Infinitely many inequivalent field theories from one Lagrangian

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Logarithmic time-like Liouville quantum field theory has a generalized $\mathcal{PT}$ invariance, where $\mathcal{T}$ is the time-reversal operator and $\mathcal{P}$ stands for an $S$-duality reflection of the Liouville field $\phi$. In Euclidean space the Lagrangian of such a theory, $L = \frac{1}{2} (\nabla \phi)^2 - ig \Phi \exp(ia\phi)$, is analyzed using the techniques of $\mathcal{PT}$-symmetric quantum field theory. It is shown that $L$ defines an infinite number of unitarily inequivalent sectors of the theory labeled by the integer $n$. In one-dimensional space (quantum mechanics) the energy spectrum is calculated in the semiclassical limit and the $n$th energy level in the $\mathcal{PT}$ sector is given by $E_{m,n} \sim (m + 1/2)^2 \alpha^2/(16n^2)$.

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Motivated by studies of time-like logarithmic Liouville quantum field theory, we examine here the interaction $-ig \Phi \exp(ia\phi)$ in field theory and its quantum-mechanical analog $-ig x \exp(iax)$. This remarkable interaction gives rise to a countably infinite number of inequivalent quantum theories.

The interaction $-ig \Phi \exp(ia\phi)$ has its origin in conformal field theory (CFT) of Liouville type, whose interaction term has a periodic component $g_{\alpha\phi}$ and thus $\exp(ia\phi)$, where $g_{\alpha\phi}$ is a scalar field. The Lagrangian (1) is not Hermitian and one cannot make such a theory Hermitian by adding its Hermitian conjugate because this would destroy the conformality property of the theory. Nevertheless, the techniques of $\mathcal{PT}$ quantum theory [13] can be used to study this field theory. The Lagrangian is not obviously $\mathcal{PT}$ invariant because in Liouville theory the field $\phi$ is assumed to transform as a scalar, so it does not change sign under space reflection. [If $\phi$ were a pseudoscalar field, the Lagrangian would be $\mathcal{PT}$ invariant because under parity reflection $\mathcal{P}$, $\phi$ would change sign $\mathcal{P}\phi(x,t)\mathcal{P} = -\phi(-x,t)$, and under time reversal $\mathcal{T}$, $\phi$ changes sign $\mathcal{T}\phi = -i\phi$.] However, we let $L$ represent an $S$ duality reflection [15],

\[ \mathcal{P}\phi(x,t)\mathcal{P} = -\phi(-x,t), \]

and with this definition of $\mathcal{P}$, $L$ is manifestly $\mathcal{PT}$ symmetric. A non-Hermitian $\mathcal{PT}$-invariant theory can have a positive real spectrum and unitary time evolution [14].

The interaction terms of (1) have a periodic component and thus $L$ bears a resemblance to some previously studied $\mathcal{PT}$-symmetric theories, including the complex Toda lattice [14], complex diffraction gratings [17], and complex crystal lattices [18]. Complex $\mathcal{PT}$-symmetric periodic potentials exhibit real-energy band structure. There have also been studies of the complex sine-Gordon equation [19], complex dynamical systems [20], and $\mathcal{PT}$-symmetric exponential potentials [21]. However, the factor of $\mathcal{P}$ multiplying $g$ in (1), which is characteristic of logarithmic CFT, leads to surprising new effects. Specifically, the partition function as a path integral over $L$,

\[ Z_n = \int D\phi \exp \left( \int d^d x \ L \right), \]

has infinitely many distinct functional integration paths labeled by $n = 1, 2, 3, \ldots$, each defining a valid but unitarily inequivalent quantum theory. This multiplicity of
Theories is not due to monodromy (there is no winding number because the integrand is entire) nor is it a topological effect (like $\theta$ vacua). The quantum-mechanical version of this time-like logarithmic CFT has discrete energy levels rather than energy bands. The $m$th energy level in the $n$th theory grows like $m^2$ as $m \to \infty$, but for fixed $m$ the energies decay like $n^{-2}$ as $n \to \infty$.

To find the integration paths on which the integral (4) converges, we must locate in field space the pairs of Stokes wedges inside which the integrand vanishes exponentially. To begin, we simplify this integral by shifting $\phi$ by a constant to eliminate the parameter $h$ in $L$. Next, we neglect the effect of the kinetic term $(\nabla \phi)^2$ because it does not affect the convergence. We also ignore the spatial integral in the exponent and study the convergence at each lattice point separately; that is, we perform an ultralocal analysis [22] and examine the convergence of

$$I = \int d\phi \exp (i g \phi e^{i a \phi}).$$

(4)

We illustrate how to locate Stokes wedges in the complex-$\phi$ plane by using monomial potentials $\phi^4$. For such potentials the angular opening of the Stokes wedges has a simple $k$ dependence. For $k = 4$ the integral $\int d\phi \exp (-\phi^4)$ converges in a pair of Stokes wedges of angular opening $45^\circ$ centered about the positive-$\phi$ and negative-$\phi$ axes (Fig. 1 left panel). The integration contour must terminate inside these Stokes wedges. For a $\mathcal{PT}$-symmetric upside-down $-\phi^4$ potential, the associated integral $\int d\phi \exp (\phi^4)$ converges in a pair of Stokes wedges of angular opening $45^\circ$ centered about $\arg \phi = -45^\circ$ and $\arg \phi = 135^\circ$ (Fig. 1 right panel).

Ref. [24] it was shown that if the pairs of Stokes wedges possess left-right symmetry ($\mathcal{PT}$ symmetry) in complex field space, the theory is physically acceptable because the masses (poles of the Green’s functions) are real and the theory is unitary. However, Ref. [24] only considered the case of a finite number of distinct physical theories, one theory for each pair of wedges.

Here, we consider the unusual case of an infinite number of inequivalent theories corresponding to pairs of infinitely thin Stokes wedges. To find the paths of integration on which $I$ converges, we introduce polar coordinates $\phi = R e^{i \theta}$ and treat $R$ as large. Then, (4) becomes

$$I = \int d\theta \exp (i g R \sin \theta e^{i \theta + i a R \cos \theta}).$$

(5)

We need to find Stokes wedges, that is, the angles at which the integrand vanishes exponentially fast as $R \to \infty$. At the center $\theta$ of a Stokes wedge, the exponent in (5) must be real and thus $\theta + a R \cos \theta \sim (n + 1/2) \pi$ as $R \to \infty$ ($n$ integer). Also, the argument of the exponential must be negative so that it vanishes as $R \to \infty$. Thus, $\sin(\theta + a R \cos \theta) \sim 1$ as $R \to \infty$ and we find that

$$\theta + a R \cos \theta \sim (\pm 2 n + 1/2) \pi \quad (R \to \infty),$$

(6)

where $n > 0$. The maximum rate of decay occurs at the center of the wedge, so $\theta$ must be close to $-\pi/2$. Hence, we substitute $\theta = -\pi/2 + \epsilon$ into (6) and obtain $\epsilon \sim (2 n + 1) \pi / (a R)$ ($R \to \infty$). We find that the centers of the Stokes wedges lie at

$$\theta_n \sim -\frac{1}{2} \pi \pm \frac{(2 n + 1) \pi}{a R} \quad (R \to \infty).$$

(7)

To summarize, for the partition function $Z_n$ in (3) the $n$th path of functional integration originates in the $-n$th Stokes wedge, terminates in the $n$th Stokes wedge, and is asymptotically parallel to the negative-imaginary axis. The path is $\mathcal{PT}$ (left-right) symmetric (see Fig. 2).

The $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonian $\mathcal{H}$ corresponding to the field-theoretic Lagrangian (1) is

$$\mathcal{H} = p^2 - i g x e^{i a x} + h e^{i a x},$$

(8)
where \( a, g, \) and \( h \) are assumed to be real and positive. As we did for the the field-theoretic model, we shift \( x \) by a constant to eliminate \( h \) and obtain the Hamiltonian

\[
\mathcal{H} = p^2 - igx e^{iax}.
\]

Both quantum-mechanical Hamiltonians \((8)\) and \((9)\) possess a singular limit. If \( a \to 0 \), \( \mathcal{H} \) in \((9)\) reduces to \( \mathcal{H} = p^2 - igx \). This limit is singular because, as was shown by Herbst, the spectrum of this Hamiltonian is null \cite{Herbst}. To explain intuitively the absence of eigenvalues, we solve Hamilton’s classical equations \( \dot{x} = \partial\mathcal{H}/\partial\dot{p} = 2p, \dot{p} = -\partial\mathcal{H}/\partial x = ig \). Combining these equations gives \( \dot{x} = 2ig \), whose solutions are parabolas in the complex plane: \( x(t) = igt^2 + at + \beta \) (\( a \) and \( \beta \) are constants.) Parabolas are open curves (see Fig. 3), so it is not possible to satisfy the Bohr-Sommerfeld (WKB) quantization condition, \( \oint dx \sqrt{E-V(x)} = (m + 1/2)\pi \) \( m = 0, 1, 2, \ldots \), which involves an integral over a closed path.

The imaginary part of \((10)\) gives

\[
E = -ig(A + iB)e^{-ia(A + iB)}. \tag{10}
\]

The imaginary part of \((10)\) gives \( B \) in terms of \( A, B = A \text{cot}(aA) \), and substituting this result into \((10)\) gives \( E \sin(aA)/(ga) = e^{-a\text{cot}(aA)} \). So, if we let \( \nu = aE/g \) and \( \alpha = aA \), we get the transcendental equation

\[
\nu \alpha^{-1} \sin \alpha = e^{-\alpha \text{cot} \alpha}. \tag{11}
\]

To solve this equation graphically, we plot the left side of \((11)\) as a solid curve and the right side as a dotted curve.

\[
\nu = \frac{\sin \alpha}{\alpha} e^{\alpha \text{cot} \alpha}.
\]

Figure 5 shows that there are two sets of solutions. The first is exactly \( x = n\pi \), but we reject this solution because it gives \( B = \infty \). The second set has intersection points near \( \alpha_n = (2n+1/2)\pi - \delta \), where \( \delta \ll 1 \) as \( n \to \infty \). Thus, for large \( n \) the turning points are located symmetrically about the imaginary-\( x \) axis at

\[
\begin{align*}
\text{Re } x &= A \sim \pm(2n+1/2)\pi/a, \\
\text{Im } x &= B \sim \frac{1}{a} \log \left[ \frac{(2n+1/2)\pi g}{aE} \right].
\end{align*} \tag{12}
\]

Table I verifies the accuracy of this asymptotic formula. Having found the turning points, we next examine the complex classical paths for \( \mathcal{H} \) in \((9)\). In general, for any given Hamiltonian, the classical energy determines a continuous family of classical paths distinguished by the initial value of \( x \). See, for example, the classical paths for the \( x^2 \) oscillator in Fig. 6 (left panel) and for the \( x^6 \) oscillator in Fig. 6 (right panel). Note that every family of classical paths encloses one pair of turning points.

The classical paths for the Hamiltonian \((8)\) are shown in Fig. 7. The turning points are enclosed in pairs. Also
the upper and lower pairs correspond to $\mathcal{PT}$-symmetric $x^6$ oscillators. Thus, since there are an infinite number of pairs of classical turning points for the model in Fig. 7, we anticipate that there will be an infinite number of pairs of Stokes wedges and have different values of $n$ only in higher-order WKB. The theories corresponding to $n$ are given in (12). For large $n$, (14) simplifies to

$$\left(\frac{m + 1/2}{2n\sqrt{E}}\right)^a \sim \int_{-1}^1 dw \sqrt{1 + iwe^{2in\pi w}} \sim 2 \quad (n \to \infty).$$

Thus, for large $n$ and large $m$, the WKB approximation to the $n$th energy in the $n$th eigenspectrum is

$$E_{m,n} \sim (m + 1/2)^2 a^2 n^{-2}/16.$$

The $m$th eigenvalue in the $n$th spectrum grows like the energies in a square well [27]. However, the $m$th eigenvalue in the $n$th spectrum behaves like the energy levels in the Balmer series for the hydrogen atom, and decays like $n^{-2}$. The parameter $g$ does not appear in (15); it appears only in higher-order WKB. The theories corresponding to different values of $n$ are all inequivalent — they are associated with different pairs of Stokes wedges and have different energy spectra. To conclude, $\mathcal{PT}$ analysis reveals the simple but astonishing structure that underlies time-like Liouville logarithmic CFT.

In future work we will investigate (i) whether it is possible to tunnel between the inequivalent theories that we have found, and (ii) whether it is possible to examine the conformal nature of two-dimensional logarithmic CFT by studying the WKB approximation to the quantum-mechanical eigenfunctions in (13) [28].

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<th>Turning point number</th>
<th>Exact</th>
<th>Approximate</th>
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<tr>
<td>$n = 0$</td>
<td>$1.3372 + 0.3181i$</td>
<td>$1.5708 + 0.4516i$</td>
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<tr>
<td>$n = 1$</td>
<td>$7.5886 + 2.0623i$</td>
<td>$7.85398 + 2.0610i$</td>
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<td>$n = 2$</td>
<td>$13.9492 + 2.6532i$</td>
<td>$14.1372 + 2.6488i$</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>$20.2725 + 3.0202i$</td>
<td>$20.4204 + 3.0165i$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>$26.5805 + 3.2878i$</td>
<td>$26.7035 + 3.2848i$</td>
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</table>

TABLE I: Good agreement between the numerically precise values of the turning points and the asymptotic approximation in (12) when $a = 1$, $g = 1$, and $E = 1$.

<table>
<thead>
<tr>
<th>Number of separatrices</th>
<th>Crossing point on the $y$ axis</th>
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<tbody>
<tr>
<td>$n = 1$</td>
<td>$0.7613i$</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>$1.3867i$</td>
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<tr>
<td>$n = 3$</td>
<td>$1.7485i$</td>
</tr>
</tbody>
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TABLE II: Points where the separatrix paths in Fig. 7 cross the imaginary axis [20].


C. M. Bender and S. P. Klevansky, Phys. Rev. Lett. 105, 031601 (2010). In this reference it is shown that a $\phi^6$ theory has two different spectra, one for the conventional theory and another for the $P\bar{T}$-symmetric theory. $A\phi^{10}$ theory has three spectra, a $A\phi^{14}$ has four, and so on.


Square-well $m^2$ growth of eigenvalues was also found in the large-$\epsilon$ behavior of the $x^2(\epsilon x)^\alpha$ $P\bar{T}$-symmetric potential. See C. M. Bender, S. Boettcher, H. F. Jones, and V. M. Savage, J. Phys. A: Math. Gen. 32, 6771 (1999). In this work the quantization paths enclosed the negative imaginary axis as in Fig. 2.