigma := (\sigma_k)_{k\in\mathbb{Z}^n}$ and $\{\Box^\Sigma_k\}_{k\in\mathbb{Z}^n}$ be defined as above. Reasoning along the lines of Estimate (2.10), we obtain

$$\begin{aligned}
\|\Box^\Sigma_k S(t)f\|_{L^p(\mathbb{R}^n)} &\leqslant \sum_{l\in\Delta(\Sigma,\Sigma,k)} \left\|\mathscr{F}^{-1}\, \sigma_k\sigma_{k+l}e^{-i|\xi|^2 t}\, \mathscr{F} f\right\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C_1 \sum_{l\in\Delta(\Sigma,\Sigma,k)} \left\|\sigma_k e^{-i|\xi|^2 t}\sigma_{k+l}\, \mathscr{F} f\right\|_{L^{p'}(\widehat{\mathbb{R}^n})} \\
&\leqslant C_1 \sum_{l\in\Delta(\Sigma,\Sigma,k)} \left\|e^{-i|\xi|^2 t}\sigma_{k+l}\right\|_{L^\infty(\widehat{\mathbb{R}^n})} \|\sigma_k\, \mathscr{F} f\|_{L^{p'}(\widehat{\mathbb{R}^n})} \\
&\leqslant C_1 C_{\Sigma,\Sigma} \cdot 1 \cdot \|\sigma_k\, \mathscr{F} f\|_{L^{p''}(\widehat{\mathbb{R}^n})} \leqslant C_2 C_1 C_{\Sigma,\Sigma} \|\Box^\Sigma_k f\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

that is,

$$\|\Box^\Sigma_k S(t)f\|_{L^p(\mathbb{R}^n)} \lesssim \|\Box^\Sigma_k f\|_{L^p(\mathbb{R}^n)} \quad \text{, and thus} \quad \|S(t)f\|_{M^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{M^{p,q}(\mathbb{R}^n)}, \tag{2.13}$$

for all $1 \leqslant q \leqslant \infty$. Here we have used the Hausdorff-Young inequality in the second and the last estimate and the fact that $\|g\|_{L^{p'}(\mathbb{R}^n)} \leqslant \|g\|_{L^{q'}(\mathbb{R}^n)}$ for all $1 \leqslant p \leqslant q \leqslant \infty$ and all compactly supported functions g on $\mathbb{R}^n$. Combining the estimates (2.12) and 2.13 gives

$$\|S(t)f\|_{M^{p,q}(\mathbb{R}^n)} \lesssim (1+|t|)^{-n(1/2-1/p)} \|f\|_{M^{p',q}(\mathbb{R}^n)} .$$

2.1.2 The Short Time Fourier Transform and Modulation Spaces as Coorbit Spaces

A quite different perspective at modulation spaces is offered by the observation that the convolution operators (respectively Fourier multipliers) from (2.5) can be rewritten the following way for $G = \mathbb{R}^n$: if we denote $\widetilde{\varphi}(x) := \overline{\varphi}(-x)$, we observe

$$\begin{aligned}
(f * M_\xi \varphi)(x) &= \int_{\mathbb{R}^n} f(y) e^{2\pi i (x-y)\xi} \varphi(x-y)\, dy && (2.14)\\
&= e^{2\pi i x \xi} \int_{\mathbb{R}^n} f(y) \overline{e^{2\pi i y \xi}\, \widetilde{\varphi}(y-x)}\, dy \\
&= e^{2\pi i x\xi} \langle f, M_\xi T_x \widetilde{\varphi} \rangle_{L^2(\mathbb{R}^n)} && (2.15)\\
&=: e^{2\pi i x\xi} V_{\widetilde{\varphi}} f(x,\xi) && (2.16)
\end{aligned}$$

for all f in $\mathscr{S}(\mathbb{R}^n)$, say. Since the modulation $e^{2\pi i x\xi}$ is not noticed in $\|\ \|_B$ and $\widetilde{\varphi}$ is also a member of $\mathscr{S}(\mathbb{R}^n)$ we can rewrite Definition 2.11 in terms of the L^p-L^q-integrability of the coefficients $V_{\widetilde{\varphi}} f(x,\xi)$.

Definition 2.15. *Let φ be a function in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. We define the short-time Fourier transform (STFT) with respect to the window φ by*

$$\begin{aligned}
&V_\varphi : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^{2n}),\\
&\qquad f \mapsto \Big((x,\xi) \mapsto e^{-2\pi i x\xi} (f * M_\xi \widetilde{\varphi})(x) \Big) = \langle f, M_\xi T_x \varphi \rangle_{\mathscr{S}'(\mathbb{R}^n)}, && (2.17)
\end{aligned}$$

where $\langle\ ,\ \rangle_{\mathscr{S}'(\mathbb{R}^n)}$ denotes the conjugate linear $\mathscr{S}'(\mathbb{R}^n)$-$\mathscr{S}(\mathbb{R}^n)$ duality.

The classical modulation spaces $M^{p,q}_s(\mathbb{R}^n)$ can thus equivalently be defined in terms of the STFT and its mixed $L^{p,q}$-integrability over $\mathbb{R}^n \times \widehat{\mathbb{R}^n} \cong \mathbb{R}^{2n}$.

Definition 2.16. *Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$, and let φ be an arbitrary, but fixed non-zero member of $\mathscr{S}(\mathbb{R}^n)$. Then we define the modulation space $M^{p,q}_s(\mathbb{R}^n)$ to be the space of all distributions f in $\mathscr{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{M^{p,q}_s(\mathbb{R}^n)} := \left(\int_{\widehat{\mathbb{R}^n}} \left(\int_{\mathbb{R}^n} |V_\varphi f(x,\xi)|^p \, dx \right)^{q/p} v_s(\xi)^q \, d\xi \right)^{1/q} < \infty \qquad (2.18)$$

for $1 \leqslant p, q < \infty$*, with the standard modifications otherwise.*

The operators involved in Identity (2.17) are so-called time-frequency shifts since they combine translations in both the "time-variable" x and the "frequency-variable" ξ. It is therefore not by chance that the application of modulation space techniques to signal processing, etc. is considered to be an integral part of the field called "time-frequency analysis".

An important yet quite immediate observation is that the STFT involves the action of the Schrödinger representation of $\mathbf{H}_n$: for $\lambda = 1$ and $(p, q, t) = (-x, \xi, 0)$ Identity (1.19) gives

$$(\rho(-x, \xi, 0)\varphi)(y) = e^{-\pi i x\xi + 2\pi i \xi y}\,\varphi(x - y) = e^{-\pi i x\xi}\,(M_\xi T_x \varphi)(y),$$

and thus

$$V_\varphi f(x, \xi) = e^{-\pi i x\xi}\langle f, \rho(-x, \xi, 0)\varphi\rangle_{\mathscr{S}'(\mathbb{R}^n)}$$

for all tempered distributions f. The STFT can hence be viewed as the matrix coefficients of ρ with respect to f and φ, and Condition (2.18) is a statement about the mixed L^p - L^q-integrability of these coefficients over $\mathbb{R}^{2n}$, and, due to compactness of $\mathbb{T}$, essentially over the reduced Heisenberg group $\mathbf{H}_{n,red}$. More precisely, we know that any integrable function $F : \mathbf{H}_{n,red} \cong \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T} \to \mathbb{C}$ can be expanded into a Fourier series in the central variable $t \in \mathbb{T}$:

$$F(p, q, t) = \sum_{k \in \mathbb{Z}} \widehat{F}_k(p, q)\, e^{2\pi i k t}.$$

But integrating over $\mathbb{T}$, only $\widehat{F}_{-1}$, the Fourier coefficient of order -1, remains; so F can be identified with a function of $(p, q) \in \mathbb{R}^{2n}$.

In the same spirit we can identify $L^{p,q}(\mathbf{H}_{n,red})$ with $L^{p,q}(\mathbb{R}^{2n})$, and the STFT ought to be viewed as an operation that transforms a given distribution into matrix coefficients of a group representation, ρ that is, which in turns acts on some test function space dense in $L^2(\mathbb{R}^n)$, and via duality on its dual space, i.e., some distribution space.

One can therefore recast Defintions 2.15 and 2.16 in the following way with a slightly more general class of moderate functions m.

Definition 2.17. *Let* $\rho = \rho_1$ *be the Schrödinger representation of the reduced Heisenberg group* $\mathbf{H}_{n,red}$ *acting* $L^2(\mathbb{R}^n)$*, and let* φ *be an arbitrary, but fixed non-zero member of*

$\mathscr{S}(\mathbb{R}^n)$. *We define the so-called ambiguity function with respect to the window* φ *by*

$$A_\varphi : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbf{H}_{n,red}),$$
$$f \mapsto \Big((p,q,t) \mapsto \langle f, \rho(p,q,t)\varphi\rangle_{\mathscr{S}'(\mathbb{R}^n)} \Big). \tag{2.19}$$

For an admissible weight w *on* $\mathbf{H}_{n,red}$*, a* w*-moderate function* m *and* $1 \leqslant p, q \leqslant \infty$ *we define the modulation spaces*

$$M_m^{p,q}(\mathbb{R}^n) := \{ f \in \mathscr{S}'(\mathbb{R}^n) \mid ||V_\varphi' f \mid L_m^{p,q}(\mathbf{H}_{n,red})|| < \infty \}$$

Remark 2.18. For $m(x,\xi) = v(s)$ we recover the classical spaces from Defintion 2.16. The name

According to [19], another family of function spaces described in terms of a coefficient transform, the Besov-Lizorkin-Triebel spaces $B_s^{p,q}(\mathbb{R}^n)$, seems to have been an additional inspiration for Feichtinger and Gröchenig's novel generalized framework [21, 22].

In the latter case another equivalent description for the spaces $B_s^{p,q}(\mathbb{R}^n)$ was found, involving $L^{p,q}$-integrability of the coefficient transform

$$W_\varphi f(a,b) := a^{-n/2} \int_{\mathbb{R}^n} f(x) \overline{\varphi(a^{-1}(x-b))}\, dx, \tag{2.20}$$

defined for complex-valued functions f and φ on $\mathbb{R}^n$, $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^n$. This transform is usually referred to as the (continuous) wavelet transform (WT) on $\mathbb{R}^n$. (Cf. [40], Chapter 10, e.g.)

As in the case of the ambiguity function A_φ in Definition 2.17, Identity (2.20) actually involves a representation of a locally compact group, the so-called $ax+b$-group or affine group. It is given as the semi-direct product

$$\mathbf{A}^n := \mathbb{R}^+ \ltimes \mathbb{R}^n, \quad (a,b)\cdot(a',b') := (aa', ab' + b)$$

with left Haar measure $db\, a^{-(n+1)} da$. Its unitary dual $\widehat{\mathbf{A}^n}$ splits up into two classes: the one-dimensional representations $\pi_\lambda(a,b) = a^{i\lambda}$ and an infinite-dimensional representation

$$\pi : \mathbf{A}^n \to \mathcal{U}(L^2(\mathbb{R}^n)),$$
$$(\pi(a,b)f)(x) := \frac{1}{\sqrt{a}^n} f(a^{-1}(x-b)) = (D_a T_b f)(x). \tag{2.21}$$

This representation obviously acts via a combination of translation by b and dilation by a.

It is therefore not surprising that π is the representation associated to the natural action of $\mathbf{A}^n$ on $\mathbb{R}^n$ (in the sense that $x \mapsto a^{-1}(x-b)$ is just the inverse function of $x \mapsto ax+b$). We conclude that the continuous wavelet transform (2.20) is another instance of coefficient transform arising from a group representation.

Supposing we have made the right choice of test function space, namely $\mathscr{S}_0(\mathbb{R}^n)$, the space of Schwartz functions with all moments vanishing, we can characterize the Besov-Lizorkin-Triebel space $B_s^{p,q}(\mathbb{R}^n)$ as follows:

Theorem 2.19. *Let $\mathscr{S}_0(\mathbb{R}^n)$ be the space of Schwartz functions with all moments vanishing and let $\mathscr{S}_0'(\mathbb{R}^n)$ be its dual, the space of tempered distributions modulo polynomials. Furthermore, let φ be an arbitrary, but fixed non-zero member of $\mathscr{S}_0(\mathbb{R}^n)$. For any $1 \leqslant p,q \leqslant \infty$ the Besov-Lizorkin-Triebel space $B_s^{p,q}(\mathbb{R}^n)$ coincides with the space*

$$\{f \in \mathscr{S}_0'(\mathbb{R}^n) \mid (a,b) \mapsto \langle f, \pi(a,b)\varphi\rangle_{\mathscr{S}_0'(\mathbb{R}^n)} = W_\varphi f \in L^{p,q}_{s+n/2-n/q}(\mathbf{A}^n)\},$$

where the $L_s^{p,q}(\mathbf{A}^n)$-norm is defined by

$$\|F\|_{L_s^{p,q}(\mathbf{A}^n)} := \left(\int_{\mathbb{R}^+}\left(\int_{\mathbb{R}^n} |f(a,b)|^p\, db\right)^{q/p} a^{-qs/2}\frac{da}{a}\right)^{1/q}.$$

According to Feichtinger's own accounts (cf. [18, 19], e.g.), it seems that the similarities in the descriptions of Besov-Lizorkin-Triebel spaces and modulation spaces on $\mathbb{R}^n$ insinuated that there might be a more general theory in the background. Iindeed, a few years later an abstract unifying approach was presented in Feichtinger and Gröchenig's seminal paper [20].

Not only did it describe so-called coorbit spaces (including Besov and Besov-Lizorkin-Triebel spaces, modulation spaces, certain Bergmann spaces, etc.) in terms of generalized "wavelet transforms", it but also provided generalized atomic decompositions. More precisely, this implies that for such a space of functions or distributions each of its members could be represented as a sum of "simple functions", called atoms. Atomic decompositions in turn would facilitate the study of properties like duality, interpolation, embeddings, operator theory on these spaces, etc.

Thus let us briefly introduce the most important notions and results of coorbit theory. We refer the interested reader to [20, 21, 22, 39, 23] for the original work by Feichtinger and Gröchenig and to [52, 58, 53, 54, 56, 57, 55, 9, 8, 31, 47] for recent generalizations and extensions of the coorbit framework.

An essential prerequisite for the theory of coorbit spaces is integrability (or the somewhat weaker square-integrability) of the involved unirreps π. In their early papers [20, 21, 22] Feichtinger and Gröchenig still highlight the use of integrable unirreps but eventually relax the condition to square-integrability in [39, 23] since they have to impose a further, independent integrability condition on the analyzing vectors (windows).

Some more recent accounts on coorbit theory (cf. [8, 9], e.g.) completely drop square-integrable unirreps in favor of (possibly reducible) cyclic representations on Fréchet spaces S that satisfy a reproducing kernel identity and some continuity conditions. In the cases of $\mathbf{H}_n$ and the $ax+b$-group $\mathbf{A}^n$, e.g., the corresponding Fréchet spaces can be identified with the spaces of admissible windows, the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ and its proper Fréchet subspace $\mathscr{S}_0'(\mathbb{R}^n)$, respectively, and one can reinterpret this approach in the context of so-called Gelfand triples. In [31] H. Führ and A. Mayeli make use of this approach in order to characterize Besov spaces on stratified Lie groups such as, e.g., the Heisenberg group.

Yet other approaches completely circumvent the use of representation theory and formulate a generalized coorbit theory in terms of continuous Banach frames [52, 58, 53, 54, 56, 57, 55]. The importance of Banach frames is also emphasized in Gröchenig's monograph [40] on time-frequency analysis in Euclidean space, where an equivalent description of modulation spaces is given in terms of discrete frames.

In this subsection we will focus on Feichtinger and Gröchenig's approach and recall some important definitions and results for the case of square-integrable unirreps.

Definition 2.20. *Let $\mathscr{G}$ be a locally compact group with left Haar measure dx and let π be a unitary representation of $\mathscr{G}$ on the Hilbert space $\mathcal{H}_\pi$. We say π is square-integrable if there exists a non-zero $u \in \mathcal{H}_\pi$ such that*

$$\int_{\mathscr{G}} \left|\langle u, \pi(x)u\rangle_{\mathcal{H}_\pi}\right|^2 \, dx < \infty,$$

respectively. We call such vectors u square-integrable.

It turns out that for a square-integrable unirrep π there exists a positive, self-adjoint (thus densely defined) operator A on $\mathcal{H}_\pi$ such that the orthogonality relation

$$\int_{\mathscr{G}} \langle v_1, \pi(x)u_1\rangle_{\mathcal{H}_\pi} \overline{\langle v_2, \pi(x)u_2\rangle_{\mathcal{H}_\pi}} \, dx = \langle Au_2, Au_1\rangle_{\mathcal{H}_\pi} \langle v_1, v_2\rangle_{\mathcal{H}_\pi} \tag{2.22}$$

holds for all v_1, v_2 in $\mathcal{H}_\pi$ and all u_1, u_2 in $\mathcal{D}(A)$. In the spirit of the identities (2.19) and (2.20) one defines the corresponding coefficient transform (CT) (also voice transform

(VT), generalized wavelet transform (GWT) or simply wavelet transform (WT)) by

$$\begin{aligned} V_u : \mathcal{H}_\pi &\to L^2(\mathscr{G}), \\ x &\mapsto \left(v \mapsto \langle v, \pi(x)u\rangle_{\mathcal{H}_\pi}\right). \end{aligned} \tag{2.23}$$

We observe that Identity (2.22) now implies

$$V_{u_1}v_1 * V_{u_2}v_2 = \langle Au_1, Av_2\rangle_{\mathcal{H}_\pi} V_{u_2}v_1,$$

for all v_1, u_2 in $\mathcal{H}_\pi$ and all u_1, v_2, with the group convolution on $\mathscr{G}$ defined as in Identity (1.6). Thus, for $u := u_1 = u_2 = v_2$ with $\|Au\|_{\mathcal{H}_\pi} = 1$, we obtain the reproducing identity

$$V_u v * V_u u = V_u v$$

for all v in $\mathcal{H}_\pi$. We can furthermore rewrite the orthogonality relation as

$$\langle V_{u_1}v_1, V_{u_2}v_2\rangle_{L^2(\mathscr{G})} = \langle Au_2, Au_1\rangle_{\mathcal{H}_\pi}\langle v_1, v_2\rangle_{\mathcal{H}_\pi}.$$

Moreover, we notice that the CT V_u intertwines the representation π and the left regular representation L of $\mathscr{G}$. More precisely, $V_u : \mathcal{H}_\pi \to L^2(\mathscr{G}) : v \mapsto V_u v$ is isometric and satisfies

$$V_u(\pi(x)v) = L(x)(V_u v)) = T_x V_u v$$

for all v in $\mathcal{H}_\pi$ and all x in $\mathcal{G}$.

The definition of coorbit spaces we will give is the original one, which presumes the existence of a aquare-integrable u in $\mathcal{H}_\pi$.

To begin with, let us assume that Y is a translation-invariant Banach function space over $\mathscr{G}$ (Definitions 2.2 and 2.7 carry over to the not necessarily Abelian case without any changes). For the weight

$$w(x) := \max\{\|T_x\|, \|T_{x^{-1}}\|, \|T_x^R\|, \|T_{x^{-1}}^R\| \cdot \Delta(x^{-1})\} < \infty$$

we define the set of analyzing vectors

$$\mathscr{A}_w := \{u \in \mathcal{H}_\pi \mid V_u u \in L^1_w(\mathscr{G})\}.$$

Assuming that $\mathscr{A}_w$ is not empty, irreducibility of π implies that it is a dense linear subspace of $\mathcal{H}_\pi$. Given an arbitrary, but fixed non-zero member u of $\mathscr{A}_w$ we furthermore define the test function space

$$\mathscr{H}_w^1 := \{v \in \mathcal{H}_\pi \mid V_u v \in L_w^1(\mathscr{G})\},$$

which we equip it with the norm $\|v\|_{\mathscr{H}_w^1} := \|V_u v\|_{L^1(\mathscr{G})w}$. The space $(\mathscr{H}_w^1, \| \ \|_{\mathscr{H}_w^1})$ is a π-invariant Banach space, dense in $(\mathcal{H}_\pi, \| \ \|_{\mathcal{H}_\pi})$, on which $\pi|_{\mathscr{H}_w^1}$ is strongly continuous. Moreover, it turns out that $\mathscr{H}_w^1$ does not depend on the particular choice of u in $\mathscr{A}_w$ and that for each such vector the set $\{\pi(x)u \mid x \in \mathscr{G}\}$ is total in $(\mathscr{H}_w^1, \| \ \|_{\mathscr{H}_w^1})$. (Note that for an isometrically translation-invariant BF-space Y, i.e., if $w = 1$, $\mathscr{H}_w^1$ is just the set of integrable vectors.)

If $(\mathscr{H}_w^1)'$ denotes the conjugate dual space of $\mathscr{H}_w^1$, called the reservoir, we finally define the coorbit space

$$\mathrm{Co}_{FG}^u(Y) := \{\phi \in (\mathscr{H}_w^1)' \mid V_u\phi \in Y\}.$$

Not surprisingly, it turns out that for modulation spaces and Besov spaces as defined/characterized in Definition 2.17 and Theorem 2.19, respectively, we have

$$M_m^{p,q}(\mathbb{R}^n) = \mathrm{Co}_{FG}^{\varphi}(L_m^{p,q}(\mathbf{H}_{n,red})) \quad \text{and} \quad B_s^{p,q}(\mathbb{R}^n) = \mathrm{Co}_{FG}^{\varphi}(L_{s+n/2-n/q}^{p,q}(\mathbf{A}^n))\,.$$

This is due to the following observation: The windows φ in $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}_0(\mathbb{R}^n)$, respectively, are analyzing vectors in the sense of their membership in $\mathscr{A}_w$. Now this implies

$$\mathscr{S}(\mathbb{R}^n) \subseteq \mathscr{A}_w \subseteq \mathscr{H}_w^1 \subseteq M_m^{p,q}(\mathbb{R}^n) \subseteq (\mathscr{H}_w^1)' \subseteq \mathscr{S}'(\mathbb{R}^n)$$

and

$$\mathscr{S}_0(\mathbb{R}^n) \subseteq \mathscr{A}_w \subseteq \mathscr{H}_w^1 \subseteq B_s^{p,q}(\mathbb{R}^n) \subseteq (\mathscr{H}_w^1)' \subseteq \mathscr{S}_0'(\mathbb{R}^n),$$

but as we will see in what follows $(\mathscr{H}_w^1)'$ is the biggest of all modulation spaces. Thus, the spaces $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}_0(\mathbb{R}^n)$ on the one hand and $\mathscr{H}_w^1$ on the other define the same coorbit spaces. Knowing this, it follows immediately that in the case of modulation spaces we have $\mathscr{H}_w^1 = M_w^{1,1}(\mathbb{R}^n)$.

Let us extend the notation π to both its restriction to $\mathscr{H}_w^1$ and its extension to $(\mathscr{H}_w^1)'$ by conjugate-linear duality. The following theorem now gives a list of important properties of $\mathrm{Co}^u_{FG}(Y)$ and V_u.

Theorem 2.21. *Let Y be a translation-invariant BF-space over $\mathscr{G}$ and let u be an arbitrary, but fixed non-zero element of $\mathscr{A}_w$. We then have*

(i) The space $\mathrm{Co}^u_{FG}(Y)$ is a π-invariant Banach space which is continuously embedded into $(\mathscr{H}^1_w)'$.

(ii) The definition of $\mathrm{Co}^u_{FG}(Y)$ is independent of the particular choice of analyzing vector, i.e., different vectors define equivalent norms and thus the same space.

(iii) The definition is furthermore independent of the reservoir $(\mathscr{H}^1_w)'$ in the sense that if w' is another weight with $w(x) \leqslant Cw'(x)$ for some positive constant C and all x in $\mathscr{G}$ with $\mathscr{A}_{w'} \neq \{0\}$, then $\mathrm{Co}^u_{FG}(Y) = \mathrm{Co}^{u'}_{FG}(Y)$.

(iv) The coefficient transform $V_u : \mathrm{Co}^u_{FG}(Y) \to Y$ intertwines π and the left regular representation, i.e., left translation.

(v) If left translation is continuous on Y, then π acts continuously on $\mathrm{Co}^u_{FG}(Y)$.

(vi) The map $V_u : \mathrm{Co}^u_{FG}(Y) \to Y$ restricts to an isometric isomorphism from $\mathrm{Co}^u_{FG}(Y)$ onto the closed subspace $Y_u := Y * V_u u$ of Y, whereas the map $F \mapsto F * V_u u$ defines a bounded projection from Y onto Y_u.

(vii) The inverse operator to the isomorphism $V_u : \mathrm{Co}^u_{FG}(Y) \to Y_u$ is given by its adjoint $V_u^* : F \mapsto \int_{\mathscr{G}} F(x)\pi^*(x)u\, dx$.

(viii) Every function F in Y_u is continuous, it belongs to $L^\infty_{1/w}(\mathscr{G})$ and the evaluation map $F \mapsto F(x)$ coincides with the map

$$F \mapsto \langle F, T_x V_u u\rangle_{L^1_w(\mathscr{G}), L^\infty_{1/w}(\mathscr{G})}.$$

(ix) $\mathrm{Co}^u_{FG}(L^\infty_{1/w}(\mathscr{G})) = (\mathscr{H}^1_w)'$.

(x) $\mathrm{Co}^u_{FG}(L^2(\mathscr{G})) = \mathcal{H}_\pi$.

We notice that for a homogenous space like $L^{p,q}(\mathbf{H}_{n,red})$, e.g., the weight w is simply 1. Also note that although we have $\mathscr{S}(\mathbb{R}^n) \subseteq \mathscr{H}^1_w$ in the case of the Heisenberg group $\mathbf{H}_n$, we never explicitly make us of it and rather work with a smaller space of test functions and a bigger reservoir. The same holds true for the $ax + b$-group. That is, although the representations involved in each case are square-integrable they seem to have inspired the more general definition in [8, 9].

Finally, we would like to discuss the above-mentioned atomic decomposition of the spaces $\mathrm{Co}^u_{FG}(Y)$. The coorbit approach makes use of BUPU's and related decompositions on the space Y, which in the case of $\mathrm{Co}^u_{FG}(Y) = M^{p,q}_m(\mathbb{R}^n)$ means discretization not only takes place on the frequency space $\widehat{G} = \mathbb{R}^n$ as described in Subsection 2.1.1 but on the whole group $\mathscr{G} = \mathbf{H}_{n,red}$ which acts on $\mathcal{H}_\pi = L^2(\mathbb{R}^n)$. (Because $\mathbf{H}_{n,red} \cong R^{2n} \times \mathbb{T}$, in practice this means on $\mathbb{R}^{2n}$.) To be more precise, the discretization will be applied to the projector $F \mapsto F * V_u u : Y \to Y_u$. In the case of the (reduced) Heisenberg group this will yield discretized time-frequency shifts and hence an equivalent description of $M^{p,q}_m(\mathbb{R}^n)$ as an $\ell^{p,q}_m$-space on a suitable lattice. (The latter will often be $\mathbb{Z}^n$ in view of $\mathbf{H}_{n,red} \cong R^{2n} \times \mathbb{T}$ and Lemma 2.24 (iii) below.)

Note that even in the more general case of modulation spaces $M^{p,q}_v(G)$ on Abelian groups G the coorbit approach is possible and yields a description equivalent to Feichtinger's original one for $B = L^p_{s_1}$. At the end of this subsection we will briefly sketch the reasoning.

Remark 2.22 (On BUPU's)**.** In the following we will make use of BUPU's Ψ of size W and well-spread sets X on $\mathscr{G}$. To this end, note that Definition 2.5 carries over almost literally literally to the setting of an arbitrary locally compact group $\mathscr{G}$ with the only difference that our multiplicative algebra $(C_0(\mathscr{G}), \| \ \|_\infty)$ is not necessarily the Fourier algebra of a pre-dual group $\mathscr{K}$. Let us furthermore denote by χ_W the characteristic function of a set measurable set W.

Let us furhermore point out that for any locally compact group $\mathscr{G}$ it is possible to construct arbitrarily fine BUPU's, i.e., BUPU's of size W for any given W. (Cf. [16], e.g.)

Definition 2.23. *Given a discrete family $X = (x_j)_{j\in J}$ in $\mathscr{G}$, a non-empty relatively compact set $W \subseteq \mathscr{G}$ and a translation-invariant Banach function space $(Y, \| \ \|_Y)$ we define the associate discrete BK-space*

$$Y_d(X) := \{\Lambda \mid \Lambda = (\lambda_j)_{j\in J}, \sum_{j\in J} \lambda_j \chi_{x_j W} \in Y\},$$

equipped with the natural norm $\|\Lambda\|_{Y_d} := \left\| \sum_{j\in J} \lambda_j \chi_{x_j W} \right\|_Y$.

It turns out that for a well-spread family X (cf. Definition 2.5) $Y_d(X)$ is independent of the particular choice of W in the sense that different sets W yield equivalent norms on $Y_d(X)$. The following lemma collects some basic properties of BK-spaces.

Lemma 2.24. *Given two discrete families $X = (x_j)_{j\in J}$ and $X' = (x'_{j'})_{j'\in J'}$ in $\mathscr{G}$, a non-empty relatively compact set $W \subseteq \mathscr{G}$ and two translation-invariant Banach function spaces $(Y, \| \, \|_Y)$ and $(Z, \| \, \|_Z)$, the following statements hold true:*

(i) If the functions of compact support are dense in Y then the set finite sequences forms a dense subspace in Y_d.

(ii) For $w(x) := \|T_x\|$ and $w(i) := w(x_j)$ we have $\ell^1_w \subseteq Y_d \subseteq \ell^\infty_{1/w}$.

(iii) If both X and X' are well-spread we have $Y_d(X) \subseteq Z_d(X)$ if and only if $Y_d(X') \subseteq Z_d(X')$, which allows us to simply write Y_d from now on.

(iv) If for well-spread X and X' over the same index set J there exists a compact set $Q \Subset \mathscr{G}$ such that $x_j^{-1}x'_j \in Q$ for all $j \in J$, then $Y_d(X) = Y_d(X')$ and the corresponding norms are equivalent.

(v) Given a weighted Lebesgue space $L^p_m(\mathscr{G})$, the associated BK-space over X is given by ℓ^p_m with $m(j) := m(x_j)$ for $j \in J$. The same is true for general rearrangement-invariant Banach function spaces over $\mathscr{G}$.

(v) For a well-spread family X and any finite partition $J_{r=1}^{r_0}$ of the index set J the projections

$$\begin{aligned} P_r : Y_d &\to Y_d, \\ \Lambda &\mapsto \Lambda_r := (\lambda_j)_{j\in J_r} \end{aligned}$$

define an partition of unity on Y_d and $\sum_{r=1}^{r_0} \left\| \sum_{j\in J_r} \lambda_j \chi_{x_j W} \right\|_Y$ defines an equivalent norm on Y_d.

As a matter of fact we could have introduced BK-spaces already in Subsection 2.1.1 in order to define Wiener Amalgam spaces $W(B, Y)$ and their discrete counterparts $W(B, Y_d)$, as Feichtinger introduced them this in [15]. Yet there were two reasons not to do so: Historically, Feichtinger introduced the classical modulation spaces in [18] only with $Y = L^q_v(\widehat{G})$. The second and probably more important reason is conceptual lucidity.

For the sake of completeness we include the following statement.

Proposition 2.25. *For two left-invariant Banach function spaces B and Y over $\mathscr{G}$ we have $f \in W(B, Y)$ if and only if $(\|\psi_j f\|_B)_{j\in J} \in Y_d$ for some BUPU Ψ and $\|(\|\psi_j f\|_B)_{j\in J}\|_{Y_d}$ defines an equivalent norm.*

Moreover, we have $W(B, Y_1) \subseteq W(B, Y_2)$ if and only if $Y_{1d} \subseteq Y_{2d}$ holds for the corresponding sequence spaces.

In order to discretize the convolutive projector

$$\begin{aligned} T : Y &\to Y, \\ F &\mapsto F * V_u u = \int_{\mathscr{G}} F(y) V_u u(y^{-1}\,.\,)\, dy, \end{aligned}$$

given as a Bochner integral in Y, we need to introduce a restricted class of analyzing vectors, the so-called better vectors

$$\mathscr{B}_w := \{u \in \mathcal{H}_\pi \mid V_u u \in W^R(C_0, L^1_w)(\mathscr{G})\}.$$

The superscript R here indicates that the control function in Definition 2.4 is defined in terms of the right translation T^R instead of the left translation T. For an arbitrary function V in $W^R(C_0, L^1_w)(\mathscr{G})$ and a given BUPU Ψ we define the approximating operator

$$\begin{aligned} T_\Psi : Y &\to Y * V, \\ F &\mapsto \sum_{j \in J} \langle F, \psi_j \rangle_{L^1_w(\mathscr{G}), L^\infty_{1/w}(\mathscr{G})} T_{x_j} V. \end{aligned}$$

As we will refine the BUPU's Ψ in size, i.e., W running a basis of neighborhoods of $e \in \mathscr{G}$, the operators T_Ψ approximate T not only in the strong operator topology, as one would expect in view of Bochner integration theory, but even in the norm topology:

Proposition 2.26. *Let Y be a translation-invariant Banach function space over $\mathscr{G}$ and let $V \in W^R(C_0, L^1_w)(\mathscr{G})$. Then T_Ψ maps Y into $Y * V$ for all BUPU's Ψ on $\mathscr{G}$ and any net $(T_\Psi)_\Psi$ running through a system of W-BUPU's is uniformly bounded by $C \cdot \|V\|_{W^R(C_0, L^1_w)(\mathscr{G})}$, C being independent of V and $\{\Psi\}$, and converges to T in the operator norm.*

We can now state the following fundamental theorem about atomic decompositions.

Theorem 2.27. *Let Y be a translation-invariant Banach function space over $\mathscr{G}$. For any $u \in \mathscr{B}_w$ there exist positive constants C and C' and a neighborhood W of $e \in \mathscr{G}$ such that for any W-dense and relatively separated family $X \subseteq \mathscr{G}$ the following holds true:*

(i) There exists a bounded linear operator

$$\begin{aligned} A : \mathrm{Co}^u_{FG}(Y) &\to Y_d(X), \\ A(v) &:= \Lambda(v) := (\lambda_j(v))_{j \in J}, \end{aligned}$$

called analysis operator, satisfying $\|\Lambda(v)\|_{Y_d(X)} \leqslant C\,\|v\|_{\mathrm{Co}^u_{FG}(Y)}$*, such that every* $v \in \mathrm{Co}^u_{FG}(Y)$ *can be represented as*

$$v = \sum_{j\in J} \lambda_j(v)\pi(x_j)u.$$

(ii) Conversely, assuming that $X \subseteq \mathscr{G}$ *is* W*-dense and relatively separated there exists a bounded linear operator*

$$\begin{aligned} S : Y_d(X) &\to \mathrm{Co}^u_{FG}(Y), \\ S(\Lambda) := v(\Lambda) &:= \sum_{j\in J} \lambda_j(v)\pi(x_j)u, \end{aligned}$$

called synthesis operator that satisfies $\|v(\Lambda)\|_{\mathrm{Co}^u_{FG}(Y)} \leqslant C'\,\|\Lambda\|_{Y_d(X)}$.

In both cases we have convergence in the $\mathrm{Co}^u_{FG}(Y)$*-norm if the finite sequences are dense in* Y_d*, and in the* w^**-sense of* $(\mathscr{H}^1_w)'$*, otherwise.*

Sufficient conditions for Theorem 2.27 to hold, say, in the case of $\mathrm{Co}^u_{FG}(Y) = M^{p,q}_m(\mathbb{R}^n)$ are usually expressed in terms of the analyzing window $u = \varphi$ belonging to certain modulation spaces (in the spirit of $u \in \mathscr{B}_w$ in Theorem 2.27) and explicit descriptions of X as a lattice $\alpha\mathbb{Z}^n \times \beta\mathbb{Z}^n \subset \mathbb{R}^n \times \mathbb{R}^n$.

Let us give the following example:

Theorem 2.28. *Let* $1 \leqslant p, q \leqslant \infty$*,* $\alpha, \beta > 0$*,* $0 \neq \varphi \in M^{1,1}_v$ *and let* m *be any* v*-moderate weight function. Let us furthermore denote by* $\langle\ ,\ \rangle$ *the conjugate-linear dual pairing* $\langle\ ,\ \rangle_{M^{1,1}_v, M^{\infty,\infty}_{1/v}}$ *and let us write* $\tilde{m}(k,l) := m(\alpha l, \beta l)$*. Then the analysis operator*

$$\begin{aligned} A_\varphi : M^{p,q}_m(\mathbb{R}^n) &\to \ell^{p,q}_{\tilde{m}}, \\ f &\mapsto \big(\langle f, T_{\alpha k} M_{\beta l}\varphi\rangle\big)_{k,l\in\mathbb{Z}^n} \end{aligned}$$

and the synthesis operator

$$\begin{aligned} S_\varphi : \ell^{p,q}_m &\to M^{p,q}_{\tilde{m}}(\mathbb{R}^n), \\ c = (c_{k,l})_{k,l\in\mathbb{Z}^n} &\mapsto \sum_{k,l\in\mathbb{Z}^n} c_{k,l} T_{\alpha k} M_{\beta l}\varphi \end{aligned} \tag{2.24}$$

are bounded, and

$$\|A_\varphi\| \leqslant C(v,\alpha,\beta)\,\|V_\varphi\varphi\|_{L^{1,1}_v(\mathbb{R}^{2n})} \quad \text{and} \quad \|S_\varphi\| \leqslant C'(v,\alpha,\beta)\,\|V_\varphi\varphi\|_{L^{1,1}_v(\mathbb{R}^{2n})},$$

the constants being positive and independent of p, q, m.

For $p, q < \infty$ *the sum in Identity* (2.24) *converges unconditionally in the* $M_m^{p,q}(\mathbb{R}^n)$-*norm, and in the* w^*-*topology of* $M_{1/v}^{\infty,\infty}(\mathbb{R}^n)$ *otherwise.*

A proof for Theorem 2.28 can be found in Gröchenig [40]. (Cf. Theorems 12.2.3 and 12.2.4)

The atomic description of modulation spaces has immediate consequences such as a quick proof for the embeddings

$$M_v^{1,1}(\mathbb{R}^n) \subseteq \ldots \subseteq M_{m_1}^{p_1,q_1}(\mathbb{R}^n) \subseteq M_{m_2}^{p_2,q_2}(\mathbb{R}^n) \subseteq \ldots \subseteq M_{1/v}^{\infty,\infty}(\mathbb{R}^n)$$

for $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$ and v-moderate $m_1 \leqslant Cm_2$ due to the easily checked statement

$$\ell_v^{1,1}(\mathbb{Z}^{2n}) \subseteq \ldots \subseteq \ell_{m_1}^{p_1,q_1}(\mathbb{Z}^{2n}) \subseteq \ell_{m_2}^{p_2,q_2}(\mathbb{Z}^{2n}) \subseteq \ldots \subseteq \ell_{1/v}^{\infty,\infty}(\mathbb{Z}^{2n}).$$

2.1.3 Abstract Heisenberg Groups and Coorbit Spaces on Locally Compact Abelian Groups

As we will see in the following, one can define a STFT, and thus modulation spaces, for an arbitrary locally compact Abelian group G, making use of a Heisenberg group construction for G. A calculation identical to (2.16) then implies that also in this more general case Feichtinger's original approach coincides with the coorbit approach.

Thus, given an arbitrary locally compact Abelian group G, we can define a locally compact non-Abelian group arising from G that in the case of $G = \mathbb{R}^n$ corresponds to the reduced version of the so-called polarized Heisenberg group. (The latter is just another realization of (CCR) in the sense that instead of using exponential coordinates, i.e., canonical coordinates of the first kind, ones uses canonical coordinates of the second kind. Cf. [28, p. 19].)

We will refer to this group as the Heisenberg group $H(G)$ of G. More precisely, it is given by the set $G \times \widehat{G} \times \mathbb{T}$ equipped with the product topology and the following group law:

$$(x, \xi, z) \cdot (x', \xi', z') := (xx', \xi\xi', zz'\xi'(x)).$$

By an argument analogous to the one in the Euclidean case, $H(G)$ possesses a family of irreducible unitary representations $\pi_j, j \in \mathbb{N}_0$, on $L^2(G)$ which exhaust all irreducible unitary representations that are non-trivial on the centre (cf. the Section "Postscripts" at the end of Chapter 1 in [28]). For $f \in L^2(G)$ the representation π_j is given by

$$(\pi_j(x, \xi, z)f)(y) = z^j\, \xi(y)^j\, f(xy).$$

Again we can define a STFT making use of π_1: For $\varphi \in \Lambda_w^K(G)$ we define the STFT by

$$\begin{aligned} &V_\varphi f(x,\xi) : G \times \widehat{G} \mapsto \mathbb{C}, \\ &\qquad V_\varphi f(x,\xi) := \big\langle f, \pi_1(x^{-1}, \xi, 1)\varphi \big\rangle_{L^2(G)} = \int_G f(y)\overline{\xi(y)\varphi(x^{-1}y)}\, dy. \end{aligned} \tag{2.25}$$

And again we can perform exactly the same calculation as above in order to rewrite the STFT as a convolution operator:

$$V_\varphi f(x,\xi) = \xi(x^{-1}) \int_G f(y)\, \xi(x)\, \xi(y^{-1})\, \widetilde{\varphi}(y^{-1}x)\, dy = \overline{\xi}(x)\, (f * \xi\widetilde{\varphi})(x).$$

As in the Euclidean case we can disregard $\mathbb{T}$ whenever we integrate over $H(G)$ and hence identify $L_v^{p,q}(H(G))$ with $L_v^{p,q}(G \times \widehat{G})$. Also note that the semi-normed space $\Lambda_w^K(G)$ is a dense subspace of the representation space $\mathcal{H}_\pi = L^2(G)$ as well as a linear subspace of the space of analyzing vector $\mathscr{A}_w := \{u \in \mathcal{H}_\pi \mid V_u u \in L_w^1(G \times \widehat{G})\}$, which we infer as follows: For $\varphi \in \Lambda_w^K(G)$ the functions $M_\xi \varphi$ and $\varphi * M_\xi \varphi$ are also members of $\Lambda_w^K(G)$. Moreover, the map

$$\xi \mapsto \int_G |\varphi * M_\xi \varphi|\; dx$$

is continuous and compactly supported since modulation acts continuously on $\Lambda_w^K(G)$ and $\varphi * M_\xi \varphi = 0$ whenever $\mathrm{supp}(\widehat{\varphi}) \cap \mathrm{supp}(T_\xi \widehat{\varphi}) = \varnothing$, which is the case for all ξ outside a compact neighborhood of e. We conclude that

$$\int_{\widehat{G}} \int_G |V_\varphi \varphi|\; dx\, w(\xi)\, d\xi = \int_{\widehat{G}} \int_G |\varphi * M_\xi \varphi|\; dx\, w(\xi)\, d\xi < \infty,$$

that is, $\varphi \in \mathscr{A}_w$. As in the Euclidean case we have

$$\Lambda_w^K(G) \subseteq \mathscr{A}_w \subseteq \mathscr{H}_w^1 \subseteq M_m^{p,q}(\mathbb{R}^n) \subseteq (\mathscr{H}_w^1)' \subseteq (\Lambda_w^K)'.$$

It thus follows that for all $1 \leqslant p, q \leqslant \infty$

$$\begin{aligned}\mathrm{Co}^{\varphi}_{FG}(L^{p,q}_v(G \times \widehat{G})) &= \{f \in (\mathscr{H}^1_w)' \mid V_\varphi \varphi \in L^{p,q}_v(G \times \widehat{G})\} \\ &= \{f \in (\Lambda^K_w)' \mid V_\varphi \varphi \in L^{p,q}_v(G \times \widehat{G})\} = M(L^p, L^q_v)(G),\end{aligned}$$

that is, Feichtinger's original modulation spaces $M(L^p, L^q_v)(G)$ coincide with the coorbit spaces $\mathrm{Co}^{\varphi}_{FG}(L^{p,q}_v(H(G)))$.

2.2 Approaches to Modulation Spaces on the Heisenberg Group

2.2.1 Motivation

The definition of modulation spaces on the Heiseberg group is motivated by various applications in the Euclidean case that we aim to study in a similar way for the Heisenberg group. One particularly important aspect is the strong and fruitful relation between modulation spaces and pseudodifferential operators.

The definition of pseudodifferential operators on $\mathbf{H}_n$ follows completely different paths depending whether one is interested in a Kohn-Nirenberg-type or a Weyl-type quantization. Different approaches to the Kohn-Nirenberg quantization have been studied by various authors, in particular Taylor [68], and Fischer and Ruzhansky [24, 26, 25].

In resemblance to the Euclidean calculus, the main idea is, roughly speaking, to express the symbol σ_K of an operator K as an operator-valued function $(x, \pi) \mapsto \sigma_K(x, \pi)$: $\mathbf{H}_n \times \widehat{\mathbf{H}}^n \to L^\infty(\widehat{\mathbf{H}}^n)$. This obviously involves the GFT and could potentially relate to an application of frequency localization techniques in the spirit of Feichtinger's classical modulation spaces. A particularly interesting problem arises with the definition of Schrödinger evolution groups on $\mathbf{H}_n$ defined in terms of the sub-Laplacian $\Delta_{\mathbf{H}_n}$.

The pursuit of a Weyl quantization on the Heisenberg group leads to a completely different underlying structure, one we could call "the Heisenberg group of the Heisenberg group." (In fact, cf. [28, p. 90].) As we will indeed see in Chapter 4 one can employ the STFT arising from this new group in order to investigate the role of modulation spaces on $\mathbf{H}_n$.

2.2.2 The Uniform Frequency Decomposition Approach

Note that throughout this subsection G stands for a locally compact Abelian group.

The approach to modulation spaces on the Heisenberg group $\mathbf{H}_n$ via uniform frequency decompositions is quite different from the Abelian case. If we simply tried to imitate the action of a convolution operator like $f \mapsto f * M_\xi\varphi$ in order to achieve frequency-shifts and frequency-localization, we would very quickly realize that the natural substitute for modulations M_ξ on Abelian groups, namely multiplication by a Schrödinger representation ρ_λ, does not yield a translation on $\widehat{\mathbf{H}}^n$. Indeed, what we get is the following: let, for the sake of simplicity, f be in $L^1(\mathbf{H}_n) \cap L^2(\mathbf{H}_n)$ and let φ be in $\mathscr{S}(\mathbf{H}_n)$. Then for $\lambda \neq 0$ we compute

$$\begin{aligned}(f * \rho_\lambda\varphi)(g) &= \int_{\mathbf{H}_n} f(h)\rho_\lambda(h^{-1}g)\varphi(h^{-1}g)\,dh = \int_{\mathbf{H}_n} f(h)\rho_\lambda(h^{-1})\widetilde{\varphi}(g^{-1}h)\,dh \cdot \rho_\lambda(g) \\ &= \int_{\mathbf{H}_n} f(h)\rho_\lambda^*(h)T_g\widetilde{\varphi}(h)\,dh\,\rho_\lambda(g) = \widehat{fT_g\widetilde{\varphi}}(\lambda)\,\rho_\lambda(g).\end{aligned}$$

Hence, we observe that the convolution with a "modulated window" is already an $\mathfrak{S}_2$-valued function (since ρ_λ is unitary it does not alter the $\mathfrak{S}_2$-norm) but not complex-valued as $f * M_\xi$, in which case one can still apply the GFT to get a translation on the unitary dual in the sense of Identity (1.8).

Apart from the problem of dimensionality, we struggle with the lack of structure on the unitary dual. The unitary dual $\widehat{G}$ of any Abelian group G also possesses an Abelian group structure, given by the pointwise product of two characters. (In fact, it is even a locally compact (Abelian) group. Cf. [30] Section 4.3, e.g.) Also, all the characters have the same representation space, that is, $H = \mathbb{C}$. Although by the Stone-von Neumann theorem any infinite-dimensional unirrep is unitarily equivalent to some ρ_λ, there is no group structure on $\widehat{\mathbf{H}}^n$, while there is one on every $\widehat{G}$.

Nevertheless we can explicitly calculate the pointwise product of two Schrödinger representations to see what it gives: let $f \in L^2(\mathbb{R}^n)$ and let $(p, q, t) \in \mathbf{H}_n$. For $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$ we compute

$$\begin{aligned}(\pi_{\lambda_2}(p,q,t)\pi_{\lambda_1}(p,q,t)f)(x) &= e^{2\pi i\lambda_2 t + 2\pi iqx + \pi i\lambda_2 p\cdot q}e^{2\pi i\lambda_1 t + 2\pi iq(x+\lambda_2 p) + \pi i\lambda_2 p\cdot q}f(x + \lambda_1 p) \\ &= e^{2\pi i(\lambda_1+\lambda_2)t + 2\pi iqx + \pi i(\lambda_1+\lambda_2)p\cdot q}f(x + (\lambda_1 + \lambda_2)p) \cdot e^{2\pi iqx + 2\pi i\lambda_2 q\cdot p} \\ &= (\pi_{\lambda_1+\lambda_2}(p,q,t)f)(x) \cdot e^{2\pi iqx + 2\pi i\lambda_2 q\cdot p}.\end{aligned}$$

A repetition of this argument shows that for $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}\backslash\{0\}$ we have

$$\begin{aligned}&(\pi_{\lambda_1}(p,q,t)\pi_{\lambda_2}(p,q,t)\ldots\pi_{\lambda_k}(p,q,t)f)(x) \\ &\qquad = e^{ikq\cdot x + ik\lambda_1 q\cdot p + i(k-1)\lambda_2 q\cdot p + \ldots + i\lambda_{k-1}q\cdot p}\,(\pi_{\lambda_1+\ldots+\lambda_k}f)(x).\end{aligned}$$

We notice two things: neither is the pointwise product a group product nor is it commutative. This is not surprising. An outcome like, say

$$\pi_{\lambda_1} \cdot \pi_{\lambda_2} = \pi_{\lambda_1+\lambda_2+r(\lambda_1,\lambda_2)}$$

with some error term $r(\lambda_1, \lambda_2) \in \mathbb{R}\backslash\{0\}$, would in fact imply commutativity of the product since $\widehat{\mathbf{H}}^n$ is parameterized by $\mathbb{R}\backslash\{0\}$ and the indices would still have to satisfy a group homorphism property. But by the Pontryagin duality theorem this would imply commutativity of $\mathbf{H}_n$, thus a contradiction.

We conclude that it does not make sense to aim at frequency shifts for functions $f : \mathbf{H}_n \to \mathbb{C}$ by convolving them with modulated windows. This is admittedly bad news for the STFT in the sense of Identity (2.25).

It is worthwhile remarking at this point that it is generally difficult to get useful operations on $\mathbf{H}_n$ defined by manipulations on $\widehat{\mathbf{H}}^n$. It is also remarkable that $\mathbf{H}_n$ does not seem to admit any concise and illustrative description of the image of $\mathscr{S}(\mathbf{H}_n)$ under the GFT (cf. D. Geller [33]).

Our observations thus imply that the frequency decompositions are not performed by a priori well-know operations on $\mathbf{H}_n$; it rather seems one would have to perform the decompositions directly on $\widehat{\mathbf{H}}^n$. To be more precise, it is not sufficient to decompose $1_{\widehat{\mathbf{H}}^n}$ but rather the "$\widehat{\mathbf{H}}^n$-unity" $\mathrm{I}_{\widehat{\mathbf{H}}^n} := \lambda \mapsto \mathrm{I}_\lambda$, where I_λ stands for the identity operator on $H_{\rho_\lambda} \cong L^2(\mathbb{R}^n)$. A Fourier multiplier defined exclusively in terms of $\lambda \in \mathbb{R}\backslash\{0\}$ would yield a convolution operator whose distributional kernel behaves like a Dirac delta in $p = q = 0$. The Fourier transform would essentially be performed in the central variable t.

Yet, it is an interesting observation that, disregarding the operator's singularity for a moment, modulation space-like semi-norms defined by continuous and discrete spectral shifts in $\lambda \in \widehat{\mathbf{H}}^n$ are formally equivalent if one only accepts a priori-weighted ℓ^q-spaces as the global component of discrete modulation spaces. The weight factors $|j|^n, j \in \mathbb{Z}$, in that case are due to the factor $|\lambda|^n$ in the Plancherel measure on $\widehat{\mathbf{H}}^n$.

It is also still not completely clear how to perform a clever frequency decomposition of $I_{\widehat{\mathbf{H}}^n}$, but one might guess that joint spectral multipliers m of the sub-Laplacian $\Delta_{\mathbf{H}_n}$ and the central derivative D_t might do the job. The latter spectrum, called the Heisenberg fan, consists of the pairs $\big(\lambda, |\lambda|\,(2\,|k| + n)\big)$ with $\lambda \in \mathbb{R}\backslash\{0\}$ and a multi-index $k \in \mathbb{N}^n$.

Joint work by F. Ricci et al [60, 61, 59] has shown that the convolution kernel of $m(L, D_t)$ is a Schwartz function if and only if m is the restriction to the Heisenberg fan of some Schwartz function on $\mathbb{R}^2$. A discrete decomposition of $\mathrm{I}_{\widehat{\mathbf{H}}^n}$ can then be performed be finding a useful BUPU-like decomposition of unity on $\mathbb{R}^2$.

It still remains an open question how much the modulation spaces defined this way would differ from homogeneous Besov spaces on $\mathbf{H}_n$ since the latter can also be characterized by a similar, namely dyadic, spectral decomposition. (Cf. [32, 31].)

2.2.3 The Coorbit Approach

A coorbit-type approach for modulation spaces on $\mathbf{H}_n$ is discussed in Chapter 4. Although originally inspired by Feichtinger and Gröchenig's paper [21], the present author and his collaborators decided to start the description in terms of an adapted framework due to Daniel and Ingrid Beltiţă [5, 6], only to resort to the original approach for technical as well as conceptual reasons.

The representation theory involved is discussed very explicitly in Chapter 3. We should point out that there is also a strong link between these modulation spaces and Weyl-quantized operators $\mathbf{H}_n$ as this is already the case on $\mathbb{R}^n$. We refer the interested reader to Chapter 5, in particular Subsection 5.2.2.

3 The Dynin-Folland Group and its Representation Theory

In [13], A. S. Dynin was apparently first to consider the Lie algebra generated by (left- or right-) invariant vector fields on the Heisenberg group $\mathbf{H}_n$ and multiplication by the $2n+1$ coordinate functions (multiplied by $2\pi i$). This Lie algebra of operators of $L^2(\mathbf{H}_n)$ is in fact finite dimensional, moreover it turns out to be nilpotent of step 3. Viewing it as an abstract nilpotent Lie algebra, the corresponding (connected simply connected) nilpotent Lie group, denoted here by $\mathbf{H}_{2,n}$, acts naturally on $L^2(\mathbf{H}_n)$. This Schrödinger-type representation of $\mathbf{H}_{2,n}$ on $L^2(\mathbf{H}_n)$ is the main ingredient in the subsequent Weyl-type quantization on $\mathbf{H}_n$ developed by Dynin. As Dynin was motivated by this quantization, his account on the group $\mathbf{H}_{2,n}$ and its Schrödinger-type representation was not very explicit.

G. B. Folland mentiones the paper [13] by Dynin in a miscellaneous remark in his monograph [28, p.90], saying that the group $\mathbf{H}_{2,n}$ might be called "the Heisenberg group of the Heisenberg group." Almost twenty years later, in [29], Folland provides a more rigorous account on such Heisenberg constructions and explores the structure of "meta-Heisenberg groups" of 2-step groups. There he also discusses how Dynin's quantization extends to an arbitrary meta-Heisenberg group and how it relates to other symbolic calculi (namely the classical Weyl and Kohn-Nirenberg correspondences in the Euclidean setting as well as the Beals-Greiner calculus on Heisenberg manifolds introduced in [3]).

Since Folland's account is quite general, this chapter aims at giving some more explicit formulas for $\mathbf{H}_{2,n}$ and its unitary irreducible representations (unirreps). Paying tribute to both its first introduction by Dynin and its more precise description by Folland, we will call $\mathbf{H}_{2,n}$ the *Dynin-Folland group.*

In Sections 3.1 and 3.2 we will construct the Lie algebra and group mentioned at the beginning of this introduction. Section 3.3 introduces some useful notation which will facilitate to express the group law and many formulas.

We will then give explicit formulas for the Schrödinger-type representations of $\mathbf{H}_{2,n}$ in Section 3.4. In Section 3.5 we cross-check our results and complete the set of unirreps (up to unitary equivalence) by classifying the co-adjoint orbits of $\mathbf{H}_{2,n}$ and constructing the corresponding representations.

We then briefly describe the semi-direct product structure of $\mathbf{H}_{2,n}$ in Section 3.6, and conclude the chapter with Section 3.7, where we discuss the Plancherel formula on the group $\mathbf{H}_{2,n}$.

3.1 The Lie Algebra $\mathfrak{h}_{n,2}$

In this subsection we study the Lie algebra generated by the left-invariant vector fields $\mathscr{D}_{p_j}, \mathscr{D}_{q_k}, \mathscr{D}_t$ on $\mathbf{H}_n$ (cf. Subsection 1.3.2), $j, k = 1, \ldots, n$, and the multiplications by coordinate functions:

$$\left.\begin{array}{rcl} \mathscr{X}_{p_j} f\,(p,q,t) & = & p_j f(p,q,t), \\ \mathscr{X}_{q_j} f\,(p,q,t) & = & q_j f(p,q,t), \\ \mathscr{X}_t f\,(p,q,t) & = & t f(p,q,t), \end{array}\right\} \tag{3.1}$$

where $j = 1, \ldots n$ and $f \in \mathscr{S}(\mathbf{H}_n)$. To this end, we compute all possible commutators between these operators, up to skew-symmetry. The symbol I will denote the identity operator on $L^2(\mathbf{H}_n)$. As in Section 1.3 we define the Lie bracket for two essentially self-adjoint operators A, B acting on $\mathscr{S}(\mathbb{R}^n)$ is defined by $2\pi i$ times the standard commutator $[A, B] := AB - BA$.

The commutator brackets between the $\mathscr{X}_{p_j}$, $\mathscr{X}_{q_j}$, and $\mathscr{X}_t$, are zero since scalar multiplication operators commute:

$$[\mathscr{X}_{p_j}, \mathscr{X}_{q_k}] = [\mathscr{X}_{p_j}, \mathscr{X}_t] = [\mathscr{X}_{q_k}, \mathscr{X}_t] = 0.$$

The commutator brackets between the left invariant vector fields $\mathscr{D}_{p_j}, \mathscr{D}_{q_k}, \mathscr{D}_t$, for $j, k \in \{1, \ldots, n\}$, can be computed directly using their expressions given in (1.18):

$$\begin{array}{rcl} (2\pi i)^2\,[\mathscr{D}_{p_j}, \mathscr{D}_{q_k}] & = & [\partial_{p_j} - \frac{1}{2}q_j\partial_t, \partial_{q_k} + \frac{1}{2}p_k\partial_t] = [\partial_{p_j}, \frac{1}{2}p_k\partial_t] + [-\frac{1}{2}q_j\partial_t, \partial_{q_k}] \\ & = & \frac{1}{2}\delta_{j,k}\partial_t + \frac{1}{2}\delta_{k,j}\partial_t = \delta_{j,k}\,\partial_t = 2\pi i\delta_{j,k}\,\mathscr{D}_t, \\ (2\pi i)^2\,[\mathscr{D}_{p_j}, \mathscr{D}_t] & = & [\partial_{p_j} - \frac{1}{2}q_j\partial_t, \partial_t] = 0, \\ (2\pi i)^2\,[\mathscr{D}_{q_j}, \mathscr{D}_t] & = & [\partial_{q_j} + \frac{1}{2}p_j\partial_t, \partial_t] = 0. \end{array}$$

Naturally we obtain that these operators satisfy also the CCR since the space of left-invariant vector fields on $\mathbf{H}_n$ form a Lie algebra of operators isomorphic to $\mathfrak{h}_n$, see Section 1.3.2. Let us compute the commutator brackets between the left-invariant vector fields and the coordinate operators, first the commutators with $\mathscr{D}_{p_j}$:

$$\begin{array}{lllllllll}(2\pi i)\,[\mathscr{D}_{p_j},\mathscr{X}_{p_k}] & = & [\partial_{p_j}-\frac{1}{2}q_j\partial_t,p_k] & = & [\partial_{p_j},p_k] & = & \delta_{j,k}\,I, & & \\ (2\pi i)\,[\mathscr{D}_{p_j},\mathscr{X}_{q_k}] & = & [\partial_{p_j}-\frac{1}{2}q_j\partial_t,q_k] & = & 0, & & & & \\ (2\pi i)\,[\mathscr{D}_{p_j},\mathscr{X}_{t}] & = & [\partial_{p_j}-\frac{1}{2}q_j\partial_t,t] & = & [-\frac{1}{2}q_j\partial_t,t] & = & -\frac{1}{2}q_j & = & -\frac{1}{2}\mathscr{X}_{q_j},\end{array}$$

then with $\mathscr{D}_{q_j}$:

$$\begin{array}{lllllll}(2\pi i)\,[\mathscr{D}_{q_j},\mathscr{X}_{p_k}] & = & [\partial_{q_j}+\frac{1}{2}p_j\partial_t,p_k] & = & 0, & & \\ (2\pi i)\,[\mathscr{D}_{q_j},\mathscr{X}_{q_k}] & = & [\partial_{q_j}+\frac{1}{2}p_j\partial_t,q_k] & = & [\partial_{q_j},q_k] & = & \delta_{j,k}\,\mathrm{I}, \\ (2\pi i)\,[\mathscr{D}_{q_j},\mathscr{X}_{t}] & = & [\partial_{q_j}+\frac{1}{2}p_j\partial_t,t] & = & [\frac{1}{2}p_j\partial_t,t]=\frac{1}{2}p_j & = & \frac{1}{2}\mathscr{X}_{p_j},\end{array}$$

and eventually with $\mathscr{D}_t$:

$$\begin{array}{lllll}(2\pi i)\,[\mathscr{D}_{t},\mathscr{X}_{p_k}] & = & [\partial_t,p_k] & = & 0, \\ (2\pi i)\,[\mathscr{D}_{t},\mathscr{X}_{q_k}] & = & [\partial_t,q_k] & = & 0, \\ (2\pi i)\,[\mathscr{D}_{t},\mathscr{X}_{t}] & = & [\partial_t,t] & = & \mathrm{I}.\end{array}$$

We have obtained that the linear space generated by the first order Lie brackets between the operators $\mathscr{D}_{p_j},\mathscr{D}_{q_j},\mathscr{D}_t$ and $\mathscr{X}_{p_j}$, $\mathscr{X}_{q_k}$, $\mathscr{X}_t$ is

$$\mathbb{R}\mathscr{D}_t\oplus\mathbb{R}\mathrm{I}\oplus\mathbb{R}\mathscr{X}_{q_1}\oplus\ldots\oplus\mathbb{R}\mathscr{X}_{q_n}\oplus\mathbb{R}\mathscr{X}_{p_1}\oplus\ldots\oplus\mathbb{R}\mathscr{X}_{p_n}.$$

The whole lot of commutators tells us that very few second order commutators remain. More precisely, the Lie brackets of $\mathscr{D}_t$, $\mathscr{X}_{p_j}$ or $\mathscr{X}_{q_k}$ with any $\mathscr{D}_{p_{j'}},\mathscr{D}_{q_{j'}},\mathscr{D}_t$ and $\mathscr{X}_{p_{j'}}$, $\mathscr{X}_{q_{k'}}$, $\mathscr{X}_t$ can only vanish or be equal to I, and the operator I clearly commutes with all operators, hence does not create any new structure. Therefore, the second order commutator brackets are all proportional to I and all third order commutators must be zero. We have obtained:

Lemma 3.1. *The real Lie algebra of operators generated by the left-invariant vector fields and the coordinate functions multiplied by i is*

$$\begin{aligned}\langle\mathscr{D}_{p_j},\mathscr{D}_{q_j},\mathscr{D}_t,\mathscr{X}_{p_j},\mathscr{X}_{q_k},\mathscr{X}_t\rangle &= \mathbb{R}\mathscr{D}_{p_1}\oplus\ldots\oplus\mathbb{R}\mathscr{D}_{p_n}\oplus\mathbb{R}\mathscr{D}_{q_1}\oplus\ldots\oplus\mathbb{R}\mathscr{D}_{q_n}\oplus\mathbb{R}\mathscr{D}_t\\ &\oplus\mathbb{R}\mathscr{X}_{p_1}\oplus\ldots\oplus\mathbb{R}\mathscr{X}_{p_n}\oplus\mathbb{R}\mathscr{X}_{q_1}\oplus\ldots\oplus\mathbb{R}\mathscr{X}_{q_n}\oplus\mathbb{R}\mathrm{I}.\end{aligned}$$

In other words, the identity operator I *is the only newly generated element. Furthermore this Lie algebra is of topological dimension* $2(2n+1)+1$ *and* 3*-step nilpotent.*

We now define the "abstract" Lie algebra that will naturally be isomorphic to

$\langle \mathscr{D}_{p_j}, \mathscr{D}_{q_j}, \mathscr{D}_t, \mathscr{X}_{p_j}, \mathscr{X}_{q_k}, \mathscr{X}_t \rangle$. First we index the standard basis of $\mathbb{R}^{2(2n+1)+1}$ as

$$(X_{u_1}, \dots, X_{u_n}, X_{v_1}, \dots, X_{v_n}, X_w, X_{x_1}, \dots, X_{x_n}, X_{y_1}, \dots, X_{y_n}, X_z, X_s).$$

Then we consider the linear isomorphism

$$d\pi : \mathbb{R}^{2(2n+1)+1} \longrightarrow \langle \mathscr{D}_{p_j}, \mathscr{D}_{q_j}, \mathscr{D}_t, \mathscr{X}_{p_j}, \mathscr{X}_{q_k}, \mathscr{X}_t \rangle \tag{3.2}$$

defined via

$$\begin{aligned} &d\pi(X_{u_j}) = 2\pi i\, \mathscr{D}_{p_j}, \quad d\pi(X_{v_j}) = 2\pi i\, \mathscr{D}_{q_j}, \quad d\pi(X_w) = 2\pi i\, \mathscr{D}_t, \\ &d\pi(X_{x_j}) = 2\pi i\, \mathscr{X}_{p_j}, \quad d\pi(X_{y_j}) = 2\pi i\, \mathscr{X}_{q_j}, \\ &d\pi(X_z) = 2\pi i\, \mathscr{X}_t, \qquad d\pi(X_s) = 2\pi i\, \mathrm{I}. \end{aligned}$$

Definition 3.2. *We denote by $\mathfrak{h}_{n,2}$ the real Lie algebra with underlying linear space $\mathbb{R}^{2(2n+1)+1}$ and Lie bracket $[\cdot,\cdot]_{\mathfrak{h}_{n,2}}$ defined so that $d\pi$ is a Lie algebra morphism.*

This means that the vectors in the standard basis of $\mathbb{R}^{2(2n+1)+1}$ satisfy the following commutator relations

$$\left. \begin{array}{rcl} [X_{u_j}, X_{v_k}]_{\mathfrak{h}_{n,2}} &=& \delta_{j,k}\, X_w \\ [X_{u_j}, X_{x_k}]_{\mathfrak{h}_{n,2}} &=& \delta_{j,k}\, X_s \\ [X_{u_j}, X_z]_{\mathfrak{h}_{n,2}} &=& -\frac{1}{2} X_{y_j} \\ [X_{v_j}, X_{y_k}]_{\mathfrak{h}_{n,2}} &=& \delta_{j,k}\, X_s \\ [X_{v_j}, X_z]_{\mathfrak{h}_{n,2}} &=& \frac{1}{2} X_{x_j} \\ [X_w, X_z]_{\mathfrak{h}_{n,2}} &=& X_s. \end{array} \right\} \tag{3.3}$$

In (3.3), we have only listed the non-vanishing Lie brackets of $\mathfrak{h}_{2,n}$, up to anti-symmetry.

Our choice of notation $\mathfrak{h}_{n,2}$ for the Lie algebra reflects the fact that we just have applied a further type of Heisenberg construction to $\mathfrak{h}_n$. We will refer to $\mathfrak{h}_{n,2}$ as the Dynin-Folland Lie algebra in recognition of Dynin's and Folland's works [13, 14] and [29], respectively.

The following properties are straightforward:

Proposition 3.3. *(i) The Lie algebra $\mathfrak{h}_{n,2}$ is nilpotent of step 3, with centre $\mathbb{R}X_s$.*

(ii) The mapping $d\pi$ is a morphism from the Heisenberg Lie algebra $\mathfrak{h}_{n,2}$ onto $\langle \mathscr{D}_{p_j}, \mathscr{D}_{q_j}, \mathscr{D}_t, \mathscr{X}_{p_j}, \mathscr{X}_{q_k}, \mathscr{X}_t \rangle$.

(iii) The subalgebra $\mathbb{R}\mathscr{D}_{p_1} \oplus \dots \oplus \mathbb{R}\mathscr{D}_{q_n} \oplus \mathbb{R}\mathscr{D}_t$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_n$, and so is the subalgebra $\mathbb{R}X_{u_1} \oplus \dots \oplus \mathbb{R}X_{v_n} \oplus \mathbb{R}X_w$. Furthermore, the restriction

of $d\pi$ to the subalgebra $\mathbb{R}X_{u_1} \oplus \ldots \oplus \mathbb{R}X_{v_n} \oplus \mathbb{R}X_w$ coincides with the infinitesimal right regular representation of $\mathbf{H}_n$ on $L^2(\mathbb{R}^n)$.

(iv) The subalgebra $\mathbb{R}\mathscr{X}_{p_1} \oplus \ldots \oplus \mathbb{R}\mathscr{X}_{q_n} \oplus \mathbb{R}\mathscr{X}_t$ is abelian and so is the subalgebra $\mathbb{R}X_{x_1} \oplus \ldots \oplus \mathbb{R}X_{y_n} \oplus \mathbb{R}X_s$.

3.2 The Lie Group $\mathbf{H}_{2,n}$

Here we describe the connected simply connected 3-step nilpotent Lie group that we obtain by exponentiating the Dynin-Folland Lie algebra $\mathfrak{h}_{n,2}$. We denote this group by $\mathbf{H}_{2,n}$.

As in the case of the Heisenberg group (cf. Subsection 1.3.1) we can again make use of the Baker-Campbell-Hausdorff formula recalled in (1.15). Since the Dynin-Folland Lie algebra is of step 3, we obtain the group law

$$\exp_{\mathbf{H}_{2,n}}(X) \odot_{\mathbf{H}_{2,n}} \exp_{\mathbf{H}_{2,n}}(X') = \exp_{\mathbf{H}_{2,n}}(Z),$$

with

$$Z := X + X' + \frac{1}{2}[X, X']_{\mathfrak{h}_{n,2}} + \frac{1}{12}[(X - X'), [X, X']_{\mathfrak{h}_{n,2}}]_{\mathfrak{h}_{n,2}}. \tag{3.4}$$

Let us compute Z more explicitly. We write

$$X = \sum_{j=1}^{n}(u_j X_{u_j} + v_j X_{v_j}) + wX_w + \sum_{j=1}^{n}(x_j X_{x_j} + y_j X_{y_j}) + zX_z + sX_s,$$

and similarly for X'. As in the Heisenberg case, we abbreviate for instance sums like $\sum_{j=1}^{n} u_j X_{u_j}$ by the dot-product-like notation uX_u. Consequently we have

$$X = uX_u + vX_v + wX_w + xX_x + yX_y + zX_z + sX_s.$$

Lemma 3.4. *With the notation above, the expression of Z given in (3.4) becomes*

$$\begin{aligned} Z = (u+u')X_u + (v+v')X_v + \Big(w + w' + \frac{uv' - vu'}{2}\Big)X_w \\ + \Big(x + x' + \frac{1}{4}(z'v - zv')\Big)X_x + \Big(y + y' - \frac{1}{4}(z'u - zu')\Big)X_y + (z+z')X_z \\ + \Big(s + s' + \frac{ux' - xu'}{2} + \frac{vy' - yv'}{2} + \frac{wz' - zw'}{2} - \frac{z - z'}{8}(uv' - vu')\Big)X_s. \end{aligned}$$

Proof. Employing the commutation relations (3.3) we compute

$$[X, X']_{\mathfrak{h}_{n,2}} = (uv' - vu')X_w - \frac{1}{2}(z'u - zu')X_y + \frac{1}{2}(z'v - zv')X_x \\ + (ux' - xu' \ + \ vy' - yv' \ + \ wz' - zw')X_s \tag{3.5}$$

for the first order commutator, and for the second commutator:

$$[(X - X'), [X, X']_{\mathfrak{h}_{n,2}}]_{\mathfrak{h}_{n,2}} \\ = -(z - z')(uv' - vu')X_s - (v - v')\frac{1}{2}(z'u - zu')X_s + (u - u')\frac{1}{2}(z'v - zv')X_s,$$

that is, the vector cX_s with

$$\begin{aligned} c &= -(z - z')(uv' - vu') + \frac{z(u'(v - v') - v'(u - u')) + z'(-u(v - v') + v(u - u'))}{2} \\ &= -(z - z')(uv' - vu') + \frac{z(u'v - v'u) + z'(uv' - vu')}{2} \\ &= -\frac{3}{2}(z - z')(uv' - vu'). \end{aligned}$$

Collecting the commutators of order 0,1 and 2 computed above and inserting them into Formula (3.4), we obtain the expression for Z stated above. □

As in the case of the Heisenberg group, we identify an element of the group with an element of the underlying vector space $\mathbb{R}^{2(2n+1)+1}$ of the Lie algebra:

$$(u, v, w, x, y, z, s) = \exp_{\mathbf{H}_{2,n}}\Big(uX_u + vX_v + wX_w + xX_x + yX_y + zX_z + sX_s\Big).$$

Proposition 3.5. *With the convention explained above, the centre of the* $\mathbf{H}_{2,n}$ *is* $\exp_{\mathbf{H}_{2,n}}(\mathbb{R}X_s) = \{(0, 0, 0, 0, 0, 0, s) : s \in \mathbb{R}\}$, *and the group law becomes*

$$\begin{aligned} &(u, v, w, x, y, z, s) \odot_{\mathbf{H}_{2,n}} (u', v', w', x', y', z', s') \\ &= \Big(u + u' \ , \ v + v' \ , \ w + w' + \frac{uv' - vu'}{2} \ , \\ &\qquad x + x' + \frac{1}{4}(z'v - zv') \ , \ y + y' - \frac{1}{4}(z'u - zu') \ , \ z + z' \ , \\ &\qquad s + s' + \frac{ux' - xu'}{2} + \frac{vy' - yv'}{2} + \frac{wz' - zw'}{2} - \frac{z - z'}{8}(uv' - vu')\Big). \end{aligned} \tag{3.6}$$

Furthermore, the subgroup $\{(u, v, w, 0, 0, 0, 0) : u, v \in \mathbb{R}^n, \ w \in \mathbb{R}\}$ *is isomorphic to the Heisenberg group* $\mathbf{H}_n$.

3.3 An Extended Notation - Ambiguities and Usefulness

In this Section, we introduce new notation to be able to perform computations in a concise manner. Unfortunately, this will mean on the one hand identifying many different objects and on the other hand having several ways for describing one and the same operation. Yet, the nature of our situation requires it.

Having identified the groups $\mathbf{H}_n$ and $\mathbf{H}_{2,n}$ with the underlying vector space (via exponential coordinates), many computations involve the variables p, q, t, u, v, w, x, y, z, s, which may refer to elements or the components of elements of the Lie algebras $\mathbb{R}^n$, $\mathfrak{h}_n$, $\mathfrak{h}_{2,k}$ as well as elements or components of elements of the Lie groups $\mathbb{R}^n$, $\mathbf{H}_n$, $\mathbf{H}_{2,n}$. Certain specific calculations moreover involve sub-indices $j, k, l, \ldots = 1, \ldots, n$ of the latter, that is, the scalar variables $p_j, q_k, t, u_l, \ldots$. Yet other formulas become not only less cumbersome but more lucid if we also introduce capital letters to denote members of $\mathbf{H}_n \cong \mathfrak{h}_n \cong \mathbb{R}^{2n+1}$ and calligraphic capital letters for either $\mathbf{H}_n$-valued or scalar-valued components of the $2\,(2n+1)+1$-dimensional elements of $\mathfrak{h}_{n,2} \cong \mathbf{H}_{2,n}$.

Let the standard variables that define the elements of the Heisenberg group $\mathbf{H}_n \cong \mathfrak{h}_n \cong \mathbb{R}^{2n+1}$ once and for all be fixed to be

$$X := (p, q, t) := (p_1, \ldots, p_n, q_1, \ldots, q_n, t), \tag{3.7}$$

and let the standard variables defining the elements of the Dynin-Folland group $\mathbf{H}_{2,n} \cong \mathfrak{h}_{n,2} \cong \mathbb{R}^{2\,(2n+1)+1}$ be denoted by

$$\begin{aligned}(\mathcal{P}, \mathcal{Q}, \mathcal{S}) &:= ((u, v, w), (x, y, z), s) \\ &:= ((u_1, \ldots, u_n, v_1, \ldots, v_n, w), (x_1, \ldots, x_n, y_1, \ldots, y_n, z), s).\end{aligned} \tag{3.8}$$

This purely notational identification of elements belonging to Lie groups, Lie algebras and Euclidean vector spaces will prove very useful in many instances. The $\mathbf{H}_n$ and $\mathbf{H}_{2,n}$-group laws, for example, can be expressed in a very convenient way. Let expressions like $p'q$ or uv', e.g., again denote the standard $\mathbb{R}^n$-inner products of the vectors p', q and u, v', respectively, whereas $\mathbb{R}^{2n+1}$-inner products will be denoted by

$$\langle\,.\,,\,.\,\rangle := \langle\,.\,,\,.\,\rangle_{\mathbb{R}^{2n+1}}.$$

Moreover, let us introduce the 'big dot-product'

$$X \bullet X' := (p, q, t) \bullet (p', q', t')$$

for elements in $\mathfrak{h}_n \cong \mathbf{H}_n \cong \mathbb{R}^{2n+1}$ as an abbreviation of the $\mathbf{H}_n$-product (1.17), and let us agree that for all such vectors we can employ the $\mathfrak{h}_n$-Lie bracket notation

$$[X, X'] := [X, X']_{\mathfrak{h}_n} := (0, 0, pq' - qp').$$

We can then rewrite the $\mathbf{H}_n$-group law as

$$\begin{aligned} X \odot_{\mathbf{H}_n} X' &= (p, q, t) \odot_{\mathbf{H}_n} (p', q', t') = \big(p + p', q + q', t + t' + \frac{1}{2}(pq' - qp')\big) \\ &= X \cdot X' = X + X' + \frac{1}{2}[X, X']. \end{aligned} \tag{3.9}$$

Let us turn our attention to the group law of $\mathbf{H}_{2,n}$. The beginning of Formula (3.6) can be rewritten as

$$\left(u + u', v + v', w + w' + \frac{1}{2}(uv' - vu')\right) = \mathcal{P} \cdot \mathcal{P}',$$

if $\mathcal{P} = (u, v, w)$ and similarly for $\mathcal{P}'$. For the rest of the formula we need to introduce the operation

$$\mathrm{ad}^*_{\mathbf{H}_\mathrm{n}}(X)(X') = (t'q, -t'p, 0), \tag{3.10}$$

if $X = (p, q, t)$ as in (3.7) and similarly for X'.

Remark 3.6. As the notation suggests, the operation $\mathrm{ad}^*_{\mathbf{H}_\mathrm{n}}$ is the co-adjoint representation, where $\mathfrak{h}_n$ and its dual have been identified with $\mathbb{R}^{2n+1}$. Indeed, the adjoint representation of $\mathfrak{h}_n$ is the Lie algebra morphism $\mathrm{ad}_{\mathbf{H}_\mathrm{n}}$ from $\mathfrak{h}_n$ to its algebra of automorphisms defined by $\mathrm{ad}_{\mathbf{H}_\mathrm{n}}(X)(Y) = [X, Y]$. This representation of $\mathfrak{h}_n$ on itself yields a dual representation of $\mathfrak{h}_n$ on its dual, called dual of the adjoint representation, or co-adjoint representation, denoted here by $\mathrm{ad}^*_{\mathbf{H}_\mathrm{n}}$. It is defined via

$$\mathrm{ad}^*_{\mathbf{H}_\mathrm{n}}(X)(\phi) = -\phi \circ \mathrm{ad}_{\mathbf{H}_\mathrm{n}}(X),$$

for X in $\mathfrak{h}_n$ and ϕ a real linear form on $\mathfrak{h}_n$. It is an easy exercise left to the reader to check that when $\mathfrak{h}_n$ and its dual of $\mathfrak{h}_n$ are identified with $\mathbb{R}^{2n+1}$ via the standard basis and its dual respectively, one finds (3.10).

With (3.10) and $\mathcal{P}$, $\mathcal{Q}$ as in (3.8) and similarly for $\mathcal{P}'$, $\mathcal{Q}'$, we have

$$(z'v, -z'u, 0) = \mathrm{ad}^*_{\mathbf{H}_\mathrm{n}}(\mathcal{P})(\mathcal{Q}') \quad \text{and} \quad (zv', -zu', 0) = \mathrm{ad}^*_{\mathbf{H}_\mathrm{n}}(\mathcal{P}')(\mathcal{Q}),$$

and we can now express the next set of coordinates in Formula (3.6):

$$\left(x + x' + \frac{1}{4}(z'v - zv')\ ,\ y + y' - \frac{1}{4}(z'u - zu'), z + z'\right)$$
$$= \mathcal{Q} + \mathcal{Q}' + \frac{1}{4}\left(\mathrm{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}') - \mathrm{ad}^*_{\mathbf{H}_n}(\mathcal{P}')(\mathcal{Q})\right).$$

For the last coordinate in Formula (3.6), we observe that

$$\begin{aligned}
&s + s' + \frac{ux' - xu'}{2} + \frac{vy' - yv'}{2} + \frac{wz' - zw'}{2} - \frac{z - z'}{8}(uv' - vu') \\
&= s + s' + \frac{\langle (u,v,w),(x',y',z')\rangle - \langle (x,y,z),(u',v',w')\rangle}{2} \\
&\quad - \frac{1}{8}\big\langle (x - x', y - y', z - z'), (0, 0, uv' - vu')\big\rangle \\
&= \mathcal{S} + \mathcal{S}' + \frac{1}{2}\big(\langle \mathcal{P}, \mathcal{Q}'\rangle - \langle \mathcal{Q}, \mathcal{P}'\rangle\big) - \frac{1}{8}\big\langle \mathcal{Q} - \mathcal{Q}', [\mathcal{P}, \mathcal{P}']\big\rangle.
\end{aligned}$$

Hence we have found the following expression for the group law of $\mathbf{H}_{2,n}$:

Lemma 3.7. *With the convention explained above, the product of two elements* $(\mathcal{P}, \mathcal{Q}, \mathcal{S})$ *and* $(\mathcal{P}', \mathcal{Q}', \mathcal{S}')$ *in* $\mathbf{H}_{2,n} \cong \mathfrak{h}_{2,n}$ *is*

$$\begin{aligned}
&(\mathcal{P}, \mathcal{Q}, \mathcal{S}) \odot_{\mathbf{H}_{2,n}} (\mathcal{P}', \mathcal{Q}', \mathcal{S}') \\
&= \Big(\mathcal{P} \cdot \mathcal{P}'\ ,\ \mathcal{Q} + \mathcal{Q}' + \frac{1}{4}(\mathrm{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}') - \mathrm{ad}^*_{\mathbf{H}_n}(\mathcal{P}')(\mathcal{Q}))\ , \\
&\qquad \mathcal{S} + \mathcal{S}' + \frac{1}{2}\big(\langle \mathcal{P}, \mathcal{Q}'\rangle - \langle \mathcal{Q}, \mathcal{P}'\rangle\big) - \frac{1}{8}\big\langle \mathcal{Q} - \mathcal{Q}', [\mathcal{P}, \mathcal{P}']\big\rangle\Big),
\end{aligned} \tag{3.11}$$

whereas their Lie bracket is given by

$$\begin{aligned}
&[(\mathcal{P}, \mathcal{Q}, \mathcal{S}), (\mathcal{P}', \mathcal{Q}', \mathcal{S}')]_{\mathfrak{h}_{2,n}} \\
&= \Big([\mathcal{P}, \mathcal{P}']_{\mathfrak{h}_n}, \frac{1}{2}\big(\mathrm{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}') - \mathrm{ad}^*_{\mathbf{H}_n}(\mathcal{P}')(\mathcal{Q})\big), \langle \mathcal{P}, \mathcal{Q}'\rangle - \langle \mathcal{Q}, \mathcal{P}'\rangle\Big).
\end{aligned} \tag{3.12}$$

Proof. The second claim follows from the above discussion and a direct comparison with Formula (3.5). □

The following technical identities will be needed later. They are best expressed and proved using the notation explained above.

Lemma 3.8. *1. Any element $(\mathcal{P},\mathcal{Q},\mathcal{S})$ in $\mathbf{H}_{2,n}$ can be written as*

$$(\mathcal{P},\mathcal{Q},\mathcal{S}) = \left(0, \mathcal{Q} + \frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}), 0\right) \odot_{\mathbf{H}_{2,n}} (\mathcal{P},0,0) \odot_{\mathbf{H}_{2,n}} \left(0,0,\mathcal{S} + \frac{1}{2}\langle \mathcal{Q},\mathcal{P}\rangle\right).$$

2. For any $X, X_1, X_2 \in \mathbb{R}^{2n+1}$, the following scalar products coincide:

$$\left\langle \operatorname{ad}^*_{\mathbf{H}_n}(X)(X_1), X_2 \right\rangle = \langle X_1, [X_2, X]\rangle .$$

3. For any $X \in \mathbb{R}^{2n+1}$ and $(\mathcal{P},\mathcal{Q},\mathcal{S}) \in \mathbf{H}_{2,n}$, we have

$$(X,0,0) \odot_{\mathbf{H}_{2,n}} (\mathcal{P},\mathcal{Q},\mathcal{S}) = (0,\mathcal{Q}',\mathcal{S}') \odot_{\mathbf{H}_{2,n}} (X \cdot \mathcal{P}, 0, 0)$$

for some $\mathcal{Q}' \in \mathbb{R}^{2n+1}$ and $\mathcal{S}' \in \mathbb{R}$ given by

$$\mathcal{S}' := \mathcal{S} + \left\langle \mathcal{Q}, X \cdot (\frac{1}{2}\mathcal{P}) \right\rangle .$$

Proof of Lemma 3.8. Part (2) could be proved using the definition of the co-adjoint explained in Remark 3.6 but we show it here by direct calculations using (3.10):

$$\left\langle \operatorname{ad}^*_{\mathbf{H}_n}(X)(X_1), X_2 \right\rangle = \langle (t_1 q, -t_1 p, 0, (p_2, q_2, t_2)\rangle = t_1 q p_2 - t_1 p q_2.$$

Let us prove Part (1). Firstly we notice that ${\operatorname{ad}^*_{\mathbf{H}_n}}^2 = 0$ since $\mathbf{H}_n$ is of step 2 or by direct calculations using (3.10):

$${\operatorname{ad}^*_{\mathbf{H}_n}}^2(X)(X') = \operatorname{ad}^*_{\mathbf{H}_n}(X)(t'q, -t'p, 0) = 0.$$

Secondly, we have

$$\left\langle \operatorname{ad}^*_{\mathbf{H}_n}(X)(X'), X \right\rangle = 0$$

as a consequence of Part (2). Now we apply the newly found expression for the group law in (3.11) to

$$\begin{aligned}
&\left(0, \mathcal{Q} + \frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}), 0\right) \odot_{\mathbf{H}_{2,n}} (\mathcal{P}, 0, 0) \\
&= \left(\mathcal{P}, \mathcal{Q} + \frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}) - \frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})\left(\mathcal{Q} + \frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q})\right),\right. \\
&\qquad \left. -\frac{1}{2}\left\langle \mathcal{Q} + \frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}), \mathcal{P}\right\rangle\right) \\
&= \left(\mathcal{P}, \mathcal{Q} - \frac{1}{16}(\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P}))(\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q})), -\frac{1}{2}\langle \mathcal{Q}, \mathcal{P}\rangle - \frac{1}{8}\langle \operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}), \mathcal{P}\rangle\right) \\
&= \left(\mathcal{P}, \mathcal{Q}, -\frac{1}{2}\langle \mathcal{Q}, \mathcal{P}\rangle\right),
\end{aligned}$$

having applied for the last line of the computations the two observations above. Since the centre of $\mathbf{H}_{n,2}$ is $\{(0,0,\mathcal{S}) : \mathcal{S} \in \mathbb{R}\}$, Part (1) is proved.

Let us prove Part (3). Using the group law expressed in (3.11) and the decomposition given in Part (1), we have

$$\begin{aligned}
&(X, 0, 0) \odot_{\mathbf{H}_{2,n}} (\mathcal{P}, \mathcal{Q}, \mathcal{S}) \\
&= \left(X \cdot \mathcal{P}, \mathcal{Q} + \frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\mathcal{Q}), \mathcal{S} + \frac{1}{2}\langle X, \mathcal{Q}\rangle + \frac{1}{8}\langle \mathcal{Q}, [X, \mathcal{P}]\rangle\right) \\
&= (0, \mathcal{Q}', 0) \odot_{\mathbf{H}_{2,n}} (X \cdot \mathcal{P}, 0, 0) \odot_{\mathbf{H}_{2,n}} (0, 0, \mathcal{S}'),
\end{aligned}$$

for some $\mathcal{Q}' \in \mathbb{R}^{2n+1}$, whose expression we do not need to compute, and for the centre component

$$\mathcal{S}' := \mathcal{S} + \frac{1}{2}\langle X, \mathcal{Q}\rangle + \frac{1}{8}\langle \mathcal{Q}, [X, \mathcal{P}]\rangle + \frac{1}{2}\left\langle \mathcal{Q} + \frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\mathcal{Q}), X \cdot \mathcal{P}\right\rangle.$$

Let us use Part (2) for the last term:

$$\begin{aligned}
\left\langle \mathcal{Q} + \frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\mathcal{Q}), X \cdot \mathcal{P}\right\rangle &= \langle \mathcal{Q}, X \cdot \mathcal{P}\rangle + \frac{1}{4}\langle \operatorname{ad}^*_{\mathbf{H}_n}(X)(\mathcal{Q}), X \cdot \mathcal{P}\rangle \\
&= \langle \mathcal{Q}, X \cdot \mathcal{P}\rangle + \frac{1}{4}\langle \mathcal{Q}, [X \cdot \mathcal{P}, X]\rangle. \qquad (3.13)
\end{aligned}$$

Now since $X \cdot \mathcal{P} = X + \mathcal{P} + \frac{1}{2}[X, \mathcal{P}]$ (see (3.9)) and the iterated bracket is zero, (3.13) becomes

$$\left\langle \mathcal{Q}, X + \mathcal{P} + \frac{1}{2}[X,\mathcal{P}] \right\rangle + \frac{1}{4}\left\langle \mathcal{Q}, \left[X + \mathcal{P} + \frac{1}{2}[X,\mathcal{P}], X \right] \right\rangle$$
$$= \left\langle \mathcal{Q}, X + \mathcal{P} + \frac{1}{2}[X,\mathcal{P}] \right\rangle + \frac{1}{4}\langle \mathcal{Q}, [\mathcal{P}, X]\rangle = \left\langle \mathcal{Q}, X + \mathcal{P} + \frac{1}{4}[X,\mathcal{P}] \right\rangle.$$

Therefore, we have for $\mathcal{S}'$:

$$\begin{aligned} \mathcal{S}' &= \mathcal{S} + \frac{1}{2}\langle X, \mathcal{Q}\rangle + \frac{1}{8}\langle \mathcal{Q}, [X,\mathcal{P}]\rangle + \frac{1}{2}\left\langle \mathcal{Q}, X + \mathcal{P} + \frac{1}{4}[X,\mathcal{P}] \right\rangle \\ &= \mathcal{S} + \left\langle \mathcal{Q}, X + \frac{1}{2}\mathcal{P} + \frac{1}{4}[X,\mathcal{P}] \right\rangle. \end{aligned}$$

This concludes the proof of Part (3). □

3.4 The Schrödinger-type representations of $\mathrm{H}_{2,n}$

In this Section, we show that the isomorphism $d\pi$ defined in (3.2) can be viewed as the infinitesimal representation of a Schrödinger-type representation π of $\mathfrak{h}_{2,n}$. We will present the argument for the whole family π_λ, $\lambda \in \mathbb{R}\backslash\{0\}$, of Schrödinger-type representations which contains $\pi_1 = \pi$.

We begin by defining for each $\lambda \in \mathbb{R}\backslash\{0\}$ the linear mapping

$$d\pi_\lambda : \mathbb{R}^{2(2n+1)+1} \longrightarrow \langle \mathscr{D}_{p_j}, \mathscr{D}_{q_j}, \mathscr{D}_t, \mathscr{X}_{p_j}, \mathscr{X}_{q_j}, \mathscr{X}_t \rangle,$$

via

$$\left.\begin{array}{lll} d\pi_\lambda(X_{u_j}) = 2\pi i\, \mathscr{D}_{p_j}, & d\pi_\lambda(X_{v_j}) = 2\pi i\, \mathscr{D}_{q_j}, & d\pi_\lambda(X_w) = 2\pi i\, \mathscr{D}_t, \\ d\pi_\lambda(X_{x_j}) = 2\pi i\lambda \mathscr{X}_{p_j}, & d\pi_\lambda(X_{y_j}) = 2\pi i\lambda\, \mathscr{X}_{q_j}, & \\ d\pi_\lambda(X_z) = 2\pi i\lambda\, \mathscr{X}_t, & d\pi_\lambda(X_s) = 2\pi i\lambda\, \mathrm{I}. & \end{array}\right\} \tag{3.14}$$

With all our conventions (see Section 3.3) we can also write

$$d\pi_\lambda(u,v,w,x,y,z,s) = 2\pi i \left(u\mathscr{D}_p + v\mathscr{D}_q + w\mathscr{D}_t + \lambda x\mathscr{X}_p + \lambda y\mathscr{X}_q + \lambda z\mathscr{X}_t + \lambda s\mathrm{I}\right).$$

The main property of this subsection is:

Proposition 3.9. *1. For any $\lambda \in \mathbb{R}\backslash\{0\}$, the linear mapping $d\pi_\lambda$ is a Lie algebra isomorphism between $\mathfrak{h}_{n,2}$ and $\langle \mathscr{D}_{p_j}, \mathscr{D}_{q_j}, \mathscr{D}_t, \mathscr{X}_{p_j}, \mathscr{X}_{q_j}, \mathscr{X}_t \rangle$.*

2. $d\pi_1 = d\pi$.

3. *Let $\lambda \in \mathbb{R}\backslash\{0\}$. The representation $d\pi_\lambda$ is the infinitesimal representation of the unitary representation π_λ of $\mathbf{H}_{2,n}$ acting on $L^2(\mathbf{H}_n)$ given by*

$$\big(\pi_\lambda(\mathcal{P}, \mathcal{Q}, \mathcal{S})f\big)(X) = e^{2\pi i\lambda\left(\mathcal{S}+\left\langle \mathcal{Q}, X\cdot(\frac{1}{2}\mathcal{P})\right\rangle\right)} f(X\cdot\mathcal{P}), \tag{3.15}$$

for $(\mathcal{P}, \mathcal{Q}, \mathcal{S}) \in \mathbf{H}_{2,n}$, $X \in \mathbf{H}_n$ and $f \in L^2(\mathbf{H}_n)$.

4. *If $\lambda \neq \lambda'$ in $\mathbb{R}\backslash\{0\}$, the representations π_λ and $\pi_{\lambda'}$ are inequivalent.*

Proof. Parts 1 and 2 are easy to check.

For Part 3, one can check by direct computations that Formula (3.15) defines a unitary representation π_λ of $\mathbf{H}_{2,n}$ and that its infinitesimal representation coincides with $d\pi_\lambda$.

Clearly each π_λ coincides with the characters $\mathcal{S} \to e^{2\pi i\lambda\mathcal{S}}$ on the centre of the group $\mathbf{H}_{2,n}$. Hence, two representations π_λ and $\pi_{\lambda'}$ corresponding to different $\lambda \neq \lambda'$ are inequivalent, and Part 4 is proved. □

Let us explain how Formula (3.15) appears by showing that the unique candidate for the representation π_λ of $\mathbf{H}_{n,2}$ on $L^2(\mathbf{H}_n) \cong L^2(\mathbb{R}^{2n+1})$ that admits $d\pi_\lambda$ as infinitesimal representation is given by (3.15).

As in Proposition 3.3 (iii) (see also Section 1.3.2), we see that the restriction of $d\pi_\lambda$ to the subalgebra $\mathbb{R}X_{u_1} \oplus \ldots \oplus \mathbb{R}X_{v_n} \oplus \mathbb{R}X_w$ coincides with the infinitesimal right regular representation of $\mathbf{H}_n$ on $L^2(\mathbb{R}^n)$. Therefore, the restriction of π_λ to $\{(\mathcal{P}, 0, 0) : \mathcal{P} \in \mathbb{R}^{2n+1}\}$ must be given by the right regular representation of $\mathbf{H}_n$:

$$\big(\pi_\lambda(\mathcal{P}, 0, 0)f\big)(X) = f(X\cdot\mathcal{P}). \tag{3.16}$$

This could also be proved with a simple argument about unitary one-parameter groups in the spirit of Stone's Theorem. The same argument also yields that such π_λ must satisfy

$$\big(\pi_\lambda(0, \mathcal{Q}, 0)f\big)(X) = e^{2\pi i\lambda\langle \mathcal{Q}, X\rangle} f(X), \tag{3.17}$$

$$\big(\pi_\lambda(0, 0, \mathcal{S})f\big)(X) = e^{2\pi i\lambda\mathcal{S}} f(X). \tag{3.18}$$

Using (3.16), (3.17) and (3.18), together with the group law and, more precisely, Part

(1) of Lemma 3.8, we must have

$$\begin{aligned}
&\big(\pi_\lambda(\mathcal{P},\mathcal{Q},\mathcal{S})f\big)(X)\\
&= \big(\pi_\lambda(0,\mathcal{Q}+\frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}),0)\pi_\lambda(\mathcal{P},0,0)\pi_\lambda(0,0,\mathcal{S}+\frac{1}{2}\langle\mathcal{Q},\mathcal{P}\rangle)f\big)(X)\\
&= e^{2\pi i\lambda\left\langle \mathcal{Q}+\frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}),X\right\rangle}\big(\pi_\lambda(\mathcal{P},0,0)\pi_\lambda(0,0,\mathcal{S}+\frac{1}{2}\langle\mathcal{Q},\mathcal{P}\rangle)f\big)(X)\\
&= e^{2\pi i\lambda\left\langle \mathcal{Q}+\frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}),X\right\rangle}\big(\pi_\lambda(0,0,\mathcal{S}+\frac{1}{2}\langle\mathcal{Q},\mathcal{P}\rangle)f\big)(X\cdot\mathcal{P})\\
&= e^{2\pi i\lambda\left\langle \mathcal{Q}+\frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}),X\right\rangle}e^{2\pi i\lambda(\mathcal{S}+\frac{1}{2}\langle\mathcal{Q},\mathcal{P}\rangle)}f(X\cdot\mathcal{P}).
\end{aligned}$$

By Lemma 3.8 Part (2), we have

$$\left\langle \operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}),X\right\rangle = \langle\mathcal{Q},[X,\mathcal{P}]\rangle,$$

thus

$$\begin{aligned}
\left\langle \mathcal{Q}+\frac{1}{4}\operatorname{ad}^*_{\mathbf{H}_n}(\mathcal{P})(\mathcal{Q}),X\right\rangle+\frac{1}{2}\langle\mathcal{Q},\mathcal{P}\rangle &= \langle\mathcal{Q},X\rangle+\frac{1}{4}\langle\mathcal{Q},[X,\mathcal{P}]\rangle+\frac{1}{2}\langle\mathcal{Q},\mathcal{P}\rangle\\
&= \left\langle \mathcal{Q},X+\frac{1}{2}\mathcal{P}+\frac{1}{2}[X,\frac{1}{2}\mathcal{P}]\right\rangle = \left\langle \mathcal{Q},X\cdot(\frac{1}{2}\mathcal{P})\right\rangle,
\end{aligned}$$

with the convention that the dot product denotes the Heisenberg group law (cf. Section 3.3). Therefore, we have obtained that the unique candidate for π_λ is given by (3.15). Conversely, one checks easily that Formula (3.15) defines a unitary representation of $\mathbf{H}_{2,n}$.

Remark 3.10. In exponential coordinates (u,v,w,x,y,z,s) the representation π_λ is given by

$$\begin{aligned}
\big(\pi_\lambda(u,v,w,x,y,z,s)f\big)(p,q,t) &= e^{2\pi i\lambda\left(\left\langle(x,y,z)^t,(p,q,t)^t\right\rangle+\frac{1}{2}\left\langle(x,y,z)^t,(u,v,w)^t\right\rangle+\frac{1}{4}z\,(pv-uq)+s\right)}\\
&\qquad\times f\big(p+u,q+v,t+w+\frac{1}{2}(pv-uq)\big).
\end{aligned}$$

In the next section we list all the unirreps of $\mathbf{H}_{2,n}$ up to unitary equivalence using the orbit method. Since π_λ will be among these, this will show its irreducibility.

3.5 The Unitary Irreducible Representations of the Dynin-Folland Group - A Classification via the Orbit Method

In this subsection we will classify the unitary irreducible representations of the Dynin-Folland group employing Kirillov's orbit method. We refer to, e.g., [10] for a description of this method. We will first give a description of the co-adjoint orbits of $\mathbf{H}_{2,n}$. Subsequently, we will construct the corresponding unirreps. Finally, for each orbit we will have a look at the corresponding jump sets.

3.5.1 The Co-adjoint Orbits

In order to classify the $\mathbf{H}_{2,n}$-co-adjoint orbits, we first we give an explicit formula for the co-adjoint representation K of $\mathbf{H}_{2,n}$ on the dual $\mathfrak{h}_{2,n}^*$ of its Lie algebra $\mathfrak{h}_{n,2}$. Recall that K is given by

$$\langle K(g)F, X\rangle = \langle F, \mathrm{Ad}(g^{-1})X\rangle , \tag{3.19}$$

if $F \in \mathfrak{h}_{n,2}^*$, $g \in \mathbf{H}_{2,n}$ and $X \in \mathfrak{h}_{n,2}$. We denote by

$$(X_{u_1}^*, \dots, X_{u_n}^*, X_{v_1}^*, \dots, X_{v_n}^*, X_w^*, X_{x_1}^*, \dots, X_{x_n}^*, X_{y_1}^*, \dots, X_{y_n}^*, X_z^*, X_s^*),$$

the dual standard basis of $\mathbb{R}^{2(2n+1)+1}$.

Lemma 3.11. *For any $X \in \mathfrak{h}_{2,n}$ and $F \in \mathfrak{h}_{n,2}^*$ written as*

$$\begin{aligned} F &= f_u X_u^* + f_v X_v^* + f_w X_w^* + f_x X_x^* + f_y X_y^* + f_z X_z^* + f_s X_s^*, \\ X &= uX_u + vX_v + wX_w + xX_x + yX_y + zX_z + sX_s, \end{aligned}$$

we have

$$\begin{aligned} K(\exp_{\mathbf{H}_{2,n}}(X))F \;=\; & \big(f_u + f_w v - \frac{z}{2} f_y + f_s x + \frac{3}{4} f_s z v\big) X_u^* \\ & + \big(f_v - f_w u + \frac{z}{2} f_x + f_s y - \frac{3}{4} f_s z u\big) X_v^* \\ & + \big(f_w + f_s z\big) X_w^* \quad + \big(f_x - f_s u\big) X_x^* \quad + \big(f_y - f_s v\big) X_y^* \\ & + \big(f_z - \frac{f_x v}{2} + \frac{f_y u}{2} - f_s w\big) X_z^* \quad + f_s X_s^*. \end{aligned}$$

Proof. We apply (3.19) to $g = \exp_{\mathbf{H}_{2,n}}(X)$. We notice that due to nilpotency of $\mathbf{H}_{2,n}$, we have

$$\begin{aligned}\mathrm{Ad}(g^{-1})X' &= \mathrm{Ad}(\exp_{\mathbf{H}_{2,n}}(X)^{-1})X' = \mathrm{Ad}(\exp_{\mathbf{H}_{2,n}}(-X))X' = e^{\mathrm{ad}(-X)}X' \\ &= X' - [X, X'] + \frac{1}{2}[X, [X, X']].\end{aligned}$$

For X as in the statement and a similar expression for X', we compute

$$\begin{aligned}&\mathrm{Ad}(\exp_{\mathbf{H}_{2,n}}(-X))(X') \\ &= \big(u', v', w' - uv' + vu', x' + \frac{1}{2}(zv' - z'v), y' + \frac{1}{2}(z'u - zu'), z', \\ &\qquad s' - ux' + xu' - vy' + yv' - wz' + zw' - \frac{3}{4}z(uv' - vu')\big),\end{aligned}$$

and hence

$$\begin{aligned}&\left\langle K(\exp_{\mathbf{H}_{2,n}}(X))F, X'\right\rangle \\ &= f_u u' + f_v v' + f_w(w' - uv' + vu') \\ &\quad + f_x\left(x' + \frac{1}{2}(zv' - z'v)\right) + f_y\left(y' + \frac{1}{2}(z'u - zu')\right) + f_z z' \\ &\quad + f_s\left(s' - ux' + xu' - vy' + yv' - wz' + zw' - \frac{3}{4}z(uv' - vu')\right).\end{aligned}$$

A reorganisation of the terms gives the stated equality. □

We can now describe the co-adjoint orbits of $\mathbf{H}_{2,n}$ by giving their representatives. Given our convention, we may write $\mathbb{R}^n X_x$ for $\mathbb{R}X_{x_1} \oplus \ldots \oplus \mathbb{R}X_{x_n}$ and similarly for $\mathbb{R}^n X_y$, $\mathbb{R}^n X_u$, $\mathbb{R}^n X_v$ etc.

Proposition 3.12. *Any co-adjoint orbit of $\mathbf{H}_{2,n}$ has exactly one representative among the following elements of $\mathfrak{h}_{n,2}^*$:*

(Case (1)) $f_s X_s^*$ *if* $f_s \neq 0$,

(Case (2)) $f_w X_w^* + f_x X_x^* + f_y X_y^* + f_z X_z^*$ *with* $f_s = 0$ *but* $f_w \neq 0$,

(Case (3)) $f_u X_u^* + f_v X_v^* + f_x X_x^* + f_y X_y^*$ *with the equality* $f_u f_y = f_v f_x$ *between the scalar products, and vanishing of the coordinates* $f_s = f_w = f_z = 0$ *but the non-vanishing of the* $\mathbb{R}^{2n}$*-vector* $(f_x, f_y) \neq 0$,

(Case (4)) $f_u X_u^* + f_v X_v^* + f_z X_z^*$ *with* $f_s = f_w = 0$, $f_x = f_y = 0$.

All the co-adjoint orbits are affine subspaces of $\mathfrak{h}_{n,2}^*$. More precisely, in Case (1), the orbit of $f_s X_s^*$ is the affine hyperplane passing through $f_s X_s^*$ given by

$$K(\mathbf{H}_{2,n})(f_s X_s^*) = f_s X_s^* \oplus \mathbb{R}^n X_u^* \oplus \mathbb{R}^n X_v^* \oplus \mathbb{R} X_w^* \oplus \mathbb{R}^n X_x \oplus \mathbb{R}^n X_y^* \oplus \mathbb{R} X_z^*. \tag{3.20}$$

The orbits $K(\mathbf{H}_{2,n})(f_s X_s^*)$, $f_s \in \mathbb{R}\backslash\{0\}$, are the generic co-adjoint orbits. They form an open dense subset of $\mathfrak{h}_{2,n}^*$.

In Case (2), the orbits are $2n$-dimensional affine subspaces:

$$\begin{aligned} &K(\mathbf{H}_{2,n})(f_w X_w^* + f_x X_x^* + f_y X_y^* + f_z X_z^*) \\ &= f_w X_w^* + f_x X_x^* + f_y X_y^* + f_z X_z^* + \{\tilde{v} X_u^* + \tilde{u} X_v^* - \frac{f_x \tilde{v} + f_y \tilde{u}}{2 f_w} X_z^* : \tilde{u}, \tilde{v} \in \mathbb{R}^n\}. \end{aligned} \tag{3.21}$$

In Case (3), the orbits are 2-dimensional affine subspaces:

$$\begin{aligned} &K(\mathbf{H}_{2,n})(f_u X_u^* + f_v X_v^* + f_x X_x^* + f_y X_y^*) \\ &\quad = f_u X_u^* + f_v X_v^* + f_x X_x^* + f_y X_y^* + \mathbb{R}(-f_y X_u^* + f_x X_v^*) + \mathbb{R} X_z^*. \end{aligned} \tag{3.22}$$

In Case (4), the orbits are singletons.

Proof. Case (1) Let $F \in \mathfrak{h}_{n,2}^*\backslash\{0\}$ be such that its component f_s is not zero. Then we choose X as in Lemma 3.11 with z, u, v such that the coordinates of $K(\exp_{\mathbf{H}_{2,n}} X)(F)$ in X_w^*, X_x^* and X_y^* are zero, that is,

$$f_w + f_s z = 0, \quad f_x - f_s u = f_y - f_s v = 0,$$

then w, x, y such that the coordinates in X_z^*, X_u^* and X_v^* are zero, that is,

$$f_z - \frac{f_x v}{2} + \frac{f_y u}{2} - f_s w = 0,$$

and

$$0 \;=\; f_u + f_w v - \frac{z}{2} f_y + f_s x + \frac{3}{4} f_s z v \;=\; f_v - f_w u + \frac{z}{2} f_x + f_s y - \frac{3}{4} f_s z u.$$

We have obtained $K(\exp(X))F = f_s X_s^*$. Therefore, the orbit $K(\mathbf{H}_{2,n})F$ describes the $2(2n+1)$-dimensional hyperplane at height f_s parallel to the subspace $\mathfrak{h}_{n,2}{}^*/\mathbb{R} X_s^*$.

Case (2). We assume $f_s = 0$ but $f_w \neq 0$, so that we have

$$\begin{aligned} K(\exp_{\mathbf{H}_{2,n}}(X))F \;=\; &\big(f_u + f_w v - \frac{1}{2} z f_y\big) X_u^* + \big(f_v - f_w u + \frac{1}{2} z f_x\big) X_v^* \\ &+ f_w X_w^* + f_x X_x^* + f_y X_y^* + \big(f_z - \frac{1}{2} f_x v + \frac{1}{2} f_y u\big) X_z^*. \end{aligned}$$

We choose u and v such that the coordinates in X_u^* and X_v^* vanish, that is,

$$v = \frac{1}{f_w}(-f_u + \frac{1}{2}zf_y) \quad \text{and} \quad u = \frac{1}{f_w}(f_v + \frac{1}{2}zf_x).$$

Then the X_z^*-coordinate of $K(\exp_{\mathbf{H}_{2,n}}(X))F$ becomes

$$f_z - \frac{1}{2}f_x v + \frac{1}{2}f_y u = f_z + \frac{1}{2f_w}(f_u f_x + f_v f_y),$$

independently of the other entries w, x, y, z, s of X. Therefore, $F' := f_w X_w^* + f_x X_x^* + f_y X_y^* + f_z' X_z^*$ with $f_z' = f_z + \frac{1}{2f_w}(f_u f_x + f_v f_y)$ is in the same orbit as F and F' is the only element of the orbit with zero coordinates in X_u^* and X_v^*. We choose F' as the representative of the co-adjoint orbit that contains F. Similar computations as above, together with setting $\tilde{v} = f_w v - \frac{z}{2}f_y \in \mathbb{R}^n$ and $\tilde{u} = -f_w u + \frac{z}{2}f_x \in \mathbb{R}^n$, yield

$$\begin{aligned} K(\exp_{\mathbf{H}_{2,n}}(X))F' &= F' + \tilde{v}X_u^* + \tilde{u}X_v^* + \frac{-f_x v + f_y u}{2}X_z^* \\ &= F' + \tilde{v}X_u^* + \tilde{u}X_v^* - \frac{f_x\tilde{v} + f_y\tilde{u}}{2f_w}X_z^*. \end{aligned}$$

This yields the description of the F'-orbit.

Case (3). We assume $f_s = 0 = f_w$. Then

$$\begin{aligned} K(\exp_{\mathbf{H}_{2,n}}(X))F &= \big(f_u - \frac{z}{2}f_y\big)X_u^* + \big(f_v + \frac{z}{2}f_x\big)X_v^* \\ &\quad + f_x X_x^* + f_y X_y^* + \big(f_z - \frac{1}{2}f_x v + \frac{1}{2}f_y u\big)X_z^*. \end{aligned}$$

We also assume $(f_x, f_y) \neq 0$. Then we can choose v or u such that the X_z^*-coordinate vanishes, and we also choose z such that the following scalar product in $\mathbb{R}^{2n}$ vanishes:

$$\langle (f_u, f_v) + \frac{z}{2}(-f_y, f_x), (-f_y, f_x)\rangle_{\mathbb{R}^{2n}} = 0.$$

This means that, in this case, F and $F' := f_u' X_u^* + f_v' X_v^* + f_x X_x^* + f_y X_y^*$ with $(f_u', f_v') \perp (-f_y, f_x)$ in $\mathbb{R}^{2n}$ are in the same orbit. Furthermore F' is the only element of this orbit with $(f_u', f_v') \perp (-f_y, f_x)$. Similar computations as above, with $\tilde{z} = \frac{z}{2} \in \mathbb{R}$ and $\tilde{a} = -\frac{1}{2}f_x v + \frac{1}{2}f_y u \in \mathbb{R}$, give

$$K(\exp_{\mathbf{H}_{2,n}}(X))F' = F' - \tilde{z}f_y X_u^* + \tilde{z}f_x X_v^* + \tilde{a}X_z^*.$$

This yields the description of the F'-orbit.

If $f_x = f_y = 0$, then $F = f_u X_u^* + f_v X_v^* + f_z X_z^* = K(\exp_{\mathbf{H}_{2,n}}(X))F$ for any $X \in \mathfrak{h}_{n,2}$. This corresponds to Case (4). This concludes the proof of Proposition 3.12. $\square$

Corollary 3.13. *If we denote by $F_1, \ldots, F_4$ the representatives of the co-adjoint orbits given by Cases* (1) - (4)*, then the corresponding stabilizer subgroups of $\mathbf{H}_{2,n}$, denoted by* $\mathrm{Stab}(F_j)$*, $j = 1, \ldots, 4$, are given by:*

(Case (1)) $\mathrm{Stab}(F_1) = \exp_{\mathbf{H}_{2,n}}(\mathbb{R}X_s)$,

(Case (2)) $\mathrm{Stab}(F_2) = \exp_{\mathbf{H}_{2,n}}\big(\mathbb{R}X_w \oplus \mathbb{R}^n X_x \oplus \mathbb{R}^n X_y \oplus \mathbb{R}X_z \oplus \mathbb{R}X_s\big)$,

(Case (3)) $\mathrm{Stab}(F_3) = \{(u, v, w, x, y, z, s) \in \mathbf{H}_{2,n} \mid z = 0, f_x v = f_y u\}$,

(Case (4)) $\mathrm{Stab}(F_4) = \mathbf{H}_{2,n}$.

Proof. Cases (1) and (4) are straight-forward in view of Lemma 3.11. To prove Case (2), we find that the necessary and sufficient condition

$$\begin{aligned} K(\exp_{\mathbf{H}_{2,n}}(X))F_2 &= \big(f_u + f_w v - \frac{1}{2}\, z f_y\big) X_u^* + \big(f_v - f_w u + \frac{1}{2}\, z f_x\big) X_v^* \\ &\quad + f_w X_w^* + f_x X_x^* + f_y X_y^* + \big(f_z - \frac{1}{2}\, f_x v + \frac{1}{2}\, f_y u\big) X_z^* \\ &= f_w X_w^* + f_x X_x^* + f_y X_y^* + f_z X_z^* \\ &= F_2, \end{aligned}$$

is equivalent to

$$\big(f_u + f_w v - \frac{1}{2}\, z f_y\big) = \big(f_v - f_w u + \frac{1}{2}\, z f_x\big) = \big(-\frac{1}{2}\, f_x v + \frac{1}{2}\, f_y u\big) = 0. \tag{3.23}$$

It is now easily seen that the largest subgroup satisfying (3.23) is the one asserted above.

In order to determine $\mathrm{Stab}(F_3)$, we observe that

$$\begin{aligned} K(\exp_{\mathbf{H}_{2,n}}(X))F_3 &= \big(f_u - \frac{z}{2} f_y\big) X_u^* + \big(f_v + \frac{z}{2} f_x\big) X_v^* \\ &\quad + f_x X_x^* + f_y X_y^* + \big(-\frac{1}{2} f_x v + \frac{1}{2} f_y u\big) X_z^* \\ &= f_u X_u^* + f_v X_v^* + f_x X_x^* + f_y X_y^* \\ &= F_3 \end{aligned}$$

holds if and only if $z = 0$ and $f_x v = f_y u$. This concludes the proof. $\square$

3.5.2 The Unirreps

To begin with, let us show that the representations corresponding to the orbits of Case (1) via the orbit method coincide with the representations π_λ constructed in Section 3.4:

Proposition 3.14. *Let $f_s = \lambda \in \mathbb{R}\backslash\{0\}$. The representation π_λ as defined by Equality (3.15) is unitarily equivalent to the unirrep corresponding to the linear form λX_s^*, and the (maximal polarising λX_s^*-subordinated) subalgebra*

$$\mathfrak{l} := \mathbb{R}^n X_x \oplus \mathbb{R}^n X_y \oplus \mathbb{R} X_z \oplus \mathbb{R} X_s.$$

Proof. One checks easily that the subspace $\mathfrak{l}$ of $\mathfrak{h}_{2,n}$ is a maximal subalgebra subordinated to $F := \lambda X_s^*$ and that its corresponding subgroup is

$$\mathbf{L} = \exp_{\mathbf{H}_{2,n}}(\mathfrak{l}) = \{(0, \mathcal{Q}, \mathcal{S}) : \mathcal{Q} \in \mathbb{R}^{2n+1},\ \mathcal{S} \in \mathbb{R}\}.$$

Let $\rho_{F,\mathbf{L}}$ be the character of the subgroup $\mathbf{L}$ with infinitesimal character iF. It is given for any $X = xX_x + yX_y + zX_z + sX_s \in \mathfrak{l}$ by

$$\rho_{F,\mathbf{L}}(\exp_{\mathbf{H}_{2,n}}(X)) = e^{2\pi i F(X)} = e^{2\pi i \lambda x_s}.$$

and also for any $(0, \mathcal{Q}, \mathcal{S}) \in \mathbf{L}$ by

$$\rho_{F,\mathbf{L}}(0, \mathcal{Q}, \mathcal{S}) = e^{2\pi i \lambda \mathcal{S}}. \tag{3.24}$$

In order to define the representation induced by $\rho_{F,\mathbf{L}}$, we consider $\mathscr{F}_0$, the space of continuous functions $\varphi : \mathbf{H}_{2,n} \to \mathbb{C}$ that satisfy

$$\varphi(\ell \odot_{\mathbf{H}_{2,n}} g) = \rho_{F,\mathbf{L}}(\ell)\, \varphi(g), \quad \text{for all } \ell \in \mathbf{L},\ g \in \mathbf{H}_{2,n}, \tag{3.25}$$

and whose support modulo $\mathbf{L}$ is compact. Let $\operatorname{ind}(\rho_{F,\mathbf{L}})_{\mathbf{L}}^{\mathbf{H}_{2,n}}$ be the representation induced by $\rho_{F,\mathbf{L}}$ on the group $\mathbf{H}_{2,n}$ that acts on $\mathscr{F}_0$. It may be realized as

$$\Big(\big(\operatorname{ind}(\rho_{F,\mathbf{L}})_{\mathbf{L}}^{\mathbf{H}_{2,n}}(g)\big)\varphi\Big)(g_1) := \varphi(g_1 \odot_{\mathbf{H}_{2,n}} g), \quad g, g_1 \in \mathbf{H}_{2,n},\ \varphi \in \mathscr{F}_0.$$

By Proposition 3.5, the subset $\{(X, 0, 0) : X \in \mathbb{R}^{2n+1}\}$ of $\mathbf{H}_{n,2}$ is a subgroup of $\mathbf{H}_{2,n}$ which is isomorphic to the Heisenberg group $\mathbf{H}_n$. Here, we allow ourselves to identify this subgroup with $\mathbf{H}_n$. Let U denote the restriction map from $\mathbf{H}_{2,n}$ to $\mathbf{H}_n$, that is,

$$U(\varphi)(X) = \varphi(X, 0, 0).$$

for any scalar function $\varphi : \mathbf{H}_{2,n} \to \mathbb{C}$. Clearly, if $\varphi \in \mathscr{F}_0$, then $U\varphi$ is in $C_c(\mathbf{H}_n)$, the space of continuous functions with compact support on $\mathbf{H}_n$. In fact, a function $\varphi \in \mathscr{F}_0$ is completely determined by its restriction to $\mathbf{H}_n$ since the Lie algebra of $\mathbf{H}_n$ within $\mathfrak{h}_{n,2}$ complements $\mathfrak{l}$. With this observation it is easy to check that U is a linear isomorphism from $\mathscr{F}_0$ to $C_c(\mathbf{H}_n)$. Since $C_c(\mathbf{H}_n)$ is dense in the Hilbert space $L^2(\mathbf{H}_n)$, the proof will be complete once we have shown that the induced representation $\mathrm{ind}(\rho_{F,\mathbf{L}})_{\mathbf{L}}^{\mathbf{H}_{2,n}}$ intertwined with U coincides with the representation π_λ acting on $C_c(\mathbf{H}_n)$, that is,

$$\forall g \in \mathbf{H}_{2,n},\ \forall \varphi \in \mathscr{F}_0 \qquad U\left[\mathrm{ind}(\rho_{F,\mathbf{L}})_{\mathbf{L}}^{\mathbf{H}_{2,n}}(g)(\varphi)\right] = \pi_\lambda(g)(U\varphi). \tag{3.26}$$

Let us prove (3.26). We fix a function $\varphi \in \mathscr{F}_0$. By Lemma 3.8 Part (3), we have for $g = (\mathcal{P}, \mathcal{Q}, \mathcal{S})$ and $g_1 = (X, 0, 0) \in \mathbf{H}_n$

$$\begin{aligned}\left(\left(\mathrm{ind}(\rho_{F,\mathbf{L}})_{\mathbf{L}}^{\mathbf{H}_{2,n}}(g)\right)\varphi\right)(X,0,0) &= \varphi\left((X,0,0) \odot_{\mathbf{H}_{2,n}} (\mathcal{P},\mathcal{Q},\mathcal{S})\right)\\ &= \varphi\left(\ell \odot_{\mathbf{H}_{2,n}} (X\cdot\mathcal{P},0,0)\right)\end{aligned}$$

with

$$\ell = \left(0, \mathcal{Q}', \mathcal{S} + \left\langle \mathcal{Q}, X\cdot(\frac{1}{2}\mathcal{P})\right\rangle\right) \in \mathbf{L},$$

for some $\mathcal{Q}' \in \mathbb{R}^{2n+1}$. Since φ is in $\mathscr{F}_0$, it satisfies (3.25) and we have

$$\begin{aligned}\varphi\left(\ell \odot_{\mathbf{H}_{2,n}} (X\cdot\mathcal{P},0,0)\right) &= \rho_{F,\mathbf{L}}(\ell)\,\varphi(X\cdot\mathcal{P},0,0)\\ &= e^{2\pi i\lambda(\mathcal{S}+\langle \mathcal{Q}, X\cdot(\frac{1}{2}\mathcal{P})\rangle)}\varphi(X\cdot\mathcal{P},0,0)\end{aligned}$$

by (3.24). We recognise $\pi_\lambda(g)f(X)$ with $f = U\varphi$ due to (3.15). Therefore, Formula (3.26) is proved, and the proof is complete. □

Let us now give concrete realizations in $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R})$ of the unirreps associated with the co-adjoint orbits of Cases (2) and (3) in Proposition 3.12:

Proposition 3.15. • (Case (2)) *Let* $F_2 := f_w X_w^* + f_x X_x^* + f_y X_y^* + f_z X_z^* \in \mathfrak{h}_{n,2}^*$ *with* $f_s = 0$ *but* $f_w \neq 0$. *A maximal (polarising) subalgebra subordinated to* F_2 *is*

$$\mathfrak{l}_2 := \mathbb{R}^n X_v \oplus \mathbb{R}X_w \oplus \mathbb{R}^n X_x \oplus \mathbb{R}^n X_y \oplus \left\{\frac{z}{2f_w} f_x X_u + zX_z : z \in \mathbb{R}\right\} \oplus \mathbb{R}X_s.$$

The associated unirrep of $\mathbf{H}_{2,n}$ *may be realized as the representation* $\pi_{(f_w,f_x,f_y,f_z)}$ *acting* *unitarily on* $L^2(\mathbb{R}^n)$ *via*

$$\begin{aligned}\left(\pi_{(f_w,f_x,f_y,f_z)}(g)\psi\right)(\tilde{u}) &= \psi(\tilde{u}+u-\frac{z}{2f_w}f_x)e^{2\pi i(wf_w+xf_x+yf_y+zf_z)}\\ &\quad \exp \pi i\Big\langle 2\tilde{u}+u-\frac{z}{2f_w}f_x, f_w v-\frac{z}{2}f_y\Big\rangle_{\mathbb{R}^n}\end{aligned}$$

for $g=(u,v,w,x,y,z,s)\in\mathbf{H}_{2,n}$, $\psi\in L^2(\mathbb{R}^n)$, *and* $\tilde{u}\in\mathbb{R}^n$.

- (Case (3)) *Let* $F_3 := f_uX_u^*+f_vX_v^*+f_xX_x^*+f_yX_y^*\in\mathfrak{h}_{n,2}^*$ *with* $f_uf_y=f_vf_x$ *and* $f_s=f_w=f_z=0$ *but* $(f_x,f_y)\neq 0$. *A maximal (polarising) subalgebra subordinated to* F_3 *is*

$$\mathfrak{l}_3 := \mathbb{R}^nX_u\oplus\mathbb{R}^nX_v\oplus\mathbb{R}X_w\oplus\mathbb{R}^nX_x\oplus\mathbb{R}^nX_y\oplus\mathbb{R}X_s.$$

The associated unirrep of $\mathbf{H}_{2,n}$ *may be realized as the representation* $\pi_{(f_u,f_v,f_x,f_y)}$ *acting unitarily on* $L^2(\mathbb{R})$ *via*

$$\begin{aligned}\left(\pi_{(f_u,f_v,f_x,f_y)}(g)\psi\right)(\tilde{z}) &= \psi(\tilde{z}+z)e^{2\pi i(f_uu+f_vv+f_xx+f_yy)}\\ &\quad \exp \pi i\Big(\frac{2\tilde{z}+z}{2}(-f_xv+f_yu)\Big),\end{aligned}$$

for $g=(u,v,w,x,y,z,s)\in\mathbf{H}_{2,n}$, $\psi\in L^2(\mathbb{R})$, *and* $\tilde{z}\in\mathbb{R}$.

Proof. In both cases, we proceed as in the proof of Proposition 3.14.

For Case (2), we have the following identity with $g=(u,v,w,x,y,z,s)\in\mathbf{H}_{2,n}$

$$\begin{aligned}(\tilde{u},0,\dots,0)\odot_{\mathbf{H}_{2,n}} g &= \Big(\tilde{u}+u,v,w+\frac{\tilde{u}v}{2},x,y-\frac{z}{4}\tilde{u},z,s_1\Big)\\ &= \Big(\frac{z}{2f_w}f_x,v,w+\frac{1}{2}v(2\tilde{u}+u-\frac{z}{2f_w}f_x),x,y-\frac{1}{4}z(2\tilde{u}+u-\frac{z}{2f_w}f_x),z,s_2\Big)\\ &\qquad \odot_{\mathbf{H}_{2,n}}\Big(\tilde{u}+u-\frac{z}{2f_w}f_x,0,\dots,0\Big),\end{aligned}$$

for some s_1 and s_2 we do not need to compute. This yields that the unirrep of $\mathbf{H}_{2,n}$ associated with F_2 and $\mathfrak{l}_2$ may be realized as the unitary representation $\pi_{(f_w,f_x,f_y,f_z)}$ acting on $L^2(\mathbb{R}^n)$ via

$$\begin{aligned}\left(\pi_{(f_w,f_x,f_y,f_z)}(g)\psi\right)(\tilde{u}) &= \psi(\tilde{u}+u-\frac{z}{2f_w}f_x)\exp 2\pi i\Big((w+\frac{1}{2}v(2\tilde{u}+u-\frac{z}{2f_w}f_x))f_w\Big)\\ &\quad \exp 2\pi i\Big(xf_x+(y-\frac{1}{4}z(2\tilde{u}+u-\frac{z}{2f_w}f_x))f_y+f_zz\Big),\end{aligned}$$

for $g=(u,v,w,x,y,z,s)\in\mathbf{H}_{2,n}$, $\psi\in L^2(\mathbb{R}^n)$, and $\tilde{u}\in\mathbb{R}^n$.

For Case (3), we have for some $s_1, s_2 \in \mathbb{R}$,

$$\begin{aligned}\exp(\tilde{z}X_z) \odot_{\mathbf{H}_{2,n}} g &= (u, v, w, x - \frac{\tilde{z}}{4}v, y + \frac{\tilde{z}}{4}u, \tilde{z} + z, s_1)\\ &= \left(u, v, w, x - \frac{1}{4}(2\tilde{z} + z)v, y + \frac{1}{4}(2\tilde{z} + z)u, 0, s_2\right) \odot_{\mathbf{H}_{2,n}} \exp_{\mathbf{H}_{2,n}}\left((\tilde{z} + z)X_z\right).\end{aligned}$$

This yields that the unirrep of $\mathbf{H}_{2,n}$ associated with F_3 and $\mathfrak{l}_3$ may be realized as the unitary representation π_{F_3} acting on $L^2(\mathbb{R})$ via

$$\begin{aligned}\left(\pi_{(f_u,f_v,f_x,f_y)}(g)\psi\right)(\tilde{z}) = \psi(\tilde{z} + z)\exp 2\pi i\Big(f_u u + f_v v + f_x(x - \frac{1}{4}(2\tilde{z} + z)v)\\ + f_y(y + \frac{1}{4}(2\tilde{z} + z)u)\Big),\end{aligned}$$

for $g = (u, v, w, x, y, z, s) \in \mathbf{H}_{2,n}$, $\psi \in L^2(\mathbb{R})$, and $\tilde{z} \in \mathbb{R}$. □

By Kirillov's orbit method [45, 10], Propositions 3.12, 3.14, and 3.15 imply the following classification of the unitary dual of the Dynin-Folland group:

Theorem 3.16. *Any unitary irreducible representation of the Dynin-Folland group $\mathbf{H}_{2,n}$ is unitarily equivalent to exactly one of the following representations:*

- π_λ *for* $\lambda \in \mathbb{R}\backslash\{0\}$, *acting on* $L^2(\mathbf{H}_n)$, *defined in Proposition 3.9,*
- $\pi_{(f_w,f_x,f_y,f_z)}$ *for any* $f_x, f_y \in \mathbb{R}^n$, $f_z \in \mathbb{R}$ *and* $f_w \in \mathbb{R}\backslash\{0\}$, *acting on* $L^2(\mathbb{R}^n)$, *defined in Proposition 3.15,*
- $\pi_{(f_u,f_v,f_x,f_y)}$ *for any* $f_u, f_v, f_x, f_y \in \mathbb{R}^n$ *with* $f_u f_y = f_v f_x$ *but* $(f_x, f_y) \neq 0$, *acting on* $L^2(\mathbb{R})$, *defined in Proposition 3.15,*
- *the characters* π_{f_u,f_v,f_z} *given by*

$$\pi_{f_u,f_v,f_z} : (u, \ldots, s) \in \mathbf{H}_{2,n} \longmapsto e^{2\pi i(uf_u + vf_v + zf_z)},$$

  for any $f_u, f_v \in \mathbb{R}^n$ *and* $f_z \in \mathbb{R}$.

3.5.3 Jump Sets

For the sake of usefulness at some later stage we will describe each orbit's set of jump indices. For a detailed account on jump indices we refer the reader to [10] Section 3.1. Our use of jump sets and related notions below essentially follows Pedersen's exposition in [51].

To start with, let us recall that for any n-dimensional nilpotent Lie group G there exists a sequence of ideals $\mathfrak{g}_j \subseteq \mathfrak{g}$, $\dim(\mathfrak{g}_j) = j$, $j = 1, \ldots, n$, with

$$\{0\} \subseteq \mathfrak{g}_1 \subseteq \ldots \subseteq \mathfrak{g}_n = \mathfrak{g}, \tag{3.27}$$

$\dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1$ and $[\mathfrak{g}, \mathfrak{g}_j] \subseteq \mathfrak{g}_{j-1}$, $j = 1, \ldots, n$.

For every such flag of ideals in fact there exists a basis $\{X_j\}_{j=1}^n$ for $\mathfrak{g}$ such that $X_j \in \mathfrak{g}_j/\mathfrak{g}_{j-1}$ for all $j = 1, \ldots, n$. Such a basis is often referred to as a Jordan-Hölder basis. (Cf. [10] Theorems 1.1.9 and 1.1.13.)

Let us recall that for a fixed Jordan-Hölder basis $\{X_j\}_{j=1}^n$, an arbitrary but fixed co-adjoint orbit $\mathcal{O}$ and some representative $F_{\mathcal{O}}$ the jump set $e_{\mathcal{O}}$ consists of those indices $j_1, \ldots, j_{2d} \in \{1, \ldots, n\}$ which satisfy $\mathfrak{g}_j \nsubseteq \mathfrak{g}_{j-1} + \operatorname{stab}(F_{\mathcal{O}})$.

Proposition 3.17. *Let $\mathcal{O}_1, \ldots, \mathcal{O}_4$ denote the co-adjoint orbits of the Dynin-Folland group (for arbitrary, but fixed constants in each case), classified in Proposition 3.12. Then the corresponding jump sets for each orbit are the following:*

(Case (1)) $e_{\mathcal{O}_1} = \{x_1, \ldots, x_n, y_1, \ldots, y_n, z, w, u_1, \ldots, u_n, v_1, \ldots, v_n\}$,

(Case (2)) $e_{\mathcal{O}_2} = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$,

(Case (3)) $e_{\mathcal{O}_3} = \{z, u_j\}$ *or* $e_{\mathcal{O}_3} = \{z, v_k\}$, *for some j or some k in $\{1, \ldots, n\}$, where the second index is determined by the vector $(f_y, f_x) = (f_{y_1}, \ldots, f_{x_n}) \neq 0 \in \mathbb{R}^{2n}$,*

(Case (4)) $e_{\mathcal{O}_4} = \varnothing$.

Proof. In the case of $G = \mathbf{H}_{2,n}$ it is easily checked that the basis

$$B := \{X_s, X_{x_1}, \ldots, X_{x_n}, X_{y_1}, \ldots, X_{y_n}, X_z, X_w, X_{u_1}, \ldots, X_{u_n}, X_{v_1}, \ldots, X_{v_n}\}, \tag{3.28}$$

form in fact a Jordan-Hölder basis.

Cases (1), (2) and (4) follow immediately from Corollary 3.13.

Case (3) also uses Corollary 3.13 and the specific order in which we nest the Jordan-Höoder flag (3.27). The order is determined up to permutations in the x-, y-, u- and v-variables. If we fix the order of $\mathfrak{g}_{\mathrm{j}}$ to match the order of vectors X_j as in (3.28), then the first non-vanishing summand $f_{x_j} v_j$ or $f_{y_k} u_k$ on the right-hand-side of the equation $f_x v - f_y u = 0$ determines second variable of $e_{\mathcal{O}_3}$. The fact that $z \in e_{\mathcal{O}_3}$ anyway follows from the condition $z = 0$. ☐

3.6 The Semi-direct Product Structure

Let us briefly discuss why the Dynin-Folland group is actually given as a semi-direct product $\mathbb{R}^{2n+2} \rtimes_\alpha \mathbf{H}_n$. For this purpose we recall that given two simply connected nilpotent Lie groups H and N, and a map $\tau : \mathfrak{h} \to \mathrm{Der}(\mathfrak{n})$, there exists a simply connected nilpotent Lie group G and a map $\alpha : H \to \mathrm{Aut}(N)$ such that $G = N \rtimes_\alpha H$ with Lie algebra $\mathfrak{g} = \mathfrak{n} \oplus_{d\bar{\alpha}} \mathfrak{h}$ and such that $d\bar{\alpha} = \tau$, where $\bar{\alpha}(h) := d(\alpha(h)(\,.\,)) \in \mathrm{Aut}(\mathfrak{n})$. We recall that in this case the Lie bracket on $\mathfrak{g}$ is given by

$$[(X_\mathfrak{h}, Y_\mathfrak{n}), (X'_\mathfrak{h}, Y'_\mathfrak{n})]_\mathfrak{g} = [X_\mathfrak{h}, X'_\mathfrak{h}]_\mathfrak{h} + d\bar{\alpha}(X)(Y') - d\bar{\alpha}(X')(Y) + [Y_\mathfrak{n}, Y'_\mathfrak{n}]_\mathfrak{n}. \tag{3.29}$$

(For details see A. Knapp [46] Theorem 1. 125.) As we seek to write $\mathbf{H}_{2,n} = \mathbb{R}^{2n+2} \rtimes_\alpha \mathbf{H}_n$, we first recall that by Lemma 3.7 Formula (3.12) we have

$$\begin{aligned} &[(\mathcal{P}, \mathcal{Q}, \mathcal{S}), (\mathcal{P}', \mathcal{Q}', \mathcal{S}')]_{\mathfrak{h}_{2,n}} \\ &\qquad = \Big([\mathcal{P}, \mathcal{P}']_{\mathfrak{h}_n}, \frac{1}{2}\big(\mathrm{ad}^*_{\mathbf{H}_\mathrm{n}}(\mathcal{P})(\mathcal{Q}') - \mathrm{ad}^*_{\mathbf{H}_\mathrm{n}}(\mathcal{P}')(\mathcal{Q})\big), \big\langle \mathcal{P}, \mathcal{Q}' \big\rangle - \big\langle \mathcal{Q}, \mathcal{P}' \big\rangle \Big) \\ &\qquad := (\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{S}}). \end{aligned} \tag{3.30}$$

Bearing $[\,.\,,\,.\,]_{\mathbb{R}^{2n+2}} = 0$ in mind, we immediately recognize that the components $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{S}}$ represent the two $d\bar{\alpha}$-terms in Equality (3.29). In view of Equality (3.10), we may hence conclude that

$$\tau_{\mathbf{H}_{2,n}} : \mathcal{P} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \\ v_1 \\ \vdots \\ v_n \\ w \end{pmatrix} \mapsto \begin{pmatrix} 0 & \cdots & & & \cdots & 0 & \frac{1}{2}v_1 & 0 \\ \vdots & \ddots & & & & \vdots & \vdots & \vdots \\ & & & & & & \frac{1}{2}v_n & \\ & & & & & & \frac{1}{2}u_1 & \\ & & & & \ddots & & \vdots & \\ \vdots & & & & & 0 & -\frac{1}{2}u_n & \\ 0 & \cdots & & & \cdots & 0 & 0 & \vdots \\ u_1 & \cdots & u_n & v_1 & \cdots & v_n & w & 0 \end{pmatrix},$$

indeed defines a linear map from $\mathfrak{h}_n$ into $\mathrm{Der}(\mathbb{R}^{2n+2})$, for wich the associated map $\alpha_{\mathbf{H}_{2,n}}$ defines a semi-direct product $\mathbb{R}^{2n+2} \rtimes_{\alpha_{\mathbf{H}_{2,n}}} \mathbf{H}_n$ with Lie bracket given by (3.30).

3.7 Group Fourier Transform and Plancherel formula

In this subsection we study the group Fourier transform on $\mathbf{H}_{2,n}$. In particular we obtain the Plancherel formula for the Folland-Dynin group.

The group Fourier transform of an integrable function $f \in L^1(\mathbf{H}_{2,n})$ is defined via the integral (convergent in norm)

$$\widehat{f}(\tau) = \int_{\mathbf{H}_{2,n}} f(g)\tau(g)^* dg$$

for any unirrep τ of $\mathbf{H}_{2,n}$.

For the Schrödinger-type representations π_λ, defined in Proposition 3.15, one can describe the corresponding group Fourier transform as follows:

Theorem 3.18. *(i) Let $f \in L^1(\mathbf{H}_{2,n})$, and let $\lambda \in \mathbb{R}\backslash\{0\}$ be fixed. Then the operator $\widehat{f}(\pi_\lambda)$ acts on $L^2(\mathbf{H}_n)$, with integral kernel given by the locally integrable distribution K^f_λ defined via*

$$K^f_\lambda(X,Y) := \iint\limits_{\mathbb{R}^{2n+2}} f(Y^{-1} \cdot X, \mathcal{Q}, \mathcal{S})\, e^{-2\pi i\lambda\mathcal{S}}\, e^{-\pi i\lambda\langle\mathcal{Q}, X+Y\rangle}\, d\mathcal{Q}\, d\mathcal{S}.$$

(ii) Furthermore, if $f \in L^1(\mathbf{H}_{2,n}) \cap L^2(\mathbf{H}_{2,n})$, then the operator $\widehat{f}(\pi_\lambda)$ is in the Hilbert-Schmidt class with Hilbert-Schmidt norm

$$\begin{aligned} \|\widehat{f}(\pi_\lambda)\|^2_{HS} &= \int_{\mathbf{H}_n \times \mathbf{H}_n} |K^f_\lambda(X,Y)|^2 dX\, dY \\ &= |\lambda|^{-(2n+1)}\, \|\mathscr{F}_{s\to\lambda} f\|_{L^2(\mathbb{R}^{2(2n+1)})} \end{aligned} \tag{3.31}$$

where $\mathscr{F}_{s\to\lambda} f$ denotes the Fourier transform of f with respect to the central variable, that is,

$$\left(\mathscr{F}_{s\to\lambda} f\right)(X,Y) := \int_{\mathbb{R}} f(X,Y,\mathcal{S})\, e^{-2\pi i\lambda\mathcal{S}}\, d\mathcal{S}.$$

Consequently,

$$\int_{\mathbf{H}_{2,n}} |f(g)|^2 dg = \int_{\mathbb{R}\backslash\{0\}} \|\widehat{f}(\pi_\lambda)\|^2_{HS} |\lambda|^{2n+1} d\lambda. \tag{3.32}$$

Proof. Let us prove Part (i). Let $\varphi \in \mathscr{S}(\mathbf{H}_n)$. We then have

$$\begin{aligned}\big(\widehat{f}(\pi_\lambda)\varphi\big)(X) &= \int_{\mathbf{H}_{2,n}} f(\mathcal{P},\mathcal{Q},\mathcal{S})\,\big(\pi_\lambda(\mathcal{P},\mathcal{Q},\mathcal{S})^*\varphi\big)(X)\,d\mathcal{P}\,d\mathcal{Q}\,d\mathcal{S}\\ &= \int_{\mathbf{H}_{2,n}} f(\mathcal{P},\mathcal{Q},\mathcal{S})\,\big(\pi_\lambda(-\mathcal{P},-\mathcal{Q},-\mathcal{S})\varphi\big)(X)\,d\mathcal{P}\,d\mathcal{Q}\,d\mathcal{S}\\ &= \int_{\mathbf{H}_{2,n}} f(\mathcal{P},\mathcal{Q},\mathcal{S})\,e^{-2\pi i\lambda\mathcal{S}}\,e^{2\pi i\lambda\left\langle -\mathcal{Q},X\cdot(-\frac{1}{2}\mathcal{P}\right\rangle}\,\varphi(X\cdot\mathcal{P}^{-1})\,d\mathcal{P}\,d\mathcal{Q}\,d\mathcal{S}.\end{aligned}$$

We now apply the change of variables $\mathcal{P}\mapsto Y := X\cdot\mathcal{P}^{-1}$. We observe that $dY = d\mathcal{P}$, and, using $\mathcal{P} = Y^{-1}\cdot X = -(X^{-1}\cdot Y)$, that

$$\begin{aligned}X\cdot\Big(-\frac{1}{2}\mathcal{P}\Big) &= X\cdot\big(\frac{1}{2}\,(X^{-1}\cdot Y)\big) = X + \frac{1}{2}(X^{-1}\cdot Y) + \frac{1}{4}[X, X^{-1}\cdot Y]\\ &= X + \frac{1}{2}\big(-X+Y-\frac{1}{2}[X,Y]\big) + \frac{1}{4}\big(-[X,X]+[X,Y]-\frac{1}{2}[X,[X,Y]]\big)\\ &= \frac{1}{2}(X+Y).\end{aligned}$$

Therefore, we obtain

$$\big(\widehat{f}(\pi_\lambda)\varphi\big)(X) = \int_{\mathbf{H}_n} K^f_\lambda(X,Y)\varphi(Y)\,dY,$$

with K^f_λ as in the statement above. We observe that K^f_λ is the composition of the Euclidean Fourier transform in the $\mathcal{Q}$-variable of $\mathscr{F}_{\mathcal{S}\to\lambda}f$ composed with the smooth diffeomorphism $X \mapsto Y^{-1}\cdot X = X'$ and then $Y \mapsto \frac{\lambda}{2}(Y\cdot X' + Y) = Y'$. Since f is integrable, the kernel $K^f_\lambda(X,Y)$ makes sense as a locally integrable distribution on $\mathbf{H}_n\times\mathbf{H}_n$ by the properties of the Euclidean Fourier transform.

In order to prove Part (ii), let us compute the L^2-norm of the kernel K^f_λ. First we apply the change of variables $X' = Y^{-1}\cdot X$, which has Jacobian determinant 1, and then $Y' = \frac{\lambda}{2}(Y \cdot X' + Y)$, which has Jacobian determinant $|\lambda|^{2n+1}$, to obtain

$$\begin{aligned}
&\int_{\mathbf{H}_n \times \mathbf{H}_n} |K_\lambda^f(X,Y)|^2 dX\, dY \\
&= \iint\limits_{\mathbb{R}^{2(2n+1)}} \left| \int_{\mathbb{R}^{2n+1}} \mathscr{F}_{\mathcal{S}\to\lambda} f(X', \mathcal{Q}) e^{-\pi i \lambda \langle \mathcal{Q}, Y\cdot X' + Y\rangle}\, d\mathcal{Q} \right|^2 dX' dY \\
&= |\lambda|^{2n+1} \iint\limits_{\mathbb{R}^{2(2n+1)}} \left| \int_{\mathbb{R}^{2n+1}} \mathscr{F}_{\mathcal{S}\to\lambda} f(X', \mathcal{Q}) e^{-2\pi i \langle \mathcal{Q}, Y'\rangle}\, d\mathcal{Q} \right|^2 dX' dY' \\
&= |\lambda|^{-(2n+1)} \iint\limits_{\mathbb{R}^{2(2n+1)}} \left| \mathscr{F}_{\mathcal{S}\to\lambda} f(X', \mathcal{Q}) \right|^2 dX' d\mathcal{Q},
\end{aligned}$$

having used the properties of the Euclidean Fourier transform on $\mathbb{R}^{2n+1}$. (Here we use our standard convention $\mathscr{F}(f)(\xi) := \int f(x) e^{-i2\pi x\xi}\, dx$.) Clearly, the L^2-norm of K_λ^f is finite since $f \in L^1(\mathbf{H}_{2,n}) \cap L^2(\mathbf{H}_{2,n})$. Equivalently, the operator $\widehat{f}(\pi_\lambda)$ is Hilbert-Schmidt with operator norm given by the L^2-norm of K_λ^f. Thus (3.31) is proved.

Now we integrate (3.31) against $|\lambda|^{2n+1}$ and use the property of the Euclidean Fourier transform to obtain (3.32). This concludes the proof. □

Formula (3.32) is the *Plancherel formula*. It implies that the definition of the group Fourier transform may be extended unitarily from $L^1(G) \cap L^2(G)$ to $L^2(G)$.

The Plancherel formula can be also deduced from the orbit method, cf [10, Theorem 4.3.9]. As expected our expression for the Plancherel formula involves only the representations of Schrödinger-type π_λ since these representations correspond to the generic orbits, see Proposition 3.12, Case (1).

4 Modulation Spaces of the Dynin-Folland Group

In this Chapter we will finally discuss one possible answer to the question about modulation spaces on the Heisenberg group. Our approach is a blend of Feichtinger and Gröchenig's original coorbit approach and Daniel and Ingrid Beltiţă's adapted framework for nilpotent Lie groups. The latter is strongly based on useful techniques developed in Pedersen [50, 51].

After a brief review of some basic ideas of the Pedersen-Beltiţă setting in Section 4.1 and some preliminary results on semi-direct product nilpotent Lie groups in Section 4.2, we will study the modulation spaces that arise from the four types of $\mathbf{H}_{2,n}$-unirreps classified by Theorem 3.16. We will proceed case-by-case following the theorem.

For the sake of convenience, we will use a simplified notation for the co-adjoint orbits of $\mathbf{H}_{2,n}$ and the corresponding unirreps throughout this chapter: orbits will be denoted by $\mathcal{O}_j$ and unirreps by π_j, where $j = 1, \dots, 4$ according to the four cases of co-adjoint orbits (cf. Proposition 3.12). Note that we have to assume arbitrary but fixed constants $f_u, f_v, f_w, f_x, f_y, f_z, f_s$ for each case as the particular choice of constants determines the individual orbit in each class.

Let us point out that many classical accounts on modulation spaces make use of the reduced Heisenberg group $\mathbf{H}_{n,red}$ in order to operate within the classical coorbit framework. (Cf. [20, 17, 40], e.g.) In contrast to this practice, we make no use of any reduced versions of the Dynin-Folland group, but instead employ the quotient group $\mathbf{H}_{2,n}/P$, where P stands for the projective kernel of the involved unirrep.

Let us furthermore remark that we denote our modulation spaces by $M^{r,s}_{\varphi}(\pi)$, where *varphi* denotes the analyzing window and π the specific unirrep which gives rise to $M^{r,s}_{\varphi}(\pi)$; the exponents r and s were chosen for the trivial reason that p and q are already in use to denote elements $(p, q, t) \in \mathbf{H}_n$.

4.1 Modulation Spaces induced by Unirreps of Nilpotent Lie Groups

In this section we want to give a brief review of Daniel and Ingrid Beltiţă's approach to modulation spaces induced by unitary irreducible representations of nilpotent Lie groups. It is strongly based on the Weyl-Pedersen calculus introduced in [50, 51] since their papers [5, 6] analyze mapping properties between modulation spaces of Weyl-quantized operators with modulation space-valued symbols.

In particular, it focuses on the relation between conjoint orbits and certain subspaces of the Lie algebra $\mathfrak{g}$ isomorphic to them as it allows to use the corresponding unirreps π without having to take care of their projective kernels, i.e., the subgroups of G on which the unirreps reduce to periodic exponential multiples of the identity. In other words, it permits the use of unirreps which are not necessarily square-integrable over the whole of G as required in the original coorbit approach by Feichtinger and Gröchenig. (Cf. [20, 21, 22].) Thus the notions employed in [5, 6] are rather based on subalgebras of $\mathfrak{g}$ than on the whole group G or subgroups of it.

Since in our case we are mainly interested in a specific instance of modulation spaces, namely those induced by the Dynin-Folland group $\mathbf{H}_{n,2}$, and whether these happen to be independent of a special parameter in the definition, the so-called analyzing window vector, we will employ equivalent but slightly altered definitions and notions more in the spirit of the original coorbit. Our techniques rely on the fact that the projective kernel Lie subalgebras of $\mathbf{H}_{2,n}$ are actually ideals, hence giving rise to normal subgroups of G. It is for this reason and due to some technical issues that we resort to work with groups as in [21], e.g.

Yet as it is a priori not clear which conditions allow for this approach, we will start with the Beltiţă-framework, adapting it to our case as we progress.

We start with a connected, simply connected nilpotent Lie group G of dimension n and an arbitrary unitary irreducible representation $\pi : G \to \mathscr{U}(\mathcal{H}_\pi)$ corresponding to a uniquely determined co-adjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$. Let us recall that each orbit is a symplectic manifold equipped with a canonical $\mathrm{Ad}^*(G)$-invariant measure, here denoted by $\beta_{\mathcal{O}}$. Note that $\beta_{\mathcal{O}}$ is uniquely determined up to a positive multiplicative constant, which is usually chosen to suit certain formulas.

An object of particular importance is the following subspace $\mathfrak{g}_e \subseteq \mathfrak{g}$, to which we will refer as the predual of $\mathcal{O}$: let us recall that for an arbitrary but fixed representative

$F_{\mathcal{O}} \in \mathfrak{g}^*$ of $\mathcal{O}$, i.e., $\mathrm{Ad}^*_{\mathrm{G}}(G)F_{\mathcal{O}} = \mathcal{O}$ its stabilizer group is defined by

$$\mathrm{Stab}(F_{\mathcal{O}}) := \{g \in G \mid \mathrm{Ad}^*_{\mathrm{G}}(g)F_{\mathcal{O}} = F_{\mathcal{O}}\}.$$

$\mathrm{Stab}(F_{\mathcal{O}})$ is obviously a subgroup of G and G can be viewed as a fibre bundle over the base $\mathcal{O} \cong G/\mathrm{Stab}(F_{\mathcal{O}})$. Let us denote the Lie algebra of $\mathrm{Stab}(F_{\mathcal{O}})$ by $\mathrm{stab}(F_{\mathcal{O}})$.

Definition 4.1. *Let G be nilpotent Lie group. Let $\mathcal{O}$ be one of its co-adjoint orbits and $2d := \dim(\mathcal{O})$. Given any Jordan-Hölder sequence*

$$\mathscr{F} : \{0\} = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_n = \mathfrak{g},$$

i.e., $\dim(\mathfrak{g}_j) = j$ and $[\mathfrak{g}, \mathfrak{g}_j] \subseteq \mathfrak{g}_{j-1}$, let $\{X_j\}_j$, with $X_j \in \mathfrak{g}_j/\mathfrak{g}_{j-1}$, be a basis of $\mathfrak{g}$, thus a so-called Jordan-Hölder basis for $\mathfrak{g}$. For any such basis $\{X_j\}_j$ we then define the set of jump indices, or simply jump set, of $\mathcal{O}$ by

$$\begin{aligned} e := e_{\mathcal{O}} :=& \{1 \leqslant j_1 \leqslant \ldots \leqslant j_{2d} \leqslant n \mid \mathfrak{g}_j \nsubseteq \mathfrak{g}_{j-1} + \mathrm{stab}(F_0)\} \\ =& \{1 \leqslant j_1 \leqslant \ldots \leqslant j_{2d} \leqslant n \mid X_j \notin \mathfrak{g}_{j-1} + \mathrm{stab}(F_0)\}. \end{aligned}$$

Then predual of $\mathcal{O}$ is $\mathfrak{g}_e$ is defined to be the linear span of $\{X_{j_k} \mid j_k \in e\}$.

Remark 4.2. Although the notion of jump indices is frequently used in representation theory the name predual seems to go back to [4].

We immediately notice that by the definition of $\mathrm{Stab}(F_{\mathcal{O}})$ the Lie algebra $\mathfrak{g}$ is given as the direct sum $\mathfrak{g} = \mathfrak{g}_e \oplus \mathrm{stab}(F_{\mathcal{O}})$.

Given an orbit $\mathcal{O}$, a representative $F_{\mathcal{O}}$ and the corresponding jump set $e = e_{\mathcal{O}}$, a useful result by Pedersen yields that the map

$$\phi : \mathcal{O} \to \mathbb{R}^{2d} : F \mapsto \big(\langle F, X_{j_1}\rangle, \ldots, \langle F, X_{j_{2d}}\rangle\big) \tag{4.1}$$

defines a global chart of the manifold $\mathcal{O}$ which pushes $\beta_{\mathcal{O}}$ forward to the $2d$-dimensional Lebesgue measure (modulo some positive multiplicative constant). That is, if we elegantly ignore the isomorphism $\mathbb{R}^{2d} \cong \mathfrak{g}_e$, ϕ yields in fact a global diffeomorphism between $\mathcal{O}$ and $\mathfrak{g}_e$. It worthwhile mentioning that this chart is in fact polynomial. (Cf. [50] Subsection 1.6 p. 521.)

Let us mention that we will frequently work with both G and $\mathfrak{g}$; our preferred coordinate system on G will be the so-called exponential coordinates, also called canonical coordinates of the first kind: given a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ the associated coordinates on G are defined by

$$\exp : \mathbb{R}^n \cong \mathfrak{g} \to G : (t_1, \dots, t_n) \mapsto \exp_G\Big(\sum_{j=1}^{n} t_j x_j\Big).$$

Modulation spaces now enter the arena via the so-called ambiguity function. In order to define the latter appropriately, we have to make use of the space of smooth vectors of π, which we denote by $\mathcal{H}_\pi^\infty$. (For further details confer the Appendix of [10], in particular Section A.1 p. 226.)

Definition 4.3. *Let G be a connected, simply connected nilpotent Lie group and let π be a unitary irreducible representation of G on $\mathcal{H}_\pi$ corresponding to some co-adjoint orbit $\mathcal{O}$. Furthermore, let $\mathfrak{g}_e$ be the predual of $\mathcal{O}$. Then for any $f \in (\mathcal{H}_\pi^\infty)'$ and any $\varphi \in \mathcal{H}_\pi^\infty$ we define the ambiguity function of f with respect to the window vector φ by*

$$\begin{aligned} A_\varphi^\pi f : \mathfrak{g}_e &\to \mathbb{C}, \\ X &\mapsto \langle f, \pi(\exp X)\varphi\rangle_{(\mathcal{H}_\pi^\infty)'}, \end{aligned}$$

where $\langle\,.\,,\,.\,\rangle_{(\mathcal{H}_\pi^\infty)'}$ denotes the sesqui-linear $(\mathcal{H}_\pi^\infty)'$-$\mathcal{H}_\pi^\infty$-duality that coincides with the $\mathcal{H}_\pi$-inner product in case $f \in \mathcal{H}_\pi$.

Definition 4.4 (Modulation Spaces Ascending from Co-adjoint Orbits)**.** *Let G and π be as in Definition 4.3. Let furthermore $\mathfrak{g}_e = \mathfrak{g}_{e1} \oplus \mathfrak{g}_{e2}$ be a direct sum decomposition of $\mathfrak{g}_e$ and let $\varphi \in \mathcal{H}_\pi^\infty \backslash \{0\}$. For $r, s \in [1, \infty]$ we then define the modulation space $M_\varphi^{r,s}(\pi)$ for the unitary irreducible representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ with respect to the decomposition $\mathfrak{g}_e = \mathfrak{g}_{e1} \oplus \mathfrak{g}_{e2}$ and the analyzing window φ to be the space of all $f \in (\mathcal{H}_\pi^\infty)'$ such that*

$$\|f\|_{M_\varphi^{r,s}(\pi)} := \big\|A_\varphi^\pi f\big\|_{L^{r,s}(\mathfrak{g}_{e1} \oplus \mathfrak{g}_{e2})} = \bigg(\int_{\mathfrak{g}_{e2}} \bigg(\int_{\mathfrak{g}_{e1}} \big|A_\varphi^\pi f(X_1, X_2)\big|^r \, dX_1\bigg)^{s/r} dX_2\bigg)^{1/s} \tag{4.2}$$

is finite, with the obvious changes whenever some $r, s = \infty$.

Remark 4.5. [On the Direct Sum Decomposition] Let us emphasize that the number of precisely two summands in the direct sum decomposition is arbitrary and can be extended to any number up to the topological dimension of $\mathfrak{g}_e$. An important reason for this specific choice is the endeavour to define spaces with properties similar to those displayed by the classical modulation spaces $M^{r,s}(\mathbb{R}^n)$.

For our specific case of modulation spaces arising from the unirreps of the Dynin-Folland group $\mathbf{H}_{2,n}$ we will decompose the preduals according to the action of its corresponding unirreps on smooth vectors, that is, by either translating them or modulating them. (Cf. Theorem 3.16 for an explicit description of the $\mathbf{H}_{2,n}$-unirreps.)

4.2 Semi-direct Products and Square-Integrability

A crucial property found in many instances of modulation spaces is their independence of the analyzing window φ. In the general framework of coorbit spaces this is guaranteed for square-integrable unirreps provided the mixed-norm space $L^{r,s}(G)$ is a Banach module over $L^1(G)$. (Cf. [21], Section 4 and in particular Theorem 4.2 (*ii*) as well as p. 311.) For a clarification of the notion of square-integrability we refer to Corwin and Greenleef [10] p. 170 and More and Wolf [48]).

In the case of nilpotent Lie groups square-integrability of a unirrep (in the sense of Moore and Wolf) is given precisely when the corresponding co-adjoint orbit is flat. But a closer look at the orbits of the Dynin-Folland group immediately reveals that they are indeed all flat. So, independence of the analyzing window should in principle, as we may hope, be given for the modulation spaces described in the following subsection. In order to prove this rigorously, however, we will have to show that our mixed-norm spaces $L^{r,s}(G)$ allow for an adopted version of Young's inequality, thus the Banach module property. Whether or not this property is given strongly depends on the decomposition $\mathfrak{g}_e = \mathfrak{g}_{e1} \oplus \mathfrak{g}_{e2}$ and, a fortiori, on the structure of the group G itself.

Let us point out that we will make no further reference to Banach modules nor will we refer to any abstract results from coorbit theory, even though the principal idea behind the proof is an adaption of the abstract coorbit approach. Instead we will prove Young's inequality under certain conditions on the group and use it to provide the crucial estimate to prove independence of the window. Although the core argument itself is classical and well-known, we will write it out for the sake of a better reading.

Certain technicalities in our proof were originally inspired by Beltiţă and Beltiţă's approach in [5] (cf. particularly Theorem 3.3), yet had to be adopted to more relaxed conditions in order to cover all possible instances of modulation spaces arising from the Dynin-Folland group $\mathbf{H}_{2,n}$. To meet our target, we will focus on groups G given as the semi-direct product $G = N \rtimes H$ of two nilpotent groups H and N.

Remark 4.6. Without loss of generality, let us work with the realization $G = NH$, writing elements of G as products nh, with $n \in N$, $h \in H$. This realization of G as a product is in fact a very natural one if we keep in mind that N is a normal subgroup of G, thus rendering H isomorphic to G/N.

Employing this realization, we can give the following definitions.

Definition 4.7. *Let G be a connected simply connected nilpotent Lie group given as the semi-direct product $G = N \rtimes_\alpha H$ of the Lie groups H and N. For any $r, s \in [1,\infty]$ we define the mixed-norm space $L^{r,s}(G)$ as the set of all $f \in \mathscr{S}'(G)$ such that*

$$\|f\|_{L^{r,s}(G)} := \Big(\int_N \Big(\int_H |f(nh)|^r \, dh\Big)^{s/r} dn\Big)^{1/s} < \infty,$$

with the usual modifications for $r = \infty$ and $s = \infty$.

A concept well-known from the theory of modulation spaces on $\mathbb{R}^n$ is the so-called short-time Fourier transform (STFT). In a nutshell, it can be viewed as the family of pointwise matrix coefficients of the combined time-frequency shifts

$$(p,q) \mapsto e^{2\pi i q \cdot} T_p^{\mathbb{R}^n} = \rho(0,q,0)\rho(p,0,0) = \rho\big((0,q,0)(p,0,0)\big),$$

where ρ again denotes the Schrödinger representation of $\lambda = 1$. But let us give the more general definition of STFT for generic unirreps π of nilpotent semi-direct product groups right away.

Definition 4.8. *Let G be as in Definition 4.7 and let π be an irreducible unitary representation of G on $\mathcal{H}_\pi$. Then the short-time Fourier transform of $f \in (\mathcal{H}_\pi^\infty)'$ with respect to the window $\varphi \in \mathcal{H}_\pi^\infty \backslash \{0\}$ is defined by*

$$\begin{aligned} V_\varphi^\pi f : G &\to \mathbb{C}, \\ nh &\mapsto \langle f, \pi(nh)\varphi \rangle_{(\mathcal{H}_\pi^\infty)'}, \end{aligned}$$

where $\langle \,.\,,\,.\, \rangle_{(\mathcal{H}_\pi^\infty)'}$ denotes the sesqui-linear $(\mathcal{H}_\pi^\infty)'$-$\mathcal{H}_\pi^\infty$-duality that coincides with the $\mathcal{H}_\pi$-inner product in case $f \in \mathcal{H}_\pi$.

Its intimate relation with the ambiguity function $A_\varphi^\pi f$ will become clear in the proof of Theorem 4.11. The following two auxiliary results set the stage for the actual proof of independence.

Proposition 4.9. *Let G be a connected, simply connected nilpotent Lie group and let π be a unitary irreducible representation of G on $\mathcal{H}_\pi$ which is square-integrable modulo the projective kernel P, that is, for the subgroup $P := \{x \in G \mid \pi(x) \in \mathbb{C}\,\mathrm{Id}_{\mathcal{H}_\pi}\}$ there exist $\psi_1, \psi_2 \in \mathcal{H}_\pi$ such that*

$$\int_{G/P} \big|\langle \psi_1, \pi(x)\psi_2 \rangle_{\mathcal{H}_\pi}\big|^2 \, d\dot{x} < \infty.$$

In this case P is a normal subgroup of G and we can realize the quotient G/P as $G_e := \exp_G(\mathfrak{g}_e)$.

Proof. The first statement is proved by simply combining Theorems 3.2.3 and 4.5.2 in [10] (see pages 99 and 171, respectively). The latter says that for a nilpotent Lie group G square-integrability modulo the projective kernel P of a unirrep $\pi_{\mathcal{O}}$ is equivalent to the "flat-orbit" condition, that is, that $\mathcal{O}$ is an affine subspace of $\mathfrak{g}^*$. The former theorem in turn says for nilpotent G and a co-adjoint orbit $\mathcal{O}$ with representative $F_{\mathcal{O}}$ we have: the flat-orbit condition $\Leftrightarrow \operatorname{stab}(F_{\mathcal{O}}) = \mathfrak{p} \Leftrightarrow \operatorname{stab}(F_{\mathcal{O}})$ is an ideal of $\mathfrak{g}$. ☐

Lemma 4.10. *Let G be a connected simply connected nilpotent Lie group given as the semi-direct product $G = N \rtimes_\alpha H$ of the nilpotent Lie groups H and N. Then the group convolution on G maps continuously from $L^{r,s}(G) \times L^1(G)$ into $L^{r,s}(G)$. In particular, Young's inequality holds true for the mixed-norm space $L^{r,s}(G)$.*

The proof is similar to the classical one for L^p-spaces and makes use of the fact that dn is invariant under the action of H.

Proof. Let us recall that we can identify N and H in $G = N \rtimes_\alpha H = NH$ with (N, e_H) and (e_N, H), respectively, writing $(n_1, h_1)(n_2, h_2) = \big(\alpha(h_2^{-1})(n_1)n_2, h_1h_2\big)$. An easy calculation now implies that $\operatorname{conj}_{(e_n,h)}\big((n, e_H)\big) = (e_n, h)(n, e_H)(e_n, h) = \big(\alpha(h^{-1})(n), e_H\big)$, i.e., that H acts on N essentially via conjugation. But conj is measure-preserving on unimodular groups, hence in particular on nilpotent Lie groups.

It is therefore easy to see that for the right regular representation R of G on the Banach space $L^{r,s}(G)$ each operator $R(g)$, $g \in G$, is an isometry on $L^{r,s}(G)$. For $f \in L^{r,s}(G), \varphi \in L^1(G)$, we can now regard the integral

$$f * \varphi = \int_G f(\,.\,g)\varphi(g^{-1})dg = \int_G \varphi(g^{-1})R(g)f dg$$

as an $L^{r,s}(G)$-valued Bochner integral which converges since φ is intregable and $\|R(g)f\|_{L^{r,s}(G)} = \|f\|_{L^{r,s}(G)} < \infty$. But a standard estimate for convergent Bochner integrals then yields

$$\left\| \int_G \varphi(g^{-1})R(g)f dg \right\|_{L^{r,s}(G)} \leqslant \int_G \left|\varphi(g^{-1})\right| \|R(g)f\|_{L^{r,s}(G)} \, dg = \|\varphi\|_{L^1(G)} \|f\|_{L^{r,s}(G)} \,.$$

Hence, we have shown $\|f * \varphi\|_{L^{r,s}(G)} \leqslant \|\varphi\|_{L^1(G)} \|f\|_{L^{r,s}(G)}$, which concludes the proof. ☐

We can finally state our main observation. One of its interesting features is an alternative definition for modulation spaces in terms of the STFT.

Theorem 4.11. *Let G be a connected simply connected nilpotent Lie group and let π be a unitary irreducible representation of G on $\mathcal{H}_\pi$ which is square-integrable modulo the projective kernel P. Furthermore, let $G_e := \exp_G(\mathfrak{g}_e)$, indentified with G/P, be given as the semi-direct product $G_{e2} \rtimes_\alpha G_{e1}$ of the nilpotent Lie groups G_{e1} and G_{e2}, and let $r, s \in [1, \infty]$.*

If $V_\varphi^\pi f$ denotes the STFT defined on G_e, we then have

$$\left\|V_\varphi^\pi f\right\|_{L^{r,s}(G_e)} = \left\|A_\varphi^\pi f\right\|_{L^{r,s}(\mathfrak{g}_{e1} \oplus \mathfrak{g}_{e2})} \tag{4.3}$$

for all $f \in M_\varphi^{r,s}(\pi)$. Thus the map $f \mapsto \left\|V_\varphi^\pi f\right\|_{L^{r,s}(G_e)}$ defines an equivalent norm on $M_\varphi^{r,s}(\pi)$, by an abuse of notation still denoted by $\|\,.\,\|_{M_\varphi^{r,s}(\pi)}$. Thus $M_\varphi^{r,s}(\pi)$ is the coorbit (in the sense of Feichtinger and Gröchenig) of $L^{r,s}(G_e)$ under the representation $\pi : G_e \to \mathscr{U}(\mathcal{H}_\pi)$.

Moreover, $M_\varphi^{r,s}(\pi)$ does not depend on the particular choice of window φ, and any two norms defined with respect to different windows $\varphi_1, \varphi_2 \in (\mathcal{H}_\pi^\infty)' \backslash \{0\}$ are equivalent.

Proof. In order to prove the first part of the statement, we make use of the identification of elements $g_1 \in G_{e1}$ with elements $(e_N, g_1) \in G_e$ and the analogous one for $g_2 \in G_{e2}$. Furthermore, we know that for each $g_1 \in G_{e1}$ there exists an $X_{g_1} \in \mathfrak{g}_{e1}$ such that $\exp_{G_e}(X_{g_1}) = g_1$, with an analogous statement for $g_2 \in G_{e2}$.

This allows us to identify $g_2 g_1$ with both $(g_2, e_{G_{e1}})(e_{G_{e1}}, g_1) = \big(\alpha(g_1^{-1})(g_2), g_1\big)$ and $\exp_{G_e}(X_{g_2}) \exp_{G_e}(X_{g_1})$, whereas we may identify $\exp_{G_e}(X_{g_1} + X_{g_2})$ with (g_2, g_1) by the use of standard exponential coordinates. But since dg_2 is G_{e1}-invariant, we hence compute

$$\begin{aligned}
\left\|(V_\varphi^\pi f)\right\|_{L^{r,s}(G_e)} &= \Big(\int_{G_{e2}} \Big(\int_{G_{e1}} \left|\langle f, \pi(g_2 g_1)\varphi\rangle_{(\mathcal{H}_\pi^\infty)'}\right|^r \, dg_1\Big)^{r/s} \, dg_2\Big)^{1/s} \\
&= \Big(\int_{G_{e2}} \Big(\int_{G_{e1}} \left|\big\langle f, \pi\big(\alpha(g_1^{-1})(g_2), g_1\big)\varphi\big\rangle_{(\mathcal{H}_\pi^\infty)'}\right|^r \, dg_1\Big)^{r/s} \, dg_2\Big)^{1/s} \\
&= \Big(\int_{G_{e2}} \Big(\int_{G_{e1}} \left|\langle f, \pi(g_2, g_1)\varphi\rangle_{(\mathcal{H}_\pi^\infty)'}\right|^r \, dg_1\Big)^{r/s} \, dg_2\Big)^{1/s} \\
&= \Big(\int_{\mathfrak{g}_{e2}} \Big(\int_{\mathfrak{g}_{e1}} \left|\big\langle f, \pi\big(\exp_{G_e}(X_{g_1} + X_{g_2})\big)\varphi\big\rangle_{(\mathcal{H}_\pi^\infty)'}\right|^r \, dX_{g_1}\Big)^{r/s} \, dX_{g_2}\Big)^{1/s} \\
&= \left\|A_\varphi^\pi f\right\|_{L^{r,s}(\mathfrak{g}_{e1} \oplus \mathfrak{g}_{e2})}.
\end{aligned}$$

This proves the first part of our theorem.

Given the first part of the theorem, its second part is a standard result of coorbit theory for square-integrable group representations (cf. [21] Theorem 4.2). A condensed version of the argument is the following: recalling that for any $\varphi_1, \varphi_2 \in \mathcal{H}_\pi$ square-integrability of $\pi|_{G_e}$ yields the reproducing identity

$$\varphi_2 = \frac{1}{\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}_\pi}} \int_{G_e} \langle \varphi_2, \pi(g)\varphi_1 \rangle_{(\mathcal{H}_\pi^\infty)'} \pi(g)\varphi_1 \, dg,$$

a straight-forward computation furthermore shows that

$$V^\pi_{\varphi_2} f = \frac{1}{\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}_\pi}} \left(V^\pi_{\varphi_1} f *_{G_e} V^\pi_{\varphi_2} \varphi_1 \right).$$

But since for $\varphi_1, \varphi_2 \in \mathcal{H}_\pi^\infty$ we have $A^\pi_{\varphi_2}\varphi_1 \in \mathscr{S}(\mathfrak{g}_e)$ (cf. [5] Corollary 2.9 (3)), equivalently we have $V^\pi_{\varphi_2}\varphi_1 \in \mathscr{S}(G_e) \subseteq L^1(G_e) = L^{1,1}(G_e)$. We can now apply Young's inequality to estimate

$$\|f\|_{M^{r,s}_{\varphi_2}} \leqslant \frac{1}{\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}_\pi}} \|\varphi_1\|_{M^{1,1}_{\varphi_2}} \|f\|_{M^{r,s}_{\varphi_1}} .$$

Since the order of φ_1 and φ_2 was arbitrary, $\| \, . \, \|_{M^{r,s}_{\varphi_1}(\pi)}$ is equivalent to $\| \, . \, \|_{M^{r,s}_{\varphi_2}(\pi)}$ and our proof is complete. □

Let us conclude this subsection with a technical lemma we will need in the following.

Lemma 4.12. *Let G be a connected simply connected nilpotent Lie group given as the semi-direct product $G = N \rtimes_\alpha H$ of the nilpotent Lie groups H and N and let π be a unitary irreducible representation of G which is square-integrable modulo the projective kernel P. Furthermore, let $\mathfrak{g}_e = \mathfrak{g}_{e_1} \oplus \mathfrak{g}_{e_2}$ be a direct sum decomposition of the predual $\mathfrak{g}_e$ of $\mathcal{O}_\pi$ such that $\mathfrak{g}_{e_1} = \mathfrak{h} \cap \mathfrak{g}_e$ and $\mathfrak{g}_{e_2} = \mathfrak{n} \cap \mathfrak{g}_e$.*

Then the semi-direct product $G = N \rtimes_\alpha H$ factorizes through P, i.e., if $P_H := \exp_H(\mathfrak{g}_{e_1})$ and $P_N := \exp_N(\mathfrak{g}_{e_1})$, then α induces a map β such that $G_e = N/P_N \rtimes_\beta H/P_H$.

Proof. To start with, we recall that we can identify G_e with G/P since $\mathcal{O}_\pi$ is flat (cf. Proposition 4.9). Let us also point out that $\mathfrak{g}_{e_1}$ and $\mathfrak{g}_{e_2}$ are ideals in $\mathfrak{h}$ and $\mathfrak{n}$, respectively. Equivalently, P_H and P_N are normal subgroups in H and N, respectively. For the quotients groups H/P_H and N/P_N one can easily verify that $\alpha(P_H)(N) \subseteq P_N$ and that $\alpha(h)(P_N) \subseteq P_N$ for every $h \in H$. Hence, the homomorphism $\alpha : H \to \mathrm{Aut}(N)$ induces a homomorphism $\beta : H/P_H \mapsto \mathrm{Aut}(N/P_N)$ such that $G_e = H/P_H \ltimes_\beta N/P_N$. □

4.3 Case (1) - Modulation Spaces on $\mathbf{H}_n$

Let us recall that by Proposition 3.17 the jump set in Case (1) was given by $e_{\mathcal{O}_1} = \{x_1, \ldots, x_n, y_1, \ldots, y_n, z, w, u_1, \ldots, u_n, v_1, \ldots, v_n\} =: e$, hence the predual of $\mathcal{O}_1$ is given by

$$\mathfrak{h}_{n,2_e} \cong \mathbb{R}^n X_u \oplus \mathbb{R}^n X_v \oplus \mathbb{R} X_w \oplus \mathbb{R}^n X_x \oplus \mathbb{R}^n X_y \oplus \mathbb{R} X_z.$$

The direct sum decomposition we employ is

$$\begin{aligned} \mathfrak{h}_{2,n_e} &:= \mathfrak{h}_{2,n_\mathcal{P}} \oplus \mathfrak{h}_{2,n_\mathcal{Q}} \\ &:= \big(\mathbb{R}^n X_u \oplus \mathbb{R}^n X_v \oplus \mathbb{R} X_w\big) \oplus \big(\mathbb{R}^n X_x \oplus \mathbb{R}^n X_y \oplus \mathbb{R} X_z\big) \end{aligned}$$

as it meets the above-mentioned meta-criterium of splitting the representation's action into right $\mathbf{H}_n$-translations in $\mathcal{P}$ and modulations in $\mathcal{Q}$. (See Remark 4.5.)

Proposition 4.13. *The modulation spaces $M^{r,s}_\varphi(\pi_\lambda)$ are independent of the particular choice of analyzing window φ.*

Proof. In order to prove independence of φ, let us recall from Subsection 3.6 that the Dynin-Folland group can be written as a semi-direct product $\mathbb{R}^{2n+2} \rtimes_\alpha \mathbf{H}_n$. Let us furthermore recall from Proposition 3.12 and Corollary 3.13 that the co-adjoint orbit corresponding to π_1 is flat and that the projective kernel P coincides with the centre of $\mathbf{H}_{2,n}$ since $P \cong \mathrm{Stab}(F_1) \cong \exp_{\mathbf{H}_{2,n}}(\mathbb{R} X_s)$. Hence, the conditions of Lemma 4.12 are obviously satisfied for the direct sum decomposition $\mathfrak{h}_{2,n_e} = \mathfrak{h}_{2,n_\mathcal{P}} \oplus \mathfrak{h}_{2,n_\mathcal{Q}}$. We can thus employ Theorem 4.11 to conclude that the modulation spaces $M^{r,s}_\varphi(\pi_\lambda)$ are independent of the particular choice of window φ. □

Conjecture 4.14. *The modulation spaces $M^{r,s}(\pi_\lambda)$ are genuinely different from any classical modulation space $M^{\tilde{r},\tilde{s}}(\mathbb{R}^{2n+1})$ for all $r, s \in [0, \infty)$.*

Although there are strong hints in this direction, there remains to be given a rigorous proof.

Remark 4.15. It also remains unclear which global diffeomorphisms of the underlying space $\mathbb{R}^{2n+1}$ leave the spaces $M^{r,s}(\pi_\lambda)$ invariant. In the classical case, i.e., for $M^{r,s}(\mathbb{R}^n)$, the only admissible diffeomorphisms are affine transformations. But this alone would already exclude the use of many coordinates charts $\phi : \mathfrak{h}_n \to \mathbf{H}_n$ different from the exponential map.

Such an answer would probably pose the unpleasant question of how much use such spaces can be if they do not admit different (coordinate) realizations of the underlying group $\mathbf{H}_n$.

Proposition 4.16. *The following properties hold true for the modulation spaces* $M^{r,s}(\pi_\lambda)$*:*

(i) For $1 \leqslant r_1 \leqslant r_2 \leqslant \infty$, $1 \leqslant s_1 \leqslant s_2 \leqslant \infty$ *we have*

$$M^{1,1}(\pi_\lambda) \subseteq M^{r_1,s_1}(\pi_\lambda) \subseteq M^{r_2,s_2}(\pi_\lambda) \subseteq M^{\infty,\infty}(\pi_\lambda).$$

(ii) Let r', s' *be the conjugate indices of* $r, s \in [1, \infty]$*. Then* $\big(M^{r,s}(\pi_\lambda)\big)' = M^{r',s'}(\pi_\lambda)$*.*

Proof. By the reasoning in the proof of Theorem 4.13, we may consider π_1 as a square-integrable representation of the group $\mathbf{H}_{2,n_e} \cong \mathbf{H}_{2,n}/P$, where P denoted the projective kernel of π_1. Square-integrability then yields $M^{r,s}(\pi_\lambda) \cap M^{\infty,\infty}(\pi_\lambda) = M^{r,s}(\pi_\lambda)$ by Feichtinger and Gröchenig [21] Corollary 4.4. Hence, the first claim is due to the general fact $(L^{r_1,s_1}_{\mathcal{P},\mathcal{Q}} \cap L^{\infty,\infty}_{\mathcal{P},\mathcal{Q}})(\mathbb{R}^{4n+2}) \subseteq (L^{r_2,s_2}_{\mathcal{P},\mathcal{Q}} \cap L^{\infty,\infty}_{\mathcal{P},\mathcal{Q}})(\mathbb{R}^{4n+2})$.

The second claim is owed to Theorem 4.9 of the same paper and the fact that the Banach dual $\big(L^{r,s}_{\mathcal{P},\mathcal{Q}}(\mathbb{R}^{4n+2})\big)'$ and the Köthe dual $\big(L^{r,s}_{\mathcal{P},\mathcal{Q}}(\mathbb{R}^{4n+2})\big)^\alpha$ both coincide with $L^{r',s'}_{\mathcal{P},\mathcal{Q}}(\mathbb{R}^{4n+2})$. □

Remark 4.17 (Atomic Decompositions)**.** The existence of arbitrarily fine BUPU's for any locally compact group (cf. Remark 2.22), thus specifically for $\mathbf{H}_{2,n}$, automatically implies the existence of atomic decompositions in $M^{r,s}(\pi_\lambda)$. A concrete example of a well-spread family of points in $\mathbf{H}_{2,n}$ has yet to be given, though.

The existence of smooth BUPU's (although not under this name) for homogeneous groups, like $\mathbf{H}_{2,n}$, is shown in [] in the subsection on the Calderón-Vaillancourt Theorem.

4.4 Case (2) - A Quasi-Classical Case in n dimensions

As in Case (1) we start with the the predual of $\mathcal{O}_2$: since the corresponding jump set is given $e_{\mathcal{O}_2} = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$, we obtain

$$\mathfrak{h}_{2,n_e} = \mathbb{R}^n X_u \oplus \mathbb{R}^n X_v. \tag{4.4}$$

But as the following shows this already reduces this case to the classical modulation spaces on $\mathbb{R}^n$.

Proposition 4.18. *Let $r, s \in [1, \infty]$. Then $M^{r,s}_\varphi(\pi_2)$ is isomorphic to $M^{r,s}(\rho^{\mathbb{R}^n}_{f_w})$, that is, the modulation spaces on $\mathbb{R}^n$ induced by the Schödinger representation of parameter f_w. In the special case of $f_w = 1$, $M^{r,s}_\varphi(\pi_2)$ even coincides with the classical space $M^{r,s}(\mathbb{R}^n)$.*

In any case, the definition of $M^{r,s}_\varphi(\pi_2)$ is independent of the particular choice of analyzing vector φ.

Proof. To start with, let us recall that if we set $g := (u, v, w, x, y, z, s) \in \mathbf{H}_{2,n}$, π_2 was given by

$$(\pi_2(g)\psi)(\tilde{u}) = \psi(\tilde{u} + u - \frac{z}{2f_w} f_x)\, e^{2\pi i(f_w w + f_x x + f_y y + f_z z)}\, e^{\pi i \left\langle 2\tilde{u} + u - \frac{z}{2f_w} f_x, f_w v - \frac{z}{2} f_y \right\rangle_{\mathbb{R}^n}}.$$

Hence, restricting π_2 to $G_e = \exp_{\mathbf{H}_{2,n}}(\mathfrak{h}_{2,n_e})$ (realized as the quotient group $\mathbf{H}_{2,n}/P$) we immediately observe that it coincides with the Schrödinger representation ρ_{f_w} restricted to $\mathbb{R}^{2n}_{u,v}$. This proves the first two assertions.

Our third claim follows from the observation that G_e decomposes as the direct product $\mathbf{H}_{2,n_{e_2}} = \exp_{\mathbf{H}_{2,n_{e_2}}}(\mathbb{R}^n X_u) \times \exp_{\mathbf{H}_{2,n_{e_2}}}(\mathbb{R}^n X_v)$ since $[\mathbb{R}^n X_u, \mathbb{R}^n X_v] = \mathbb{R}X_w \subseteq \mathfrak{p}$. Since this is a special instance of semi-direct product (with $\alpha = \mathrm{id} : H \to \mathrm{Aut}(N)$), we can apply Theorem 4.11 to $\mathbf{H}_{2,n_{e_2}}$. This concludes our proof. ☐

4.5 Case (3) - A Quasi-Classical Case in 1 dimension

Let us first recall that for the orbit $\mathcal{O}_3$ the jump set is given by either $e_{\mathcal{O}_3} = \{z, u_j\}$ or $e_{\mathcal{O}_3} = \{z, v_k\}$, for some j or some k in $\{1, \ldots, n\}$, depending on the vector $(f_y, f_x) \in \mathbb{R}^{2n}$. (For more details see Proposition 3.17.)

This implies that the predual of $\mathcal{O}_2$ is given by either

$$\mathfrak{h}_{2,n_e} = \mathbb{R}X_z \oplus \mathbb{R}X_{u_j} \text{ or } \mathfrak{h}_{2,n_e} = \mathbb{R}X_z \oplus \mathbb{R}X_{v_k}$$

for some j or some k in $\{1, \ldots, n\}$. Without loss of generality, let us focus on the second case, for which we obtain the following.

Proposition 4.19. *Let $r, s \in [1, \infty]$. Then $M^{r,s}_\varphi(\pi_3)$ is isomorphic to $M^{r,s}(\rho_{-f_{x_k}/2})$, that is, the modulation spaces on $\mathbb{R}$ induced by the Schödinger representation of parameter $-f_{x_k}/2$. In the special case of $-f_{x_k}/2 = 1$, $M^{r,s}_\varphi(\pi_3)$ coincides with the classical space $M^{r,s}(\mathbb{R})$.*

In any case, the definition of $M^{r,s}_\varphi(\pi_3)$ is independent of the particular choice of analyzing vector φ.

Proof. The proof is similar to the proof of Proposition 4.18. For $g := (u, v, w, x, y, z, s) \in \mathbf{H}_{2,n}$ we have

$$(\pi_3(g)\varphi)(\tilde{z}) = \varphi(\tilde{z} + z)\, e^{2\pi i(f_u u + f_v v + f_x x + f_y y)} \exp \pi i\Big(\frac{2\tilde{z} + z}{2}(-f_x v + f_y u)\Big).$$

Hence the restriction to $G_e = \exp_{\mathbf{H}_{2,n}}(\mathfrak{h}_{2,n_e})$ of π_3 coincides with the representation $e^{2\pi i f_{v_k} v_k}\rho_{-f_{x_k}/2}$ restricted to $\mathbb{R}^2_{z,v_k}$. Since the factor $e^{2\pi i f_{v_k} v_k}$ is of modulus 1 and hence plays no role in the modulation space norm (4.2), the rest of the proof reduces to a 2-dimensional special case of the proof of Proposition 4.18. □

4.6 Case (4) - The Trivial Case

Since the corresponding orbits are singletons, the whole group stabilises each of them, i.e., $\mathfrak{h}_{2,n_e} \cong \{0\}$. Recall that the functional dimension (being half the dimension of the orbits) equals zero and hence equivalenty the corresponding representation spaces H_{π_3} are isomorphic to $\mathbb{C}$. As the flat-orbit condition is trivially satisfied, all modulation spaces $M^{r,s}_{\varphi}(\pi_4)$ are all isomorphic to $\mathbb{C}$.

5 Weyl-Quantized Operators on the Heisenberg Group

This Chapter is mainly concerned with properties of the Weyl calculus on the Heisenberg group first proposed by Dynin [13]. Seemingly little known to the scientific community, Dynin's rather brief account was picked up and studied more extensively by Folland [29] almost two decades later. Dynin's Weyl quantization is defined in terms of the generic unitary irreducible representation of a 3-step nilpotent group which relates to the Heisenberg group the same way the Heisenberg group $\mathbf{H}_n$ relates to $\mathbb{R}^n$. This relation can be realized by constructing the big group's Lie algebra as the set of commutator relations of left-invariant vector fields and multiplication by coordinates. Briefly sketched by Dynin, Folland gives a full account on the construction while extending it to all (fully non-Abelian) 2-step nilpotent Lie groups G; he refers to the groups thus obtained as meta-Heisenberg groups $H(G)$. In the special case of $G = \mathbf{H}_n$, we will refer to $H(\mathbf{H}_n)$ as the Dynin-Folland group, denoting it by $\mathbf{H}_{2,n}$.

As Folland's account aims at proving the usefulness of meta-Heisenberg groups in general rather than studying the induced operator calculus in every detail, the author restricts himself to establishing some important basics, in particular formulas as well as a link to other existing results such as Beals and Greiner's calculus on Heisenberg manifolds [3].

Another yet more abstract approach to Weyl quantization via the unirreps of nilpotent Lie groups is found in Pedersen [50, 51]. The author of the present text and his collaborators became aware of Pedersen's work while studying the co-adjoint orbits and unirreps of $\mathbf{H}_{2,n}$ in their attempt at making sense of modulation spaces on $\mathbf{H}_n$. In a very general framework, Pedersen establishes an elegant and remarkably explicit approach to geometric quantization of the co-adjoint orbits of nilpotent Lie groups, which in its extended setting provides strong results for the Weyl quantization of symbols that are Schwartz class or tempered distributions.

For the special case of the Lie group being $\mathbf{H}_n$, Pedersen recovers the classical Weyl quantization on $\mathbb{R}^n$; for $\mathbf{H}_{2,n}$ Pedersen's Weyl corresponde in fact coincides with Dynin's, although it appears that so far the connection between the two approaches has not been mentioned in the literature. Despite being elegant, Pedersen's account is focused on representation theory and abstains from any applications to PDE theory.

It is thus the purpose of our account to investigate a little further some ΨDO-related questions. Sections 5.1 - 5.3 motivate the quantization from a pseudodifferential point of view and present a few basic results, most of which are already present in [29]. In particular, Section 5.2 provides some useful ways to rewrite the quantization, which makes it easily applicable in ignorance of any representation theoretic background.

In Section 5.4 we give a brief account on left-invariant operators, especially differential operators, from a rather Lie group theoretic point of view, while Section 5.5 returns to the ΨDO perspective, introducing a type of global non-isotropic Hörmander symbol classes first suggested by Dynin (and in a localized version used by Beals and Greiner). We justify the definition by showing that their elements quantize continuous operators on the Schwartz space $\mathscr{S}(\mathbf{H}_n)$.

Section 5.6 is focussed on the natural Moyal product of symbols, i.e., the product of symbols which quantizes the product of two ΨDO's. We derive a formal asymptotic expansion for it and discuss its limited use for the non-isotropic symbol classes we consider. Another representation of the Moyal is given in form of an oscillatory integral, which we employ to show the expected mapping properties between symbol classes.

Finally, we revisit the link with Beals and Greiner's calculus on Heisenberg manifolds first established in [29]. Following Folland's example, we discuss the special case when the Heisenberg manifold is given by $\mathbf{H}_n \times \mathbb{R}^{2n+1}$, the phase space of the $\mathbf{H}_n$-Weyl quantization, and compare some of Beals and Greiner's results with ours.

5.1 The Quantization Problem for the Heisenberg Group

Our approach to pseudodifferential operators on $\mathbf{H}_n$ is motivated by Hermann Weyl's quantization procedure on $\mathbb{R}^n$, which can be expressed as follows: How can one associate an operator S on $L^2(\mathbb{R}^n)$ (or some dense subspace of it) to a given function σ defined on the classical phase space $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ such that the coordinate projections

$$\xi = (\xi_1, \dots, \xi_n, x_1, \dots, x_n) \mapsto \xi_j \quad \text{and} \quad x = (\xi_1, \dots, \xi_n, x_1, \dots, x_n) \mapsto x_k,$$

$j, k \in \{1, \dots, n\}$, correspond to the self-adjoint operators

$$D_{x_j} = (2\pi i)^{-1} \frac{\partial}{\partial x_j} \quad \text{and} \quad X_k = f \mapsto x_k f,$$

respectively? Weyl's quantization correspondence suggests

$$\sigma(D,X) := \iint_{\mathbb{R}^{2n}} \Big(\iint_{\mathbb{R}^{2n}} \sigma(\xi,x)\, e^{-2\pi i(p\xi+qx)}\, d\xi\, dx \Big) e^{2\pi i(pD+qX)}\, dp\, dq. \tag{5.1}$$

(Cf. Weyl [70] p. 27 and p. 33 for the original German version as well as [71] p. 274 and p. 280 for an English translation.)

As we recognize the unitary operators $e^{2\pi i(pD+qX)}$ to be an instance of the Schrödinger representation $\rho = \rho_1$, introduced in Sections 1.3.3 and 1.3.4, we can equally rewrite Weyl's correspondence (5.1) as

$$\mathrm{Op}^{\rho}(\sigma) := \sigma(D,X) = \iint_{\mathbb{R}^{2n}} \hat{\sigma}(p,q)\rho(p,q,0)\, dp\, dq.$$

Let us note that if we do not want to make use of representation theory at all, but rather study the convergence of the defining integral in dependence of σ, e.g., we can easily arrive at another useful representation:

$$(\sigma(D,X)f)(x) = \iint \sigma\big(\xi, \frac{1}{2}(x+y)\big)\, e^{2\pi i\xi(x-y)} f(y)\, d\xi\, dy. \tag{5.2}$$

With this at hand, it is now easily seen that the Weyl correspondence indeed solves the above quantization problem.

Examples of other quantizations which satisfy the above criterium are the Kohn-Nirenberg quantization, any extrapolated quantization between the Kohn-Nirenberg and Weyl quantizations or beyond (cf. Shubin's τ-calculus, [67] Subsection 23.3), the Born-Jordan quantization (cf. [11]), etc. We shall say no more about these in the following.

On non-Abelian Lie groups G like $\mathbf{H}_n$ it is a priori not clear what the according phase space should look like and how one should quantize. One thing we know for sure is that the differential operators D_{x_j} should be replaced by the standard left (or right) invariant vector fields, i.e., the ones whose left (or right) trivialization at $e \times T_e G$ coincides with the standard basis of the Lie algebra $\mathfrak{g}$. Let us, without loss of generality, focus on the left-invariant vector fields. Moreover, the quantization correspondence should incorporate multipliers in the group variable g.

A very satisfying answer to the quantization problem on compact groups is given by Ruzhansky and Turunen [64, 65, 66]. Their approach makes use of the groups' representation theory and defines a Kohn-Nirenberg-type quantization in terms of the natural group Fourier transform. More precisely, the quantization employs representation-valued, i.e., matrix-valued symbols, which are defined on $G \times \widehat{G}$. An approach of similar build has been successfully applied to graded Lie groups G such as the Heisenberg group by Fischer and Ruzhansky [24, 26, 25].

Another quite different approach, the one we will study, is to employ a Heisenberg-type structure based upon the group G and to find an adequate unitary representation of that structure that eventually quantizes ΨDO's. For $G = \mathbf{H}_n$ the required meta structure is precisely the Dynin-Folland group $\mathbf{H}_{2,n}$, which we introduced in Chapter 3.

Since in our case we have to quantize the $2n+1$ left-invariant vector fields defined in Subsection 1.3.2 as well as multiplication by the $2n+1$ coordinates of $\mathbf{H}_n$, we postulate that our phase space is isomorphic to $\mathbb{R}^{4n+2}$.

Definition 5.1. *Let us define the $\mathbf{H}_n$-phase space to be the Euclidean space $\mathbb{R}^{4n+2}$, whose elements we denote by*

$$
\begin{aligned}
(\Xi, \mathcal{X}) &:= (\xi_u, \xi_v, \xi_w, \chi_x, \chi_y, \chi_z) \\
&:= (\xi_{u_1}, \dots, \xi_{u_n}, \xi_{v_1}, \dots, \xi_{v_n}, \xi_w, \chi_{x_1}, \dots, \chi_{x_n}, \chi_{y_1}, \dots, \chi_{y_n}, \chi_z).
\end{aligned}
$$

We can now rephrase the quantization problem on $\mathbf{H}_n$ as the following task:

How can we associate an operator S on $L^2(\mathbf{H}_n)$ (or some dense subspace of it) to a given function σ on the $\mathbf{H}_n$-phase space $\mathbb{R}^{4n+2}$ such that the following correspondences are included:

$$\xi_{u_j} \rightsquigarrow \mathscr{D}_{p_j}, \qquad \xi_{v_k} \rightsquigarrow \mathscr{D}_{q_k}, \qquad \xi_w \rightsquigarrow \mathscr{D}_t, \tag{5.3}$$

$$\chi_{x_l} \rightsquigarrow \mathscr{X}_{p_l}, \qquad \chi_{y_m} \rightsquigarrow \mathscr{X}_{q_m}, \qquad \chi_z \rightsquigarrow \mathscr{X}_t, \tag{5.4}$$

for $j, k, l, m = 1, \dots, n$, where the above operators are defined as in (1.18) and (3.1).

An answer, namely the one we shall study more closely throughout this chapter, is given by what we call the $\mathbf{H}_n$-Weyl quantization.

5.2 The H_n-Weyl-Quantization

In this section we first revisit the Weyl quantization on $\mathbf{H}_n$ proposed by Dynin [13]. After giving the original definition, we review some more useful representations of the operator-valued integral defining it. Most of the useful formulas can in fact already be found in Folland's discussion of Dynin's results (cf. [29] Section 4).

At the end of the section we discuss why Dynin's Weyl quantization coincides with Pedersen's in this specific case. As we will see this is not mere coincidence but rather a reflection of the fact that Weyl quantizations are an instance of geometric quantization. In order not to have to verbally distinguish between the two quantizations, we will from now only refer to the $\mathbf{H}_n$-Weyl quantization.

The proof that this quantization in fact solves the quantization problem posed in Section 5.1 is given in Section 5.3.

Notation 5.2. Just as for the Euclidean Weyl quantization we will have to consider symbols σ defined $\mathbf{H}_n$-phase space $\mathbb{R}^{4n+2}$ as well their Euclidean Fourier transforms $\hat{\sigma}$. For reasons that will be obvious in a moment, we will impose that the symbols $\sigma : \mathbb{R}^{4n+2} \to \mathbb{C}$ be functions of the variables

$$\begin{aligned}
(\Xi, \mathcal{X}) &:= (\xi_u, \xi_v, \xi_w, \chi_x, \chi_y, \chi_z) \\
&:= (\xi_{u_1}, \dots, \xi_{u_n}, \xi_{v_1}, \dots, \xi_{v_n}, \xi_w, \chi_{x_1}, \dots, \chi_{x_n}, \chi_{y_1}, \dots, \chi_{y_n}, \chi_z), \\
(\Xi', \mathcal{X}') &:= (\xi'_u, \xi'_v, \xi'_w, \chi'_x, \chi'_y, \chi'_z) \\
&:= (\xi'_{u_1}, \dots, \xi'_{u_n}, \xi'_{v_1}, \dots, \xi'_{v_n}, \xi'_w, \chi'_{x_1}, \dots, \chi'_{x_n}, \chi'_{y_1}, \dots, \chi'_{y_n}, \chi'_z) \\
&\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad ,
\end{aligned}$$

or

$$\begin{aligned}
(\Xi, X) &:= (\xi_u, \xi_v, \xi_w, p_X, q_X, t_X) \\
&:= (\xi_{u_1}, \dots, \xi_{u_n}, \xi_{v_1}, \dots, \xi_{v_n}, \xi_w, p_{1X}, \dots, p_{nX}, q_{1X}, \dots, q_{nX}, t_X), \\
(\Xi, Y) &:= (\xi_u, \xi_v, \xi_w, p_Y, q_Y, t_Y) \\
&:= (\xi_{u_1}, \dots, \xi_{u_n}, \xi_{v_1}, \dots, \xi_{v_n}, \xi_w, p_{1Y}, \dots, p_{nY}, q_{1Y}, \dots, q_{nY}, t_Y), \\
&\vdots \qquad\qquad \vdots \qquad\qquad ,
\end{aligned}$$

whereas their Fourier transforms $\hat{\sigma} : \mathbb{R}^{4n+2} \to \mathbb{C}$ shall be functions of

$$\begin{aligned}
(\mathcal{P}, \mathcal{Q}) &:= (u, v, w, x, y, z) \\
&:= (u_1, \dots, u_n, v_1, \dots, v_n, t, x_1, \dots, x_n, y_1, \dots, y_n, z), \\
(\mathcal{P}', \mathcal{Q}') &:= (u', v', w', x', y', z') \\
&:= (u'_1, \dots, u'_n, v'_1, \dots, v'_n, t', x'_1, \dots, x'_n, y'_1, \dots, y'_n, z'), \\
&\vdots \qquad\qquad \vdots \qquad\qquad .
\end{aligned}$$

The strange convention of using both $\mathcal{X}$ (to be read as an uppercase χ) and X is chosen for the merit of reflecting the phase space character of $\mathbb{R}^{4n+2}_{\Xi,\mathcal{X}}$ on the one hand and offering the possibility of keeping the standard functional variable $X = (p, q, t)$ on $\mathbf{H}_n$ on the other hand.

5.2.1 Dynin's Weyl Quantization

In the following f will denote a unspecified complex valued function of $X \in \mathbf{H}_n$.

Definition 5.3. *Let π denote the generic representation (of $\lambda = 1$) of the Dynin-Folland group $\mathbf{H}_{2,n}$ defined by Proposition 3.9 To a given symbol $\sigma : \mathbb{R}^{4n+2} \to \mathbb{C}$ we formally define the corresponding $\mathbf{H}_n$-Weyl-quantized pseudodifferential operator on $L^2(\mathbf{H}_n)$*

$$\begin{aligned}
\sigma(\mathscr{D}, \mathscr{X}) &:= \iint\limits_{\mathbb{R}^{4n+2}} \hat{\sigma}(\mathcal{P}, \mathcal{Q})\, e^{2\pi i(\langle \mathcal{P}, \mathscr{D}\rangle + \langle \mathcal{Q}, \mathscr{X}\rangle)}\, d\mathcal{P}\, d\mathcal{Q} \\
&= \iint\limits_{\mathbb{R}^{4n+2}} \hat{\sigma}(\mathcal{P}, \mathcal{Q})\, \pi(\mathcal{P}, \mathcal{Q}, 0)\, d\mathcal{P}\, d\mathcal{Q}. \qquad (5.5)
\end{aligned}$$

For the sake of a convenient reading, we will denote $\sigma(\mathscr{D}, \mathscr{X})$ also by $\mathrm{Op}^{\pi}(\sigma)$.

Remark 5.4. The notation $(\mathcal{P}, \mathcal{Q}) \mapsto e^{2\pi i(\langle \mathcal{P}, \mathscr{D}\rangle + \langle \mathcal{Q}, \mathscr{X}\rangle)}$ is explicitly used in [13] for the representation we usually denote by $\pi = \pi_1$ (for $\lambda = 1$). We recall that π is indeed generated by the left $\mathbf{H}_n$-translations $e^{2\pi i\langle \mathcal{P},\mathscr{D}\rangle}$ and the Euclidean modulations $e^{2\pi i\langle \mathcal{Q},\mathscr{X}\rangle}$. (Cf. Section 3.4.)

In Dynin [13] Identity (5.5) is a priori understood in terms of Anderson's Weyl functional calculus for non-commuting self-anoint operators (cf. [1]).

The integral defining Identity (5.5) can in fact be viewed as an ordinary Bochner integral (converging in the strong operator topology) provided that $\hat{\sigma} \in L^1(\mathbb{R}^{4n+2})$. By applying the operator to a function $f \in \mathscr{S}(\mathbf{H}_n)$ we can rewrite the integrals and obtain

$$
\begin{aligned}
(\sigma(\mathscr{D},\mathscr{X})f)(X) &= \iint \hat{\sigma}(\mathcal{P},\mathcal{Q})\, e^{2\pi i\left\langle \mathcal{Q}, X\cdot(\frac{1}{2}\mathcal{P})\right\rangle} f(X\cdot\mathcal{P})\, d\mathcal{P}\, d\mathcal{Q} \\
&= \iiiint \sigma(\Xi,\mathcal{X})\, e^{-2\pi i\langle \Xi,\mathcal{P}\rangle}\, e^{-2\pi i\langle \mathcal{X},\mathcal{Q}\rangle}\, e^{2\pi i\left\langle \mathcal{Q}, X\cdot(\frac{1}{2}\mathcal{P})\right\rangle} \\
&\quad f(X\cdot\mathcal{P})\, d\Xi\, d\mathcal{X}\, d\mathcal{P}\, d\mathcal{Q} \\
&= \iiint \sigma(\Xi,\mathcal{X})\, e^{-2\pi i\langle \Xi,\mathcal{P}\rangle}\, \delta_0\big(\mathcal{X} - X\cdot(\tfrac{1}{2}\mathcal{P})\big)\, f(X\cdot\mathcal{P}) && (5.6)\\
&\quad d\Xi\, d\mathcal{X}\, d\mathcal{P} && (5.7)\\
&= \iint \sigma\big(\Xi, X\cdot(\tfrac{1}{2}\mathcal{P})\big)\, e^{-2\pi i\langle \Xi,\mathcal{P}\rangle}\, f(X\cdot\mathcal{P})\, d\Xi\, d\mathcal{P} && (5.8)\\
&= \iint \sigma\big(\Xi, X\cdot\tfrac{1}{2}(X^{-1}\cdot Y)\big)\, e^{-2\pi i\left\langle \Xi, X^{-1}\cdot Y\right\rangle}\, f(Y)\, d\Xi\, dY && (5.9)\\
&= \iint \sigma\big(\Xi, \tfrac{1}{2}(X+Y)\big)\, e^{2\pi i\left\langle \Xi, Y^{-1}\cdot X\right\rangle}\, f(Y)\, d\Xi\, dY && (5.10)\\
&= \iint \sigma\big(\Xi, \tfrac{1}{2}(X+Y)\big)\, e^{2\pi i\left\langle \Xi, X-Y-\frac{1}{2}[X,Y]\right\rangle}\, f(Y)\, d\Xi\, dY. && (5.11)
\end{aligned}
$$

Here we have used the Euclidean Fourier Inversion Theorem in line (5.7), and in line (5.9) the fact that $X\cdot\frac{1}{2}(X^{-1}\cdot Y) = \frac{1}{2}(X+Y)$. Let us keep in mind that $X^{-1} = -X$ for all $X \in \mathbf{H}_n$ due to its nilpotent structure. The change of variables applied in line (5.8) does not affect the measure since on $\mathbb{R}^{2n+1} \cong \mathbf{H}_n$ the Lebesgue measure coincides with the $\mathbf{H}_n$-Haar measure, which in turn is invariant under group multiplication and inversion.

Remark 5.5. For the case we wish to completely forget about the $\mathbf{H}_n$-group structure, we can rewrite Integral (5.11) in terms of Notation 5.2:

$$(\sigma(\mathscr{D},\mathscr{X})f)(p_X,q_X,t_X) = \iint\limits_{\mathbb{R}^{4n+2}} \sigma\Big(\xi_u,\xi_v,\xi_w,\frac{1}{2}(p_X+p_Y),\frac{1}{2}(q_X+q_Y),\frac{1}{2}(t_X+t_Y)\Big) \times e^{2\pi i\langle(\xi_u,\xi_v,\xi_w)^t,(p_X-p_Y,q_X-q_Y,t_X-t_Y)^t\rangle}\, e^{-\pi i(p_Xq_Y-q_Xp_Y)} \times f(p_Y,q_Y,t_Y)\,d\xi_u\,d\xi_v\,d\xi_w\,dp_Y\,dq_Y\,dt_Y. \tag{5.12}$$

We notice that Integral (5.12) is very similar to the $2n+1$-dimensional version of its Euclidean counterpart (5.2), which defines the Weyl quantization on $\mathbb{R}^n$. The main difference lies in the additional factor $e^{-\pi i(p_Xq_Y-q_Xp_Y)}$; as we will see in Section 5.6, it yields a quite different Moyal product structure on the $\mathbf{H}_n$-phase space in comparison with its $\mathbb{R}^{4n+2}$ equipped with the standard Moyal product. (Cf. Definition 5.35.)

To complete what has been said, let us observe that the $\mathbf{H}_n$-Weyl-quantized operator $\sigma(\mathscr{D},\mathscr{X})$ can be expressed as an integral operator with kernel

$$K_\sigma(X,Y) = \int\limits_{\mathbb{R}^{2n+1}} \sigma\big(\Xi,\frac{1}{2}(X+Y)\big)\,e^{2\pi i\langle\Xi,Y^{-1}\cdot X\rangle}\,d\Xi$$
$$= \big(\mathscr{F}_1\sigma\big)\big(Y^{-1}\cdot X,\frac{1}{2}(X+Y)\big).$$

Thus the kernel is obtained from the symbol σ by applying a partial Fourier transform in the first variable, followed by the measure-preserving change of variables

$$T:\mathbb{R}^{4n+2}\to\mathbb{R}^{4n+2}:(X,Y)\mapsto\big(Y^{-1}\cdot X,\frac{1}{2}(X+Y)\big).$$

Since $\mathscr{F}_1$ and the pullback T^* are isomorphisms on $\mathscr{S}(\mathbb{R}^{4n+2})$, and a fortiori on $\mathscr{S}'(\mathbb{R}^{4n+2})$, as well as unitary isomorphisms on $L^2(\mathbb{R}^{4n+2})$, the integral kernel K_σ is a member of $\mathscr{S}'(\mathbb{R}^{4n+2})$ or $L^2(\mathbb{R}^{4n+2})$ precisely when σ belongs to the respective spaces. But the Schwarz kernel theorem classifies the continuous linear operators from $\mathscr{S}(\mathbb{R}^{2n+1})$ into $\mathscr{S}'(\mathbb{R}^{2n+1})$ as those that possess an $\mathscr{S}'(\mathbb{R}^{4n+2})$-kernel. Another classical statement classifies the Hilbert-Schmidt operators on $L^2(\mathbb{R}^{2n+1})$ as those that possess an integral kernel in $L^2(\mathbb{R}^{4n+2})$. This can be summarized by the following statement.

Proposition 5.6. *The $\mathbf{H}_n$-Weyl quantization defined for symbols $\sigma\in\mathscr{F}L^1(\mathbb{R}^{4n+2})$ extends uniquely to a quantization calculus for tempered distributions and square-integrable functions:*

(i) A linear operator from $\mathscr{S}(\mathbf{H}_n)$ to $\mathscr{S}'(\mathbf{H}_n)$ is continuous if and only if it is given as the $\mathbf{H}_n$-Weyl quantization of a symbol in $\mathscr{S}'(\mathbb{R}^{4n+2})$.

(ii) A bounded linear operator on $L^2(\mathbf{H}_n)$ is Hilbert-Schmidt if and only if it is given as the $\mathbf{H}_n$-Weyl quantization of a symbol in $L^2(\mathbb{R}^{4n+2})$.

5.2.2 Pedersen's Weyl Quantization

We conclude the section with a brief account on the link between Dynin's and Pedersen's Weyl quantizations. To do so, we will recall a few important aspects of Pedersen's machinery.

As usual, let G be a nilpotent Lie group and $\mathcal{O}$ one of its co-adjoint orbits, which corresponds uniquely to a unirrep $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$. Let furthermore, $F_\mathcal{O}$ be a representative of $\mathcal{O}$, and let the set of jump indices $e = e_\mathcal{O}$, the predual $\mathfrak{g}_e$ and the global chart ϕ be as in Defintion 4.1 and Identity (4.1), respectively.

The Schwartz space $\mathscr{S}(\mathcal{O})$ is well-defined: the orbit is equipped with a smooth structure via the identification $\mathcal{O} \cong G/\operatorname{Stab}(F_\mathcal{O})$, and also the polynomial structure defined on $G/\operatorname{Stab}(F_\mathcal{O})$ carries over to $\mathcal{O}$. (Cf. [51] Subsection 4.1 p. 31 and [50] Subsection 1.6 p. 521.)

If $\beta_\mathcal{O}$ denotes the canonical measure on $\mathcal{O}$, one can now define a Fourier transform on $\mathscr{S}(\mathcal{O})$ by

$$\widehat{\sigma}(X) := \int_{\mathcal{O}} \sigma(F) e^{-2\pi i \langle F, X\rangle}\, d\beta_\mathcal{O}(F) \tag{5.13}$$

for $\sigma \in \mathscr{S}(\mathcal{O})$ and $X \in \mathfrak{g}_e$. Here $\langle\, .\, ,\, .\,\rangle$ denotes the standard $\mathfrak{g}^*$-$\mathfrak{g}$-duality.

It turns out that the map $\sigma \mapsto \widehat{\sigma}|_{\mathfrak{g}_e}$ defines a topological isomorphism from $\mathscr{S}(\mathcal{O})$ onto $\mathscr{S}(\mathfrak{g}_e)$, which satisfies the identity

$$\int_{\mathfrak{g}_e} \widehat{\sigma}(X)\, \overline{\widehat{\tau}(X)}\, dX = C_\mathcal{O} \int_{\mathcal{O}} \sigma(F)\, \overline{\tau}(F)\, d\beta_\mathcal{O}$$

for all $\sigma, \tau \in \mathscr{S}(\mathcal{O})$. The constant $C_\mathcal{O}$ only depends on the particular orbit $\mathcal{O}$. In fact, $C_\mathcal{O}^{-1}$ equals the absolute value of the Pfaffian of the symplectic form of $\mathcal{O}$. (Cf. [51] Subsection 4.1 p. 31 as well as p. 10.) In the case of the non-degenerate orbits of $\mathbf{H}_n$, $C_{\mathcal{O}_\lambda}$ is given by $|\lambda|^{-n}$, whereas for the generic orbits of $\mathbf{H}_{2,n}$ it equals $|\lambda|^{-2n-1}$.

With this at hand, we can now give Pedersen's definition of Weyl quantization.

Definition 5.7. *Let $\sigma \in \mathscr{S}(\mathcal{O})$. The Weyl-Pedersen quantization of σ is then defined* *to be the operator*

$$\mathrm{Op}_P^{\pi}(\sigma) := C_{\mathcal{O}} \int_{\mathfrak{g}_e} \widehat{\sigma}(X)\pi(\exp(X))\, dX, \tag{5.14}$$

which is bounded linear on the representation space $\mathcal{H}_\pi$.

Remark 5.8. Like the integral in Defintion 5.3 Integral (5.14) can be viewed as Bochner integral converging in the strong operator topology on $\mathcal{B}(\mathcal{H}_\pi)$, the space of bounded linear operators on $\mathcal{H}_\pi$. The fact that $\mathrm{Op}^{\pi}(\sigma)$ is bounded is due to $\widehat{\sigma} \in L^1(\mathfrak{g}_e)$ and a standard estimate for Bochner integrals converging in the strong operator topology.

We now easily conclude the following:

Proposition 5.9. *The Weyl quantization on the Heisenberg group defined by Dynin [13] coincides with the Weyl quantization defined by Pedersen [51] for the special case when G is the Dynin-Folland group $\mathbf{H}_{2,n}$ and the unirrep employed is the generic representation $\pi = \pi_1$ defined in Section 3.4.*

Proof. We recall from Proposition 3.12 that for the generic unirrep π_l, $\lambda \neq 0$, the corresponding co-adjoint orbit $\mathcal{O}_{\pi_\lambda}$ is given by

$$\mathcal{O}_{\pi_\lambda} = \lambda X_s^* \oplus \mathbb{R}^n X_u^* \oplus \mathbb{R}^n X_v^* \oplus \mathbb{R} X_w^* \oplus \mathbb{R}^n X_x \oplus \mathbb{R}^n X_y^* \oplus \mathbb{R} X_z^*.$$

Moreover, we recall from Section 4.3 that the corresponding predual is

$$\begin{aligned} \mathfrak{h}_{2,n_e} &:= \mathfrak{h}_{2,n_{\mathcal{P}}} \oplus \mathfrak{h}_{2,n_{\mathcal{Q}}} \\ &:= \big(\mathbb{R}^n X_u \oplus \mathbb{R}^n X_v \oplus \mathbb{R} X_w\big) \oplus \big(\mathbb{R}^n X_x \oplus \mathbb{R}^n X_y \oplus \mathbb{R} X_z\big). \end{aligned}$$

It follows that for any $\lambda \neq 0$, the Fourier transform defined by (5.13) coincides with the standard Euclidean Fourier transform restricted to $\mathscr{S}(\mathbb{R}^{4n+2})$ times $C_{\mathcal{O}_\lambda} = |\lambda|^{-2n-1}$. Hence, for $\lambda = 1$ this immediately implies that $\sigma(\mathscr{D}, \mathscr{X}) = \mathrm{Op}^{\pi_1}(\sigma) = \mathrm{Op}_P^{\pi_1}(\sigma)$. ☐

From now on we will make no further distinction between the two quantizations and drop the subindex P in Op_P^{π}.

Remark 5.10. As we have mentioned in the introduction to this chapter Pedersen's calculus for $\mathbf{H}_n$ agrees with the classical Weyl quantization. For a comparison between (5.14) for generic ρ_λ and $\mathrm{Op}^{\rho_\lambda}(\sigma) = \sigma(\lambda D, X)$ we refer to [28, p. 109].

Pedersen's account [51] indeed features some very interesting properties of the quantization Op^π, some of which we will list here. (The list is essentially the same as the one given by [51] Theorem 4.1.4.)

Let us denote by $\mathfrak{S}_1(\mathcal{H}_\pi)$ the the Schatten-von Neumann 1-class, or trace class, of operators acting on $\mathcal{H}_\pi$ and by $\mathfrak{S}_2(\mathcal{H}_\pi)$ the Schatten-von Neumann 2-class, or Hilbert-Schmidt class, on $\mathcal{H}_\pi$. The inner product on $\mathfrak{S}_2(\mathcal{H}_\pi)$ will be denoted by $\langle\, .\, ,\, .\,\rangle_{HS}$.

Furthermore, let $\mathfrak{u}\,(\mathfrak{g}_\mathbb{C})$ denote the universal enveloping algebra of the complexification of $\mathfrak{g}$. We then have

Theorem 5.11. *For the Weyl correspondence* Op^π *defined by* (5.13) *the following properties hold true for all* $\sigma, \tau \in \mathscr{S}(\mathcal{O})$*:*

(i) $\mathrm{Op}^\pi(\sigma) \in \mathfrak{S}_1(\mathcal{H}_\pi) \cap \mathfrak{S}_2(\mathcal{H}_\pi)$ *and* $\mathrm{Tr}\,(\mathrm{Op}^\pi(\sigma)) = \int_{\mathcal{O}} \sigma(F)\, d\beta_{\mathcal{O}}(F)$*,*

(ii) $\langle \mathrm{Op}^\pi(\sigma), \mathrm{Op}^\pi(\tau)\rangle_{HS} = \mathrm{Tr}\,(\mathrm{Op}^\pi(\tau)^*\mathrm{Op}^\pi(\sigma)) = \int_{\mathcal{O}} \sigma(F)\bar{\tau}(F)\, d\beta_{\mathcal{O}}(F)$*,*

(iii) $\mathrm{Op}^\pi(\sigma)^* = \mathrm{Op}^\pi(\bar{\sigma})$*.*

(iv) By duality Op^π *extends to* $\mathscr{S}'(\mathcal{O})$ *and the image under* Op^π *of the polynomial functions on* $\mathcal{O}$*,* $\mathscr{P}(\mathcal{O})$*, coincides with* $d\pi(\mathfrak{u}\,(\mathfrak{g}_\mathbb{C}))$*.*

(v) $\mathrm{Op}^\pi(1) = I$*.*

(vi) $\mathrm{Op}^\pi(2\pi i\langle\, .\, , X\rangle) = d\pi(X)$ *for all* $X \in \mathfrak{g}$*,*

(vii) $\mathrm{Op}^\pi(e^{2\pi i\langle\, .\, , X\rangle}) = \pi(\exp_G(X))$ *for all* $X \in \mathfrak{g}$*.*

Remark 5.12. Theorem 5.11 *(viii)* implies, in particular, that $\mathrm{Op}^\pi_P = \mathrm{Op}^\pi$ solves the quantization problem for $\mathbf{H}_n$, without assuming any prior knowledge about Dynin's quantization. (Cf. Identities (3.14).)

Remark 5.13. The strong link between Pedersen's Weyl quantization and modulation spaces as defined in the framework of [5, 6] is due to the relation

$$\langle \mathrm{Op}^\pi(\sigma)f, \varphi\rangle_{\mathcal{H}_\pi} = \left\langle \hat{\sigma}, A^\pi_\varphi f\right\rangle_{L^2(\mathfrak{g}_e)}, \tag{5.15}$$

which holds true for all $\sigma \in \mathscr{S}(\mathcal{O})$ and $f, \varphi \in \mathcal{H}_\pi$. Identity (5.46) is of particular interest for operators $\mathrm{Op}^\pi(\sigma)$ whose symbols a members of some modulation space on $G \rtimes G$. (Cf. [5] Corollary 2.25 p. 306, e.g.)

Let us point, however, that this link has been widely used in the context of modulation spaces on $\mathbb{R}^n$, where the link exists for the classical Weyl quantization Op^ρ induced by $\mathbf{H}_n$. (Cf. [40, 41, 44] as well as [42] Chapter 8.)

5.3 A First Justification of Usefulness

In this section we finally show that the $\mathbf{H}_n$-Weyl quantization indeed solves the quantization problem posed in Section 5.1. More generally, we will see that a finite order polynomial in either ξ_u, ξ_v or ξ_w quantizes the corresponding polynomial of either $\mathscr{D}_p$, $\mathscr{D}_q$ or $\mathscr{D}_t$, respectively, and that a function of X quantizes multiplication by that function.

We provide elementary proofs for the formulas of adjoint and transposed operators of any given $\sigma(\mathscr{D}, \mathscr{X})$. Moreover, we show that for appropriately restricted symbols σ, $\mathrm{Op}^\pi(\sigma)$ agrees with $\mathrm{Op}^\rho(\sigma)$, its Euclidean Weyl quantization acting on $L^2(\mathbb{R}^n)$.

Let us first check that Op^π solves the quantization problem.

Proposition 5.14. *Suppose that $\sigma, \tau \in \mathscr{S}'(\mathbb{R}^{4n+2})$ are such that the following two conditions are satisfied:*

(i) $\sigma(\Xi, \mathcal{X})$ is a polynomial function of finite degree in either ξ_u, ξ_v or ξ_w only.

(ii) $\tau(\Xi, \mathcal{X}) = \tau(\mathcal{X})$.

The $\mathbf{H}_n$-Weyl quantization then yields:

(i) $\sigma(\mathscr{D}, \mathscr{X})$ defines the corresponding polynomial in either $\mathcal{D}_p$, $\mathcal{D}_q$ or $\mathcal{D}_t$ defined by the spectral calculus for self-adjoint operators.

(ii) $(\tau(\mathscr{D}, \mathscr{X})f)(X) = \tau(X)\, f(X)$ for all $f \in \mathscr{S}(\mathbf{H}_n)$.

This implies, in particular, that the $\mathbf{H}_n$-Weyl quantization solves the quantization problem for Heisenberg group posed in Section 5.1.

Proof. In order to prove Case (i) we can take advantage of the $\mathscr{S}$-$\mathscr{S}'$-duality and show the result for $\sigma, \tau \in \mathscr{S}(\mathbb{R}^{4n+2})$ without loss of generality. We can now rewrite the exponent in Equality (5.11) as

$$
\begin{aligned}
\left\langle \Xi, Y^{-1} \cdot X \right\rangle &= \left\langle \Xi, X - Y - \frac{1}{2}[X, Y] \right\rangle \\
&= \left\langle (\xi_u, \xi_v, \xi_w)^t, (p_X, q_X, t_X)^t - (p_Y, q_Y, t_Y)^t - (0, 0, \frac{1}{2}(p_X q_Y - q_X p_Y))^t \right\rangle \\
&= (\xi_u p_X + \xi_v q_X + \xi_w t_X) - (p_Y(\xi_u + \frac{1}{2}\xi_w q_X) + q_Y(\xi_v - \frac{1}{2}\xi_w p_X) + t_Y \xi_w).
\end{aligned}
$$

Inserting some $f \in \mathscr{S}(\mathbf{H}_n)$ into Equality (5.11) yields

$$
\begin{aligned}
(\mathrm{Op}^\pi(\sigma)f)(X) = & \iiint \sigma(\xi_u, \xi_v, \xi_w)\, e^{2\pi i(\xi_u p_X + \xi_v q_X + \xi_w t_X)} \iiint f(p_Y, q_Y, t_Y) \\
& \times e^{-2\pi i(p_Y(\xi_u + \frac{1}{2}\xi_w q_X) + q_Y(\xi_v - \frac{1}{2}\xi_w p_X) + t_Y \xi_w)}\, dp_Y\, dq_Y\, dt_Y \\
& d\xi_u\, d\xi_v\, d\xi_w \\
= & \iiint \sigma(\xi_u, \xi_v, \xi_w)\, e^{2\pi i(\xi_u p_X + \xi_v q_X + \xi_w t_X)} \\
& \hat{f}(\xi_u + \frac{1}{2}\xi_w q_X, \xi_v - \frac{1}{2}\xi_w p_X, \xi_w)\, d\xi_u\, d\xi_v\, d\xi_w.
\end{aligned}
$$

An application of the measure-preserving change of variables

$$
(\xi_u, \xi_v, \xi_w) \mapsto (\xi_u - \frac{1}{2}\xi_w q_X, \xi_v + \frac{1}{2}\xi_w p_X, \xi_w),
$$

which leaves the exponent $(\xi_u p_X + \xi_v q_X + \xi_w t_X)$ unchanged, yields

$$
\begin{aligned}
(\mathrm{Op}^\pi(\sigma)f)(X) = & \iiint \sigma(\xi_u - \frac{\xi_w}{2} q_X, \xi_v + \frac{\xi_w}{2} p_X, \xi_w) \hat{f}(\xi_u, \xi_v, \xi_w) \\
& \times e^{2\pi i(\xi_u p_X + \xi_v q_X + \xi_w t_X)}\, d\xi_u\, d\xi_v\, d\xi_w.
\end{aligned}
$$

Since σ is a polynomial of only one of the variables ξ_u, ξ_v, ξ_w, and w.l.o.g we can assume it is ξ_u, we know that $\sigma(\xi_u - \frac{\xi_w}{2} q_X)$ is a polynomial in ξ_u and ξ_w. Hence via the inverse Fourier transform it acts as joint Fourier multiplier in p and t, or equivalently as a joint spectral multiplier of the self-adjoint operators $D_{p_j} = (2\pi i)^{-1}\partial_{p_j}$, $j = 1, \ldots, n$, and $D_t = (2\pi i)^{-1}\partial_t$, thus as the polynomial σ in $\mathscr{D}_p = (D_p - \frac{1}{2} q D_t)$.

The proof of Case (ii) is even shorter and only makes use of the fact $\mathscr{F}(1) = \delta_0$. For an arbitrary $f \in \mathscr{S}(\mathbf{H}_n)$ we compute

$$
\begin{aligned}
(\tau(\mathscr{D}, \mathscr{X})f)(X) &= \iint \hat{\tau}(\mathcal{Q})\, \delta_0(\mathcal{P})\, e^{2\pi i \langle \mathcal{Q}, X \cdot (\frac{1}{2}\mathcal{P}) \rangle} f(X \cdot \mathcal{P})\, d\mathcal{P}\, d\mathcal{Q} \\
&= \int \hat{\tau}(\mathcal{Q})\, e^{2\pi i \langle \mathcal{Q}, X \rangle} f(X)\, d\mathcal{Q} \\
&= \tau(X)\, f(X).
\end{aligned}
$$

This proves the proposition. □

Remark 5.15. The operators discussed by Proposition 5.14 (i) are clearly left-invariant differential operators $\mathbf{H}_n$. So, two questions that immediately arise from this result are: What are the operators that correspond to arbitrary polynomials in Ξ? And how exhaustive is this method of quantizing polynomials? In other words, do we obtain all left-invariant differential operators this way? We will give a full answer in Subsection 5.4.

The $\mathbf{H}_n$-Weyl quantization also allows us to readily obtain the adjoint and the transposed operators:

Proposition 5.16. *For all symbols $\sigma \in \mathscr{S}'(\mathbb{R}^{4n+2})$ we have*

$$(\sigma(\mathscr{D}, \mathscr{X}))^* = \bar{\sigma}(\mathscr{D}, \mathscr{X}) \quad \textit{and} \quad (\sigma(\mathscr{D}, \mathscr{X}))^t = \sigma(-\mathscr{D}, \mathscr{X}).$$

Proof. Once again we argue by an formal derivation for $\sigma \in \mathscr{S}(\mathbb{R}^{4n+2})$ and conclude the result for distributional σ by duality. To this end, let $f, g \in \mathscr{S}(\mathbf{H}_n)$ and let $\langle\,.\,,\,.\,\rangle_{\mathscr{S}'}$ denote the $\mathscr{S}$-$\mathscr{S}'$-duality that extends the restriction to $\mathscr{S}(\mathbf{H}_n) \times \mathscr{S}(\mathbf{H}_n)$ of the $L^2(\mathbf{H}_n)$-inner product $\langle\,.\,,\,.\,\rangle_{L^2(\mathbf{H}_n)}$. Due to Bochner integration theory we can repeatedly interchange integration and dual action to compute

$$\begin{aligned}
\langle(\sigma(\mathscr{D}, \mathscr{X}))^* f, g\rangle_{\mathscr{S}'} &= \left\langle f, \iint \hat{\sigma}(\mathcal{P}, \mathcal{Q})\, \pi(\mathcal{P}, \mathcal{Q}, 0) g \, d\mathcal{P}\, d\mathcal{Q} \right\rangle_{\mathscr{S}'} \\
&= \iint \langle f, \hat{\sigma}(\mathcal{P}, \mathcal{Q})\, \pi(\mathcal{P}, \mathcal{Q}, 0) g\rangle_{\mathscr{S}'} \, d\mathcal{P}\, d\mathcal{Q} \\
&= \iint \left\langle \overline{\hat{\sigma}(\mathcal{P}, \mathcal{Q})}\, f, \pi(\mathcal{P}, \mathcal{Q}, 0) g \right\rangle_{\mathscr{S}'} d\mathcal{P}\, d\mathcal{Q} \\
&= \iint \left\langle \pi(-\mathcal{P}, -\mathcal{Q}, 0)\, \hat{\bar{\sigma}}(-\mathcal{P}, -\mathcal{Q})\, f, g \right\rangle_{\mathscr{S}'} d\mathcal{P}\, d\mathcal{Q} \\
&= \left\langle \iint \hat{\bar{\sigma}}(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}})\, \pi(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, 0) f \, d\tilde{\mathcal{P}}\, d\tilde{\mathcal{Q}}, g \right\rangle_{\mathscr{S}'} \\
&= \langle \bar{\sigma}(\mathscr{D}, \mathscr{X}) f, g\rangle_{\mathscr{S}'}.
\end{aligned}$$

Under the same assumptions let $(\,.\,,\,.\,)_{\mathscr{S}'(\mathbf{H}_n)}$ denote the standard $\mathscr{S}'(\mathbf{H}_n)$-$\mathscr{S}(\mathbf{H}_n)$-dual action. The second claim then follows from

$$
\begin{aligned}
&\big((\sigma(\mathscr{D},\mathscr{X}))^t f, g\big)_{\mathscr{S}'(\mathbf{H}_n)} \\
&\quad = \int f(X) \iint \hat{\sigma}(\mathcal{P},\mathcal{Q}) e^{2\pi i \left\langle \mathcal{Q}, X \cdot (\frac{1}{2}\mathcal{P})\right\rangle} g(X\cdot \mathcal{P})\, d\mathcal{P}\, d\mathcal{Q}\, dX \\
&\quad = \int f(Y\cdot \mathcal{P}^{-1}) \iint \hat{\sigma}(\mathcal{P},\mathcal{Q})\, e^{2\pi i\lambda \left\langle \mathcal{Q}, Y\cdot \mathcal{P}^{-1}\cdot (\frac{1}{2}\mathcal{P})\right\rangle}\, g(Y)\, d\mathcal{P}\, d\mathcal{Q}\, dY \\
&\quad = \int \iint \hat{\sigma}(\mathcal{P},\mathcal{Q})\, e^{2\pi i\lambda \left\langle \mathcal{Q}, Y\cdot (-\frac{1}{2}\mathcal{P})\right\rangle} f(Y\cdot(-\mathcal{P}))\, d\mathcal{P}\, d\mathcal{Q}\, g(Y)\, dY \\
&\quad = \int \iint \hat{\sigma}(-\tilde{\mathcal{P}},\mathcal{Q})\, e^{2\pi i\lambda \left\langle \mathcal{Q}, Y\cdot (\frac{1}{2}\tilde{\mathcal{P}})\right\rangle} f(Y\cdot \tilde{\mathcal{P}})\, d\tilde{\mathcal{P}}\, d\mathcal{Q}\, g(Y)\, dY \\
&\quad = (\sigma(-\mathscr{D},\mathscr{X}) f, g)_{\mathscr{S}'(\mathbf{H}_n)}.
\end{aligned}
$$

This concludes the proof. □

The usefulness of this quantization has to pass another hurdle of some importance: Since $\mathbf{H}_n$ forms a subgroup of $\mathbf{H}_{2,n}$ we would like to confirm that Op^π in some way or another defines an extension of Op^ρ on $\mathbb{R}^n$, i.e., the standard Weyl quantization on $\mathbb{R}^n$.

Proposition 5.17. *Let $\sigma \in \mathscr{S}'(\mathbb{R}^{4n+2})$ be such that $\sigma(\xi_u,\xi_v,\xi_w,\chi_x,\chi_y,\chi_z) = \sigma_0(\xi_u,\chi_x)$ for some $\sigma_0 \in \mathscr{S}'(\mathbb{R}^{2n})$. Then the $\mathbf{H}_n$-Weyl quantization coincides with the Euclidean Weyl quantization if its action is restricted to Schwartz functions of $p \in \mathbb{R}^n$.*

Proof. By a standard duality argument it suffices to prove our claim for $\sigma \in \mathscr{S}(\mathbb{R}^{4n+2})$. It then follows from a straight-forward computation:

$$
\begin{aligned}
\Big(\sigma(\mathscr{D},\mathscr{X}) f(\,.\,,0,0)\Big)(p) &= \int \dots \int \hat{\sigma}_R(u,x)\, \delta(v)\, \delta(w)\, \delta(y)\, \delta(z) \\
&\qquad \big(\pi(u,v,w,x,y,z,0) f(\,.\,,0,0)\big)(p)\, du\, dv\, dw\, dx\, dy\, dz \\
&= \int \dots \int \hat{\sigma}_R(u,x)\, \delta(v)\, \delta(w)\, \delta(y)\, \delta(z) e^{2\pi i (xp + \frac{1}{2}xu)} \\
&\qquad \times e^{2\pi i \frac{1}{4} zpv} f(p+u, v, w + \frac{1}{2}pv)\, du\, dv\, dw\, dx\, dy\, dz \\
&= \iint \hat{\sigma}_R(u,x) e^{2\pi i(xp+\frac{1}{2}xu)} f(p+u,0,0)\, du\, dx \\
&= \iint \hat{\sigma}_R(u,x) \Big(\rho(u,x,0) f|_{\mathbb{R}^n\times\{0\}^{n+1}}\Big)(p)\, du\, dx \\
&= \Big(\sigma_0(D_p,X_p) f|_{\mathbb{R}^n\times\{0\}^{n+1}}\Big)(p),
\end{aligned}
$$

where $f \in \mathscr{S}(\mathbf{H}_n)$ was arbitrary. □

Remark 5.18. The same result holds true for σ_0 being a function or distribution of ξ_v, χ_y or ξ_w, χ_z, where the latter case yields the Weyl quantization on $\mathbb{R}$.

5.4 Left-invariant Operators

A very natural question which needs to be answered affirmatively is whether the $\mathbf{H}_n$-Weyl quantization covers all left-invariant differential operators on $\mathbf{H}_n$. Fortunately our answer is positive: since any left-invariant differential operator T is continuous as an operator $T : \mathscr{S}(G) \to \mathscr{S}'(G)$, its distributional kernel $\kappa \in \mathscr{S}'(G)$ given by the Schwartz kernel theorem is automatically a right convolution kernel due to left invariance on G, i.e., $Tf = f * \kappa$ for all $f \in \mathscr{S}(G)$. (Cf. [25] Subsection 2.5, e.g.)

Remark 5.19. So let us note that in fact we can quantize every left invariant continuous operator from $\mathscr{S}(\mathbf{H}_n)$ to $\mathscr{S}'(\mathbf{H}_n)$ via Op^π: For any symbol $\sigma(\Xi, \mathcal{X}) = \sigma(\Xi) = \sigma(\xi_u, \xi_v, \xi_w)$ in $\mathscr{S}'(\mathbf{H}_n)$ the operator $\mathrm{Op}^\pi(\sigma)$ can be expressed as the $\mathbf{H}_n$-group convolution with the inverse Euclidean Fourier transform $\check{\sigma}$. Indeed, for $\sigma \in \mathscr{S}(\mathbf{H}_n)$ we compute

$$
\begin{aligned}
\big(\mathrm{Op}^\pi(\sigma)f\big)(X) &= \iint \hat{\sigma}(\mathcal{P})\, \delta(\mathcal{Q})\, e^{2\pi i \langle \mathcal{Q}, X \cdot (\frac{1}{2}\mathcal{P})\rangle} f(X \cdot \mathcal{P})\, d\mathcal{P}\, d\mathcal{Q} \\
&= \iint \hat{\sigma}(\mathcal{P})\, f(X \cdot \mathcal{P})\, d\mathcal{P}\, d\mathcal{Q} \\
&= \iint \check{\sigma}(\mathcal{P}^{-1} \cdot X)\, f(\mathcal{P})\, d\mathcal{P} \\
&= \big(f *_{\mathbf{H}_n} \check{\sigma}\big)(X),
\end{aligned}
$$

and a standard duality argument extends the last identity to $\mathscr{S}'(\mathbf{H}_n)$. But since the inverse Fourier transform is an isomorphism on both $\mathscr{S}(\mathbf{H}_n)$ and $\mathscr{S}'(\mathbf{H}_n)$, the above version of the Schwartz kernel theorem asserts our claim.

The symbol class $\mathscr{S}'(\mathbf{H}_n)$ is, of course, very general and covers all sensible classes of pseudodifferential operators, but the obtained set of operators is too big and thus often not very useful. We will not elaborate much more on $\mathscr{S}'(\mathbf{H}_n)$ but instead introduce more convenient symbol classes in Section 5.5.

Having reassured ourselves that all left-invariant differential operators are covered by the $\mathbf{H}_n$-Weyl quantization still leaves the question to what sort of symbols they correspond. An educated guess, drawn in analogy to the Euclidean case, would be precisely the polynomials in Ξ.

Our pursuit to check this will make use of the concept of symmetrization in the sense of Helgason [43], e.g.: if we can show that Op^π is equivariant under the action of $U(n) = \mathcal{U}(\mathbb{C}^n)$ on the variables $(u, v) \in \mathbb{R}^{2n} \cong \mathbb{C}^n$, we can easily conclude that Op^π coincides, modulo some powers of $2\pi i$, with the so-called symmetrization λ on $\mathbf{H}_n$. The latter is a linear bijection between the symmetric algebra $\mathcal{S}(\mathfrak{h}_n)$ over $\mathfrak{h}_n$ and the algebra $\mathbf{D}(\mathbf{H}_n)$ of left-invariant differential operators on $\mathbf{H}_n$ that is uniquely determined by the following property: If $\tilde{X}$ is a left-invariant vector field on $\mathbf{H}_n$, that is, $\tilde{X} = dR(X)$ for some $X \in \mathfrak{h}_n$ (cf. Subsection 1.3.2), then $\lambda(X^m) = \tilde{X}^m$ for all $m \in \mathbb{N}$. In other words, the polynomials in $\xi_{u_j}, \xi_{v_k}, \xi_w$, $j, k = 1, \ldots, n$, do "quantize" (via λ) all left-invariant differential operators on $\mathbf{H}_n$. (For details see [43], Chapter II Section 4 and in particular Theorem 4.3.)

Remark 5.20. We recall that the symmetrization λ satisfies two other interesting properties:

(i) If $X_1, \ldots, X_{2n+1}$ forms a basis of $\mathfrak{h}_n$ and $P \in \mathcal{S}(\mathfrak{h}_n)$, then

$$(\lambda(P)f)(g) = \big(P(\partial_{p_1}, \ldots, \partial_t)|_{(p,q,t)=0} f\big)\big(g \exp_{\mathbf{H}_n}(p_1 X_1 + \ldots + t X_{2n+1})\big)$$

for all $f \in \mathscr{S}(\mathbf{H}_n)$ and all $g \in \mathbf{H}_n$.

(ii) If $Y_1, \ldots, Y_p \in \mathfrak{h}_n$, then

$$\lambda(Y_1 \cdots Y_p) = \frac{1}{p!} \sum_{\sigma \in S_p} \tilde{Y}_{\sigma(1)} \cdots \tilde{Y}_{\sigma(p)},$$

which gives an account for the origin of the name.

To prove equivariance for Op^π, we need to agree on a few conventions: To start with, let us consider (u, v) as a vector in $\mathbb{C}^n$ whenever it seems convenient. By an abuse of notation we will subsequently treat the $\mathbb{R}^{2n}$-inner product as the $\mathbb{C}^n$-inner product. Since we will have to employ orthogonal transformations on $\mathbb{R}^{2n}$ as well as symplectic matrices (as autormorphisms of $\mathbf{H}_n$), this convention is very useful in combination with the fact that $U(n) = O(2n, \mathbb{R}) \cap Sp(2n, \mathbb{R})$. Following another convention (found in Folland [28], e.g.), we will then write $A(p, q, t) = AX$ for the $\mathbf{H}_n$-automorphism $(p, q, t)^t \mapsto (A \times I_{\mathbb{R}}) \cdot (p, q, t)^t$ with $A \in Sp(2n, \mathbb{R})$, in particular for unitary A.

Thus, for any symbol σ let us set $\sigma \circ A$ to be the symbol $(\Xi, X) \mapsto \sigma(A\Xi, AX)$.

Lemma 5.21. *For each U in $U(n)$ there exists an operator $\tilde{U}$ in $\mathcal{U}(L^2(\mathbf{H}_n))$ such that*

$$\mathrm{Op}^\pi(\sigma \circ U) = \tilde{U} \mathrm{Op}^\pi(\sigma) \tilde{U}^*$$

for all symbols σ in $\mathscr{S}'(\mathbf{H}_n)$ for which $\operatorname{Op}^\pi(\sigma)$ is bounded on $L^2(\mathbf{H}_n)$.

Proof. As usual we will prove the statement for Schwartz class σ and extend it by duality. Thus, considering the symplectic group $Sp(2n,\mathbb{R})$ as a subgroup of $\operatorname{Aut}(\mathbf{H}_n)$, we can use the above conventions to write

$$\Big\langle U^*Q, X\cdot(\tfrac{1}{2}U^*\mathcal{P})\Big\rangle = \Big\langle Q, U\big(X\cdot(\tfrac{1}{2}U^*\mathcal{P})\big)\Big\rangle = \Big\langle Q, UX\cdot(\tfrac{1}{2}\mathcal{P})\Big\rangle,$$

which holds true for all $\mathcal{P},\mathcal{Q}\in\mathbf{H}_n$. If we now keep in mind that a unitary change of variables in (u,v) is measure-preserving on $\mathbf{H}_n$, then $(\tilde{P},\tilde{Q}) := (U\mathcal{P},U\mathcal{Q})$ implies

$$\begin{aligned}
&\big(\operatorname{Op}^\pi(\sigma\circ U)f\big)(X)\\
&\quad= \iint \hat{\sigma}(U\mathcal{P},U\mathcal{Q})\, e^{2\pi i\langle \mathcal{Q},X\cdot(\frac{1}{2}\mathcal{P})\rangle} f(X\cdot\mathcal{P})\, d\mathcal{P}\, d\mathcal{Q}\\
&\quad= \iint \hat{\sigma}(\tilde{\mathcal{P}},\tilde{\mathcal{Q}})\, e^{2\pi i\langle U^*\tilde{\mathcal{Q}},X\cdot(\frac{1}{2}U^*\tilde{\mathcal{P}})\rangle} f(X\cdot U^*\tilde{\mathcal{P}})\, d\tilde{\mathcal{P}}\, d\tilde{\mathcal{Q}}\\
&\quad= \iint \hat{\sigma}(\tilde{\mathcal{P}},\tilde{\mathcal{Q}})\, e^{2\pi i\langle \tilde{\mathcal{Q}},UX\cdot(\frac{1}{2}\tilde{\mathcal{P}})\rangle} (f\circ U^*)(UX\cdot\mathcal{P})\, d\tilde{\mathcal{P}}\, d\tilde{\mathcal{Q}}\\
&\quad= \Big(\operatorname{Op}^\pi(\sigma)(f\circ U^*)\Big)(UX)
\end{aligned}$$

for any $f\in\mathscr{S}(\mathbf{H}_n)$, $X\in\mathbf{H}_n$. Finally, if for any $U\in U(n)$ we define $\tilde{U}$ in $\mathcal{U}(L^2(\mathbf{H}_n))$ to be the map $f\mapsto f\circ U$, we have just proved our claim as $\widetilde{U^*} = \tilde{U}^*$ clearly holds true. □

With this in hand we can prove our claim.

Theorem 5.22. *The $\mathbf{H}_n$-Weyl quantization Op^π restricted to complex coefficient polynomials in the frequency variables ξ_u,ξ_v,ξ_w agrees with the symmetrization λ on the Heisenberg group $\mathbf{H}_n$ (modulo powers of $2\pi i$). In particular, it is precisely these polynomials which quantize all left-invariant differential operators on $\mathbf{H}_n$.*

Proof. To prove that Op^π coincides with the symmetrization for all elements of the symmetric algebra $\mathcal{S}(\mathfrak{h}_n)$, we have to show that for every polynomial P in ξ_u,ξ_v,ξ_w the operator $\operatorname{Op}^\pi(P)$ yields a symmetrized left-invariant differential operator on $\mathscr{S}(\mathbf{H}_n)$. More precisely, we will show that it agrees with the operator $\lambda(\tilde{P})$, where

$$\tilde{P}(\xi_u,\xi_v,\xi_w) := P\Big((2\pi i)^{-1}\xi_u,(2\pi i)^{-1}\xi_v,(2\pi i)^{-1}\xi_w\Big).$$

(Cf. Identities (1.18).) We proceed in three reductive steps.

First we restrict the problem to polynomials in ξ_u,ξ_v only since any operator given as the quantization of a polynomial in ξ_w only, i.e., a polynomial in X_t, commutes with all other left-invariant operators on $\mathbf{H}_n$. (For further clarification we refer to the asymptotic expansion of the natural Moyal product arising from the $\mathbf{H}_n$-Weyl quantization. Confer Subsection 5.6.1 and in particular Formula 5.25. The Moyal product of two polynomials $P(\xi_u, \xi_v)$ and $Q(\xi_w)$, say, reduces to the simple pointwise product $P(\xi_u, \xi_v)Q(\xi_w)$.)

Second, we notice that polynomials of the form $\xi_u^\alpha \xi_v^\beta = \xi_{u_1}^{\alpha_1} \cdots \xi_{u_n}^{\beta_n}$ can be expressed in terms of sums of the polynomials $(\mu\xi_u + \nu\xi_v)^m = (\mu_1\xi_{u_1} + \ldots + \nu_n\xi_{v_n})^m$, with $\mu, \nu \in \mathbb{R}^n$ and $m \leqslant |\alpha| + |\beta|$. So, if we can show that for $\sigma(\xi_u, \xi_v) = (2\pi i)^m(\mu\xi_u + \nu\xi_v)^m$, $m \in \mathbb{N}$, $\mathrm{Op}^\pi(\sigma)$ agrees with $\lambda\Big((\mu_1 X_{p_1} + \ldots + \nu_n X_{q_n})^m\Big)$ on $\mathbf{H}_n$, then we are done.

To start with, we observe that $\sigma_{\tilde{X}}(\xi_u, \xi_v) = 2\pi i(\mu\xi_u + \nu\xi_v)$ is the symbol of the left-invariant vector field $\tilde{X} = (\mu\tilde{X}_p + \nu\tilde{X}_q)$. We furthermore know that there exists some $U \in U(n)$ such that $U^*(\mu, \nu)^t = (\alpha, 0, \ldots, 0)^t$ for some $\alpha \in \mathbb{R}$. Let us assume, without loss of generality, that $\alpha = 1$. We now find that

$$\frac{1}{2\pi i}(\sigma_{\tilde{X}} \circ U)(\xi_u, \xi_v) = \Big\langle \begin{pmatrix} \mu \\ \nu \end{pmatrix}, U \begin{pmatrix} \xi_u \\ \xi_v \end{pmatrix} \Big\rangle = \Big\langle U^* \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \begin{pmatrix} \xi_u \\ \xi_v \end{pmatrix} \Big\rangle = \xi_{u_1} = \frac{1}{2\pi i}\sigma_{\tilde{X}_{p_1}}(\xi_u, \xi_v)$$

and hence

$$\begin{aligned}(2\pi i\, \xi_{u_1})^m &= \sigma_{\tilde{X}_{p_1}^m}(\xi_u, \xi_v) = \big((\sigma_{\tilde{X}} \circ U)(\xi_u, \xi_v)\big)^m \\ &= (\sigma_{\tilde{X}})^m(U(\xi_u, \xi_v)) = \big((\sigma_{\tilde{X}})^m \circ U\big)(\xi_u, \xi_v) = (\sigma \circ U)(\xi_u, \xi_v).\end{aligned}$$

Lemma 5.21 now implies the existence of an operator $\tilde{U} \in \mathcal{U}(L^2(\mathbf{H}_n))$ such that

$$\tilde{U}\mathrm{Op}^\pi(\sigma)\tilde{U}^* = \mathrm{Op}^\pi(\sigma \circ U) = (2\pi i)^m \,\mathrm{Op}^\pi(\xi_{u_1}^m) = \tilde{X}_{p_1}^m = \lambda(X_{p_1}^m).$$

Rewriting the above, we obtain

$$\begin{aligned}(2\pi i)^m \,\mathrm{Op}^\pi((\mu\xi_u + \nu\xi_v)^m) &= \mathrm{Op}^\pi(\sigma) = \mathrm{Op}^\pi(\sigma_{\tilde{X}_{p_1}^m} \circ U^*) \\ &= \tilde{U}^*\mathrm{Op}^\pi(\sigma_{\tilde{X}_{p_1}^m})\tilde{U} = \tilde{U}^*\tilde{X}_{p_1}^m\tilde{U} = \Big(\tilde{U}^*\tilde{X}_{p_1}\tilde{U}\Big)^m \\ &= \big(\mathrm{Op}^\pi(\sigma_{\tilde{X}_{p_1}} \circ U^*)\big)^m = \big(\mathrm{Op}^\pi(\sigma_{\tilde{X}})\big)^m = \tilde{X}^m = \lambda(X^m)\end{aligned}$$

for $X = \mu X_p + \nu X_q$. Since $m \in \mathbb{N}$ was arbitrary this concludes the proof. □

Corollary 5.23. *Part (i) of Proposition 5.14 now follows as a special case of Theorem 5.22.*

Remark 5.24 (On the Difference between Right and Left Invariance)**.** If we had chosen the representation π_λ to be generated by Euclidean modulations and left $\mathbf{H}_n$-translations, the quantization defined by Formula (5.5) would require $\hat{\sigma}(-\mathcal{P},-\mathcal{Q})$ instead of $\hat{\sigma}(\mathcal{P},\mathcal{Q})$ and would yield right-invariant operators for symbols $(\Xi \mapsto \sigma(\Xi)) \in \mathscr{S}'(\mathbf{H}_n)$.

Let us note that due to the lack of difference between left and right-invariant vector fields in the Euclidean setting, and hence between right and left translations, there is but one Schrödinger representation ρ that can be used for the Weyl quantization.

5.5 Hörmander-Type Symbol Classes and ΨDO's

In this Section we will have a look at symbols that belong to some type of non-isotropic Heisenberg analogues of the classes $S^m_{1,0}(\mathbb{R}^{2n+1})$, for which the usual decay estimates are given with respect to an $\mathbf{H}_n$-homogeneous norm.

Definition 5.25. *The homogeneous norm $|\,.\,|_{\mathbf{H}_n}$ on the Heisenberg group $\mathbf{H}_n$ is defined by*

$$\begin{aligned} &|\,.\,|_{\mathbf{H}_n} : \mathbf{H}_n \to [0,\infty), \\ &(p,q,t) = X \mapsto |X|_{\mathbf{H}_n} := \Big(\big(|p|^2 + |q|^2 \big)^2 + t^2 \Big)^{1/4}. \end{aligned}$$

We will furthermore define the $\mathbf{H}_n$-Japanese brackets by

$$\prec X \succ := \big(1 + |X|^4_{\mathbf{H}_n}\big)^{1/4}.$$

Definition 5.26. *The class of $\mathbf{H}_n$-symbols of order $m \in \mathbb{R}$, denoted by $S^m(\mathbf{H}_n)$, is defined to be the set of all functions $\sigma \in C^\infty(\mathbb{R}^{2n+1} \times \mathbf{H}_n)$ for which for all multi-indices $\alpha = (\alpha_p, \alpha_q, \alpha_t)$, $\beta = (\beta_p, \beta_q, \beta_t) \in (\mathbb{N} \cup \{0\})^{2n+1}$ there exists $C_{\alpha,\beta} > 0$ such that*

$$\sup_{X \in \mathbf{H}_n} \left| \left(D^\alpha_\Xi \mathscr{D}^\beta_X \sigma \right)(\Xi, X) \right| \leqslant C_{\alpha,\beta} \prec \Xi \succ^{m - \langle \alpha \rangle} \tag{5.16}$$

if $\langle \alpha \rangle := |\alpha_p| + |\alpha_q| + 2\alpha_t$.
We set

$$S^\infty(\mathbf{H}_n) := \bigcup_{m \in \mathbb{R}} S^m(\mathbf{H}_n) \qquad \textit{and} \qquad S^{-\infty}(\mathbf{H}_n) := \bigcap_{m \in \mathbb{R}} S^m(\mathbf{H}_n)$$

and define $\mathrm{Op}^\pi S^m(\mathbf{H}_n)$ to be the space of $\mathbf{H}_n$-Weyl quantized operators with symbols in $S^m(\mathbf{H}_n)$.

Remark 5.27. Note that the Ξ-derivative is indeed the Euclidean derivative in $2n+1$ dimensions, whereas the X-derivative is understood to be a higher order application of the standard left-invariant vector fields on $\mathbf{H}_n$, defined as in Subsection 1.3.2.

Remark 5.28. As usual, the symbol classes $S^m(\mathbf{H}_n)$ are in fact Fréchet spaces if their topology is defined by the semi-norms

$$\|\sigma\|_{[j]} := \max_{|\alpha|+|\beta|\leqslant j} \sup_{\Xi, X\in\mathbb{R}^{4n+2}} \left|\left(D_\Xi^\alpha \mathscr{D}_X^\beta \sigma\right)(\Xi, X)\right| \prec \Xi \succ^{-m+\langle\alpha\rangle},$$

where $j \in \mathbb{N} \cup \{0\}$. Occasionally, we will also consider them as topological subspaces of the Fréchet space $C^\infty(\mathbb{R}^{4n+2})$.

Examples 5.29. (i) The left-invariant vector fields. For the standard left-invariant vector fields on $\mathbf{H}_n$ we have

$$dR(X_{p_j}) \in \mathrm{Op}^\pi S^1(\mathbf{H}_n), \quad dR(X_{q_k}) \in \mathrm{Op}^\pi S^1(\mathbf{H}_n), \quad dR(X_t) \in \mathrm{Op}^\pi S^2(\mathbf{H}_n) \tag{5.17}$$

as we recall

$$\left.\begin{array}{rcl} 2\pi i\, dR(X_{p_j}) &=& \mathscr{D}_{p_j} = \mathrm{Op}^\pi(\xi_{u_j}), \\ 2\pi i\, dR(X_{q_k}) &=& \mathscr{D}_{q_k} = \mathrm{Op}^\pi(\xi_{v_j}), \\ 2\pi i\, dR(X_t) &=& \mathscr{D}_t = \mathrm{Op}^\pi(\xi_w) \end{array}\right\} \tag{5.18}$$

from the quantization problem in Section 5.1. The orders given by (5.17) agree conveniently with the natural homogeneous degrees of the left-invariant vector fields. (Cf. [27] p. 916)

(ii) The sub-Laplacian. The sub-elliptic operator

$$\Delta_{\mathbf{H}_n} := -\sum_{j=1}^n \Big(\big(\frac{\partial}{\partial p_j} - \frac{1}{2}q_j\frac{\partial}{\partial t}\big)^2 + \big(\frac{\partial}{\partial q_j} + \frac{1}{2}p_j\frac{\partial}{\partial t}\big)^2\Big)$$

has the symbol $\sigma_{\Delta_{\mathbf{H}_n}}(\xi_u, \xi_v) = -(2\pi i)^2(|\xi_u|^2 + |\xi_v|^2)$, which is obviously a member of $S^2(\mathbf{H}_n)$. (We remark that the negative sign was chosen in accordance with works by Folland, Stein et al. We can equally choose the positive sign more in the spirit of Hörmander's sums of squares, e.g.)

(iii) Differential operators. Any polynomial function in Ξ of finite degree is a member of some $S^m(\mathbf{H}_n)$. Hence by Theorem 5.22 all left-invariant differential operators on $\mathbf{H}_n$ belong to $\mathrm{Op}^\pi S^\infty(\mathbf{H}_n)$.

(iv) Adjoint and transpose. Let us recall from by Proposition 5.16 that for given $\sigma \in \mathscr{S}'(\mathbf{H}_n)$ the symbols of the adjoint and transpose of $\mathrm{Op}^\pi(\sigma)$ are given by $\bar{\sigma}$ and $\sigma^t := (\Xi, X) \mapsto \sigma(-\Xi, X)$. Hence, if σ belongs to some $S^m(\mathbf{H}_n)$, so do $\bar{\sigma}$ and σ^t.

Remark 5.30 (Euclidean vs. $\mathbf{H}_n$-symbol classes)**.** For the usual Hörmander symbol classes $S^m_{1,0}(\mathbb{R}^{2n+1})$ we have neither $S^m(\mathbf{H}_n) \subseteq S^m_{1,0}(\mathbb{R}^{2n+1})$ nor $S^m(\mathbf{H}_n) \supseteq S^m_{1,0}(\mathbb{R}^{2n+1})$: the rate of decay in ξ_w in Condition (5.16) is weaker than usual for positive exponents m, but weaker for negative ones. This leaves a possibility for the first inclusion for positive m and for the second inclusion for negative m. But the growth in p and q of the left-invariant derivatives $\mathscr{D}_p$ and $\mathscr{D}_q$ is not necessarily compensated by the behaviour in $X = (p, q, t)$ of $\sigma \in S^m_{1,0}(\mathbb{R}^{2n+1})$. This necessarily excludes either of the two inclusions.

The following Proposition guarantees that symbols in $S^m(\mathbf{H}_n)$ define continuous operators on $\mathscr{S}(\mathbf{H}_n)$. Moreover, it assures us that convergent sequences of symbols quantize convergent nets of operators.

Proposition 5.31. *The following assertions hold true:*

(*a*) *For any $\sigma \in S^m(\mathbf{H}_n)$, $m \in \mathbb{R}$, the operator $\mathrm{Op}^\pi(\sigma)$ is continuous from $\mathscr{S}(\mathbf{H}_n)$ into itself.*

Part (*b*) *Let σ_k be a sequence of symbols in $S^m(\mathbf{H}_n)$, $m \in \mathbb{R}$ which satisfy the symbol estimates 5.16 uniformly in k and which converge to some σ in the topology of $C^\infty(\mathbb{R}^{4n+2})$. Then $\sigma \in S^m(\mathbf{H}_n)$ and $\mathrm{Op}^\pi(\sigma_k)f \xrightarrow{\mathscr{S}(\mathbf{H}_n)} \mathrm{Op}^\pi(\sigma)f$ for all $f \in \mathscr{S}(\mathbf{H}_n)$.*

Proof. (*a*) Let us recall from Equalities 5.8 and 5.10 that the $\mathbf{H}_n$-Weyl-quantization of some symbol $\sigma \in \mathscr{S}(\mathbf{H}_n)$ can be expressed via the integral

$$\begin{aligned}\big(\mathrm{Op}^\pi(\sigma)f\big)(X) &= \iint \sigma\big(\Xi, X \cdot (\tfrac{1}{2}\mathcal{P})\big)\, e^{-2\pi i \langle \Xi, \mathcal{P}\rangle}\, f(X \cdot \mathcal{P})\, d\mathcal{P}\, d\Xi \\ &= \iint \sigma\big(\Xi, \tfrac{1}{2}(X+Y)\big)\, e^{2\pi i \langle \Xi, Y^{-1}\cdot X\rangle}\, f(Y)\, dY\, d\Xi. \qquad (5.19)\end{aligned}$$

In order to show that this iterated integral converges absolutely for $\sigma \in S^m(\mathbf{H}_n)$, we will make use of the function

$$\begin{aligned}g(\Xi, X) &:= \int \sigma\big(\Xi, X \cdot (\tfrac{1}{2}\mathcal{P})\big)\, e^{-2\pi i \langle \Xi, \mathcal{P}\rangle}\, f(X \cdot \mathcal{P})\, d\mathcal{P} \\ &= \int \sigma\big(\Xi, \tfrac{1}{2}(X+Y)\big)\, e^{2\pi i \langle \Xi, Y^{-1}\cdot X\rangle}\, f(Y)\, dY,\end{aligned}$$

applying the usual techniques of integration by parts, etc. To this aim we define the operator

$$\mathscr{L}_{\mathcal{P}} := \frac{1}{4}\Big(|D_u|^2 + |D_v|^2\Big)^2 + \frac{1}{2}D_w^2, \qquad (5.20)$$

for we which we observe the relation

$$(1+\mathscr{L}_{\mathcal{P}})e^{-2\pi i\langle\Xi,\mathcal{P}\rangle} = \big(1+|\Xi|_{\mathbf{H}_n}\big)^4 e^{-2\pi i\langle\Xi,\mathcal{P}\rangle} = \prec \Xi \succ^4 e^{-2\pi i\langle\Xi,\mathcal{P}\rangle}.$$

We then compute

$$\begin{aligned} g(\Xi,X) &= \int \frac{(1+\mathscr{L}_{\mathcal{P}})^N e^{-2\pi i\langle\Xi,\mathcal{P}\rangle}}{\prec \Xi \succ^{4N}} \sigma\big(\Xi, X\cdot(\tfrac{1}{2}\mathcal{P})\big)\, f(X\cdot\mathcal{P})\, d\mathcal{P} \\ &= \int \frac{e^{2\pi i\langle\Xi,Y^{-1}\cdot X\rangle}}{\prec \Xi \succ^{4N}} (1+\mathscr{L}_{\mathcal{P}})^N \Big(\sigma\big(\Xi, X\cdot(\tfrac{1}{2}\mathcal{P})\big)\, f(X\cdot\mathcal{P})\Big)\, d\mathcal{P} \\ &= \sum_{<\alpha+\beta>\leqslant 4N} C_{\alpha,\beta} \int \frac{e^{2\pi i\langle\Xi,Y^{-1}\cdot X\rangle}}{\prec \Xi \succ^{4N}} D^{\alpha}_{\mathcal{P}}\sigma\big(\Xi, X\cdot(\tfrac{1}{2}\mathcal{P})\big) D^{\beta}_{\mathcal{P}} f(X\cdot\mathcal{P})\, d\mathcal{P} \\ &= \sum_{<\alpha+\beta>\leqslant 4N} C_{\alpha,\beta} \int \frac{e^{2\pi i\langle\Xi,Y^{-1}\cdot X\rangle}}{\prec \Xi \succ^{4N}} \frac{1}{2}\big(\mathscr{D}^{\alpha}_{X}\sigma\big)\big(\Xi, X\cdot(\tfrac{1}{2}\mathcal{P})\big)(\mathscr{D}^{\beta}_{X} f)(X\cdot\mathcal{P})\, d\mathcal{P}, \end{aligned}$$

which due to the definition of the symbols classes $S^m(\mathbf{H}_n)$ yields

$$\begin{aligned} |g(\Xi,X)| &\leqslant C_N \sum_{<\alpha+\beta>\leqslant 4N} \int \prec \Xi \succ^{-4N} \Big|\big(\mathscr{D}^{\alpha}_{X}\sigma\big)\big(\Xi, X\cdot(\tfrac{1}{2}\mathcal{P})\big)\Big| \Big|(D^{\beta}_{\mathcal{P}} f)(X\cdot\mathcal{P})\Big|\, d\mathcal{P} \\ &\leqslant C'_N \prec \Xi \succ^{-4N+m} . \end{aligned}$$

We thus conclude that $g(\,.\,,X)$ is a member of $L^1(\mathbb{R}^{2n+1})$ uniformly in X if only N is large enough. The estimate furthermore shows that $\big(\mathrm{Op}^{\pi}(\sigma)f\big)(X) = \int g(\Xi,X)\, d\Xi$ is uniformly bounded in $X \in \mathbf{H}_n$.

In order to check that $\mathrm{Op}^{\pi}(\sigma)f$ is indeed Schwartz class, we will scrutinize the cases $X^{\alpha}\mathrm{Op}^{\pi}(\sigma)f$ and $\mathcal{D}^{\beta}_{X}\mathrm{Op}^{\pi}(\sigma)f$ for each of the vector components $p_{j_X}, q_{k_X}, t_X,\ j,k = 1,\dots,n$, of $X = (p_X, q_X, t_X)$. A simple induction argument can finally be employed to obtain full generality.

Let us first have a look at multiplication by polynomials in X. A straight-forward computation yields

$$p_{j_X} e^{2\pi i\langle\Xi,Y^{-1}\cdot X\rangle} = D_{\xi_u} e^{2\pi i\langle\Xi,Y^{-1}\cdot X\rangle} - p_{j_Y} e^{2\pi i\langle\Xi,Y^{-1}\cdot X\rangle},$$

and an analogous relation for q_{k_X}, whereas for t_X one obtains

$$t_X e^{2\pi i\langle\Xi,Y^{-1}\cdot X\rangle} = \big(D_{\xi_w} + p_{j_Y} + \frac{1}{2}(p_Y q_X - q_Y p_X)\big) e^{2\pi i\langle\Xi,Y^{-1}\cdot X\rangle}.$$

In view of Integral 5.19, this implies that $X^\alpha \mathrm{Op}^\pi(\sigma)f$ translates into sums of $D_\Xi^\beta(\sigma)$ and $Y^\gamma f$ inside 5.19. Neither of these terms harms the rate of convergence; to the contrary, $D_\Xi^\beta(\sigma)$ even improve the decay in Ξ. Thus by the same argument as above, the integrals defining $X^\alpha \mathrm{Op}^\pi(\sigma)f$ are both bounded and absolutely convergent uniformly in $X \in \mathbf{H}_n$.

In the case of $\mathcal{D}_X^\beta \mathrm{Op}^\pi(\sigma)f$, three simple calculations yield

$$\left.\begin{array}{lcl} \mathcal{D}_{p_j X} e^{2\pi i \langle \Xi, Y^{-1}\cdot X\rangle} & = & \xi_{u_j} e^{2\pi i \langle \Xi, Y^{-1}\cdot X\rangle}, \\ \mathcal{D}_{q_j X} e^{2\pi i \langle \Xi, Y^{-1}\cdot X\rangle} & = & \xi_{v_j} e^{2\pi i \langle \Xi, Y^{-1}\cdot X\rangle}, \\ \mathcal{D}_t e^{2\pi i \langle \Xi, Y^{-1}\cdot X\rangle} & = & \xi_w e^{2\pi i \langle \Xi, Y^{-1}\cdot X\rangle}. \end{array}\right\}$$

Hence the absolute convergence of Integral 5.19, allows us to compute

$$\begin{aligned} \big(\mathcal{D}_{p_j X} \mathrm{Op}^\pi(\sigma)f\big)(X) &= \iint \xi_{u_j} \sigma\big(\Xi, \tfrac{1}{2}(X+Y)\big)\, e^{2\pi i \langle \Xi, Y^{-1}\cdot X\rangle} f(Y)\, dY\, d\Xi \\ &\quad + \iint \big(\mathcal{D}_{p_j X}\sigma\big)\big(\Xi, \tfrac{1}{2}(X+Y)\big)\, e^{2\pi i \langle \Xi, Y^{-1}\cdot X\rangle} f(Y)\, dY\, d\Xi, \end{aligned}$$

and similar expressions for $\mathcal{D}_{q_k X} \mathrm{Op}^\pi(\sigma)f$ and $\mathcal{D}_{t_X} \mathrm{Op}^\pi(\sigma)f$. By the same arguments as above, the corresponding oscillatory integrals involved are absolutely convergent and bounded uniformly in X. Thus we have shown that $\mathrm{Op}^\pi(\sigma)f$ is indeed Schwartz on $\mathbf{H}_n$. This proves part (a).

(b) essentially follows from an application of the latter arguments to show that the limit in k in the C^∞-topology interchanges with the oscillatory integrals. That σ must be a member of $S^m(\mathbf{H}_n)$ follows from the uniformity in k of the symbol estimates 5.16. This concludes our proof. □

5.6 The Heisenberg-Moyal Product

As we are naturally interested in the composition of two $\mathbf{H}_n$-Weyl quantized operators, we would like to know whether a composite operator can be assigned a symbol by the $\mathbf{H}_n$-Weyl calculus. If we suppose that, say, $\sigma_1(\mathscr{D}, \mathscr{X})$ and $\sigma_2(\mathscr{D}, \mathscr{X})$ map $\mathscr{S}(\mathbf{H}_n)$ into itself, then their composition does so and is the $\mathbf{H}_n$-Weyl quantization of a uniquely determined symbol in $\mathscr{S}(\mathbb{R}^{4n+2})$ due to Proposition 5.6. That is, if the latter symbol is denoted by $\sigma_1 \circledast \sigma_1$, we have

$$\sigma_1(\mathscr{D}, \mathscr{X})\, \sigma_2(\mathscr{D}, \mathscr{X}) = (\sigma_1 \circledast \sigma_2)(\mathscr{D}, \mathscr{X}). \tag{5.21}$$

We will call $\sigma_1 \circledast \sigma_2$ the Heisenberg-Moyal or twisted product of σ_1 and σ_1 or for the sake of abbreviation throughout this text also $\circledast$-product.

It turns out that just like in the Euclidean case the symbol $\sigma_1 \circledast \sigma_2$ is given by a specific convolution type product of the symbols σ_1 and σ_1, which, at least formally, bear some similarity with the standard $\mathbb{R}^{2n+1}$-Moyal product, i.e., the $\sharp$-product of two $\mathbb{R}^{2n+1}$-Weyl-quantized pseudo-differential operators, yet obviously reflects the Heisenberg group structure.

Given existence, uniqueness and continuity of the $\circledast$-product as a map from $\mathscr{S}(\mathbb{R}^{4n+2}) \times \mathscr{S}(\mathbb{R}^{4n+2})$ to $\mathscr{S}(\mathbb{R}^{4n+2})$, we will compute an explicit formula for this case and show that it extends as a continuous map from $S^{m_1}(\mathbf{H}_n) \times S^{m_2}(\mathbf{H}_n)$ to $S^{m_1+m_2}(\mathbf{H}_n)$.

5.6.1 An Asymptotic Expansion

Before we derive an integral formula that will help us to prove the latter mapping properties, we will attempt a formal derivation of an asymptotic expansion for $\sigma_1 \circledast \sigma_2$. For the sake of convenience through convergent integrals, let us again suppose that $\sigma_1, \sigma_2 \in \mathscr{S}(\mathbb{R}^{4n+2})$. We commence our calculations by writing out Formula (5.21) explicitly:

$$\begin{aligned}
\sigma_1(\mathscr{D},\mathscr{X})\,\sigma_2(\mathscr{D},\mathscr{X})f &= \iint \hat{\sigma}_1(\tilde{\mathcal{P}},\tilde{\mathcal{Q}})\,\pi(\tilde{\mathcal{P}},\tilde{\mathcal{Q}}) \iint \hat{\sigma}_2(\mathcal{P}',\mathcal{Q}')\,\pi(\mathcal{P}',\mathcal{Q}')f \\
&\quad d\mathcal{P}'\,d\mathcal{Q}'\,d\tilde{\mathcal{P}}\,d\tilde{\mathcal{Q}} \\
&= \iiiint \hat{\sigma}_1(\tilde{\mathcal{P}},\tilde{\mathcal{Q}})\,\hat{\sigma}_2(\mathcal{P}',\mathcal{Q}') \\
&\quad \pi\big((\tilde{\mathcal{P}},\tilde{\mathcal{Q}},0) \odot_{\mathbf{H}_{2,n}} (\mathcal{P}',\mathcal{Q}',0)\big) f\,d\mathcal{P}'\,d\mathcal{Q}'\,d\tilde{\mathcal{P}}\,d\tilde{\mathcal{Q}} \\
&= \iint (\widehat{\sigma_1 \circledast \sigma_2})(\mathcal{P},\mathcal{Q})\,\pi(\mathcal{P},\mathcal{Q},0)f\,d\mathcal{P}\,d\mathcal{Q}.
\end{aligned}$$

For the last identity to make sense we require two necessary and sufficient conditions, the first of which can be expressed as the existence of some uniquely determined element

$s = \mathcal{S} = \mathcal{S}(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \mathcal{P}', \mathcal{Q}') \in \mathbb{R}$ such that

$$\begin{aligned}(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, 0) \odot_{\mathbf{H}_{2,n}} (\mathcal{P}', \mathcal{Q}', 0) = \ &(\tilde{u} + u', \tilde{v} + v', \tilde{w} + w' + \frac{1}{2}(\tilde{u}v' - \tilde{v}u'),\\ &\tilde{x} + x' + \frac{1}{4}(\tilde{v}z' - \tilde{z}v'), \tilde{y} + y' - \frac{1}{4}(\tilde{u}z' - \tilde{z}u'), \tilde{z} + z',\\ &\frac{1}{2}(\tilde{u}x' - \tilde{x}u') + \frac{1}{2}(\tilde{v}y' - \tilde{y}v') + \frac{1}{2}(\tilde{w}z' - \tilde{z}w')\\ &+ \frac{1}{8}z'(\tilde{u}v' - \tilde{v}u') - \frac{1}{8}\tilde{z}(\tilde{u}v' - \tilde{v}u'))\\ = \ &(u, v, w, x, y, z, s) = (\mathcal{P}, \mathcal{Q}, \mathcal{S}).\end{aligned}$$

Here we see that $\mathcal{P}$ and $\mathcal{Q}$ relate to $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ via a Haar measure-preserving, and thus Lebesgue measure-preserving change of variables, with the central variable $\mathcal{S}$ carrying some additional information about the $\circledast$-product. But existence and uniqueness of such an element $(\mathcal{P}, \mathcal{Q}, \mathcal{S})$ simply follows from the relation

$$(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, 0) = (\mathcal{P}, \mathcal{Q}, \mathcal{S}) \odot_{\mathbf{H}_{2,n}} (\mathcal{P}', \mathcal{Q}', 0)^{-1}.$$

The second condition now states the Fourier transform of the symbol must be given by

$$\widehat{(a \circledast b)}(\mathcal{P}, \mathcal{Q}) = \iint \hat{a}(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}})\, \hat{b}(\mathcal{P}', \mathcal{Q}')\, \pi(0, 0, \mathcal{S})\, d\mathcal{P}'\, d\mathcal{Q}'. \tag{5.22}$$

Employing the measure-preserving change of variables

$$\begin{aligned}\tilde{u} &= u - u',\\ \tilde{v} &= v - v',\\ \tilde{w} &= w - w' - \frac{1}{2}\left((u - u')v' - (v - v')u'\right) = w - w' - \frac{1}{2}\left(uv' - vu'\right),\\ \tilde{x} &= x - x' - \frac{1}{4}\left((v - v')z' - (z - z')v'\right) = x - x' - \frac{1}{4}(vz' - zv'),\\ \tilde{y} &= y - y' + \frac{1}{4}\left((u - u')z' - z(u - u')\right) = y - y' + \frac{1}{4}\left(uz' - zu'\right),\\ \tilde{z} &= z - z',\end{aligned}$$

with

$$\begin{aligned}\mathcal{S} = s &= 1/2\,(\tilde{u}x' - \tilde{x}u') + 1/2\,(\tilde{v}y' - \tilde{y}v') + 1/2\,(\tilde{w}z' - \tilde{z}w') + 1/8\,z'(\tilde{u}v' - \tilde{v}u') \\ &\quad - 1/8\,\tilde{z}(\tilde{u}v' - \tilde{v}u') \\ &= \frac{1}{2}\left(ux' - xu'\right) + \frac{1}{2}\left(vy' - yv'\right) + \frac{1}{2}\left(wz' - zw'\right) + \frac{1}{4}z'\left(uv' - vu'\right) \\ &\quad - \frac{1}{8}z\left(uv' - vu'\right),\end{aligned}$$

we can rewrite Formula (5.22) equivalently as

$$\begin{aligned}(\widehat{\sigma_1 \circledast \sigma_2})(\mathcal{P}, \mathcal{Q}) = \int \cdots \int &\hat{\sigma}_1\big(u - u', v - v', w - w' - \frac{1}{2}\left(uv' - vu'\right), \\ &x - x' - \frac{1}{4}\left(vz' - zv'\right), y - y' + \frac{1}{4}\left(uz' - zu'\right), z - z'\big) \\ &\times \hat{\sigma}_2(u', v', w', x', y', z') \qquad (5.23) \\ &\times e^{2\pi i\left(\frac{1}{2}(ux'-xu') + \frac{1}{2}(vy'-yv') + \frac{1}{2}(wz'-zw')\right)} \\ &\times e^{2\pi i\left(\frac{1}{4}z'(uv'-vu') - \frac{1}{8}z(uv'-vu')\right)}\, du'\, dv'\, dw'\, dx'\, dy'\, dz'. \qquad (5.24)\end{aligned}$$

To obtain a neat formula and the above-mentioned asymptotic expansion, we need to write Formula (5.24) as an $\mathbb{R}^{2(2n+1)}$-convolution product of $\hat{\sigma}_1$ and $\hat{\sigma}_2$ twisted by some exponential factor. To this end, we rearrange the terms and express the disturbing translations of $\hat{a}$ as exponentials in D_t, D_x, D_y. This yields

$$\begin{aligned}(\widehat{\sigma_1 \circledast \sigma_2})(u, v, w, x, y, z) = \int \cdots \int &e^{\pi i\left((ux'-xu') + (vy'-yv') + (wz'-zw') + \frac{1}{2}z'(uv'-vu')\right)} \\ &\times e^{-\frac{1}{4}\pi i z(uv'-vu')}\, \hat{\sigma}_1(u', v', w', x', y', z') \\ &\times e^{2\pi i\left(-\frac{1}{2}(uv'-vu')D_w - \frac{1}{4}(vz'-zv')D_x + \frac{1}{4}(uz'-zu')D_y\right)} \\ &\hat{\sigma}_2\left(u - u', v - v', w - w', x - x', y - y', z - z'\right) \\ &du'\, dv'\, dw'\, dx'\, dy'\, dz'.\end{aligned}$$

If we now formally invert the Fourier transform of this expression, we obtain an asymptotic power series of derivatives in $\xi_u, \xi_v, \xi_w, \chi_x, \chi_y, \chi_z$ and $\xi'_u, \xi'_v, \xi'_w, \chi'_x, \chi'_y, \chi'_z$ of products of σ_1, σ_2 and monomials in ξ_w, χ_x, χ_y:

$$
\begin{aligned}
&(\sigma_1 \circledast \sigma_2)(\xi_u, \xi_v, \xi_w, \chi_x, \chi_y, \chi_z) \\
&\quad \sim \quad \Bigg(e^{\pi i \left(D_{\xi_u} D_{\chi'_x} - D_{\chi_x} D_{\xi'_u} \right)} \, e^{\pi i \left(D_{\xi_v} D_{\chi'_y} - D_{\chi_y} D_{\xi'_v} \right)} \\
&\qquad e^{\pi i \left(D_{\xi_w} D_{\chi'_z} - D_{\chi_z} D_{\xi'_w} \right)} \, e^{\frac{\pi i}{2} D_{\chi'_z} \left(D_{\xi_u} D_{\xi'_v} - D_{\xi_v} D_{\xi'_u} \right)} \\
&\qquad e^{-\frac{\pi i}{4} D_{\chi_z} \left(D_{\xi_u} D_{\xi'_v} - D_{\xi_v} D_{\xi'_u} \right)} \, e^{\pi i \xi_w \left(D_{\xi_u} D_{\xi'_v} - D_{\xi_v} D_{\xi'_u} \right)} \\
&\qquad e^{\frac{\pi i}{2} \chi_x \left(D_{\xi_v} D_{\chi'_z} - D_{\chi_z} D_{\xi'_v} \right)} \, e^{-\frac{\pi i}{2} \chi_y \left(D_{\xi_u} D_{\chi'_z} - D_{\chi_z} D_{\xi'_u} \right)} \\
&\qquad \sigma_1(\xi_u, \xi_v, \xi_w, \chi_x, \chi_y, \chi_z) \, \sigma_2(\xi'_u, \xi'_v, \xi'_w, \chi'_x, \chi'_y, \chi'_z) \Bigg) \Bigg|_{(\Xi, \mathcal{X}) = (\Xi', \mathcal{X}')} \\
&= \Bigg(\sum_{j_1=0}^{\infty} \frac{(\pi i)^{j_1}}{j_1!} \left(D_{\xi_u} D_{\chi'_x} - D_{\chi_x} D_{\xi'_u} \right)^{j_1} \sum_{j_2=0}^{\infty} \frac{(\pi i)^{j_2}}{j_2!} \left(D_{\xi_v} D_{\chi'_y} - D_{\chi_y} D_{\xi'_v} \right)^{j_2} \\
&\qquad \sum_{j_3=0}^{\infty} \frac{(\pi i)^{j_3}}{j_3!} \left(D_{\xi_w} D_{\chi'_z} - D_{\chi_z} D_{\xi'_w} \right)^{j_3} \sum_{j_4=0}^{\infty} \frac{(\pi i)^{j_4}}{2^{j_4} j_4!} D_{\chi'_z}^{j_4} \left(D_{\xi_u} D_{\xi'_v} - D_{\xi_v} D_{\xi'_u} \right)^{j_4} \\
&\qquad \sum_{j_5=0}^{\infty} \frac{(-\pi i)^{j_5}}{4^{j_5} j_5!} D_{\chi_z}^{j_5} \left(D_{\xi_u} D_{\xi'_v} - D_{\xi_v} D_{\xi'_u} \right)^{j_5} \sum_{j_6=0}^{\infty} \frac{(\pi i)^{j_6}}{j_6!} \xi_w^{j_6} \left(D_{\xi_u} D_{\xi'_v} - D_{\xi_v} D_{\xi'_u} \right)^{j_6} \\
&\qquad \sum_{j_7=0}^{\infty} \frac{(\pi i)^{j_7}}{2^{j_7} j_7!} \chi_x^{j_7} \left(D_{\xi_v} D_{\chi'_z} - D_{\chi_z} D_{\xi'_v} \right)^{j_7} \sum_{j_8=0}^{\infty} \frac{(-\pi i)^{j_8}}{2^{j_8} j_8!} \chi_y^{j_8} \left(D_{\xi_u} D_{\chi'_z} - D_{\chi_z} D_{\xi'_u} \right)^{j_8} \\
&\qquad \sigma_1(\xi_u, \xi_v, \xi_w, \chi_x, \chi_y, \chi_z) \, \sigma_2(\xi'_u, \xi'_v, \xi'_w, \chi'_x, \chi'_y, \chi'_z) \Bigg) \Bigg|_{(\Xi, \mathcal{X}) = (\Xi', \mathcal{X}')} . \qquad (5.25)
\end{aligned}
$$

Remark 5.32 (Preservation of Algebraic Structure)**.** As the $\circledast$-product preserves the $\mathbf{H}_{2,n}$-group structure, it also preserves all the commutator brackets that define $\mathfrak{h}_{2,n}$, and in particular the Heisenberg commutation relations

$$
\begin{aligned}
[D_{x_j}, X_l] = (2\pi i)^{-1} \delta_{j,l} \, I_{L^2(\mathbb{R}^n)} \;&\Leftrightarrow\; [L_{p_j}, L_{q_l}] = \delta_{j,l} \, L_t \\
&\Leftrightarrow\; [\mathscr{D}_{p_j}, \mathscr{D}_{q_l}] = (2\pi i)^{-1} \delta_{j,l} \, \mathscr{D}_t.
\end{aligned}
$$

But let us prove this last equation directly from the composition formula. We recall that

$$
\mathrm{Op}^{\pi}(\xi_{u_j}) = \mathscr{D}_{u_j}, \quad \mathrm{Op}^{\pi}(\xi_{v_l}) = \mathscr{D}_{q_l}.
$$

For the composition of the symbols ξ_{u_j} and ξ_{u_l} we compute

$$\begin{aligned}\xi_{u_j} \circledast \xi_{v_l} &= \left(\xi_u\, \xi'_v + \pi i\, \xi_w \left(D_{\xi_u}\, D_{\xi'_v}\, \xi_{u_j}\, \xi'_{v_l} - D_{\xi_v}\, D_{\xi'_u}\, \xi_{u_j}\, \xi'_{v_l}\right) + 0\right)\Big|_{\Xi=H} \\ &= \left(\xi_{u_j}\, \xi'_{v_l} + \pi i\, \xi_w\, \left((2\pi i)^{-2} \delta_{j,l} - 0\right)\right)\Big|_{(\Xi,\mathcal{X})=(\Xi',\mathcal{X}')} \\ &= \xi_{u_j}\, \xi_{v_l} + \delta_{j,l}\, \frac{\pi i}{(2\pi i)^2}\, \xi_w,\end{aligned}$$

and analogously

$$\xi_{v_l} \circledast \xi_{u_j} = \xi_{u_j}\, \xi_{v_l} - \delta_{j,l}\, \frac{\pi i}{(2\pi i)^2}\, \xi_w.$$

Hence, we recover

$$\begin{aligned}[\mathscr{D}_{p_j}, \mathscr{D}_{q_l}] &= \mathrm{Op}^\pi(\xi_{u_j})\, \mathrm{Op}^\pi(\xi_{v_l}) - \mathrm{Op}^\pi(\xi_{v_l})\, \mathrm{Op}^\pi(\xi_{u_j}) \\ &= \mathrm{Op}^\pi(\xi_{u_j} \circledast \xi_{v_l}) - \mathrm{Op}^\pi(\xi_{v_l} \circledast \xi_{u_j}) \\ &= (2\pi i)^{-1} \delta_{j,l}\, \mathrm{Op}^\pi(\xi_w) \\ &= (2\pi i)^{-1} \delta_{j,l}\, \mathscr{D}_t.\end{aligned}$$

Remark 5.33 (Clash with Symbol Classes)**.** It is a priori not clear how the appearance of $D_{\mathcal{X}}$ and $D_{\mathcal{X}'}$ instead of $\mathcal{D}_{\mathcal{X}}$ and $\mathcal{D}_{\mathcal{X}'}$ in Formula 5.25 affects the symbol estimates (5.16) for $\sigma_1 \circledast \sigma_2$, although the terms

$$\chi_x^{j_7} \left(D_{\xi_v} D_{\chi'_z} - D_{\chi_z} D_{\xi'_v}\right)^{j_7}_{(\Xi,\mathcal{X})=(\Xi',\mathcal{X}')} \quad \text{and} \quad \chi_y^{j_8} \left(D_{\xi_u} D_{\chi'_z} - D_{\chi_z} D_{\xi'_u}\right)^{j_8}_{(\Xi,\mathcal{X})=(\Xi',\mathcal{X}')}$$

may account for some of these differences.

But the structure of Formula 5.25 itself poses a problem for a useful asymptotic expansion due to the third-last term

$$\xi_w^{j_6} \left(D_{\xi_u} D_{\xi'_v} - D_{\xi_v} D_{\xi'_u}\right)^{j_6}_{(\Xi,\mathcal{X})=(\Xi',\mathcal{X}')}$$

which neither decreases nor increases the order of $\sigma_1 \circledast \sigma_2$ by Defintion 5.26.

So, while this quantization via $\mathbf{H}_{2,n}$ respects the $\mathbf{H}_n$-structure and homogeneity, it creates a severe problem for formal asymptotic expansions.

5.6.2 An Oscillatory Integral Representation

Since asymptotic expansion does not provide the adequate tool to study the $\mathbf{H}_n$-Moyal product for the symbol classes $S^m(\mathbf{H}_n)$, we may try to find more clarity through an integral formula for it. In order to express the $\mathbf{H}_n$-Moyal product in such manner, we will employ the integral formula of the Euclidean Moyal product, usually denoted by $\tau_1 \sharp \tau_2$, i.e., the product of symbols for the standard Weyl quantization on the phase space $\mathbb{R}^{4n+2}$. By Theorem 5.6, we know that any pseudo-differential operator A acting continuously on $\mathscr{S}(\mathbb{R}^{4n+2})$ can be expressed both as the $\mathbb{R}^{2n+1}$-Weyl quantization of some symbol τ and as the $\mathbf{H}_n$-Weyl quantization of some symbol σ, i.e., $\tau(D, X) = A = \sigma(\mathscr{D}, \mathscr{X})$. In the following we will make use of the fact that a modification of σ, say σ_M, can be $\mathbb{R}^{2n+1}$-Weyl quantized to yield the same operator. We cast this relation into the following definition.

Definition 5.34. *Let two symbols $\tau, \sigma \in \mathscr{S}'(\mathbb{R}^{4n+2})$ be such that $\tau(D, X) = \sigma(\mathscr{D}, \mathscr{X})$. We then denote τ also by σ_M and call it the modified symbol associated to σ.*

The following statement provides a handy formula for well behaved symbols to start with. Its proof will thus not be compromised by the concern over convergence and interchangeability of occurring integrals. Using this formula, we will subsequently show that the $\mathbf{H}_n$-Moyal product in fact extends to a continuous map from $S^{m_1}(\mathbf{H}_n) \times S^{m_2}(\mathbf{H}_n)$ to $S^{m_1+m_2}(\mathbf{H}_n)$.

But let us first recall the definition of the Euclidean Moyal product for operators acting on (some subspace of) $L^2(\mathbb{R}^{2n+1})$. (See for example Folland [28] p. 103.)

Definition 5.35. *For two symbols $\tau_1, \tau_2 : \mathbb{R}^{4n+2} \mapsto \mathbb{C}$ the (Euclidean) Moyal product is formally defined by*

$$\begin{aligned}
\big(\tau_1 \sharp \tau_2\big)(\Xi, X) &= 4^{2n+1} \iint \tau_1(\Psi, U) \tau_2(\Phi, V) e^{4\pi i \left(\langle \Xi - \Phi, X - U\rangle - \langle \Xi - \Psi, X - V\rangle\right)} \\
&\qquad d\Phi \, dU \, d\Psi \, dV \qquad (5.26)\\
&= 4^{2n+1} \iint \tau_1(\Psi, U) \tau_2(\Phi, V) e^{-4\pi i \omega \left((\Xi - \Psi, X - U), (\Xi - \Phi, X - V)\right)} \\
&\qquad d\Phi \, dU \, d\Psi \, dV, \qquad (5.27)
\end{aligned}$$

where ω denotes the standard symplectic form on $\mathbb{R}^{4n+2}$.

Proposition 5.36. *For any two symbols $\sigma_1, \sigma_2 \in \mathscr{S}(\mathbb{R}^{4n+2})$ their $\mathbf{H}_n$-Moyal product is* *given by*

$$\big(\sigma_1 \circledast \sigma_2\big)(\Xi, X) = 4^{2n+1} \iiiint \sigma_1(\Psi, U)\sigma_2(\Phi, V) e^{4\pi i\left(\left\langle \Xi-\Phi, U^{-1}\cdot X\right\rangle - \left\langle \Xi-\Psi, V^{-1}\cdot X\right\rangle\right)} \times e^{2\pi i\left\langle \Psi+\Phi, [U^{-1}\cdot X, V^{-1}\cdot X]\right\rangle}\, d\Phi\, dU\, d\Psi\, dV. \tag{5.28}$$

Proof. To begin with, let us recall that for the pseudo-differential operator defined by σ or σ_M, respectively, there exists exactly one integral kernel $K \in \mathscr{S}(\mathbb{R}^{4n+2})$ such that

$$(\sigma_M(D, X)f)(X) = (\sigma(\mathscr{D}, \mathscr{X})f)(X) = \int K(X, X')f(X')\, dX'$$

for all $f \in \mathscr{S}(\mathbb{R}^{2n+1})$ and all $X \in \mathbb{R}^{2n+1}$. The symbols τ and σ in turn can be expressed in terms of K and vice versa:

$$\sigma_M(\Xi, X) = \int e^{2\pi i\langle \Xi, Y\rangle} K(X + \frac{1}{2}Y, X - \frac{1}{2}Y)\, dY,$$
$$\sigma(\Xi, X) = \int e^{2\pi i\langle \Xi, Y\rangle} K(X \cdot (\frac{1}{2}Y), X \cdot (-\frac{1}{2}Y))\, dY,$$

whereas

$$K(X, Y) = \int e^{2\pi i\langle \Xi, X-Y\rangle} \sigma_M(\Xi, \frac{1}{2}(X+Y))\, dY$$
$$= \int e^{2\pi i\langle \Xi, Y^{-1}\cdot X\rangle} \sigma(\Xi, \frac{1}{2}(X+Y))\, dY.$$

Making use of these formulas, we can express τ in terms of σ and vice versa with the help of the following four observations.

First, we compute

$$(X - \frac{1}{2}Y)^{-1} \cdot (X + \frac{1}{2}Y) = (-X + \frac{1}{2}Y) \cdot (X + \frac{1}{2}Y)$$
$$= -X + \frac{1}{2}Y + X + \frac{1}{2}Y + \frac{1}{4}[Y, X] - \frac{1}{4}[-Y, X]$$
$$= Y + \frac{1}{2}[Y, X].$$

Second, if we define $Y' := Y + \frac{1}{2}[Y, X]$, we equivalently have $Y = Y' - \frac{1}{2}[Y, X] = Y' - \frac{1}{2}[Y', X]$ since $[Y', X] = [Y + \frac{1}{2}[Y, X], X] = [Y, X]$. We also know that $Y' = Y \cdot X - X$, from which we conclude that $dY' = dY$. (Recall that the $\mathbb{R}^{2n+1}$-Lebesgue measure dY coincides with the $\mathbf{H}_n$-Haar measure. It is therefore bi-invariant under both group actions.)

Third, if $X = (p_X, q_X, t_X)$ and $Y' = (p_{Y'}, q_{Y'}, t_{Y'})$, then

$$\Big\langle \Xi, Y' - \frac{1}{2}[Y', X] \Big\rangle = \Big(\xi_u - \frac{1}{2}\xi_w q_X\Big) p_{Y'} + \Big(\xi_v + \frac{1}{2}\xi_w p_X\Big) q_{Y'} + \xi_w t_{Y'}$$
$$= \Big\langle \Xi - \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi), Y' \Big\rangle,$$

and similarly,

$$\Big\langle \Xi, Y + \frac{1}{2}[Y, X] \Big\rangle = \Big\langle \Xi + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi), Y \Big\rangle.$$

Fourth, for $\Xi' := \Xi - \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi)$ we observe that

$$\Xi = \Xi' + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi) = \Xi' + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi'), \tag{5.29}$$

as $\operatorname{ad}^*_{\mathbf{H}_n}(X)^2 = 0$ for any $X \in \mathfrak{h}_n \cong \mathbb{R}^{2n+1}$. This yields

$$\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi') = \operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi) - \frac{1}{2}\left(\operatorname{ad}^*_{\mathbf{H}_n}(X)\right)^2(\Xi') = \operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi).$$

It is furthermore easily seen that the change of variables $\Xi \mapsto \Xi'$ is measure-preserving.

We now combine the above formulas for σ, σ_M and K and the first two observations to give an explicit description of σ_M in terms of σ. Thus we compute

$$\begin{aligned}
\sigma_M(\Xi, X) &= \int e^{2\pi i\langle \Xi, Y\rangle} K(X + \tfrac{1}{2}Y, X - \tfrac{1}{2}Y)\, dY \\
&= \iint e^{2\pi i\langle \Xi, Y\rangle} \int e^{2\pi i\left\langle \Theta, (X - \frac{1}{2}Y)^{-1}\cdot(X + \frac{1}{2}Y)\right\rangle} \\
&\quad \times \sigma(\Theta, \tfrac{1}{2}\big(X + \tfrac{1}{2}Y + X - \tfrac{1}{2}Y\big))\, d\Theta\, dY \\
&= \iint e^{-2\pi i\left(\langle \Xi, Y\rangle + \left\langle \Theta, Y + \frac{1}{2}[Y,X]\right\rangle\right)} \sigma(\Theta, X)\, d\Theta\, dY \\
&= \int e^{-2\pi i\langle \Xi, Y\rangle}\Big(\mathscr{F}_1^{-1}\sigma\Big)(Y + \tfrac{1}{2}[Y, X], X)\, dY \\
&= \int e^{-2\pi i\left\langle \Xi, Y' + \frac{1}{2}[Y', X]\right\rangle}\Big(\mathscr{F}_1^{-1}\sigma\Big)(Y', X)\, dY \\
&= \sigma(\Xi + \tfrac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi), X).
\end{aligned} \tag{5.30}$$

Our fourth observation additionally yields

$$\sigma_M(\Xi - \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi), X) = \sigma(\Xi, X), \tag{5.31}$$

Formula (5.28) now follows from a calculation in which we rewrite the $\circledast$-product as a modified $\sharp$-product of "anti"-modified symbols as in Formula (5.31). Thus, employing Formulas (5.30), (5.31) and our fourth observation we obtain

$$\begin{aligned}
\big(\sigma_1 \circledast \sigma_2\big)(\Xi, X) &= \big(\sigma_{1M} \sharp \sigma_{2M}\big)(\Xi - \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi)) \\
&= 4^{2n+1} \iiiint \sigma_{1M}(\Psi, U)\sigma_{2M}(\Phi, V) \\
&\quad \times e^{4\pi i \left\langle \Xi + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi) - \Phi, X - U\right\rangle} e^{-4\pi i \left\langle \Xi + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi) - \Psi, X - V\right\rangle} \qquad (5.32) \\
&\quad d\Phi\, dU\, d\Phi\, dV \\
&= 4^{2n+1} \iiiint \sigma_1(\Psi - \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(U)(\Psi), U) \\
&\quad \times \sigma_2(\Phi - \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(V)(\Phi), V) e^{4\pi i \left\langle \Xi + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi) - \Phi, X - U\right\rangle} \\
&\quad e^{-4\pi i \left\langle \Xi + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi) - \Psi, X - V\right\rangle} d\Phi\, dU\, d\Phi\, dV \\
&= 4^{2n+1} \iiiint \sigma_1(\Psi, U)\sigma_2(\Phi, V) \\
&\quad \times e^{4\pi i \left\langle \Xi + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi) - \Phi - \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(V)(\Phi), X - U\right\rangle} \qquad (5.33) \\
&\quad \times e^{-4\pi i \left\langle \Xi + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi) - \Psi - \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(U)(\Psi), X - V\right\rangle} d\Phi\, dU\, d\Phi\, dV. \qquad (5.34)
\end{aligned}$$

A few more auxiliary results will eventually yield Formula (5.28). To this end, we first notice that

$$\begin{aligned}
[U^{-1} \cdot X, V^{-1} \cdot X] &= [-U + X - \frac{1}{2}[U, X], -V + X - \frac{1}{2}[V, X]] \\
&= [U, V] - [U, X] + [V, X].
\end{aligned}$$

With this at hand, we scrutinize the exponent in Equality (5.33) and find that

$$\begin{aligned}
&\left\langle \Xi + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi) - \Phi - \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(V)(\Phi), X - U \right\rangle \qquad (5.35) \\
&\quad = \left\langle \Xi, X - U + \frac{1}{2}[X - U, X] \right\rangle - \left\langle \Phi, X - U + \frac{1}{2}[X - U, V] \right\rangle.
\end{aligned}$$

We furthermore observe that

$$X - U + \frac{1}{2}[X - U, X] = X - U + \frac{1}{2}[-U, X] = (-U) \cdot X = U^{-1} \cdot X$$

and that

$$\begin{aligned} X - U + \frac{1}{2}[X - U, V] &= X - U + \frac{1}{2}[X - U, U + (V - U)] \\ &= -U + X + \frac{1}{2}[X - U, X] + \frac{1}{2}[X - U, V - U] \\ &= -U + X + \frac{1}{2}[-X, U] - \frac{1}{2}\big([U, V] - [U, X] + [V, X]\big) \\ &= U^{-1} \cdot X - \frac{1}{2}[U^{-1} \cdot X, V^{-1} \cdot X]. \end{aligned}$$

Hence, the exponent (5.35) equals

$$\begin{aligned} &\big\langle \Xi, U^{-1} \cdot X \big\rangle - \big\langle \Phi, U^{-1} \cdot X \big\rangle + \frac{1}{2}\big\langle \Phi, [U^{-1} \cdot X, V^{-1} \cdot X] \big\rangle \\ &\qquad = \big\langle \Xi - \Phi, U^{-1} \cdot X \big\rangle + \frac{1}{2}\big\langle \Phi, [U^{-1} \cdot X, V^{-1} \cdot X] \big\rangle. \end{aligned}$$

Analogously we obtain for the other exponent in Equality (5.34) that

$$\begin{aligned} &\Big\langle \Xi + \frac{1}{2}\,\mathrm{ad}^*_{\mathbf{H}_n}(X)(\Xi) - \Psi - \frac{1}{2}\,\mathrm{ad}^*_{\mathbf{H}_n}(U)(\Psi), X - V \Big\rangle \\ &\qquad = \big\langle \Xi - \Psi, V^{-1} \cdot X \big\rangle - \frac{1}{2}\big\langle \Psi, [U^{-1} \cdot X, V^{-1} \cdot X] \big\rangle. \end{aligned}$$

If we employ these simplifications, we finally obtain Formula (5.28). □

Remark 5.37 (On the Relation Moyal Product-Symplectic Form)**.** A reasonable and justified question at this point is whether the $\mathbf{H}_n$-Moyal product $\circledast$ bears the same relation with the symplectic form $\Omega = \Omega_1$ on the generic orbit $\mathcal{O}_1 \cong \mathbb{R}^{4n+2}$, i.e., the orbit $\mathcal{O}_\lambda$ for $\lambda = 1$, as the $\mathbb{R}^{2n+1}$-Moyal product $\sharp$ does with ω via Equality (5.27). A short answer to this question is: not quite. If one bears in mind that for $\lambda \in \mathbb{R}\backslash\{0\}$ the map

$$\begin{aligned} \phi_\lambda : \mathbb{R}^{4n+2} &\to \mathcal{O}_\lambda \subseteq \mathfrak{h}^*_{2,n} \cong \mathbb{R}^{4n+3}, \\ (\Xi, X) &\mapsto (\Xi - \frac{1}{2}\,\mathrm{ad}^*_{\mathbf{H}_n}(X)(\Xi), \frac{1}{\lambda}X, \lambda) \end{aligned}$$

is a symplectomorphism that maps ω to Ω_λ (cf. Folland [29] p. 19), then it is easy to see that via the identification of $\mathcal{O}_\lambda \cong \mathrm{T}_{(\Xi,X)}\,\mathcal{O}_\lambda \cong \mathbb{R}^{4n+2}$ we obtain

$$\Omega_\lambda(\Xi,X)\big((\Psi,U),(\Phi,V)\big) = \frac{1}{\lambda}\Big(\langle\Psi,V\rangle - \langle\Phi,U\rangle\Big) + \frac{1}{2\lambda^2}\Big(\langle\Phi,[X,U]\rangle - \langle\Psi,[X,V]\rangle + \langle\Xi,[U,V]\rangle\Big).$$

Moreover, as one readily observes, we have $\phi_1^*\sigma_M = \sigma$.

For $\lambda = 1$, an expansion of the $\mathbf{H}_n$-group products in the exponent of (5.28), abbreviated by $4\pi i E(\Xi,X,\Psi,U,\Phi,V)$, now yields

$$E(\Xi,X,\Psi,U,\Phi,V) = -\Omega(\Xi,X)\big((\Xi-\Psi,X-U),(\Xi-\Phi,X-V)\big) - \frac{1}{2}\langle\Xi-(\Psi+\Phi),[X-U,X-V]\rangle,$$

which is not quite as nice as Formula (5.27).

In the following we will prove that the $\mathbf{H}_n$-Moyal product $\circledast$ extends to a continuous map from $S^{m_1}(\mathbf{H}_n)\times S^{m_1}(\mathbf{H}_n)$ to $S^{m_1+m_2}(\mathbf{H}_n)$. The proof by and large follows Folland's proof for the analogous statement for $\sharp$ in the Euclidean case (cf. [28] Theorem 2.47). We will even adopt some notation Folland has introduced for a similar adaption of a Euclidean statement to a meta-Heisenberg case (cf. Folland [29] Proposition 5). Although Folland has already outlined there many of the ideas and changes that recur in our proof, we will still give a complete proof with all required details for the sake of a convenient reading.

As in Folland's proof for the Euclidean case, the proof is divided into three main steps to which we will add a preliminary section, where we introduce some notation and provide a few observations that will be used throughout the proof. Drawing this analogy between the Euclidean and the Heisenberg cases, the gist of the proof is again to sneak the right choice of differential operators into the oscillatory integral given by (5.28) to render it absolutely convergent uniformly in Ξ, X, while providing the required bounds.

Theorem 5.38. *The Heisenberg-Moyal product $(\sigma_1,\sigma_2)\mapsto\sigma_1\circledast\sigma_1$ is continuous from $S^{m_1}(\mathbf{H}_n)\times S^{m_2}(\mathbf{H}_n)$ to $S^{m_1+m_2}(\mathbf{H}_n)$ for all $m_1,m_2\in\mathbb{R}$.*

Proof. As mentioned above, the first section of the proof concerns preliminary observations and the introduction of necessary notation. So, let us first note that

$$E_1(\Xi,\Phi,U,X) := \big\langle\Xi-\Phi,U^{-1}\cdot X\big\rangle = (\xi_u+\varphi_u)(p_X-p_U) + (\xi_v-\varphi_v)(q_X-q_U) + (\xi_w-\varphi_w)(t_X-t_U-\frac{1}{2}(p_Uq_X-q_Up_X)),$$

and hence

$$\begin{aligned}(\partial_{p_U} E_1)(\Xi, \Phi, U, X) &= -(\xi_u - \varphi_u) - \frac{1}{2}(\xi_w - \varphi_w) q_X \\ &= -\big((\xi_u - \varphi_u) - \frac{1}{2}(\xi_w - \varphi_w) q_X\big), \\ (\partial_{q_U} E_1)(\Xi, \Phi, U, X) &= -\big((\xi_v - \varphi_v) - \frac{1}{2}(\xi_w - \varphi_w) p_X\big), \\ (\partial_{t_U} E_1)(\Xi, \Phi, U, X) &= -(\xi_w - \varphi_w).\end{aligned}$$

Equivalently, we may express this system of linear equations as

$$-\Big((\partial_{p_U}, \partial_{q_U}, \partial_{t_U}) E_1\Big)(\Xi, \Phi, U, X) = \Xi - \Phi + \frac{1}{2}\operatorname{ad}^*_{\mathbf{H}_n}(X)(\Xi - \Phi),$$

whence

$$-\Big((\partial_{p_U} - \frac{1}{2} q_X \partial_{t_U}, \partial_{q_U} + \frac{1}{2} p_X \partial_{t_U}, \partial_{t_U}) E_1\Big)(\Xi, \Phi, U, X) = \Xi - \Phi$$

follows by (5.29). The occurring entries in the vector field are not quite the left-invariant standard basis vector fields on $\mathbf{H}_n$, but rather some entangled versions of the latter. We therefore define

$$\begin{aligned}\mathscr{D}_{p_U, X} &:= \frac{1}{2\pi i}(\partial_{p_U} - \frac{1}{2} q_X \partial_{t_U}) = \mathscr{D}_{p_U} + \frac{1}{2}(q_U - q_X)\mathscr{D}_{t_U}, \\ \mathscr{D}_{q_U, X} &:= \frac{1}{2\pi i}(\partial_{q_U} + \frac{1}{2} p_X \partial_{t_U}) = \mathscr{D}_{q_U} - \frac{1}{2}(p_U - p_X)\mathscr{D}_{t_U}, \text{ and} \\ \mathcal{N}_{U,X} &:= \frac{1}{16}\Big(|\mathscr{D}_{p_U,X}|^2 + |\mathscr{D}_{q_U,X}|^2\Big)^2 + \frac{1}{4}\mathscr{D}_{t_U}^2.\end{aligned}$$

Applying the latter operator to $e^{4\pi i E_1}$, we obtain

$$\mathcal{N}_{U,X} e^{4\pi i E_1(\Xi,\Phi,U,X)} = |\Xi - \Phi|^4_{\mathbf{H}_n}\, e^{4\pi i E_1(\Xi,\Phi,U,X)},$$

and by changing some variable names also

$$\mathcal{N}_{V,X} e^{4\pi i E_1(\Xi,\Psi,V,X)} = |\Xi - \Psi|^4_{\mathbf{H}_n}\, e^{4\pi i E_1(\Xi,\Psi,V,X)}.$$

In much the same spirit we (re)define

$$\mathcal{L}_\Phi := \frac{1}{16}\Big(|D_{\varphi_u}|^2 + |D_{\varphi_v}|^2\Big)^2 + \frac{1}{4} D_{\varphi_w}^2,$$

(cf. the operator $\mathscr{L}_\Phi$ defined as in 5.20, which differs from $\mathcal{L}_\Phi$ only by a normalizing factor) and we immediately obtain

$$\mathcal{L}_\Phi e^{4\pi i E_1(\Xi,\Phi,U,X)} = \left|U^{-1}\cdot X\right|^4_{\mathbf{H}_n} e^{4\pi i E_1(\Xi,\Phi,U,X)}, \text{ and}$$
$$\mathcal{L}_\Psi e^{4\pi i E_1(\Xi,\Psi,V,X)} = \left|V^{-1}\cdot X\right|^4_{\mathbf{H}_n} e^{4\pi i E_1(\Xi,\Psi,V,X)}.$$

For the sake of completeness, let us remark that

$$0 = \mathcal{N}_{U,X} e^{4\pi i E_1(\Xi,\Psi,V,X)} = \mathcal{L}_\Phi e^{4\pi i E_1(\Xi,\Psi,V,X)} = \mathcal{N}_{V,X} e^{4\pi i E_1(\Xi,\Phi,U,X)}$$
$$= \mathcal{L}_\Psi e^{4\pi i E_1(\Xi,\Phi,U,X)}.$$

What now remains to be looked at is the action of $\mathcal{N}_{U,X}, \mathcal{N}_{V,X}, \mathcal{L}_\Psi, \mathcal{L}_\Phi$ on

$$E_2(\Psi,\Phi,U,V,X) := \left\langle \Psi+\Phi, [U^{-1}\cdot X, V^{-1}\cdot X]\right\rangle$$

and $e^{2\pi i E_2(\Psi,\Phi,U,V,X)}$. To this end, let us compute E_2 in terms of the coordinates $\psi_u, \psi_v, \dots, q_X, t_X$. Since

$$\begin{aligned}[U^{-1}\cdot X, V^{-1}\cdot X] &= \Big[\big(p_X-p_U, q_X-q_U, t_X-t_U-\tfrac{1}{2}(p_Uq_X-q_Up_X)\big),\\ &\qquad \big(p_X-p_V, q_X-q_V, t_X-t_V-\tfrac{1}{2}(p_Vq_X-q_Vp_X)\big)\Big]\\ &= \big(0,0,(p_X-p_U)(q_X-q_V)-(q_X-q_U)(p_X-p_V)\big),\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{L}_\Phi e^{2\pi i E_2(\Psi,\Phi,U,V,X)} &= \mathcal{L}_\Phi e^{2\pi i\Big((\psi_w+\varphi_w)\big((p_X-p_U)(q_X-q_V)-(q_X-q_U)(p_X-p_V)\big)\Big)}\\ &= \frac{1}{4}\big((p_X-p_U)(q_X-q_V)-(q_X-q_U)(p_X-p_V)\big)^2\\ &= \frac{1}{4}\left|[U^{-1}\cdot X, V^{-1}\cdot X]\right|^2\\ &= \mathcal{L}_\Psi e^{2\pi i E_2(\Psi,\Phi,U,V,X)}.\end{aligned}$$

To determine the action of $\mathcal{N}_{U,X}$ and $\mathcal{N}_{V,X}$, we observe that for $j\in 1,\dots,n$

$$\mathscr{D}^2_{p_{j_U}} e^{2\pi i E_2(\Psi,\Phi,U,V,X)} = (\psi_w+\varphi_w)^2(q_{j_X}-q_{j_V})^2 e^{2\pi i E_2(\Psi,\Phi,U,V,X)},$$
$$\mathscr{D}^2_{q_{j_U}} e^{2\pi i E_2(\Psi,\Phi,U,V,X)} = (\psi_w+\varphi_w)^2(p_{j_X}-p_{j_V})^2 e^{2\pi i E_2(\Psi,\Phi,U,V,X)},$$
$$\mathscr{D}_{t_U} e^{2\pi i E_2(\Psi,\Phi,U,V,X)} = 0.$$

Hence, we have

$$\mathcal{N}_{U,X} e^{2\pi i E_2(\Psi,\Phi,U,V,X)} = \frac{1}{16}(\psi_w + \varphi_w)^4 \Big(|p_X - p_V|^2 + |q_X - q_V|^2 \Big)^2$$

and the analogous result for $\mathcal{N}_{V,X}$.

Let us finally start with the actual proof. To prove continuity of $\circledast$ we need to show that for every $j \in \mathbb{N} \cup \{0\}$ there exists $k(j) \in \mathbb{N} \cup \{0\}$ and a constant $C_j > 0$ such that

$$\|\sigma_1 \circledast \sigma_2\|_{[j]} \leqslant C_j \, \|\sigma_1\|_{[k(j)]} \, \|\sigma_2\|_{[k(j)]} \,. \tag{5.36}$$

Yet, as the following calculation will show, we only have to consider derivatives in Ξ since the left $\mathbf{H}_n$-translations $T^{\mathbf{H}_n}_{X'}$, and hence their generators $\mathscr{D}_{e_j}$, $j = 1, \ldots, 2n+1$, commute with $\sigma_1 \circledast \sigma_2$. Thus, let $\Xi', X' \in \mathbb{R}^{2n+1}$, and let $T^{\mathbb{R}^{2n+1}}_{\Xi'}$ denote the Euclidean translation by Ξ' in the variable Ξ. We then compute

$$\begin{aligned}
&\big(T^{\mathbb{R}^{2n+1}}_{\Xi'} T^{\mathbf{H}_n}_{X'} \sigma_1 \circledast \sigma_2\big)(\Xi, X) = (\sigma_1 \circledast \sigma_2)(\Xi - \Xi', X'^{-1} \cdot X) \\
&= 4^{2n+1} \iiiint \sigma_1(\Psi, U) \sigma_2(\Phi, V) e^{4\pi i \left\langle \Xi - \Xi' - \Phi, U^{-1} \cdot X'^{-1} \cdot X \right\rangle} \\
&\quad \times e^{-4\pi i \left\langle \Xi - \Xi' - \Psi, V^{-1} \cdot X'^{-1} \cdot X \right\rangle} e^{2\pi i \left\langle \Psi + \Phi, [U^{-1} \cdot X'^{-1} \cdot X, V^{-1} \cdot X'^{-1} \cdot X] \right\rangle} \, d\Psi \, dU \, d\Phi \, dV \\
&= 4^{2n+1} \iiiint \sigma_1(\Psi, U) \sigma_2(\Phi, V) e^{4\pi i \left\langle \Xi - (\Phi + \Xi'), (X' \cdot U)^{-1} \cdot X \right\rangle} \\
&\quad \times e^{-4\pi i \left\langle \Xi - (\Psi + \Xi'), (X' \cdot V)^{-1} \cdot X \right\rangle} e^{2\pi i \left\langle \Psi + \Phi, [(X' \cdot U)^{-1} \cdot X, (X' \cdot V)^{-1} \cdot X] \right\rangle} \, d\Psi \, dU \, d\Phi \, dV,
\end{aligned}$$

which after a measure-preserving change of variables equals

$$\begin{aligned}
&4^{2n+1} \iiiint \sigma_1(\Psi - \Xi', X'^{-1} \cdot U) \sigma_2(\Phi - \Xi', X'^{-1} \cdot V) e^{4\pi i \left\langle \Xi - \Phi, U^{-1} X \right\rangle} \\
&\quad \times e^{-4\pi i \left\langle \Xi - \Psi, V^{-1} \cdot X \right\rangle} e^{2\pi i \left\langle \Psi + \Phi - 2\Xi', [U^{-1} \cdot X, V^{-1} \cdot X] \right\rangle} \, d\Psi \, dU \, d\Phi \, dV.
\end{aligned}$$

Hence $\circledast$ commutes with $T^{\mathbf{H}_n}_{X'}$, but not quite with $T^{\mathbb{R}^{2n+1}}_{\Xi'}$, thus with the derivatives $\mathscr{D}^{\beta}_{X}$, but not with D^{α}_{Ξ} for $\langle \alpha \rangle, \langle \beta \rangle > 0$. For each step of the proof we will therefore mainly prove the case $\langle \alpha \rangle = \langle \beta \rangle = 0$ and eventually indicate the slight changes required to cover the case $\langle \alpha \rangle > 0$. To this end, it is useful to note that

$$\begin{aligned}
(D^{\alpha}_{\Xi} \sigma_1 \circledast \sigma_2)(\Xi, X) = &\iiint \big(U^{-1} \cdot X + V^{-1} \cdot X\big)^{\alpha} \\
&\times \mathrm{Integrand}(\Xi, X, \Psi, U, \Phi, V) \, d\Phi \, dU \, d\Psi \, dV
\end{aligned} \tag{5.37}$$

if $\mathrm{Integrand}(\Xi, X, \Psi, U, \Phi, V)$ denotes the untouched integrand from (5.28).

Step 1)
Here we show the result, i.e., the required semi-norm estimates, for the special case that $\sigma_1, \sigma_1 \in \mathscr{S}(\mathbb{R}^{4n+2})$. Let us pick some $\phi \in C_c^\infty(\mathbb{R}^{2n+1}, [0,1])$ such that $\phi(\Theta) = 1$ for all $|\Theta|_{\mathbf{H}_n} \leqslant 1$ and $\phi(\Theta) = 0$ for all $|\Theta|_{\mathbf{H}_n} \geqslant 2$. For an arbitrary but fixed $\Xi \in \mathbb{R}^{2n+1}$ we then define

$$\phi_\Xi(\Theta) := \phi\big(\delta_{<\Xi>^{-1}}(\Xi - \Theta)\big),$$

so that, $\phi = 1$, whenever $|\Xi - \Theta|_{\mathbf{H}_n} \leqslant < \Xi >$, and $\phi = 0$, whenever $|\Xi - \Theta|_{\mathbf{H}_n} \geqslant 2 < \Xi >$. Similarly we define

$$\tilde{\phi}_X(Z) := \phi\big(\delta_{<X>^{-1}}(Z^{-1} \cdot X)\big).$$

As one can easily check, $\phi_\Xi, \tilde{\phi}_X \in S^0(\mathbf{H}_n)$ uniformly in Ξ and X, respectively, and for

$$\begin{aligned}\kappa_1(\Psi, U, \Phi, V) &:= \sigma_1(\Psi, U)\sigma_2(\Phi, V)\phi_\Xi(\Psi)\tilde{\phi}_X(U)\phi_\Xi(\Phi)\tilde{\phi}_X(V),\\ \kappa_2(\Psi, U, \Phi, V) &:= \sigma_1(\Psi, U)\sigma_2(\Phi, V)(1 - \phi_\Xi(\Psi)\tilde{\phi}_X(U)\phi_\Xi(\Phi)\tilde{\phi}_X(V)),\end{aligned}$$

we have $\kappa_1, \kappa_2 \in S^{m_1+m_2}(\mathbf{H}_n)$ with

$$[\kappa_1]_{[j]} \leqslant c_j[\sigma_1]_{[j]}[\sigma_2]_{[j]} \text{ and } [\kappa_2]_{[j]} \leqslant c_j[\sigma_1]_{[j]}[\sigma_2]_{[j]},$$

uniformly in Ξ and X. In order to prove the required continuity estimates for $\sigma_1 \circledast \sigma_2$, we will split up the integral into two parts and prove the estimates for the two summands corresponding to κ_1 and κ_2, respectively.

As above, let us denote by $4\pi i E(\Xi, X, \Psi, U, \Phi, V)$ or, if there is no danger of confusion, simply $4\pi i E$, the exponent in Formula (5.28). Moreover, let us set

$$F(U, X, V) := \left|U^{-1} \cdot X\right|_{\mathbf{H}_n}^4 + \frac{1}{4}\left|[U^{-1} \cdot X, V^{-1} \cdot X]\right|^2,$$

for which the order of U and V will be crucial in the following estimates. For the first integral we then observe that an integration by parts yields

$$\begin{aligned}
&\iiiint \kappa_1(\Psi,U,\Phi,V)e^{4\pi iE}\,d\Psi\,dU\,d\Phi\,dV\\
&\quad=\iiiint \kappa_1(\Psi,U,\Phi,V)\frac{\big(1+<\Xi>^{4N}\,\mathcal{L}_\Psi^N\big)\big(1+<\Xi>^{4N}\,\mathcal{L}_\Phi^N\big)e^{4\pi iE}}{\big(1+<\Xi>^{4N}\,F(U,X,V)^N\big)\big(1+<\Xi>^{4N}\,F(V,X,U)^N\big)}\\
&\qquad d\Psi\,dU\,d\Phi\,dV\\
&\quad=\iiiint \frac{e^{4\pi iE}}{\big(1+<\Xi>^{4N}\,F(U,X,V)^N\big)\big(1+<\Xi>^{4N}\,F(V,X,U)^N\big)}\\
&\qquad\times\Big[\big(1+<\Xi>^{4N}\,\mathcal{L}_\Psi^N\big)\big(1+<\Xi>^{4N}\,\mathcal{L}_\Phi^N\big)\kappa_1\Big](\Psi,U,\Phi,V)\,d\Psi\,dU\,d\Phi\,dV,
\end{aligned}$$

for which the latter integral is dominated by

$$C<\Xi>^{m1}<\Xi>^{m2}\;\times\iiiint\limits_{\substack{\max\{|\Xi-\Psi|_{\mathbf{H}_n},|\Xi-\Phi|_{\mathbf{H}_n}\}\leqslant 2<\Xi>,\\ \max\{|U^{-1}\cdot X|_{\mathbf{H}_n},|V^{-1}\cdot X|_{\mathbf{H}_n}\}\leqslant 2<X>}}\frac{d\Psi\,dU\,d\Phi\,dV}{\big(1+<\Xi>^{4N}\,F(U,X,V)^N\big)\big(1+<\Xi>^{4N}\,F(V,X,U)^N\big)}.$$

The replacement of the factors $<\Psi>^{m1}<\Phi>^{m2}$ by $<\Xi>^{m1}<\Xi>^{m2}$ is due their comparability on the compact set characterized by

$$\max\{|\Xi-\Psi|_{\mathbf{H}_n},|\Xi-\Phi|_{\mathbf{H}_n}\}\leqslant 2<\Xi>.$$

We also observe that the integral can be bounded from above by

$$C'\iiiint\frac{d\Psi\,dU\,d\Phi\,dV}{\big(1+<\Psi>^{4N}\big)\big(1+<\Phi>^{4N}\big)\big(1+F(U,X,V)^N\big)\big(1+F(V,X,U)^N\big)}<\infty,$$

provided $N>\frac{2n+1}{2}$. Let us furthermore note that the constant C can be a very generous bound in principle and will always encompass the according semi-norms of σ_1,σ_2. By taking $<\Xi>^{m1}<\Xi>^{m2}$ to the other side and combining the appropriate constants, we hence conclude that $[\kappa_1]_{[j]}\leqslant C_j[\sigma_1]_{[4N]}[\sigma_2]_{[4N]}$.

We now focus on the second integral. Let

$$G(U,V,X):=\big(|p_X-p_U|^2+|q_X-q_U|^2\big)^2+\big(|p_X-p_V|^2+|q_X-q_V|^2\big)^2,$$

for which the order of U and V is irrelevant. Again we employ integration by parts and compute

$$\iiiint \kappa_2(\Psi,U,\Phi,V)e^{4\pi iE}\,d\Psi\,dU\,d\Phi\,dV$$
$$= \iiiint_{\substack{|\Xi-\Psi|^4_{\mathbf{H}_n}+|\Xi-\Phi|^4_{\mathbf{H}_n}\geqslant 16<\Xi>^4,\\ |U^{-1}\cdot X|^4_{\mathbf{H}_n}+|V^{-1}\cdot X|^4_{\mathbf{H}_n}\geqslant 16<X>^4}} \frac{e^{4\pi iE}}{\Big(|\Xi-\Psi|_{\mathbf{H}_n}{}^4+|\Xi-\Phi|_{\mathbf{H}_n}{}^4+\frac{1}{16}(\psi+\varphi)^4G(U,V,X)\Big)^N}$$
$$\big(\mathcal{N}_{U,X}+\mathcal{N}_{V,X}\big)^N\Bigg[\Big(\big(1+\mathcal{L}_\Psi^N\big)\big(1+\mathcal{L}_\Phi^N\big)\kappa_2\Big)(\Psi,U,\Phi,V)\big(1+F(U,X,V)^N\big)^{-1}$$
$$\big(1+F(V,X,U)^N\big)^{-1}\Bigg]\,d\Psi\,dU\,d\Phi\,dV.$$

Provided $4N>2n+2$, the homogeneous dimension of $\mathbf{H}_n$, the latter integral is dominated by

$$C\iint_{|\Xi-\Psi|^4_{\mathbf{H}_n}+|\Xi-\Phi|^4_{\mathbf{H}_n}\geqslant 16<\Xi>^4} \frac{<\Psi>^{m_1-4N}<\Phi>^{m_2-4N}}{\Big(|\Xi-\Psi|_{\mathbf{H}_n}{}^4+|\Xi-\Phi|_{\mathbf{H}_n}{}^4+\frac{1}{16}(\psi+\varphi)^4\Big)^N}\,d\Psi\,d\Phi \tag{5.38}$$

for the following reasons: First, the compact region K_ϕ^Ξ outside which $(\Psi,\Phi)\mapsto\phi_\Xi(\Psi)\phi_\Xi(\Phi)$ vanishes is given by $K_\phi^\Xi=\{(\Psi,\Phi)\in\mathbb{R}^{4n+2}\mid\max\{|\Xi-\Psi|_{\mathbf{H}_n},|\Xi-\Phi|_{\mathbf{H}_n}\}\leqslant 2<\Xi>\}$. Hence $\{(\Psi,\Phi)\in\mathbb{R}^{4n+2}\mid|\Xi-\Psi|^4_{\mathbf{H}_n}+|\Xi-\Phi|^4_{\mathbf{H}_n}\geqslant 16<\Xi>^4\}\supseteq(K_\phi^\Xi)^c$ and the domain of integration is fine. Second, if we use the analogous notation K_ϕ^X for the (U,V)-integral, then $(U,V)\mapsto G(U,V,X)$ is bounded from below on $(K_\phi^X)^c$ and can therefore be disregarded in the remaining integral (5.38).

For the latter we will split up the domain of integration into $|\Psi|^4_{\mathbf{H}_n}+|\Phi|^4_{\mathbf{H}_n}\geqslant 4^4<\Xi>^4$ and $|\Psi|^4_{\mathbf{H}_n}+|\Phi|^4_{\mathbf{H}_n}\leqslant 4^4<\Xi>^4$. For the first region we observe that (5.38) is dominated by

$$C_1<\Xi>^{-4N}\times\iint_{|\Psi|^4_{\mathbf{H}_n}+|\Phi|^4_{\mathbf{H}_n}\geqslant 4^4<\Xi>^4}\big(<\Psi>^4+<\Phi>^4\big)^{\frac{|m_1|+|m_2|}{4}-2N}\,d\Psi\,d\Phi$$
$$\leqslant C_2<\Xi>^{-4N}<\Xi>^{|m_1|+|m_2|-2N}$$

if only N is large enough. Without difficulty we can shift the factor $<\Xi>^{m_1+m_2}$ to the left-hand side and require the constant C_2 to encompass the product $[\sigma_1]_{[4N]}[\sigma_2]_{[4N]}$. The same can be expected of the constant in the second estimate for which the integral is dominated by

$$C_3 \prec \Xi \succ^{-4N} \times \iint\limits_{|\Psi|^4_{\mathbf{H}_n}+|\Phi|^4_{\mathbf{H}_n} \leqslant 4^4 \prec\Xi\succ^4} |\Psi|^{m_1-4N}_{\mathbf{H}_n} |\Phi|^{m_2-4N}_{\mathbf{H}_n} \, d\Psi \, d\Phi$$
$$\leqslant C_2 \prec \Xi \succ^{-4N} \prec \Xi \succ^{m_+ m_2 - 8N + 4n + 2} .$$

Hence we conclude that $[\kappa_2]_{[j]} \leqslant C_j[\sigma_1]_{[4N]}[\sigma_2]_{[4N]}$. This proves Step 1 for $\langle\alpha\rangle = 0$. Formula (5.37) shows that for $\langle\alpha\rangle > 0$ we simply have to choose N accordingly larger to retain our estimates.

Step 2)

This step is concerned with a pointwise estimate that will become important once we stick our pieces together at the end of the proof. Let $\sigma_1, \sigma_2 \in \mathscr{S}(\mathbb{R}^{4n+2})$ and such that

$$\left.\begin{array}{l} \sigma_1 \text{ is supported where } |\Psi|_{\mathbf{H}_n} + |U|_{\mathbf{H}_n} \geqslant K > 0, \\ \sigma_2 \text{ is supported where } |\Phi|_{\mathbf{H}_n} + |V|_{\mathbf{H}_n} \geqslant K. \end{array}\right\} \tag{5.39}$$

We show that for every $a > 0$ there exist $j_a, M \in \mathbb{N}$ and $C_a > 0$ such that

$$\left|\left(D^\alpha_\Xi \mathscr{D}^\beta_X \sigma_1 \circledast \sigma_2\right)(\Xi, X)\right| \leqslant C_a[\sigma_1]_{[j_a+\langle\alpha\rangle+\langle\beta\rangle]}[\sigma_2]_{[j_a+\langle\alpha\rangle+\langle\beta\rangle]}K^{-a} \tag{5.40}$$

for all $|\Xi|_{\mathbf{H}_n} + |X|_{\mathbf{H}_n} \leqslant \frac{K}{M}$. Again because $\circledast$ commutes with $T^{\mathbf{H}_n}_X$ we only have to consider the $\langle\alpha\rangle > 0, \langle\beta\rangle = 0$. And as above we will first treat the case $\langle\alpha\rangle = 0$ and eventually remark on the required changes in the proof.

To start with, we observe that

$$\begin{aligned}
&(1 + \mathcal{N}_{V,X} + \mathcal{L}_\Psi)e^{4\pi i E} \\
&= e^{4\pi i E}\Big(1 + \big(|\Xi - \Psi|^4_{\mathbf{H}_n} + \frac{1}{16}(\psi+\varphi)^4\big(|p_X - p_U|^2 + |q_X - q_U|^2\big)^2\big)^N |U^{-1} \cdot X|^{4N}_{\mathbf{H}_n}\Big) \\
&:= e^{4\pi i E} F_1(\Xi, X, \Psi, U),
\end{aligned}$$

and furthermore that

$$F_1(\Xi, X, \Psi, U)^{-1} \leqslant C\big(1 + |\Xi - \Psi|^{4N}_{\mathbf{H}_n} + |U^{-1} \cdot X|^{4N}_{\mathbf{H}_n}\big)^{-1}$$

for some $C > 0$ if Conditions (5.39) are satisfied. An analogous estimate holds of course for Ψ, U. Hence,

$$(\sigma_1 \circledast \sigma_2)(\Xi, X) = 4^{2n+1} \iiiint e^{4\pi i E} \qquad (1 + \mathcal{N}_{V,X}^N + \mathcal{L}_\Psi^N)(1 + \mathcal{N}_{V,X}^N + \mathcal{L}_\Psi^N) \left(\frac{\sigma_1(\Psi, U)\sigma_2(\Phi, V)}{F_1(\Xi, X, \Psi, U)F_1(\Xi, X, \Phi, V)} \right) d\Psi\, dU\, d\Phi\, dV,$$

which is dominated by

$$C[\sigma_1]_{[4N]}[\sigma_2]_{[4N]} \times \iiint_{\substack{|\Psi|_{\mathbf{H}_n} + |U|_{\mathbf{H}_n} \geqslant K, \\ |\Phi|_{\mathbf{H}_n} + |V|_{\mathbf{H}_n} \geqslant K}} \frac{\prec \Psi \succ^{m_1 - 4N} \prec \Phi \succ^{m_1 - 4N} \; d\Psi\, dU\, d\Phi\, dV}{\left(1 + |\Xi - \Phi|_{\mathbf{H}_n}^{4N} + |V^{-1} \cdot X|_{\mathbf{H}_n}^{4N}\right)\left(1 + |\Xi - \Phi|_{\mathbf{H}_n}^{4N} + |V^{-1} \cdot X|_{\mathbf{H}_n}^{4N}\right)}.$$

Now, since for $|\Psi|_{\mathbf{H}_n} + |U|_{\mathbf{H}_n} \geqslant K$, $|\Phi|_{\mathbf{H}_n} + |V|_{\mathbf{H}_n} \geqslant K$ there exists $M \in \mathbb{N}$ such that for $\prec \Xi \succ + \prec X \succ \leqslant \frac{K}{M}$ we have

$$|\Xi - \Psi|_{\mathbf{H}_n} + \left|U^{-1} \cdot X\right|_{\mathbf{H}_n} \geqslant \frac{1}{2}\left(|\Psi|_{\mathbf{H}_n} + |U|_{\mathbf{H}_n}\right),$$
$$|\Xi - \Phi|_{\mathbf{H}_n} + \left|V^{-1} \cdot X\right|_{\mathbf{H}_n} \geqslant \frac{1}{2}\left(|\Phi|_{\mathbf{H}_n} + |V|_{\mathbf{H}_n}\right),$$

the last integral is dominated by

$$C \iiint_{\substack{|\Psi|_{\mathbf{H}_n} + |U|_{\mathbf{H}_n} \geqslant K, \\ |\Phi|_{\mathbf{H}_n} + |V|_{\mathbf{H}_n} \geqslant K}} \left(1 + |\Psi|_{\mathbf{H}_n} + |U|_{\mathbf{H}_n}\right)^{|m_1| - 8N} \left(1 + |\Phi|_{\mathbf{H}_n} + |V|_{\mathbf{H}_n}\right)^{|m_2| - 8N} d\Psi\, dU\, d\Phi\, dV$$
$$\leqslant C_N K^{|m_1| + |m_2| + 16N + 8n + 4}.$$

Choosing N appropriately large, we obtain Estimate (5.40).

In case $\langle \alpha \rangle > 0$ we can again compensate for additional factors $\left(U^{-1} \cdot X + V^{-1} \cdot X\right)^\alpha$ in the integral by choosing N large enough.

Step 3)

For $\sigma_1 \in S^{m_1}(\mathbf{H}_n), \sigma_2 \in S^{m_2}(\mathbf{H}_n)$ and ϕ as above, let us define

$$\sigma_{1,\varepsilon}(\Psi, U) := \phi(\delta_\varepsilon \Psi)\phi(\delta_\varepsilon U)\sigma_1(\Psi, U),$$
$$\sigma_{2,\varepsilon}(\Psi, U) := \phi(\delta_\varepsilon \Phi)\phi(\delta_\varepsilon V)\sigma_2(\Phi, V).$$

Then $\sigma_{1,\varepsilon} \to \sigma_1$ and $\sigma_{2,\varepsilon} \to \sigma_2$ in the C^∞-topolgy as $\varepsilon \to 0$, and $\|\sigma_{1,\varepsilon}\|_{[j]} \leqslant c_j \|\sigma_1\|_{[j]}$, $\|\sigma_{2,\varepsilon}\|_{[j]} \leqslant c_j \|\sigma_2\|_{[j]}$, for c_j independent of ε, follows from an estimate similar to the reasoning at the beginning of Step 1). Step 1) hence implies that for each $j \in \mathbb{N}$ there exist $k(j) \in \mathbb{N}$ and $C_j > 0$, both independent of ε, such that $\|\sigma_{1,\varepsilon} \circledast \sigma_{2,\varepsilon}\|_{[j]} \leqslant C_j \|\sigma_1\|_{[k(j)]} \|\sigma_2\|_{[k(j)]}$. Furthermore, Step 2) yields the Cauchy property in C^∞ for $(\sigma_{1,\varepsilon} \circledast \sigma_{2,\varepsilon})_{\varepsilon>0}$. Hence $(\sigma_{1,\varepsilon} \circledast \sigma_{2,\varepsilon})_\varepsilon$ converges in C^∞ to some symbol $\kappa \in S^{m_1+m_2}(\mathbf{H}_n)$. On the other hand, by Proposition 5.31, $\sigma_{1,\varepsilon}(\mathscr{D}, \mathscr{X})$ and $\sigma_{2,\varepsilon}(\mathscr{D}, \mathscr{X})$ converge strongly as operators on $\mathscr{S}(\mathbf{H}_n)$ to $\sigma_1(\mathscr{D}, \mathscr{X})$ and $\sigma_2(\mathscr{D}, \mathscr{X})$, respectively, and hence $\sigma_{1,\varepsilon}(\mathscr{D}, \mathscr{X})\sigma_{2,\varepsilon}(\mathscr{D}, \mathscr{X})$ to the operator $\sigma_1(\mathscr{D}, \mathscr{X})\sigma_2(\mathscr{D}, \mathscr{X})$, while $(\sigma_{1,\varepsilon} \circledast \sigma_{2,\varepsilon})(\mathscr{D}, \mathscr{X})$ converges strongly to $\kappa(\mathscr{D}, \mathscr{X})$. But by Proposition 5.6 (i) the operators' action on $\mathscr{S}(\mathbf{H}_n)$ uniquely determines their symbols, thus we must have $\sigma_1 \circledast \sigma_2 = \kappa$. This proves convergence and hence membership of $\sigma_1 \circledast \sigma_2 = \kappa$ in $S^{m_1+m_2}(\mathbf{H}_n)$. A standard application of the open mapping theorem for Fréchet spaces finally yields continuity of $\circledast$. This completes the proof. □

5.7 The Link with the Beals-Greiner-Quantization

We conclude this chapter with a few remarks on the connection between the $\mathbf{H}_n$-Weyl quantization and the work of Beals and Greiner, which has been pointed out before in Folland [29]. In their monograph [3] on pseudodifferential operators on Heisenberg manifolds, i.e., manifolds M locally diffeomorphic to $\mathbf{H}_n \times \mathbb{R}^{d-2n}$, for fixed $d \in \mathbb{N}$ but variable dimension n, Beals and Greiner employ a modified version of Euclidean Kohn-Nirenberg-quantization which is adapted to non-isotropic Heisenberg structure.

5.7.1 The Natural Semi-direct Product Approach

In case the Heisenberg manifold M coincides $\mathbf{H}_n \times \mathbb{R}^{2n+1}$ their quantization of a symbol $\sigma \in \mathscr{S}(\mathbf{H}_n \times \mathbb{R}^{2n+1})$ is given by the $2n+1$-dimensional Euclidean Kohn-Nirenberg quantization of

$$\begin{aligned}(\mathcal{R}\sigma)(\Xi, X) &= \mathcal{R}(\sigma)(\xi_u, \xi_v, \xi_w, \chi_x, \chi_y, \chi_z) \\ &:= \sigma(\xi_u - \frac{\xi_w}{2}\chi_v, \xi_v + \frac{\xi_w}{2}\chi_u, \xi_w, \chi_x, \chi_y, \chi_z) \\ &= \sigma(\Xi - \frac{1}{2}\,\mathrm{ad}^*_{\mathbf{H}_n}(X)(\Xi), X).\end{aligned} \tag{5.41}$$

Let us point out that the symbols corresponding to the Euclidean Weyl quantization bear precisely the inverse relation to their $\mathbf{H}_n$-Weyl analogues, that is, $\mathcal{R}(\sigma_M) = \sigma$ (cf. Identity (5.31)). Bearing in mind that the Euclidean Kohn-Nirenberg quantization of a symbol τ (applied to a function f) is given by

$$\begin{aligned}\big(\mathrm{Op}^{\pi}_{KN}(\tau)f\big)(X) &= \int \widehat{\tau}(\mathcal{P},\mathcal{Q}) e^{2\pi i\langle \mathcal{Q},X\rangle}\big(e^{2\pi i\langle \mathcal{P},D\rangle} f\big)(X)\,d\mathcal{P}\,d\mathcal{Q} \\ &= \int \widehat{\tau}(\mathcal{P},\mathcal{Q})\, e^{2\pi i\langle \mathcal{Q},X\rangle} f(X+\mathcal{P})\,d\mathcal{P}\,d\mathcal{Q} \\ &= \int \tau(\Xi,X)\, e^{2\pi i\langle \Xi,X\rangle}\widehat{f}(\Xi)\,d\Xi \\ &= \iint \tau(\Xi,Y)\, e^{2\pi i\langle \Xi,X-Y\rangle} f(Y)\,dY\,d\Xi,\end{aligned} \tag{5.42}$$

we thus compute

$$\mathrm{Op}^{\pi}_{BG}(\sigma) = \iint \sigma(\Xi - \tfrac{1}{2}\,\mathrm{ad}^*_{\mathbf{H}_n}(X)(\Xi),X)\, e^{2\pi i\langle \Xi,X-Y\rangle} f(Y)\,dY\,d\Xi$$

$$= \iint \sigma(\Xi,X)\, e^{2\pi i\left\langle \Xi+\frac{1}{2}\,\mathrm{ad}^*_{\mathbf{H}_n}(X)(\Xi),X-Y\right\rangle} f(Y)\,dY\,d\Xi \tag{5.43}$$

$$= \iint \sigma(\Xi,X)\, e^{2\pi i\left\langle \Xi,Y^{-1}\cdot X\right\rangle} f(Y)\,dY\,d\Xi \tag{5.44}$$

$$= \iint \sigma(\Xi,X)\, e^{-2\pi i\left\langle \Xi,X^{-1}\cdot Y\right\rangle} f(Y)\,dY\,d\Xi. \tag{5.45}$$

In Equality (5.43) we have re-used the change of variables from Identities (5.30) and (5.31), and Equality (5.45) makes use of the following short calculation, which essentially reduces to the same argument, too:

$$\begin{aligned}\left\langle \Xi + \frac{1}{2}\,\mathrm{ad}^*_{\mathbf{H}_n}(X)(\Xi), X-Y\right\rangle &= \left\langle \big(I + \frac{1}{2}\,\mathrm{ad}^*_{\mathbf{H}_n}(X)\big)(\Xi), X-Y\right\rangle \\ &= \left\langle \Xi, \big(I - \frac{1}{2}\,\mathrm{ad}_{\mathbf{H}_n}\big)(X)(X-Y)\right\rangle \\ &= \left\langle \Xi, X-Y-\frac{1}{2}[X,X-Y]\right\rangle \\ &= \left\langle \Xi, Y^{-1}\cdot X\right\rangle \\ &= -\left\langle \Xi, X^{-1}\cdot Y\right\rangle.\end{aligned}$$

On the other hand, we may dare an educated guess based on the close relation between the $2n+1$-dimensional Euclidean Kohn-Nirenberg quantization and the Beals-Greiner-quantization, i.e., (5.41), suspecting that the latter expressed by Identity (5.42) but with the $\mathbb{R}^{2n+1}$-gradient D replaced by its $\mathbf{H}_n$-version $\mathscr{D}$. Indeed we compute

$$\begin{aligned}
\int \widehat{\sigma}(\mathcal{P},\mathcal{Q})e^{2\pi i\langle \mathcal{Q},X\rangle}\big(e^{2\pi i\langle \mathcal{P},D\rangle}f\big)(X)\,d\mathcal{P}\,d\mathcal{Q}& \\
&= \int \widehat{\tau}(\mathcal{P},\mathcal{Q})\,e^{2\pi i\langle \mathcal{Q},X\rangle}f(X\cdot\mathcal{P})\,d\mathcal{P}\,d\mathcal{Q}\\
&= \iiiint \sigma(\Xi,\mathcal{X})\,e^{-2\pi i\langle \Xi,\mathcal{P}\rangle}e^{-2\pi i\langle \mathcal{X},\mathcal{Q}\rangle}e^{2\pi i\langle \mathcal{Q},X\rangle}f(X\cdot\mathcal{P})\,d\Xi\,d\mathcal{X}\,d\mathcal{P}\,d\mathcal{Q}\\
&= \iint \sigma(\Xi,X)\,e^{-2\pi i\langle \Xi,\mathcal{P}\rangle}f(X\cdot\mathcal{P})\,d\Xi\,d\mathcal{P}\\
&= \iint \sigma(\Xi,X)\,e^{-2\pi i\left\langle \Xi,X^{-1}\cdot Y\right\rangle}f(Y)\,d\Xi\,dY.
\end{aligned}$$

Thus, the Beals-Greiner-quantization can be expressed by

$$\begin{aligned}
\mathrm{Op}^{\pi}_{BG}(\sigma) &= \int \widehat{\sigma}(\mathcal{P},\mathcal{Q})\pi(0,\mathcal{Q},0)\pi(\mathcal{P},0,0)\,d\mathcal{P}\,d\mathcal{Q}\\
&= \int \widehat{\sigma}(\mathcal{P},\mathcal{Q})\pi\big((0,\mathcal{Q},0)\odot_{\mathbf{H}_{2,n}}(\mathcal{P},0,0)\big)\,d\mathcal{P}\,d\mathcal{Q},
\end{aligned}$$

that is, via integrating against π as a representation of $\mathbf{H}_{2,n}\cong\mathbb{R}^{2n+1}\rtimes\mathbf{H}_n$ whose elements are written as $(0,\mathcal{Q},\mathcal{S})\odot_{\mathbf{H}_{2,n}}(\mathcal{P},0,0)$.

Remark 5.39. The representation

$$\tilde{\pi}:(\mathcal{P},\mathcal{Q})\mapsto\pi(0,\mathcal{Q},0)\pi(\mathcal{P},0,0)=e^{2\pi i\langle \mathcal{Q},X\rangle}e^{2\pi i\langle \mathcal{P},D\rangle}$$

of the quotient group $\mathbb{R}^{2n+1}\rtimes\mathbf{H}_n\cong\mathbf{H}_{2,n_e}$, defined in Sections 4.2 and 4.3, is precisely the one which establishes the relation

$$\langle \mathrm{Op}^{\pi}_{BG}(\sigma)f,\varphi\rangle_{\mathcal{H}_\pi}=\big\langle\hat{\sigma},V^{\pi}_{\varphi}f\big\rangle_{L^2(\mathfrak{h}_{2,n_e})},\tag{5.46}$$

between the Beals-Greiner quantization on $\mathbf{H}_n$ and the STFT V^{π}_{φ} on $\mathbf{H}_n$ defined by Defintion 4.8.

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