PARTIAL WAVE ANALYSIS OF \( \pi \pi N \)

FINAL STATES FROM \( \pi^+ p \) INTERACTIONS AROUND 1 GeV

by

Robert Stevens

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of Doctor of Philosophy of the University of London

Department of Physics
Imperial College
London S.W.7

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ABSTRACT

This work describes a study of positive pion proton interactions, performed by the author, at around 1 GeV/c pion laboratory momentum. The data was obtained during two hydrogen bubble chamber exposures at the Rutherford Laboratory.

The major part of this thesis concerns a very detailed analysis of the final states $\pi^+ p \pi^0$, $\pi^+ \pi^+ n$ in the incident pion momentum range $895 - 1040$ MeV/c. The Isobar Model has been used as a framework to fit a partial wave series to both these channels simultaneously, the method employing all the variables required to specify such final states and the correlations between them. Amplitudes and phases of the various transitions fitted are presented and their resonance structure analysed. In particular strong resonant behaviour is seen in the $S_{31}$ and $D_{33}$ waves as found in elastic analyses but in addition a strong signal in the $D_{35}$ wave was noticed.

New data in the range $1100 - 1400$ MeV/c has been examined and reaction cross-sections are detailed. The differential cross-sections for elastic data in this range are presented and these fitted by a partial wave series. Finally simple phenomenological studies of the single pion production channels are discussed.
And what you thought you came for
Is only a shell, a husk of meaning
From which the purpose breaks only when it is fulfilled
If at all.

Little Gidding
T.S. Eliot
And what you thought you came for
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Little Gidding
T.S. Eliot
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INTRODUCTION

Film on this experiment was taken in three separate hydrogen bubble chamber exposures at the Rutherford Laboratory. The first of these in 1966 used the K1 beam line and Saclay 81 cm. bubble chamber: pictures were taken at 4 separate nominal pion laboratory momenta 895, 945, 995, 1040 MeV/c. The kinematic reconstruction of events in this batch was undertaken jointly by Imperial and Westfield Colleges, London. The remaining exposures employed the K9 beam line to transport positive pions to the 1.5m. British National Hydrogen Bubble Chamber. The original collaboration was expanded to include Cambridge University and pictures were taken at 1.1, 1.2, 1.3, 1.4, 1.45, 1.5, 1.55 GeV/c and 0.80, 0.85, 1.15, 1.25 GeV/c respectively. The distribution of this film between the various collaborators is shown in Table I. and the Westfield and Cambridge data handling is described in References 4 and 5.

At the time of writing the experiment was still incomplete in that data from the last of the exposures remained largely in the form of raw film measurements as did the bulk of the data at 1.45, 1.5, 1.55 GeV/c. These measurements are currently being digested by the various kinematics programs (qv.) and will figure in another forthcoming thesis by Chris Mier-Jedrejowicz.

As for the remainder the events in the first exposure have undergone the rigours of identification and reconstruction.
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*ABANDONED

Table I
and are now on Data Summary Tapes: the procedures are
described and thoroughly discussed in the Ph.D. theses
of the time$^{1,2}$. The Imperial College share of the data
between 1.1 and 1.4 GeV/c was largely the concern of the
author, and the contribution made by him in the kinematic
analysis stage was as follows:
Scanning, measurement, computing of 1.3 and 1.4 GeV/c
momenta and similar processing of the remeasures at 1.2
GeV/c; merging of the DST's prepared by the various col-
laborators at all the momenta 1.1 to 1.4 GeV/c and prelimi-
nary checks to discover any discrepancies and differences
between the various institutions, both in terms of philo-
sophy and errors which might have been present. The study
of the data described in this thesis were almost entirely
the author's own work and included:
(1) Determination of cross-section for the various reaction
channels and their correction for any biases discovered.
(2) Angular momentum analysis of the elastic interactions,
the correction of the forward elastic data for losses,
the determination of the backward differential cross-
sections for such scatters and the comparison of these
with other published determinations. The programs
used to determine the legendre coefficients in the
expansion of the differential cross-section were
standard and not written by the author.
(3) Preliminary phenomenological study of the data between
1.1 and 1.4 GeV/c as described in Chapter VI.
Detailed study of the single pion production channels as described in Chapters IV, V. The work presented in Chapter V is entirely original. Of that in the proceeding chapter the fitting of the $\pi^0\pi^+p$ and $\pi^+\pi^+n$ mass plots employed programs not written by the author but the debugging and correcting of such programs, detailed examination and corrections of the underlying theory and the correct application of such to the data were carried out personally.

Theoretical Background and Motivation of the Experiment

The idea of isospin and the assignment of the particles observed in nature into well defined $I$-spin 'multiplets' arose from the experimentally observed fact that strong interactions are charge independant; an example is the approximate equality of proton and neutron masses. Translating this into the language of group theory, an invariance principle (charge independence) may be described by a group of symmetry operations (SU(2)) and the Hamiltonian $H$ of the strong interaction then commutes with all the group generators $\{I_k, k=1,2,3\}$

$$[H, I_i] = 0$$

Taking appropriate combinations of the group generators one is able to construct operators which commute with every generator: in SU(2) there is only one such operator I defined by
\[ I^2 = I_1^2 + I_2^2 + I_3^2 \]

\[ \{I^2, I_k\} = 0 \quad k = 1,2,3 \]

The irreducible representations of the group may then be labelled by the I-spin as this is invariant to any group operation, and the isospin multiplets of particles \((p\pi, \pi^+,\pi^0)\) are then identified with these irreducible representations. Other labels than the I-spin are however required to label these representations because particles in nature are described by other quantum numbers such as spin, parity, strangeness \((J,P,S)\), and these labels do not correspond to any of the SU(2) operators i.e. SU(2) is not the complete symmetry one seeks to describe the strong interaction.

A great advance was made when it was discovered that the hypercharge \(Y(=B+S)\) could be incorporated with I into a higher symmetry scheme described by the group SU(3) and that strong interactions were still invariant under this symmetry, at least to a good approximation. The only extra labelling of the particle representations required is \(J^P\) and one finds that mesons fall into the singlet and octet representations, baryons into singlets, octets and decuplets.

Practically SU(3) is not such a perfect symmetry as SU(2) (even within a particular \(J^P\) multiplet for example quite large mass splittings may be observed), but its predictive power is still very useful particularly if one regards the SU(3)-breaking interaction as a perturbation on
a stronger SU(3) invariant interaction and makes simplifying assumptions as to the nature of this. Most of the predictions concerning such diverse observations as masses, magnetic moments, couplings etc. have now been validated experimentally and the agreement is extraordinarily convincing.

A question arises as to why the observed particles should lie only in singlets, octets and decuplets. A possible explanation is the so-called quark model. There are two fundamental representations of SU(3) denoted \( \{3\} \), \( \{\bar{3}\} \). By the group properties one may show

\[
\{3\} \oplus \{\bar{3}\} = \{1\} \oplus \{8\}
\]

\[
\{3\} \oplus \{3\} \oplus \{3\} = \{1\} \oplus \{8\} \oplus \{8\} \oplus \{10\}
\]

The quark model is that the representation \( \{3\} \) may be identified with 3 hitherto undiscovered particles 'quarks': the \( \{\bar{3}\} \) is then identified with the anti-quarks. The mesons would then be composed of a quark-antiquark pair and the baryons of a 3-quark system (anti-baryons 3 antiquarks).

SU(3) by no means exhausts the range of higher symmetries appropriate to particle physics. Leaving behind pure group theory for the moment and considering quarks as physical particles a reasonable assumption would be that they are fermions with spin \( \frac{1}{2} \). Then the mesons would automatically have integral spin and the baryons half integral spin. Instead therefore of our basic representation of dimension \( \{3\} \) each quark has itself two degrees of freedom,
spin-up and spin-down, and leads to a representation carried in a 6-dimensional spinor-space. The group describing this is SU(6). One may write

\[ \{6\} \times \{\bar{6}\} = \{1\} + \{35\} \]

\[ \{6\} \times \{6\} \times \{6\} = \{20\} + \{56\} + \{70\} + \{70\} \]

These larger representations may be written explicitly as a combination of spin-unitary spin. For example

\[ \{56\} = \{10, 4\} \oplus \{8, 2\} \]

10 - SU(3) decuplet
4 - (2J+1) spin \(3/2\) rep \(n\) of SU(2)
8 - SU(3) octet
2 - SU(2) doublet

and one sees that the \(\{56\}\) is composed of the \(3/2^+\) decuplet \((\Delta, Y, Z, \Xi, \Omega)\) and the \((N, \Sigma, \Xi, \Lambda)\) \(1^+\) octet. Similarly the \(\{35\}\) meson is made up of the \(0^-\) and \(1^-\) pseudoscalar and pseudovector meson multiplets. This and other unifications of the particles and their properties make SU(6) a very successful scheme for classifying the mesons and baryons. However unlike SU(3), SU(6) is \underline{not} a valid symmetry for describing particle-particle interactions, not even approximately. It predicts for example that the decay of the \(\Delta(1236)\) into nucleon-pion is forbidden.

\[ \Delta \not\to N\pi \]
Quark spin $\frac{3}{2}$ object cannot decay to a quark spin $\frac{1}{2}$ and quark spin 0 objects.

Similarly the decay
\[ \rho \rightarrow \pi\pi \]
is also forbidden.

This is not surprising. In the physical world it is not spin $S$ that is conserved but $J$ and furthermore the decomposition of $J$ into a spin and an orbital part is not valid since these are not relativistically invariant. One might expect therefore that in any situation where the orbital motion of quarks becomes important SU(6) would fail. This led Lipkin and Meshkov\textsuperscript{44} to consider an SU(6) symmetry in which the spin operators $S_i$ are replaced by operators $W_i$ having identical commutation relations but admitting a relativistic interpretation. These operators act in $W$-spin space and the SU(6) generated is referred to as SU(6)$_W$.

SU(6)$_W$ is a viable vertex symmetry in that it allows the decays $\Delta \rightarrow N\pi$, $\rho \rightarrow \pi\pi$ and is expected to be applicable for collinear processes. (i.e. Any 1 + 2 body decay since this is always collinear in a suitably chosen reference frame.) There are still notable discrepancies with experiment however, one of the most spectacular being the $B \rightarrow \omega\pi$ meson decay, where SU(6)$_W$ predicts pure $\lambda = 0$ amplitudes whereas $\lambda = \pm 1$ dominate\textsuperscript{45}. Essentially the good and bad SU(6)$_W$ predictions arise because the model predicts $\Delta S_z = 0$ (quark spin) and consequently that $\Delta L_z = 0$ also. This is not expected to be a good approximation when transverse
motion of the quarks inside the hadron is important. The allowance of $\Delta L_z = \pm 1$ transitions has given rise to a spate of broken $SU(6)_W$ models, some of which have a definite prediction of the magnitude of the breaking, some of which allow the ratio $\Delta L_z = \pm 1: \Delta L_z = 0$ to be a free parameter determined by experiment.

The motivation of studying the single-pion production channels is severalfold. In the first place many of the resonances in the $70^-$ multiplet produced in $\pi N$ collisions are expected to be strongly inelastic so that corroboration of their existence (determined by elastic analyses) requires analysis of the $\pi\pi N$ channels. Furthermore the strong two-body couplings present in the $\pi\pi N$ states ($\rho, \Delta$ etc.) should enable one to determine the partial widths and phases of the transitions. Many of the $SU(6)_W$ schemes make firm predictions as to the coupling signs in

$$70^1- \rightarrow 56^{0+} + 35^{0-}$$

decays, and comparison of these with experiment will enable many of the schemes to be ruled out.

Whether $SU(6)_W$ provides a consistent explanation of the physics involved is the current goal of the major partial wave analyses and in particular is the ultimate aim of much of the work described in this thesis.
CHAPTER I

1.1 Introduction

This chapter describes some features of the Imperial College scanning-measuring-kinematics chain used to identify the various $p^+p$ interactions seen on the film. Much of the discussion is somewhat brief as this system had already been turned to high fidelity before the intervention of the author. The topics covered are mainly those which are less well known and have only briefly been described elsewhere; in particular some quirks peculiar to the Imperial College part of the $p^+p$ experiment.

1.2 Scanning and Digitising

All the film at 1.2, 1.3, 1.4 GeV/c was initially prepared for measurement on the I.C. flying spot digitiser (HPD). The complete flow diagram for film processing is displayed in Fig. 1.1. The film has first to be scanned for all the interactions of interest and these must then be rough digitised to provide guidance for the HPD. In fact these tasks are combined and performed simultaneously. Only those events visible on all three views within the fiducial volume are digitised, this volume being chosen so that a sufficient length of all the tracks from an interaction could be measured to give good chance of kinematic reconstruction. (In practice about 10 cms.) Rough digitising includes measuring and labelling all points of interest (primary and secondary interactions, decay points, the end
points of stopping tracks), two predetermined fiducials and two points per track. Secondary scatter vertices are not usually measured unless the length of the track connecting them with the primary is so short that the sagitta is \(< 1\) cm.: it would give poor momentum reconstruction if the sagitta were less than this. The data from these rough digitisations is then tidied and edited by various computer programs before it is converted to HPD instructions by the CERN program MIST\(^6\).

Further discussion of any scanning biases inherent in this procedure is postponed until later in this chapter.

1.3 Kinematic Processing

The HPD measures the three views separately and these measurements are then merged by program SMOG to produce a single input tape to THRESH\(^6\). The operation of the CERN kinematics program THRESH\(^6\) and GRIND\(^6\) is too well known to require much exposition here. The magnetic field in the experiment was determined by fitting tracks from \(K_s^0\) decays until the mass agreed with the \(K^0\) mass. This was performed in another I.C. experiment using \(K^+d\) interactions. The geometry constants were determined by fitting, with these as parameters, many measurements of fiducials made by the HPD to the known position of these fiducials as measured on the chamber. This was performed by the program ADDER\(^7\).

The program GRIND has the facility to impose a central beam momentum, dip, and azimuth at the beam entry window to
the bubble chamber. These values are then 'swung' to the primary vertex in a manner which takes into account, for example, the changing dip and the retardation of the beam due to ionisation losses. The beam momentum is weighted towards these values from those measured in the hypothesis fitting of GRIND, and is also used to impose starting values in those cases where the beam track has failed to reconstruct. This latter is important since 20% of all beam tracks fail, usually because they are too short for adequate momentum determination. The procedure adopted to obtain these central beam values was as follows:

(1) A reference point was chosen with coordinates \( (x_D, y_D, z_D) \) to which all measurements may be referred, the beam being described at this point by momentum, dip, azimuth \( P_D, \lambda_D, \phi_D \). The values of these kinematic variables at the interaction point \( (x, y, z) \) are then given by

(a) \( P = P_D + (x-x_D) \times 0.00024 \)

Where the constant represents the energy lost in GeV/cm. by pions in liquid hydrogen.

(b) \( \lambda = \lambda_D + c_z (z' - z_D) \)

\( c_z \) is a parameter to be determined and \( z' \) is the \( z \) coordinate of the point at which the track intersects the \( x = x_D \) plane.

(c) \( \phi = \phi_D + c_y (y' - y_D) + c_L L_D \)

\( y' \) is the \( y \) coordinate of the intersection of the track with the plane \( x = x_D \).
c_y and c_L are to be determined; c_L = \langle \rho \rangle \text{ where } \rho 
newline is the curvature of the beam track, and L_D is the distance between the vertex of the event and the plane x=x_D.

These formulae arise in a natural way out of the beam optics involved. An ideogram of \frac{1}{\rho} is plotted and the mean and standard deviation found. This determines c_L and then scattergrams of (\lambda \text{ vs. } z') and (\phi-c_L L_D \text{ vs. } y') determine c_y and c_z respectively. Finally the mean values of P_D, \lambda_D, \phi_D are determined from ideograms of these quantities.

GRIND is run on a sample of the data both with and without the beam block present, to check the consistency of the parameters. Finally plots are made of the stretch function for 1c and 4c fits to check for possible biases at this early stage, thus reducing the work involved should these be found to be incorrect.

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<td></td>
<td></td>
<td>$\pi^+ p \rightarrow \pi^+ n$</td>
<td>103</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$pp \rightarrow pp$</td>
<td>004</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$pp \rightarrow \pi^+$</td>
<td>104</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\pi^+ p \rightarrow \pi^+ n$</td>
<td>105, 106</td>
</tr>
<tr>
<td>100</td>
<td>51000 S</td>
<td>$\Sigma^+ \rightarrow \pi^+ n$</td>
<td>101</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$p \rightarrow \pi^+$</td>
<td>102</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k^+ \rightarrow \pi^+ \pi^0$</td>
<td>103</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\mu^+ \nu$</td>
<td>104</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\pi^+ \rightarrow \mu^+ \nu$</td>
<td>105</td>
</tr>
<tr>
<td>200</td>
<td>51010 S</td>
<td>$\Lambda^0 \rightarrow p \pi^-$</td>
<td>001</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k^0 \rightarrow \pi^+ \pi^-$</td>
<td>002</td>
</tr>
</tbody>
</table>

*Table 1.2*
For those cases where the measured beam momentum is more
than 3 standard deviations from the central value GRIND
makes no use of the beam titles. This can result in
events being fitted which are significantly off beam as
well as 'fitted' events with a beam other than a pion,
but such events may be cut out of the final data sample
by judicious cuts on the incident pion momentum. The
values used in the GRIND 'BEAM' title block are given in
Table 1.1.

The mass hypotheses of the various event topologies
are given in Table 1.2 together with the type and hypothesis
numbers used by GRIND to identify such candidates and the
possible track permutations.

1.4 Auto and Hypothesis Selection

The philosophy of the $\pi p$ experiment has been that no
event should be rejected, for whatever reason, until the
stage of checking the various ambiguous and failing events
on the scan table. Thus even those events failing for
reasons as trivial as a format error appear in the output
at the choosing stage. This obviates the necessity for
too-elaborate bookkeeping and at the same time enables
possible errors and biases in the data handling to be more
easily discovered.

AUTO examines the possible GRIND fits to an event
and makes more stringent tests to decide between these.
In particular if ionization information is present a fit
to the bubble density of these tracks is made in a way which is more fully described in Ref. 3.

The logic of AUTO is as follows.

(1) Check whether event has failed.

This may be because of a data format error or perhaps because a track could not be reconstructed in THRESH. Such events go for remeasurement.

(2) Proceed to make the ionisation fit for all the possible mass assignments to a given track, for all tracks. The conditions under which AUTO is able to make this fit are described fully in Ref. 3.

(3) If no mass assignment has a probability to produce the ionization observed \( > 6\% \) then no further ionization comparisons are made. For those cases where ionization information is available the probabilities for the hypothesis on the 3 different views are multiplied together. By comparing these various probabilities \( P \) for different hypotheses AUTO selects that ionization fit with the greatest probability. It then rejects any hypothesis \( i \) such that

\[
\frac{P_i}{P_{\text{max}}} < 0.025
\]

This logic is not statistically justifiable but was found empirically to produce suitable criteria for deciding between conflicting fits. In addition to these ionization decisions AUTO also makes further checks on the kinematic fits:
(1) Reject any inelastic hypothesis where, because 1 physical quantity is missing (e.g. momentum for a short track), the fit was downgraded from 1c to a 0c solution.

(2) For an event to quality as a nofit

$$MM2 + 3\Delta(MM2) \geq (M^2_{\pi^0} + M^2_{\eta})$$

where $M_i$ is the mass of the neutral particle for the hypothesis under consideration.

(3) No 1c fit is accepted with a probability $< 10^{-3}$. No 4c fit is accepted with probability $< 10^{-9}$. This latter cut off is several orders of magnitude lower than is quoted typically for a bubble chamber expt. In practice it was found to make a difference of only 1 or 2 events per roll and the 4c fit was invariably correct.

On the basis of this (simplified) logic AUTO classified all events into the following categories -

(1) Pre-Thresh Failure.

(2) Bad Test.

In THRESH one track, other than the beam, completely failed to reconstruct.

(3) Less than three views.

Some experiments accept events if they are well measured on two views but fail on the third. However, because of the otherwise high statistics available, it was decided to remeasure events of this type and so avoid a possible bias if a track happened to lie along the line joining two of the cameras.
(4) Decided.

(5) Neither fits nor nofits found.

No hypothesis satisfying all the tests was found. These events are examined on the scan table, particularly to check on the correctness of the ionization test. The physicist can overrule the program on the basis of a judicious 'eyeball' fit.

(6) Check Ionization.

AUTO was unable to decide between two (or more) possible kinematic fits on the basis of ionization. The physicist decides upon a particular hypothesis or several fits are written to the data summary tape each with a suitably calculated weight.

(7) 4c Candidate.

A 4c fit was rejected in preference to a lc fit on the basis of ionisation or the 4c fit was rejected in GRIND as the initial probability was slightly too low.

AUTO punches a card for every tenable hypothesis and these are edited in the light of physicist intervention. Cards for those hypotheses required on the DST are passed to SLICE which writes a DST in the format requested. Those events which have been classified as remeasured are passed to REINS which produces a list of events with the coordinates and topologies for each. These are conventionally measured and the data re-cycled through the system with minor changes (see Fig. 1.1). Any events which still do not reconstruct
are rejected and the rest are merged with the original data sample.

1.5 Ambiguities and Biases

The methods just outlined have been developed over several years to maximise the efficiency of event reconstruction both in reducing the number of physicist hours required and in the reliable identification of interactions. Biases do however still exist; these are highlighted below and in addition some details presented of situations where the programs have been found to make unequivocally correct choices.

(a) $p\pi^+\pi^0$ vs. $\pi^+p$

These two types of event can be resolved at the energies involved. This is demonstrated in a plot of MM2 vs. missing energy for both channels lumped together and the elastic channel in isolation (Figs. 1.2 and 1.3). It is seen that there is very little overlap in this plot thus confirming that our momentum resolution is good enough to distinguish these events.

(b) $\pi^+p\pi^0$ vs. $\pi^+\pi^+n$

This ambiguity can nearly always be resolved on ionization information alone since throughout almost the entire momentum range of the secondary tracks the proton is several times darker than the pion. However if the ambiguity involves a slow track then this is sometimes not possible, but in this case a pion is often seen to decay or else the measured momentum would
be inconsistent with the range-momentum distribution of a proton. Mass Dependant Thresh (MDT) has also helped in removing ambiguities in this region.

(c) $\Sigma^+ K^+$

$$\begin{cases} p\pi^0 \\ n\pi^+ \end{cases}$$

The threshold for this channel corresponds to an incident pion laboratory momentum of approximately 1.02 GeV/c. The $\Sigma$ then decays within 1-3 cms giving such events a very definite signature but as the $\Sigma$ track has an unmeasured momentum these are difficult to fit in the latter stages of the processing. In particular Westfield College and Imperial College used a slightly faulty version of GRIND which made these particular difficult to handle. Westfield abandoned all such events and certain corrections to establish the cross-section for this reaction were necessitated (q.v.).

(d) 4-prongs

It was found that when film was conventionally measured ambiguities occurred between $p\pi^+\pi^+\pi^-$ and $p\pi^+\pi^+\pi^-$ and also between $p\pi^+\pi^+\pi^-\gamma$ and $p\pi^+\pi^+\pi^-\pi^0$. The fairly large track measurement errors make it possible to swim the momenta during fitting so that the extra neutral may be incorporated. This problem did not exist to an appreciable extra with film measured on the H.P.D.

At Imperial College the 4c fit was almost invariably
chosen and in those cases where 2 lc fits were possible both were included as possible interactions, each with weight one half.
Elastic & πρπ events at 1.3 Gev/c lab. momentum

Fig 1.2
Elastic events at 1.3 GeV/c.

Fig 1.3
CHAPTER II

2.1 Introduction

Data from the three collaborators was merged to produce one final DST. Described in this chapter are the preliminary studies made of such data. In particular the data from the three separate institutions was studied separately to discover any obvious discrepancies between them. Cuts were made on the data to separate genuine $\pi^+p$ interactions from possible contamination by 'off beam' particles, and the data then examined for any possible systematic biases in the kinematic fitting or misidentification of interactions. Finally the procedure adopted to obtain cross-section is described. This was somewhat involved owing to certain slight differences in priorities at the various institutions.

2.2 Studies of Simple Kinematic Observables and Examination for Systematic Effects

The distribution of beam momentum shows typically a gaussian shape with a spread of about 16 MeV/c. However the tail of the distribution is very long and certainly contains some events where the interaction was initiated by an 'off beam' track. These of course should have been ignored at scan time but some contamination is inevitable. For all these reasons it is desirable to make judiciously chosen cuts on beam momenta and for similar reasons a cut was also made on the beam dip. Those eventually decided upon
are presented in Table 2.1. The beam momentum profile after removal of events in the wings is presented in Figure 2.1 and shows a gaussian shape slightly distorted towards lower momenta. This is due to energy loss of beams in the chamber, and may be understood by recalling that a beam with a perfectly defined momentum will become spread over a lower momentum range as it travels through the hydrogen medium, and a long tail of low momentum beam tracks will be seen. The profile presented in 2.1 is typical of that observed at other momenta. Having made these cuts the data was then examined for systematic biases in the processing in the following ways.

(a) The probability distributions for the elastic and single pion production channels is plotted in Fig. 2.2. The expected distributions are flat and the experimental distributions too are flat apart from an excess of very low probability events. This is not due to misidentification of events for two reasons: the \( \pi^+p \) and \( \pi^+p \pi^0 \) events occupy different regions in the MM2 vs. ME space (Fig. 1.3), and plotting the angular distributions for the lowest probability bins does not lead to distributions significantly different from the total data samples.

(b) The stretch function for a variable \( X \) and standard deviation \( \sigma \) defined by

\[
S_x = \sqrt{\frac{\sigma_M^2 - \sigma_F^2}{\sigma^2}}
\]

where \( \sigma_M^2 \) and \( \sigma_F^2 \) are the variances for the measured and fitted distributions respectively.
is expected to be normally distributed with a mean of zero and standard deviation of unity for any variable $x$ which is normally distributed. This quantity is plotted for the momentum, dip and azimuth of beam tracks for events which reconstructed as 4c fits. Any tendency for the measured quantities to be swum asymmetrically will show up in a non zero mean. This might happen, for example, if the central beam momentum had been incorrectly determined or if the THRESH geometry constants had been incorrectly determined with the result that the chamber would appear distorted. Similarly an underestimation of the measurement error on a track would tend to make the stretches too broad. As is apparent the stretches are not significantly distorted (Fig. 2.3).

(c) Elastic Coplanarity Check.

Momentum conservation requires

$$P_{\pi_i} - (P_{\pi_f} - P_{p_f})$$

to be zero for any elastic event. This quantity is histogrammed in Figure 2.4 with the measured momenta inserted into the formula for all events which subsequently were fitted as 4c elastics as well as for a subset of such events with probabilities less than 0.04. The latter is wider as expected since the sample contains a higher proportion of events with a substantial momentum imbalance, and the measured variables will have to be swung more to satisfy moment-
tum-energy conservation. The mean of both distributions is zero and there is no apparent skewness, so one is reasonably confident that this sample is unbiased.

(d) Scattergrams and histograms of $MH^2$ and ME for the elastic events have already been given in Fig. 1.3. The slightly negative mean of the former is expected.

2.3 Cross Sections

The ivory tower principle has resulted in discrepancies between the various collaborators in the scanning and measuring procedures adopted. As a consequence the distribution of events in the various channels is not identical in the various subsamples from different institutions. In particular $E\gamma$ events were not measured by Westfield College and Cambridge University made no distinction between the various categories of multineutral events, nor did all these events arrive at the DST. The makeup of the DST's at various momenta is different and consequently different procedures were adopted to determine the various partial cross sections for all interaction types observed. (Table 2.2 gives a breakdown of the data into the institutions responsible.) In all cases however the total number of events was normalised to the total cross section measurement of Carter et al.: thus it was not necessary to make explicit correction for contamination of the beam by muons.

It is well known that there exists a dearth of events
Table 2.2

<table>
<thead>
<tr>
<th>Final State</th>
<th>NOMINAL $p_{LAB}$ $\pi^+$ GeV/c</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.100</td>
</tr>
<tr>
<td>$\pi^+p$</td>
<td></td>
</tr>
<tr>
<td>I.C.</td>
<td>-</td>
</tr>
<tr>
<td>W.C.</td>
<td>1778</td>
</tr>
<tr>
<td>CAMB.</td>
<td>1139</td>
</tr>
<tr>
<td>$\pi^+p\pi^0$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1450</td>
</tr>
<tr>
<td>$\pi^+\pi^+n$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>310</td>
</tr>
<tr>
<td>$\pi^+p\eta^0$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>23</td>
</tr>
<tr>
<td>$\pi^+p\pi^0$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>94</td>
</tr>
<tr>
<td>$\pi^+\pi^+\pi^-\pi^-$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>91</td>
</tr>
<tr>
<td>$\pi^+\pi^+\pi^-\pi^0$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>$\pi^+\pi^+\pi^-\pi^+$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>56</td>
</tr>
<tr>
<td>$\pi^+\pi^+\pi^-\pi^-\pi^-$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>-</td>
</tr>
<tr>
<td>$\Sigma^+K^+$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>
in the forward elastic scattering region because of the impaired scanning efficiencies at these angles. This is discussed fully in the next chapter where the methods adopted to correct for these biases are detailed, but the cross sections presented in this chapter have been corrected for this effect.

(a) 1.1 GeV/c

Data here was prepared by Cambridge and Westfield. The ΣK events in the Westfield sample were not measured and the various types of multineutral in the Cambridge sample were not distinguished nor were all of them measured. It was decided to use the Westfield data alone in determining cross sections. A further complication arose here out of the ambiguities observed between the final states $p\pi^+\pi^\pm\pi^0$ and $p\pi^+\pi^\pm\gamma$ in conventionally measured film, as discussed in Chapter I. It was assumed in the following that the 4c fit was correct (to conform to the I.C. treatment), and this was the philosophy adopted at all other momenta. Table 2.2 summarises the ambiguities. In the event of an ambiguity between $p\pi^+\pi^\pm\gamma$ and $p\pi^+\pi^\pm\pi^0$ each was given a weight of a half. An estimate of the ΣK events lost was made from the Cambridge data weighted by the ratio of the data samples at the two institutions - hence the large error on the ΣK cross section. Results are presented in Table 2.3.
### Table 2.3

<table>
<thead>
<tr>
<th>PLAB + GeV/c</th>
<th>AMBIGUITY</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p\pi^+\pi^-\pi^+/\pi^+\pi^-\pi^+\gamma$</td>
</tr>
<tr>
<td>1.1</td>
<td>CAMBRIDGE</td>
</tr>
<tr>
<td></td>
<td>WESTFIELD</td>
</tr>
<tr>
<td>1.2</td>
<td>IMPERIAL</td>
</tr>
<tr>
<td></td>
<td>WESTFIELD</td>
</tr>
<tr>
<td>1.3</td>
<td>IMPERIAL</td>
</tr>
<tr>
<td></td>
<td>WESTFIELD</td>
</tr>
<tr>
<td>1.4</td>
<td>CAMBRIDGE</td>
</tr>
<tr>
<td></td>
<td>IMPERIAL</td>
</tr>
<tr>
<td></td>
<td>WESTFIELD</td>
</tr>
</tbody>
</table>
(b) 1.2 GeV/c

Imperial/Westfield. The complete sample was used to determine cross sections; this momentum had been carefully studied in the original de-bugging of AUTO$^3$.

(c) 1.3 GeV/c

Imperial/Westfield. The scan lists for two rolls were thoroughly examined and it was discovered that, of the events failing to reconstruct, there was a bias towards 4-prongs and strange events. After further detailed examination of the scan lists and final DST sample an estimated correction was applied to such data.

(d) 1.4 GeV/c

The major bulk of this film ($> 60\%$) was prepared by Cambridge. As the Imperial College data is awaiting completion by addition of the remeasurement events, and the Westfield College data alone is so meagre, the cross section quoted are those determined by Cambridge on their data sample alone$^5$. These are included for completeness.

The total cross section quoted at each of these momenta were obtained by interpolating the measurements of Ref. 8 and the errors quoted are those on the nearest measured data point.

The cross sections for the single pion production channels are plotted in Figs. 2.5 and 2.6 together with a selection of the other published determinations. The agree-
ment in the $\pi^+\pi^+n$ channel is pretty good and confirms the increased contribution of this channel relative to the $\pi^+p\pi^0$. The agreement of our results for this latter channel is also good although the error bars in these other references are rather large. The only discrepancy is with the determinations of Chavanon and this is rather disturbing since the error bars are quite small compared to the difference in result, but the agreement gets better at the higher energies.

<table>
<thead>
<tr>
<th>NOMINAL MOMENTA MEV/C</th>
<th>1100</th>
<th>1200</th>
<th>1300</th>
<th>1400</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEAN $P_{LAB}$ MeV/c</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1092</td>
<td>1186</td>
<td>1283</td>
<td>1394</td>
<td></td>
</tr>
<tr>
<td>$P_{LAB}$ CUTS MeV/c</td>
<td>±35 MeV</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MEAN DIP $\lambda$ Rads</td>
<td>0.00429</td>
<td>0.00423</td>
<td>0.0044</td>
<td>0.00335</td>
</tr>
<tr>
<td>$\lambda_L$ Rads</td>
<td>-0.018</td>
<td>-0.018</td>
<td>-0.014</td>
<td>-0.021</td>
</tr>
<tr>
<td>$\lambda_U$ Rads</td>
<td>0.026</td>
<td>0.026</td>
<td>0.024</td>
<td>0.026</td>
</tr>
</tbody>
</table>

Table 2.1
<table>
<thead>
<tr>
<th>Final State</th>
<th>$p_{LAB}$ $\pi^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma \pi^+ p$</td>
<td>12.82 ± 0.62</td>
</tr>
<tr>
<td>$\sigma \pi^+ p\pi^0$</td>
<td>9.36 ± 0.46</td>
</tr>
<tr>
<td>$\sigma \pi^+ \pi^- n$</td>
<td>2.00 ± 0.16</td>
</tr>
<tr>
<td>$\sigma \pi^+ \pi^- n \eta$</td>
<td>0.76 ± 0.075</td>
</tr>
<tr>
<td>$\sigma \pi^+ \pi^- n \pi^+ \pi^- mm$</td>
<td>0.36 ± 0.057</td>
</tr>
<tr>
<td>$\sigma \pi^+ \pi^- n \pi^+ \pi^- \pi^0$</td>
<td>0.58 ± 0.08</td>
</tr>
<tr>
<td>$\sigma \pi^+ \pi^- n \pi^+ \pi^- \gamma$</td>
<td>0.013 ± 0.013</td>
</tr>
<tr>
<td>$\sigma \pi^+ \pi^- n \pi^+ \pi^- n$</td>
<td>0.058 ± 0.031</td>
</tr>
<tr>
<td>$\sigma \Sigma^+ k^+$</td>
<td>0.045 ± 0.033</td>
</tr>
<tr>
<td>$\sigma_{TOT}$</td>
<td>26.0 ± 0.2</td>
</tr>
</tbody>
</table>
Fig 2.1
PROBABILITY DISTRIBUTION ELASTIC 4C EVENTS

PROBABILITY DISTRIBUTION 3-BODY FINAL STATES
Fig 2.3

STRETCH PHI BEAM ELASTIC 4C FITS

STRETCH I/P ELASTIC 4C FITS
Fig 2.3
Fig 2.4
Fig 2.5

\( \sigma(\pi^+ p \pi^-) \) \text{mb.}

\( \pi^+ \pi^- p \) Cross section

plab \text{ gev/c}
Fig 2.6

\[ \sigma(\pi^+\pi^-n)_{mb} \]

- Ref. 12
- Ref. 14
- This Expt.
CHAPTER III

3.1 Introduction

Invariably in a bubble chamber experiment forward elastic scatters are difficult to detect in scanning due to the small deflections suffered by the incident beam. The first section of this chapter describes the biases involved and the procedure adopted to correct for this loss. Subsequent sections present elastic differential cross-sections between 1.1 and 1.4 GeV/c and a description of such data in terms of a legendre polynomial expansion. The coefficients of the various polynomials are tabulated, and the underlying physics discussed. Finally the backward differential cross-sections are compared with those obtained in other experiments; particular attention is drawn to the continuing severe discrepancies between bubble chamber and counter experiments in the evaluation of this quantity.

3.2 Scanning and Measurement Biases

Elastic scattering may be described by two variables $\theta$, $\phi$: the polar angle between the incident beam and its deflected direction after scattering, and an azimuthal angle measuring the orientation of the scattering plane with respect to a fixed plane. In an experiment with unpolarized target the theoretical distribution of $\phi$ will be flat - any deviation from this is attributable to various inherent
biases in the measurement.

Two kinematic regions of the $\theta, \phi$ space are particularly inaccessible for observation, viz.:

(a) Momentum conservation dictates that pure forward scattering will result in a final state of a zero length proton and an undeflected beam as described in the laboratory frame. Such events are, clearly, unobservable and losses might be expected throughout a small angular range about the forward direction.

(b) An event such that the plane of the scatter is along the stereo-axis (i.e. a plane perpendicular to the plane of the cameras) will mean that the deflection of the beam is not obvious in any picture taken and further that the two deflected tracks overlay one another. (Fig. 3.1)

An inspection of a scatter plot of $\cos \theta_{\text{lab}}$ vs. $\phi_{\text{lab}}$ at 1.3 GeV/c (Fig. 3.2) and of the one dimensional projections of these variables reveal that a depletion of events in these regions is indeed observed, and that the two types of bias are by no means uncorrelated. The loss in $\phi$ is particularly pernicious in the forward region ($\cos \theta = 1$) and is more important generally towards forward scattering, (i.e. the slower the proton). Figure 3.3 presents the corresponding variable in the centre of mass system (CMS).

(A short digression here is in order concerning the definition of the azimuthal angle $\phi$. The angle between the plane of scattering and the stereo axis $\alpha$ is not the definition of the physical azimuth $\phi$, and only becomes so...
when the beam track has no dip. In fact the relation is

\[ \tan \alpha = \frac{\sin \theta \sin \phi}{\cos \lambda \sin \theta \cos \phi - \sin \lambda \cos \lambda} \]

where \( \lambda \) is the dip of the beam track. The distribution in \( \alpha \) is, therefore, not flat. However, if we make all cuts on \( \phi \), and take them sufficiently large, then we may overcome this complication since \( \lambda \) is in reality quite close to zero. A further point is that in all the figures in this chapter \( \phi \) has been twice folded so as to lie between 0 and \( \pi/2 \) (Fig. 3.1).

Clearly any attempt to fit a polynomial to \( \frac{d\sigma}{d\cos \theta} \) over the whole angular range \((-1.0 \leq \cos \theta \leq 1.0)\) is doomed to failure, the forward points being completely inaccurate. The data was cut in \( \cos \theta \) and the coefficients of the expansion obtained were used to extrapolate the expression to determine the number of events lost in the forward direction and hence forward differential cross-sections. The possible correlation of \( \phi \) and \( \theta \) means that a phi cut too must be made. The magnitude of these cuts was determined by examining the distributions of the variables at all momenta concerned.

The values chosen were:

<table>
<thead>
<tr>
<th>( P_{\text{LAB}} )</th>
<th>( \cos \theta )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>&lt; 0.90</td>
<td>&lt; 0.42\pi</td>
</tr>
<tr>
<td>1.2</td>
<td>&lt; 0.95</td>
<td>&lt; 0.42\pi</td>
</tr>
<tr>
<td>1.3</td>
<td>&lt; 0.95</td>
<td>&lt; 0.40\pi</td>
</tr>
<tr>
<td>1.4</td>
<td>&lt; 0.95</td>
<td>&lt; 0.40\pi</td>
</tr>
</tbody>
</table>

Table 3.1
The cuts of phi were fairly generous and could probably be reduced but it was felt that the fairly good statistics warranted the attempt to remove any possible bias. Examination of the results of the fit (Fig. 3.6 and discussion in 3.4) showed that a smaller cut could conceivably have been taken since in the range \(0.90 > \cos \theta > -1.0\). \(\phi\) does not appear to be too strongly correlated with this variable.

3.3 Fitting Procedure

The data was divided into 40 \(\cos \theta\) bins 0.05 wide, and those bins in the forward direction were discarded as described previously. This data was then fitted by a series of the form

\[
\sum_{n=1}^{N_{\text{max}}} B_n P_n (\cos \theta)
\]

by minimising the chi-squared \((\chi^2)\)

\[
\chi^2 = \sum_{i=1}^{N \text{ bins}} \left( \frac{N_{f}^{i} - N_{e}^{i}}{\Delta N_{e}^{i}} \right)^2
\]

where \(N_{f}^{i}\) - No. of events in the \(i^{th}\) bin from the fit. 
\(N_{e}^{i}\) - Experimental no. of events in the \(i^{th}\) bin. 
\(\Delta N_{e}^{i}\) - Error on \(N_{e}^{i}\).

With the assumption of Poisson statistics \(\Delta N_{e}^{i} = \sqrt{N_{e}^{i}}\). The order of the fit \(N_{\text{max}}\) required to explain the data satisfactorily is decided by the \(\chi^2\) and the Fischer F ratio test.
When F becomes small and the $\chi^2/D.F.$ (Degrees of Freedom) does not become smaller on the addition of another term the expansion was terminated. Integrating the resulting expression over the $\cos \theta$ range neglected and scaling up the result appropriately to the full $\phi$ range we obtain the corrected total no. of elastic events, and may then make the appropriate alteration in the no. of events observed. The total cross-section data of Carter et al. is used to determine the microbarn equivalent per event and elastic differential cross-sections $\frac{d\sigma}{d\Omega}$ may then be determined. Finally $\frac{d\sigma}{d\Omega}$ was fitted by a polynomial of the tyre

$$\frac{d\sigma}{d\Omega} = \frac{1}{q^2} \sum A_n P_n (\cos \theta) \quad q = \text{incident pion momentum}$$

The statistical data on the solution is presented in Table 3.2 and the experimental cross-sections and fitted curves in Figure 3.4.

3.4 Legendre Coefficients

Tables of the differential cross-sections are given in 3.3, and the values of the coefficients $A_i$ as well as normalization independent coefficients $\frac{A_i}{A_o}$ are presented in Table 3.4. Table 3.3 also contains the results from the earlier momenta in the $\pi^+p$ experiment, which were first given in Ref.1 and are mentioned here for completeness. The
coefficients $A_i/A_0$ are plotted in Figure 3.5.

The most prominent features of these coefficients are the rise in $A_3$, $A_4$ above 1100 MeV/c a result which is also clearly seen in the very recent Bristol/Southampton/RHEL collaboration data\(^{(13)}\). Coefficient $A_5$ changes sign at about 1.33 GeV/c. $A_2$ and $A_6$ maintain a steady positive increase over our momentum range.

The interpretation of the behaviour of these coefficients is fraught with danger, since each and every $A$ has a contribution from several different partial wave amplitudes. The values for the coefficients of the particular terms contributing to any $A_i$ are given, for example, in the paper by Tripp\(^{15}\). From these the absence of any contribution from $A_7$ or higher suggests that waves higher than $F_{7/2}$ may be neglected. This being so the rise in $A_6$ is due to the increasing $F_{7/2}$ contribution, and indeed there is the well established Regge recurrence of the $\Delta(1236)$ at a mass of about 1900 MeV/c. The change in sign of $B_5$ is then in the $D_{5/2} - F_{5/2}$ interference or more likely a rising contribution from the $D_{5/2} F_{7/2}$. The rise in $A_4$ has its most likely origin in the increasing amplitude of the $D_{5/2}$ and/or the $F_{5/2}$ contributions.

To test the consistency of the results the effect of different phi-cuts on the fitted values of the Legendre coefficients was investigated, and the results of this shown in Figure 3.6 for the 1.2 GeV/c data. Here the variation of the values (\(A_i/A_0\)) is plotted as a function of the cut,
and the differences in the values of the coefficients are seen to be
well within their error bars, furthermore there appears to be no systematic change in the values of these coefficients.

3.5 Backward Scattering

The most alarming features in our data are the continuing discrepancies between the backward elastic differential cross-sections and those determined in counter experiments. Our values at eight momenta compared with a selection of several other experiments are plotted in Figure 3.7. The most accurate counter experiment quoted is that of Rothschild et al. whose backward cross-section shows a pronounced dip at -0.74 GeV/c. All our bubble chamber results differ from these by many standard deviations. Even when the agreement improves at higher energies the difference is still a factor of two. It is hard to explain a loss of such events in a bubble chamber experiment. The backward bins in the CMS involved in such a loss come from a much wider angular range in the laboratory, and earlier studies of our data\(^\text{(1)}\) have shown that there is no appreciable scanning loss for such events, nor indeed any bias in reconstruction. It is encouraging therefore to note that the bubble chamber experiment of Kalmus et al.\(^\text{17}\) agrees fairly well with ours over the small momentum range available for comparison, and that the latest counter experiment by Bristol/RHEL/Southampton collaboration have produced results in much better agreement with our own\(^\text{13}\).
The importance of this measurement to partial wave analyses should be emphasized: it is in precisely such areas of phase space that one is able to separate the effects of interference between opposite parity resonances. Further elastic cross-sections from this experiment at lower momenta will enable more definite conclusions to be made about the variation of \( \frac{d\sigma}{d\Omega} \cos \theta = -1.0 \) at about 0.8 GeV/c, and present data will be carefully studied in an attempt to clear up or confirm these serious discrepancies.

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Table 3.2
\[ \frac{d \cos \theta}{d \cos \theta} \]

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*Extrapolated from polynomial fit

Table 3.3
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Fig 3.1
Fig 3.2
Fig 3.3
Fig 3.4

Elastic Differential Cross Sections
Elastic Differential Cross Sections

Fig 3.4
Fig 3.5a
Fig 3.5b
Fig 3.5c
Fig 3.6
Fig 3.7

Backward Elastic Differential Cross Sections

PLAB GEV./C
CHAPTER IV

FORMALISM OF 3-BODY PARTIAL WAVE ANALYSIS
AND DALITZ PLOT FITTING

4.1 Introduction

The total $\pi^+p$ cross-section at around 1 GeV/c displays detailed structure, but without the presence of any well separated peaks as are apparent in, say, $\pi^-p$ total cross-sections. This indicates that, while resonances structure is almost certainly present, the separation of the mass peaks of the resonances must be smaller than their widths. Various analyses\textsuperscript{19,20} of the elastic $\pi^+p \rightarrow \pi^+p$ scattering do indeed confirm this view, and amplitudes and phases of the various transitions now available in the literature are considered as virtually gospel. There are however discrepancies between the conclusions of the major 'phase shifters', notably CERN\textsuperscript{20} and SACLAY\textsuperscript{19}, and since many waves are strongly inelastic it would seem to be highly profitable to examine the single pion production channels, which account for $\approx 90\%$ of the total inelasticity at these energies. Early attempts made at this concentrated on fitting the Dalitz Plots of the final states $\pi^+p\,\pi^0,\,\pi^+\pi^-n$ and this is described here after a somewhat detailed discussion of the theory and assumptions involved.

4.2 Isobar Model - Theory and Assumptions

The detailed dynamics of a $2 \rightarrow 3$ body process are still unknown but even a very rudimentary phenomenological
study of the data suggest that the mechanism for many processes is almost entirely described by superposition of various pseudo $2 \to 2$ body reactions, followed by a strong decay of one of these particles - the isobar. In particular in the reactions

$$\pi^+ p \rightarrow \begin{cases} \pi^+ p \pi^0 \\ \pi^+ \pi^+ n \end{cases}$$

at a laboratory momentum of $\approx 1$ GeV/c, inspection of the Dalitz-plots of the final states shows enormous enhancements at pion-nucleon masses of 1200 - 1220 MeV (Fig. 4.4): roughly the peak of the $\Delta_{33} (1236)$ in elastic $\pi N$ scattering. A closer inspection of the data reveals other two-body couplings (q.v) so we assume that the entire dynamics are described by these isobar production reactions, followed almost immediately by a decay of the isobar which, however, does not involve re-scattering from the 'spectator' particle in the final state.

Symbolically one assumes the dynamics are:

$$a \rightarrow \sum_{i=1}^{3} \left( b_{i}^{j} \right)$$

and that higher order contributions in the expansion of the amplitude, such as for example the simplest re-scattering term:
may be neglected. Notice that such simplifications in approach can only, at present, be justified by phenomenology.

A further criticism of the model is that an amplitude constructed as described above fails to satisfy quite general unitarity constraints \(^{21}\), (which was recognized for some time), but the magnitude of the correction has recently been the subject of considerable study although no definite results concerning \(\pi\pi N\) states have yet appeared. Ducking such questions for the moment we go on to describe the model in a more detailed way, and to the attempts made to fit \((\pi\pi N)\) effective mass data using it.

4.3 Practical Formulation of the Model

Isobars Involved in \(\pi^+ p \rightarrow \pi\pi N\).

As well as the contribution of \(\Delta_{33} (1236)\) already discussed certain other competing reactions must also be present. At 1.3 GeV/c and higher lab momenta there is a strong peak in the \((\pi^+ \pi^0)\) mass distributions corresponding to the \(\rho (770)\). Even at our lowest studied momentum of \(\sim 0.895\) GeV/c careful study shows an enhancement of \((\pi^+ \pi^0)\) at high masses even though this momentum is nominally below the \(\rho P\) threshold. These two isobars are all that seems to be required to describe the \(\pi^+ p \pi^0\) channel but
there must nevertheless be at least one more competing process as witnessed by comparison of $\pi^+\pi^-n$ to $\pi^+p\pi^0$ data. By $I$-spin conservation $p$ is not present in the $\pi^+\pi^-n$ channel so that one might expect the $\pi^+\pi^-n$ data to be entirely dominated by $\Delta$. That this cannot be so may be seen by considering the isospin decomposition of the amplitudes for $\Delta$-production:

\[
\pi^+p \rightarrow \begin{cases} 
\Delta^+ & \pi^0 \\
\Delta^+ & \pi^0 
\end{cases}
\]

\[
\begin{align*}
|\frac{3}{2}, \frac{1}{2}\rangle &= \sqrt{\frac{3}{5}} |\frac{3}{2}, \frac{3}{2}, 10\rangle - \sqrt{\frac{2}{5}} |\frac{3}{2}, \frac{1}{2}, 11\rangle \\
|\frac{3}{2}, \frac{3}{2}, 10\rangle &= |\frac{3}{2}, \frac{3}{2}, 10\rangle = |(p\pi^+)\pi^0\rangle \\
|\frac{1}{2}, \frac{1}{2}, 11\rangle &= \sqrt{\frac{1}{3}} |\frac{1}{2}, \frac{1}{2}, 11\rangle + \sqrt{\frac{2}{3}} |\frac{1}{2}, \frac{1}{2}, 11\rangle \\
&= \sqrt{\frac{1}{3}} |(\pi^+)\pi^+\rangle + \sqrt{\frac{2}{3}} |(\pi^0p)\pi^+\rangle \\
\end{align*}
\]

i.e.

\[
|\frac{3}{2}, \frac{3}{2}\rangle = \frac{3}{\sqrt{15}} |(p\pi^+)\pi^0\rangle - \frac{2}{\sqrt{15}} |(p\pi^0)\pi^+\rangle - \frac{2}{\sqrt{15}} |(\pi^+)\pi^+\rangle
\]
Now if we assume that we are able to distinguish the reactions \((p\pi^+)\pi^0\) and \((p\pi^0),\pi^+\) then from this formula we find

\[
R = \frac{\sigma(p\pi^+\pi^0)}{\sigma(n\pi^+\pi^+)} = \frac{13}{2} = 6.5
\]

This latter assumption becomes valid for high energies (i.e. above about 1.5 GeV/c) because there the two '\(\Delta\)' bands on a Dalitz plot do not overlap and their production may be considered incoherent. Furthermore any \(\rho\) production, while it could theoretically decrease the \(p\pi^+\pi^0\) contribution by negative interference, will in fact almost certainly boost the \(\pi^+\pi^0\) contribution. Thus one might expect that

\[R \gtrsim 6.5\]

In fact, referring ahead to Table 6.1, \(R\) continually decreases with energy and has become as low as 3 at pion laboratory momenta of 1.4 GeV/c. Thus we require isobar production which will give a higher contribution to the \(\pi^+\pi^+\) channel. Any \(I = \frac{1}{2}\) \((\pi N)\) will do, as well as an \(I = 2\) \((\pi^+\pi^+)\) isobar. Since the latter is an exotic state it cannot be a resonance and must be parametrized by \(\pi^+ - \pi^+\) phase shifts. This has been done by Bowler and Cashmore\(^{22}\), but seems a rather arbitrary procedure as the spirit of the isobar model moves us to consider resonant states only. At our energies there are three possible \(I = \frac{1}{2}\) pion nucleon candidates

\[
\begin{align*}
N^* (1470) & \quad P_{11} \\
N^* (1520) & \quad D_{13} \\
N^* (1535) & \quad S_{11}
\end{align*}
\]
Of these the last has a strong $N\eta^0$ decay mode, but very little $\eta$ production is observed at low energies ($\lesssim 1$ GeV/c). Simple resonance fitting using one or other of the remaining isobars was carried out by V. Taylor who showed that the $N^* (1470)$ gave significantly better fits and without the need to include any straight 3-body phase space contribution. The large width of this object would explain why no resonance peak is observed at the mass values concerned, and we have used this in our Dalitz plot fitting and also in the complete analysis of Chapter V. The acid test of such procedures must come by examination of the fit quality obtained (q.v).

All our isobars have been parametrised by relativistic Breit-Wigners (q.v.) and the various parameters for masses, widths etc. are given in Table 4.1.

Angular Momentum Decomposition.

As in the two body case considerable simplification results from extracting from the matrix elements those effects concerned with angular momentum conservation, and resulting from this we obtain a partial wave series. The earliest attempts at this were made by Lindenbaum and Sternheimer who assumed $s$-wave production and decay of incoherent $\Delta$ resonances (incorrect); subsequent minor improvements were made by Ollson -Yodh, and Bergia-Bonsignon-Stanghellini. Deler and Valladas (DV) published an expressions for the angular momentum decomposition for reaction types.
\[ a + b \rightarrow i j k \]

Spin:
\[
\begin{array}{ccc}
0 & 1 & 0 \\
\pi^+ & p & \pi_i \pi_j N
\end{array}
\]

which leant themselves to the Isobar Model. It is this formalism which has been used throughout the work described hereafter, and a résumé of this formalism follows.

We require 6 independent variables to specify a 3-body final state in a 2 \rightarrow 3 body collision, but for energy independent analysis with unpolarized initial state particles we may reduce them to 4. DV chose two effective masses \((w_1^2, w_2^2)\) and the polar angles of the beam in a coordinate system depicted in Fig. 4.1. The \(z\)-axis is chosen as

\[
\hat{z} = (Q_{\pi_1} \wedge Q_{\pi_2})
\]

and \(\hat{x} = Q_{\text{Nucleon}}\)

(This last definition differs from that adopted by DV who chose the bisector of the directions \(Q_{\pi_1}, Q_{\pi_2}\). For the \(\pi^0\pi^+ p\) we make the assignment \(\pi^0 \equiv \pi_1, \pi^+ \equiv \pi_2\). For \(\pi^+\pi^- n\) it does not matter which way we choose the pions so long as we remember to symmetrize the data by interchanging them.

With these definitions:

\[
\langle \pi R | T_7 | \pi R \rangle = \langle T_7 N | T_0 | R \rangle
\]

\[
= \sum_{j \neq l} \frac{1}{\sqrt{\frac{4\pi}{Q_{\pi_j}}} \sqrt{\frac{4\pi}{Q_{\pi_l}}}} \mathcal{T}_{j \wedge l} \mathcal{T}_{\pi_j \pi_l} (\mathbf{w}, \mathbf{w}_j) \int_{\mu_i \mu_j} (\theta, \phi, \theta_j, \phi_j)
\]
Notation is described in Appendix I, and the various terms involved are:

1. \( \mathcal{F}^{jL',L}_{\mu_i \mu_f} (\Theta, \delta', \theta, \phi) \)

Contains all the angular dependence for the production of an isobar in the \((\pi_2N)\) combination with an orbital angular momentum state \(L'\) resulting from an initial \(\pi^+_p\) orbital angular momentum \(L\), total angular momentum \(J\), and its subsequent decay into a pion nucleon state \(k\), with the isobar having a total spin \(j\). The indices \(\mu_i, \mu_f\) are the spin projection of the initial and final state nucleon respectively, on to the normal to the 3-particle plane. For s-channel diagram see Fig. 4.2.

Explicitly

\[
\mathcal{F}^{jL',L}_{\mu_i \mu_f} (\Theta, \delta', \theta, \phi) = (-1)^{\frac{J+1}{2}} \sum_{\lambda, \lambda'} \left\{ 1 + (-1)^{L'+\lambda'} \right\} (-1)^{j+\frac{1}{2}} \\
\times \left( \frac{j-j-L}{2J+1} \right) \left( \frac{j+1}{2J} \right)^{\lambda'} \left( \frac{1}{2} \right)^{\lambda'} \left( \frac{1}{2} \right)^{\lambda'} \\
\times \left( \frac{j-j-L}{2J} \right) \left( \frac{j+1}{2J} \right)^{\lambda'} \left( \frac{1}{2} \right)^{\lambda'} \left( \frac{1}{2} \right)^{\lambda'} \\
\times d^{j', \lambda'}_{m', -\nu} \left( \frac{\pi}{2} \right) \rho^\nu (\Theta, \phi) \left( \frac{1}{2} \right)^{\lambda'} \left( \frac{1}{2} \right)^{\lambda'} \left( \frac{1}{2} \right)^{\lambda'} \left( \frac{1}{2} \right)^{\lambda'} \left( \frac{1}{2} \right)^{\lambda'} \left( \frac{1}{2} \right)^{\lambda'} \left( \frac{1}{2} \right)^{\lambda'} \\
\times \gamma^{\mu \nu \lambda \lambda'}_{\mu'} (\Theta, \phi) 
\]
and note the following:

(a) The definition of \( P^\nu \) is slightly different from that normally encountered:

\[
\gamma^\nu_{\mu}(\phi) = (-1)^{\nu} \frac{P^\nu(\phi)}{\sqrt{2\pi}}
\]

(b) The angle \( \delta \) is a rotation about the z-axis which takes the isobar onto the x-axis i.e. the nucleon.

(c) \( \Omega \) is generally referred to as the Stapp angle. It arises because the decay matrix element is calculated in the isobar rest frame, whereas the total matrix element must be evaluated in the CMS. A Lorentz transformation connecting these two rotates the spin of the nucleon. At the energies involved \( \Omega \) is only \( O(2^0) \) and is neglected in all subsequent discussion.

(d) A similar expression holds for the production and decay of an isobar in the \((\pi N)\) subsystem if the z-axis is redefined as \((\pi_2 \times \pi_1)\) i.e. we essentially interchange the two pions. To express the complete matrix element for the simultaneous production of both isobars we then introduce a further phase factor derived from the rotation necessary to transform one set of axes to the other. When this is done we obtain

\[
\int_{\mu_1 \mu_2} \gamma_{\mu_1 \mu_2} (-\phi_1, -\phi_2, \phi, \phi)
\]

Finally if we wish to express the mechanism of isobar formation in the \((\pi\pi)\) diparticle we have a somewhat
more complex situation in that the spin of the iso-
bar has to be combined with that of the recoil nucleon
leading to several different invariant spins in the
final state. Nonetheless the formalism may be
developed along similar lines and the angular matrix
element now becomes:
\[
\langle \Phi, \Theta, \omega | \mu_1 \mu_2 \rangle \]

Where \( \Theta_3 \) is the angle between the pion labelled 2
and the \((\pi\pi)\) isobar and \( j \) now becomes the total in-
variant spin rather than the isobar spin.

(e) The \( f \)'s so constructed satisfy various symmetry re-
lations and normalization properties. These relations
are summarized in Appendix 2.

2. \( T_{j j L L} \) \((\omega, \omega)\)

Contains all the dynamics of the interaction; the
angular dependence of the \( f \)'s being that part of the matrix
element concerned with angular momentum conservation

\[
T_{j j L L} \left( \omega, \omega \right) = A_{j j L L} \left( \omega, \omega \right) B_{j L L} \left( \omega \right)
\]

where \( A \) represents the amplitude for production of the
resonance, \( B \) containing the dynamics for the decay.
B may be parametrized in a variety of ways e.g.

1. The phase shifts for elastic scattering of the pair (jk) leading to the Watson-formula\textsuperscript{27}.

2. A relativistic or non-relativistic Breit-Wigner, assuming that the isobar is a resonance.

We have chosen the relativistic Breit-Wigner formulation given in Jackson\textsuperscript{28}

\[
B_{jk}(\omega) = \frac{1}{\pi} \frac{\sqrt{\omega_0^2 - \omega^2}}{i\omega_0} \cdot \left( I_k I_k^* I_k^* I_k | I I^* \right)
\]

The Clebsch-Gordon coefficient is necessary if we wish to compare the amplitudes for, say, $\Delta^{++} \rightarrow \pi^+ p$ or $\Delta^+ \rightarrow \pi^0 p$.

(This is really saying that the width in the numerator is a partial rather than a total width.)

$\Gamma$ is a mass dependent width varying with energy ($\omega$) of the diparticle

\[
\Gamma = \Gamma_0 \left( \frac{\rho}{\rho_0} \right)^{2L+1} \frac{\rho(\omega)}{\rho(\omega_0)}
\]

$\rho$ is a form factor for the decay, various expressions for which have been discussed by Jackson\textsuperscript{28}. For Baryon resonances we choose a parametrization first used by Rosenfeld and Glashow\textsuperscript{30}.\textsuperscript{30}
\[ \rho(w) = w^{-1} (0.1225 + q^2)^{-\frac{1}{2}} \]

For the \( \rho \) meson we choose \( \rho(w) = w^{-1} \). Clearly the better prescription is to use some kind of 'look up' procedure to evaluate the phase of the elastic \((jk) \rightarrow (jk)\) scattering process but this leads to difficulties in the case of \((\pi\pi)\) isobars. With the parametrization adopted the decay amplitude should be accurately described near the resonance mass, but it tends to have rather too long a tail at mass separations of 2 or 3 widths from the peak value.

The production amplitude \( A_{j'j\pm L'\pm L} (W,w) \) is constructed with a partial wave amplitude \( X(W) \) and various barrier factors \( Y \) depending upon the initial and final wave momenta.

\[ A_{j'j\pm L'\pm L} (W,w) = X_{j\pm j',\pm L,\pm L'} (W) Y_{\L}(\phi, \zeta) Y_{\L'} (Q) \]

The \( X_{j'j\pm L'\pm L} (W) \) are the various transition amplitudes we are trying to determine and depend only upon the total CMS energy. Centrifugal barrier terms \( Y \) are certainly present, as they are needed to account for the shift in the observed peaks of the \( \Lambda \) from its nominal mass of 1236 MeV to masses of 1200 - 1215 MeV. This is too large a discrepancy to be accounted for merely by the mass dependant width, and must be concerned with the mechanism at the primary interaction.

Two possibilities were considered:

\[ Y_L (P) = P^L \]
\[ Y_L (P) = \mathcal{J}_L (P^R) \]
The $J_L^{(pR)}$ are polynomials in $(pR)$ and are tabulated in Table 4.2. They are derived in Blatt-Weisskopf where they are shown to be the penetration factor of a wave function from a spherically symmetric potential of radius $R$.

The effect of these two types of barriers on the projections of the $\Delta_{33}^{(1236)}$, $N^*_{11}(1470)$, $\rho_{21}(770)$ is shown in Figure 4.3 for a beam momentum of 895 MeV/c and radius $R$ of 1 fermi. Clearly it makes very little difference to the shape of the $\Delta$ which of these two are chosen, but for broad resonances whose masses are very near the top edge of phase space there is a considerable distortion from the undamped Breit-Wigner shape (i.e. $s$-wave production figure). It was decided that since the difference in shape between the two barriers was less significant than the difference between damped and undamped waves it was simpler to choose the minimal power dependance. This however is a weakness in the model employed as one really has very little idea how to parametrise such an effect.

The production amplitude may now be written as

$$A_{j\ell m\nu}(w,\omega) = \frac{I^\nu}{I^m} q^\nu \delta_{m\nu} \left( I^2 I^2 I^2 I^2 \right)^{\frac{1}{2}}$$

where we again modify the amplitude with a Clebsch-Gordon coefficient to make $X$ charge independent.

The target is unpolarized, no polarizations are measured in the final state, so the differential cross section
becomes

\[
\frac{d\sigma}{d\Omega d\cos \theta} = \sum_{\text{finj}} \left| \sum_{JW} \sum_{i} T_{j'jW'}(\omega, \nu, c) \right|^2 \]

The sum over \( i \) is intended to indicate all possible mass combinations in which the isobar figures.

4.4 Dalitz Plot Fitting

The method employed is to minimise a chi-squared generated from data bins on the Dalitz-plots of the three body channels.

For the \( \pi^0 \pi^+ p \) channel a 15 x 15 grid was superimposed on the scatterplot \( M^2(\pi^+ p) \) vs. \( M^2(\pi^+ \pi^0) \), the upper and lower bounds on each axis being chosen appropriately. The rather poorer statistics of the \( \pi^+ \pi^+ n \) channel dictated the use of a 10 x 10 grid: this Dalitz plot was first symmetrized with respect to interchange of the two identical pions. The chi-square is formed by summing over the data bins so constructed (q.v.).

The isobars which seem to be necessary to describe the data have already been described: it merely remains to indicate how the partial wave series were truncated. We cite the following evidence for the assumptions made -

(a) Detailed study of the \( \Delta_{33}^{++} (1236) \) band in the data reveals that partial waves up to \( L_{ij} = D_{35} \) are
statistically non zero and some of these are very strong. Some $F_{35}$ contribution is indicated at the highest momentum. Ref. 1 gives the details of this analysis between 895 and 1040 MeV/c. Similar studies between 1100 - 1400 MeV/c are presented in Chapter VI and suggest the continued importance of these waves.

(b) The latest elastic analyses of SACLAY\textsuperscript{19}, CERN\textsuperscript{20} corroborate this evidence: waves up to $D_{35}$ are found to be significantly inelastic and the $F_{35}$ is becoming inelastic at 1 GeV/c.

(c) Single resonance model fits to the data indicate the presence of $\rho$ and $N^*$ contributions\textsuperscript{1,2}.

Thus the $\Delta$-transition included were:

$\Delta_{33}(1236)$: SD1, PP1, PF3, DS3, DD3, DD5, FP5

(Notation $LL'_{2J}$: given JL there are 4 possible values of $L'$, parity removes two of these. Hence the necessity for the extra index.)

(d) The $\rho$ and $N^*$ are damped by the reduction in phase space available at 1 GeV/c. General arguments concerning production momenta etc. would indicate s-wave production only so we chose the following waves:

$N^*(1470)$: SS1

$\rho(770)$: SS1

These assumptions are rather drastic and the arguments somewhat unconvincing. Certainly more transitions than these should be included but were deferred until a complete analysis of the data was made using the production
as well and as the decay information.

The formulae given in the previous section for various pure angular momentum states may be greatly simplified and the reduced forms of these are given in Appendix III for those values of \((JLL'j\ell)\) pertinent to this chapter and the next. Examination of these formulae show that, due to the orthogonality of the \(Y^M_L(\theta, \phi)\), waves of different spin parity \(J^P\) (or equivalently \(JL\)) do not interfere on the Dalitz plot, thus only the following phases have to be considered:

\[
\begin{align*}
\Delta P F 3/\Delta P P 3 & : \phi_1 \\
\Delta D D 3/\Delta D S 3 & : \phi_2 \\
\Delta S D 1/\pi^* S S 1 & : \psi_1 - \psi_2 = : \phi_3 \\
\Delta S D 1/\pi S S 1 & : \psi_1 - \psi_3 = : \phi_4 \\
N^* S S 1/\pi S S 1 & : \psi_2 \psi_3 = : \phi_4 = \phi_3
\end{align*}
\]

So there are 14 parameters in total: 10 amplitudes and 4 independent phases. The Dalitz plot density distributions constructed as

\[
\sum_{\mu_i \mu_f} \left| \sum_\alpha \sum_\xi T^n (\omega_i, \omega_j) \frac{f_{\mu_i \omega_i}}{\mu_i \mu_f} \right|^2
\]

where we have abbreviated \((JLL'j\ell)\) to a single index \(n\).

If from the matrix \(T^n\) we extract the partial wave amplitudes \(X_n\) we have

\[
\sum_{\mu_i \mu_j} X_{\alpha} X^*_{\beta} \sum_{i,j} T^n (\omega_i, \omega_j) T^{\alpha*} (\omega_i, \omega_j) \frac{f_{\mu_i \omega_i}}{\mu_i \mu_j} \frac{f_{\mu_j \omega_j}}{\mu_j \mu_i}
\]
The Dalitz plot density distributions $R_{nm}$ are given in Table 4.3, for the diagonal elements (real) and for those interferences which are non-zero.

**Construction of the chi-squared.**

For each experimental data bin the partial wave series predicts a total no. of events $N_f$ as

$$N_f = C \int \int \int \sum_{n} \alpha \beta \gamma \delta \; R_{nm}(\omega^L, \omega^T) \; d\omega^L \; d\omega^T$$

The coordinates $(\alpha, \beta, \gamma, \delta)$ are the limits of the data bin. $C$ is a normalization constant. For a data bin lying entirely within the Dalitz-Plot boundary the integral is performed by subdividing the bin into a 3 x 3 mesh and evaluating $N_{ij}$ at the central point of each of these sub-rectangles. Summing these and multiplying by the area of integration we find an unnomalized integral. For bins which straddle the phase space boundary this was not sufficiently accurate and a 6 x 6 sub division was used. Moreover the boundary is smeared because at each energy data within ±35 MeV of the nominal beam momentum was included in the sample. This dictated that bins on the boundary containing fewer than 5 $\pi^0 \pi^+ \pi^-$ events (or 10 $\pi^+ \pi^- \pi^0$ events)
events because of symmetrization) were discarded.

The normalization content \( C \) is determined by equating
the integral over all phase space and summed over both
channels to the total no. of events contributing (i.e. with
the provisos above). A chi-squared is then formed

\[
\chi^2_{\text{tot.}} = \sum_{c=1}^{2} \sum_{i=1}^{N_c} \left\{ \frac{N_s^i - N_c^i}{\Delta N_s^i} \right\}^2 + \left( \frac{R_f - R_e}{\Delta R_f} \right)^2
\]

\( c = 1,2 \) Sum over 2-charge channels \( \pi^0 \pi^+ p, \pi^+ \pi^+ n \)

\( N_f^i \) Fitted no. of events in bin \( i \)

\( N_e^i \) Experimental no. of events in bin \( i \)

\( \Delta N_f^i \) Error on \( N_f^i \)

\( R_f \) Fitted ratio : \( \frac{N_{\pi^0 \pi^+ p}}{N_{\pi^+ \pi^+ n}} \)

\( R_e \) Experimental ratio \( \frac{\sigma_{\pi^0 \pi^+ p}}{\sigma_{\pi^+ \pi^+ n}} \)

\( \Delta R_f \) Error on \( R_f \)

We make the usual assumptions

\[
\Delta N_f^i = \Delta N_e^i = \sqrt{N_e^i}
\]

\[
\Delta R_f = \Delta R_e
\]

The parameters \( X_i \) are adjusted to minimise this sum using
the CERN program MINUIT. This merely ensures that we have
found a minimum in the chi-squared function but says nothing
about the possibility of further local minima. Clearly we
desire the universal minimum and so the following procedure
was adopted -
(a) At 895 MeV 200 sets of starting values were generated and were allowed to vary with weak convergence criteria to attain 200 approximate solutions. Of these the best 10 were chosen, and the fit performed with very tight convergence criteria. Finally a selection of the poorer solution were refitted with tight convergence criteria. All of these converged to the same solution.

(b) At 945 a similar procedure was adopted, with only 50 random starting sets however.

At the remaining two momenta 995, 1040 the solution at 945 were taken as starting values and 5 random starts were made. These too converged to essentially identical solutions at each momentum separately, so one was confident in stating that the solutions presented in the next section are indeed unique, given the assumptions made.

4.5 Results and Discussion

At each energy 895, 945, 995, 1040 MeV/c the solutions showed that the transitions P31, D35, F35 were negligible. These waves were then set to zero and the fits re-performed with minute changes found in the remaining parameters. The results establish the enormous importance of the transitions S31, P33, D33 in this energy region, and the cross-sections found are compared with the predictions of the SACLAY¹⁹ '74 and CERN²⁰ solutions in Fig. 4.4. Invariably the cross-sections from our fit are larger than the elastic predictions, and indeed at 1040 MeV/c the S31 violates the theoretical uni-
Nevertheless this distribution is encouraging: all waves that elastic phase shifters predict strongly inelastic do indeed couple strongly to the single pion channels, where \( \approx 90\% \) of the inelastic cross-section resides. The measure of the quality of the fits is the statistical data tabulated in 4.4. The

\[
\frac{\chi^2}{\text{NDF}} = \chi^2 \text{ per degrees of freedom}
\]

are all close to unity: the largest being 1.37 at 995 MeV/c. Figure 4.4 shows the Dalitz plots at each of these energies and the mass distribution along each of the 3 axes. Superposed are the fitted lines: these reproduce the essential features of the data. Table 4.5 gives the T matrix elements for the transition, defined by

\[
\sigma^{Jj\lambda LL'} = 4\pi \lambda^2 (J+\frac{1}{2}) |T|^2
\]

S matrix unitarity dictates the cross-section for any inelastic reaction is

\[
\sigma = \pi \lambda^2 (J+\frac{1}{2}) (1-\eta^2) \quad 0 \leq \eta \leq 1
\]

Hence the maximum value of T is 0.5. An argand representation of T lies within a circle centred on the origin and radius 0.5 – this will be relevant in the next chapter.

No errors are presented: their determination is complicated due to the redundant parametrization: – A solution \( \{X_i\} \) is invariant to scale changes \( c \), and so

\( \{cX_i\} \)
is an equally good solution. So in parameter space there
is always one-direction in which the function remains flat.
This has the effect that normal parabolic error estimates
are invalid. Since this was a pilot study for a 4-variable
analysis no attempt was made to determine the confidence
level for the parameters.

Noticeable is the rise in the $N^*(1470)$ s-wave cross-
sections with energy corresponding to a fall in the ratio $R$
between the two channels. The s-wave cross-section is
largely composed of the transition
$$\pi^+ p \rightarrow \rho^+ p$$

Very little contribution is found from the SD1 $\Delta-\pi$ transition.

Both $P_{33}$ transitions are strong and in the $D_{33}$ case
the $\Delta-\pi$ DD3 is rather more important than the DS3.

The phases presented in Table 4.5 seem to be rather
discontinuous especially between 995 and 1040 MeV/c.
Certainly these phases are rather poorly determined in a
mass plot fit, but they could be suggestive of other im-
portant mechanisms at the higher energies, not present in
the theoretical input, and consequently distorting the results
found.

The conclusions and criticisms of the method are
summarised below:
(a) Strong contributions are present from those waves where
elastic analyses suggest large inelasticities.
(b) Other waves are consistent with zero: this is clearly
a fault in the method. Finer binning of the data
would almost certainly reveal more structure than could be explained by the observed strong transitions. Indeed the Dalitz plot method is probably rather poor in determining the magnitude of small waves.

(c) Increasing \( N^* \) (1470) production with energy: possibly more transitions would lead to a higher angular momentum contribution.

(d) Very strong \( \rho \) production at all energies: very little s-wave \( \Delta \) production.

(e) Rather structureless total cross-sections in \( S_{31}, P_{33} \) and \( D_{33} \), but large. It is really impossible to make any definite conclusion as to resonance content and this must await the more complete analysis.

(f) Such phases as are determined seem rather discontinuous. The interferences between waves of the same \( J^P \) are very strong around the edges of the Dalitz plot and the procedure of discarding bins near this boundary is very suspect. A better analysis would be to weight events according to their fitted beam momentum and then use, say, a likelihood method in two variable.

In short this Dalitz-plot analysis confirms the essential applicability of the isobar model but obscures some, perhaps many, of the features present in the data and consequently in the partial wave amplitudes themselves. With this in mind we turn to a more complete description of the full 4-variable fit.
Table 4.2

<table>
<thead>
<tr>
<th>L</th>
<th>$J_L(x)$</th>
</tr>
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<tr>
<td>0</td>
<td>$\frac{1}{1 + x^2}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{x}{9 + 3x^2 + x^4}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{x^6}{225 + 45x^2 + 6x^4}$</td>
</tr>
</tbody>
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Table 4.1

<table>
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<tr>
<th>Resonance</th>
<th>Mass GeV/c</th>
<th>Width GeV/c</th>
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</thead>
<tbody>
<tr>
<td>$\Delta$</td>
<td>1.236</td>
<td>0.120</td>
</tr>
<tr>
<td>$N^*$</td>
<td>1.470</td>
<td>0.195</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.770</td>
<td>0.146</td>
</tr>
<tr>
<td>WAVE</td>
<td>DALITZ PLOT DENSITY DISTRIBUTION</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>-------------------------------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>SD1(Δ)</td>
<td>$\alpha \left</td>
<td>T(ω_1) \right</td>
</tr>
<tr>
<td>PP1(Δ)</td>
<td>$\alpha \left</td>
<td>T(ω_1) \right</td>
</tr>
<tr>
<td>PP3(Δ)</td>
<td>$\alpha \left</td>
<td>T(ω_1) \right</td>
</tr>
<tr>
<td>PF3(Δ)</td>
<td>$\alpha \left</td>
<td>T(ω_1) \right</td>
</tr>
<tr>
<td>DS3(Δ)</td>
<td>$\alpha \left</td>
<td>T(ω_1) \right</td>
</tr>
<tr>
<td>DD3(Δ)</td>
<td>$\alpha \left</td>
<td>T(ω_1) \right</td>
</tr>
<tr>
<td>DD5(Δ)</td>
<td>$\alpha \left</td>
<td>T(ω_1) \right</td>
</tr>
<tr>
<td>FP5(Δ)</td>
<td>$\alpha \left</td>
<td>T(ω_1) \right</td>
</tr>
<tr>
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<td>$\alpha \left</td>
<td>T(ω_1) \right</td>
</tr>
<tr>
<td>SSI(p)</td>
<td>$\alpha \left</td>
<td>T(ω_3) \right</td>
</tr>
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<td>DALITZ PLOT DENSITY DISTRIBUTION</td>
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<td>--------------</td>
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| PP3*(Δ)/PF3(Δ) | \[ T_p^*(\omega_1)T_F(\omega_1) \{ 1-3\cos^2\theta_1 \} + T_p^*(\omega_2)T_F(\omega_1) \{ 1-3\cos^2\theta_2 \} \\
- \frac{1}{2} T_p^*(\omega_1)T_F(\omega_2) \{ \cos\theta_2\cos(\theta_1-2\theta_2)+\cos\theta_1\cos\theta_2+2\cos(\theta_1-\theta_2-2\theta_2) \} \\
- \frac{1}{2} T_p^*(\omega_2)T_F(\omega_1) \{ \cos\theta_1\cos(\theta_2-2\theta_1)+\cos\theta_1\cos\theta_2+2\cos(\theta_1-\theta_2+2\theta_2) \} \] |
| DS3*(Δ)/DD3(Δ) | \[ T_S^*(\omega_1)T_D(\omega_1) \{ 1-3\cos^2\theta_2 \} + T_S^*(\omega_2)T_D(\omega_2) \{ 1-3\cos^2\theta_2 \} \\
- T_S^*(\omega_1)T_D(\omega_2) \{ \cos(\theta_1-\theta_2-\theta_12)+\cos\theta_1\cos(\theta_1-\theta_12) \} \\
- T_S^*(\omega_2)T_D(\omega_1) \{ \cos(\theta_2-\theta_1-\theta_12)+\cos\theta_1\cos(\theta_2-\theta_12) \} \] |
| SD1*(Δ)/SS1(N*) | \[ T_D^*(\omega_1) T_S(\omega_1) \{ 1-3\cos^2\theta_1 \} + T_D^*(\omega_2) T_S(\omega_2) \{ 1-3\cos^2\theta_2 \} \\
- T_D^*(\omega_1) T_S(\omega_2) \{ \cos(\theta_2-\theta_1-\theta_12)+\cos\theta_1\cos(\theta_2-\theta_12) \} \\
- T_S^*(\omega_2) T_D(\omega_1) \{ \cos(\theta_1-\theta_2-\theta_12)+\cos\theta_1\cos(\theta_1-\theta_12) \} \] |
| SD1*(Δ)/SS1(ρ) | \[ -T_D^*(\omega_1) T_S(\omega_3) \{ \cos(\theta_1+\theta_3+\delta_1-\pi) + \cos\theta_1\cos(\theta_1+\delta_1-\pi) \} \\
- T_D^*(\omega_2) T_S(\omega_3) \{ \cos(\theta_3-\theta_2-\delta_2-\pi) + \cos\theta_2\cos(\theta_3-\delta_2-\pi) \} \] |
| SS1*(N*)/SS1(ρ) | \[ T_S^*(\omega_1) T_S(\omega_3) \cos(\theta_3-\theta_1+\delta_1-\pi) \\
+ T_S^*(\omega_2) T_S(\omega_3) \cos(\theta_2+\theta_3-\pi-\delta_2) \] |
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Table 4.5
Fig 4.1

\[ \Theta = \Theta_1 + \Theta_2 \]
Fig 4.2
Barrier Modified Isobar Mass Projections

Fig 4.3a.
Fig 4.3b.
Cont.

Fig 4.3c.
895 MeV/c

Fig 4.4a.
Mass Spectra 895 Mev/c

Fig 4.4b
Mass Spectra 895 MeV/c

Fig 4.4c
Fig 4.4d
Mass Spectra 945 Mev/c.

Fig 4.4e.
Mass Spectra 945 Mev/c

Fig 4.4f
Fig 4.4g
Mass Spectra 995 Mev/c.

Fig 44h
Mass Spectra 995 Mev/c
Fig 44i
1040 MeV/c

Fig 44j
Mass Spectra 1040 Mev/c

Fig 4.4k
Mass Spectra 1040 Mev/c

Fig. 4.41
Fig 4.5
CHAPTER V

5.1 Introduction

The previous chapter described the simplified analysis of the 3-body states involved in this experiment which utilised only the information discernible from the Dalitz plot density distributions, and this is related to isobar decays. The motivation for undertaking an analysis involving all the variables has been explained in the introduction: relative phases of the transitions, resonance parameters etc. require the more detailed analysis of combined production and decay information. The method and results to date of this study are described in this chapter.

5.2 Definitions and Preliminary Remarks

The dynamical assumptions have already been explained in the previous chapter and the parametrizations of these in the four variable fit are identical to those of the mass plot analysis. We introduce here a rather more simplified notation which will be of use in discussing the construction of the likelihood function (q.v.).

The spin indices \((\mu_1,\mu_f)\) are abbreviated to a single index \(\mu\) and the set of quantum numbers \((Jj\ell \ell'L'S')\) represented by the abbreviation \(\{n\}\). Then if \(F^n_\mu\) contains all the functional dependance of a particular transition upon the independent variables \((\omega_1^2, \omega_2^2, \theta, \phi)\).
where the \( X_n \) are the (complex) partial wave amplitudes to be determined by the fit. Finally an extra superscript \( \{a\} \) on the \( F_n^a \)'s be introduced to specify a particular charge channel (i.e. either \( \pi^0 \pi^+ p \) or \( \pi^+ \pi^- n \)).

Integrating the expression (1) for the differential cross section, the cross section for a specific channel becomes

\[
\sigma^a = \sum_{nm} R_{nm}^a X_n X_m^* \]

with

\[
R_{nm}^a = \sum_{\mu} F_{\mu}^{na} F_{\mu}^{ma*} \]

The quantities \((\omega_1^2, \omega_2^2, \Theta, \phi)\) will be represented by a vector \( y \) and the set of such quantities for a particular event \( \{i\} \) denoted by \( y_i \). Where emphasis on the functional dependance of \( F_{\mu}^n \) is required this will be written as \( F_{\mu}^n(y) \). For a particular event \( \{i\} \) observed in charge channel \( \{a\} \) this becomes \( F_{\mu}^{na_i}(y_i) \).

We define the modulus of the T matrix element for a particular transition \( \{n\} \) by

\[
\sigma^n = 4\pi \lambda^2 (J+\frac{1}{2}) |T^n|^2 \tag{2}
\]

and

\[
\sigma^n = \left\{ \frac{\sum a R_{\alpha}^a |X_n|^2}{\sum_{\alpha, p, \nu} R_{\alpha}^{p, \nu} X_p X_{\nu}^*} \right\} \sigma_{tot}. \]
where $\sigma_{TOT}$ is the total 3-body cross section observed in the experiment (i.e. $\sigma = \sigma(\pi^0 \pi^+ p) + \sigma(\pi^+ \pi^- n)$).

The cross section for a transition for a particular $J^P$ is made up of the various decays into the differing isobar-recoil particle subsystems having the appropriate quantum numbers $(JL)$. We may then write

$$\sigma(J^P) = \left\{ \frac{\sum \sum R_{nm}^a X_n X_m^*}{\sum \sum R_{p\nu}^a X_p X_\nu^*} \right\} \sigma_{TOT}.$$ 

Note the $\sigma(J^P) \neq \sum_{n \in JP} \sigma_n$ because of the interferences between the different isobars.

The phase of $T_n$ is given by the corresponding $X_n$ and with the definition above an argand plot of $T_n$ is confined within a circle centred on $(0,0)$ and of radius 0.5.

5.3 Maximum Likelihood Method

At each energy we have some 7000 $\pi\pi N$ events and a method involving construction of a chi-squared function is impossible; even to bin as coarsely as $10 \times 10 \times 10 \times 10$ would leave on average fewer than 1 event per bin. The method that has been chosen is that of maximum likelihood, which utilises the data in the most efficient manner, but which requires a vast amount of computing power to be successful.
Of the other workers involved in similar analyses SLAC-BERKELEY also use this method whereas SAACLAY bin the data in the mass variables and construct moments in each bin for $(\Theta', \Phi')$.

Theory of the Likelihood Method.

The likelihood $L$ is a function defined over a given set of observations $S$ for a particular theory $\Theta$. If for any observation $\{i\}$ the measurements required to specify that observation are $y_i$ then

$$L(\Theta|S) = \frac{1}{\pi} \prod_{i=1}^{N} G_{\Theta}(y_i)$$

$G_{\Theta}$ is the functional dependance of the 'theory' $\Theta$ on coordinates $y$. The likelihood function $L$ becomes the joint probability density of obtaining a particular set of measurements $y_i$ (assuming $\Theta$ to be true) in the case that

$$\int L(\Theta|y) dy = 1$$

Becoming less abstract, applied to the problem in hand $\Theta$ is specified by the set of partial wave amplitudes $\{X_n\} \equiv \{\vec{X}\}$ and the observations consist of the distribution of events over the 4-variable space as measured in the experiment. This distribution is determined by the differential cross-section $d\sigma/d$ (phase space) and thus

$$L = \sum_{f=1}^{N} \left\{ \frac{\sum_{\mu, \nu} X_n X_m^* F_{\mu}^{\frac{d\sigma}{d}}} {\sum_{p, q} X_p X_q^* R_{pq}} \right\}$$

(3)
\( R_{pq} \) is the sum over all the charge channels of \( R_{pq}^a \).

Tacitly assumed in this method of constructing the likelihood is the fact that the event sample is representative of the cross-sections for the different charge channels \( \sigma^a \)

\[
i.e. \quad N^a = \sigma^a \quad \text{i.e.} \quad N^a = l\sigma^a \quad \text{and independent of} \quad a
\]

This is indeed the case for the data gathered by our experiment. For the correct procedure when \( N^a \neq l\sigma^a \) see e.g. L. Miller Ph.D. Thesis and also the review paper by Solmitz. The expression quoted by Miller reduces to (3) in the special case \( N^a = l\sigma^a \).

The presence of the normalization integral on the bottom line is to ensure that the function is indeed a genuine joint probability density function. The procedure of fitting the parameters \( X_n \) becomes the problem of determining a special set \( \{X_\lambda\} \) such that \( \mathcal{L} \) is a maximum and this is performed numerically. \( \mathcal{L} \) may have several distinct maxima however and time consuming explorations of parameter space have to be undertaken to ensure that all of these have been discovered. In fact rather than maximise \( \mathcal{L} \) the negative of the log likelihood was minimised.

\[
L = -\ln \mathcal{L}
\]

\[
= - \sum_{i=1}^{N} \ln J(X, y_i) + N \ln R(X)
\]

\[
J(X, y_i) = \sum_{\mu} \sum_{nm} X_n X_m^* P_{\mu i}^{na}(y_i) P_{\mu i}^{ma*}(y_i)
\]

\[
R(X) = \sum_{nm} X_n X_m R_{nm}
\]
Minimising Techniques

A variety of pre-packaged programs were tested for their effectiveness in minimising the function $L$. CERN program MINUIT$^6$ was found to be quite unsatisfactory for this purpose as the interaction algorithms employed were too simple minded to be able to deal with poorly behaved functions. The programs used in all the work to be described were from the HARWELL Subroutine Library, VA01A and VA06A, author M.J. Powell. A full description of these programs may be found in the Harwell Program Library write up. The theory behind these methods is described in Refs. 33 and 34 but briefly the theory is as follows:

VA01A uses the Davidon method which is to calculate the direction in which the function descends most steeply and then to vary the parameters along that direction. VA06A uses a quasi-Newton algorithm. The estimated parameters at the $(k+1)^{th}$ iteration $X_{k+1}$ are derived from those at the $k^{th}$ iteration by

$$X_{k+1} = X_k - H^{-1} G$$

where $G$ is the column matrix of first derivatives

$H$ is the matrix of second derivatives

at the $k^{th}$ iteration

and $\alpha$ is a scale factor.

The vector $G$ must be provided by the user and an estimate of $H$ is made at each iteration and then inverted. The problem of estimating $H$ is discussed in the reference cited$^{34}$: the
starting approximation is to assume a diagonal matrix. As the program converges upon the solution a more accurate approximation to $H$ is built up and ideally should be exact at the minimum. The parameter $\alpha$ should be unity for ultimate rapid convergence but initially, when $H$ is a very poor approximation to the second derivatives, a much smaller number is chosen. The methods of choosing $\alpha$ effect the overall efficiency of performance of the optimization: that in VA06A is explained in Ref. 34.

As explained VA06A requires explicit first derivatives (as does VA01A). The (complex) $X_n$ are expressed in terms of two real numbers: an amplitude $A_n$ and phase $\phi_n$. Then:

$$\frac{\partial L}{\partial A_j} = -\sum_{i=1}^{N} \frac{2}{J(x, y_i)} \sum_{\mu \lambda} A_i \text{Re} \left\{ e^{i(\phi_i - \phi_j)} \gamma_{\mu i} F_{\mu} (y_i) F_{\lambda}^{*} (y_i) \right\}$$

$$+ \frac{2N}{R} \sum_{\alpha e} A_i \text{Re} \left\{ e^{i(\phi_i - \phi_j)} \alpha_{ij} \right\}$$

$$\frac{\partial L}{\partial \phi_j} = -\sum_{i=1}^{N} \frac{2}{J(x, y_i)} \sum_{\mu \lambda} A_i A_{\lambda} \text{Im} \left\{ e^{i(\phi_i - \phi_j)} \gamma_{\mu i} F_{\mu} (y_i) F_{\lambda}^{*} (y_i) \right\}$$

$$+ \frac{2N}{R} \sum_{\alpha e} A_i A_{\lambda} \text{Im} \left\{ e^{i(\phi_i - \phi_j)} \alpha_{ij} \right\}$$

are calculated afresh at each iteration.
The performance of VA01A was good for up to 30 parameters and assuming the initial estimate of the solution $X$ was a reasonably accurate one. It essentially failed when these two conditions were not met. VA06A is an extremely reliable rugged program and tests showed it to be quite capable of minimizing in 40 parameters even from rather poor starting estimates. In those cases where both programs succeeded VA01A was the quicker. The convergence criterion of VA01A took the form of a tolerance vector $E$ on the parameters $X$. When

$$X_{n}^{k+1} - X_{n}^{k} \leq E_{n} \quad \text{(all n)}$$

the solution was terminated. VA06A was much harder to use in this respect the iteration being stopped when

$$\sum_{n} \left| \frac{\partial L}{\partial X_{n}} \right|^{2} < \text{EPS}$$

where EPS is specified by the user. In practice this never occurred as EPS was set so tiny that machine rounding errors introduced perturbations upon the derivatives of $O(\text{EPS})$. However since the likelihood did not change by more than 1 part in $10^5$ for many iterations before this the success of the solution was assured.

Programming Procedure

The normalization coefficients $R_{nm}^{a}$ do not depend upon the parameters $\{X\}$ and as such need to be calculated once only. With the simplification that states of different $J^P$ do not interfere on the Dalitz Plot the integration in
the angles $\Theta$, $\Phi$ is quite straightforward to perform analytically. The resulting expressions are then integrated over $\omega_1^2$, $\omega_2^2$ numerically. The accuracy obtained in these constants was typically 0.2% and this was considered sufficiently good to work with. The value $\frac{R_{1i}^1}{R_{2i}^2}$ is the ratio of the predicted cross sections $\sigma(\pi^0\pi^+\rho_i^{-})$; $\sigma(\pi^+\pi^-\pi^0)$ for partial wave $\{i\}$ and these are tabulated in 5.1.

For each real event $\{i\}$ the various $F_{\mu_{Zi}}^n(y_i)$ are calculated and stored on tape. Prior to a new solution being attempted a specified selection of these are off-loaded to disk, and the data set so created is read by the programs at each iteration.

Experience has shown that the complex nature of the formalism and programming invariably induce errors and an important task was the thorough checking at the several stages of the calculations as follows:

(a) The functions $f_n^{\mu_i\mu_f}(\Theta,\delta,\Theta,\Phi)$ are required to satisfy various normalization and symmetry properties, detailed in Appendix II. The normalization was checked by explicit calculation and the orthogonality of the $f$'s by numerical computation (i.e. the integrals should be zero within errors). The amplitude was also constructed by interchanging the labels of the two pions and the differential cross section found to be invariant to this change.

(b) The BERKELEY-SLAC collaboration use a formalism which is quite different to that employed by ourselves and in
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Table 5.1

Predicted Ratio \( R = \frac{\sigma(\pi^{0}\pi^{+}p)}{\sigma(\pi^{+}\pi^{+}n)} \)
particular construct single- and di-particle states using the so-called helicity convention rather than the canonical convention. This helicity formalism, is described in Ref. 35 and an independent program based on this was written at Imperial College by Andy White. The two programs were checked against each other on a wave by wave and event by event basis. Unfortunately an immediate comparison of amplitudes is not possible due to the different phase factors inherent in the formalisms, but the differential cross-sections should be identical and this was found to be the case in nearly all combinations. The only discrepancy was found to be in the $j = \frac{1}{2}$ even parity (odd $L$) waves. Studies showed this to be due to a phase difference of $\pi$ in the contributions from such waves. The reason for the discrepancy has not yet been determined; the phase discrepancy remains and comparison of the phases determined by the different institutions for this effect must be taken into account.

5.4 Error Matrix

The covariance of two parameters $a_i$, $a_j$ describing the likelihood, about the values which maximise the likelihood, $a_i^{\text{max}}$, $a_j^{\text{max}}$ is

$$< (a_i - a_i^{\text{max}})(a_j - a_j^{\text{max}}) >$$

$$= \frac{\int (a_i - a_i^{\text{max}})(a_j - a_j^{\text{max}}) L(a|S) \, da}{\int L(a|S) \, da}$$
measures the extent to which the various parameters are correlated with each other. The quantity

\[ \sqrt{\langle (a_i - a_{i}^{\text{max}})^2 \rangle} \]

is generally referred to as the standard deviation on the parameter \( a_i \). Assuming that the likelihood function has been constructed properly then, in the limit of sufficiently good statistics, \( \mathcal{L} \) is a hyper-gaussian in parameter space. In this case the hypersurface over which the log likelihood function \( L \) is defined as

\[ L = L_{\text{max}} - 0.5 \]

is known as the error ellipse. The standard deviation on a parameter \( a_i \) is then

\[ \Delta a_i = \left( H^{-1} \right)_{ii}^{1/2} \]

where \( H \) is the second derivative matrix of \( L \)

\[ H_{ij} = \frac{\partial^2 L}{\partial a_i \partial a_j} \]

The likelihood function here defined does not satisfy such properties: it is invariant to changes in scale of all the amplitudes \( A_n \rightarrow s A_n \) and in matrix language the second derivative matrix has a zero eigenvalue. (Similarly a change of all phases \( \phi_n \rightarrow \phi_n + \psi \) leaves the likelihood unchanged.) Consequently the second derivative matrix is singular: it cannot be inverted. The explanation lies in the redundant parametrization of the likelihood, and a usual
solution (not adopted here) is to 'freeze' one amplitude and one phase. However the asymmetry in the treatment of the parameters introduced by this can lead to trouble and a particular instance, noted also by Miller, was that the programs take longer converging to a solution. We have allowed all the parameters to vary simultaneously and estimated the errors assuming that the likelihood has a gaussian shape—the second derivative matrix is an estimate only and can in consequence be inverted. This has been checked in two ways:

(a) Fix one amplitude and phase at the minimum and determine a proper second derivative matrix.

(b) Find the maximum $A_i$ such that

$$L(A_i \pm \Delta A_i, \sum_{n \neq i} A_n \phi_n) - L(A_n^{\text{max}}) = -0.5$$

where the remaining parameters $A_n$ are allowed to vary to minimise the perturbed likelihood. To do this subroutine MINOS from the program package MINUIT was used.

The results of these two methods were found to be very similar.

5.5 Fitting Procedure

Data has been analysed at the 4 lab. momenta 895, 945, 995, 1040 MeV/c. The beam momentum cuts made to obtain a reasonable no. of events at each momentum were ±25, ±25, ±25, ±20 MeV/c respectively. This and other statistical
information is summarised in Table 5.2.

Neither of two other major groups involved in similar studies, SACLAY and the BERKELEY-SLAC (B-S) collaboration, have attempted to fit the $\pi^+\pi^-n$ channel and the inclusion of this channel and the presence of the $N^*(1470)$ isobar distinguishes the Imperial College analysis from these. The various solutions obtained by B-S and SACLAY are compared and contrasted with our own in Section 5.6. No solution can be regarded as definitive until the $\pi^+\pi^-n$ data has been satisfactorily described, and we have made definite progress in doing this. Further improvements to the model such as the corrections dictated by 3-body unitarity and the peripheral nature of $\rho$ production at higher energies have not been incorporated to date, but better future analyses must definitely take this into account.

Two solutions have been obtained by the author and are referred to hereafter as the '10-wave' and the '14-wave' solutions. The practical manner in which these were obtained is described immediately below. All discussion pertaining to the conclusions is deferred until section 5.6.

10-wave solution

The earlier Dalitz-plot fitting suggested that waves of $D_{5/2}$ and greater were unimportant at these energies and this seemed to be born out by B-S who did not discover significant amounts of such waves particular in solution B(qv.)\textsuperscript{36}. The data was therefore
<table>
<thead>
<tr>
<th>QUANTITY</th>
<th>895 MeV/c</th>
<th>945 MeV/c</th>
<th>995 MeV/c</th>
<th>1040 MeV/c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δp BEAM</td>
<td>±25</td>
<td>±25</td>
<td>±25</td>
<td>±20</td>
</tr>
<tr>
<td>No. (\pi^0\pi^+p) events</td>
<td>5537</td>
<td>6596</td>
<td>5965</td>
<td>5207</td>
</tr>
<tr>
<td>No. (\pi^+\pi^+n) events</td>
<td>900</td>
<td>1116</td>
<td>1115</td>
<td>989</td>
</tr>
<tr>
<td>Ratio (\sigma(\pi^0\pi^+p):\sigma(\pi^+\pi^+n))</td>
<td>6.15</td>
<td>5.91</td>
<td>5.35</td>
<td>5.27</td>
</tr>
<tr>
<td>R. fitted</td>
<td>5.90</td>
<td>5.63</td>
<td>5.12</td>
<td>5.04</td>
</tr>
<tr>
<td>ECMS MeV.</td>
<td>1612</td>
<td>1641</td>
<td>1669</td>
<td>1694</td>
</tr>
<tr>
<td>(\sigma(\pi^0\pi^+p) + *)</td>
<td>11.92</td>
<td>12.49</td>
<td>12.63</td>
<td>12.60</td>
</tr>
<tr>
<td>(\sigma(\pi^+\pi^+n))</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*From Ref.1

Table 5.2
fitted with a very limited number of partial waves (given in Table 5.3) and the highest angular momentum transition considered was the $\Delta-\pi$ DD3.

Partial wave amplitudes and phases were generated by a Monte-Carlo program. The ASD1 was set to have an amplitude of unity and phase zero and the other $A_k, \phi_k$ were chosen such that

$$10^2 \gg A_k \gg 10^{-2}$$

$$\pi \gg \phi_k \gg -\pi$$

At 895 and 945 MeV/c approximately 150 random sets of amplitudes were generated and those dozen or so having the best likelihood were used as starting values for the full minimisation process. At 995 MeV/c there were fewer random sets chosen ($\approx 50$) and only 3 of these were used in the full minimisation process.

At each momentum all of these starting sets converged to the same solution and reasonable continuity between solutions at different momenta was apparent. The T matrix elements defined in equation (2) are presented in Table 5.3. The phase of the ASD1 is set to zero because there is an overall phase ambiguity at each momentum: the phases determined are relative phases between the transitions. To obtain the overall phase at each energy a common procedure is to minimise the net path length between the points determined at different energies in the multidimensional space of phases. This often obscures any resonance structure.
### Table 5.3

<table>
<thead>
<tr>
<th>MOMENTUM</th>
<th>'10-WAVE SOLUTION'</th>
<th>'14-WAVE SOLUTION'</th>
</tr>
</thead>
<tbody>
<tr>
<td>895</td>
<td>$\Delta \pi$ SD1, PP1, PP3, PF3, DS3, DD3</td>
<td>$\Delta \pi$ SD1, PP1, PP3, PF3, DS3, DD3, DD5, FP5</td>
</tr>
<tr>
<td></td>
<td>$N^\pi$ SS1, PP1, PP3</td>
<td>$N^\pi$ SS1, PP1, PP3, DD3, DD5, FF5</td>
</tr>
<tr>
<td></td>
<td>$\rho_1 N$ SS1, PP1, PP3</td>
<td>$\rho_1 N$ SS1, PP1, PP3, DD3, DD5, FF5</td>
</tr>
<tr>
<td></td>
<td>$\rho_3 N$ SD1, PP1, PP3, DS3, DD3</td>
<td>$\rho_3 N$ SD1, PP1, PP3, PF3, DS3, DD3, DD5, FP5, FF5</td>
</tr>
<tr>
<td>945</td>
<td>$\Delta \pi$ SD1, PP1, PP3, PF3, DS3, DD3</td>
<td>d.o.</td>
</tr>
<tr>
<td></td>
<td>$N^\pi$ SS1, PP1, PP3, DD3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\rho_1 N$ SS1, PP1, PP3, DD3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\rho_3 N$ SD1, PP1, PP3, PF3, DS3, DD3</td>
<td></td>
</tr>
<tr>
<td>995</td>
<td>d.o.</td>
<td>d.o.</td>
</tr>
<tr>
<td>1040</td>
<td>d.o.</td>
<td>d.o.</td>
</tr>
</tbody>
</table>

Starting sets of partial waves.

N.B. Unimportant waves may subsequently be removed from the fit.
that might be present in the waves and in any case we have not sufficient data points to make this viable. A more complete description is afforded by a multi-channel $K$-matrix formalism which has the advantage of including the information known about the elastic channel. The construction of a $K$-matrix to describe a $2 \to 3$ body reaction rather more complicate that in the simple $2 \to 2$ body case and the particular approach adopted by B-S is described in Ref. 46. Again our lack of data points precludes an independant $K$-matrix fit but the phase variation obtained by B-S for the $\Delta SD1$ wave (from solution B) has been used to rotate the phases of our fit at each energy appropriately. The result is depicted in the Argand Plot of Fig. 5.1.

14-wave solution

For the reasons to be discussed in 5.6 the fit obtained with so limited no. of partial waves had some unsatisfactory features which it was thought could well be symptomatic of the lack of any high partial wave. The data at all 4 energies was re-fitted with the 30-wave set of Table 5.3. The starting values were the amplitudes and phases of the 10-wave solution plus higher amplitudes initially set to a small positive value. Very few extra starting sets were generated using random numbers although a few were made at 995 MeV/c. There is some evidence to suggest that the statistics and/or programs are not quite good enough to allow a reasonable fit in so many variables: the random starting sets at 995 MeV/c often converged to several
<table>
<thead>
<tr>
<th></th>
<th>895</th>
<th>945</th>
<th>995</th>
<th>1040</th>
</tr>
</thead>
<tbody>
<tr>
<td>ΔSD1 T</td>
<td>0.36 ± 0.05</td>
<td>0.38 ± 0.06</td>
<td>0.39 ± 0.07</td>
<td>0.41 ± 0.06</td>
</tr>
<tr>
<td>φ</td>
<td>0.0 ± 0.14</td>
<td>0.0 ± 0.12</td>
<td>0.0 ± 0.12</td>
<td>0.0 ± 0.10</td>
</tr>
<tr>
<td>ΔPP1</td>
<td>0.33 ± 0.07</td>
<td>0.40 ± 0.07</td>
<td>0.10 ± 0.03</td>
<td>0.35 ± 0.05</td>
</tr>
<tr>
<td></td>
<td>-1.20 ± 0.12</td>
<td>-1.17 ± 0.09</td>
<td>-2.10 ± 0.23</td>
<td>-0.61 ± 0.14</td>
</tr>
<tr>
<td>ΔPP3</td>
<td>0.16 ± 0.04</td>
<td>0.23 ± 0.05</td>
<td>0.44 ± 0.09</td>
<td>0.08 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>-2.85 ± 0.28</td>
<td>-3.06 ± 0.18</td>
<td>2.37 ± 0.18</td>
<td>-2.94 ± 0.35</td>
</tr>
<tr>
<td>ΔDS3</td>
<td>0.29 ± 0.04</td>
<td>0.31 ± 0.04</td>
<td>0.24 ± 0.04</td>
<td>0.31 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>3.08 ± 0.16</td>
<td>-3.09 ± 0.12</td>
<td>2.90 ± 0.18</td>
<td>-2.46 ± 0.15</td>
</tr>
<tr>
<td>ΔDD3</td>
<td>0.10 ± 0.03</td>
<td>0.14 ± 0.03</td>
<td>0.14 ± 0.03</td>
<td>0.20 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>-3.01 ± 0.41</td>
<td>3.07 ± 0.23</td>
<td>1.49 ± 0.31</td>
<td>3.11 ± 0.21</td>
</tr>
<tr>
<td>N*SS1</td>
<td>0.04 ± 0.04</td>
<td>0.15 ± 0.04</td>
<td>0.23 ± 0.05</td>
<td>0.19 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>-1.53 ± 1.47</td>
<td>-0.31 ± 0.34</td>
<td>-0.42 ± 0.27</td>
<td>-0.34 ± 0.26</td>
</tr>
<tr>
<td>N*PP1</td>
<td>0.15 ± 0.04</td>
<td>0.12 ± 0.04</td>
<td>0.04 ± 0.04</td>
<td>0.17 ± 0.03</td>
</tr>
<tr>
<td></td>
<td>-2.71 ± 0.30</td>
<td>-2.45 ± 0.33</td>
<td>0.11 ± 0.56</td>
<td>-1.79 ± 0.28</td>
</tr>
<tr>
<td>ρ1SS1</td>
<td>0.28 ± 0.05</td>
<td>0.19 ± 0.05</td>
<td>0.20 ± 0.03</td>
<td>0.28 ± 0.03</td>
</tr>
<tr>
<td></td>
<td>1.74 ± 0.18</td>
<td>1.32 ± 0.23</td>
<td>0.73 ± 0.13</td>
<td>0.14 ± 0.15</td>
</tr>
<tr>
<td>ρ1PP1</td>
<td>0.09 ± 0.05</td>
<td>0.03 ± 0.03</td>
<td>0.05 ± 0.04</td>
<td>0.09 ± 0.03</td>
</tr>
<tr>
<td></td>
<td>0.87 ± 0.52</td>
<td>-1.72 ± 0.97</td>
<td>1.76 ± 0.59</td>
<td>-0.55 ± 0.52</td>
</tr>
<tr>
<td>ρ3PP1</td>
<td>0.14 ± 0.05</td>
<td>0.24 ± 0.05</td>
<td>0.19 ± 0.03</td>
<td>0.25 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>-0.28 ± 0.34</td>
<td>0.40 ± 0.19</td>
<td>0.39 ± 0.19</td>
<td>0.87 ± 0.15</td>
</tr>
</tbody>
</table>

'10-Wave Solution

Table 5.4
<table>
<thead>
<tr>
<th>WAVE</th>
<th>395</th>
<th>945</th>
<th>995</th>
<th>1040</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASDL</td>
<td>0.32 ± 0.04</td>
<td>0.34 ± 0.04</td>
<td>0.35 ± 0.04</td>
<td>0.39 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>0.0 ± 0.11</td>
<td>0.0 ± 0.09</td>
<td>0.00 ± 0.12</td>
<td>0.0 ± 0.12</td>
</tr>
<tr>
<td>APP1</td>
<td>0.33 ± 0.04</td>
<td>0.32 ± 0.04</td>
<td>0.26 ± 0.04</td>
<td>0.18 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>0.26 ± 0.15</td>
<td>-0.15 ± 0.10</td>
<td>-0.33 ± 0.12</td>
<td>-0.27 ± 0.21</td>
</tr>
<tr>
<td>APP3</td>
<td>0.17 ± 0.03</td>
<td>0.20 ± 0.02</td>
<td>0.16 ± 0.02</td>
<td>0.16 ± 0.03</td>
</tr>
<tr>
<td></td>
<td>1.86 ± 0.15</td>
<td>1.63 ± 0.12</td>
<td>1.60 ± 0.20</td>
<td>2.04 ± 0.27</td>
</tr>
<tr>
<td>AD53</td>
<td>0.21 ± 0.03</td>
<td>0.24 ± 0.03</td>
<td>0.24 ± 0.03</td>
<td>0.17 ± 0.03</td>
</tr>
<tr>
<td></td>
<td>-2.42 ± 0.15</td>
<td>-2.52 ± 0.12</td>
<td>-2.56 ± 0.12</td>
<td>-2.56 ± 0.18</td>
</tr>
<tr>
<td>ADD3</td>
<td>0.16 ± 0.02</td>
<td>0.15 ± 0.02</td>
<td>0.10 ± 0.03</td>
<td>0.08 ± 0.02</td>
</tr>
<tr>
<td></td>
<td>2.73 ± 0.14</td>
<td>2.68 ± 0.13</td>
<td>2.34 ± 0.18</td>
<td>1.06 ± 0.39</td>
</tr>
<tr>
<td>ADD5</td>
<td>0.17 ± 0.03</td>
<td>0.22 ± 0.03</td>
<td>0.28 ± 0.03</td>
<td>0.31 ± 0.03</td>
</tr>
<tr>
<td></td>
<td>2.18 ± 0.13</td>
<td>1.90 ± 0.09</td>
<td>1.70 ± 0.08</td>
<td>1.72 ± 0.09</td>
</tr>
<tr>
<td>N SS1</td>
<td>0.05 ± 0.02</td>
<td>0.09 ± 0.02</td>
<td>0.11 ± 0.03</td>
<td>0.13 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>-1.62 ± 0.49</td>
<td>-0.38 ± 0.54</td>
<td>-0.04 ± 0.39</td>
<td>0.40 ± 0.33</td>
</tr>
<tr>
<td>N PP1</td>
<td>0.17 ± 0.03</td>
<td>0.09 ± 0.02</td>
<td>0.11 ± 0.03</td>
<td>0.08 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>-1.36 ± 0.22</td>
<td>-1.22 ± 0.36</td>
<td>-1.37 ± 0.30</td>
<td>-1.53 ± 0.46</td>
</tr>
<tr>
<td>N DD3</td>
<td>0.13 ± 0.02</td>
<td>0.14 ± 0.02</td>
<td>0.17 ± 0.02</td>
<td>0.19 ± 0.02</td>
</tr>
<tr>
<td></td>
<td>0.26 ± 0.20</td>
<td>-0.08 ± 0.16</td>
<td>-0.50 ± 0.16</td>
<td>-0.66 ± 0.15</td>
</tr>
<tr>
<td>p1SS1</td>
<td>0.24 ± 0.03</td>
<td>0.25 ± 0.02</td>
<td>0.21 ± 0.03</td>
<td>0.21 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>1.07 ± 0.20</td>
<td>0.78 ± 0.16</td>
<td>0.46 ± 0.23</td>
<td>0.85 ± 0.29</td>
</tr>
<tr>
<td>p1PP1</td>
<td>0.12 ± 0.03</td>
<td>0.15 ± 0.03</td>
<td>0.12 ± 0.04</td>
<td>0.05 ± 0.05</td>
</tr>
<tr>
<td></td>
<td>-1.17 ± 0.32</td>
<td>-1.45 ± 0.30</td>
<td>-2.69 ± 0.37</td>
<td>-2.46 ± 0.98</td>
</tr>
<tr>
<td>p3PP1</td>
<td>0.15 ± 0.03</td>
<td>0.07 ± 0.03</td>
<td>0.18 ± 0.04</td>
<td>0.17 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>1.07 ± 0.29</td>
<td>1.91 ± 0.49</td>
<td>1.63 ± 0.21</td>
<td>1.66 ± 0.21</td>
</tr>
<tr>
<td>p3DS3</td>
<td>0.14 ± 0.01</td>
<td>0.19 ± 0.02</td>
<td>0.18 ± 0.02</td>
<td>0.17 ± 0.02</td>
</tr>
<tr>
<td></td>
<td>1.50 ± 0.13</td>
<td>1.06 ± 3.12</td>
<td>0.92 ± 0.14</td>
<td>1.27 ± 0.17</td>
</tr>
<tr>
<td>p3DD3</td>
<td>0.11 ± 0.03</td>
<td>3.15 ± 3.02</td>
<td>0.16 ± 0.03</td>
<td>0.18 ± 0.02</td>
</tr>
<tr>
<td></td>
<td>-3.10 ± 0.21</td>
<td>-2.89 ± 0.14</td>
<td>3.10 ± 0.15</td>
<td>3.00 ± 0.17</td>
</tr>
</tbody>
</table>

14-WAVE SOLUTION

Table 5.5
solutions all with the same gross characteristics but with discrepancies inconsistent with the errors determined on the parameters. This is suggestive of a likelihood space having many subsidiary maxima all lying rather close together.

The solutions obtained were examined for consistency and continuity among the 4 energies. Except between 895 and 945 MeV/c they were very discontinuous particularly in phase. All those waves which had significant magnitudes at a minimum of 2 energies were retained and the rest were eliminated. The resultant 14-wave set was then used at each energy to fit the data. T matrix elements for this set are tabulated in 5.5. The prescription for the 10-wave set was again used to fix the phase variation of the 14-wave set. The argand diagram are presented as are single plots for each energy in Figure 5.2. The behaviour of the amplitudes and phases showed discrepancies with Solution B\textsuperscript{36} of B-S which was used to provide a reference phase and so one should regard the overall phases with caution.
5.6 Discussion and Conclusions

An idea of the quality of fit to the data may be gained by examination of the histograms in \((\omega_1^2, \omega_2^2, \Theta - \Phi)\) and the superimposed fitted curves (Fig. 5.3). A few remarks concerning these are necessary.

At all energies the distributions of the \(\pi^0 \pi^+ p\) channel are reproduced with high fidelity and this is encouraging since some features require the contribution of several different partial waves for their explanation (e.g. the flattening of \(\Theta\) around \(\Theta = \pi/2\)). Things are not quite so good in the \(\pi^+ n\) channel. The \(\pi^+ n\) mass distribution, particularly separation of the \(\Delta^+\) and its reflection are well reproduced, but the flattish shape of the \(\pi^+\pi^+\) mass is not observed in the data, neither is the slight enhancement shown towards low and high \(\pi^+\pi^+\) masses. At all energies but 1040 MeV/c the fit predicts more events with \(\phi = 0\) than are observed, and in general the \(\Theta\) distribution of the data rises rather sharper in the range \(\pi/4 \rightarrow 2\pi/3\) that the fit, with a flattening around \(\pi/2\). (N.B. Because of the \(\pi^+ - \pi^+\) folding \(\phi\) and \(\Theta\) are symmetric about 0 and \(\pi/2\) respectively).

These effects in the angular distribution are equivalent to an inability to make the neutron peripheral enough (\(\cos \Theta\) production = \(\cos \Phi \sin \Theta\)). One hypothesis is that a \((\pi^+\pi^+)\) enhancement is required since this would tend to correct both these failings, but this is disturbingly contrary to the spirit of the isobar model in that there are no known exotic resonances.
Some quantitative estimates on the goodness of fit have been obtained by binning the 1-d distributions and determining the $\chi^2$/Degrees of Freedom for the fit. Typical values obtained were in the range 1.2 $\rightarrow$ 1.6. These nos. are not too meaningful however since they ignore the correlations between the variables, but even with our comparatively excellent statistics binning in 4-variables is profitless.

The Argand Plots of Figs. 5.1 and 5.2 were obtained by fixing the undetermined overall phases as described in section 5.4. The solutions obtained for our $I = \frac{3}{2}$ amplitudes may be compared with other values from the recent analyses by Berkeley-SLAC and Saclay.

B-S have found two solutions which describe their $\pi^-p$ and $\pi^+p$ data. The first of these, now generally referred to as solution A, is described in Ref. 37. This solution depended rather critically on the phase continuation of the waves through a 100 MeV. wide gap in the data between 1540 - 1640 MeV. CMS energy. The partial wave amplitudes and phases reported by Saclay at the Aix-en-Provence Conference suggested that another continuation might be possible and that some other waves might also be important: namely the $\pi^-\Delta$ (SD11, DD33, FF15) and $\rho_1N$ (PP11), and the presence of these waves was also strongly suggested by theory. B-S subsequently obtained another solution 'B' which incorporated the new waves, agreed quite well with the Saclay results, and had a rather more desirable theoretical
interpretation in that the relative coupling signs of the $\Delta-\pi$ DD13(1520) and DD15(1600) agreed with $\xi$-broken SU(6)$_W$ (which had derived considerable theoretical support from the work of Melosh\textsuperscript{40}), as did the phases in the higher energy range (1640 - 1900 MeV). Both our 10 and 14 wave solutions tend to confirm the correctness of solution B. The overall phase at each energy was determined assuming this to be true but the discrimination between A and B did not depend upon this. The relative phases between $\Delta-\pi$ SD31, $\Delta-\pi$ DS33 and $\rho_1-N$ SS31 are quite different in the two solutions and these phases determined in our fit are much closer to B than to A. The Imperial College $\pi^+\rho + \pi^+\pi^0p$ data at 895 MeV/c has been analysed by B-S and also confirms the correctness of B\textsuperscript{41}. The continuous line on some of the Argand plots is the result of the K-matrix fit B-S made to their "B" energy independent points over a comparable energy range and is reproduced for comparison.

The "10-wave" solution exhibits some discouraging features. Many of the waves are discontinuous in phase and the $\Delta-\pi$ PP1 and PP3 have a random looking amplitude variation. The limitation of the partial wave series below $D_{3/2}^+$ was suggested by the fact that B-S and Saclay did not require higher waves at these energies. The situation improves dramatically with the 14-wave fit due largely to the addition of the $\Delta-\pi$ DD5 amplitude, and hence we have some major discrepancies with the other analyses. The sub-
sequent remarks will be confined to the 14-wave solution. Both the $\Delta-\pi$ PP1 and DD5 show large amplitudes with strikingly good phase continuity. Because of their strong contributions the P31 and D35 cross-sections both violate the predictions of the elastic analyses established by unitarity. (Fig. 5.4 compares the $J^P$ cross-sections with the SACLAY$^{19}$ and CERN$^{20}$ solutions.) The phase behaviour of these waves is suggestive of non-resonant background. The larger waves contributing to the S31 and D33 cross-sections show strong resonance-like behaviour of their amplitudes and phases, and confirms the existence of resonances with these $J^P$'s at around 1650 MeV/c. (A K-matrix fit to our results in these channels is, unfortunately, not possible, because even a minimal K-matrix parametrization requires more free variables than we have data points, and so we must await the further energy independent work currently in progress at I.C.). The $\Delta-\pi$ DS33 has a coupling opposite to that of the $\Delta-\pi$ SD31 but of the same sign as the $\Delta-\pi$ DD33. This latter observation is predicted by $L$-broken SU(6)$^{42}_W$ and is contrary to naive SU(6)$_W$. The behaviour of the $\rho_4^-$-$N$ SS31 amplitude is indicative of there being background underneath this wave and so it is not possible to determine the sign of this coupling at present. The behaviour of both the $\Delta-\pi$ DS33 and $\rho-N$ SS31 is similar to B-S solution B although the former is rather shifted in phase. The $N^*(1470)-\pi$ SS31 transition exhibits convincing resonance behaviour with a coupling of the same sign as the $\Delta-\pi$ SD31. The $N^*-\pi$ DD33
shows no phase movement at all within the errors quoted. (The phases of some transitions are very close - Figure 5. gives a separate argand diagram at each energy showing the relative phases of the transitions.) The remaining D33 transitions of the $\rho_{3/2}^-\bar{N}$ type show a somewhat ambiguous behaviour but the DS33 is in good agreement with the B-S solution. The only P$_{33}$ transition required was the $\Delta^-\pi$ PP33 and the cross-section for this wave was found to be fairly small. No resonance behaviour is observed over our energy range and so the possible claims of a P33 resonance at 1700 MeV. by B-S are not confirmed by us.

To summarise: for those waves where we have an unambiguous behaviour $\Delta\pi SD31$, $\Delta^-\pi$ DS33, $\Delta^-\pi$ DD33 the predictions of "anti-" SU(6)$_W$ appear to be valid. Another feature of the B-S solution that we reproduce are the phase variation of the $\rho_{3/2}^-\bar{N}$ DS33 wave. Strong contributions are observed in the $\Delta^-\pi$ P31 and D35 waves which are not seen elsewhere: this might be due to the elasticity constraint incorporated into the fitting programs of B-S and Saclay. The $N^*(1470)-\pi$ SS31 shows convincing resonant behaviour and is of the same sign as $\Delta^-\pi$ SD31. No P$_{33}$ resonance is observed in our analysis.

The analysis presented here is being continued to higher and lower energies and will be the result of further papers and theses.
10 Wave Solution

Fig 5.1a.
10 Wave Solution

Fig 5.1
<table>
<thead>
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<th>#</th>
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</tr>
<tr>
<td>2</td>
<td>(1.641)</td>
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</tr>
<tr>
<td>3</td>
<td>(1.669)</td>
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</tr>
<tr>
<td>4</td>
<td>(1.694)</td>
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14 Wave Solution

Fig 5.2a
Fig 5.2b
14. Wave Solution

Fig 5.2c
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<thead>
<tr>
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<th>Symbol</th>
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<td>$\Delta^{-\pi}$</td>
<td>SD1</td>
</tr>
<tr>
<td>2</td>
<td>$\Delta^{-\pi}$</td>
<td>PP1</td>
</tr>
<tr>
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</tr>
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<td>4</td>
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<td>6</td>
<td>$\Delta^{-\pi}$</td>
<td>DD5</td>
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<tr>
<td>7</td>
<td>$N^{*-\pi}$</td>
<td>SS1</td>
</tr>
<tr>
<td>8</td>
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<td>PP1</td>
</tr>
<tr>
<td>9</td>
<td>$N^{*-\pi}$</td>
<td>DD3</td>
</tr>
</tbody>
</table>

**A** $\rho_1^{-N}$ SS1  
**B** $\rho_1^{-N}$ PP1  
**C** $\rho_3^{-N}$ PP1  
**D** $\rho_3^{-N}$ DS1  
**E** $\rho_3^{-N}$ DD3
Fig. 5.2d
14 Wave Solution
Fig 5.2e
895 Mev/c

Fig 5.3a

EFFECTIVE MASS SQUARED $\pi^+ p$ 0.895 GeV/c

EFFECTIVE MASS SQUARED OF $\pi^+ \pi^0$ 0.895 GeV/c
895 Mev/c

**Fig 5.3b**

**EFFECTIVE MASS SQUARED PI N 0.895 GEV/C**

**EFFECTIVE MASS SQUARED PI PI 0.895 GEV/C**
Fig 5.3c
Fig 5.3d

895 MeV/c

(THETA) -- POLAR ANGLE OF BEAM PI+ P PIO CHANNEL

(PHI) -- AZIMUTHAL ANGLE OF BEAM PI+ P PIO CHANNEL
895 Mev./c.

Fig 5.3e
945 Mev/c

Fig 5.3f
Fig 5.3g
Fig 5.3h

$94_5$ MeV/c

$\theta, \pi, \phi, \pi$ CHANNEL

$\theta, \pi, \phi, \pi$ CHANNEL
Fig 53i

94.5 MeV/c

NO. EVENTS

Theta - Polar angle of beam Pi^+ Pi^- Pi^0 channel

Phi - Azimuthal angle of beam Pi^+ Pi^- Pi^0 channel
Fig 5.3j

945 MeV/c

(THETA) -- POLAR ANGLE OF BEAM PI+ PI+ N CHANNEL

(PHI) -- AZIMUTHAL ANGLE OF BEAM PI+ PI+ N CHANNEL
Fig 5.3k

995 Mev/c

EFFECTIVE MASS SQUARED OF PI+ PI0 0.995 GEV/C
THETA VS. PHI PI+ P PIO CHANNEL

THETA VS. PHI PI+ PI+ N CHANNEL

Fig 5.3n
Fig 5.30

995 Mev/c

$(\Theta) \quad \text{POLAR ANGLE OF BEAM } \pi^+ \pi^- \pi^0 \text{ CHANNEL}

$(\Phi) \quad \text{AZIMUTHAL ANGLE OF BEAM } \pi^+ \pi^- \pi^0 \text{ CHANNEL}

NO. EVENTS

-250.0
-200.0
-150.0
-100.0
-50.0
0.0
0.250
0.500
0.750
1.000

NO. EVENTS

-300.0
-200.0
-100.0
0.0
100.0
200.0
300.0

$(\Theta) \quad \frac{\pi}{\pi}

$(\Phi) \quad \frac{\pi}{\pi}$
Fig 53p
Fig 5.3q, 1040 Mev/c

EFFECTIVE MASS SQUARED P1+P 1.040 GEV/C

NO. EVENTS

100.0
200.0
300.0
400.0

1.10  1.40  1.70  2.00  2.30

EFFECTIVE MASS SQUARED OF P1+PIO 1.040 GEV/C

NO. EVENTS

0.05  0.15  0.25  0.35  0.45  0.55

Fig 53q.
Fig 53r
Fig 53s
1040 Mev/c
Fig 53t
Fig 5.3u

1040 MeV/c

THETA -- POLAR ANGLE OF BEAM PI+ PI+ N CHANNEL

PHI -- AZIMUTHAL ANGLE OF BEAM PI+ PI+ N CHANNEL
Fig 5.4a
Fig 5.4b
6.1 Introduction

In this chapter the mass plots and production angular distributions are presented for the single pion production channels in the momentum range 1.1 to 1.4 GeV/c. This data has not so far been studied by the techniques developed in the last two chapters, but the phenomenology of events in the $\Delta^{++}$ band has been used to determine the various partial waves contributing in the production of this isobar.

6.2 Phenomenology of Single Pion Production

Comparison of the histograms $M^2(\pi^+\pi^0)$ at the 4 momenta 1100, 1200, 1300, 1400 MeV/c show the increasing importance of the $\rho$ at the higher energies. At 1.1 GeV/c the distribution does not display any clear bump corresponding to the resonance but an enhancement at high $\pi^+\pi^0$ mass values is clearly distinguishable. The classical Breit-Wigner shape of the $\rho$ will be distorted by the phase space cut off, and the effect of the production angular momentum barriers will be to introduce a further skewing to low masses. At 1.3 and 1.4 GeV/c, where the barriers and phase space effects have diminished effect, the $\rho$ peak becomes well separated and seems to be close to its normal value of $\approx 770$ MeV.

The $\pi^+p$ mass spectra show again the enormous $\Delta^{++}$ signal but apart from this and reflections of the $\Delta^+$ very little other structure is visible.
Throughout this energy range the \( \pi^+\pi^+n \) cross section is rising more rapidly than the \( \pi^+p\pi^0 \) and at 1400 MeV/c the ratio \( \sigma(\pi^+\pi^+n)/\sigma(\pi^+p\pi^0) \) has risen to \( \frac{1}{3} \). As discussed in Section 4.3 this clearly indicates the importance of other mechanisms in this channel besides

\[
\pi^+p \rightarrow \pi^+\Delta^+ \rightarrow \pi^+ (\pi^+n)
\]

The peaks of the \( \Delta^+ \) in the \( \pi^+n \) mass distribution are visible but between these there is considerable background. (One \( \Delta^+ \) peak is a reflection from the 'other' \( \pi^+n \) mass combination). This is too large to be explained by positive interference between the deltas, which do not overlap to any considerable degree at these higher momenta. At the very highest momentum there is some suggestion of a further peak in the \( \pi^+n \) spectrum at a mass of \( \approx 1.7 \text{ GeV}/c \), but the statistics and proximity of the \( \Delta \) reflection could be responsible for this. The \( \pi^+\pi^+ \) spectrum seems to be roughly consistent with the phase space prediction, as might be expected from the generally observed absence of exotic states. All the spectra described here are depicted in Figure 6.1.

Production Angular Distributions

Figure 6.2 shows the various production angle spectra for the two channels. In \( p\pi^+\pi^0 \) the \( \pi^0 \) is emitted preferentially in a forward direction but with a smaller backward peak at 1100 MeV/c becoming much stronger with increasing energy. This peak remains even when events in the \( \Delta^{++} \) band are selected and also when the Eberhard-Pripstein 49
separation is made to eliminate the rho (q.v.). Such an effect is not explicable by t-channel exchanges and some \( \omega \)-channel baryon exchange is required to account for it.

The \( \pi^+ \) production angles show an essentially flat distribution in \( \cos 0 \) with a slight tendency to peak in the forward and backward directions. This is noticeable at 1.1 GeV/c, slightly more pronounced at 1.2 GeV/c but the backward peak virtually disappears at 1.3 and 1.4 GeV/c. At the lower momenta there is presumably a substantial contribution from a \( \cos^2 0 \) term in the differential cross section which becomes less important at the higher momenta. The proton is becoming more peripheral as the energy increases (the production angle in the figure is measured w.r.t. the incident proton in CMS) and the backward peak becomes sharper. At 1.3 and 1.4 GeV/c there is a slight tendency for a forward peak to develop in the \( \cos 0 \) distribution.

The production angles in the \( \pi^+ \pi^+ n \) channel also show interesting structure. As with the proton the neutron becomes increasingly peripheral with energy. There also appears to be some evidence for a forward enhancement which again is similar to the proton production distributions. The \( \pi^+ \) production angle shows pronounced forward/backward peaks at all energies and the entire distribution looks very similar from energy to energy.
Considerable interest centres in establishing the mechanism of Δ production and such work often takes the form of, for example, establishing the Δ matrix elements for various values of the CMS scattering angle θ. We are primarily concerned with the study of resonance production in the s-channel and subsequently would like to determine which of the many possible angular momentum transitions are strong, and to cut off the partial wave expansion above some maximum value L' for the Δ-π orbital angular momentum. The data may then be subjected to a far more detailed analysis as described in Chapter V.

The Δ is partially polarized by the production process and this effect may be described by the elements of the density matrix ρ in spin space. If then the Δ decays to p-π the angular distribution (θ*, φ*) of these particles in

<table>
<thead>
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<th>Momentum (MeV/c)</th>
<th>R</th>
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<tbody>
<tr>
<td>895</td>
<td>5.96 ± 0.21</td>
</tr>
<tr>
<td>945</td>
<td>5.69 ± 0.18</td>
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<td>995</td>
<td>5.04 ± 0.16</td>
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<tr>
<td>1040</td>
<td>4.76 ± 0.16</td>
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<tr>
<td>1092</td>
<td>4.24 ± 0.26</td>
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<tr>
<td>1182</td>
<td>4.05 ± 0.21</td>
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<tr>
<td>1283</td>
<td>3.60 ± 0.17</td>
</tr>
<tr>
<td>1394</td>
<td>2.92 ± 0.12</td>
</tr>
</tbody>
</table>

Table 6.1
the Δ rest frame is given by:

\[
I(\theta^*, \phi^*) \propto \left\{ \begin{array}{c}
\frac{1}{\sqrt{\pi}} \left( \rho_{ss} + \rho_{uu} \right) Y_0^0(\theta^*, \phi^*) - \frac{1}{\sqrt{\pi}} \left( \rho_{uu} - \rho_{ss} \right) Y_0^0(\phi^*, \theta^*) \\
+ \frac{g}{\sqrt{\pi}} \Re \rho_{l3} \Re Y_L^l(\theta^*, \phi^*) - \frac{g}{\sqrt{\pi}} \Re \rho_{l1} \Re Y_L^l(\phi^*, \theta^*)
\end{array} \right. \]

(1)

where the angles $\theta^*$, $\phi^*$ describing the decay are measured in any frame with the z-axis in the production plane. The elements $\rho_{nm}$ will be dependant on the Δ production angle $\theta$ in a manner which depends upon the mechanism involved. This formula may be derived as follows:

If $\rho_f$ is the density matrix of the decay nucleon then

\[ I(\theta^*, \phi^*) = \text{Tr}(\rho_f^f). \]

Now the amplitude to produce a nucleon in helicity state $\lambda$ from the rest decay of a resonance of spin $J$ and spin projection $M$, mass $\omega$ is

\[ <g\lambda \rightarrow |T|JM> = T_{\lambda M}^J(\theta^*, \phi^*) \]
and $T$ may be expanded in the formalism propounded by Jacob
and Wick as:

$$
\mathcal{T}^J_{\lambda M}(\Theta, \Psi, \omega) = \int \frac{4\omega}{q} \int \frac{2^{J+1}}{4\pi} \mathcal{D}_{M \lambda}^{J \ast}(\phi, \theta, \omega) \mathcal{T}^J_{\lambda M}(\omega)
$$

Now, since the initially $A$ is a statistical ensemble of spin states described by its density matrix $\rho_i$ we have

$$\rho_f = T \rho_i T^+ \quad \text{(neglecting normalisations)}$$

i.e.

$$
\text{Tr} \rho_f = \sum_{MM'} \mathcal{T}^J_{\lambda M} \rho^i_{MM'} \mathcal{T}^J_{\lambda M'}
$$

and this may be simplified by the following.

(a) Parity conservation in both production and decay (strong interaction) requires

$$
\mathcal{T}^J_{\lambda M} = \mathcal{T}^J_{-\lambda M} \quad (\lambda = \pm \frac{1}{2} \text{ only})
$$

and $\rho_{MM'} = (-1)^{M-M'} \rho_{-M-M'}$ respectively.

(b) $J = \frac{3}{2}$

(c) $\mathcal{D}_{\Theta^J}^{\phi}(R) \mathcal{D}_{\Theta^J}^{\phi}(R) = \sum_{MM'} \left( \mathcal{D}_{\Theta^J}^{\phi} \mathcal{D}_{\Theta^J}^{\phi} \right) \mathcal{D}_{M M'}^{\phi}(R)$

(Clebsch-Gordon series)

Inserting all these into the formula above we arrive at equation (1) above.
In a very similar manner one may determine the spin density matrix elements for the $\Delta$ in production $p^\gamma$.

Using the notation of Roberts, these may be written

$$\rho_{M,M'} = \sum_{\alpha} \xi_{M-M',0} (\Theta) \xi_{M,M'}$$

(2)

with

$$\xi_{M,M'} = \sum_{J L L', J' L' L'^*} R_{n, M M'} T (w) T (w)$$

The coefficients $R$ are then given by (in the case of $\Delta$ production)

$$R_{n, M M'} = (-1)^{2M+1} \frac{1}{2} \{1 + (-1)^{L+L'+n}\} (2J+1)(2J'+1)$$

$$\times \left( \frac{3}{2} \right)^{J M-M' |L^* O} \left( \frac{3}{2} \right)^{J' M'-M' |L^* O} \left( J M-M' \right) \left( J' M-M' \right)$$

The notation in the paper by Roberts differs from that here in that he only considers the production of the resonant $N^*$ and does not consider the decay process. In that case the coefficients $R$ only enter in the combinations where $M = M'$ and the values $M$ are summed over

i.e. $R_{n, J L L', J' L' L'^*} = R_{n, J L L', J' L' L'^*} + R_{n, J L L', J' L' L'^*}$

Combining equations (1) and (2)
\[
\frac{d^3 \sigma}{d \omega \, d \phi \, d \cos \theta'} = \frac{\pi}{q^2} \frac{2 \lambda}{4 \pi} (C_{3L}^0 \gamma_L + C_{1L}^0 \gamma_L) P_\alpha^0 (\cos \theta) \gamma_\alpha^0 (\vec{r}, \vec{r}')
\]
\[
- \frac{1}{\sqrt{4 \pi}} (C_{3L}^0 - C_{1L}^0) P_\beta^0 (\cos \theta) \gamma_\beta^0 (\vec{r}, \vec{r}')
\]
\[
+ \frac{\sqrt{2}}{\sqrt{4 \pi}} \frac{\mathrm{Re} \; C_{3L}^0 \gamma_2^0 \left. P_\alpha^0 (\cos \theta) \right| \mathrm{Re} \; C_{\alpha}^0 (\vec{r}, \vec{r}')}{\sqrt{\pi(n+1)}}
\]
\[
- \frac{\sqrt{2}}{\sqrt{4 \pi}} \frac{\mathrm{Re} \; C_{3L}^0 \frac{1}{2} \left. P_\alpha^0 (\cos \theta) \right| \mathrm{Re} \; C_{\alpha}^0 (\vec{r}, \vec{r}')}{\sqrt{\pi(n+1)(n-1)(n+2)}}
\]

(Note that the x-section does not depend upon \( \phi \) in the instance of an unpolarized target nucleon.)

The normalisation of the previous expression is not absolutely determined but by using

\[
\int d^4 \sigma \, d \omega \, d \phi \, d \phi' = \frac{4 \pi}{q^2} \left\{ C_{3L}^0 \gamma_L \gamma_L + C_{1L}^0 \gamma_L \gamma_L \right\}
\]

the average value of any function \( f \) may be written

\[
\langle f \rangle = \frac{\int f \, d^4 \sigma}{\int d^4 \sigma} = \frac{\int f \, d^4 \sigma}{\frac{4 \pi}{q^2} \left( C_{3L}^0 \gamma_L \gamma_L + C_{1L}^0 \gamma_L \gamma_L \right)}
\]

Then we have

\[
\langle \gamma_{\alpha}^0 \gamma_{\beta}^0 \rangle = \frac{1}{(2\eta+1)} \frac{1}{\sqrt{4 \pi}} \left\{ \frac{C_{3L}^0 \gamma_L \gamma_L + C_{1L}^0 \gamma_L \gamma_L}{C_{3L}^0 \gamma_L \gamma_L + C_{1L}^0 \gamma_L \gamma_L} \right\}
\]

\[
\left\{ \frac{C_{3L}^0 \gamma_L \gamma_L + C_{1L}^0 \gamma_L \gamma_L}{C_{3L}^0 \gamma_L \gamma_L + C_{1L}^0 \gamma_L \gamma_L} \right\}
\]
\[
\begin{align*}
\langle P^0 \gamma^0 \rangle &= -\frac{1}{(2n+1)} \frac{1}{\sqrt{\pi}} \left\{ \frac{C^0_{3/2} \gamma^0_{3/2}}{C^0_{1/2} \gamma^0_{1/2}} \right\} \\
\langle P^1 \gamma^1 \rangle &= \frac{1}{4} \frac{\sqrt{g}}{\sqrt{\pi}} \frac{\sqrt{(n+1)}}{(2n+1)} \frac{\text{Re} \ C^0_{3/2} \gamma^0_{3/2}}{\left\{ C^0_{3/2} + C^0_{1/2} \right\}} \\
\langle P^2 \gamma^2 \rangle &= -\frac{1}{4} \frac{\sqrt{g}}{\sqrt{\pi}} \frac{\sqrt{(n+1)(n+2)}}{(2n+1)} \frac{\text{Re} \ C^0_{3/2} \gamma^0_{3/2}}{\left\{ C^0_{3/2} + C^0_{1/2} \right\}}
\end{align*}
\]

6.4 Data Sample

Isospin Clebsch-Gordon coefficients predict that the ratio

\[
\sigma \left\{ \pi^+ \rho \rightarrow \Delta^{++} \pi^0 \right\} \overset{\text{E}_{\rho \pi^+}}{=} \frac{q}{4r}
\]

\[
\sigma \left\{ \pi^+ \rho \rightarrow \Delta^{++} \pi^0 \right\}
\]

Events from the stronger \( \Delta^{++} \) band are selected such that

\[
1.16 < M^2(\rho \pi^+) < 1.28
\]

The \( \Delta^{++} \) bands overlap with the \( \Delta^+ \) and \( \rho^+ \) bands on the Dalitz plot, heavily contaminating this sample. Fortunately at our
energies these interferences may be removed by the Eberhard-Pripstein\textsuperscript{49} separation. The cosine of the decay angle $\theta^*$ of the $\pi^+$ relative to the $\Delta^{++}$ direction is linearly related to $M^2(\pi^+\pi^0)$

$$\cos \theta^*_{\pi^+} \propto M^2(\pi^+\pi^0)$$

All the interferences in our data occur in the angular range $0 < \cos \theta < 1.0$ but the differential cross sections for decays at angles $\theta^* = \alpha$ and $\theta^* = \pi - \alpha$ are related by parity, so that by restricting $\cos \theta < 0$ we are able to discard the interference regions at the cost of statistics only. These cuts were made and the final data samples remaining for analysis were 536, 599, 596, 838 events at respectively 1.1, 1.2, 1.3, 1.4 GeV/c.

6.5 Experimental Results and Discussion

The moments of the data are tabulated in 6.2. Significantly non-zero values are obtained for values of Legendre coefficients up to $\ell=6$. Values for $\ell > 7$ are consistent with zero, all the moments being within 1.5 standard deviations of zero and most being much closer than this. Values of the coefficients $R$ are not tabulated here but are easy to calculate. The absence of terms with $\ell > 7$ means that transitions with incident $G$ waves are insignificant and the highest waves one needs to consider are $FF7$, $FH7$ (Not $^HLL_{2J}^*$).

Interpretation of the low order moments is very difficult as all the high partial waves contribute to them,
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<td>-0.0270±0.0073</td>
<td>-0.0286±0.0077</td>
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<td>$\langle p_1 \gamma \rangle$</td>
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<td>0.0219±0.0056</td>
<td>0.0043±0.0047</td>
<td>-0.0007±0.0041</td>
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<td>$\langle p_1 \gamma^2 \rangle$</td>
<td>0.0624±0.0073</td>
<td>0.0691±0.0072</td>
<td>0.0679±0.0079</td>
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<td>$\langle p_2 \gamma \rangle$</td>
<td>-0.0041±0.0054</td>
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<td>-0.0263±0.0053</td>
<td>-0.0100±0.0047</td>
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<tr>
<td>$\langle p_2 \gamma^2 \rangle$</td>
<td>0.0339±0.0095</td>
<td>0.0509±0.0094</td>
<td>0.0162±0.0088</td>
<td>0.0339±0.0074</td>
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<tr>
<td>$\langle p_2 \gamma^2 \rangle$</td>
<td>-0.0568±0.0181</td>
<td>-0.0477±0.0151</td>
<td>-0.0406±0.0148</td>
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<tr>
<td>$\langle p_2 \gamma \rangle$</td>
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<td>0.0572±0.0049</td>
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<tr>
<td>$\langle p_3 \gamma \rangle$</td>
<td>0.0084±0.0044</td>
<td>0.0092±0.0042</td>
<td>-0.0138±0.0043</td>
<td>-0.0000±0.0037</td>
</tr>
<tr>
<td>$\langle p_3 \gamma \rangle$</td>
<td>0.0140±0.0108</td>
<td>0.0329±0.0116</td>
<td>0.0078±0.0109</td>
<td>0.0137±0.0096</td>
</tr>
<tr>
<td>$\langle p_3 \gamma \rangle$</td>
<td>-0.0667±0.0390</td>
<td>-0.0727±0.0375</td>
<td>-0.0273±0.0380</td>
<td>-0.0258±0.0314</td>
</tr>
<tr>
<td>$\langle p_3 \gamma \rangle$</td>
<td>-0.0148±0.0046</td>
<td>-0.0139±0.0044</td>
<td>0.0090±0.0047</td>
<td>0.0109±0.0036</td>
</tr>
<tr>
<td>$\langle p_4 \gamma \rangle$</td>
<td>0.0087±0.0038</td>
<td>0.0072±0.0038</td>
<td>0.0066±0.0038</td>
<td>0.0115±0.0033</td>
</tr>
<tr>
<td>$\langle p_4 \gamma \rangle$</td>
<td>0.0085±0.0126</td>
<td>0.0311±0.0125</td>
<td>0.0203±0.0121</td>
<td>0.0321±0.0101</td>
</tr>
<tr>
<td>$\langle p_4 \gamma \rangle$</td>
<td>0.0120±0.0062</td>
<td>-0.1239±0.0626</td>
<td>-0.0624±0.0631</td>
<td>-0.1809±0.0545</td>
</tr>
<tr>
<td>$\langle p_4 \gamma \rangle$</td>
<td>-0.0117±0.0041</td>
<td>-0.0206±0.0039</td>
<td>-0.0137±0.0041</td>
<td>-0.0126±0.0035</td>
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<tr>
<td>$\langle p_5 \gamma \rangle$</td>
<td>0.0067±0.0034</td>
<td>0.0068±0.0035</td>
<td>0.0069±0.0035</td>
<td>0.0062±0.0030</td>
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<tr>
<td>$\langle p_5 \gamma \rangle$</td>
<td>0.0211±0.0141</td>
<td>0.0111±0.0133</td>
<td>0.0033±0.0130</td>
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<td>$\langle p_5 \gamma \rangle$</td>
<td>-0.0020±0.0042</td>
<td>0.0302±0.0085</td>
<td>0.0872±0.0091</td>
<td>0.0034±0.0075</td>
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<td>$\langle p_5 \gamma \rangle$</td>
<td>-0.0041±0.0037</td>
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<td>-0.0015±0.0037</td>
<td>0.0011±0.0032</td>
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<tr>
<td>$\langle p_6 \gamma \rangle$</td>
<td>0.0081±0.0032</td>
<td>0.0079±0.0032</td>
<td>0.0068±0.0032</td>
<td>0.0079±0.0027</td>
</tr>
<tr>
<td>$\langle p_6 \gamma \rangle$</td>
<td>0.0094±0.0144</td>
<td>-0.0094±0.0148</td>
<td>0.0195±0.0141</td>
<td>0.0309±0.0124</td>
</tr>
<tr>
<td>$\langle p_6 \gamma \rangle$</td>
<td>-0.4067±0.1101</td>
<td>-0.0062±0.1075</td>
<td>-0.0522±0.1168</td>
<td>-0.1846±0.0966</td>
</tr>
<tr>
<td>$\langle p_6 \gamma \rangle$</td>
<td>0.0062±0.0035</td>
<td>-0.0144±0.0032</td>
<td>-0.0147±0.0034</td>
<td>-0.0227±0.0028</td>
</tr>
</tbody>
</table>
but simplifications result if one considers the highest orders, when only a few waves are present. Even with all the information available a detailed analysis may only be made with computer techniques and the following conclusions are somewhat handwaving but give a reasonable general description of the processes likely to be occurring.

\(<p_6^0 y_0^o>\) is negative and strongly decreasing throughout the energy range. The coefficients R show that FF7, FH7 contribute directly to this moment in the following ratio:

\[-4.04|FF7|^2 : 1.62|FH7|^2\]

Hence it is clear that the FF7 is vastly more important and in what follows the FH7 is considered to be zero. The fact that

\[<p_6^0 y_3^o> = \frac{1}{13} \frac{1}{\sqrt{4\pi}} \left\{ \frac{C_{\tau \nu}^\prime \gamma_\nu^\tau + C_{\gamma_\nu}^\prime \gamma_\tau^\nu}{C_{3\nu}^\prime \gamma_\nu^\tau + C_{\gamma_\nu}^\prime \gamma_\nu^\tau} \right\}\]

together with the knowledge that FF7 contributes to \(C^0\) with magnitude \(4|FF7|^2\) shows that

\[<p_6^0 y_0^o> = -\frac{1}{13} \frac{1}{\sqrt{4\pi}}\]

for the case when the available cross section is saturated by this resonance. This is nearly true at 1.4 GeV/c but there is substantial cross section available at the other momenta i.e. \(ECMS < 1800\text{ MeV}\).
\(<p_6^0 y_2^0>\) - positive and practically the same at all energies.

The term in \(|FF7|^2\) predicts a true contribution to this moment, but the lack of variation suggests a decreasing interference over the energy range. This is possible by interference with the FF5 and FP5 waves the terms being

\[
\begin{align*}
\alpha + 5.75 \text{ Re } (FF7 \cdot FF5^X) \\
\alpha + 1.17 \text{ Re } (FF7 \cdot FF5^X)
\end{align*}
\]

so that one or either of these waves may be contributing and the effect becomes less important at higher energies. This could be because the F35 amplitudes 'peak' lower than the FF7.

\(<p_6^1 y_2^1>\) zero at lowest momenta but significantly +ve at 1.3 and 1.4 GeV/c. The contributions to this are -1.88|FF7|^2, 2.20 re(FF7.FF5), -7.15 Re(FF7.FP5).

Clearly a significant interference term is required to make this go positive. The observation that \(<p_6^2 y_2^2>\) is negative, coupled with the appropriate coefficients, shows that FF7/FF5 will be the dominant term, and these two waves must be in phase.

The coefficients for P5 are nearly all consistent with zero indicating the absence of any strong interference between F and D waves. (The odd coefficients result from interference between waves of different parity whereas the even coefficients may arise by direct contributions or by interference between waves of the same parity.)
\[ \langle p^4 y^0 \rangle \] is large and negative at all energies but since the direct contributions for FF7 and FF5 have negative coefficients this does not provide much new information. Similarly \[ \langle p^4 y^0 \rangle \] is also of a sign which is the same as these direct contributions. \[ \langle p^2 y^2 \rangle \] seems to wander about and any is probably zero. \[ \langle p^1 y^1 \rangle \] may be explained by FF5 and the interference with FF7.

Most of the moments of order 3 are zero, the only one showing any consistent behaviour is \[ \langle p^3 y^0 \rangle \] which changes from being strongly -ve to large and +ve at 1.4 GeV/c. The most important terms here are the following:

\[ - 5.37 \text{ Re} (\text{FF5.DS3}^*) \]
\[ 3.80 \text{ Re} (\text{FF5.SD1}^*) \]
\[ 4.62 \text{ Re} (\text{FF7.DS3}^*) \]
\[ - 3.27 \text{ Re} (\text{FF7.SD1}^*) \]

Quite a likely explanation for this therefore could be a DS3 contribution in phase with the FF5 and FF7 transitions. As these interferences appear with opposite coefficients one might expect the sign change to occur with the FF7 becoming larger more rapidly than the FF5.

Of the remaining coefficients the \[ \langle p^1 y^0 \rangle \] and \[ \langle p^2 y^0 \rangle \], \[ \langle p^2 y^2 \rangle \], \[ \langle p^1 y^1 \rangle \] show that most structure, but these are impossibly difficult to interpret by methods other than full scale computer analysis with parametrisation of the various waves by resonances and background.

The most important conclusions seems to be the signi-
ficance of the FF7 and FF5 waves and the fact that they are in phase over the energy region considered. This has been noticed by Mehtani et al. at rather higher energies who also showed that one might expect transitions of the type \((LL)_{2J}\) to be more important than \((L, L\pm 2)_{2J}\) on duality grounds.
1100 Mev/c

Fig 6.1
Fig 6.1b
NO. EVENTS

EFFECTIVE MASS SQUARED \( \pi \cdot N \) 1100 MEV/C

EFFECTIVE MASS SQUARED \( \pi \cdot \pi \) 1100 MEV/C

Fig 6.1c
1200 Mev/c

Fig 61d
Fig 61e

EFFECTIVE MASS SQUARED OF $\pi^+ p$ 1200 MEV/C

EFFECTIVE MASS SQUARED OF $\pi^+ \pi^0$ 1200 MEV/C
Fig 61g
Fig 6.1h
Fig 6.1i
Fig 6.1j
Fig 61k
Fig 6.11
1100 Mev/c

Fig 6.2a
Fig 6.2b
Fig 62c
NO. EVENTS

Fig 62e

COSINE CMS PRODUCTION ANGLE P P PI, PIO CHANNEL 1200 MEV/C
Fig 62f

COSINE CMS PRODUCTION ANGLE $\pi^+\pi^-\pi^+\pi^- N$ CHANNEL
1200 MeV/c
Fig 62g
Fig 6.24
Fig 6.2j

1400 Mev/c
Fig 621

Graph 1: COSINE CMS PRODUCTION ANGLE $\pi^+\pi^+\pi^-N$ CHANNEL 1400 MeV/c

Graph 2: COSINE CMS PRODUCTION ANGLE $N\pi^+\pi^-N$ CHANNEL 1400 MeV/c
APPENDIX I

Notation and Symbols

\( J \)  
Total angular momentum

\( L, L' \)  
Orbital angular momentum between the combinations \((\pi_i^0 - N), (\text{Isobar-Recoil})\) respectively

\( j \)  
Total invariant spin of the final state

\( \lambda \)  
Orbital angular momentum of isobar decay particles

\( M_f^* \)  
Projection of nucleon spin on Isobar direction in overall CMS

\( \lambda \)  
Helicity of Isobar

\( \nu \)  
Projection of orbital angular momentum of isobar decay particles on CMS isobar direction

\( m \)  
Projection of orbital angular momentum on z-axis

\( \delta_1 \)  
Angle between \((\pi_2^0 N)\) and nucleon (+ve)

\( \delta_2 \)  
Angle between \((\pi_1^0 N)\) and nucleon (+ve)

\( \theta_1 \)  
Helicity decay angle to proton from \((\pi_2^0 N)\)

\( \theta_2 \)  
Helicity decay angle to proton from \((\pi_1^0 N)\)

\( \Theta, \phi \)  
Polar angles of beam in fixed (xyz) frame

\( \mu_i, \mu_f \)  
Projection of initial and final spin of nucleon on z-axis

\( q_i \)  
Relative CMS momentum of incident state

\( Q_k \)  
CMS momentum of recoil particle

\( q_k \)  
Momentum of decay particle from the isobar, in the isobar rest frame \((k = 1 \text{ to } 3)\)

\( \omega_k \)  
Invariant mass of the two body subsystem comprising the isobar \((k = 1, 2, 3)\)
$\omega_0$ Nominal mass of the resonance
$\Gamma, \Gamma_0$ Full width and nominal width of the resonance
$q_0$ Decay momentum of the isobar when the mass equals the nominal resonance mass

Isospin indices.

$I_k, I^z_k, I^z, I^z_\ell$ Isospin and third components of particles comprising the isobar
$I, I^z$ Isospin, third component of the isobar
$I_j, I^z_j$ Isospin, third component of recoil particle
APPENDIX II

The functions $f_{i}^{\mu_{i} \nu_{f}}$ are defined such that they satisfy the following properties:

(1) $\sum_{\mu_{i} \mu_{f}} \int \int \int \int f_{\mu_{i} \mu_{f}}^{\tau_{x_{i}} \tau_{y_{i}} \tau_{z_{i}} \tau_{\omega_{i}}} (\theta_{i}, \phi_{i}, \phi, \phi) \, d\cos \theta_{i} \, d\phi_{i} = 1$

(2) $\sum_{\mu_{i} \mu_{f}} \int \int \int \int f_{\mu_{i} \mu_{f}}^{\tau_{x_{i}} \tau_{y_{i}} \tau_{z_{i}} \tau_{\omega_{i}}} (\theta_{i}, \phi_{i}, \phi, \phi) \, f_{\mu_{i} \mu_{f}}^{\tau_{x_{i}} \tau_{y_{i}} \tau_{z_{i}} \tau_{\omega_{i}}} \, d\cos \theta_{i} \, d\phi_{i} = \delta_{x_{i} x_{i}} \delta_{y_{i} y_{i}} \delta_{z_{i} z_{i}} \delta_{\omega_{i} \omega_{i}} \delta_{\omega_{i} \omega_{i}}$

These two relations may be obtained from the expression for $f$ by using the orthogonality of the spherical harmonics $Y_{L}^{M}$ and then the Clebsch-Gordon series may be used to reduce the summations still further. These expressions are particularly useful in checking the functional forms of the $f$'s as detailed in Appendix III.

With these normalizations the differential cross section for production and decay of an isobar $n$ in mass combination $i$ is given by

$$d\sigma = \frac{\pi}{\mathcal{Q}_{n}^{2}} \sum_{\omega_{i} \omega_{c}} (J + \frac{1}{2}) \int \left| T_{\omega_{i} \omega_{c}}^{\tau_{x_{i} \tau_{y_{i}} \tau_{z_{i}}}} (\omega, \omega_{c}) \right|^{2} d\omega_{c}$$
In the limit of an isobar with zero width (i.e. stable particle) the expression for the Breit-Wigner reduces to a delta function

$$\delta(\omega^2 - \omega_0^2)$$

and we obtain

$$d\sigma = \frac{\pi}{p^2} \sum_{JL} (J+\frac{1}{2}) \left| A \right|^2$$

and comparison with the expression for e.g. elastic $0^- \rightarrow 0^- \frac{1}{2}^+$ shows that these are indeed identical.

The fact that two of the final state particles are identical, obeying Bose-statistics, has important consequences on the symmetry properties of the amplitude. Thus consider the combination from the two baryon isobars represented as:

$$\tau(\omega) \frac{f_{\mu_1 \mu_2}}{A_1 A_2} (\Theta_1, \Omega_1, \Theta, \Omega) + \tau(\omega_2) f(-\Theta_1, -\Omega_1, \Theta, \Omega)$$

Then interchanging the labels on the two pions we find the expression becomes:

$$\tau(\omega) \frac{f_{\mu_1 \mu_2}}{A_1 A_2} (\Theta_2, \Omega_2, \pi-\Theta, \pi-\Omega) + \tau(\omega) \frac{f_{\mu_1 \mu_2}}{A_1 A_2} (-\Theta_1, -\Omega_1, \pi-\Theta, \pi-\Omega)$$

The change $\phi \rightarrow \pi - \phi$, $\phi \rightarrow -\phi$ comes about because of the inversion of the $z$-axis $= \pi_1 \cdot \pi_2$

Now it is not very difficult to show, using the expression for $f$ in Chapter IV that

$$f_{\mu_1 \mu_2} (\Theta_1, \Omega_1, \pi-\Theta, \pi-\Omega) = (-1)^{1+\mu_1+\mu_2} f_{\mu_1 \mu_2} (-\Theta_1, -\Omega_1, \Theta, \Omega)$$

$$= (-1)^{1+2\mu_1} f_{-\mu_1 -\mu_2} (-\Theta_2, -\Omega_2, \pi-\Theta, \pi-\Omega)$$

$$= f_{-\mu_1 -\mu_2} (-\Theta_2, -\Omega_2, \pi-\Theta, \pi-\Omega)$$
So the expression becomes

\[ T(\omega_3) \frac{f}{-\mu_i \rightarrow \mu_f} (\theta_{3}, -\pi, \pi, \phi) + T(\omega_1) \frac{f}{-\mu_i \rightarrow \mu_f} (\theta_{1}, \pi, \pi, \phi) \]

which is precisely what would have been obtained by inverting the z-axis.

Now consider a meson (\(\pi-\pi\)) resonance. It contributes a term

\[ T(\omega_3) \frac{f}{\mu_i} (\theta_{3}, \pi, \pi, \phi) \]

to the amplitude.

Interchanging the labels of the pions (and ignoring for the moment any Clebsch-Gordon coefficients) this will give

\[ T(\omega_3) \frac{f}{\mu_i} (\pi-\theta_{3}, -\pi, \pi-\theta_{1}) \]

And inspection of the formulae for \(f_{\mu_i} \mu_f\) shows that this may be written as

\[ T(\omega_3) \frac{f}{-\mu_i \rightarrow \mu_f} (\theta_{3}, \pi, \pi, \phi) (-1)^{j} \]

So that, for example, in the case of the \(\rho\) resonance there is a sign change w.r.t. the contributions from the nucleon-pion isobars. However the Clebsch-Gordon coefficient is

\[ C_{\pi_1 \pi_2 \phi} \]
and by the well known properties of these coefficients

\[ C_{\pi_L \pi_H \alpha} = C_{\pi_H \pi_L \alpha} \]

so that one obtains the same formula for the partial differential cross section.
### APPENDIX III

\[ J_{LL'}^{j \ell} (\theta, \phi, \theta, \phi): j = 3/2, \ell = 1 \]

<table>
<thead>
<tr>
<th>WAVE ( LL'2j )</th>
<th>SPIN-PROTECTIONS</th>
<th>AMPLITUDE</th>
</tr>
</thead>
<tbody>
<tr>
<td>SD1 m=1</td>
<td>NSF</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>SF</td>
<td>( \frac{1}{2\sqrt{2}} (2\cos \theta + i \sin \theta) e^{i\delta} Y_0^0 (\theta, \phi) )</td>
</tr>
<tr>
<td>PP1 m=0</td>
<td>NSF</td>
<td>- ( \frac{1}{2\sqrt{6}} (2\cos \theta + i \sin \theta) Y_1^0 (\theta, \phi) )</td>
</tr>
<tr>
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<td>SF</td>
<td>- ( \frac{1}{2\sqrt{3}} (2\cos \theta + i \sin \theta) Y_1^1 (\theta, \phi) )</td>
</tr>
<tr>
<td>PP3 m=0</td>
<td>NSF</td>
<td>- ( \frac{1}{2\sqrt{30}} (\cos \theta + 5 \sin \theta) Y_1^0 (\theta, \phi) )</td>
</tr>
<tr>
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<td>SF</td>
<td>( \frac{1}{2\sqrt{60}} (\cos \theta + 5 \sin \theta) Y_1^1 (\theta, \phi) )</td>
</tr>
<tr>
<td>m=2 SF</td>
<td>( \frac{1}{2\sqrt{20}} (\sin \theta - \cos \theta)e^{2i\delta} Y_1^0 (\theta, \phi) )</td>
<td></td>
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<tr>
<td>PF3 m=0</td>
<td>NSF</td>
<td>( \frac{\sqrt{3}}{\sqrt{400}} \cos \theta Y_1^0 )</td>
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<tr>
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<td>SF</td>
<td>( \frac{\sqrt{3}}{\sqrt{80}} \cos \theta Y_1^1 )</td>
</tr>
<tr>
<td>m=2 SF</td>
<td>( \frac{\sqrt{3}}{\sqrt{80}} (3 \cos \theta + 2 \sin \theta)e^{2i\delta} Y_1^1 (\theta, \phi) )</td>
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</tr>
<tr>
<td>DS3 m=-1</td>
<td>NSF</td>
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</tr>
<tr>
<td></td>
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<td>( \frac{\sqrt{3}}{\sqrt{20}} (\sin \theta + \cos \theta)e^{-i\delta} Y_2^{-2} (\theta, \phi) )</td>
</tr>
<tr>
<td>m=+1 SF</td>
<td>( \frac{\sqrt{3}}{\sqrt{80}} (\sin \theta - \cos \theta)e^{i\delta} Y_2^1 (\theta, \phi) )</td>
<td></td>
</tr>
<tr>
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<td>SF</td>
<td>( \frac{\sqrt{3}}{\sqrt{40}} (\sin \theta - \cos \theta)e^{i\delta} Y_2^0 (\theta, \phi) )</td>
</tr>
<tr>
<td>WAVE</td>
<td>SPIN-PROTECTIONS</td>
<td>AMPLITUDE</td>
</tr>
<tr>
<td>------</td>
<td>------------------</td>
<td>-----------</td>
</tr>
<tr>
<td></td>
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<td>$\frac{\sqrt{3}}{\sqrt{80}} \cos \theta e^{-i\delta Y_{2}^{-1}}(\theta,\phi)$</td>
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<td>$\frac{\sqrt{3}}{\sqrt{20}} \cos \theta e^{-i\delta Y_{2}^{-2}}(\theta,\phi)$</td>
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<td>NSF m=+1</td>
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<td>$\frac{\sqrt{3}}{\sqrt{112}} (\cos \theta + i \sin \theta) e^{-2i\delta Y_{3}^{-1}(\theta,\phi)}$</td>
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<td></td>
<td>SF m=0</td>
<td>$\frac{3}{\sqrt{280}} \cos \theta Y_{3}^{0}(\theta,\phi)$</td>
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<td>SF m=0</td>
<td>$\frac{\sqrt{3}}{\sqrt{70}} \cos \theta Y_{3}^{-1}(\theta,\phi)$</td>
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<tr>
<td></td>
<td>SF m=2</td>
<td>$-\frac{\sqrt{3}}{\sqrt{280}} (\sin \theta - \cos \theta) e^{2i\delta Y_{3}^{1}(\theta,\phi)}$</td>
</tr>
<tr>
<td>WAVE</td>
<td>SPIN PROTECTION</td>
<td>AMPLITUDE</td>
</tr>
<tr>
<td>------</td>
<td>-----------------</td>
<td>-----------</td>
</tr>
<tr>
<td>FF5</td>
<td></td>
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<tr>
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<td>NSF</td>
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<td>SF</td>
<td>$- \frac{1}{8\sqrt{7}}(4\cos\theta - i\sin\theta)e^{-2i\delta}Y_{3}^{0}(\theta,\phi)$</td>
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<td>NSF</td>
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<td>SF</td>
<td>$- \frac{1}{8\sqrt{35}}(4\cos\theta + 1i\sin\theta)e^{2i\delta}Y_{3}^{1}(\theta,\phi)$</td>
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<tr>
<td>SS1</td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
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</tr>
<tr>
<td>PP1</td>
<td></td>
<td></td>
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<tr>
<td>m=0</td>
<td>NSF</td>
<td>$\frac{1}{12}(4\cos\theta - i\sin\theta)Y_{1}^{0}(\theta,\phi)$</td>
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<tr>
<td></td>
<td>SF</td>
<td>$\frac{1}{\sqrt{6}}(4\cos\theta - i\sin\theta)Y_{1}^{-1}(\theta,\phi)$</td>
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<td>PP3</td>
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<td>$- \frac{1}{2\sqrt{6}}(4\cos\theta - i\sin\theta)Y_{1}^{0}(\theta,\phi)$</td>
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<tr>
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<td>m=2</td>
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<td>$- \frac{1}{4\sqrt{3}}(4\cos\theta - i\sin\theta)e^{2i\delta}Y_{1}^{1}(\theta,\phi)$</td>
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<td>DD3</td>
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<tr>
<td>m=1</td>
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</tr>
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<td>$- \frac{1}{2\sqrt{10}}(4\cos\theta - i\sin\theta)e^{i\delta}Y_{2}^{0}(\theta,\phi)$</td>
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<tr>
<td>WAVE</td>
<td>SPIN PROTECTION</td>
<td>AMPLITUDE</td>
</tr>
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<td>------</td>
<td>-----------------</td>
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<td>DD5</td>
<td>m=-1 NSF</td>
<td>$-\frac{1}{2\sqrt{10}}(\cos\theta-\text{i}\sin\theta)e^{-\text{i}Y_2^{-1}(\theta,\phi)}$</td>
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<td>$\frac{1}{4\sqrt{10}}(\cos\theta-\text{i}\sin\theta)e^{-\text{i}Y_2^{-2}(\theta,\phi)}$</td>
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<td>$\frac{1}{2\sqrt{10}}(\cos\theta-\text{i}\sin\theta)e^{\text{i}Y_2^{1}(\theta,\phi)}$</td>
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<td>$-\frac{3}{4\sqrt{5}}(\cos\theta-\text{i}\sin\theta)e^{\text{i}Y_2^{0}(\theta,\phi)}$</td>
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<td>m=3 NSF</td>
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<td>$\frac{5}{4\sqrt{10}}(\cos\theta-\text{i}\sin\theta)e^{3\text{i}Y_2^{2}(\theta,\phi)}$</td>
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### Table

<table>
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<th>WAVE</th>
<th>SPIN-PROJECTION</th>
<th>AMPLITUDE</th>
</tr>
</thead>
</table>
| FFS  | \( m = -2 \)  | \[
\begin{align*}
    \text{NSF} & \quad \frac{1}{2^2} \frac{\sqrt{5}}{14} (\cos \theta - i \sin \theta) e^{-2i \delta} y_{-2}^0 (\hat{\Phi}) \\
    \text{SF} & \quad \frac{1}{2^2} \frac{\sqrt{5}}{14} (\cos \theta - i \sin \theta) e^{2i \delta} y_{-3}^0 (\hat{\Phi})
\end{align*}
\] |
|      | \( m = 0 \)   | \[
\begin{align*}
    \text{NSF} & \quad -\frac{1}{2^2} \frac{\sqrt{1}}{7} (\cos \theta - i \sin \theta) y_{-3}^0 (\hat{\Phi}) \\
    \text{SF} & \quad -\frac{1}{2^2} \frac{\sqrt{1}}{7} (\cos \theta - i \sin \theta) y_{-3}^{-1} (\hat{\Phi})
\end{align*}
\] |
|      | \( m = 2 \)   | \[
\begin{align*}
    \text{NSF} & \quad \frac{1}{2^2} \frac{\sqrt{5}}{14} (\cos \theta - i \sin \theta) e^{2i \delta} y_{-2}^0 (\hat{\Phi}) \\
    \text{SF} & \quad \frac{1}{2^2} \frac{\sqrt{5}}{14} (\cos \theta - i \sin \theta) e^{2i \delta} y_{-3}^{1} (\hat{\Phi})
\end{align*}
\] |

### Notes:

1. The 4 possible spin projections give rise to amplitudes which are not independent (by parity conservation). The two non-spin flip and the two spin flip amplitudes are related by the following relation

\[
f_{-\mu - \nu} = (-1)^{\nu + \nu} f_{\mu \nu}^*\]

The values quoted are \( \mu f = \frac{1}{2} \); \( \mu i = \pm \frac{1}{2} \)

2. The value quoted for \( m \) is the projection of the orbital angular momentum of the final state on the z-axis.
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\[ M^2 p\pi^+ \quad \sigma_{NN}^2 \]

\[ M^2 \pi^+\pi^- \quad \sigma_{NN}^2 \]
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