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ADVANCES IN THE THEORY OF STRONG GRAVITY

by

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ABSTRACT

This thesis is concerned with the way in which ideas normally only encountered in the context of gravity may be relevant to hadron physics. In models of this kind the spacetime metric to which hadrons respond (the "strong" metric) is different from that experienced by leptons (the "weak" metric).

In Chapter One a short review of the f-g theory of Isham, Salam and Strathdee, and Wess and Zumino, is presented.

In Chapter Two a class of exact, spherically symmetric, classical solutions of the coupled f-g field equations is found and their properties are discussed. The f and g metrics each effectively induce a cosmological constant in the field equations of the other with the result that the solutions involve de Sitter and anti-de Sitter spacetimes.

The anti-de Sitter case may be able to provide a mechanism for quark confinement. For this and other reasons one is led to ask whether or not sensible quantum field theories exist in such a spacetime. Chapter Three describes an investigation of this problem. The usual quantisation procedures are inapplicable to anti-de Sitter spacetime because it is not globally hyperbolic. Nevertheless a consistent quantisation scheme can be devised by carefully controlling information entering the spacetime through its timelike spatial infinity.

In Chapter Four a model is presented in which the strong and weak metrics are conformally related by a scalar field. Its basis is an adaptation of the Brans-Dicke scalar-tensor theory of gravity. An interesting consequence of the model is that when the separation of two quarks becomes very small the interaction between them switches off except inasmuch as they lower each other's effective mass.

Finally, in Chapter Five, a few remarks are made concerning the future development of strong gravity theories.

PREFACE

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College London, between October 1976 and June 1978, under the supervision of Dr. C.J. Isham. Except where otherwise stated, this work is original and has not been submitted for a degree of this or any other university.

The work described in Chapters Two and Three was originally reported in Refs. 1 and 2 respectively. It was carried out in collaboration with my supervisor and, in the case of Chapter Three, S.J. Avis. I am grateful to them for permission to include this material in my thesis.

Thanks are due to all members of the Theoretical Physics Group for providing a suitable working environment, and particularly to Professor Abdus Salam who is largely responsible for my interest in strong gravity. Above all, I especially thank Chris Isham for assistance, advice and encouragement at all times.

Finally, the financial support of the Science Research Council is gratefully acknowledged.

TO MY PARENTS

CONTENTS

NOTATION AND CONVENTIONS

" $\equiv$ " means "equals by definition".

The metric, $g_{\mu\nu}$, has signature -2.

$\eta_{\mu\nu} \equiv \mathrm{diag}(1,-1,-1,-1)$ or $\mathrm{diag}(1,-1,-r^2,-r^2\sin^2\theta)$.

$$g \equiv \det g_{\mu\nu} \ ,$$

$$\Gamma^{\alpha}_{\mu\nu} \equiv \tfrac{1}{2}g^{\alpha\lambda}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}) \ ,$$

$$R^{\alpha}{}_{\mu\beta\nu} \equiv \Gamma^{\alpha}_{\mu\nu,\beta} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\alpha}_{\sigma\beta}\,\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\sigma\nu}\,\Gamma^{\sigma}_{\mu\beta} \ ,$$

$$R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu} \ .$$

With these conventions the usual Einstein equations, including a cosmological term, are

$$G_{\mu\nu} - \Lambda\, g_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R - \Lambda\, g_{\mu\nu} = +\,\kappa_g^2\, T_{\mu\nu} \ ,$$

where $\kappa_g^2 = 8\pi G$.

$\hbar = c = 1$, but $\hbar$ is sometimes explicitly shown in Chapter 3.

$\overleftrightarrow{\partial}_\mu$ is defined by $\alpha \overleftrightarrow{\partial}_\mu \beta \equiv \alpha\, \partial_\mu \beta - \beta\, \partial_\mu \alpha$.

$$\theta(x) \equiv \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases} \ .$$

CHAPTER ONE

REVIEW OF f-g THEORY

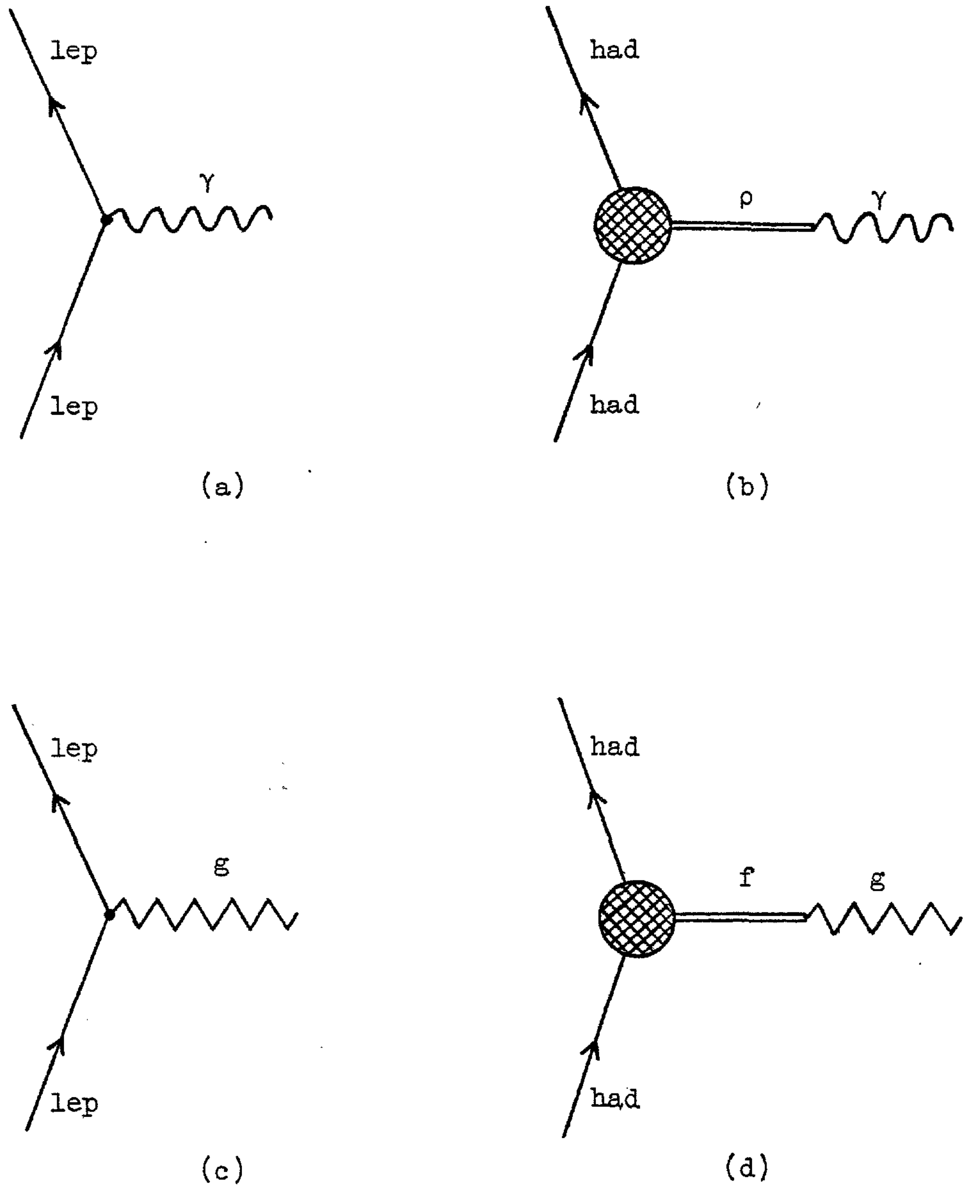
1.1 Why have a radically new massive spin-2 field theory?

Isham, Salam and Strathdee's f-g theory[3] (independently invented in a slightly different form by Wess and Zumino[4]; see Sec. 1.4) describes a spin-2 hadron field coupling universally to other hadrons, and related to gravity in a fundamental way. Some of the reasons for trying to find such a theory may be summarised as follows:

(i) Tensor dominance

The vector dominance hypothesis for the interaction of electromagnetism with charged hadronic matter has had some success, for low energy processes at least. The basic idea here is that photons should only couple to hadrons by virtue of a mixing of the Maxwell field with vector hadron fields of negative C-parity, the lowest mass candidates being the ρ^0, ω and ϕ mesons. In the simplest models only a single field, ρ, is considered. Of course the leptons are assumed to interact directly with the photon, since quantum electrodynamics is known to be a good theory. Diagrammatically, these couplings are represented by Figs. 1.1(a),(b).

If gravity is treated in an analogous way one obtains the tensor dominance hypothesis[5–7] in which hadrons are assumed to interact with gravity via tensor mesons, such as the f, f' and A2, coupled to the hadronic energy-momentum tensor. In the simplest versions of the theory only one such field is considered and is denoted by f, though it is not necessarily supposed to represent the experimentally observed f-meson. It might instead be associated with the pomeron Regge trajectory. (At present the pomeron and f trajectories are generally believed to be distinct - see e.g. Ref. 8.) Figs. 1.1(c),(d) depict the appropriate diagrammatical representations of gravity coupling to leptons and hadrons.



<u>Fig. 1.1</u>

(a) Direct photon-lepton coupling.

(b) Photon-hadron coupling via ρ-γ mixing.

(c) Direct graviton-lepton coupling.

(d) Graviton-hadron coupling via f-g mixing.

(ii) Hadrons and geometry

If the f meson field is to be closely related to gravity then the resulting theory is likely to have a high geometrical content. The differential geometry of fibre bundles has already appeared in particle physics in the context of gauge theories. In addition, however, there is evidence that spacetime geometry could play a rôle in hadron phenomena. There are two main lines of thought here.

Firstly it has been suggested[9] that the ideas of "temperature"[10] and "fireballs" in hadron physics might be related to Hawking radiation[11] from horizons of "strong" spacetime. Of course the correspondence, if any, would be with the very last stage of quantum evaporation, the nature of which is not yet clear.

Secondly, it might be possible to use geometry to provide an interesting mechanism for quark confinement[12,13]. This idea will be discussed in more detail later on.

(iii) Problems with massive spin-2 field theory

A mass M, linear, spin-2 field in Minkowski space (with metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$) is represented by a symmetric tensor $\psi_{\mu\nu}$ satisfying the wave equation

$$(\Box + M^2)\psi_{\mu\nu} = 0 \tag{1.1.1}$$

and the subsidiary conditions

$$\psi_{\mu\nu,\mu} = 0 \tag{1.1.2}$$

and

$$\psi \equiv \psi_{\mu\mu} = 0 \tag{1.1.3}$$

(contractions made using $\eta^{\mu\nu}$), which freeze out the spin-1 and spin-0 parts of $\psi_{\mu\nu}$. All three equations may be derived from the Fierz-Pauli lagrangian density:

$$\mathcal{L}_{FP} = \tfrac{1}{4}\psi_{\mu\nu,\lambda}\,\psi_{\mu\nu,\lambda} - \tfrac{1}{2}\psi_{\lambda\mu,\mu}\,\psi_{\lambda\nu,\nu} + \tfrac{1}{2}\psi_{,\mu}\,\psi_{\mu\nu,\nu} - \tfrac{1}{4}\psi_{,\mu}\,\psi_{,\mu} - \frac{M^2}{4}(\psi_{\mu\nu}\,\psi_{\mu\nu} - \psi^2) \quad . \qquad (1.1.4)$$

What happens if we try to generalise this to a theory in curved spacetime, replacing ordinary derivatives by covariant derivatives and making contractions using $g^{\mu\nu}$? It transpires that the subsidiary conditions may still be derived in the case of a Ricci-flat spacetime, but not otherwise[14]. The problem is that the manipulations used in deriving (1.1.2) and (1.1.3) make use of the commutativity of ordinary derivatives. In curved spacetime the commutation of covariant derivatives gives rise to extra terms involving curvatures.

So in non-Ricci-flat spacetimes the scalar and vector parts of $\psi_{\mu\nu}$ are not suppressed. Some of these degrees of freedom will be "ghosts", describing fields whose energy is of the wrong sign, making the physical interpretation somewhat difficult.

One might ask why it is that spin-2 suffers in this way whereas spins 0 and 1 do not. Of course for spin-0 it is not so surprising since there are no subsidiary conditions. The reason that the massive spin-1 (Proca) field generalises to curved spacetime in a satisfactory way is that the derivative part of its lagrangian is based on the exterior calculus. In particular it makes use of the exterior derivative which does not depend on the metric at all. Indeed this derivative can be defined perfectly well on a nonriemannian manifold. Thus the spin-1 field maintains a certain degree of independence of the metric. For spin-2 on the other hand there is no way of avoiding use of the covariant derivative in the curved spacetime generalisation of $\mathcal{L}_{FP}$, with the result that the metric and its derivatives become entangled with the would-be constraints in an altogether inextricable way.

This comparison may be thought of as a guide to the construction of an improved spin-2 theory in curved spacetime. One should attempt to use derivatives which are independent of $g_{\mu\nu}$.

(iv) Experimental quantum gravity

Due to the small size of the gravitational coupling constant, quantum effects in gravity (perturbative effects at least) are probably irrelevant for energies much smaller than the Planck energy. This is roughly 10^{19} GeV, compared with present day experimental energies of order 10 GeV. So if hadron physics does involve a "strong" gravitational field of some description then it may provide the only experimental testing ground for quantum gravity theories.

1.2 The f-g lagrangian

In (iii) above it was suggested that the derivative part of the spin-2 lagrangian should be constructed without using the covariant derivative. Let us survey the various possibilities.

The exterior derivative has already been mentioned. It is ideal for spin-1, but is unusable in the present case since there is no form capable of describing a spin-2 field.

The other coordinate independent derivative which can be defined on a nonriemannian manifold is the Lie derivative. This, however, requires a vector field for its definition, and on detailed investigation seems to offer little hope.

There is only one alternative left. We already know of a spin-2 theory constructed without prior metric, namely general relativity. Although the derivatives in the definitions of the curvature tensor cannot themselves be written in a coordinate independent way, the resulting object <u>is</u> coordinate independent. Thus it makes sense to postulate an Einstein type lagrangian for the f-field which is now represented by the metric tensor $f_{\mu\nu}$. This introduction of the "strong" metric, $f_{\mu\nu}$, realises the desire to incorporate geometry into hadron physics, as discussed in (ii) above.

As a first stage in incorporating tensor dominance ideas ((i) above) it is postulated that whereas leptons (the photon is included in this category) respond to the metric $g_{\mu\nu}$, hadrons live in the world of $f_{\mu\nu}$. Thus we may write the lagrangian density (as yet incomplete) as

$$\mathcal{L}_0 = \mathcal{L}_g + \mathcal{L}(\text{leps.}, g_{\mu\nu}) + \mathcal{L}_f + \mathcal{L}(\text{hads.}, f_{\mu\nu}) \tag{1.2.1}$$

where

$$\mathcal{L}_g = -\frac{1}{\kappa_g^2}\sqrt{-g}\, R^g \quad , \tag{1.2.2}$$

and

$$\mathcal{L}_f = -\frac{1}{\kappa_f^2}\sqrt{-f}\, R^f \quad . \tag{1.2.3}$$

R^g and R^f are the Ricci scalar curvatures constructed from $g_{\mu\nu}$ and $f_{\mu\nu}$ respectively. The constant κ_g is related to the gravitational constant, G, by

$$\kappa_g = \sqrt{8\pi G} \simeq 2 \times 10^{-22}\, m_e^{-1} \tag{1.2.4}$$

whereas κ_f, the strong gravity coupling constant, is assumed to have a size characteristic of the strong interactions:

$$\kappa_f \sim m_f^{-1} \quad . \tag{1.2.5}$$

In fact the reasons given above for the use of an Einstein lagrangian for $f_{\mu\nu}$ are not the only ones. It is known that the universality of the coupling of gravity to matter causes gravity to interact with <u>itself</u> as a consistency requirement, and by an iterative process starting from linear spin-2 theory one is led to the Einstein lagrangian[15]. The same argument can be used for the strong gravity field if it is to be coupled universally to hadronic matter.

The lagrangian is not yet complete. From (1.2.1) it is clear that the lepton and hadron worlds do not communicate with each other, and that the f-meson is massless. Both defects are rectified by the introduction of a generally covariant mixing term, $\mathcal{L}_{fg}$, which provides the f-g mixing for the tensor dominance hypothesis. The invariance of the lagrangian is reduced by $\mathcal{L}_{fg}$. Whereas before, independent coordinate transformations could be carried out in the f and g worlds, $f_{\mu\nu}$ and $g_{\mu\nu}$ must now be treated as tensors on the same manifold, and only a single set of coordinate transformations makes sense.

1.3 The f-g mixing term

The criteria which the f-g mixing term is required to satisfy are as follows:

(1) General covariance. $\mathcal{L}_{fg}$ must be a scalar density.

(2) The field equations should admit the "vacuum solution" $f_{\mu\nu} = g_{\mu\nu} = \eta_{\mu\nu}$, corresponding to Minkowski spacetime with a vanishing "physical" f-meson field (associated with the difference of $f_{\mu\nu}$ and $g_{\mu\nu}$, as discussed later).

(3) The theory should be ghost free.

Suppose the fields are close to their vacuum values and write

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa_g h_{\mu\nu} \qquad (1.3.1)$$

$$f_{\mu\nu} = \eta_{\mu\nu} + \kappa_f e_{\mu\nu} \qquad (1.3.2)$$

To within 4-divergences, the expansions of $\mathcal{L}_f$ and $\mathcal{L}_g$ to bilinear order in $e_{\mu\nu}$ and $h_{\mu\nu}$ are identical to the derivative part of the Fierz-Pauli lagrangian density, $\mathcal{L}_{FP}$ (eqn. (1.1.4)). So what is needed is a mixing term which reduces to the Fierz-Pauli mass term of (1.1.4) in the bilinear approximation. Now it is well known (see e.g. Ref. 16) that a mass cannot be attached to a single gravitational field in a generally covariant way. At lowest order a mass term for $e_{\mu\nu}$ say would not respect invariance under the gauge transformation

$$\begin{aligned} \kappa_f\, e_{\mu\nu} &\to \kappa_f\, e_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} \\ \kappa_g\, h_{\mu\nu} &\to \kappa_g\, h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} \quad . \end{aligned} \tag{1.3.3}$$

Such a gauge transformation is the lowest order part of a coordinate transformation in the full theory. However, an invariant mass term is possible for the difference $\kappa_f\, e_{\mu\nu} - \kappa_g\, h_{\mu\nu}$. This means $\mathcal{L}_{fg}$ is of the form

$$\mathcal{L}_{fg} = \frac{M^2}{4\kappa_f^2}(\kappa_f\, e_{\mu\nu} - \kappa_g\, h_{\mu\nu})(\kappa_f\, e_{\alpha\beta} - \kappa_g\, h_{\alpha\beta})(\eta^{\mu\alpha}\,\eta^{\nu\beta} - \eta^{\mu\nu}\,\eta^{\alpha\beta}) + \text{+ higher order terms.} \tag{1.3.4}$$

Diagonalisation for mass now shows that there is a massive ghost-free field $(\kappa_f\, e_{\mu\nu} - \kappa_g\, h_{\mu\nu})$, with mass

$$m_f = (1 + \kappa_g^2\, \kappa_f^{-2})^{\frac{1}{2}}\, M \simeq M \quad , \tag{1.3.5}$$

and a massless field $(\kappa_g\, e_{\mu\nu} + \kappa_f\, h_{\mu\nu})$, to be regarded as the linearised f-meson and graviton fields respectively.

The mixing term $\mathcal{L}_{fg}$ is not uniquely determined by its bilinear structure and general covariance. Some suggested forms are as follows:

(a) A straightforward covariantisation[3] of the Fierz-Pauli mass term for $(f_{\mu\nu} - g_{\mu\nu})$ or $(f^{\mu\nu} - g^{\mu\nu})$, e.g.

$$\mathcal{L}_{fg} = -\frac{M^2}{4\kappa_f^2}(f^{\mu\nu} - g^{\mu\nu})(f^{\alpha\beta} - g^{\alpha\beta})(g_{\mu\alpha}\, g_{\nu\beta} - g_{\mu\nu}\, g_{\alpha\beta})\sqrt{-g} \, . \tag{1.3.6}$$

Many simple modifications are possible e.g. exchanging g's for f's in the third bracket, or using $\sqrt{-f}$ instead of $\sqrt{-g}$. It is this type of mass term which will receive most attention in Chapter Two.

(b) The "cosmological" mixing term[3]

$$\mathcal{L}_{fg} = -\lambda\sqrt{-g} - \lambda'\sqrt{-f} + (\lambda + \lambda')(-f)^{\alpha}(-g)^{\beta}\left[-\det\tfrac{1}{2}(f^{\mu\nu} + g^{\mu\nu})\right]^{\alpha+\beta-\frac{1}{2}} \quad (1.3.7)$$

where

$$\alpha = \tfrac{1}{2}(2\lambda + \lambda')\,\lambda'(\lambda + \lambda')^{-2} \quad (1.3.8)$$

and

$$\beta = \tfrac{1}{2}\lambda\,(\lambda + 2\lambda')(\lambda + \lambda')^{-2} \quad (1.3.9)$$

and the f mass is given by

$$M^2 = \tfrac{1}{2}(\kappa_f^2 + \kappa_g^2)\lambda\lambda'(\lambda + \lambda')^{-1} \quad . \quad (1.3.10)$$

Again many simple modifications are possible.

(c) Some attractively simple mixing terms are possible in the vierbein version of f-g theory due to Wess and Zumino[4] which will be discussed in the next section.

From the discussion above it should be clear that the physical f-meson field is actually associated with something like the difference of the two metrics (precisely the difference in the linear approximation). When the metrics are identical the theory reduces to the usual Einstein lagrangian for general relativity.

As regards the constraints problem, the improvement over the linear theory described in Sec. 1.1(iii) is as follows. Because $G^{g\mu\nu}$ and $G^{f\mu\nu}$ satisfy the Bianchi identities

$$\nabla^g_\mu G^{g\mu\nu} = \nabla^f_\mu G^{f\mu\nu} = 0 \quad (1.3.11)$$

(where ∇^g_μ and ∇^f_μ denote the covariant derivatives associated with $g_{\mu\nu}$ and $f_{\mu\nu}$ respectively), we apparently have eight first order constraints, namely

$$\nabla^g_\mu \left[\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}_{fg}}{\delta g_{\mu\nu}} \right] = \nabla^f_\mu \left[\frac{1}{\sqrt{-f}} \frac{\delta \mathcal{L}_{fg}}{\delta f_{\mu\nu}} \right] = 0 \quad . \tag{1.3.12}$$

Four combinations of these are satisfied identically due to general covariance, which is associated with the reduction in the degrees of freedom of the massless field from ten to two.

The other four are the analogue of (1.1.2). Thus the four corresponding degrees of freedom, which cause trouble in the usual linear massive spin-2 theory as discussed in Sec. 1.1(iii), are completely removed in f-g theory.

Regrettably an analogue of (1.1.3) cannot be found, so this one (possibly ghost) degree of freedom remains, and casts doubt on the boundedness below of the energy when an ADM decomposition is made[16,17]. It is an attractive idea that there might be a restricted class of mixing terms for which the ghost is exorcised, but arduous investigations along these lines have not yet proved fruitful. Nevertheless the ghost is certainly eliminated at lowest order at least, by virtue of the Fierz-Pauli form of the linearised theory. Salam and Strathdee have suggested[18] that the problem might be resolved by generating the f-meson mass dynamically, constructing the mixing term using Yang-Mills fields which may develop c-number parts[19]. Even in this case though the mixing term is still effectively of the form already discussed, and will be kept as such in the sequel.

1.4 Vierbein version of f-g theory

In the version of f-g theory due to Wess and Zumino[4] the emphasis is shifted from metrics to vierbeins. Thus rather than using $f_{\mu\nu}$ and $g_{\mu\nu}$ as the fundamental fields one uses orthonormal tetrads of vectors,

$K_a{}^\mu$ and $L_a{}^\mu$ respectively. Latin indices label the member of the tetrad and are raised or lowered using $\eta^{ab} = \eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ whereas Greek indices label the components of each vector and are raised or lowered using the respective metric as usual. The metrics may be expressed in terms of the vierbeins by the formulas

$$f_{\mu\nu} = K^a{}_\mu K_{a\nu} \quad , \quad g_{\mu\nu} = L^a{}_\mu L_{a\nu} \quad . \tag{1.4.1}$$

The derivative part of the lagrangian is the same as before, but with the above substitutions. In fact this introduces an extra invariance into the theory, corresponding to local Lorentz (SO(3,1)) transformations of the tetrads. So we begin with the lagrangian density

$$\mathcal{L}_0 = -\frac{1}{2\kappa_f^2} K R^K - \frac{1}{2\kappa_g^2} L R^L \quad , \tag{1.4.2}$$

where

$$K \equiv \det K_{a\mu} \quad , \quad L \equiv \det L_{a\mu} \quad . \tag{1.4.3}$$

A mixing term $\mathcal{L}_{KL}$ is introduced just as in the metric theory. This has the effect that $K_a{}^\mu$ and $L_a{}^\mu$ must now be regarded as vierbeins on the <u>same</u> manifold and must be treated accordingly when performing coordinate and local Lorentz transformations. $\mathcal{L}_{KL}$ is required to be of the Fierz-Pauli form for $(\kappa_f k_{a\mu} - \kappa_g \ell_{a\mu})$ in the linearised limit in which

$$K_a{}^\mu = \delta_a^\mu + \kappa_f k_a{}^\mu \quad , \quad L_a{}^\mu = \delta_a^\mu + \kappa_g \ell_a{}^\mu \quad . \tag{1.4.4}$$

Wess and Zumino found four particularly simple mixing terms with this property, namely

$$-\frac{M^2}{2\kappa_f^2}(3K - K K_a{}^\mu L^a{}_\mu + L) \tag{1.4.5}$$

$$-\frac{M}{2\kappa_f^2}(K - L K^a{}_\mu L_a{}^\mu + 3L) \tag{1.4.6}$$

$$-\frac{M}{2\kappa_f^2}(K - K^{\frac{1}{2}} L^{\frac{1}{2}} K_a{}^\mu L^a{}_\mu + 3L) \tag{1.4.7}$$

$$-\frac{M}{2\kappa_f^2}(3K - K^{\frac{1}{2}} L^{\frac{1}{2}} K^a{}_\mu L_a{}^\mu + L) \tag{1.4.8}$$

The ability to construct such simple mixing terms is a consequence of the expansion of determinants:

$$K = -1 - k_{aa} + \tfrac{1}{2}(k_{ab}\, k_{ab} - k_{aa}\, k_{bb}) + \ldots.. \qquad (1.4.9)$$

(contractions made using η^{ab}). The bilinear part is of precisely the form required, and it is this fact which is exploited in constructing the above expressions. In the metric theory it is the square roots of determinants which must occur in order that the mixing term be a scalar density of the correct weight. These do not have the above property and so simple mixing terms of this type cannot be found in that version of the theory.

An extra twelve degrees of freedom have been introduced in the vierbein theory and these must be removed again if the particle content is to be the same. The SO(3,1) invariance allows six of them to be fixed by choice, for example by stipulating that one of the vierbeins be symmetric. The other six are removed by the antisymmetric parts of the field equations since the derivative parts of these equations are necessarily symmetric. The way in which these constraints appear will be seen more clearly in Sec. 2.6 in the context of trying to find spherically symmetric classical solutions.

More realistic theories should incorporate internal symmetry and the use of vierbeins offers an attractive way of doing so. Isham, Salam and Strathdee[20] showed that the SO(3,1) (or more appropriately $SL(2,\mathbb{C})$) invariance could be extended in a natural way to $SL(2,\mathbb{C}) \times SU(2)$, $SL(4,\mathbb{C})$, $SL(2,\mathbb{C}) \times SU(3)$, $SL(6,\mathbb{C})$, etc.

Attempts have also been made to put internal symmetry into the metric version, although more difficulties seem to arise here. The U(3) model of Isham and Tucker[21] describes a nonet of tensor mesons aquiring their masses through a mixing with gravity.

For the purposes of the next chapter, however, only the simple theory without internal symmetry will be considered.

CHAPTER TWO

EXACT SPHERICALLY SYMMETRIC CLASSICAL SOLUTIONS FOR f-g THEORY

2.1 Why look for spherically symmetric classical solutions?

In this chapter spherically symmetric solutions of the f-g field equations are investigated with the emphasis on the metric version of the theory, for which explicit solutions can be found.

The importance of classical solutions for field theories has recently been enhanced by the discovery of the relevance in a quantum context of solitons, instantons and related objects. An important class in a 3+1 dimensional theory is that composed of static, spherically symmetric solutions, the t'Hooft monopole being a prime example.

In general relativity the Schwarzschild metric is perhaps the most discussed solution of the field equations other than Minkowski space. It was the desire to find the f-g analogue of this which prompted the investigation described in this chapter, but the results are rather different, as will be seen. The possible relevance to hadron physics has already been discussed in Sec. 1.1(ii) and so will not be repeated here.

The first major attempts to find such solutions were made by Aragone and Chela-Flores[22,23]. More recently an explicit solution was found by Salam and Strathdee[24], but only in the approximation that the g-metric is that of Minkowski space. Although this might seem a physically reasonable approximation, many important questions cannot be satisfactorily resolved within this framework. For example, the rôle played by coordinate singularities is difficult to discuss when the g-metric is not completely known. In any case, the solutions found here certainly do not possess "almost flat" g-metrics in general.

It is worth mentioning in passing that solutions with other symmetries have been discussed in Refs. 25-29.

2.2 The mixing term, field equations and imposition of symmetry

The mixing term chosen here is of the type discussed in Sec. 1.3(a). It is

$$\mathcal{L}_{fg} = -\frac{M^2}{4\kappa_f^2}(-g)^u(-f)^v(f^{\alpha\beta} - g^{\alpha\beta})(f^{\sigma\tau} - g^{\sigma\tau})(g_{\alpha\sigma}\, g_{\beta\tau} - g_{\alpha\beta}\, g_{\sigma\tau}) \tag{2.2.1}$$

where

$$u + v = \tfrac{1}{2} \tag{2.2.2}$$

so that $\mathcal{L}_{fg}$ is a tensor density of the correct weight.

Only solutions of the matter-free equations are sought, so the lagrangian is

$$L = \int \left\{ -\frac{1}{\kappa_g^2}\sqrt{-g}\, R^g - \frac{1}{\kappa_f^2}\sqrt{-f}\, R^f + \mathcal{L}_{fg} \right\} d^4x \quad . \tag{2.2.3}$$

Salam and Strathdee[24] have considered the case in which $u = \tfrac{1}{2}$ and $v = 0$, with the approximation $g_{\mu\nu} = \eta_{\mu\nu}$ (i.e. $\kappa_g \to 0$). No such approximation is made here, but their methods are extended to solve the coupled equations for $f_{\mu\nu}$ and $g_{\mu\nu}$.

The value of the parameter u is left unspecified for the sake of generality. It will be seen that it plays an important rôle in determining the "cosmological" (i.e. large r) behaviour of the solutions.

Upon varying $f^{\mu\nu}$, the action principle $\delta L = 0$ gives the f-field equations

$$G^f_{\mu\nu} \equiv R^f_{\mu\nu} - \tfrac{1}{2} f_{\mu\nu} R^f = \kappa_f^2\, T^f_{\mu\nu} \tag{2.2.4}$$

where

$$T^f_{\mu\nu} = \frac{M^2}{4\kappa_f^2}\left(\frac{g}{f}\right)^u \Big[v\, f_{\mu\nu}(f^{\alpha\beta} - g^{\alpha\beta})(f^{\sigma\tau} - g^{\sigma\tau})(g_{\alpha\sigma}\, g_{\beta\tau} - g_{\alpha\beta}\, g_{\sigma\tau}) - -2(f^{\alpha\beta} - g^{\alpha\beta})(g_{\alpha\mu}\, g_{\beta\nu} - g_{\alpha\beta}\, g_{\mu\nu}) \Big] \quad . \tag{2.2.5}$$

A useful way of re-expressing (2.2.4) is

$$R^f_{\mu\nu} = \kappa_f^2\, (T^f_{\mu\nu} - \tfrac{1}{2} f_{\mu\nu}\, T^f) \tag{2.2.6}$$

with

$$T^f \equiv f^{\mu\nu} T^f_{\mu\nu} \quad . \tag{2.2.7}$$

Variation of $g^{\mu\nu}$ yields the Einstein equations

$$G^g_{\mu\nu} \equiv R^g_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R^g = \kappa_g^2 \, T^g_{\mu\nu} \tag{2.2.8}$$

where

$$T^g_{\mu\nu} = \frac{M^2}{4\kappa_f^2}\left(\frac{f}{g}\right)^V \Big[2(f^{\alpha\beta} - g^{\alpha\beta})(g_{\alpha\mu}\, g_{\beta\nu} - g_{\alpha\beta}\, g_{\mu\nu}) + (f^{\alpha\beta} - g^{\alpha\beta})(f^{\sigma\tau} - g^{\sigma\tau}) \times$$
$$\times \, (u\, g_{\mu\nu}\, g_{\alpha\sigma}\, g_{\beta\tau} - u\, g_{\mu\nu}\, g_{\alpha\beta}\, g_{\sigma\tau} + 2 g_{\alpha\mu}\, g_{\sigma\nu}\, g_{\beta\tau} - 2 g_{\alpha\mu}\, g_{\beta\nu}\, g_{\sigma\tau})\Big] \tag{2.2.9}$$

Only the spherically symmetric "static" (see below) case is to be investigated. Then, without loss of generality, the metrics may be written in the form

$$f_{\mu\nu}\, dx^\mu\, dx^\nu = C\, dt^2 - 2D\, dt\, dr - A\, dr^2 - B(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.2.10}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = J\, dt^2 - K\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.2.11}$$

with inverses

$$f^{\mu\nu}\, \partial_\mu\, \partial_\nu = \frac{A}{\Delta}\partial_t^2 - \frac{2D}{\Delta}\partial_t\, \partial_r - \frac{C}{\Delta}\partial_r^2 - \frac{1}{B}(\partial_\theta^2 + \sin^{-2}\theta\, \partial_\phi^2) \tag{2.2.12}$$

$$g^{\mu\nu}\, \partial_\mu\, \partial_\nu = \frac{1}{J}\partial_t^2 - \frac{1}{K}\partial_r^2 - \frac{1}{r^2}(\partial_\theta^2 + \sin^{-2}\theta\, \partial_\phi^2) \tag{2.2.13}$$

where

$$\Delta \equiv AC + D^2 > 0 \tag{2.2.14}$$

and A,B,C,D,J,K are functions of r only. As explained in Sec. 1.2 the theory is invariant under coordinate transformations applied simultaneously to both metrics, and these have been used to optimally simplify the form of $g_{\mu\nu}$.

The above use of the word "static" requires further elucidation. Each metric is static in the sense that for each of them one can find a hypersurface-orthogonal timelike Killing vector field. In general it will not be possible to choose the same vector field for both and in this sense they are not relatively static. However, they are relatively stationary since a timelike Killing vector field for both metrics can be found. In the coordinate system chosen above the vector ∂_t has this property.

2.3 Curvature computations

The Ricci curvature components may be conveniently and efficiently computed using the method of curvature 2-forms (see e.g. Ref. 30). To begin with, an orthonormal frame of 1-forms must be found. A convenient choice for the f-metric is ω^a (f superscripts will be dropped for a while) where

$$\omega^0 = C^{\frac{1}{2}}\, dt - D\, C^{-\frac{1}{2}}\, dr \qquad \omega^1 = \Delta^{\frac{1}{2}}\, C^{-\frac{1}{2}}\, dr$$
$$\omega^2 = B^{\frac{1}{2}}\, d\theta \quad , \qquad \omega^3 = B^{\frac{1}{2}}\, \sin\theta\, d\phi \tag{2.3.1}$$

so that

$$f_{\mu\nu}\, dx^\mu\, dx^\nu = (\omega^0)^2 - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2 \quad . \tag{2.3.2}$$

The connection 1-forms $\omega^a{}_b$ are deduced from the structure equations

$$d\omega^a = -\,\omega^a{}_b \wedge \omega^b \tag{2.3.3}$$

and

$$\omega^{ab} = -\,\omega^{ba} \quad . \tag{2.3.4}$$

The nonvanishing connection 1-forms are

$$\omega^0{}_1 = \omega^1{}_0 = \tfrac{1}{2}\, C'\, (C\Delta)^{-\frac{1}{2}}\, \omega^0$$

$$-\,\omega^1{}_2 = \omega^2{}_1 = \frac{B'}{2B}\left(\frac{C}{\Delta}\right)^{\frac{1}{2}} \omega^2$$

(ctd.)

$$- \omega^1{}_3 = \omega^3{}_1 = \frac{B'}{2B}\left(\frac{C}{\Delta}\right)^{\frac{1}{2}} \omega^3$$

$$- \omega^2{}_3 = \omega^3{}_2 = B^{-\frac{1}{2}} \cot\theta \, \omega^3 \qquad (2.3.5)$$

The curvature 2-forms are computed from the defining formula

$$\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \qquad (2.3.6)$$

and in this way one finds

$$\mathcal{R}^0{}_1 = \mathcal{R}^1{}_0 = \left[\frac{C'\Delta'}{4\Delta^2} - \frac{C''}{2\Delta}\right] \omega^0 \wedge \omega^1$$

$$\mathcal{R}^0{}_2 = \mathcal{R}^2{}_0 = - \frac{B'C'}{4B\Delta} \omega^0 \wedge \omega^2$$

$$\mathcal{R}^0{}_3 = \mathcal{R}^3{}_0 = - \frac{B'C'}{4B\Delta} \omega^0 \wedge \omega^3 \qquad (2.3.7)$$

$$\mathcal{R}^1{}_2 = -\mathcal{R}^2{}_1 = \left[- \frac{B''C}{2B\Delta} + \frac{B'^2C}{4B^2\Delta} - \frac{B'C'}{4B\Delta} + \frac{B'\Delta'C}{4B\Delta^2}\right] \omega^1 \wedge \omega^2$$

$$\mathcal{R}^1{}_3 = -\mathcal{R}^3{}_1 = \left[- \frac{B''C}{2B\Delta} + \frac{B'^2C}{4B^2\Delta} - \frac{B'C'}{4B\Delta} + \frac{B'\Delta'C}{4B\Delta^2}\right] \omega^1 \wedge \omega^3$$

$$\mathcal{R}^2{}_3 = -\mathcal{R}^3{}_2 = \left[\frac{1}{B} - \frac{B'^2C}{4B^2\Delta}\right] \omega^2 \wedge \omega^3 \quad .$$

The nonvanishing Riemann tensor components can then be read off from the relation

$$\mathcal{R}^a{}_b = R^a{}_{bcd} \, \omega^c \wedge \omega^d \qquad \text{(summation over } c > d \text{ only).} \qquad (2.3.8)$$

They are as follows, omitting those which are obtainable from the given components using the symmetry properties of the Riemann tensor:

$$R^0{}_{101} = \frac{C'\Delta'}{4\Delta^2} - \frac{C''}{2\Delta}$$

$$R^0{}_{202} = R^0{}_{303} = - \frac{B'C'}{4B\Delta} \qquad (2.3.9)$$

$$R^1{}_{212} = R^1{}_{313} = - \frac{B''C}{2B\Delta} + \frac{B'^2C}{4B^2\Delta} - \frac{B'C'}{4B\Delta} + \frac{B'\Delta'C}{4B\Delta^2}$$

$$R^2{}_{323} = \frac{1}{B} - \frac{B'^2C}{4B^2\Delta} \quad .$$

Contracting to form the Ricci tensor and converting back to the coordinate frame produces the following nonvanishing components (f superscripts are now restored) :

$$R^f_{tt} = \frac{C}{2\Delta}\left[C'' + \frac{B'C'}{B} - \frac{C'\Delta'}{2\Delta}\right]$$

$$R^f_{tr} = R^f_{rt} = -\frac{D}{2\Delta}\left[C'' + \frac{B'C'}{B} - \frac{C'\Delta'}{2\Delta}\right] \tag{2.3.10}$$

$$R^f_{rr} = -\frac{B''}{B} + \frac{B'^2}{2B^2} - \frac{A}{2\Delta}\left[C'' + \frac{B'C'}{B} - \frac{B'\Delta'}{B\Delta} - \frac{C'\Delta'}{2\Delta}\right]$$

$$R^f_{\theta\theta} = \sin^{-2}\theta\, R^f_{\phi\phi} = 1 - \frac{C}{2\Delta}\left[B'' + \frac{B'C'}{C} - \frac{B'\Delta'}{2\Delta}\right] \quad .$$

Then the components of $R^g_{\mu\nu}$ are simply obtained by making the replacements

$$C \to J, \quad A \to K, \quad D \to 0, \quad \Delta \to JK, \quad B \to r^2, \tag{2.3.11}$$

and the nonvanishing components are

$$R^g_{tt} = \frac{J''}{2K} + \frac{J'}{rK} - \frac{J'^2}{4JK} - \frac{J'K'}{4K^2}$$

$$R^g_{rr} = -\frac{J''}{2J} + \frac{K'}{rK} + \frac{J'K'}{4JK} + \frac{J'^2}{4J^2} \tag{2.3.12}$$

$$R^g_{\theta\theta} = \sin^{-2}\theta\, R^g_{\phi\phi} = 1 - \frac{1}{K} - \frac{3rJ'}{2JK} - \frac{rK'}{2K^2} \quad .$$

2.4 Solving the equations

Expressions (2.3.10) display the simple algebraic identity

$$D\, R^f_{tt} + C\, R^f_{tr} = 0 \quad . \tag{2.4.1}$$

Hence, from the f-field equations (2.2.6)

$$D\, T^f_{tt} + C\, T^f_{tr} = 0 \tag{2.4.2}$$

which becomes, upon substituting the explicit form of the metrics, (2.2.10) - (2.2.14),

$$\left\{ \frac{2r^2}{B} - 3 \right\} D = 0 \quad . \tag{2.4.3}$$

It transpires that exactly the same result is obtained from the identity

$$T^g_{tr} = 0 \tag{2.4.4}$$

which is a consequence of $R^g_{tr} = 0$ and the Einstein equations (2.2.8). Thus we can consistently set either

$$B = \frac{2}{3} r^2 \tag{2.4.5}$$

or

$$D = 0 \tag{2.4.6}$$

and the resulting solutions will be labelled Type I and Type II respectively, following Salam and Strathdee[24].

Regrettably no explicit Type II solution has yet been found, even in the approximation $g_{\mu\nu} = \eta_{\mu\nu}$. The large r asymptotic structure has been investigated in detail by Aragone and Chela-Flores[22,23], using this approximation, and has a Yukawa-like behaviour for asymptotically flat solutions.

Here we will only consider the Type I solutions. It is already clear from (2.4.5) that there will be no weak field region, since $f^{\mu\nu} - g^{\mu\nu} \simeq 0$ requires $B \simeq r^2$, <u>not</u> $\frac{2}{3} r^2$.

At this point it is convenient to display the nonvanishing components of $T^f_{\mu\nu}$ and $T^g_{\mu\nu}$ using the explicit form of the metrics (2.2.10) - (2.2.14) and setting $B = \frac{2}{3} r^2$:

$$C^{-1} T^f_{tt} = -D^{-1} T^f_{tr} = -A^{-1} T^f_{rr} = \frac{M^2}{4\kappa_f^2} \left(\frac{9JK}{4\Delta}\right)^u \left[\frac{3v}{2} + \frac{2JK}{\Delta}(1 - v)\right] \tag{2.4.7}$$

$$T^f_{\theta\theta} = \sin^{-2}\theta \, T^f_{\phi\phi} = \frac{M^2 r^2}{4\kappa_f^2} \left(\frac{9JK}{4\Delta}\right)^u \left[\frac{4vJK}{3\Delta} - v + 3 - \frac{2}{\Delta}(JA + KC)\right] \tag{2.4.8}$$

$$J^{-1} T^g_{tt} = -K^{-1} T^g_{rr} = \frac{M^2}{4\kappa_f^2} \left(\frac{4\Delta}{9JK}\right)^v \left[\frac{3u}{2} - \frac{2JK}{\Delta}(1 + u)\right] \tag{2.4.9}$$

$$T^g_{\theta\theta} = \sin^{-2}\theta \, T^g_{\phi\phi} = \frac{M^2 r^2}{4\kappa_f^2} \left(\frac{4\Delta}{9JK}\right)^v \left[\frac{2uJK}{\Delta} + \frac{3}{\Delta}(JA + KC) - \frac{3u}{2} - \frac{9}{2}\right] . \tag{2.4.10}$$

Using the field equations, the simple relations $A\ T^f_{tt} + C\ T^f_{rr} = 0$ and $K\ T^g_{tt} + J\ T^g_{rr} = 0$ become

$$A\ R^f_{tt} + C\ R^f_{rr} = 0 \tag{2.4.11}$$

and

$$K\ R^g_{tt} + J\ R^g_{rr} = 0 \quad . \tag{2.4.12}$$

Now (2.3.10), (2.3.12) and (2.4.5) are used to substitute for R^f_{tt} , etc., and following a fair amount of algebra it is found that

$$\Delta' = (JK)' = 0 \tag{2.4.13}$$

i.e. Δ and JK are constants of integration. It is convenient to choose JK = 1 by a suitable rescaling of the time parameter t, this being the last remaining degree of freedom in our choice of coordinates.

Using these results, the general solution of the remaining f-field equations is found:

$$JA + J^{-1}C = \frac{3}{2}\Delta + \frac{2}{3} \tag{2.2.14}$$

$$C = \frac{3}{2}\Delta\left(1 - \frac{2\mu_f}{r} - \frac{2\lambda r^2}{9}\right) \tag{2.2.15}$$

where λ is a constant given by

$$\lambda = \frac{M^2}{4}\left(\frac{9}{4\Delta}\right)^u \left[\frac{3v}{2} + \frac{2}{\Delta}(1 - v)\right] \tag{2.4.16}$$

and μ_f is an integration constant.

The constant λ seems very much like a cosmological constant and in fact, substituting (2.4.14) back into the expressions for $T^f_{\mu\nu}$ one finds

$$\kappa_f^2\ T^f_{\mu\nu} = \lambda\ f_{\mu\nu} \tag{2.4.17}$$

and consequently

$$G^f_{\mu\nu} - \lambda\ f_{\mu\nu} = 0 \quad . \tag{2.4.18}$$

A very similar result is found for $T^g_{\mu\nu}$. The substitution of JK = 1 and (2.4.14) into (2.4.9) and (2.4.10) yields

$$\kappa_g^2 \, T^g_{\mu\nu} = \Lambda g_{\mu\nu} \tag{2.4.19}$$

where

$$\Lambda = \frac{M^2\kappa_g^2}{4\kappa_f^2}\left(\frac{4\Delta}{9}\right)^v \left[\frac{3u}{2} - \frac{2}{\Delta}(1 + u)\right] \quad . \tag{2.4.20}$$

What has happened is that the f-field configuration has induced a cosmological constant on the right hand side of the g-field equations and vice versa.

The g-field equations

$$G^g_{\mu\nu} = \Lambda g_{\mu\nu} \tag{2.4.21}$$

in the chosen coordinate system have the standard general spherically symmetric solution

$$J = 1 - \frac{2\mu_g}{r} - \frac{\Lambda r^2}{3} \quad , \tag{2.4.22}$$

μ_g being another integration constant.

Summarising:

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = \left(1 - \frac{2\mu_g}{r} - \frac{\Lambda r^2}{3}\right) dt^2 - \left(1 - \frac{2\mu_g}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.4.23}$$

$$f_{\mu\nu}\, dx^\mu\, dx^\nu = \frac{3\Delta}{2}\left(1 - \frac{2\mu_f}{r} - \frac{2\lambda r^2}{9}\right) - 2\,D\,dt\,dr - A\,dr^2 - \frac{2}{3}\,r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.4.24}$$

where

$$D^2 = \Delta\,(1 - X)(1 - \frac{9\Delta X}{4}) \quad , \tag{2.4.25}$$

$$A = \left(1 - \frac{2\mu_g}{r} - \frac{\Lambda r^2}{3}\right)^{-1} \left[\frac{2}{3} + \frac{3\Delta}{2}(1 - X)\right] \quad , \tag{2.4.26}$$

$$X = \left(1 - \frac{2\mu_f}{r} - \frac{2\lambda r^2}{9}\right)\left(1 - \frac{2\mu_g}{r} - \frac{\Lambda r^2}{3}\right)^{-1} \quad , \qquad (2.4.27)$$

and Λ, λ are given by respectively equations (2.4.20) and (2.4.16). This is the general Type I solution.

2.5 Properties of the solutions

The f and g equations are in a sense decoupled, each set becoming the Einstein equations with (related) cosmological terms. The coordinate system was chosen to simplify the structure of the g-metric by expressing it in a conventional diagonal form. Because the general covariance of the theory refers to simultaneous coordinate transformations for both metrics, the functional form of $f_{\mu\nu}$ is already determined by that of $g_{\mu\nu}$. It is perhaps not surprising that $f_{\mu\nu}$ appears in a form which, were it the only metric in the theory, would be regarded as being associated with a rather unconventional choice of coordinates. This "locking together" of the two metrics in the chosen coordinate system is manifested in equations (2.4.5) and (2.4.14) and in the relation between the two cosmological constants.

Each metric corresponds to a Schwarzschild (μ_f or $\mu_g > 0$) or anti-Schwarzschild (μ_f or $\mu_g < 0$) plus de Sitter (Λ or $\lambda > 0$) or anti-de Sitter (Λ or $\lambda < 0$) spacetime. For a given value of u the cosmological constants are not independent but are related to each other through their dependence on the integration constant Δ. On the other hand, the Schwarzschild masses μ_f and μ_g are completely independent. This suggests that it is the cosmological structure that is the most important aspect of the solution, particularly if interpreting it as being "solitonic".

An interesting special case results from the choice

$$\mu_g = 0 \ , \quad \Delta = \frac{4}{3}\,(1 + u^{-1}) \Rightarrow \Lambda = 0 \quad . \tag{2.5.1}$$

Then $g_{\mu\nu} = \eta_{\mu\nu}$ and so g-spacetime is simply Minkowski spacetime. (This shows that the Salam-Strathdee solution[24] happens to be exact for $\Delta = 4$.) The f cosmological constant is fixed by (2.5.1), taking the value

$$\lambda \Big|_{\Lambda=0} = \frac{3M^2}{16(u+1)}\left(\frac{27u}{16(u+1)}\right)^u \quad . \tag{2.5.2}$$

It is interesting that the object described by this special case has no mass in the gravitational sense, since $T^g_{\mu\nu}$ vanishes everywhere. This feature is shared by the Yang-Mills instantons of Belavin et al.[31] whose gravitational energy-momentum tensor also vanishes. Similarly the "ghost neutrino" solutions of Davies and Ray[32] propagate without gravitational mass in a plane-symmetric spacetime. Massless fermions in a spatially flat Robertson-Walker universe also possess non-trivial zero-energy solutions[33].

It is noteworthy that the function D, as defined by (2.4.25), will in general become imaginary for some range(s) of values of r. This indicates the presence of coordinate singularities, removable by a suitable coordinate transformation. In the flat g-spacetime case such coordinate transformations will produce a g-metric representing Minkowski space, but in a peculiar coordinate system. All of this is avoided in the particularly simple special case $\Delta = \frac{4}{9}$, which gives

$$D = \pm\,\frac{2}{3}\,(1 - X) \tag{2.5.3}$$

Of course we can only have $\Delta = \frac{4}{9}$ and a flat g-metric by postulating $u = -\frac{3}{2}$. Then from (2.5.2)

$$\lambda \Big|_{\Lambda=0,\,u=-\frac{3}{2}} = -\frac{8M^2}{243} \quad . \tag{2.5.4}$$

(A plus sign on the right hand side also appears to be possible, but would correspond to choosing either $\sqrt{-f}$ or $\sqrt{-g}$ negative which seems undesirable.) Such a fixing of the parameter λ may be of relevance to the physical interpretation of solutions of the Klein-Gordon equation in f-spacetime (Ref. 13 ; see also Sec. 2.7). It should be emphasised, however, that $\Delta = \frac{4}{9}$ is not the only way of making D real everywhere. Other more complicated possibilities exist, involving restrictions on the ranges of values of the parameters μ_f, μ_g and Δ.

2.6 Other mixing terms

The solutions discussed above were found using the particular class of mixing terms given by (2.2.1). It is clearly important to know to what extent the nature of the solutions depends on the choice of $\mathcal{L}_{fg}$. Since $\mathcal{L}_{fg}$ is only completely fixed in the bilinear approximation, whereas the Type I solutions seem to lie far away from the usual vacuum, this dependence might be expected to be large. Nevertheless it will be seen that when solutions can be found at all for other mixing terms they are at least qualitatively similar to those of Sec. 2.4.

One different class of mixing terms can be dealt with immediately namely

$$\mathcal{L}_{fg} = - \frac{M^2}{4\kappa_f^2} (-g)^u (-f)^v (f^{\alpha\beta} - g^{\alpha\beta})(f^{\sigma\tau} - g^{\sigma\tau})(f_{\alpha\sigma} f_{\beta\tau} - f_{\alpha\beta} f_{\sigma\tau}) \quad , \tag{2.6.1}$$

in which the contractions are now made with $f_{\mu\nu}$'s instead of $g_{\mu\nu}$'s. Clearly the Type I solutions can be simply obtained from those of Sec. 2.4, by making the replacements

$$f_{\mu\nu} \leftrightarrow g_{\mu\nu} \ , \ \kappa_f \leftrightarrow \kappa_g \ , \ u \leftrightarrow v \ , \ M \rightarrow M \frac{\kappa_g}{\kappa_f} \tag{2.6.2}$$

in (2.4.33) - (2.4.27), (2.4.20) and (2.4.16). Of course it will now be $f_{\mu\nu}$ which is diagonal. A coordinate transformation will be needed if it is desired to express $g_{\mu\nu}$ in conventional diagonal form.

Other mixing terms of this type do not seem so easy to deal with. Consider, for example

$$\mathcal{L}_{fg} = -\frac{M^2}{4\kappa_f^2} (-g)^u(-f)^v(f^{\alpha\beta} - g^{\alpha\beta})(f^{\sigma\tau} - g^{\sigma\tau})(f_{\alpha\sigma}\, g_{\beta\tau} - f_{\alpha\beta}\, g_{\sigma\tau}) \tag{2.6.3}$$

in which the metrics enter in a symmetrical way. The equation taking the place of (2.4.3) is

$$\left\{3(\Delta + JK) + 2\left(AJ + KC + \frac{2\Delta r^2}{B} + \frac{2JKB}{r^2}\right)\right\} D = 0 \quad . \tag{2.6.4}$$

Again D = 0 is a possibility but leads to intractable equations. Neither does setting the expression in curly brackets equal to zero give a useful simplification as it did in Sec. 2.4.

Solutions for "cosmological" type mixing terms were found by Salam and Strathdee (Ref. 24, Addendum). The trick they used to simplify the equations was first to generalise (1.3.7) to the form

$$\mathcal{L}_{fg} = -\lambda\sqrt{-g} - \lambda'\sqrt{-f} + (\lambda + \lambda')(-f)^{\alpha}(-g)^{\beta}\left(-\det(xg^{\mu\nu}+(1-x)f^{\mu\nu})\right)^{\alpha+\beta-\frac{1}{2}} \tag{2.6.5}$$

with the constraints

$$2(-x\alpha + (1-x)\beta)(\lambda + \lambda') = -x\lambda' + (1-x)\lambda \tag{2.6.6}$$

$$(2 - 4\alpha - 4\beta)\, x\, (x-1)(\lambda + \lambda')^2 = \lambda\lambda' \tag{2.6.7}$$

$$(\kappa_f^2 + \kappa_g^2)\, \lambda\lambda' = M^2(\lambda + \lambda') \tag{2.6.8}$$

and then to restrict attention to the case $\alpha + \beta - \frac{1}{2} > 1$. The ansatz

$$\det(xg^{\mu\nu}+(1-x)f^{\mu\nu}) = 0 \tag{2.6.9}$$

now effectively decouples the f and g equations. They become Einstein equations with cosmological terms and hence have de Sitter or anti-de Sitter solutions. As before, Schwarzschild masses may also be introduced and are completely independent of each other. Of course the actual functional forms of $f_{\mu\nu}$ and $g_{\mu\nu}$ must be such that (2.6.9) is satisfied, and the metrics are "locked together" in this sense. Note that (2.6.9) is more likely to be reasonable if x lies outside the interval [0,1]. So despite the big difference in mixing term, the solutions are qualitatively similar to those of Sec. 2.4. A disadvantage of the present case, however, is that g-spacetime cannot be flat since $\lambda = 0$ is inconsistent with (2.6.8).

Finally the possibility of solutions for the vierbein theory is investigated. A suitable ansatz for the vierbeins $K_{a\mu}$ and $L_{a\mu}$ is

$$K_{0t} = T, \quad K_{1r} = -S, \quad K_{0r} = P, \quad K_{1t} = Q, \quad K_{2\theta} = -Ur, \quad K_{3\phi} = -Ur\sin\theta,$$

$$L_{0t} = W, \quad L_{1r} = -X, \quad L_{2\theta} = -r, \quad L_{3\phi} = -r\sin\theta, \tag{2.6.10}$$

where T,S,P,Q,U,W and X are functions of r only. Note that to maintain generality only one of the vierbeins can be chosen to be symmetric, so there is one more unknown function than in the metric theory.

Choosing the mass term (1.4.5), the K field equations are

$$M^{-2} R^{K}_{00} = -\frac{3}{2} + \frac{(3SW + TX)}{4(ST - PQ)} + \frac{1}{2U} \tag{2.6.11}$$

$$M^{-2} R^{K}_{01} = -\frac{PW}{2(ST - PQ)} \tag{2.6.12}$$

$$M^{-2} R^{K}_{10} = \frac{QX}{2(ST - PQ)} \tag{2.6.13}$$

$$M^{-2} R^{K}_{11} = \frac{3}{2} - \frac{(SW + 3TX)}{4(ST - PQ)} - \frac{1}{2U} \tag{2.6.14}$$

$$M^{-2} R^{K}_{22} = M^{-2} R^{K}_{33} = \frac{3}{2} - \frac{(SW + TX)}{4(ST - PQ)} - \frac{1}{U} \tag{2.6.15}$$

Since $R^{K}_{01} = R^{K}_{10}$, (2.6.12) and (2.6.13) produce the algebraic constraint

$$PW + QX = 0 \tag{2.6.16}$$

which compensates for having an extra function to begin with. This exemplifies the way in which the extra degrees of freedom in the vierbein theory are removed by the antisymmetric parts of the field equations, as discussed in Sec. 1.4.

When the curvature components are calculated one finds the identity

$$(Q^2 + T^2)R^K_{01} + QT(R^K_{00} + R^K_{11}) = 0 \qquad (2.6.17)$$

which is equivalent to (2.4.1). Substituting (2.6.11), (2.6.12) and (2.6.14) and making use of (2.6.16) eventually produces

$$WQ = 0 \quad . \qquad (2.6.18)$$

If $g_{\mu\nu}$ is to be nonsingular then W and X cannot be zero. Hence

$$Q = P = 0 \qquad (2.6.19)$$

so there are no Type I solutions at all. Precisely the same result is found for the other three mass terms (1.4.6) - (1.4.8). We conclude that the Wess-Zumino theory with these mass terms admits only Type II solutions (which once again seem to involve intractable equations).

2.7 Quantised matter in classical f-g backgrounds

It seems that when spherically symmetric classical solutions of the f-g equations exist and can actually be found then de Sitter and/or anti-de Sitter spacetimes are involved, irrespective of the particular choice of $\mathcal{L}_{fg}$. This may mean that the appearance of these spacetimes in classical solutions is a basic feature of the theory, although too few examples have been found as yet to make such a statement with any degree of certainty.

The matter terms in the lagrangian have been ignored up till now in this chapter. The first stage in remedying this is to consider the behaviour of matter fields propagating in the background provided by the classical solutions. We are interested in quantum fields of course. Eventually one would hope to include the back reaction of the matter on the f and g fields and ultimately to quantise $f_{\mu\nu}$ and $g_{\mu\nu}$ themselves, but a programme as advanced as this lies far in the future. For now the first stage will provide quite enough problems!

Thus we are faced with the task of constructing quantum field theories in de Sitter and anti-de Sitter spacetimes. Quantum theory in de Sitter space has received much attention; in anti-de Sitter space, very little. This is a pity since the latter spacetime provides some challenging and entertaining problems for the quantum field theorist. These problems, due basically to a lack of global hyperbolicity, are difficult but not insurmountable. Precisely how they can be overcome forms the subject matter of the next chapter.

Salam and Strathdee[12] have pointed out that f-g solutions involving an anti-de Sitter f-metric have properties suggestive of confinement. The effects of this metric are in some ways similar to those of an r^2 potential well, and a straightforward solution of the Klein-Gordon equation in this background produces a discrete set of wavefunctions of increasing energies, with no continuum limit. In this sense the scalar particles are totally confined. Thus there is hope that if more realistic models are developed an interesting quark confinement mechanism might result.

This hope gives added motivation for the investigation described in Chapter Three. However, in view of the more general interest of the problem the emphasis will be on anti-de Sitter space on its own, rather than as part of an f-g solution, and more convenient coordinates than those encountered in this chapter will be used. In applying the results to either of the metrics in an f-g solution one must of course take carefully into account the effects of the corresponding coordinate transformations on the other metric.

CHAPTER THREE

QUANTUM FIELD THEORY IN ANTI-DE SITTER SPACETIME

3.1 Why consider quantum fields in anti-de Sitter spacetime?

In the previous chapter it was found that the classical spherically symmetric solutions of the f-g field equations, when they can be found at all, seem to involve de Sitter and anti-de Sitter spacetimes irrespective of the particular mass term chosen. There are also Schwarzschild terms in the metrics but these are less important and will not be considered in this chapter. Thus in the presence of this kind of solution, the hadrons find themselves in a de Sitter or anti-de Sitter world, and the corresponding field theories must be defined on this curved background spacetime.

Quantum field theory in de Sitter spacetime has been extensively studied (see e.g. Refs. 34-39). One of the main problems encountered is the lack of a positive-definite energy operator[39].

Anti-de Sitter spacetime, on the other hand, is relatively unexplored territory. However, it is just as important as de Sitter space in the f-g theory solutions and might even be relevant to confinement, as discussed in Sec. 2.7. Hence this chapter is devoted to the ways in which quantum field theories can be constructed in anti-de Sitter spacetime.

As it happens, there are other good reasons for carrying out such a programme. The current interest in quantum field theories in general curved spacetimes (for revues see Refs. 40-42) has been confined almost without exception to the case of globally hyperbolic spacetimes. Such spacetimes possess spatial hypersurfaces, called Cauchy surfaces, on which classical initial value data for a wave equation may be freely specified, uniquely determining the solution of the equation at all other points. (For rigorous definitions see e.g. Ref. 43.) However, many spacetimes do not possess this property. Indeed globally hyperbolic spacetimes are necessarily of the form $\mathbb{R}$(time) × Σ(space) (where Σ is a three dimensional riemannian space) and so are in many respects uninteresting. Anti-de Sitter spacetime is a famous example of non-hyperbolicity. It possesses both closed timelike curves and a timelike boundary at spatial infinity through which data can propagate. The latter property is also possessed by its universal covering space and is the main cause of the lack of hyperbolicity. These features will be described in greater detail in Sec. 3.3.

Anti-de Sitter space has also appeared in another context recently, namely as the natural background in certain supergravity models[44,45].

How, then, are we to proceed?

Methods for quantising a field propagating in a fixed, but curved, spacetime have been studied at length during the last few years. Attention has in general been focussed on linear field theories, as will be the case in this chapter, but even with this restriction there is (at least for most spacetimes) no unique quantisation scheme. Various approaches have been suggested but here we will be mainly interested in the "covariant quantisation" method in which the Heisenberg fields manifest themselves in the traditional way as operators defined on a single Hilbert space. This approach has its origins in the work of Segal on quantising arbitrary linear systems. Segal's methods relied heavily on the existence and structure of classical solutions of the field equations. For the problem of fields in curved spacetimes it is here that global hyperbolicity would normally be assumed, and so this is where new techniques will be needed in the present case.

Since anti-de Sitter space is a homogeneous space of the group O(3,2) it might perhaps seem natural to adopt a group oriented approach to quantisation. Such a study has in fact been made by Fronsdal et al. in a comprehensive series of papers[46-51]. However, from the point of view taken here the emphasis is not ideally placed. Indeed the rôle played by the timelike infinity is not readily discussed in this approach. Note that even in the well understood case of de Sitter spacetime, the group theoretic SO(4,1) treatment misses thermal radiation associated with the event horizon of an inertial observer[52]. In view of the discussion of Sec. 1.1(ii) it is especially important not to lose such effects in the f-g context. Thus the sequel is concerned mainly with finding an analogue of the covariant quantisation scheme by coming directly to grips with the problem of controlling information entering the spacetime through timelike infinity. The results obtained in this way may be regarded as complementary to those found using group theory.

The emphasis will be on anti-de Sitter space itself as opposed to its universal covering space. The latter has no closed timelike loops but is much less intersting topologically, being homeomorphic to $\mathbb{R}^4$. From the technical point of view the closed timelike loops are not too much of a problem. From the physical point of view one could think of anti-de Sitter spacetime as a "periodic system".

For the sake of simplicity only scalar fields will be considered, since it is expected that spinor and vector fields could be dealt with in a similar manner. To begin with, a short resumé of the standard covariant methods will be given.

3.2 The covariant quantisation scheme

The aim of the covariant approach to quantisation is to construct a quantum field $\hat{\psi}(x)$ satisfying both the classical field equation

$$(\Box + u(x)\ )\ \hat{\psi}(x) = 0 \tag{3.2.1}$$

and the covariant commutation relation

$$\left[\hat{\psi}(x)\ ,\ \hat{\psi}(x')\right] = -i\hbar\ \tilde{G}(x,x')\quad . \tag{3.2.2}$$

In (3.2.1), $\Box$ is the d'Alembertian operator associated with the background spacetime, and u(x) is a smooth c-number function which, if constant, may be very loosely interpreted as the "mass squared" of the field. The unique classical commutator function $\tilde{G}(x,x')$, defined as the difference of the advanced and retarded Green's functions, evolves classical Cauchy data specified on a Cauchy hypersurface Σ according to

$$\psi(x) = \int_\Sigma \tilde{G}(x,x') \overleftrightarrow{\partial}_\mu \psi(x') \, d\sigma^\mu(x') \qquad (3.2.3)$$

and it is here that global hyperbolicity is seen to be an essential prerequisite.

For our purposes we will only need to consider static spacetimes, i.e. spacetimes possessing a globally defined, hypersurface orthogonal, timelike Killing vector field. (Definitions of these and other differential geometry concepts used in this chapter can mostly be found in Ref. 43.) This is fortunate because for these special cases there is a natural and essentially unique way of proceeding.

One begins by finding a complete orthonormal (in the sense defined below) set of positive frequency classical solutions of the field equation (3.2.1), of the form

$$f_j(x) = \exp(-i\omega t)\, h_j(\underline{x}) \quad , \qquad \omega_j > 0 \quad . \qquad (3.2.4)$$

Here t is a time coordinate such that ∂_t is a global, hypersurface orthogonal, timelike Killing vector field, and $h_j(\underline{x})$ are a complete set of functions of the spatial coordinates only. The f_j form an orthonormal basis of a Hilbert space $\mathcal{H}$ having the positive definite Klein-Gordon inner product

$$B(\alpha,\beta) \equiv i \int_\Sigma \alpha^* \overleftrightarrow{\partial}_\mu \beta \, d\sigma^\mu = i \int_{t=\text{const}} \alpha^* \overleftrightarrow{\partial}_o \beta \, g^{oo} \sqrt{-g}\, d^3\underline{x} \quad ; \quad \alpha,\beta \in \mathcal{H} \qquad (3.2.5)$$

which is independent of Σ by virtue of the field equations. For convenience Σ is often chosen to be a surface of constant t.

The f_j are also required to satisfy

$$\sum_j \; f_j(x)\, f_j^*(x') - f_j^*(x)\, f_j(x') \quad = -i\, \tilde{G}(x,x') \quad . \qquad (3.2.6)$$

If now the real classical field is expanded as

$$\psi(x) = \sum_j \quad a_j\, f_j(x) + a_j^*\, f_j^*(x) \qquad , \quad a_j \in \mathbb{C} \qquad (3.2.7)$$

and the a_j are promoted to the rank of operators $\hat{a}_j$ satisfying

$$\left[\hat{a}_j\, ,\, \hat{a}_k\right] = \left[\hat{a}_j^*\, ,\, \hat{a}_k^*\right] = 0\ , \quad \left[\hat{a}_j\, ,\, \hat{a}_k^*\right] = \hbar\, \delta_{jk} \qquad (3.2.8)$$

then the resulting hermitian field operator $\hat{\psi}(x)$ will automatically satisfy (3.2.2).

The $\hat{a}_j$ and $\hat{a}_j^*$ are interpreted as annihilation and creation operators on the Fock space constructed in the usual way as an infinite tensor product of simple harmonic oscillator Hilbert spaces. The Fock representation is almost inevitably used in these circumstances, since when it exists it provides the unique quantisation for which the spectrum of the hamiltonian operator (the generator of time translations) is positive definite.

The Hilbert space $\mathcal{H}$ automatically carries a unitary representation of the time translation group. One might further require that any other isometries of the background spacetime be placed on the same footing in this respect.

3.3 Problems associated with anti-de Sitter spacetime

Anti-de Sitter spacetime ("AdS" for brevity) may be realised as the four dimensional hyperboloid

$$(\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 + (\xi^4)^2 = K^{-1} \qquad (3.3.1)$$

in a five dimensional space with metric

$$ds^2 = \eta^{(5)}_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta = (d\xi^0)^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2 + (d\xi^4)^2 \tag{3.3.2}$$

(see Fig. 3.1). AdS is a pseudoriemannian space of constant curvature K, related to the Ricci scalar R and cosmological constant λ (when it is regarded as a solution of the Einstein equations with cosmological term) by

$$K = \frac{R}{12} = -\frac{\lambda}{3} \quad . \tag{3.3.3}$$

With the conventions used here K is positive.

The isometry group of AdS is O(3,2) which is simply the "Lorentz group" of the embedding space.

AdS has the topology $\mathbb{S}^1$(time) $\times\, \mathbb{R}^3$(space) and hence contains closed timelike curves. "Unwrapping" the $\mathbb{S}^1$ gives the universal covering space ("CAdS") which has the topology of $\mathbb{R}^4$ and contains no closed timelike curves.

For the purposes of this chapter the metric of AdS, or CAdS, is most usefully written using the following parametrisation:

$$\xi^0 = K^{-\frac{1}{2}}\cos\tau\,\sec\rho\ ,\ \xi^1 = K^{-\frac{1}{2}}\tan\rho\,\cos\theta\ ,\ \xi^2 = K^{-\frac{1}{2}}\tan\rho\,\sin\theta\,\cos\phi\ ,$$

$$\xi^3 = K^{-\frac{1}{2}}\tan\rho\,\sin\theta\,\sin\phi\ ,\ \xi^4 = K^{-\frac{1}{2}}\sin\tau\,\sec\rho\ , \tag{3.3.4}$$

$$ds^2 = K^{-1}\sec^2\rho\,\left\{d\tau^2 - d\rho^2 - \sin^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right\} \tag{3.3.5}$$

$$0 \leqslant \rho < \frac{\pi}{2}\ ,\quad 0 \leqslant \theta \leqslant \pi\ ,\quad 0 \leqslant \phi < 2\pi \quad .$$

For AdS $-\pi < \tau \leqslant \pi$ with $\tau = -\pi$ and $\tau = \pi$ identified.

For CAdS $-\infty < \tau < \infty$.

These dimensionless coordinates cover the whole of AdS and CAdS, except for the usual polar type singularities.

In this coordinate system spatial infinity has finite coordinate

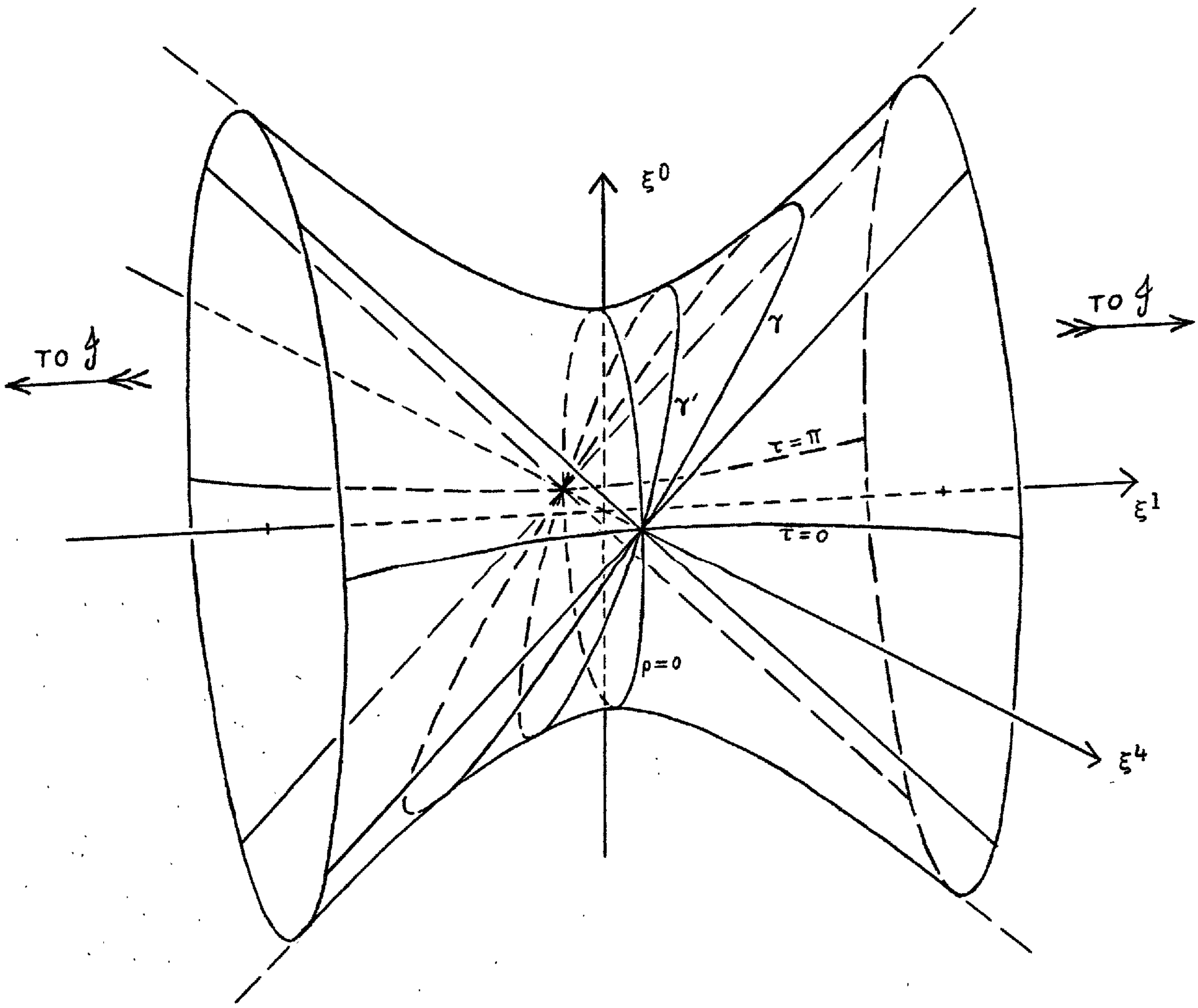


Fig. 3.1

Anti-de Sitter spacetime as a four dimensional hyperboloid embedded in five dimensions. Two dimensions (ξ^2 and ξ^3) are suppressed in the diagram.

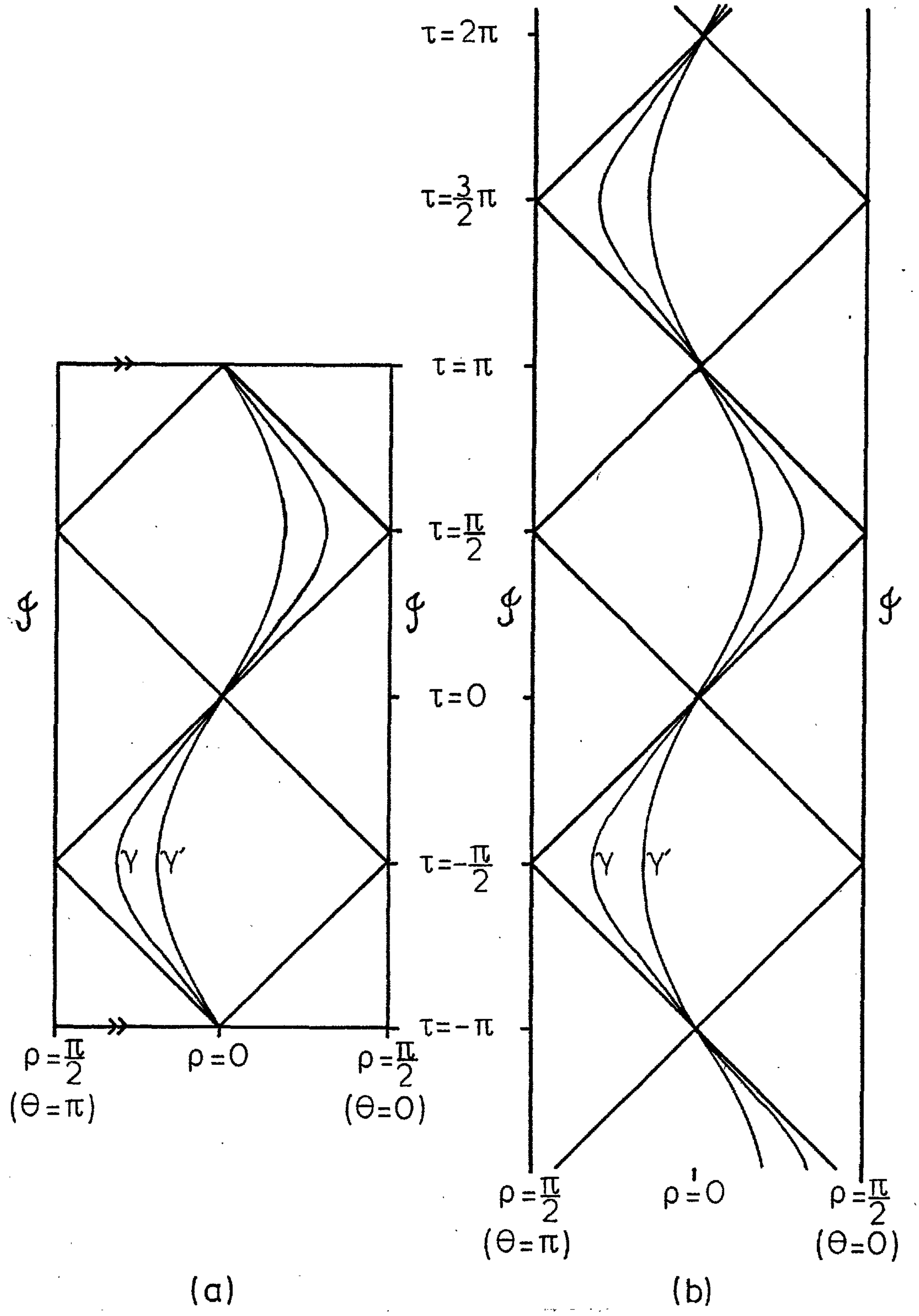


Fig. 3.2

Penrose diagrams for

(a) anti-de Sitter spacetime (top and bottom surfaces identified)

(b) its universal covering spacetime.

See text for discussion.

(Diagram drawn by S.J. Avis.)

values ($\rho = \frac{\pi}{2}$) and AdS and CAdS are conveniently represented using Penrose diagrams[53,43] as in Fig. 3.2 (compare with Fig. 3.1). The coordinates θ and ϕ are suppressed. The null lines at $\pm 45^{\circ}$ are drawn to clarify the conformal structure; a light ray crosses AdS within half the natural period. Some timelike geodesics (γ, γ') are also indicated, showing that in CAdS there is a residual effect of the time periodicity in AdS. In fact timelike geodesics emanating from any point in CAdS, which may be taken to be $\tau = \rho = 0$ since CAdS is a homogeneous space, reconverge at $\rho = 0$ for $\tau = \pi$, 2π, 3π, etc. It is in this sense that the spacetime acts rather like an r^2 potential, as mentioned in Sec. 2.7.

These Penrose diagrams show clearly the two striking features of the AdS causal structure which preclude global hyperbolicity.

Firstly, AdS contains closed timelike curves, a feature lost in CAdS as already discussed.

Secondly, the surface at $\rho = \frac{\pi}{2}$ (i.e. at spatial infinity) is timelike, a feature shared with CAdS. The effect of this is that information may be lost to, or gained from, spatial infinity in finite coordinate time. A change of coordinates is of no avail here since any time coordinate for which this is not so will not be globally defined (and will not give a manifestly static metric). It is this loss and gain of information which has the most disruptive effect on the Cauchy problem, and the closed timelike curves are in many ways a lesser evil.

There is another related possible source of trouble in this context, which is of a more technical nature. Rigorous quantisation schemes in a globally hyperbolic spacetime attach considerable importance to Cauchy data of compact support. As a consequence of global hyperbolicity the Cauchy data on any Cauchy hypersurface will then possess this property. However, it is easily seen that in our case "initial value" data with compact support on one spacelike hypersurface will in general evolve in such a way that it becomes noncompact on many other spacelike hypersurfaces.

Some of the difficulties mentioned above are similar to those encountered when considering quantisation in a box in Minkowski spacetime. If the box is "transparent", information may escape or be thrown in from outside, and the Cauchy data within the box at a given time obviously does not uniquely determine that at other times.

Of course when dealing with boxes one usually ascribes special physical properties to the walls. Typically the field, or perhaps its normal derivative, is required to vanish there, so that information is reflected and not lost. The time evolution of the Cauchy data is then unique. However, in less simple examples great care must be taken regarding the self-consistency of such mixed boundary conditions. In any case the "walls" of AdS are at infinity and so the concept of reflecting boundary conditions is somewhat obscure. This will be clarified in Secs. 3.5 and 3.6.

Returning to the transparent box, one way of establishing a well defined Cauchy problem is simply to accept that the box constitutes an incomplete manifold, and require that Cauchy data be specified on a Cauchy surface of the surrounding spacetime, not just within the box. But unlike the box AdS <u>is</u> complete and there is no such surrounding spacetime. Nevertheless, an analogue <u>can</u> be constructed, as explained in the next section.

3.4 Conformally coupled massless field -"Transparent" boundary conditions

To clarify the analogy between AdS and a box in Minkowski space it is convenient to begin by considering a massless scalar field, conformally coupled to the background metric. The appropriate wave equation is

$$(\Box - \tfrac{R}{6})\,\psi = (\Box - 2K)\,\psi = 0 \quad . \tag{3.4.1}$$

The d'Alembertian operator, $\Box$, is given by

$$\Box\psi = g^{\mu\nu}\,\nabla_\mu\,\partial_\nu\,\psi = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\,\psi) \tag{3.4.2}$$

in general, and

$$K^{-1}\Box\psi = \cos^2\rho\,\frac{\partial^2\psi}{\partial\tau^2} - \cot^2\rho\left\{\cos^2\rho\,\frac{\partial}{\partial\rho}(\tan^2\rho\,\frac{\partial\psi}{\partial\rho}) + \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}(\sin\theta\,\frac{\partial\psi}{\partial\theta}) + \frac{1}{\sin^2\theta}\,\frac{\partial^2\psi}{\partial\phi^2}\right\} \tag{3.4.3}$$

in particular, for the AdS metric (3.3.5).

Now it so happens that CAdS may be conformally mapped into half of the Einstein static universe[43,53] ("ESU"), as depicted in Fig. 3.3. ESU may be realised as the four dimensional cylinder

$$(\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2 + (\eta^4)^2 = K^{-1} \tag{3.4.4}$$

in a five dimensional space with metric

$$ds^2 = (d\eta^0)^2 - (d\eta^1)^2 - (d\eta^2)^2 - (d\eta^3)^2 - (d\eta^4)^2 \tag{3.4.5}$$

and hence it has the topology $\mathbb{R}(\text{time}) \times \mathbb{S}^3(\text{space})$. (In Fig. 3.3 two spatial dimensions are suppressed so that ESU appears as $\mathbb{R} \times \mathbb{S}^1$.) The scalar curvature is

$$R^E = -6K \quad . \tag{3.4.6}$$

The ESU metric may be written in the globally defined form

$$(ds^E)^2 = K^{-1}\left\{d\tau^2 - d\rho^2 - \sin^2\rho\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right\} \tag{3.4.7}$$

$$-\infty < \tau < \infty\,,\ \underline{0 \leq \rho \leq \pi}\,,\ 0 \leq \theta \leq \pi\,,\ 0 \leq \phi < 2\pi.$$ (compare (3.3.5).)

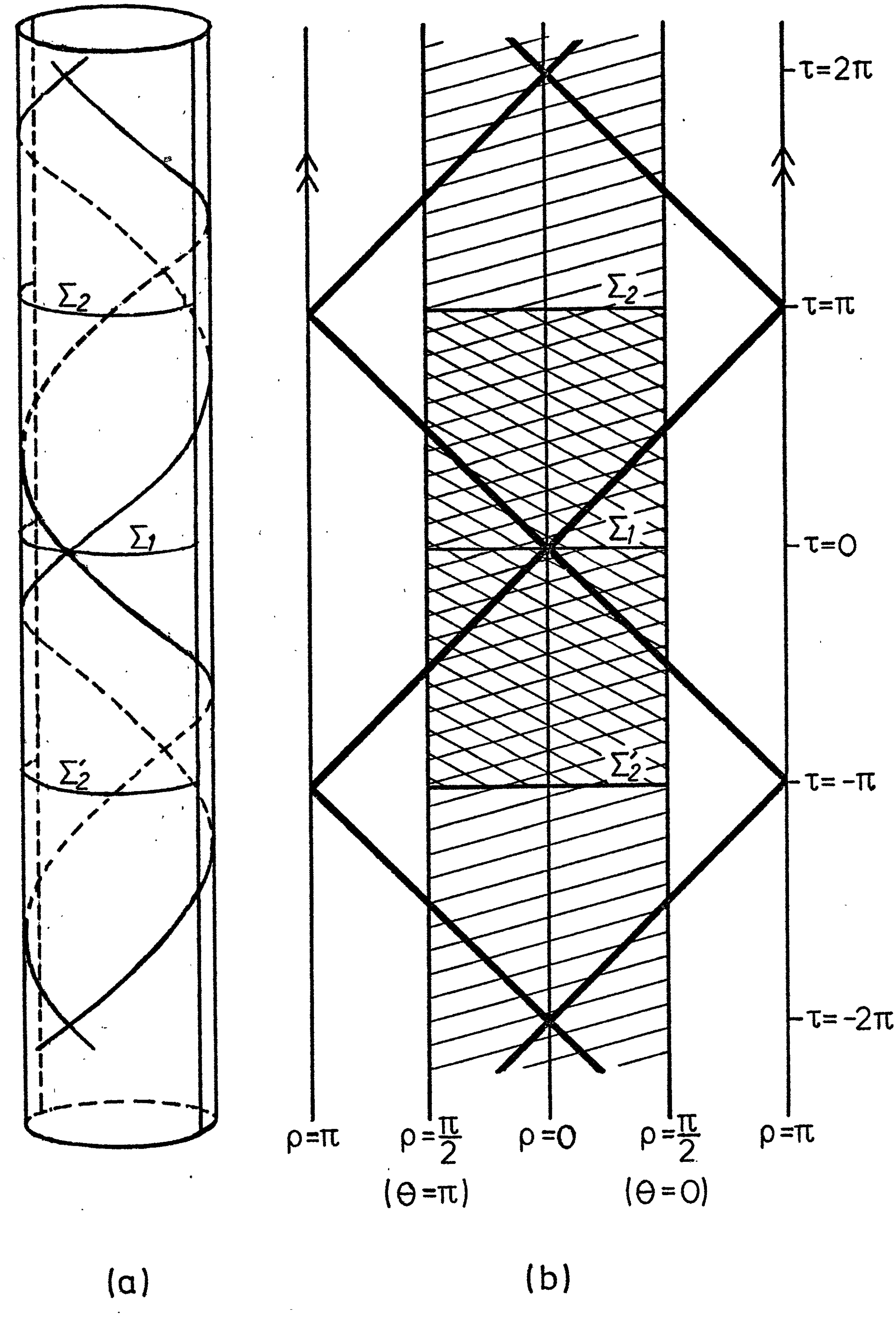


Fig. 3.3

(Caption overleaf.)

Fig. 3.3

(a) The Einstein Static Universe with two spatial dimensions suppressed is the cylinder $\mathbb{R}(\text{time}) \times \mathbb{S}^1(\text{space})$.

(b) As above, cut along $\rho = \pi$ and flattened out, showing the images under conformal mapping of CAdS (shaded) and AdS (double shaded, Σ_2 and Σ_2' identified). The null lines at $\pm 45^\circ$ are the support of $\tilde{G}^E(x,0)$. When restricted to the image of AdS they are the image of the support of $\tilde{G}^T(x,0)$. Note that the identification of Σ_2 with Σ_2' is commensurate with the periodicity of $\tilde{G}^E(x,0)$ (and all other nonsingular, finite norm solutions in ESU).

The coordinate systems have been chosen to make the conformal mapping as simple as possible. In fact

$$g^E_{\mu\nu} = \Omega^2 \, g_{\mu\nu} \tag{3.4.8}$$

where Ω, the conformal factor, is given by

$$\Omega = \cos\rho \quad . \tag{3.4.9}$$

The field equation (3.4.1) is invariant under conformal mappings provided the field is assigned a conformal weight of -1, i.e.

$$\psi^E = \Omega^{-1} \, \psi \quad . \tag{3.4.10}$$

So if ψ is a solution of (3.4.1) in CAdS then ψ^E is a solution of

$$(\Box^E - \frac{R^E}{6}) \, \psi^E = (\Box^E + K) \, \psi^E = 0 \tag{3.4.11}$$

in the appropriate half of ESU, where

$$K^{-1} \Box^E \psi^E = \frac{\partial^2 \psi^E}{\partial \tau^2} - \frac{1}{\sin^2\rho} \left\{ \frac{\partial}{\partial\rho}(\sin^2\rho \, \frac{\partial \psi^E}{\partial\rho}) + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta \, \frac{\partial\psi^E}{\partial\theta}) + \right.$$
$$\left. + \frac{1}{\sin^2\theta} \frac{\partial^2\psi^E}{\partial\phi^2} \right\} \quad . \tag{3.4.12}$$

Now ESU is a globally hyperbolic spacetime, and quantisation therein is well known and follows the pattern of Sec. 3.2. A summary will presently be given. It is proposed to use this quantisation, mapped back, to give an acceptable quantum field theory in AdS.

Separation of variables yields the following collection of positive frequency, finite B-norm (eqn. (3.2.5)) solutions of (3.4.11) defined on the whole of ESU:

$$\psi^E_{\omega\ell m} = N_{\omega\ell} \, e^{-i\omega\tau} \, (\sin\rho)^\ell \, C^{\ell+1}_{\omega-\ell-1}(\cos\rho) \, Y^m_\ell(\theta,\phi) \tag{3.4.13}$$

where ω, ℓ and m are integers such that $\omega-1 \geq \ell \geq |m|$. Here $C^p_q(z)$ are Gegenbauer polynomials[54], $Y^m_\ell(\theta,\phi)$ are the usual spherical harmonics and $N_{\omega\ell}$ are normalisation constants.

The $\psi^E_{\omega\ell m}$ form an orthonormal basis for the Hilbert space $\mathcal{H}^E$ of

<u>all</u> finite norm, positive frequency solutions of (3.4.11), with inner product defined by

$$B^E(\alpha,\beta) \equiv i \int_{\tau=\mathrm{const}} \alpha^* \overleftrightarrow{\partial}_0 \beta \sqrt{-g^E}\, d^3\underline{x} \quad , \qquad \alpha,\beta \in \mathcal{H}^E \tag{3.4.14}$$

(c.f. (3.2.5) and note $g^{E00} = 1$). Hence all such solutions are periodic in τ with period 2π. This is related to the fact that, in the absence of interactions, a classical massless particle passing through the point (τ,ρ,θ,ϕ) will also pass through the points $(\tau+2\pi n,\rho,\theta,\phi)$ for $n = \pm1,\pm2$, etc. So the spatial "periodicity" of ESU has induced an effective temporal periodicity. Moreover, upon restricting the solutions to the image of CAdS, and mapping back using (3.4.10), this periodicity is seen to be precisely that which allows the functions to be defined on AdS.

In addition to the periodicity discussed above, we also have

$$\psi^E(\tau,\rho,\theta,\phi) = -\,\psi^E(\tau+(2n+1)\pi,\pi-\rho,\pi-\theta,\phi+\pi) \quad , \quad n = 0, \pm1, \pm2, \ldots \tag{3.4.15}$$

(A classical massless particle passing through (τ,ρ,θ,ϕ) must not only pass through $(\tau+2\pi n,\rho,\theta,\phi)$ but also through $(\tau+(2n+1)\pi,\pi-\rho,\pi-\theta,\phi+\pi)$. It is more difficult to find an intuitive classical explanation of the minus sign!) It follows that the specification of Cauchy data on the complete surface $\tau = 0$ is equivalent to its specification on the <u>pair</u> of <u>incomplete</u> surfaces $\left\{\tau = 0,\ \rho < \frac{\pi}{2}\right\}$ and $\left\{\tau = \pi,\ \rho < \frac{\pi}{2}\right\}$ in the following sense. If the solution is C^∞ then so is the induced data on these partial surfaces. However the converse is not strictly true since there is a consistency condition on the boundary values of the partial data to ensure that the induced solution in ESU really is C^∞. On the other hand if distributional solutions are considered there is no such restriction, but it is now necessary to include the boundary at $\rho = \frac{\pi}{2}$ on one of the partial Cauchy surfaces in order to obtain a complete specification of the solution in terms of this partial data.

The quantisation schemes in AdS being developed here employ only those solutions in AdS whose ESU counterparts are everywhere C^∞ solutions of the wave equation (3.4.11). In the sense defined above they are specified by their "initial value" data on the pair of surfaces $\left\{ \tau = 0,\ \rho < \frac{\pi}{2} \right\}$ and $\left\{ \tau = \pi,\ \rho < \frac{\pi}{2} \right\}$ in AdS, denoted by Σ_1 and Σ_2 respectively (see Fig. 3.3). (Note that with respect to the AdS metric these are complete surfaces.) The set of all such solutions generates a Hilbert space $\mathcal{H}^T$ with inner product

$$B^T(\alpha,\beta) = i \int_{\Sigma_1 \cup \Sigma_2} \alpha^* \overleftrightarrow{\partial}_o \beta \ g^{oo} \sqrt{-g}\, d^3\underline{x} \quad , \qquad \alpha,\beta \in \mathcal{H}^T \quad . \tag{3.4.16}$$

Of course by construction $\mathcal{H}^T$ is identical to $\mathcal{H}^E$, the Hilbert space of solutions in ESU equipped with the B^E-norm of (3.4.14). Indeed this norm maps conformally into (3.4.16) with the integration region being transferable from the single Cauchy surface in ESU to the pair of surfaces in AdS by virtue of (3.4.15).

To actually reconstruct the AdS solution from its "Cauchy data" we require $\tilde{G}^T(x,x')$, the analogue of the classical commutator function. Just as for the basis functions this is obtained from the ESU commutator function, $\tilde{G}^E(x,x')$, by restriction and mapping back, using (3.4.10). Since AdS and ESU are both homogeneous spaces, $\tilde{G}^E(x,x')$ and $\tilde{G}^T(x,x')$ are characterised by their behaviour as functions of a single variable x, with x' chosen to be the coordinate origin for convenience. The commutator function $\tilde{G}^E(x,0)$ is readily constructed from the well known Feynman function[55] (propagator) and may be written in the form

$$\tilde{G}^E(x,0) = -\frac{K}{4\pi}\,\delta(\cos\rho - \cos\tau)\ \mathring{\varepsilon}(\tau) \quad , \tag{3.4.17}$$

where

$$\mathring{\varepsilon}(\tau) \equiv \mathrm{sign}(\sin\tau) \quad . \tag{3.4.18}$$

Hence (noting $\cos\rho > 0$ in AdS)

$$\tilde{G}^T(x,0) = -\frac{K}{4\pi}\,\delta(1 - \cos\tau\,\sec\rho)\,\mathring{\varepsilon}(\tau) \quad . \qquad (3.4.19)$$

The supports of $\tilde{G}^E(x,0)$ and $\tilde{G}^T(x,0)$ are concentrated on the light cones through the origin in ESU and AdS respectively (see Fig. 3.3). This "Huygens' principle" is in fact a major reason for referring to the field as "massless"[55].

The classical solution may now be constructed from the "effective Cauchy data" on Σ_1 and Σ_2 using

$$\psi(x) = \int_{\Sigma_1 \cup \Sigma_2} \tilde{G}^T(x,x')\,\overleftrightarrow{\partial}_0\psi(x')\;g^{00}\,\sqrt{-g}\;d^3\underline{x}' \qquad (3.4.20)$$

and so $\Sigma_1 \cup \Sigma_2$ will be called an "effective Cauchy surface" for AdS.

Now that the classical Cauchy problem is under control, quantisation is fairly straightforward and follows the pattern outlined in Sec. 3.2, based on the field operator

$$\hat{\psi} = \sum_{\omega\ell m} (\psi_{\omega\ell m}\,\hat{a}_{\omega\ell} + \psi^*_{\omega\ell m}\,\hat{a}^*_{\omega\ell m}) \quad , \qquad (3.4.21)$$

where the $\psi_{\omega\ell m}$ are given by

$$\psi_{\omega\ell m} = \Omega\,\psi^E_{\omega\ell m} = N_{\omega\ell}\,e^{-i\omega\tau}\cos\rho\,(\sin\rho)^\ell\,C^{\ell+1}_{\omega-\ell-1}(\cos\rho)\,Y^m_\ell(\theta,\phi) \qquad (3.4.22)$$

and are regarded now as functions on AdS. It may be checked explicitly that the relation (3.2.6) survives the restriction and mapping back. This completes the quantisation since we have constructed a quantum field on AdS satisfying both the field equation and our analogue of the covariant commutation relation ((3.2.2), using $\tilde{G}^T$).

An alternative way of completely specifying a quantum field theory is to construct a Feynman function. Hence it is of interest to try to do so for AdS, and in particular to see if any meaning can be attached to the term "time-ordered product" in a space containing closed timelike curves.

In Minkowski space the commutator function is simply related to the real part of the Feynman function which in turn is the boundary value of a unique analytic function of the Minkowskian invariant distance, satisfying the wave equation with a single δ-function source. To look for an analogous function in AdS it is advantageous to introduce the invariant distance $\sigma(x,x')$. This is the analogue of $\frac{1}{2}\left[(t-t')^2 - (\underline{x}-\underline{x}')^2\right]$ in Minkowski space and in fact is half the distance from x to x′ in the embedding space:

$$\sigma(x,x') \equiv \tfrac{1}{2}\,\eta^{(5)}_{\alpha\beta}\,(\xi^{\alpha} - \xi'^{\alpha})(\xi^{\beta} - \xi'^{\beta}) \qquad (3.4.23)$$

(c.f.(3.3.2)). In particular

$$K\,\sigma(x,0) = 1 - \cos\tau\,\sec\rho \quad . \qquad (3.4.24)$$

The points x satisfying $\sigma(x,x') = 0$ lie on the "light cone" through $x' = (\tau',\rho',\theta',\phi')$ whilst those satisfying $\sigma(x,x') = 2K^{-1}$ lie on the "light cone" through the antipodal point $x'_A = (\tau'+\pi,\rho',\pi-\theta',\phi'+\pi)$.

Expressed in terms of σ, (3.4.1) becomes

$$\left\{\sigma(2 - K\sigma)\frac{d^2}{d\sigma^2} + 4(1 - K\sigma)\frac{d}{d\sigma} - 2K\right\} G(\sigma) = 0\ , \quad \sigma \neq 0, \frac{2}{K}\ . \qquad (3.4.25)$$

The most general analytic solution of (3.4.25) is an arbitrary linear combination of $(K\sigma)^{-1}$ and $(K\sigma - 2)^{-1}$. In Minkowski space the correct function is uniquely determined by demanding that the real part be causal. Although causality is an obscure notion in AdS it is nevertheless reasonable to require that the prospective Feynman function must at least look locally like the Minkowski one. With this in mind we take

$$G^T(\sigma) = \left(\frac{iK}{8\pi^2}\right)\frac{1}{K\sigma - i0} \qquad (3.4.26)$$

as the Feynman function for "transparent" boundary conditions, which in fact solves the inhomogeneous equation (σ real)

$$(\Box - 2K)\,G^T(\sigma) = -\,\delta^4(x,x') \quad . \qquad (3.4.27)$$

With this choice the commutator function $\tilde{G}^T$ is related to G^T by

$$\tilde{G}^T(x,0) = 2\, \mathring{\varepsilon}(\tau)\ \mathrm{Re}\ G^T(x,0) \qquad (3.4.28)$$

in close analogy with the relationship in Minkowski space.

The way in which G^T can be related to a suitably defined "time-ordered product" will be explained in Sec. 3.6, since our remarks will also apply to the Feynman functions constructed in Secs. 3.5 and 3.6.

Likewise, discussion of the extent to which the Hilbert space $\mathcal{H}^T$ carries a representation of the AdS isometry group will be postponed until then. However, it is convenient to discuss the related topic of conservation laws at this stage. In view of the loss of energy, angular momentum, etc. to infinity, as discussed in Sec. 3.3, this will be of particular interest in AdS. To begin with some remarks on the definitions of energy-momentum tensors are in order.

The lagrangian density for a conformally coupled scalar field is

$$\mathcal{L} = \tfrac{1}{2}\sqrt{-g}\,\left\{ g^{\mu\nu}\,\partial_\mu\psi\,\partial_\nu\psi - (\mu^2 - \tfrac{R}{6})\,\psi^2 \right\} \quad . \qquad (3.4.29)$$

(A mass μ has been included for later use.) There are two distinct energy-momentum tensors associated with this lagrangian density.

(1) The variational ("new improved"[56]) energy-momentum tensor obtained by varying the action S with respect to the metric:

$$\delta S = \delta \int \mathcal{L}\, d^4x = \int \tfrac{1}{2}\, T_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}\, d^4x \quad . \qquad (3.4.30)$$

From (3.4.29)

$$T_{\mu\nu} = \partial_\mu\psi\,\partial_\nu\psi - \tfrac{1}{2}\, g_{\mu\nu}\left\{ g^{\lambda\sigma}\,\partial_\lambda\psi\,\partial_\sigma\psi - (\mu^2 - \tfrac{R}{6})\,\psi^2 \right\} + R_{\mu\nu}\,\frac{\psi^2}{6} + \frac{1}{6}\left\{ g_{\mu\nu}\Box - \nabla_\mu\,\partial_\nu \right\}\psi^2 \quad . \qquad (3.4.31)$$

For $\mu = 0$, $T_{\mu\nu}$ has conformal weight -2 and is traceless.

(2) The canonical energy-momentum tensor

$$t_{\mu\nu} = \partial_\mu\psi\, \partial_\nu\psi - \tfrac{1}{2} g_{\mu\nu}\left\{g^{\lambda\sigma}\, \partial_\lambda\psi\, \partial_\sigma\psi - (\mu^2 - \tfrac{R}{6})\, \psi^2\right\} \quad . \qquad (3.4.32)$$

This may also be obtained by variation of S with respect to the metric but in this case R is treated as though it were independent of $g^{\mu\nu}$. Thus $t_{\mu\nu}$ is just the same as for a minimally coupled theory with mass $(\mu^2 - \frac{R}{6})^{\frac{1}{2}}$ (which may be imaginary in the case of AdS).

Let ξ^μ_a , a = 0, 1, ... , 9 , be a complete set of global Killing vector fields on AdS such that ξ^μ_0 corresponds to time translation, ξ^μ_1, ξ^μ_2 and ξ^μ_3 to spatial rotations, and the other six to "Lorentz boosts" in the five dimensional embedding space. Table 3.1 gives a suitable set. Define

$$Q_a(\tau) \equiv \int_{\tau=\text{const}} T_{\mu\nu}\, \xi^\nu_a\, d\sigma^\mu = \int_{\tau=\text{const}} T_{o\nu}\, \xi^\nu_a\, g^{oo} \sqrt{-g}\, d\rho\, d\theta\, d\phi \quad . \qquad (3.4.33)$$

The $Q_a(\tau)$ will not be independent of τ in general.

Since $T_{\mu\nu}$ has conformal weight -2 for a massless field, the integrand of (3.4.33) is conformally invariant and so (3.4.33) is equivalent to

$$Q_a(\tau) = \int_{\substack{\tau=\text{const}\\ \rho < \frac{1}{2}\pi}} T^E_{o\nu}\, \xi^{E\nu}_a \sqrt{-g^E}\, d\rho\, d\theta\, d\phi \quad . \qquad (3.4.34)$$

The $\xi^{E\nu}_a$ are the vector fields induced on half of ESU by the action of the conformal mapping on the ξ^ν_a. The components remain unchanged by this mapping and so are as listed in Table 3.1. It is clear from this table that they can be extended in a simple manner to the whole of ESU. However, whereas $\xi^{E\nu}_0$, $\xi^{E\nu}_1$, $\xi^{E\nu}_2$ and $\xi^{E\nu}_3$ still generate isometries the other six do not, but rather correspond to proper conformal motions of ESU, i.e.

$$\nabla^{E\mu}\, \xi^{E\nu}_a + \nabla^{E\nu}\, \xi^{E\mu}_a = \lambda_a\, g^{\mu\nu} \qquad (3.4.35)$$

Table 3.1 Killing vector fields for anti-de Sitter spacetime

	τ component	ρ component	θ component	ϕ component
ξ_0^μ	1	0	0	0
ξ_1^μ	0	0	$\sin\phi$	$\cot\theta \cos\phi$
ξ_2^μ	0	0	$\cos\phi$	$-\cot\theta \sin\phi$
ξ_3^μ	0	0	0	1
ξ_4^μ	$\cos\tau \sin\rho \sin\theta \cos\phi$	$\sin\tau \cos\rho \sin\theta \cos\phi$	$\sin\tau \operatorname{cosec}\rho \cos\theta \cos\phi$	$-\sin\tau \operatorname{cosec}\rho \operatorname{cosec}\theta \sin\phi$
ξ_5^μ	$\cos\tau \sin\rho \sin\theta \sin\phi$	$\sin\tau \cos\rho \sin\theta \sin\phi$	$\sin\tau \operatorname{cosec}\rho \cos\theta \sin\phi$	$\sin\tau \operatorname{cosec}\rho \operatorname{cosec}\theta \cos\phi$
ξ_6^μ	$\cos\tau \sin\rho \cos\theta$	$\sin\tau \cos\rho \cos\theta$	$-\sin\tau \operatorname{cosec}\rho \sin\theta$	0
ξ_7^μ	$\sin\tau \sin\rho \sin\theta \cos\phi$	$-\cos\tau \cos\rho \sin\theta \cos\phi$	$-\cos\tau \operatorname{cosec}\rho \cos\theta \cos\phi$	$\cos\tau \operatorname{cosec}\rho \operatorname{cosec}\theta \sin\phi$
ξ_8^μ	$\sin\tau \sin\rho \sin\theta \sin\phi$	$-\cos\tau \cos\rho \sin\theta \sin\phi$	$-\cos\tau \operatorname{cosec}\rho \cos\theta \sin\phi$	$-\cos\tau \operatorname{cosec}\rho \operatorname{cosec}\theta \cos\phi$
ξ_9^μ	$\sin\tau \sin\rho \cos\theta$	$-\cos\tau \cos\rho \cos\theta$	$\cos\tau \operatorname{cosec}\rho \sin\theta$	0

where $\lambda_a = 0$ for a = 0,1,2,3, but $\lambda_a \neq 0$ for a = 4,5,6,7,8,9. Now

$$\nabla^{E\mu}(T^E_{\mu\nu}\ \xi_a^{E\nu}) = 0 \qquad (3.4.36)$$

by virtue of (3.4.35) and the fact that $T^E_{\mu\nu}$ is traceless and has vanishing four-divergence. Thus, integrating (3.4.36) over the compact region between two constant τ hypersurfaces of ESU and applying Gauss's theorem, it follows that

$$P_a \equiv \int_{\tau=\mathrm{const}} T^E_{o\nu}\ \xi_a^{E\nu} \sqrt{-g^E}\, d\rho\ d\theta\ d\phi \qquad (3.4.37)$$

is independent of τ. Indeed these are the usual conserved quantities for a globally hyperbolic manifold. But now (3.4.15), along with the symmetry properties of the ξ_a^E, allows P_a to be decomposed as

$$P_a = Q_a(\tau) + Q_a(\tau + \pi) \quad . \qquad (3.4.38)$$

In other words, although in general the one hypersurface quantities $Q_a(\tau)$ are not τ-independent, the sums $Q_a(\tau) + Q_a(\tau + \pi)$ are. Such sums are equal to P_a, conserved quantities corresponding to global conformal motions of ESU.

Thus the effect of the "transparent" boundary conditions obtained by conformally mapping into ESU is to recirculate the energy, angular momentum, etc. lost to timelike infinity, resulting in a well defined, if rather unusual, conservation law.

In the next section the possibility of a "closed" quantisation, analogous to a box in Minkowski space with reflecting walls, will be considered. This is achieved in practice by demanding conservation of the Q_a i.e. conservation of quantities integrated over a <u>single</u> hypersurface.

3.5 Conformally coupled massless fields -"Reflective" boundary conditions

In the preceding section a quantisation was discussed which involved the specification of effective Cauchy data on a suitable pair of spacelike hypersurfaces, and it was shown that most field configurations did not have conserved energy, momentum, etc. as calculated by integrating the appropriate density over only one surface.

In this section two alternative quantisation schemes will be obtained by finding those maximal subsets of the positive frequency solutions (3.4.22) which have the property that all finite linear combinations

$$\psi(x) = \sum_{\omega\ell m} \left\{ c_{\omega\ell m}\, \psi_{\omega\ell m}(x) + c^*_{\omega\ell m}\, \psi^*_{\omega\ell m}(x) \right\} , \quad c_{\omega\ell m} \in \mathbb{C} \qquad (3.5.1)$$

give $Q_a(\tau)$ (defined in (3.4.33)) independent of τ i.e. we are looking for conservation laws based on a single hypersurface.

First note that from (3.4.36), (3.4.34) and Gauss's theorem

$$0 = \int_{\tau_1 \leqslant \tau \leqslant \tau_2} \nabla^{E\mu}(T^E_{\mu\nu}\, \xi^{E\nu}_a)\, dv = Q_a(\tau_2) - Q_a(\tau_1) + \int_{\tau_1}^{\tau_2} X_a\, d\tau \qquad (3.5.2)$$

where

$$X_a = \int_{\rho=\frac{1}{2}\pi} T^E_{\mu 1}\, g^{E11} \sqrt{-g^E}\, d\theta\, d\phi \quad . \qquad (3.5.3)$$

The requirement that $Q_a(\tau_1) = Q_a(\tau_2)$ for all τ_1 and τ_2 is equivalent to $X_a = 0$ (i.e. no net flux across $\rho = \frac{1}{2}\pi$). The minimal conditions imposed on the $c_{\omega\ell m}$ by setting $X_0 = 0$ (energy conservation) is that for each ℓ independently either all the $c_{\omega\ell m}$ with ω odd must vanish or all the $c_{\omega\ell m}$ with ω even must vanish. No further restriction is imposed by demanding $X_1 = X_2 = X_3 = 0$ (angular momentum conservation). Finally, on requiring $X_4 = X_5 = X_6 = X_7 = X_8 = X_9 = 0$, the complete restriction is that either all the $c_{\omega\ell m}$ with $\omega - \ell$ odd must vanish or all the $c_{\omega\ell m}$ with $\omega - \ell$ even must vanish.

Thus the requirement that all the $\dot{Q}_a(\tau)$ be independent of τ decomposes the basis functions $\psi_{\omega\ell m}$ into two disjoint classes which are listed below together with their principal properties:

$$(1) \quad \psi^1_{\omega\ell m} = \sqrt{2}\, N_{\omega\ell}\, e^{-i\omega\tau} \cos\rho\, (\sin\rho)^\ell\, C^{\ell+1}_{2n}(\cos\rho)\, Y^m_\ell(\theta,\phi) \tag{3.5.4}$$

where $\omega = \ell + 2n + 1$ and n is a non-negative integer.

$$\psi^1_{\omega\ell m}(x_A) = -\psi^1_{\omega\ell m}(x) \tag{3.5.5}$$

$$\frac{\partial}{\partial\rho}(\sec\rho\, \psi^1_{\omega\ell m}) \to 0 \quad \text{as} \quad \rho \to \tfrac{1}{2}\pi \quad . \tag{3.5.6}$$

$$(2) \quad \psi^2_{\omega\ell m} = \sqrt{2}\, N_{\omega\ell}\, e^{-i\omega\tau} \cos\rho\, (\sin\rho)^\ell\, C^{\ell+1}_{2n+1}(\cos\rho)\, Y^m_\ell(\theta,\phi) \tag{3.5.7}$$

where $\omega = \ell + 2n + 2$ and n is a non-negative integer.

$$\psi^2_{\omega\ell m}(x_A) = \psi^2_{\omega\ell m}(x) \tag{3.5.8}$$

$$\sec\rho\, \psi^2_{\omega\ell m} \to 0 \quad \text{as} \quad \rho \to \tfrac{1}{2}\pi \quad . \tag{3.5.9}$$

Each class corresponds to a definite "parity" under the point to antipodal point transformation and a well defined behaviour at spatial infinity.

Let $\mathcal{H}^1$ and $\mathcal{H}^2$ denote the Hilbert spaces formed from the functions (1) and (2) respectively. It is clear that <u>all</u> elements of $\mathcal{H}^1$ or $\mathcal{H}^2$ have the same definite parity in the above sense and it follows that a solution in one of these spaces is completely determined by its initial value data on one spatial section, Σ_1 say. Indeed in view of this parity it is clear that the classical commutator functions to be used for evolving data on Σ_1 uniquely forward in time are

$$\tilde{G}^j(x,0) = \tilde{G}^T(x,0) - (-1)^j\ \tilde{G}^T(x_A,0) \qquad j = 1,2 \qquad (3.5.10)$$

$$= -\frac{K}{4\pi}\ \mathring{\varepsilon}(\tau)\left\{\delta(K\sigma) - (-1)^j\ \delta(K\sigma - 2)\right\} \qquad (3.5.11)$$

where $\tilde{G}^j(x,x')$ is the commutator function associated with $\mathcal{H}^j$. The support of $\tilde{G}^j(x,0)$ is indicated in Fig. 3.4.

The $\mathcal{H}^j$ norm may be defined in a natural way as in (3.2.5) but now integrated over Σ_1 only (hence the extra normalisation factor $\sqrt{2}$ in the $\psi^j_{\omega\ell m}$).

Just as for Sec. 3.4 the quantisation is implemented without difficulty now that the Cauchy problem has been taken care of. The relationship (3.2.6) follows easily from its "transparent" counterpart, using the symmetries of the commutatator functions and basis functions. The field operator

$$\hat{\psi}^j = \sum_{\omega\ell m}(\psi^j_{\omega\ell m}\ \hat{a}^j_{\omega\ell m} + \psi^{j*}_{\omega\ell m}\ \hat{a}^{j*}_{\omega\ell m}) \qquad (3.5.12)$$

satisfies both the field equation and covariant commutation relation as required.

To make clear the analogy with the box in Minkowski space it is only necessary to point out that the image of CAdS (and hence AdS under identification) is effectively the interior of a box in ESU with a "wall" at $\rho = \frac{1}{2}\pi$. For the two schemes of this section the ESU counterparts of the fields satisfy

$$\frac{\partial}{\partial\rho}\ \psi^{1E} \to 0 \quad \text{as} \quad \rho \to \tfrac{1}{2}\pi \qquad (3.5.13)$$

in one case and

$$\psi^{2E} \to 0 \quad \text{as} \quad \rho \to \tfrac{1}{2}\pi \qquad (3.5.14)$$

in the other. These are precisely the conditions usually imposed on the boundary of a box with reflecting walls in Minkowski space, hence the

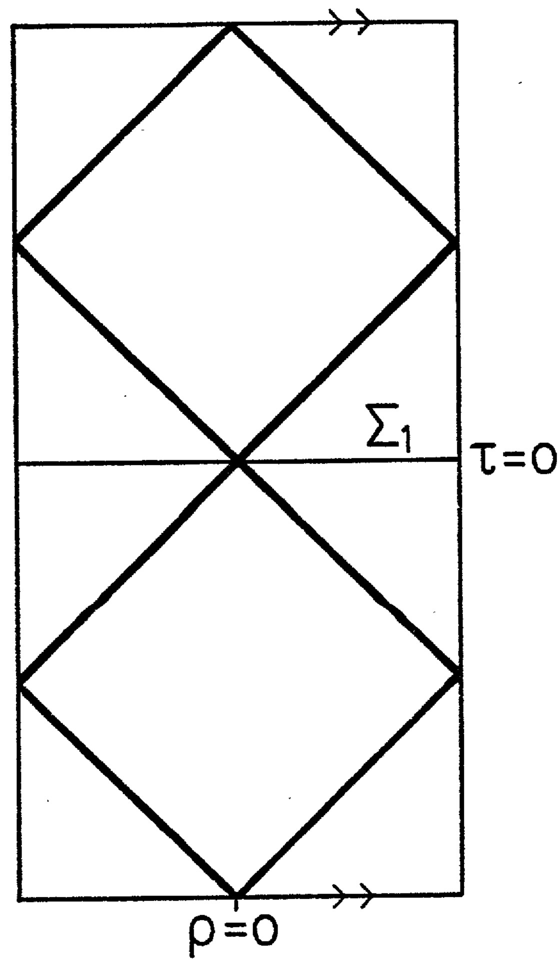


Fig. 3.4

The "reflective" conformal massless case. Single spacelike hypersurfaces, e.g. Σ_1, form effective Cauchy surfaces. The null lines (at $\pm 45^{\circ}$) are the support of the commutator functions $\tilde{G}^1(x,0)$ and $\tilde{G}^2(x,0)$.

description of the boundary conditions as "reflective". Of course the boundary conditions on the AdS fields themselves, (3.5.6) and (3.5.9), are more complicated, and their meaning would be much less clear in any naïve approach to the problem not involving ESU.

As in the transparent case, each of the two commutator functions $\tilde{G}^j$ can be related to the real part of its corresponding "Feynman" function G^j via

$$\tilde{G}^j(x,0) = 2\,\mathring{\varepsilon}(\tau)\ \mathrm{Re}\ G^j(x,0) \qquad (3.5.15)$$

where

$$G^j(x,0) = \frac{iK}{8\pi^2}\left\{\frac{1}{K\sigma - i0} - (-1)^j \frac{1}{K\sigma - 2 - i0}\right\} \qquad (3.5.16)$$

with σ as in (3.5.24). The G^j satisfy the inhomogeneous equation

$$(\Box - 2K)\ G^j(x,0) = -\,\delta^4(x) - (-1)^j\ \delta^4(x_A) \quad . \qquad (3.5.17)$$

The appearance of <u>two</u> sources here is another manifestation of the fact that in the "reflective" schemes effective Cauchy data can only be consistently set on <u>one</u> constant τ hypersurface.

The relationship between the three quantisations for the massless field in AdS is essentially summarised by the decomposition of the "transparent" one-particle Hilbert space in terms of those of the "reflective" cases:

$$\mathcal{H}^T = \mathcal{H}^1 \oplus \mathcal{H}^2 \quad . \qquad (3.5.18)$$

Consequently, the Fock spaces are related by

$$\mathcal{F}^T = \mathcal{F}^1 \otimes \mathcal{F}^2 \quad . \qquad (3.5.19)$$

Thus an n-particle "transparent" state may be written, rather symbolically, as

$$|n\rangle^T = \lambda_0 |n\rangle^1 \otimes |0\rangle^2 \oplus \lambda_1 |n-1\rangle^1 \otimes |1\rangle^2 \oplus \ldots$$
$$\ldots \oplus \lambda_{n-1} |1\rangle^1 \otimes |n-1\rangle^2 \oplus \lambda_n |0\rangle^1 \otimes |n\rangle^2 \qquad (3.5.20)$$

where

$$\sum_{i=0}^{n} |\lambda_i|^2 = 1 \quad . \tag{3.5.21}$$

From the point of view of either "reflective" scheme this state will in general appear as a mixture of n, n-1, ..., 1, 0 -particle states. The "transparent" vacuum is an exception in this respect, corresponding only to pure "reflective" vacuum states:

$$|0\rangle^T = |0\rangle^1 \otimes |0\rangle^2 \quad . \tag{3.5.22}$$

A more typical example would be a one particle "transparent" state, interpreted in the $\mathcal{H}^1$ or $\mathcal{H}^2$ scheme as a mixture of one particle and vacuum states.

3.6 Massive scalar fields

The equation of motion for a "conformally" coupled, massive, spin-zero field in AdS is

$$(\Box + \mu^2 - \tfrac{R}{6})\ \psi = (\Box + \mu^2 - 2K)\ \psi = 0 \quad , \quad \mu^2 > 0 \quad . \tag{3.6.1}$$

Most of this section also applies to a minimally coupled field with mass

$$\mu' = +\sqrt{\mu^2 - 2K} \quad , \quad \text{provided } \mu^2 \geqslant 2K \ . \tag{3.6.2}$$

The only significant difference is that for the minimal theory the canonical and variational energy-momentum tensors are identical.

Unlike the conformally coupled massless case, (3.6.1) is not conformally invariant. The corresponding equation in ESU is

$$(\Box^{\mathcal{E}} + \mu^2\Omega^2 - \tfrac{R^E}{6})\ \psi = (\Box^{\mathcal{E}} + \mu^2\cos^2\rho + K)\ \psi = 0 \tag{3.6.3}$$

and has a position dependent "mass". Thus the method of conformal mapping employed in Secs. 3.4 and 3.5 is slightly less appropriate here. Nevertheless it still proves useful in providing a concrete realisation of spatial infinity and in simplifying calculations related to conservation laws, as demonstrated for the massless case.

We will begin by considering separable, positive frequency solutions of (3.6.1) in AdS itself. These are of the form $e^{-i\omega\tau}\, h(\rho,\theta,\phi)$ where ω is required to be an integer to ensure that the solutions are single-valued in AdS. It is convenient to write

$$\mu^2 = K(M-1)(M-2) \quad , \quad M > 2 \quad . \tag{3.6.4}$$

Then it is found that nonsingular, finite B-norm, separable solutions can only exist if M satisfies either (i) $2 < M < \frac{5}{2}$ or (ii) $M = 3,4,5,\ldots$. So we have something resembling a "mass spectrum" consisting of a small continuum and an unbounded discrete part. The corresponding solutions are

$$\text{(i)} \quad \zeta^M_{\omega\ell m} = N^M_{\omega\ell}\, e^{-i\omega\tau}\, (\cos\rho)^M\, (\sin\rho)^\ell \times$$
$$\times\; {}_2F_1(\tfrac{1}{2}(\ell+M-\omega), \tfrac{1}{2}(\ell+M+\omega); \tfrac{3}{2}+\ell; \sin^2\rho)\; Y^m_\ell(\theta,\phi) \tag{3.6.5}$$

where ω, ℓ and m are integers such that $\ell \geqslant |m|$ and ${}_2F_1(a,b;c;z)$ are hypergeometric functions[54].

$$\text{(ii)} \quad \psi^M_{\omega\ell m} = N^M_{\omega\ell}\, e^{-i\omega\tau}\, (\cos\rho)^M\, (\sin\rho)^\ell\, P_n^{(\ell+\frac{1}{2},\, M-\frac{3}{2})}(\cos 2\rho)\; Y^m_\ell(\theta,\phi) \tag{3.6.6}$$

where $\omega = M + \ell + 2n$ and ℓ, m and n are integers such that $\ell \geqslant |m|$ and $n \geqslant 0$. The normalisation constants in this case are

$$N^M_{\omega\ell} = \frac{n!\,(n+\ell+M-1)!}{\Gamma(n+\ell+\frac{3}{2})\,\Gamma(n+M-\frac{1}{2})} \tag{3.6.7}$$

and the $P_n^{(\alpha,\beta)}(z)$ are Jacobi polynomials[54].

If, as in Sec. 3.4, we were to require that the ESU counterparts of these functions be C^∞ on <u>all</u> of ESU then (i) would be lost. Nor would (i) occur if only the minimally coupled case is considered. In any event, our attention will be focussed mainly on the solutions (ii).

For each M = 3,4,5,... all the solutions have the <u>same</u> definite parity under the point to antipodal point transformation, and hence so do their linear combinations. In particular,

$$\psi^M(x_A) = (-1)^M \psi^M(x) \quad . \tag{3.6.8}$$

When restricted to a single spacelike hypersurface the $\psi^M_{\omega\ell m}$ form a complete set and it is found that energy, angular momentum, etc. are conserved when integrated over such a surface.

It is clear then that for M odd these cases are analogous to the massless reflective case (1) (c.f. (3.5.5)) while for M even they are analogous to the massless reflective case (2) (c.f. (3.5.8)). Hence the quantisations of these massive fields may be modelled on the quantisations of Sec. 3.5. The $\psi^M_{\omega\ell m}$ form an orthonormal basis for the Hilbert space $\mathcal{H}^M$ with inner product (3.2.5), the integration region being Σ_1 say. They also satisfy (3.2.6) where the classical commutator function, which evolves "effective Cauchy data" specified on a single hypersurface, is given by

$$\begin{aligned}\tilde{G}^M(x,0) = \frac{K}{4\pi}\, \mathring{\varepsilon}(\tau) \Big\{ &\delta(K\sigma) - (-1)^M \delta(K\sigma - 2) + \\ &+ \left[\theta(-K\sigma) - \theta(2 - K\sigma)\right] P'_{M-2}(1 - K\sigma) \Big\}\end{aligned} \tag{3.6.9}$$

where $P'_N(z)$ denotes the derivative of the Legendre polynomial of degree N.

The support of $\tilde{G}^M(x,0)$ is shown in Fig. 3.5 and reflects in a striking way the behaviour of classical massive particles in AdS. All timelike geodesics through $\tau = \rho = 0$ lie entirely within the shaded

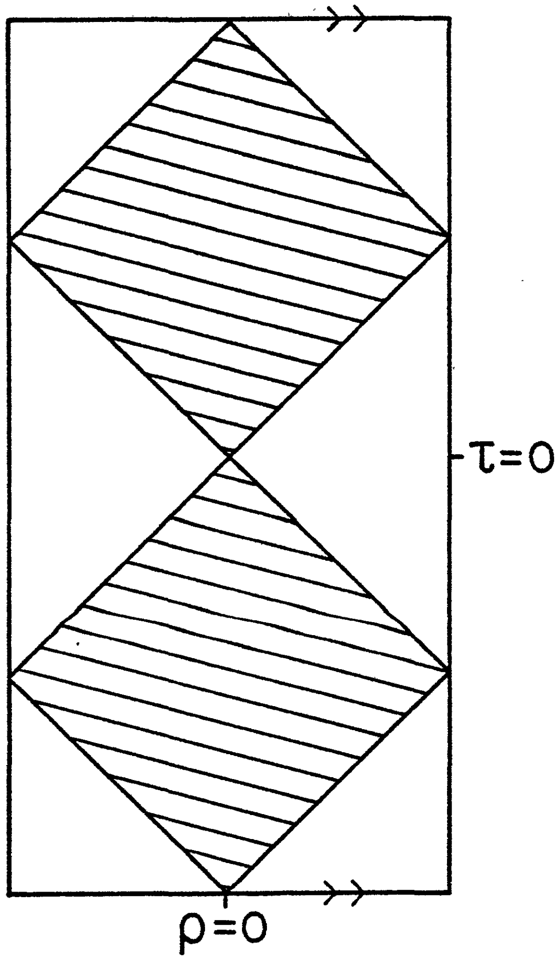


Fig. 3.5

The support of $\tilde{G}^M(x,0)$ for a massive field. It is regular within the shaded regions, singular on their boundary, and zero elsewhere.

regions. The observation that such geodesics reconverge and do not reach spatial infinity also offers a heuristic classical explanation for the lack of a "transparent" quantisation scheme for massive fields.

In fact the massless reflective cases fit into the present scheme in a very natural way. Comparing (3.5.4) with (3.6.6) and using Equation 10.9(21) of Ref. 54 it is seen that the $\mathcal{H}^1$ basis functions of Sec. 3.5 correspond to M = 1. Likewise, comparing (3.5.7) with (3.6.6) and using Equation 10.9(22) of Ref. 54, the $\mathcal{H}^2$ basis functions correspond to M = 2. This identification is clear cut, despite the fact that M = 1 and M = 2 are indistinguishable from the point of view of the wave equation (3.6.1), bearing in mind (3.6.4).

Thus the quantisation of these "special mass" fields (M = 1,2,3,...) is completed and it is convenient to briefly mention the relationship with the group theory approach at this stage. Fronsdal[47] has shown, by group theoretic arguments, that there exists a collection of irreducible representations of the universal covering group of SO(3,2), labelled by a positive number M (E_0 in his terminology) which correspond to solutions of the wave equation (3.6.1) in CAdS. Those which may be defined on <u>AdS</u> correspond to M integral and reduce to ours, but the representation is now only faithful for SO(3,2) itself. Thus $\mathcal{H}^M$ does carry the desired representation of the AdS isometry group.

The Feynman function generalising those of Sec. 3.5 is found by solving (3.4.25) with a suitable mass term included. It is given by

$$G^M(x,0) = \frac{iK}{4\pi^2} Q'_{M-2}(1 - K\sigma + i0) \tag{3.6.10}$$

$$= \frac{iK}{8\pi^2}\left\{\left[\frac{1}{K\sigma - i0} - \frac{1}{K\sigma - 2 - i0}\right] P_{M-2}(1 - K\sigma) + \right.$$

$$\left. + \left[\ln(K\sigma - i0) - \ln(K\sigma - 2 - i0)\right] P'_{M-2}(1 - K\sigma) + 2W'_{M-3}(1 - K\sigma)\right\} \tag{3.6.11}$$

where $Q_N(z)$ is a Legendre function of the second kind and $W_N(z)$ is a polynomial of degree N given by Christoffel's formula[54]

$$W_N(z) = \sum_{m=0}^{[\frac{1}{2}N]} \frac{2N - 4m + 1}{(N + 1 - m)(2m + 1)} P_{N-2m}(z) \ , \qquad (3.6.12)$$

in which the symbol $[\frac{1}{2}N]$ denotes the greatest integer less than or equal to $\frac{1}{2}N$. $G^M(x,0)$ satisfies the inhomogeneous wave equation

$$(\Box + \mu^2 - 2K)\, G^M(x,0) = -\,\delta^4(x) - (-1)^M\, \delta^4(x_A) \qquad (3.6.13)$$

and is related to the commutator function by (3.5.15), where M and j are now interchangeable.

Despite the existence of closed timelike curves, $G^M(x,x')$ can be related to the vacuum expectation value of a "time-ordered" product in the following sense:

$$-\,i\hbar\, G^M(x,x') = \langle 0|\mathring{T}\{\hat{\psi}^M(x)\ \hat{\psi}^M(x')\}|0\rangle \qquad (3.6.14)$$

where

$$\mathring{T}\{\hat{\psi}^M(x)\ \hat{\psi}^M(x')\} \equiv \mathring{\theta}(\tau-\tau')\ \hat{\psi}^M(x)\ \hat{\psi}^M(x') - (1-\mathring{\theta}(\tau-\tau'))\ \hat{\psi}^M(x')\ \hat{\psi}^M(x) \qquad (3.6.15)$$

and

$$\mathring{\theta}(\tau-\tau') \equiv \theta(\sin(\tau-\tau')) \quad . \qquad (3.6.16)$$

This applies equally well to the "transparent" massless case, i.e. in (3.6.14) M = 1, 2, 3, 4, ..., or T. In effect the time ordering is carried out using the smaller angle between τ and τ'.

3.7 Summary and further remarks

Three quantisations for a conformally coupled massless scalar field have been constructed by conformally mapping AdS into ESU. One scheme is associated with "transparent" boundary conditions in which information flows freely out of the image of AdS, passes through the other parts of ESU, and re-enters the image of AdS elsewhere. The other two schemes correspond to "reflective" boundary conditions in which the image of AdS may be thought of as a box with reflecting walls in ESU. These latter two schemes generalise to include a sequence of massive fields for each of which there is a unique natural quantisation.

The Feynman functions for these schemes were all constructed from first principles. An alternative procedure would be to try to obtain them from de Sitter space Feynman functions by analytic continuation. De Sitter space may be realised as the hyperboloid

$$(\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - (\xi^4)^2 = -K^{-1} \qquad (3.7.1)$$

in a space with metric

$$ds^2 = \eta^{(5)}_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta = (d\xi^0)^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2 - (d\xi^4)^2 \qquad (3.7.2)$$

and so the continuation required is

$$\xi^4 \to i\xi^4 \,, \qquad K \to -K \quad . \qquad (3.7.3)$$

In fact it is more convenient to take

$$\xi^j \to i\xi^j \quad \text{for} \quad j = 0,1,2,3 \qquad (3.7.4)$$

instead. This changes the signature so the mass squared will also have to be continued to the opposite sign.

An expression for the Feynman function of a minimally coupled field in de Sitter space is given in Ref. 37. Converting to conformal coupling this becomes

$$G^{dS}(x,x') = \frac{iK}{16\pi^2}\,\Gamma(3-M)\,\Gamma(M)\;{}_2F_1(3-M,M;2;1-\tfrac{1}{2}K(\sigma+i0)) \tag{3.7.5}$$

where

$$\sigma = \tfrac{1}{2}\,\eta'^{(5)}_{\alpha\beta}\,(\xi^\alpha - \xi'^\alpha)(\xi^\beta - \xi'^\beta) \quad , \tag{3.7.6}$$

$$M = \tfrac{1}{2}\left[3 + \left(1 + \frac{4\mu^2}{K}\right)^{\frac{1}{2}}\right] \tag{3.7.7}$$

and $(-\mu^2)$ is the de Sitter mass squared. Now σ and μ^2 must be continued from their de Sitter values to the appropriate AdS values.

For $\mu^2 = 0$ this is straightforward and yields precisely the "transparent" AdS Feynman function given in (3.4.26). The "reflective" massless AdS Feynman functions are obtained as analytic continuations of de Sitter Green functions solving the de Sitter inhomogeneous wave equation with two sources, one at x' and the other at the de Sitter antipodal point to x'.

The expression (3.7.5) has simple poles in M at the points 3, 4, 5, ..., and so the AdS Feynman functions (3.6.10) corresponding to these "special masses" are not related by analytic continuation to (3.7.5). So the overall conclusion is that analytic continuation from de Sitter space is not a particularly useful tool for investigating quantum field theories in AdS.

A further point of interest is that both Minkowski and de Sitter spacetimes may also be mapped into ESU, in a similar manner to AdS (see Fig. 3.6). Moreover the four-volumes of the images of all three spacetimes are the same. The solutions of the conformal massless wave equation in ESU are periodic in such a way that they are uniquely determined by their behaviour in any of these images. Thus a basis for such functions in ESU may be mapped back to form a basis in anti-de Sitter, de Sitter, or Minkowski space. In particular, mapping back the basis (3.4.13) to Minkowski space results in the "elementary states"

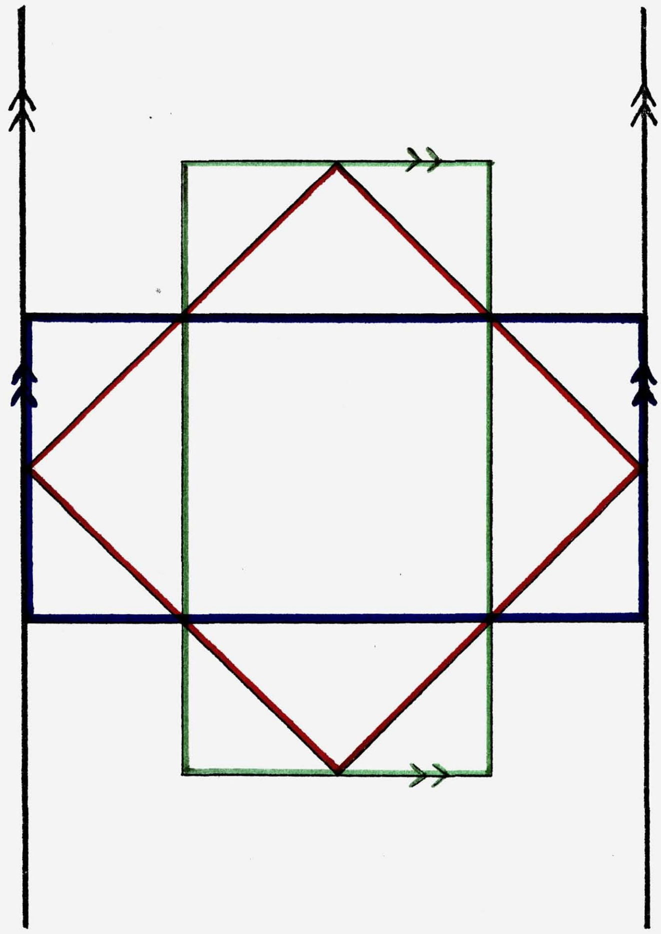


Fig. 3.6

The images of anti-de Sitter space (green), de Sitter space (blue) and Minkowski space (red) conformally mapped into the Einstein static universe (black). Two dimensions are suppressed.

of twistor theory.

To conclude, there are many problems to be faced in the construction of quantum field theories in anti-de Sitter spacetime, but none of these is insurmountable. Of course AdS is a very special spacetime with a high degree of symmetry. Whether techniques similar to those used here can be applied to more general non globally hyperbolic spacetimes is an open question.

CHAPTER FOUR

SCALAR STRONG GRAVITY

4.1 Why consider scalar strong gravity?

The tensor meson dominance hypothesis, summarised in Sec. 1.1(i), postulates that the hadronic stress-energy tensor should act as a source for a universal spin-2 field. It has also been suggested (Refs. 4, 56 and refs. therein) that the trace of the hadronic energy-momentum tensor should be related in a similar way to a universal scalar field.

Linearised spin-2 field theory is easily generalised to include a spin-0 field by simply choosing the mass term to be other than in Fierz-Pauli form. This scalar would be ideal for the present purposes if it were not for the fact that it is a ghost field. As discussed in Sec. 1.3 this means that it must be eliminated in linearised f-g theory. Of course the spin-0 part of the lagrangian can be given the correct sign, but only at the expense of making the spin-2 field into a ghost. Hence it seems desirable to consider the scalar field on its own, and such a theory will be in a sense orthogonal to f-g theory in that at the linearised level it describes the very field which is eliminated in f-g theory.

The idea, then, is to look for a theory of strong and weak metrics in which the only independent fields are one metric and one scalar. Now for ordinary gravity there is a well known theory with precisely this field content namely the Brans-Dicke[57](-Jordan[58] -Thiry[59]) theory, and its generalisations due to Bergmann[60] and Wagoner[61]. This theory will be adapted to our needs by re-interpreting the Brans-Dicke scalar as a hadron field. The strong and weak metrics will be conformally related by this scalar.

The Brans-Dicke theory was formulated as an attempt to incorporate Mach's principle into a field theory of gravitation. Loosely speaking, Mach's principle asserts that the inertia of an object should depend on the distribution of surrounding matter. Is there any evidence for such an effect in hadron physics? In fact there is. The effective masses of quarks in close proximity to each other, i.e. inside hadrons, appear to be rather small (about 300 MeV) whereas it is postulated, to explain confinement, that their masses when they are alone are large (partial confinement) or even infinite (total confinement). This is the so-called "Archimedes effect". The way in which an effect of this kind occurs in the model to be considered here will be described in Sec. 4.5.

Having conformally related weak and strong metrics contrasts sharply with f-g theory in which the metrics are, a priori at least, completely independent of one another. An advantage of having such related metrics is that it may be easier to regard one of them as being in a sense the "real" metric of spacetime. This will be particularly clear for the model discussed in the sequel. In f-g theory, by way of contrast, there seems to be no combination of $f_{\mu\nu}$ and $g_{\mu\nu}$ which plays this rôle, except in the linearised approximation where the fields can be diagonalised for mass.

One might ask if other theories of related metrics are possible. For example, they could perhaps be related through a vector field describing some kind of universal spin-1 hadron. However, as will be seen in the next section these other theories do not seem so attractive for various reasons.

4.2 Problems with other related-metric theories

Let us begin by considering vector-related metrics. (In fact covectors have been used but this is unimportant.) There are two obvious possibilities here:

(i) $\tilde{g}_{\mu\nu} = g_{\mu\nu} + k(\nabla_\mu V_\nu + \nabla_\nu V_\mu)$ (where k is a constant)

(ii) $\tilde{g}_{\mu\nu} = g_{\mu\nu} + kV_\mu V_\nu$ (Kerr-Schild related metrics[62]) .

In this chapter the strong metric is denoted by $\tilde{g}_{\mu\nu}$, to avoid f-meson connotations.

In (i), ∇_μ could denote the covariant derivative with respect to either $g_{\mu\nu}$ or $\tilde{g}_{\mu\nu}$. In fact the presence of derivatives makes (i) a rather unpleasant prospect since objects like $\sqrt{-\tilde{g}}$ must occur in the coupling to hadronic matter, even if they are somehow avoided in the source-free part of the lagrangian. If terms like $\sqrt{-\tilde{g}}\, R^{\tilde{g}}$ occur then even the linearised field equations will involve third derivatives.

Case (ii) has the attractive feature that if the constant k is chosen to be negative then the $\tilde{g}_{\mu\nu}$ light cones will always lie within the $g_{\mu\nu}$ light cones, assuming V_μ is never so large that $\tilde{g}_{\mu\nu}$ becomes riemannian, as opposed to pseudoriemannian. So if $g_{\mu\nu}$ is regarded as the "real" spacetime metric then the world lines of hadrons, i.e. curves which are timelike with respect to $\tilde{g}_{\mu\nu}$, will always be causal in the sense of being timelike with respect to $g_{\mu\nu}$. Construction of a suitable lagrangian poses some problems however. It would be easy to write down the standard Maxwell or Proca lagrangian for V_μ in the curved spacetime with metric $g_{\mu\nu}$, but this seems rather ad hoc. In particular it bears no relation to the way V_μ has been coupled to matter. It would be more natural to try to construct the lagrangian from $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ alone, for example by using $\sqrt{-g}\, R^g$ and $\sqrt{-\tilde{g}}\, R^{\tilde{g}}$. Unfortunately this has the consequence that the field equations at lowest order are cubic, rather than linear, in V_μ and so we drop this idea as well.

As regards scalar-related metrics, the only possibility not involving derivatives is to have conformally related metrics. These possess the same causality property as (ii) above, since in this case their light cones are identical. The use of derivatives, e.g.

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \nabla_\mu \partial_\nu \phi \quad ,$$

suffers from the same drawbacks as in (i) above.

Thus attention will be confined to the case of conformally related strong and weak metrics. As discussed in the previous section, the Brans-Dicke scalar-tensor theory of gravity is a natural starting point for such an investigation. A brief resumé of this theory is presented in the next section.

4.3 A brief review of the Brans-Dicke theory

The Brans-Dicke lagrangian density for (ordinary) gravity and its coupling to matter is

$$\begin{aligned}\mathcal{L}_{BD} &= \mathcal{L}_{grav} + \mathcal{L}_{matter} \\ &= \frac{\sqrt{-g}}{\kappa_g^2}(-\,\phi\, R^g + \omega\, g^{\mu\nu}\, \phi_{,\mu}\, \phi_{,\nu}\, \phi^{-1}) + \mathcal{L}_{matter} \quad . \qquad (4.3.1)\end{aligned}$$

The dynamical field $\phi\,\kappa_g^{-2}$ replaces the constant κ_g^{-2}, the Brans-Dicke interpretation of Mach's principle being that the gravitational "constant" G ($= \kappa_g^2/8\pi$) should not be constant, but should be determined by the stress-energy tensor of matter. In fact ϕ is coupled to the trace of this tensor. Of course it is assumed that in our region of the universe ϕ is unity to a good approximation.

In the term of (4.3.1) involving derivatives of ϕ inclusion of the factor ϕ^{-1} makes the matter-free equations invariant under scale changes of ϕ. The constant ω is then dimensionless and is assumed to be of order unity.

It is well known[63] that in the absence of matter the theory is equivalent to general relativity plus a massless scalar field. This is seen by rewriting the lagrangian in terms of the new fields

$$g'_{\mu\nu} \equiv \phi \, g_{\mu\nu} \quad \text{and} \quad \psi \equiv \kappa_g^{-1} \ln \phi \quad . \tag{4.3.2}$$

Using standard formulas for conformally related metrics (see e.g. Ref. 43), and ignoring four-divergences, the result is

$$\mathcal{L}_{grav} = \sqrt{-g'} \left(- \frac{R^{g'}}{\kappa_g^2} + \left(\frac{3}{2} + \omega\right) \psi_{,\mu} \, \psi_{,\nu} \, g'^{\mu\nu} \right) \quad . \tag{4.3.3}$$

It now becomes transparent that ω had better be greater than or equal to $-\frac{3}{2}$, since otherwise ψ will be a ghost. This same requirement was made by Brans and Dicke to ensure that matter acts as a source for the scalar with the correct sign.

In the Brans-Dicke theory the metric $g'_{\mu\nu}$ is not very important since it is $g_{\mu\nu}$ which matter is coupled to. But in the weak and strong gravity theory introduced in the next section both are used, one being the metric seen by leptons, which can be interpreted as the "real" spacetime metric, the other being the "strong" metric.

4.4 The scalar strong gravity theory

We begin with the lagrangian density for gravity plus the universal hadron field ψ :

$$\mathcal{L}_0 = \sqrt{-g} \left(- \frac{R^g}{\kappa_g^2} + \tfrac{1}{2} \psi_{,\mu} \, \psi_{,\nu} \, g^{\mu\nu} \right) \quad . \tag{4.4.1}$$

Leptons are coupled to the "real" metric, $g_{\mu\nu}$, but hadrons will be made to respond to $\tilde{g}_{\mu\nu}$ where

$$\tilde{g}_{\mu\nu} = \phi^{-1} \, g_{\mu\nu} = e^{-\lambda\psi} \, g_{\mu\nu} \tag{4.4.2}$$

The constant λ is a strong interaction coupling constant. As will be seen later it plays a rôle similar to that of the constant κ_f in f-g theory. Written in terms of $\tilde{g}_{\mu\nu}$ and ϕ the lagrangian density becomes

$$\mathcal{L}_0 = \frac{\sqrt{-\tilde{g}}}{\kappa_g^2}\left\{ -\phi R^{\tilde{g}} + \left(\frac{\kappa_g^2}{2\lambda^2} - \frac{3}{2}\right) \tilde{g}^{\mu\nu} \frac{\phi_{,\mu}\phi_{,\nu}}{\phi} \right\} , \qquad (4.4.3)$$

i.e. a Brans-Dicke lagrangian with ω very slightly greater than $-\frac{3}{2}$. (Taking $\lambda > 0$ for reasons given in the preceding section.)

To illustrate the coupling to matter, consider a massive real scalar field χ coupled, for the sake of generality, to $g_{\mu\nu}^{(h)}$, where

$$g_{\mu\nu}^{(h)} \equiv h\, \tilde{g}_{\mu\nu} + (1-h)\, g_{\mu\nu} = (h\phi^{-1} + 1 - h)\, g_{\mu\nu} \qquad (4.4.4)$$

and the constant h is a measure of the "hadron-ness" of the field. A lepton has h = 0 whereas a "pure" hadron has h = 1. It will turn out later to be sensible to restrict h to the range $0 \leqslant h \leqslant 1$. The lagrangian density for χ is

$$\mathcal{L}_\chi = \tfrac{1}{2}\sqrt{-g^{(h)}}\left\{ g^{(h)\mu\nu}\chi_{,\mu}\chi_{,\nu} - m^2\chi^2 \right\} \qquad (4.4.5)$$

$$= \tfrac{1}{2}\sqrt{-g}\,(he^{-\lambda\psi} + 1 - h)\left\{ g^{\mu\nu}\chi_{,\mu}\chi_{,\nu} - m^2(he^{-\lambda\psi} + 1 - h)\chi^2 \right\} . \qquad (4.4.6)$$

The field equations which result from the action principle

$$\delta \int (\mathcal{L}_0 + \mathcal{L}_\chi) d^4x = 0 \qquad (4.4.7)$$

are as follows:

g field equations:

$$G_{\mu\nu} = \kappa_g^2\, T_{\mu\nu} = \kappa_g^2 (T_{\mu\nu}^{(\psi)} + T_{\mu\nu}^{(\chi)}) \qquad (4.4.8)$$

where

$$T_{\mu\nu}^{(\psi)} = \tfrac{1}{2}(\psi_{,\mu}\psi_{,\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}) \qquad (4.4.9)$$

and

$$T^{(\chi)}_{\mu\nu} = \tfrac{1}{2}(he^{-\lambda\psi}+1-h)\left\{\chi_{,\mu}\,\chi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\,g^{\alpha\beta}\chi_{,\alpha}\chi_{,\beta} - m^2(he^{-\lambda\psi}+1-h)\chi^2\right\} . \tag{4.4.10}$$

<u>ψ field equation:</u>

$$g^{\mu\nu}\,\psi_{;\mu\nu} = \frac{\lambda h}{2}\,e^{-\lambda\psi}\left\{-\chi_{,\mu}\,\chi_{,\nu}\,g^{\mu\nu} + 2m^2(he^{-\lambda\psi}+1-h)\chi^2\right\} . \tag{4.4.11}$$

<u>χ field equation:</u>

$$g^{(h)\mu\nu}\,\nabla^{(h)}_{\mu}\,\chi_{,\nu} + m^2\chi = 0 \quad , \tag{4.4.12}$$

where $\nabla^{(h)}_{\mu}$ denotes the covariant derivative with respect to $g^{(h)}_{\mu\nu}$.

Note that for h = 1 the source of ψ is $T^{(\chi)\,\mu}_{\ \ \ \ \mu}$, i.e. the trace of the hadron stress-energy tensor, and the coupling constant is λ, justifying the remarks made about this constant earlier on. Of course ψ is not coupled to leptons (h = 0) at all.

The lagrangian written using $g_{\mu\nu}$ as opposed to $\tilde{g}_{\mu\nu}$ looks just like, and is, a lagrangian for standard general relativity including matter fields, albeit with rather strange matter interactions. As usual, the Bianchi identity $G^{\mu\nu}{}_{;\mu} = 0$ is compatible with the conservation law $T^{\mu\nu}{}_{;\mu} = 0$ which follows from the ψ and χ field equations. It is in this sense that the weak metric $g_{\mu\nu}$ can actually be regarded as the "real" metric of spacetime.

In view of this it is reasonable to consider the approximation $\kappa_g \to 0$, $g_{\mu\nu} \to \eta_{\mu\nu}$, with the consequence that $g^{(h)}_{\mu\nu}$ is now conformally flat. The lagrangian density reduces to

$$\mathcal{L} = \tfrac{1}{2}\,\psi_{,\mu}\,\psi_{,\mu} + \tfrac{1}{2}(he^{-\lambda\psi}+1-h)\left\{\chi_{,\mu}\,\chi_{,\mu} - m^2(he^{-\lambda\psi}+1-h)\chi^2\right\} \tag{4.4.13}$$

$$= \frac{\phi_{,\mu}\,\phi_{,\mu}}{2\lambda^2\phi^2} + \tfrac{1}{2}(h\phi^{-1}+1-h)\left\{\chi_{,\mu}\,\chi_{,\mu} - m^2(h\phi^{-1}+1-h)\chi^2\right\} \tag{4.4.14}$$

$$= \frac{\phi'_{,\mu}\,\phi'_{,\mu}}{2\lambda^2\phi'^2} + \tfrac{1}{2}(h\phi'+1-h)\left\{\chi_{,\mu}\,\chi_{,\mu} - m^2(h\phi'+1-h)\chi^2\right\} \tag{4.4.15}$$

where $\phi' \equiv \phi^{-1}$ and index contractions are made using $\eta^{\mu\nu}$. Written in terms of ϕ', this lagrangian is similar in appearance to that considered by Freund and Nambu[64] but in fact has some quite different properties as well as having been derived in a completely different way. For example, the Freund-Nambu lagrangian can be transformed into polynomial form whereas (4.4.15) cannot. The attractive features of the present model will become apparent in the next section.

It may seem a little distressing that the hadron field ψ is massless. Generalisations of (4.4.13) in which ψ is given a mass, based on the Bergmann-Wagoner generalisation of the Brans-Dicke theory, will be discussed in Sec. 4.6. Such generalisations do not qualitatively alter the phenomena described in the next section, in which the lagrangian is kept as it is to simplify calculations.

4.5 The "Archimedes effect" and confinement

From the lagrangian density (4.4.13) it is seen that the effective mass of the field χ (which from now on will sometimes be referred to as the "quark" field) is

$$m_e = m(he^{-\lambda\psi}+1-h)^{\frac{1}{2}} \quad . \tag{4.5.1}$$

If a non-derivative quark self interaction was included then precisely the same remarks could be made for the corresponding coupling constant(s). In addition there is an overall ψ-dependent factor in front of the χ lagrangian density, adding an extra term to the χ field equations.

The field equations following from (4.4.13) are

$$\Box\psi = \tfrac{1}{2}\lambda h\ e^{-\lambda\psi}\left\{-\ \chi_{,\mu}\ \chi_{,\mu} + 2m^2(he^{-\lambda\psi}+1-h)\chi^2\right\} \tag{4.5.2}$$

$$= \lambda h\ e^{-\lambda\psi}(he^{-\lambda\psi}+1-h)^{-1}\ T^{(\chi)}_{\mu\mu} \tag{4.5.3}$$

and

$$\Box\chi - \lambda e^{-\lambda\psi}(he^{-\lambda\psi}+1-h)^{-1}\ \psi_{,\mu}\ \chi_{,\mu} + m^2(he^{-\lambda\psi}+1-h)\chi = 0 \quad . \tag{4.5.4}$$

It will be of interest to consider the (static) field ψ in the vicinity of a point source at the origin. Since $\Box\psi = 0$ in the absence of hadronic matter, such a field will be given by

$$\psi = \frac{c}{\lambda r} \quad , \tag{4.5.5}$$

which actually satisfies

$$-\ \nabla^2\psi = \frac{4\pi c}{\lambda}\ \delta^3(\underline{r}) \tag{4.5.6}$$

where c is a measure of the strength of the source. (The tacit assumption $\psi \to 0$ as $r \to \infty$ will be discussed later.) A point source may be realised as a singular contribution to $T^{(\chi)}_{oo}$, and must be positive if the source particle is of positive mass. Hence, since the factor multiplying $T^{(\chi)}_{\mu\mu}$ in (4.5.3) is positive (with h restricted to $0 < h \leqslant 1$), the constant c is positive.

The effective mass of the χ-field in the vicinity of this source is now given by

$$m_e = m(he^{-c/r} + 1 - h)^{\frac{1}{2}} \quad . \tag{4.5.7}$$

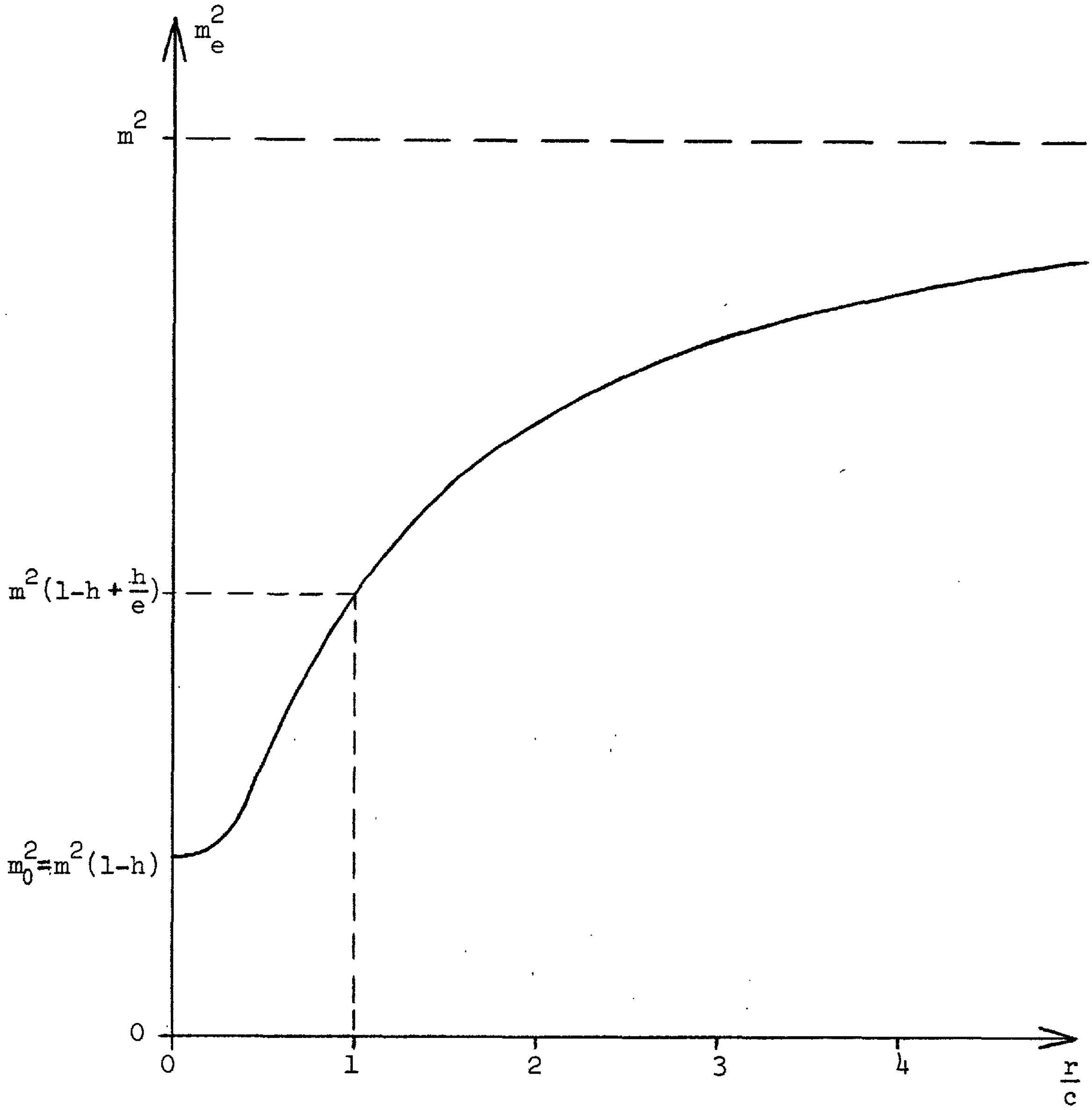


Fig. 4.1

The dependence of effective mass squared on the distance from a point source (sketched for $h = 0.8$).

This is sketched in Fig. 4.1. Such a reduction of mass in the vicinity of matter is characteristic of Brans-Dicke theory[63].

As $r \to \infty$, (4.5.4) becomes a free field equation for χ, with mass m. The most important feature, however, is the $r \to 0$ behaviour in which $m_e \to m(1-h)^{\frac{1}{2}} \equiv m_0$ and the χ field equation becomes

$$\Box\chi + m_0^2\chi = 0 \qquad \text{for} \quad h < 1 , \tag{4.5.8}$$

$$\text{or} \quad \Box\chi - \frac{ch}{r^2}\frac{\partial\chi}{\partial r} = 0 \qquad \text{for} \quad h = 1 , \tag{4.5.9}$$

this limit being reached exponentially fast. For $h < 1$ the field equation in the vicinity of $r = 0$ is simply that for a free field of mass m_0. For $h = 1$ the mass vanishes in this region but a strange first derivative term appears. Perhaps the biggest surprise is the total absence of any kind of singularity in (4.5.8) at $r = 0$.

Does this mean that two quarks in close proximity ignore each other except inasmuch as they lower each other's effective mass?

There is a convenient way of checking this effect by looking at it in a different way. We can ask what the path of a point-like test quark would be in the field of the source quark. Since the test quark lives in the world of $g^{(h)}_{\mu\nu}$ the classical paths we want to look at are precisely the timelike geodesics of the connection ${}^{(h)}\Gamma^{\alpha}_{\beta\gamma}$ corresponding to $g^{(h)}_{\mu\nu}$. They are solutions of the geodesic equations

$$\frac{d^2x^\alpha}{d\tau^2} + {}^{(h)}\Gamma^{\alpha}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{4.5.10}$$

where x^α are the particle coordinates and τ is the affine parameter of the geodesic.

For the sake of simplicity only radial geodesics will be considered since these provide all the information required at the moment. When the explicit form of the metric,

$$g^{(h)}_{\mu\nu} = (he^{-c/r} + 1 - h)\eta_{\mu\nu} \quad , \qquad (4.5.11)$$

is inserted the geodesic equations become

$$\frac{d^2t}{d\tau^2} + \frac{che^{-c/r}}{r^2}(he^{-c/r}+1-h)^{-1}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0 \quad , \qquad (4.5.12)$$

$$\frac{d^2r}{d\tau^2} + \frac{che^{-c/r}}{2r^2}(he^{-c/r}+1-h)^{-1}\left\{\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2\right\} = 0 \quad . \qquad (4.5.13)$$

It is possible to integrate once and eliminate τ, eventually yielding

$$\beta \equiv \frac{dr}{dt} = \left\{1 - k(he^{-c/r} + 1 - h)\right\}^{\frac{1}{2}} \qquad (4.5.14)$$

where k is an integration constant.

The final integration, which would allow r to be given as a function of t, is a bit daunting but in fact is unnecessary. The nature of the geodesics is easily deduced from (4.5.14). For $h < 1$ they are of six types (sketched in Fig. 4.2(a)):

(i) $k < 0$: spacelike geodesics.

(ii) $k = 0$: lightlike geodesics.

(iii) $0 < k < 1$: timelike geodesics reaching infinity.

(iv) $k = 1$: timelike geodesics "only just" reaching infinity.

(v) $1 < k < (1-h)^{-1}$: "confined" geodesics, turning back when $k(he^{-c/r}+1-h) = 1$.

(vi) $k = (1-h)^{-1}$: the geodesic $r = 0$.

The same categorisation applies to the case $h = 1$ (Fig. 4.2(b)) but now all geodesics except (vi) become lightlike at the origin.

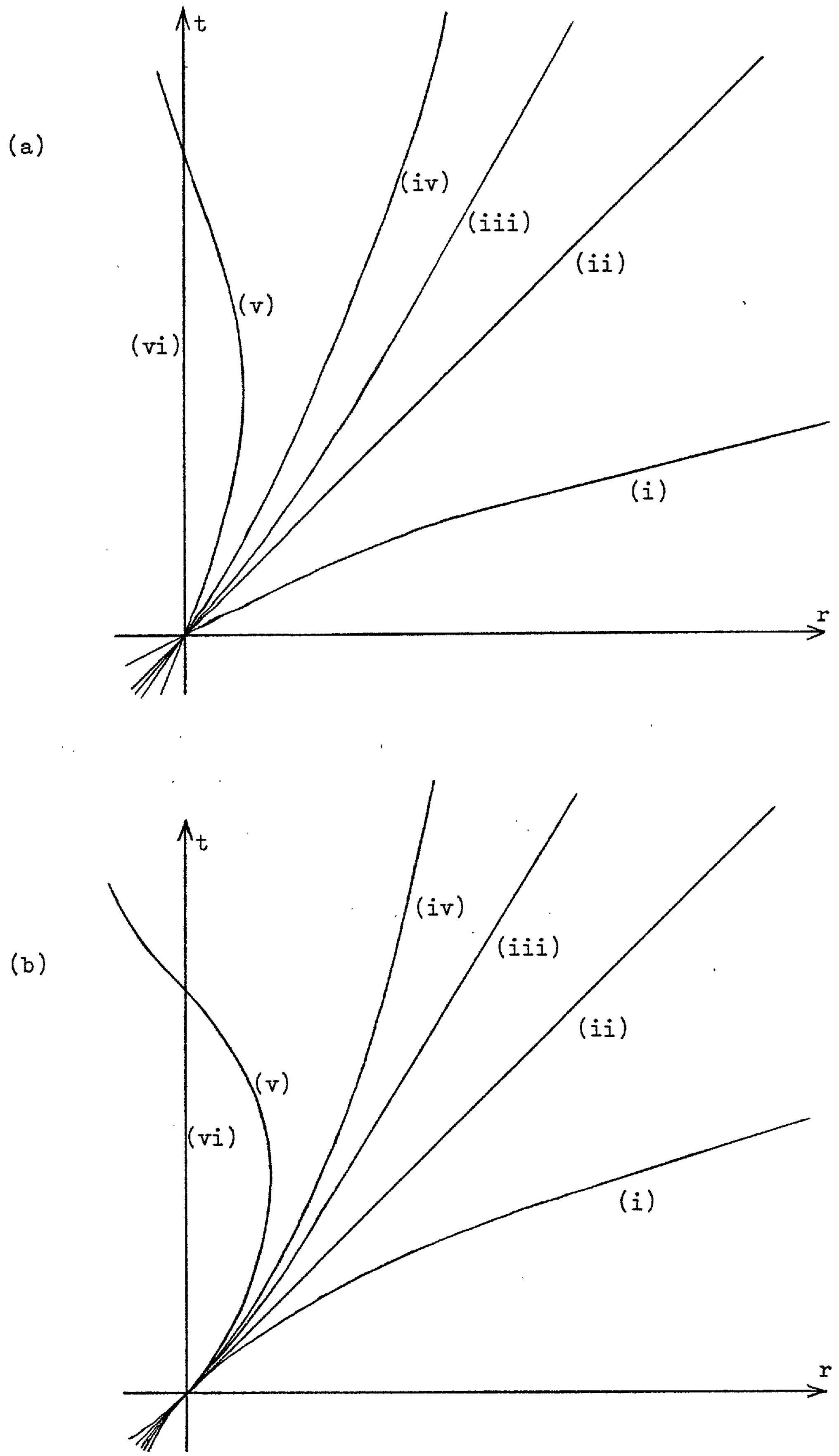


Fig. 4.2

(a) Geodesics of $g_{\mu\nu}^{(h)}$ for $h < 1$.

(b) Geodesics of $g_{\mu\nu}^{(1)}$.

We note that

$$\beta \rightarrow \beta_0 \equiv \left\{1 - k(1-h)\right\}^{\frac{1}{2}} \quad \text{as} \quad r \rightarrow 0 \quad , \tag{4.5.15}$$

the limit being reached exponentially fast. The geodesics are straight (from the Minkowski space viewpoint) in the vicinity of $r = 0$, corresponding to free particles.

It is not difficult to check that these paths can also be interpreted as the paths of particles in Minkowski space whose mass varies according to (4.5.7) and whose four-momentum is conserved.

Thus the behaviour of the χ field is corroborated by the nature of the motion of classical particles. The mass of a hadron in the vicinity of other hadronic matter is reduced from the point of view of the weak metric, which is Minkowski space in this case.

An alternative way of viewing this mass decrease, closer in spirit to the original Brans-Dicke interpretation of Mach's principle, is as follows. From the point of view of the hadron metric, the mass in the χ field equation is constant but the effective strong gravitational constant (the strong analogue of G), which from (4.4.3) is seen to be associated with $G\phi^{-1}$, decreases as $r \rightarrow 0$. Thus the "effective local strong Planck mass" increases. Now as discussed in Ref. 57 only mass ratios, not masses, can be compared at different points, and a natural way to obtain an unambiguous definition of the mass of a particle is to use its ratio to the Planck mass. The dimensionless "mass" defined in this way decreases as $r \rightarrow 0$.

The "confinement" exhibited here is of the "partial confinement" variety in which the free quark mass m may be very large while that of bound states can be very low, of the order of m_0. In (4.4.5) the boundary condition $\psi \rightarrow 0$ as $r \rightarrow \infty$ has been applied. This is somewhat ad hoc unless ψ is given a mass as discussed in the next section. In

Brans-Dicke theory itself a natural boundary condition appears to be $\phi \to 0$ as $r \to \infty$. From (4.4.14) this would seem to correspond to an infinite quark mass as $r \to \infty$, i.e. "total confinement". However, the idea is rather problematical since it implies $\psi \to -\infty$ as $r \to \infty$ so that an infinite constant must be subtracted from (4.5.5). Hence it will not be pursued further. It is notable that difficulties are also experienced in trying to apply this boundary condition in the Brans-Dicke theory itself[65].

Other models using scalar fields to produce partial confinement are to be found in Refs. 66-74. In particular Refs. 69 and 73 discuss the validity of the quasiclassical approximation in this type of model. Such an approximation would be the next step up from the simple classical methods used here.

4.6 Generalisations and internal symmetry

The hadron field ψ discussed above is massless. Since there are no observed massless hadrons, a useful simple generalisation of the theory would be to insert such a mass, attaching

$$\mathcal{L}_\mu = \tfrac{1}{2} \mu^2 \psi^2 \qquad (4.6.1)$$

to the lagrangian density (4.4.13). The boundary condition $\psi \to 0$ as $r \to \infty$ now has a proper physical basis, whereas before it was slightly ad hoc.

As regards the static solution in the presence of a point source, (4.5.5) is replaced by

$$\psi = \frac{c}{\lambda} \frac{e^{-\mu r}}{r} \quad . \qquad (4.6.2)$$

The main point to be made here is that the results of the previous section are not qualitatively altered. For small r the exponential in (4.6.2) is of no consequence and m_e still approaches m_0 exponentially fast. For large r the only difference is that m_e approaches m more rapidly than before.

Similarly a self interaction could be added if desired, especially if looking for "nontopological solitons"[70,72]. All these modifications correspond to special cases of the generalisations of the Brans-Dicke theory studied by Bergmann[60] and Wagoner[61].

The next stage in improved realism is to replace the "scalar quark" field χ by a spinor field (still denoted by χ). A vierbein corresponding to $g^{(h)}_{\mu\nu}$ must be found, and the natural choice is

$$L^{(h)}{}_a{}^{\mu} = (he^{-\lambda\psi} + 1 - h)^{-\frac{1}{2}} \delta_a^{\mu} \quad . \qquad (4.6.3)$$

The appropriate lagrangian density is then

$$\mathcal{L}_{\text{spinor}} = (he^{-\lambda\psi}+1-h)^{3/2} \left\{ \frac{i}{2} \overline{\chi} \gamma^{\mu} \partial_{\mu}\chi - m(he^{-\lambda\psi}+1-h)^{\frac{1}{2}} \overline{\chi} \chi \right\} . \qquad (4.6.4)$$

Note that the contribution due to the spin connection vanishes identically. The effective mass is related to m, h and ψ in exactly the same way as before.

Eventually the problem of internal symmetry will have to be faced but only a brief general discussion will be given here. The symmetry relevant to confinement is colour symmetry. The confinement forces should be such that only colour singlets are observed (at present day energies at least). In this theory there is an easy way and a difficult way to proceed.

The easy way is to regard ψ as a colour singlet and turn the hadron part of the theory into a colour gauge theory in the usual way.

As in Ref. 74 the idea would be to try to realise a suggestion made by Nambu, that the vector gluons should be used to unglue colour nonsinglets. If the vector gluons are given a mass by spontaneous symmetry breaking then the effective mass will have just the same behaviour as for other fields, arising in this case from the factor $(he^{-\lambda\psi}+1-h)$ in front of the derivative part of the Higgs scalar lagrangian.

The more difficult thing to try to do would be to have ψ transforming in a non-trivial way under colour symmetry transformations. One of the problems here is deciding whether to attach internal symmety indices to ψ, or to ϕ, or to ϕ^p for some power p. Then the problem of deciding which way to make contractions arises. Choosing to convert $\phi' \equiv \phi^{-1}$ into a U(3) nonet would give the scalar analogue of the U(3) version of f-g theory[21]. The scalar theory would be simpler of course. For example the matrix-valued strong metric ${}^{A}_{B}\tilde{g}_{\mu\nu} = {}^{A}_{B}\phi'\ g_{\mu\nu}$ would now have just one inverse namely ${}^{A}_{B}\phi\ g^{\mu\nu}$, symmetric in μ and ν, whereas in the f-g theory case there are two natural inverses, not symmetric in μ and ν in general. However, the problem of ordering the various factors in the lagrangian, before multiplying and taking traces, still remains. One advantage of this approach is that the gauge vectors could be given a mass using the nonvanishing vacuum expectation value of ${}^{A}_{B}\phi$ (which would be δ^{A}_{B}), without the necessity of introducing Higgs scalars especially for that purpose. Such a mass would in fact be position independent.

An alternative approach might be to try to extend the Weyl group of vierbein transformations, by analogy with the SL(6,$\mathbb{C}$) version of f-g theory[20].

In conclusion, it is fair to say that the inclusion of internal symmetry is a non-trivial task. Neither is quantisation straightforward since the theory is nonpolynomial. An important point worth stressing is that this nonpolynomial feature is essential to the phenomena described in Sec. 4.5. The function exp(-c/r) is finite at the origin but no polynomial in 1/r is.

CHAPTER FIVE

CONCLUDING REMARKS

This thesis has hopefully shown that strong gravity theories may be of considerable importance in hadron physics. Nevertheless, much work remains to be done since only the most simplified theories have been investigated in any depth so far. Of all the generalisations necessary, perhaps the most important is the introduction of internal symmetry.

It is of interest to know what happens to the results of Chapter Two when internal symmetry is included. Salam and Strathdee[75,13] have shown how one can use the classical solutions of the simplified theory to give solutions for a vierbein version of f-g theory with the vierbein invariance extended from $SL(2,\mathbb{C})$ to $SL(2,\mathbb{C}) \times SU(2)$. In these extended solutions the triplet part of the vierbein, $L_i^{a\mu}$ $(i = 1,2,3)$, may be written in the form $n^i L^{a\mu}$ where n^i points in a fixed direction in isospin space. It would also be interesting if a monopole-like solution, with $n^i \sim x^i/r$, could be found.

Even as they stand the results of Chapter Two invite further questions. For example, how can the manifold on which $f_{\mu\nu}$ and $g_{\mu\nu}$ are defined be analytically extended? One seeks an analogue of the well known extensions found for the standard solutions in general relativity. Another important topic is the stability or otherwise of the f-g solutions, and an investigation of this problem has been begun by Baran[76]. It would also be useful to know more about the Type II solutions, whose explicit form has proved so elusive.

A merging of f-g theory with another recently developed theory, namely supergravity, may well prove advantageous. Particles of other spins, notably fermions, would thus be introduced. Whereas the f-g theory itself attempts to unify gravity with strong interaction forces there would now be the possibility of incorporating the weak and electromagnetic interactions. An amusing fact, whose significance is not yet clear, is that anti-de Sitter spacetime makes an appearance in both f-g theory and certain supergravity models. A beginning in such a merging of the two theories has in fact already been made[77]. One encouraging finding is that the requirement of supersymmetry seems to reduce the arbitrariness of the f-g mixing part of the lagrangian.

When quantisation is considered, all of the theories described in this thesis suffer in the same way. They are all nonpolynomial and nonrenormalisable. This is an inevitable consequence of their kinship with gravity. Although techniques are now being developed to cope with such difficulties it is likely to be a long time before there is a complete, consistent, quantum theory of gravity. Should such a theory eventually appear then to find out whether or not it is the "correct" theory will be an awesome problem. This brings us back to a remark made in Sec. 1.1. It may be that the only hope of experimentally testing quantum gravity lies in strong gravity theories.

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