SOME UNSTEADY VISCOUS FLOWS AND THEIR STABILITY

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ABSTRACT

The work in this thesis is concerned with some unsteady viscous flows and their stability.

In Part 1 the steady streaming induced by such flows in pipes of slowly varying cross-section and channels of slowly varying depth is considered. A purely oscillatory pressure difference is maintained across the ends of each of the fluid containers and it is assumed that a modified Reynolds number associated with each of the flows is small. The first order steady streaming is evaluated for both flows in the limits of the frequency of the oscillatory pressure difference tending to zero and infinity.

In Part 2 the stability of some unsteady viscous flows is considered. In particular the stability of the flow between concentric cylinders when the outer cylinder is at rest and the inner one has angular speed $\Omega_1\{1 + \varepsilon \cos\omega t\}$ is investigated. Solutions of the linear and non-linear stability problems are obtained in the limits of ω tending to zero and infinity. The related Bénard convection problem is also investigated using the same methods as for the cylinder problem.

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GENERAL INTRODUCTION

In Part 1 of this thesis we investigate the steady streaming associated with some oscillatory viscous flows. It is well known that when such a flow is set up adjacent to a rigid boundary, a Stokes layer forms near the boundary. In this layer the flow adjusts itself so as to satisfy the no-slip condition at the boundary. Suppose ω is the frequency of the oscillatory flow, then if ν is the kinematic viscosity, this layer will be typically of thickness order $(\frac{\nu}{\omega})^{\frac{1}{2}}$. In some flows of this type, which depend also on the coordinate parallel to the boundary, the Reynolds stresses associated with the net transfer of momentum in the Stokes layer, are such that a steady component of velocity is induced there. This steady velocity persists away from the layer because of the action of viscosity. The magnitude of the induced steady velocity is much smaller than that of the basic oscillatory flow velocity, but, since it leads to a migration of fluid particles, this steady flow can be important.

The first theoretical work on this topic was done by Lord Rayleigh (1884), who considered the flow in a Kundt dust tube. Dvorak (1874) had previously considered the flow in such a tube experimentally. When a standing sound wave is set up in such a tube any small particles present are carried to the velocity nodes of the wave. In this problem the role of the steady streaming is to carry the particles to the nodes of the dominant oscillatory velocity field where they settle. Using perturbation methods Rayleigh was able to explain some of the phenomena associated with this flow.

The next problem investigated was that of determining the steady streaming induced near a body placed in a vibrating viscous fluid. The first experiments on this problem were performed by Carrière (1929), Andrade (1931) and Schlichting (1932). In particular they were concerned with the case of the body being a circular cylinder, although it should be said that many of the results for a circular cylinder are applicable to bodies of different shape. Schlichting was also able to verify some of the results of his experiments theoretically using first order boundary layer theory in the high frequency limit. More recent work on this topic has been done by Riley (1965, 1966, 1967), Stuart (1966) and Davidson & Riley (1972). There has also been some work done on the related problem of determining the steady streaming induced by a disc performing torsional vibrations. (See Rosenblat (1959), Benney (1964) and Jones & Rosenblat (1969)).

A problem which has been considered more recently is that of determining the steady streaming induced by an oscillatory viscous flow adjacent to a wavy wall. Lyne (1971) has studied this problem using a conformal transformation to change the wavy wall into a plane wall.

In Chapter I we use what is often known as 'lubrication theory' to investigate the steady streaming induced in a two-dimensional channel by an oscillatory viscous flow. In order to use this method we require that the channel depth varies slowly and that a modified Reynolds number associated with the flow is small. We solve the problem by expanding in powers of this Reynolds number, R_M, and a parameter δ which represents the order of magnitude of the rate of change of the depth of the channel. In Chapter II we use the same kind of approach to investigate the steady streaming induced by an oscillatory viscous flow in a pipe of slowly varying cross-section. Manton (1971) has considered steady flow in an axisymmetric pipe of slowly varying radius. In the Appendix we show that our formulation of the unsteady problem can be used to reproduce Manton's results for the steady problem by putting the frequency of the oscillations equal to zero.

In Part 2 we consider the stability of some unsteady viscous flows. Since the experimental work of Donnelly (1964), there has been much theoretical work in this field. The motivation for Donnelly's work was to see if fluid flows could be stabilized by modulation in the same way as can be done in other physical systems. The best example of this behaviour is that of an inverted simple pendulum composed of a heavy bob and a light rod. It can be made to stand on its end if its support is suitably oscillated (see Corben & Stehle (1960)). The stability of the pendulum is in fact governed by Mathieu's equation. This equation also governs the stability of a fluid surface in a contour performing vertical oscillation. This problem was discussed by Benjamin & Ursell (1954). The equations which arise in our work are, unlike Mathieu's equation, partial differential equations but we shall see that they have, in many ways, properties very similar to the latter equation.

The particular problem investigated by Donnelly was the stability of a viscous fluid between concentric cylinders when the outer cylinder was at rest and the inner one had angular velocity $\Omega_1'\{1 + \varepsilon \cos\omega t\}$. When ε is zero the flow first becomes unstable when the Taylor number, proportional to Ω_1^2, reaches a critical value T_0. The instability is in the form of toroidal Taylor vortices spaced periodically along the axis of the cylinders. The appearance of these vortices is predicted by linear stability theory as given by Taylor (1923). According to linear theory if the Taylor number, T, is greater than T_o the disturbance to the flow, i.e. the Taylor vortex flow, grows exponentially in time. However, this is not observed in practice, and for a range of values of T slightly greater than T_o it is found that an equilibrium flow exists, and the amplitude of the Taylor vortex velocity is then proportional to $\{T-T_o\}^{\frac{1}{2}}$. This can be explained theoretically by taking non-linear effects into account. (see Stuart (1958), Davey (1962)). If T is increased further the vortices are modified by a waviness in the azimuth and become waves travelling in that direction (see Coles (1965)). Davey, DiPrima & Stuart (1968) and Eagles (1971) have given some analysis describing this type of flow.

With ε non-zero Donnelly found that the critical Taylor number T, based on the steady part of the angular velocity of the inner cylinder, at which instability first occurred was increased from its unmodulated value for all ε,ω. Moreover, for fixed ε, there was a certain value of ω independent of ε, at which this enhancement of stability was a maximum.

The first theoretical work on this problem was done by Meister & Munzner (1966). They used a Galerkin method to solve the linearized differential equations governing the stability. of the flow. In their problem the outer cylinder also had a steady velocity. They considered the evolution in time of the kinetic energy of a disturbance imposed on the basic flow at time $\omega t = 0$. For several values of the Taylor number they found that at a certain time after the disturbance was imposed the kinetic energy was a minimum for a particular value of ω. However, they do not say whether this was the case for all time. The next and last investigation of this problem was by Rosenblat (1968). He found that, if viscosity was ignored, modulation tended to destabilize a stable mean flow and stabilize an unstable one. However, as stated by Rosenblat, it is necessary to include viscosity, to find the increment in the critical Taylor number.

Since Rosenblat's paper most of the research has been concentrated on the related Bénard convection problem. Venezian (1969) considered the stability of a fluid between parallel planes when one or both of the planes had their temperatures modulated about a non-zero mean. The mean temperatures of the planes were different and large compared to the oscillatory parts of their temperatures. Using a perturbation method he was able to show that in some cases modulation could stabilize a flow, but, unlike Donnelly's results, the maximum enhancement was always in the limit of zero frequency. A similar result was obtained by Rosenblat & Herbert (1970), who used a Galerkin method for the case when only the lower plane had its temperature modulated. Other work on this problem has been done by Davis (1970), Grescho & Sani (1970) and Yih & Li (1972). There has also been some work done on the stability of oscillatory plane flows. (See Yih (1968), Grosch & Salwen (1968), Kelly & Cheers (1970) and Kerczek & Davis (1972)).

In Part 2 of this thesis we consider theoretically the problem investigated by Donnelly and the related thermal convection problem. We consider separately the low and high frequency limits for both the linear and non-linear stability problems. The low frequency calculations follow closely the work of DiPrima & Stuart (1972, 1973) who considered the stability of the flow between eccentric rotating cylinders. In the high frequency limit we obtain a solution using the method of matched asymptotic expansions. The type of analysis used is in fact very similar to that used for example by Stuart (1966), Riley (1967) when investigating the steady streaming induced near an oscillating cylinder.

PART 1

THE STEADY STREAMING ASSOCIATED WITH SOME OSCILLATORY VISCOUS FLOWS

CHAPTER I

UNSTEADY VISCOUS FLOW IN A TWO-DIMENSIONAL CHANNEL

1.1 Introduction

We consider the steady streaming generated when a purely oscillatory pressure difference is maintained across the ends of a two-dimensional channel which is defined in Cartesian coordinates (x,y) by

$$|y| \leqslant D_0 F\left(\frac{x}{L}\right), \quad 0 \leqslant x \leqslant KL \tag{1.1.1}$$

Thus D_0, L are characteristic lengths in the y and x directions respectively, and K is taken to be a positive constant. We assume that the depth of the channel varies slowly in the sense that if we define δ by

$$\delta = \left(\frac{D_0}{L}\right)^2 \tag{1.1.2}$$

then we have that

$$\delta \ll 1 \tag{1.1.3}$$

We further assume that if we take U_0 to be a characteristic velocity along the channel, and ν to be the kinematic viscosity, then the parameter R_M, defined below, is also small compared to unity.

$$R_M = U_0 D_0^2 / L\nu \tag{1.1.4}$$

We see from (1.1.4) that R_M is just the usual Reynolds number based on the length L multiplied by δ . We shall seek a solution by expanding the velocity and pressure in powers of the parameters R_M, δ . The dominant steady streaming will first appear at order $R_M \delta^0$ in these expansions when the basic velocity field is of order $R_M^0 \delta^0$.

We shall find it useful to define a frequency parameter σ by

$$\sigma = \omega D_0^2 / \nu \tag{1.1.5}$$

where ω is the frequency of the pressure oscillations. Thus $\sigma^{1/2}$ represents the ratio of the typical channel depth compared to the thickness of the Stokes layer associated with the oscillatory motion of the fluid. We shall consider in detail the special limits of σ tending to zero and infinity, and we shall refer to the corresponding solutions as the low and high frequency solutions respectively.

In the high frequency limit we find that the solution takes a similar form to that found by Lyne (1971) if the ends of the channel are of the same depth. However, if the ends of the channel are not of the same depth, and K is of order unity, then the steady streaming velocity field is of larger order of magnitude and always directed towards the deepest end of the channel. If, on the other hand, K is allowed to tend to infinity, this part of the steady streaming will become unimportant and the steady streaming will be dominated by that corresponding to flow in a channel whose ends are of the same depth.

In the low frequency solution the steady streaming is again characterized by two parts, one of which is zero or unimportant if the ends of the channel are of the same depth or K is infinitely large. Otherwise this part of the steady streaming represents a steady flow towards the deepest end of the channel. The other part of the steady streaming gives no net flux through the channel. A related result has been reported by Mei & Liu/Kata (1970) in the context of mass transport in water waves.

1.2 Equations of motion and the Stokes flow

We consider viscous incompressible flow in the channel defined in Cartesian coordinates, (x,y), by (1.1.1). We take (u,v) to be the corresponding velocity vector and p(x,y,t), ρ, t to be the pressure, density, and time respectively. We assume throughout that the kinematic viscosity, ν, is constant. The momentum and continuity equations for the flow may be written in the form

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \qquad \begin{array}{c}(1.2.1)\\ a,b,c\end{array}$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

We assume that the pressure difference between the ends, evaluated at the upper wall, is given by

$$p(KL, F(K), t) - p(0, F(0), t) = C_0 \sin \omega t \qquad (1.2.2)$$

and the boundary conditions on the velocity required to completely specify the problem for u,v,p are

$$u = v = 0 \quad , \quad y = \pm D_0 F\left(\frac{x}{L}\right) \qquad \begin{array}{c}(1.2.3)\\ a,b\end{array}$$

so that there is no relative velocity at the boundaries.

We now define dimensionless variables X,Y, τ by

$$X = x/L \ , \ Y = y/D_0 \ , \ \tau = \omega t \qquad \begin{array}{c}(1.2.4)\\ a,b,c\end{array}$$

We assumed in §1.1that a typical value of u was U_0, and so we can infer from the equation of continuity, (1.2.1)c, that a typical value of v is $U_0 D_0 / L$. Hence we define a deminsionless velocity vector (U,V) by

$$(U,V) = \left(u/U_0,\ vL/U_0 D_0\right) \tag{1.2.5}$$

and we define a dimensionless pressure, P, by

$$P = p D_0^2 / \rho \nu U_0 \tag{1.2.6}$$

We can then use (1.2.4), (1.2.5), (1.2.6) to rewrite (1.2.1), (1.2.2), (1.2.3) in the form

$$\left.\begin{aligned} \sigma \frac{\partial U}{\partial \tau} + R_M \left\{ U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} \right\} &= -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} + \delta \frac{\partial^2 U}{\partial X^2} \\ \delta \sigma \frac{\partial V}{\partial \tau} + R_M \delta \left\{ U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} \right\} &= -\frac{\partial P}{\partial Y} + \delta \frac{\partial^2 V}{\partial Y^2} + \delta^2 \frac{\partial^2 V}{\partial X^2} \\ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} &= 0 \end{aligned}\right\} \qquad \begin{matrix}(1.2.7)\\ a,b,c\end{matrix}$$

$$P(K, F(K), \tau) - P(0, F(0), \tau) = \alpha \sin \tau \tag{1.2.8}$$

$$U = V = 0, \quad Y = \pm F(X) \qquad \begin{matrix}(1.2.9)\\ a,b\end{matrix}$$

where σ, R_M, δ are as defined by (1.1.5), (1.1.4), (1.1.2) respectively and α is given by

$$\alpha = C_0 D_0^2 / \rho \nu L U_0 \tag{1.2.10}$$

We now determine the so called 'Stokes Flow' which is obtained by putting the parameters R_M, δ equal to zero everywhere. We first expand U,V,P in the form

$$\left.\begin{aligned} U &= U_0 + U_1 R_M + O(R_M^2, \delta) \\ V &= V_0 + V_1 R_M + O(R_M^2, \delta) \\ P &= P_0 + P_1 R_M + O(R_M^2, \delta) \end{aligned}\right\} \qquad \begin{matrix}(1.2.11)\\ a,b,c\end{matrix}$$

and we then write U_o, V_o, P_o in the form

$$\left.\begin{aligned} U_0 &= \tfrac{1}{2}\left\{U^* e^{i t} + \tilde{U}^* e^{-i t}\right\} \\ V_0 &= \tfrac{1}{2}\left\{V^* e^{i t} + \tilde{V}^* e^{-i t}\right\} \\ P_0 &= \tfrac{1}{2}\left\{P^* e^{i t} + \tilde{P}^* e^{-i t}\right\} \end{aligned}\right\} \qquad \begin{array}{c}(1.2.12)\\ a,b,c\end{array}$$

where the functions U^*, V^*, P^* are independent of t and $\sim$ denotes complex conjugate. If we now substitute for U_o, V_o, P_o from (1.2.12)a,b,c into (1.2.11) and then substitute the resulting expressions into (1.2.7), (1.2.8), (1.2.9) and equate terms proportional to e^{it} and independent of R_M, δ, we obtain

$$\left.\begin{aligned} \frac{\partial P^*}{\partial X} &= \left\{\frac{\partial^2}{\partial Y^2} - i\sigma\right\} U^* \\ \frac{\partial P^*}{\partial Y} &= 0 \\ \frac{\partial U^*}{\partial X} + \frac{\partial V^*}{\partial Y} &= 0 \end{aligned}\right\} \qquad \begin{array}{c}(1.2.13)\\ a,b,c\end{array}$$

$$U^* = V^* = 0 \,, \quad Y = \pm F(X) \qquad \begin{array}{c}(1.2.14)\\ a,b\end{array}$$

$$P^*(K, F(K)) - P^*(0, F(0)) = -i\alpha \qquad (1.2.15)$$

and so it follows that P^* is a function of X only.

We can write the solution of (1.2.13)a which satisfies (1.2.14)a in the form

$$U^* = -\frac{P^{*\prime}}{i\sigma}\left\{1 - \frac{\cosh \lambda Y}{\cosh \lambda F}\right\} \qquad (1.2.16)$$

where $\lambda = \sqrt{i\sigma}$ (1.2.17)

and a dash denotes a derivative with respect to X. Substituting for $\dfrac{\partial U^*}{\partial X}$ from (1.2.16) into (1.2.13)c and then integrating from $Y=0$ to $Y=Y$ we obtain

$$V^*(X,Y) - V^*(X,0) = \frac{P^{*''}}{i\sigma}\left\{Y - \frac{\sinh\lambda Y}{\lambda\cosh\lambda F}\right\} + \frac{P^{*'}F'\sinh\lambda F\sinh\lambda Y}{i\sigma\cosh^2\lambda F} \qquad (1.2.18)$$

and by symmetry $V^*(X,0)$ must be zero. If we now put $Y=F$ in (1.2.18) and use (1.2.14) we obtain the Reynolds equation for the pressure

$$\left\{F - \frac{\sinh\lambda F}{\lambda\cosh\lambda F}\right\} P^{*''} + \frac{F'\sinh^2\lambda F}{\cosh^2\lambda F} P^{*'} = 0 \qquad (1.2.19)$$

and using this equation we can rewrite (1.2.18) in the form

$$V^*(X,Y) = \frac{P^{*''}}{i\sigma}\left\{Y - \frac{F\sinh\lambda Y}{\sinh\lambda F}\right\} \qquad (1.2.20)$$

We can integrate (1.2.19) once to give

$$P^{*'} = A\Big/\left\{F - \frac{\tanh\lambda F}{\lambda}\right\} \qquad (1.2.21)$$

where A is a constant, which, after integrating both sides of (1.2.21) from $X=0$ to $X=K$ and using (1.2.15), is found to be given by

$$A = -i\alpha\Big/\left\{\int_0^K dX\Big/\left(F - \frac{\tanh\lambda F}{\lambda}\right)\right\} \qquad (1.2.22)$$

1.3 Calculation of the steady streaming

We now evaluate the order R_M correction to the Stokes flow. If we substitute for U,V,P from (1.2.11) into (1.2.7), (1.2.8), (1.2.9) and equate terms of order R_M we obtain

$$\left.\begin{aligned} \frac{\partial P_1}{\partial X} &= \left\{\frac{\partial^2}{\partial Y^2} - \sigma\frac{\partial}{\partial t}\right\} U_1 - \left\{U_0\frac{\partial U_0}{\partial X} + V_0\frac{\partial U_0}{\partial Y}\right\} \\ \frac{\partial P_1}{\partial Y} &= 0 \\ \frac{\partial U_1}{\partial X} + \frac{\partial V_1}{\partial Y} &= 0 \end{aligned}\right\} \qquad \begin{matrix}(1.3.1)\\ a,b,c\end{matrix}$$

$$U_1 = V_1 = 0 \quad , \quad Y = \pm F(X) \tag{1.3.2 a,b}$$

$$P_1(K, F(K), \tau) - P_1(0, F(0), \tau) = 0 \tag{1.3.3}$$

and we can see from (1.2.12) that U_1, V_1, P_1 will have components independent of τ and components proportional to $\cos 2\tau$, $\sin 2\tau$. If we denote the steady parts of U_1, V_1, P_1 by U_s, V_s, P_s respectively we can use (1.2.12), (1.3.1), (1.3.2), (1.3.3) to show that

$$\left.\begin{aligned} \frac{\partial P_s}{\partial X} &= \frac{\partial^2 U_s}{\partial Y^2} - \frac{1}{4}\left\{U^*\frac{\partial \tilde{U}^*}{\partial X} + \tilde{U}^*\frac{\partial U^*}{\partial X} + V^*\frac{\partial \tilde{U}^*}{\partial Y} + \tilde{V}^*\frac{\partial U^*}{\partial Y}\right\} \\ \frac{\partial P_s}{\partial Y} &= 0 \\ \frac{\partial U_s}{\partial X} &+ \frac{\partial V_s}{\partial Y} = 0 \end{aligned}\right\} \tag{1.3.4 a,b,c}$$

$$U_s = V_s = 0 \, , \quad Y = \pm F(X) \tag{1.3.5 a,b}$$

$$P_s(K, F(K)) - P_s(0, F(0)) = 0 \tag{1.3.6}$$

and it follows immediately from above that P_s is a function of X only. If we substitute for U^*, V^* from (1.2.16), (1.2.20) into (1.3.4)a and solve the resulting differential equation subject to (1.3.5)a we obtain

$$\begin{aligned} U_s = \tilde{\Phi}\left\{\frac{Y^2 - F^2}{2}\right\} + \Bigg[\frac{p^{*\prime}\tilde{p}^{*\prime\prime}}{4\sigma^2}\Bigg\{\frac{Y\sinh\lambda Y}{\lambda\cosh\lambda F} - \frac{F\cosh\tilde{\lambda}Y}{\tilde{\lambda}\sinh\tilde{\lambda}F} - \frac{3\cosh\lambda Y}{\lambda^2\cosh\lambda F} \\ - F\left(\frac{i\cosh\sqrt{2\sigma}\,Y + \cos\sqrt{2\sigma}\,Y}{2\sqrt{2\sigma}\cosh\lambda F\sinh\tilde{\lambda}F}\right)\Bigg\} + \begin{matrix}\text{Complex}\\ \text{Conjugate}\end{matrix}\Bigg] \\ + B(X) + \tilde{B}(X) \end{aligned} \tag{1.3.7}$$

where

$$\left.\begin{aligned} \tilde{\Phi} &= P_s' + \frac{1}{4\sigma^2}\left\{\tilde{p}^{*\prime}p^{*\prime\prime} + p^{*\prime}\tilde{p}^{*\prime\prime}\right\} \\ \text{and} \quad B(X) &= \frac{1}{4\sigma^2}p^{*\prime}\tilde{p}^{*\prime\prime}\Bigg\{\frac{3}{\lambda^2} + \frac{F\coth\tilde{\lambda}F}{\tilde{\lambda}} - \frac{F\tanh\lambda F}{\lambda} \\ &\qquad + F\left(\frac{i\cosh\sqrt{2\sigma}\,F + \cos\sqrt{2\sigma}\,F}{2\sqrt{2\sigma}\cosh\lambda F\sinh\tilde{\lambda}F}\right)\Bigg\} \end{aligned}\right\} \tag{1.3.8 a,b}$$

If we now substitute for U_s from above into (1.3.4)c and integrate from $Y=0$ to $Y=Y$ we obtain the following expression for V_s after noting that by symmetry V_s is zero at $Y=0$.

$$
\begin{aligned}
V_s = & -\tilde{\Phi}'\left\{\frac{Y^3-3YF^2}{6}\right\} + FF'\tilde{\Phi}Y - B'Y - \tilde{B}'Y \\
& -\Bigg[\left(\frac{p^{*\prime}\tilde{p}^{*\prime\prime}}{4\sigma^2}\right)'\Bigg\{-\frac{3\sinh\lambda Y}{\lambda^3\cosh\lambda F} - \frac{F\sinh\tilde{\lambda}Y}{\tilde{\lambda}^2\sinh\tilde{\lambda}F} + \frac{Y\cosh\lambda Y}{\lambda^2\cosh\lambda F} \\
& \quad - \frac{\sinh\lambda Y}{\lambda^3\cosh\lambda F} - F\left(\frac{i\sinh\sqrt{2\sigma}\,Y + \sin\sqrt{2\sigma}\,Y}{4\sigma\cosh\lambda F\,\sinh\tilde{\lambda}F}\right)\Bigg\} \\
& + \frac{p^{*\prime}\tilde{p}^{*\prime\prime}}{4\sigma^2}\Bigg\{\frac{-3\sinh\lambda Y}{\lambda^3}\left(\frac{1}{\cosh\lambda F}\right)' - \frac{\sinh\tilde{\lambda}Y}{\tilde{\lambda}^2}\left(\frac{F}{\sinh\tilde{\lambda}F}\right)' \\
& \quad + \frac{Y\cosh\lambda Y}{\lambda^2}\left(\frac{1}{\cosh\lambda F}\right)' - \frac{\sinh\lambda Y}{\lambda^3}\left(\frac{1}{\cosh\lambda F}\right)' \\
& \quad - \left(i\sinh\sqrt{2\sigma}\,Y + \sin\sqrt{2\sigma}\,Y\right)\left(\frac{F}{4\sigma\sinh\tilde{\lambda}F\cosh\lambda F}\right)'\Bigg\} \\
& + \text{Complex Conjugate}\Bigg]
\end{aligned}
\qquad (1.3.9)
$$

and if we now put $Y=F$ and use (1.3.5)b we obtain

$$
\begin{aligned}
0 = & \left\{\frac{\tilde{\Phi}F^3}{3}\right\}' - FB' - F\tilde{B}' \\
& -\Bigg[\left(\frac{p^{*\prime}\tilde{p}^{*\prime\prime}}{4\sigma^2}\right)'\Bigg\{-\frac{4\tanh\lambda F}{\lambda^3} - \frac{F}{\tilde{\lambda}^2} + \frac{F}{\lambda^2} \\
& \quad - F\left(\frac{i\sinh\sqrt{2\sigma}F + \sin\sqrt{2\sigma}F}{4\sigma\cosh\lambda F\,\sinh\tilde{\lambda}F}\right)\Bigg\} \\
& + \left(\frac{p^{*\prime}\tilde{p}^{*\prime\prime}}{4\sigma^2}\right)\Bigg\{-\frac{4\sinh\lambda F}{\lambda^3}\left(\frac{1}{\cosh\lambda F}\right)' - \frac{\sinh\tilde{\lambda}F}{\tilde{\lambda}^2}\left(\frac{F}{\sinh\tilde{\lambda}F}\right)' \\
& \quad + \frac{F\cosh\lambda F}{\lambda^2}\left(\frac{1}{\cosh\lambda F}\right)' \\
& \quad - \left(i\sinh\sqrt{2\sigma}F + \sin\sqrt{2\sigma}F\right)\left(\frac{F}{4\sigma\cosh\lambda F\sinh\tilde{\lambda}F}\right)'\Bigg\} \\
& + \text{Complex Conjugate}\Bigg]
\end{aligned}
\qquad (1.3.10)
$$

If we substitute for B from (1.3.8)b into the above expression we find that the resulting expression is an exact differential which we integrate once to give

$$D = \frac{F^3\tilde{\Phi}}{3} - \Bigg[\frac{p^{*\prime}\tilde{p}^{*\prime\prime}}{4\sigma^2} \Bigg\{ -\frac{4\tanh\lambda F}{\lambda^3} + \frac{5F}{\lambda^2} + \frac{F^2\coth\tilde{\lambda}F}{\tilde{\lambda}} - \frac{F^2\tanh\lambda F}{\lambda}$$

$$+ F\left(\frac{iF\cosh\sqrt{2\sigma}F + F\cos\sqrt{2\sigma}F - i(\sqrt{2\sigma})^{-1}\sinh\sqrt{2\sigma}F - (\sqrt{2\sigma})^{-1}\sin\sqrt{2\sigma}F}{2\sqrt{2\sigma}}\right)\Bigg\}$$

$$+ \text{Complex Conjugate}\Bigg] \qquad (1.3.11)$$

where D is a constant which can be determined using (1.3.6).

Suppose we define a stream function, ψ_s, by

$$U_s = \frac{\partial\psi_s}{\partial Y} \quad , \quad V_s = -\frac{\partial\psi_s}{\partial X} \qquad (1.3.12)\ \text{a,b}$$

then we can use (1.3.7), (1.3.9) to show that ψ_s is given by

$$\psi_s = \tilde{\Phi}\left\{\frac{Y^3 - 3YF^2}{6}\right\} + BY + \tilde{B}Y$$

$$+ \Bigg[\frac{p^{*\prime}\tilde{p}^{*\prime\prime}}{4\sigma^2} \Bigg\{ -\frac{4\sinh\lambda Y}{\lambda^3\cosh\lambda F} - \frac{F\sinh\tilde{\lambda}Y}{\tilde{\lambda}^2\sinh\tilde{\lambda}F} + \frac{Y\cosh\lambda Y}{\lambda^2\cosh\lambda F}$$

$$- \frac{F\left(\sinh\sqrt{2\sigma}Y + \sin\sqrt{2\sigma}Y\right)}{4\sigma\cosh\lambda F\sinh\tilde{\lambda}F} \Bigg\}$$

$$+ \text{Complex Conjugate} \Bigg] \qquad (1.3.13)$$

where $\tilde{\Phi}$, B are as given by (1.3.8)a,b respectively

<u>The high frequency limit</u>

We now consider the nature of the steady streaming in the limit of σ tending to infinity. This corresponds to the Stokes layers associated with the oscillatory motion of the fluid being thin compared to the typical depth, D_0, of the channel. When σ is large, we can use (1.2.21), (1.2.22) to show that $P^{*\prime}$ is then expressible in the form

$$P^{*\prime} = -\frac{i\alpha}{F}\left\{\beta_0 + \frac{\beta_0}{\lambda F} + \frac{\beta_1}{\lambda} + O(\sigma^{-1})\right\} \tag{1.3.14}$$

where

$$\beta_0 = \left\{\int_0^K \frac{dX}{F^2}\right\}^{-1}, \quad \beta_1 = \beta_0^{-1}\left\{\int_0^K \frac{dX}{F}\right\}^{-2} \tag{1.3.15 a,b}$$

and in order to simplify the following algebra we choose α such that

$$\alpha = \beta_0^{-1}$$

This is equivalent to redefining the typical value of the velocity along the pipe, U_0, in terms of the amplitude of the applied pressure difference. With the above choice of α we can write $P^{*\prime}$ in the form

$$P^{*\prime} = \frac{-i}{F}\left\{1 + \frac{1}{\lambda F} + \frac{\beta_1}{\beta_0 \lambda} + O(\sigma^{-1})\right\} \tag{1.3.16}$$

We now define a Stokes layer variable, Y', for the upper Stokes layer by

$$Y' = \{F - Y\}\sqrt{\frac{\sigma}{2}} \tag{1.3.17}$$

and so Y' is of order unity in the upper Stokes layer. We can use (1.2.16), (1.2.20) to show that for large σ U^*, V^* have the following form in this layer

$$\left.\begin{aligned} U^* &= \left\{\frac{1 - e^{-(1+i)Y'}}{\sigma F}\right\} + O(\sigma^{-3/2}) \\ V^* &= F'\left\{\frac{1 - e^{-(1+i)Y'}}{\sigma F}\right\} + O(\sigma^{-3/2}) \end{aligned}\right\} \tag{1.3.18 a,b}$$

and as we might expect the dominant terms of these expressions represent a velocity parallel to the channel wall. If we put F' equal to zero in these expressions the Stokes flow in the upper layer reduces to Stokes shear wave solution for oscillatory viscous flow adjacent to a plane wall.

Suppose now that we let σ tend to infinity in (1.3.11) with ζ as given by (1.3.8)a. We thus obtain

$$\frac{3D}{F^3} = P_s' - \frac{F'}{2\sigma^2 F^3} + O(\sigma^{-5/2}) \tag{1.3.19}$$

and if we integrate both sides of this equation from X=0 to X=K and then use (1.3.6) we obtain

$$D = \left\{F^{-2}(K) - F^{-2}(0)\right\} \Big/ 12\sigma^2 \int_0^K \frac{dX}{F^3} + O(\sigma^{-5/2}) \tag{1.3.20}$$

If ends of the channel are of the same depth then the dominant term of the right hand side above is zero. In fact, it can be shown that all higher order terms also vanish in this case. If the ends of the channel are not of the same depth then the dominant steady streaming, as given by (1.3.13), has no Stokes layer type of behaviour and we can use (1.3.8), (1.3.11), (1.3.13), (1.3.20) to show that ψ_s can be written in the form

$$\psi_s = \left\{F^{-2}(K) - F^{-2}(0)\right\}\left\{Y^3 - 3YF^2\right\} \Big/ 24\sigma^2 F^3 \int_0^K \frac{dX}{F^3} + O(\sigma^{-5/2}) \tag{1.3.21}$$

and this represents a steady flow which is always directed towards the deepest end of the end of the channel. When the ends of the channel are of the same depth the dominant steady streaming does have a Stokes layer type of behaviour and we can write the following form for ψ_s in the upper Stokes layer

$$\psi_s = \frac{F'}{4\sqrt{2}\sigma^{7/2}F^3}\left\{13 - 6Y' - e^{-2Y'} - 12\cos Y' e^{-Y'} - 8\sin Y' e^{-Y'} - 4Y'\sin Y' e^{-Y'} + O(\sigma^{-1/2})\right\} \tag{1.3.22}$$

and a similar expression can be obtained in the lower Stokes layer. The appropriate form for ψ_s away from the Stokes layer is

$$\psi_s = \frac{3F'}{8\sigma^3 F^5}\left\{Y^3 - YF^2\right\} + O(\sigma^{-7/2}) \tag{1.3.23}$$

and so we see that the steady streaming persists throughout the channel.

Lyne (1971) has considered oscillatory viscous flow adjacent to a wavy wall, the amplitude of which was small compared to the thickness of the Stokes layer formed at the wall. If we take our channel walls to be wavy with amplitude small compared to the Stokes layers' thickness, then the expression (1.3.22) is identical to that found by Lyne in one particular limit. Our results correspond to the wavelength of Lyne's wall being much greater than both the Stokes layer at the wall and the amplitude of oscillation of a fluid particle far from the wall. The steady streaming found by Lyne persisted a distance of the order of magnitude of the wall's wavelength away from the Stokes layer at the wall. Since the depth of our channel is small compared to the typical length, L, a similar decay is not exhibited in our problem.

The low frequency limit

We now consider the limit of σ tending to zero. In this case the Stokes layers associated with the oscillatory motion of the fluid completely fill the channel. When σ is small we can use the series expansions sinh, cosh in (1.2.21), (1.2.22) to show that $P^{*\prime}$ can be written in the form

$$P^{*\prime} = -\frac{i\alpha}{F^3}\left\{\gamma_0 + \frac{2\gamma_0\lambda^2F^2}{5} - \gamma_1\lambda^2 + O(\sigma^2)\right\} \qquad (1.3.24)$$

where $$\gamma_0 = \left\{\int_0^K \frac{dX}{F^3}\right\}, \quad \gamma_1 = \frac{2\gamma_0^2}{5}\int_0^K \frac{dX}{F} \qquad (1.3.25)$$

and for convenience we choose $\alpha = \gamma_0^{-1}$ which is again equivalent to redefining U_0 in terms of the amplitude of the applied pressure difference. We can then write $P^{*\prime}$ in the form

$$P^{*\prime} = \frac{-i}{F^3}\left\{1 + \frac{2\lambda^2F^2}{5} - \frac{\gamma_1\lambda^2}{\gamma_0} + O(\sigma)\right\} \qquad (1.3.26)$$

and so it follows that for small σ we can write U^*, V^*, given by (1.2.16), (1.2.20), in the form

$$U^* = -i\left\{\frac{Y^2 - F^2}{2F^3}\right\} + O(\sigma)$$

$$V^* = -iF'\left\{\frac{Y^3 - YF^2}{2F^4}\right\} + O(\sigma)$$

We see immediately that in the low frequency limit the dominant terms in the Stokes flow pressure and velocity are in phase. This contrasts with the high frequency limit where they were $\pi/2$ radians out of phase

If we let σ tend to zero in (1.3.11), where $\mathcal{I}$ is as given by (1.3.8)a, we obtain

$$\frac{3D}{F^3} = P_s' - \frac{19F'}{280F^3} + O(\sigma) \tag{1.3.27}$$

which we integrate from X=0 to X=K. If we then use (1.3.6) we can show that

$$D = 19\left\{F^{-2}(K)-F^{-2}(0)\right\}\Big/1680\int_0^K \frac{dX}{F^3} + O(\sigma) \tag{1.3.28}$$

and so it follows that the dominant term on the right hand side is zero if the ends of the channel are of the same depth. It can again be shown that all higher order terms vanish in this case.

If we let σ tend to zero in (1.3.13) and use (1.3.8)a, (1.3.27), (1.3.28) we can show that ψ_s is of the following form for small σ

$$\psi_s = \frac{F'}{10080}\left\{-10\left(Y/F\right)^7 + 63\left(Y/F\right)^5 - 96\left(Y/F\right)^3 + 43\left(Y/F\right)\right\}$$

$$+\frac{19}{3360}\left\{F^{-2}(K)-F^{-2}(0)\right\}\left\{\left(Y/F\right)^3-3\left(Y/F\right)\right\}\Big/\int_0^K\frac{dX}{F^3} + O(\sigma) \tag{1.3.29}$$

If the ends of the channel are of the same depth we can write

$$\psi_s = \frac{F'}{10080}\left\{-10\left(Y/F\right)^7 + 63\left(Y/F\right)^5 - 96\left(Y/F\right)^3 + 43\left(Y/F\right)\right\} + O(\sigma) \tag{1.3.30}$$

Since ψ_s, given by (1.3.30), is zero at Y=0 and at Y=F there is no net flux associated with this steady streaming. However this is not the case with the steady streaming specified by (1.3.29). The extra term in this expression corresponds to a steady flow towards the deepest end of the channel.

1.4 Discussion of results

We have seen that in both the low and high frequency limits the geometry of the channel is crucial in determining the nature of the induced steady streaming. In particular the difference between the depths of the ends of the channel has an important role.

When this difference is zero the steady streaming is specified in the high frequency limit by (1.3.22), (1.3.23). If the terms of order $\sigma^{-7/2}$ in (1.3.23) are evaluated explicitly it is found that ψ_s is zero up to order $\sigma^{-7/2}$ at Y=0. Thus, since ψ_s evaluated at Y=F is also zero at this order, there is no net flux through the channel up to order $\sigma^{-7/2}$. However,we can easily show that there is no contribution to the flux from higher order terms by returning to the expression (1.3.13) for ψ_s. Using (1.3.8)b, (1.3.11) we can show that $\left\{\psi_s\big|_{Y=F} - \psi_s\big|_{Y=0}\right\}$ is equal to -D, and in the high frequency limit D is identically zero if F(0)=F(K). Similarly there is no flux through such a channel in the low frequency limit. In both limits the steady streaming is confined between the points F'=0 of such channels. In Figs 1, 2 we have shown the steady streaming in a wavy channel for the high and low frequency limits respectively. In Fig 1 there are four regions of recirculation, one in each of the Stokes layers, and one between each Stokes layer and the line Y=0. In Fig 2 there are two regions of recirculation, one either side of the line Y=0.

When F(O) and F(K) are not equal the steady streaming described above, which is induced directly by the Reynolds stresses associated with the Stokes flow, gives a resultant steady pressure difference between the ends of the pipe. In order to balance this pressure difference a suitable multiple of the flow velocity given below is required.

$$(u, v) = \frac{[Y^2 - F^2]}{F^3}\left(1, \frac{YF'}{F}\right)$$

In both the high and low frequency limits it is found that this component of the steady flow is always directed towards the deepest end of the channel. In the high frequency limit the steady streaming is in fact dominated by this component. If the ends of the channel are not of the same depth but K tends to infinity then we can show that D as given by (1.3.20), (1.3.28) tends to zero like K^{-1}. The steady streaming is then dominated by that corresponding to flow in a channel with $F(K)=F(O)$.

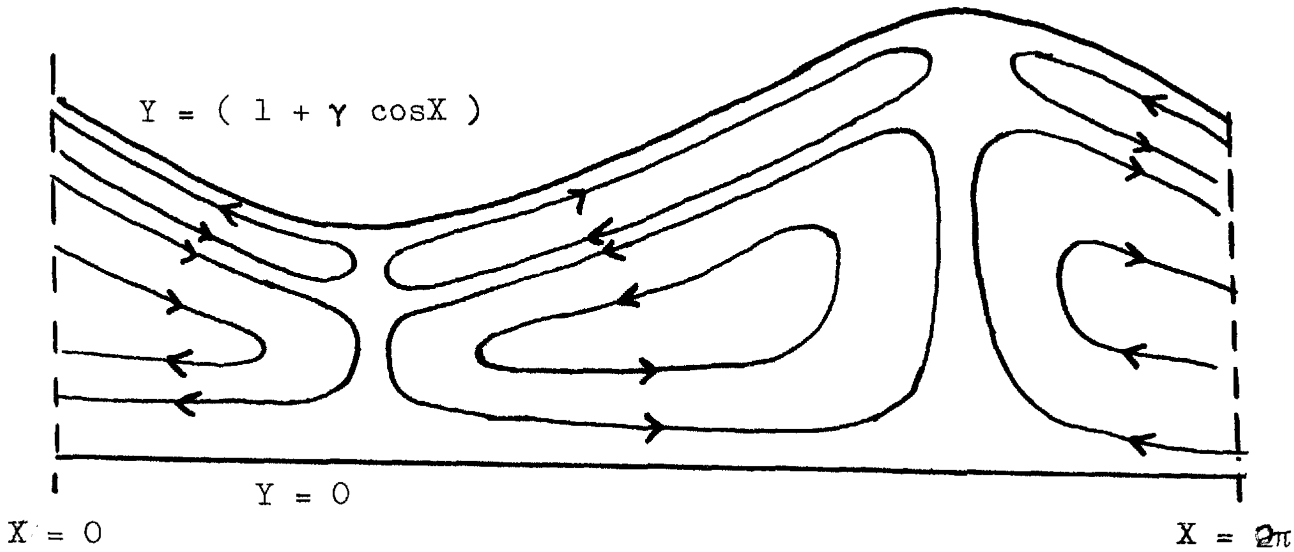

Fig. 1 : The steady streaming in the high frequency limit for a two-dimensional channel of the form $Y = \mp(1 + \gamma \cos X)$, $0 \leqslant X \leqslant 2\pi$

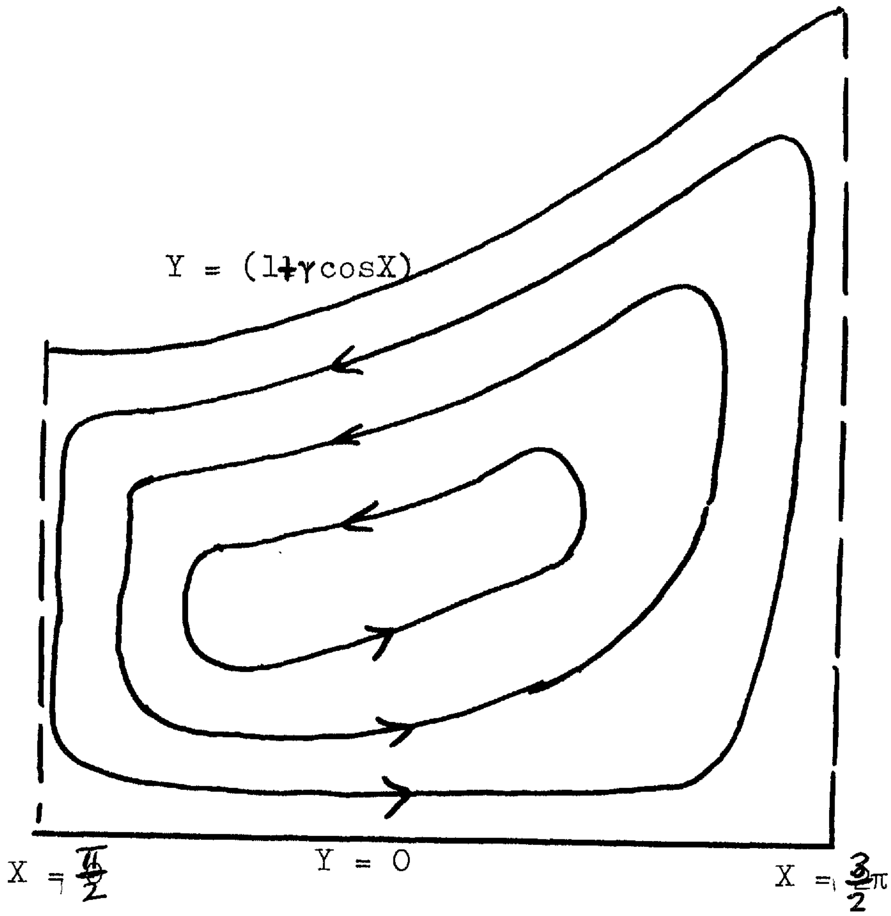

Fig. 2 : The steady streaming in the low frequency limit for a two-dimensional channel of the form $Y = \mp (1 + \gamma \cos X)$, $\frac{\pi}{2} \leqslant X \leqslant \pi \cdot \frac{3}{2}$

CHAPTER II

UNSTEADY VISCOUS FLOW IN A PIPE OF SLOWLY VARYING CROSS-SECTION

2.1 Introduction

In this chapter we use the methods of the first chapter to investigate the unsteady viscous flow in a pipe of slowly varying cross-section when a purely oscillatory pressure difference is maintained across its ends. In order to use lubrication theory again we also require that a modified Reynolds number associated with the flow is small.

Suppose that the radius of the pipe in cylindrical polar coordinates (r, θ, z) is given by

$$r = \mathcal{D}_0 \left\{ R\left(\tfrac{z}{L}\right) + \epsilon S\left(\tfrac{z}{L}\right) \cos M\theta \right\} \tag{2.1.1}$$

where M clearly must be an integer which we take to be positive. We define the parameter δ by

$$\delta = \frac{\mathcal{D}_0^2}{L^2} \tag{2.1.2}$$

so that if the radius of the pipe varies slowly we require that

$$\delta \ll 1 \tag{2.1.3}$$

and if U_0 is a typical axial velocity of the fluid, and ν is the kinematic viscosity, we require that R_M, defined below, is also small compared to unity.

$$R_M = \frac{U_0 \mathcal{D}_0^2}{L \nu} \tag{2.1.4}$$

We see from (2.1.4) that R_M is just the usual Reynolds number multiplied by δ .

If the frequency of the pressure oscillations is ω , we define a frequency parameter, σ , by

$$\sigma = \omega \mathcal{D}_0^2 / \nu \tag{2.1.5}$$

and so $\sigma^{\frac{1}{2}}$ represents the ratio of a typical radius of the pipe to the thickness of the Stokes layer associated with the oscillatory motion of the fluid. We again consider the special limits of σ tending to zero and infinity. In the latter case we consider the Stokes layer at the pipe wall and the region away from this layer separately. We solve for the velocity in each region and match corresponding components where different regions meet.

We assume throughout that the perturbation of the pipe wall in the (r , θ) plane is small compared to the thickness of the Stokes layer at the wall. Thus we require that

$$\epsilon \sigma^{1/2} \ll 1 \tag{2.1.6}$$

The procedure adopted in this chapter is as follows. In §2.2 we derive the non-dimensional partial differential system and solve for the so-called 'Stokes flow' by putting R_M, δ equal to zero everywhere. In §2.3, 2.4 we evaluate the first order steady streaming in the high and low frequency limits respectively. This first order steady streaming first appears in the order R_M correction to the Stokes flow in both limits. In §2.5 we discuss our results and their relevance to some physiological flows.

2.2 Equations of motion and the Stokes flow

We consider viscous incompressible flow in a pipe defined in cylindrical polar coordinates, (r, θ ,z), by

$$0 \leq r \leq \mathcal{D}_0 \left\{ R(\tfrac{z}{L}) + \epsilon S(\tfrac{z}{L}) \cos N\theta \right\} = r^*(\theta, z)$$

$$0 \leq z \leq KL$$

We define p, ν, ρ, t to be the pressure, kinematic viscosity density, and time respectively. We assume that the pressure between the ends is given by

$$p(r^*, \theta, KL) - p(r^*, \theta, 0) = C_0 \sin \omega t \qquad (2.2.1)$$

and we introduce dimensionless variables τ, η, ζ as follows.

$$\tau = \omega t, \quad \eta = r/\mathcal{D}_0, \quad \zeta = z/L \qquad (2.2.2)\ \text{a,b,c}$$

If U_0 is a typical velocity along the pipe then the equation of continuity shows that the other components of velocity are of order $U_0 \mathcal{D}_0 / L$. We therefore make the velocity vector (u,v,w) dimensionless by writing

$$(u, v, w) = U_0 \left(\mathcal{D}_0 g/L,\ \mathcal{D}_0 h/L,\ f \right) \qquad (2.2.3)$$

and we make the pressure dimensionless by writing

$$p = \rho \nu L U_0 p^+ / \mathcal{D}_0^2 \qquad (2.2.4)$$

We can then write the momentum and continuity equations in the form

$$\sigma\delta \frac{\partial g}{\partial \tau} + R_M \delta \left\{ g \frac{\partial g}{\partial \eta} + \frac{h}{\eta}\frac{\partial g}{\partial \theta} + f \frac{\partial g}{\partial \zeta} - \frac{h^2}{\eta} \right\} = -\frac{\partial p^+}{\partial \eta} + \delta \left\{ \nabla^2 - \frac{1}{\eta^2} \right\} g - \frac{2\delta}{\eta^2}\frac{\partial h}{\partial \theta} + \delta^2 \frac{\partial^2 g}{\partial \zeta^2}$$

$$\sigma\delta \frac{\partial h}{\partial \tau} + R_M \delta \left\{ g \frac{\partial h}{\partial \eta} + \frac{h}{\eta}\frac{\partial h}{\partial \theta} + f \frac{\partial h}{\partial \zeta} + \frac{gh}{\eta} \right\} = -\frac{1}{\eta}\frac{\partial p^+}{\partial \theta} + \delta \left\{ \nabla^2 - \frac{1}{\eta^2} \right\} h + \frac{2\delta}{\eta^2}\frac{\partial g}{\partial \theta} + \delta \frac{\partial^2 h}{\partial \zeta^2}$$

$$\sigma \frac{\partial f}{\partial \tau} + R_M \left\{ g \frac{\partial f}{\partial \eta} + \frac{h}{\eta}\frac{\partial f}{\partial \theta} + f \frac{\partial g}{\partial \zeta} \right\} = -\frac{\partial p^+}{\partial \zeta} + \nabla^2 f + \delta \frac{\partial^2 f}{\partial \zeta^2}$$

$$\frac{\partial}{\partial \eta}(\eta g) + \frac{\partial h}{\partial \theta} + \eta \frac{\partial f}{\partial \zeta} = 0 \qquad (2.2.5)\ \text{a,b,c,d}$$

where $\nabla^2 \equiv \dfrac{\partial^2}{\partial \eta^2} + \dfrac{1}{\eta}\dfrac{\partial}{\partial \eta} + \dfrac{1}{\eta^2}\dfrac{\partial^2}{\partial \theta^2}$

and δ, σ, R_M are as defined by (2.1.2), (2.1.5), (2.1.4) respectively. These equations must be solved subject to there being no relative velocity at the pipe wall. Thus we require that

$$g = h = f = 0, \quad \eta = R + \epsilon S \cos M\theta \tag{2.2.6}$$

and from (2.2.1) it follows that

$$p^{+}(R+\epsilon S\cos M\theta, \theta, K, \tau) - p^{+}(R+\epsilon S\cos M\theta, \theta, 0, \tau) = \alpha \sin \tau \tag{2.2.7}$$

where
$$\alpha = \frac{C_0 D_0^{2}}{\mu L U_0} \tag{2.2.8}$$

The remaining conditions required to completely specify the problem are kinematical in origin. At the inner boundary, $\eta = 0$, we require that p,f must be independent of θ, whilst g,h must vary with $\cos\theta$, $\sin\theta$ there. A useful reference where these conditions are discussed in more detail is Batchelor (1962) and Gill.

We now assume that the parameters ϵ, R_M, δ are all small and we seek a solution by expanding g, h, f, p^{+} as follows

$$\left.\begin{aligned}
g &= G_{00} + \epsilon G_{01} + R_M G_{10} + \epsilon R_M G_{11} + O(R_M^2, \delta, \epsilon^2) \\
h &= H_{00} + \epsilon H_{01} + R_M H_{10} + \epsilon R_M H_{11} + O(R_M^2, \delta, \epsilon^2) \\
f &= F_{00} + \epsilon F_{01} + R_M F_{10} + \epsilon R_M F_{11} + O(R_M^2, \delta, \epsilon^2) \\
p^{+} &= P_{00} + \epsilon P_{01} + R_M P_{10} + \epsilon R_M P_{11} + O(R_M^2, \delta, \epsilon^2)
\end{aligned}\right\} \quad \begin{matrix}(2.2.9)\\ a,b,c,d\end{matrix}$$

The so-called 'Stokes flow' is obtained by putting the parameters R_M, δ equal to zero everywhere. We now solve for this flow up to and including terms of order ϵ. We first write

$$\left.\begin{aligned}
G_{00} &= \tfrac{1}{2}\{ g_{00} e^{i\tau} + \hat{g}_{00} e^{-i\tau} \} \\
H_{00} &= \tfrac{1}{2}\{ h_{00} e^{i\tau} + \hat{h}_{00} e^{-i\tau} \} \\
F_{00} &= \tfrac{1}{2}\{ f_{00} e^{i\tau} + \hat{f}_{00} e^{-i\tau} \} \\
P_{00} &= \tfrac{1}{2}\{ p_{00} e^{i\tau} + \hat{p}_{00} e^{-i\tau} \}
\end{aligned}\right\} \quad \begin{matrix}(2.2.10)\\ a,b,c,d\end{matrix}$$

where $\sim$ denotes 'complex conjugate' and the functions g_{oo}, h_{oo}, f_{oo}, p_{oo} are all independent of τ. If we substitute for G_{oo}, etc from above into (2.2.9), substitute the resulting expressions into (2.2.5), and equate terms proportional to $e^{i\tau}$ and independent of ϵ, R_M, δ we obtain

$$\frac{\partial p_{oo}}{\partial \eta} = 0$$

$$\frac{\partial p_{oo}}{\partial \theta} = 0$$

$$\frac{\partial p_{oo}}{\partial \zeta} = \{\nabla^2 - i\sigma\} f_{oo} \qquad \text{(2.2.11) a,b,c,d}$$

$$\frac{\partial}{\partial \eta}(\eta g_{oo}) + \frac{\partial h_{oo}}{\partial \theta} + \eta \frac{\partial f_{oo}}{\partial \zeta} = 0$$

Thus p_{oo} is a function of ζ only, and we can use (2.2.11)c to show that, if solutions of Bessels equation which are singular at $\eta = 0$, are rejected, f_{oo} is given by

$$f_{oo} = \frac{-p'_{oo}}{i\sigma} + \sum_{\lambda=0}^{\infty} \left[A_{\lambda 1}\cos\lambda\theta + A_{\lambda 2}\sin\lambda\theta\right] J_\lambda(s\eta) \qquad (2.2.12)$$

where a dash denotes a derivative with respect to ζ and s is defined by

$$s = \sqrt{-i\sigma} \qquad (2.2.13)$$

The coefficients $A_{\lambda 1}$, $A_{\lambda 2}$ appearing in (2.2.12) are in fact functions of ζ and will be determined later. If we substitute for f_{oo} from (2.2.12) into (2.2.11)c we can show that

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta g_{oo}) + \frac{1}{\eta}\frac{\partial h_{oo}}{\partial \theta} = \frac{p''_{oo}}{i\sigma} - \sum_{\lambda=0}^{\infty} \left[A'_{\lambda 1}\cos\lambda\theta + A'_{\lambda 2}\sin\lambda\theta\right] J_\lambda(s\eta) \qquad (2.2.14)$$

In order to solve for g_{oo}, h_{oo} we clearly require another equation linking these quantities.

If we eliminate the pressure p^+ from (2.2.5)a,b, substitute for g,h,f from (2.2.9), equate terms of order δ and take G_{oo}, H_{oo} as in (2.2.10), we can show that

$$\left\{\nabla^2 - i\sigma\right\}\left[\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta h_{oo}) - \frac{1}{\eta}\frac{\partial g_{oo}}{\partial\theta}\right] = 0$$

which we solve to give

$$\frac{\partial}{\partial\eta}(\eta h_{oo}) - \frac{\partial g_{oo}}{\partial\theta} = \sum_{\lambda=1}^{\infty}\left[B_{\lambda 1}\cos\lambda\theta + B_{\lambda 2}\sin\lambda\theta\right]\eta\, J_\lambda(s\eta) \tag{2.2.15}$$

where $B_{\lambda 1}$, $B_{\lambda 2}$ are functions of ζ to be determined later. The solutions of Bessels equation which have been neglected above lead to terms in h_{oo}, g_{oo} which are singular at η=0. We can eliminate g_{oo} from (2.2.14), (2.2.15) to obtain an equation for h_{oo} whose solution which is regular at η =0 is

$$\begin{aligned} h_{oo} = & \sum_{\lambda=1}^{\infty}\frac{\lambda}{i\sigma}\left[A'_{\lambda 1}\sin\lambda\theta - A'_{\lambda 2}\cos\lambda\theta\right]\frac{J_\lambda(s\eta)}{\eta} \\ & + \sum_{\lambda=0}^{\infty}\frac{1}{i\sigma}\left[B_{\lambda 1}\cos\lambda\theta + B_{\lambda 2}\sin\lambda\theta\right]\frac{dJ_\lambda(s\eta)}{d\eta} \\ & + \sum_{\lambda=1}^{\infty}\left[C_{\lambda 1}\cos\lambda\theta + C_{\lambda 2}\sin\lambda\theta\right]\eta^{\lambda-1} \end{aligned} \tag{2.2.16}$$

where the coefficients $C_{\lambda 1}$, $C_{\lambda 2}$ are functions of ζ . If we substitute for h_{oo} from (2.2.16) into (2.2.14) then, after multiplying by η , we can integrate from η =0 to $\eta=\eta$ to show that

$$\begin{aligned} g_{oo} = & -\sum_{\lambda=0}^{\infty}\frac{1}{i\sigma}\left[A'_{\lambda 1}\cos\lambda\theta + A'_{\lambda 2}\sin\lambda\theta\right]\frac{dJ_\lambda(s\eta)}{d\eta} \\ & + \sum_{\lambda=1}^{\infty}\frac{\lambda}{i\sigma}\left[B_{\lambda 1}\sin\lambda\theta - B_{\lambda 2}\cos\lambda\theta\right]\frac{J_\lambda(s\eta)}{\eta} \\ & + \sum_{\lambda=1}^{\infty}\left[C_{\lambda 1}\sin\lambda\theta - C_{\lambda 2}\cos\lambda\theta\right]\eta^{\lambda-1} + \frac{P''_{oo}\eta}{2i\sigma} \end{aligned} \tag{2.2.17}$$

where we have used the fact that g_{oo} is regular at η =0 to show that η g_{oo} is zero there. We can repeat the above calculations to show that the order ϵ terms in the Stokes flow are given by

$$\left.\begin{aligned}
g_{01} &= -\sum_{\lambda=0}^{\infty} \frac{1}{i\sigma}\left[a'_{\lambda 1}\cos\lambda\theta + a'_{\lambda 2}\sin\lambda\theta\right]\frac{dJ_\lambda(s\eta)}{d\eta} \\
&\quad + \sum_{\lambda=1}^{\infty} \frac{\lambda}{i\sigma}\left[b_{\lambda 1}\sin\lambda\theta - b_{\lambda 2}\cos\lambda\theta\right]\frac{J_\lambda(s\eta)}{\eta} \\
&\quad + \sum_{\lambda=1}^{\infty}\left[c_{\lambda 1}\sin\lambda\theta - c_{\lambda 2}\cos\lambda\theta\right]\eta^{\lambda-1} + \frac{p''_{01}\,\eta}{2i\sigma} \\
h_{01} &= \sum_{\lambda=1}^{\infty} \frac{\lambda}{i\sigma}\left[a'_{\lambda 1}\sin\lambda\theta - a'_{\lambda 2}\cos\lambda\theta\right]\frac{J_\lambda(s\eta)}{\eta} \\
&\quad + \sum_{\lambda=0}^{\infty} \frac{1}{i\sigma}\left[b_{\lambda 1}\cos\lambda\theta + b_{\lambda 2}\sin\lambda\theta\right]\frac{dJ_\lambda(s\eta)}{d\eta} \\
&\quad + \sum_{\lambda=1}^{\infty}\left[c_{\lambda 1}\cos\lambda\theta + c_{\lambda 2}\sin\lambda\theta\right]\eta^{\lambda-1} \\
f_{01} &= \frac{-p'_{01}}{i\sigma} + \sum_{\lambda=0}^{\infty}\left[a_{\lambda 1}\cos\lambda\theta + a_{\lambda 2}\sin\lambda\theta\right]J_\lambda(s\eta)
\end{aligned}\right\} \qquad \begin{matrix}(2.2.18)\\ a,b,c\end{matrix}$$

where $a_{\lambda 1}$, etc are arbitrary functions of ζ to be determined later and g_{o1}, h_{o1}, f_{o1} are defined by

$$\left.\begin{aligned}
G_{01} &= \frac{1}{2}\left\{g_{01}e^{it} + \tilde{g}_{01}e^{-it}\right\} \\
H_{01} &= \frac{1}{2}\left\{h_{01}e^{it} + \tilde{h}_{01}e^{-it}\right\} \\
F_{01} &= \frac{1}{2}\left\{f_{01}e^{it} + \tilde{f}_{01}e^{-it}\right\} \\
P_{01} &= \frac{1}{2}\left\{p_{01}e^{it} + \tilde{p}_{01}e^{-it}\right\}
\end{aligned}\right\} \qquad \begin{matrix}(2.2.19)\\ a,b,c,d\end{matrix}$$

In order to solve for the unknown functions of ζ in the expressions for f_{oo}, f_{o1}, etc we must consider the boundary conditions on the velocity and pressure. From (2.2.6), (2.2.7), (2.2.9), (2.2.10), (2.2.19) it follows that these may be written in the form

$$g_{00} + \epsilon g_{01} = h_{00} + \epsilon h_{01} = f_{00} + \epsilon f_{01} = O(\epsilon^2),\ \eta = R + \epsilon S \cos\lambda\theta \qquad \begin{matrix}(2.2.20)\\ a,b,c\end{matrix}$$

$$p_{00}(K) - p_{00}(0) = -i\alpha$$

$$p_{01}(K) - p_{01}(0) = 0 \qquad (2.2.20)\ d,e$$

We can use (2.2.12), (2.2.18)c to show that the condition (2.2.20)a gives

$$O(\epsilon^2) = \frac{-1}{i\sigma}\left\{p_{00}' + \epsilon p_{01}'\right\} + \sum_{\lambda=0}^{\infty}\left[(A_{\lambda 1} + \epsilon a_{\lambda 1})\cos\lambda\theta + (A_{\lambda 2} + \epsilon a_{\lambda 2})\sin\lambda\theta\right]\left[J_\lambda(sR) + \epsilon s S \cos M\theta J_\lambda'(sR)\right]$$

where from now on a dash on a Bessel function denotes a derivative with respect to its argument, and we have replaced $J_\lambda(sR + \epsilon s S \cos M\theta)$ by its Taylor series expansion about sR. The validity of this expansion is ensured by (2.1.6). The coefficients $A_{\lambda 1}$, $a_{\lambda 1}$, etc are then found by equating terms proportional to $\cos\lambda\theta$, $\epsilon\cos\lambda\theta$, etc. We find that

$$A_{01} = \frac{p_{00}'}{i\sigma J_0(sR)}, \quad a_{M1} = \frac{-sRJ_0'(sR)A_{01}}{J_M(sR)} \qquad (2.2.21)\ a,b$$

and all other coefficients are zero. Thus we can write

$$f_{00} = \frac{-p_{00}'}{i\sigma}\left[1 - \frac{J_0(s\eta)}{J_0(sR)}\right]$$

$$f_{01} = \frac{-p_{01}'}{i\sigma}\left[1 - \frac{J_0(s\eta)}{J_0(sR)}\right] - \frac{sSp_{00}'J_0'(sR)J_M(s\eta)\cos M\theta}{i\sigma J_0(sR)J_M(sR)} \qquad (2.2.22)\ a,b$$

If we now use the conditions (2.2.20)b,c and take g_{00}, h_{00}, g_{01}, h_{01} as given by (2.2.17), (2.2.16), (2.2.18)a,b respectively we can show that

$$O(\epsilon^2) = -\sum_{\lambda=0}^{\infty}\left[(A_{\lambda 1}' + \epsilon a_{\lambda 1}')\cos\lambda\theta + (A_{\lambda 2}' + \epsilon a_{\lambda 2}')\sin\lambda\theta\right]\left[sRJ_\lambda'(sR) + \epsilon\cos M\theta\left(Ss J_\lambda'(sR) + Ss^2RJ_\lambda''(sR)\right)\right]$$
$$+ \sum_{\lambda=1}^{\infty}\lambda\left[(B_{\lambda 1} + \epsilon b_{\lambda 1})\sin\lambda\theta - (B_{\lambda 2} + \epsilon b_{\lambda 2})\cos\lambda\theta\right]\left[J_\lambda(sR) + \epsilon s S J_\lambda'(sR)\cos M\theta\right]$$
$$+ i\sigma\sum_{\lambda=1}^{\infty}\left[(C_{\lambda 1} + \epsilon c_{\lambda 1})\sin\lambda\theta - (C_{\lambda 2} + \epsilon c_{\lambda 2})\cos\lambda\theta\right]\left[R^\lambda + \epsilon\lambda S R^{\lambda-1}\cos M\theta\right]$$
$$+ \frac{1}{2}\left[p_{00}'' + \epsilon p_{01}''\right]\left[R^2 + 2\epsilon RS\cos M\theta\right]$$

$$
\begin{aligned}
0(\epsilon^2) = &\sum_{\lambda=1}^{\infty} \lambda \Big[(A'_{\lambda 1} + \epsilon a'_{\lambda 1}) \sin\lambda\theta - (A'_{\lambda 2} + \epsilon a'_{\lambda 2}) \cos\lambda\theta \Big]\Big[J_\lambda(sR) + \epsilon s S J'_\lambda(sR) \cos M\theta \Big] \\
&+ \sum_{\lambda=0}^{\infty} \Big[(B_{\lambda 1} + \epsilon b_{\lambda 1}) \cos\lambda\theta + (B_{\lambda 2} + \epsilon b_{\lambda 2}) \sin\lambda\theta \Big]\Big[sRJ'_\lambda(sR) + \epsilon s^2 RSJ''_\lambda(sR) \cos M\theta \\
&\qquad\qquad + \epsilon s SJ'_\lambda(sR) \cos M\theta \Big] \\
&+ i\sigma \sum_{\lambda=0}^{\infty} \Big[(C_{\lambda 1} + \epsilon c_{\lambda 1}) \cos\lambda\theta + (C_{\lambda 2} + \epsilon c_{\lambda 2}) \sin\lambda\theta \Big]\Big[R^\lambda + \epsilon S R^{\lambda-1} \cos M\theta \Big]
\end{aligned}
$$

(2.2.23) a,b

where we have again expanded the Bessel functions in Taylor series. If we equate terms independent of ϵ and θ in the above expressions we obtain

$$
\left.
\begin{aligned}
&\left\{ \frac{R^2}{2} - \frac{R J_1(sR)}{s J_0(sR)} \right\} p''_{00} - \frac{R R' p'_{00} J_1^2(sR)}{J_0^2(sR)} = 0 \\
&\text{and} \quad B_{01} = 0
\end{aligned}
\right\}
\qquad (2.2.24)\ \text{a,b}
$$

where we have replaced A_{01} in (2.2.24)a using (2.2.21)a. The equation (2.2.24)a is the so-called 'Reynolds' equation for the pressure and can be integrated once to give

$$
p'_{00} = \frac{E}{\left\{ \frac{R^2}{2} - \frac{R J_1(sR)}{s J_0(sR)} \right\}} \qquad (2.2.25)
$$

where E is an unknown constant which we determine by integrating both sides of the above equating from $\zeta = 0$ to $\zeta = K$ and then using (2.2.20)d. We thus obtain

$$
E = \frac{-i\alpha}{\displaystyle\int_0^K \frac{d\zeta}{\left\{ \frac{R^2}{2} - \frac{R J_1(sR)}{s J_0(sR)} \right\}}} \qquad (2.2.26)
$$

The remaining unknown functions of ζ , i.e. $B_{\lambda 1}$, $B_{\lambda 2}$, $b_{\lambda 1}$, $b_{\lambda 2}$, $C_{\lambda 1}$, $C_{\lambda 2}$, $c_{\lambda 1}$, $c_{\lambda 2}$, for $\lambda \geq 1$, are obtained by equating terms proportional to $\sin\lambda\theta$, $\cos\lambda\theta$, $\sin\lambda\theta$, $\cos\lambda\theta$, in (2.2.23)a,b The order ϵ pressure, p_{o1}, is found by equating terms of order ϵ in (2.2.23)a. We find that

$$p_{o1}' = \frac{F}{\left\{ \frac{R^2}{2} - \frac{R J_1(sR)}{s J_0(sR)} \right\}} \tag{2.2.27}$$

and using (2.2.20)c it follows that F is zero. The only remaining unknown function of ζ is b_{o1}. If we equate terms of order ϵ, independent of θ in (2.2.23)b we find that b_{o1} is in fact zero. Having determined all these functions of ζ we can write g_{oo}, g_{o1}, etc as follows

$$g_{oo} = \frac{p_{oo}''}{2i\sigma}\left[\eta - \frac{R J_1(s\eta)}{J_1(sR)}\right]$$

$$g_{o1} = \Bigg[\left(\eta/R\right)^{M-1}\left(s a_{M1}' J_{M-1}(sR) + \frac{A_{o1}' s S}{R}\left[J_0'(sR) - sR J_0''(sR)\right]\frac{J_M'(sR)}{J_{M+1}(sR)}\right) - a_{M1}' s J_{M-1}(s\eta) + A_{o1}' s S\left[J_0'(sR) - sR J_0''(sR)\right]\frac{M J_M(s\eta)}{\eta R s J_{M+1}(sR)}\Bigg]\frac{\cos M\theta}{i\sigma}$$

$$h_{oo} = 0$$

$$h_{o1} = \Bigg[-\left(1/R\right)^{M-1}\left(s a_{M1}' J_{M-1}(sR) + \frac{A_{o1}' s S}{R}\left[J_0'(sR) - sR J_0''(sR)\right]\frac{J_M'(sR)}{J_{M+1}(sR)}\right) + a_{M1}' s J_{M-1}(s\eta) + A_{o1}' s S\left[J_0'(sR) - sR J_0''(sR)\right]\frac{\partial J_M(s\eta)}{\partial\eta}\Bigg]\frac{\sin M\theta}{i\sigma}$$

(2.2.28) a,b,c,d

where A_{o1}, a_{M1} are as given by (2.2.21)a,b.

2.3 Calculation of the steady streaming for large σ

When σ is large the Stokes layer associated with the oscillatory motion of the fluid is very thin compared to a typical radius of the pipe. We first discuss the nature of the 'Stokes flow' for large σ. If we use the asymptotic expansion of a Bessel function of large argument in (2.2.25), (2.2.26) we can show that

$$p_{oo}' = \frac{-i\alpha}{R^2}\left\{\beta_o + \frac{2\beta_o}{R\sqrt{i\sigma}} \frac{-2\beta_1}{\sqrt{i\sigma}} + O(\sigma^{-1})\right\} \tag{2.3.1}$$

where $\beta_o = \left\{\int_o^K \frac{d\xi}{R^2}\right\}^{-1}, \beta_1 = \beta_o^2 \int_o^K \frac{d\xi}{R^3}$ (2.3.2) a,b

and for the sake of convenience we choose $\beta_o = \alpha^{-1}$.
This is equivalent to redefining the typical axial velocity, U_o, in terms of the amplitude of the applied pressure difference. We can then write p_{oo}' in the form

$$p_{oo}' = \frac{-i}{R^2}\left\{1 + \frac{2}{R\sqrt{i\sigma}} - \frac{2\beta_1}{\beta_o\sqrt{i\sigma}} + O(\sigma^{-1})\right\} \tag{2.3.3}$$

If we define a Stokes layer variable η' by

$$\eta' = [R - \eta]\sqrt{\frac{\sigma}{2}} \tag{2.3.4}$$

then if we expand the Bessel functions in (2.2.22), (2.2.28), for large $|s\eta|$, $|sR|$ and use (2.3.3) we can show that in the Stokes layer

$$\left.\begin{aligned}
f_{oo} &\sim \frac{1}{\sigma R^2}\left[1 - e^{-(1+i)\eta'}\right] + O(\sigma^{-3/2}) \\
g_{oo} &\sim \frac{R'}{\sigma R^2}\left[1 - e^{-(1+i)\eta'}\right] + O(\sigma^{-3/2}) \\
f_{o1} &\sim \frac{S}{\sqrt{-i\sigma}\,R^2} e^{-(1+i)\eta'} + O(\sigma^{-1}) \\
g_{o1} &\sim \frac{SR'}{\sqrt{-i\sigma}\,R^2} e^{-(1+i)\eta'} + O(\sigma^{-1}) \\
h_{o1} &\sim O(\sigma^{-1})
\end{aligned}\right\} \tag{2.3.5 a,b,c,d,e}$$

If we put $S=R'=0$ in (2.3.5) we see that the 'Stokes flow' in the Stokes layer reduces to Stokes shear wave solution for flow in a circular pipe. We can also see from (2.3.5) that the order ϵ corrections to the axisymmetric flow have a dominant term which decays to zero at the edge of the Stokes layer. We shall refer to the region away from this layer as the 'outer' layer

If we substitute for g, h, f, p^+ from (2.2.9) into (2.2.5) and equate terms of order R_M we obtain

$$\left.\begin{aligned} &\frac{\partial P_{10}}{\partial \eta} = 0 \\ &\frac{\partial P_{10}}{\partial \theta} = 0 \\ &\left\{\nabla^2 - \sigma\frac{\partial}{\partial \tau}\right\} F_{10} = \frac{\partial P_{10}}{\partial \xi} + \left\{F_{00}\frac{\partial F_{00}}{\partial \xi} + \frac{H_{00}}{\eta}\frac{\partial F_{00}}{\partial \theta} + G_{00}\frac{\partial F_{00}}{\partial \eta}\right\} \\ &\frac{\partial}{\partial \eta}(\eta G_{10}) + \frac{\partial H_{10}}{\partial \theta} + \eta\frac{\partial F_{10}}{\partial \xi} = 0 \end{aligned}\right\} \quad \begin{matrix}(2.3.6)\\ a,b,c,d\end{matrix}$$

and similarly by equating terms of order ϵR_M we have

$$\left.\begin{aligned} &\frac{\partial P_{11}}{\partial \eta} = 0 \\ &\frac{\partial P_{11}}{\partial \theta} = 0 \\ &\left\{\nabla^2 - \sigma\frac{\partial}{\partial \tau}\right\} F_{11} = \frac{\partial P_{11}}{\partial \xi} + \left\{F_{00}\frac{\partial F_{01}}{\partial \xi} + F_{01}\frac{\partial F_{00}}{\partial \xi} + \frac{H_{00}}{\eta}\frac{\partial F_{01}}{\partial \theta}\right. \\ &\qquad\qquad \left. + \frac{H_{01}}{\eta}\frac{\partial F_{00}}{\partial \theta} + G_{00}\frac{\partial F_{01}}{\partial \eta} + G_{01}\frac{\partial F_{00}}{\partial \eta}\right\} \\ &\frac{\partial}{\partial \eta}(\eta G_{11}) + \frac{\partial H_{11}}{\partial \theta} + \eta\frac{\partial F_{11}}{\partial \xi} = 0 \end{aligned}\right\} \quad \begin{matrix}(2.3.7)\\ a,b,c,d\end{matrix}$$

We recall that in §(2.2) F_{00}, G_{00} represented the axisymmetric solution and F_{01}, G_{01}, H_{01} gave the order ϵ non-axisymmetric correction to this solution. Similarly F_{10}, G_{10} will represent the order R_M axisymmetric solution and F_{11}, G_{11}, H_{11} will give the order ϵR_M non-axisymmetric correction to this solution. Thus we drop the θ dependence of (2.3.6) and put $H_{1o} = 0$. The relevant boundary conditions for F_{1o}, G_{1o} are

$$F_{1o} = G_{1o} = 0 \,, \quad \eta = R \tag{2.3.8 a,b}$$

and the conditions for F_{11}, G_{11}, H_{11} are

$$F_{1o} + \epsilon F_{11} = G_{1o} + \epsilon G_{11} = \epsilon H_{11} = O(\epsilon^2) \,, \quad \eta = R + \epsilon S \cos M\theta \tag{2.3.9 a,b,c}$$

where F_{1o}, G_{1o} are assumed known. We also require that F_{1o}, F_{11}, P_{1o}, P_{11} are independent of θ at $\eta = 0$ and that G_{1o}, G_{11}, H_{1o}, H_{11} vary with $\cos\theta$, $\sin\theta$ there.

We first solve for the axisymmetric solution. If we substitute for F_{oo}, G_{oo}, P_{oo} from (2.2.10) into (2.3.6) we can see that F_{1o}, G_{1o}, P_{1o} will have both steady terms and terms proportional to $\cos 2\tau$, $\sin 2\tau$. Suppose that we denote the steady parts of F_{1o}, G_{1o}, P_{1o} by f_s, g_s, p_s respectively, then we can use (2.2.10), (2.3.6) to show that

$$\left.\begin{aligned} &\left\{\frac{\partial^2}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial}{\partial\eta}\right\} f_s = p_s' + \frac{1}{4}\left\{f_{oo}\frac{\partial \tilde{f}_{oo}}{\partial \zeta} + \tilde{f}_{oo}\frac{\partial f_{oo}}{\partial \zeta} + g_{oo}\frac{\partial \tilde{f}_{oo}}{\partial \eta} + \tilde{g}_{oo}\frac{\partial f_{oo}}{\partial \eta}\right\} \\ &\frac{\partial}{\partial\eta}(\eta g_s) + \eta\frac{\partial f_s}{\partial \zeta} = 0 \end{aligned}\right\} \tag{2.3.10 a,b}$$

We shall obtain the solutions of (2.3.10)a,b in the Stokes and outer layers separately and match the solutions where the layers meet. We denote (g_s, f_s) in the Stokes and outer layers by (g_s^i, f_s^i) and (g_s^o, f_s^o) respectively. We can use (2.2.22), (2.2.28), (2.3.3), (2.3.4) to show that f_s^i, f_s^o satisfy the equations

$$\left.\begin{aligned} &\left\{\frac{\partial^2}{\partial\eta'^2} - \frac{1}{R - \sqrt{\frac{2}{\sigma}}\,\eta'}\frac{\partial}{\partial\eta'}\right\} f_s^i = \frac{2}{\sigma}\mathcal{L}(\zeta,\sigma) \\ &\qquad + \frac{R'}{\sigma^3 R^5}\Big\{\eta'[\cos\eta' + \sin\eta']e^{-\eta'} + 4\cos\eta' e^{-\eta'} - \sin\eta' e^{-\eta'} \\ &\qquad\qquad - 2e^{-2\eta'} + O(\sigma^{-1/2})\Big\} \\ &\left\{\frac{\partial^2}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial}{\partial\eta}\right\} f_s^o = \mathcal{L}(\zeta,\sigma) \end{aligned}\right\} \tag{2.3.11 a,b}$$

where $\tilde{\mathcal{P}}(\xi,\sigma)$ is as defined below and has no Stokes layer type of behaviour

$$\tilde{\mathcal{P}}(\xi,\sigma) = p_s' + \frac{1}{4\sigma^2}\left\{p_{00}'\tilde{p}_{00}'' + \tilde{p}_{00}'p_{00}''\right\} \tag{2.3.12}$$

and it follows from (2.3.8) that we require that

$$f_s^i = g_s^i = 0 \quad , \eta' = 0 \tag{2.3.13 a,b}$$

and from (2.2.7), (2.2.9) we see that

$$p_s(K) - p_s(0) = 0 \tag{2.3.14}$$

and we also require that g_s^o, f_s^o, p_s are regular at $\eta = 0$.
We can write the solutions of (2.3.11)a,b in the form

$$f_s^i = \frac{\tilde{\mathcal{P}}}{2\sigma}\left\{\eta'^2 - \sqrt{2\sigma}\,\eta' R\right\} + A(\xi,\sigma) + B(\xi,\sigma)\left\{\log R\sqrt{\frac{\sigma}{2}} - \frac{\eta'}{R}\sqrt{\frac{2}{\sigma}} + \frac{1}{2}\left(\frac{\eta'}{R}\sqrt{\frac{2}{\sigma}}\right)^2 - \cdots\right\}$$
$$+ \frac{R'}{2\sigma^3 R^5}\left\{\eta'\left[\cos\eta' - \sin\eta'\right]e^{-\eta'} - 6\sin\eta' e^{-\eta'} - \cos\eta' e^{-\eta'} - e^{-2\eta'} + O(\sigma^{-1/2})\right\}$$

$$f_s^o = \frac{\tilde{\mathcal{P}}}{4}\left\{\eta^2 - R^2\right\} + C(\xi,\sigma) \tag{2.3.15 a,b}$$

where A,B,C for the moment are unknown functions of ξ, σ and we have rejected solutions of (2.3.11)b which are singular at $\eta = 0$. If f_s^i, f_s^o are to match at the edge of the Stokes layer it follows that

$$A = C, \quad B = 0 \tag{2.3.16}$$

and using the boundary condition (2.3.13)a on f_s^i we find that

$$A = \frac{R'}{\sigma^3 R^5} + O(\sigma^{-7/2})$$

and so we can write f_s^i, f_s^o as follows

$$f_s^i = \frac{\tilde{\Phi}}{2\sigma}\left\{\eta'^2 - \sqrt{2\sigma}R\eta'\right\} + \frac{R'}{2\sigma^3 R^5}\left\{\eta'\left[\cos\eta' - \sin\eta'\right]e^{-\eta'} - 6\sin\eta' e^{-\eta'} - \cos\eta' e^{-\eta'} - e^{-2\eta'} + 2 + O(\sigma^{-7/2})\right\}$$

$$f_s^o = \frac{\tilde{\Phi}}{4}\left\{\eta^2 - R^2\right\} + \frac{R'}{\sigma^3 R^5} + O(\sigma^{-7/2}) \qquad \text{(2.3.17) a,b}$$

Suppose now that we denote g_s in the Stokes and outer layers by g_s^i and g_s^o respectively. In the Stokes layer the equations of continuity (2.3.10)b can be written in the form

$$-\frac{\partial}{\partial\eta'}\left\{\left(R - \sqrt{\tfrac{2}{\sigma}}\eta'\right)g_s^i\right\} + \left[R - \sqrt{\tfrac{2}{\sigma}}\eta'\right]\left[R'\frac{\partial f_s^i}{\partial\eta'} + \sqrt{\tfrac{2}{\sigma}}\frac{\partial f_s^i}{\partial\xi}\right] = 0 \qquad \text{(2.3.18)a}$$

and in the outer layer we have

$$\frac{\partial}{\partial\eta}\left(\eta g_s^o\right) + \eta\frac{\partial f_s^o}{\partial\xi} = 0 \qquad \text{(2.3.18)b}$$

If we substitute for f_s^i from (2.3.17)a into (2.3.18)a and integrate from $\eta' = 0$ to a point $\eta' = \eta'$ in the Stokes layer we obtain

$$\begin{aligned}\left(R - \sqrt{\tfrac{2}{\sigma}}\eta'\right)g_s^i = {} & -\frac{\tilde{\Phi}'}{16}\left\{\left(R - \sqrt{\tfrac{2}{\sigma}}\eta'\right)^4 - 2R^2\left(R - \sqrt{\tfrac{2}{\sigma}}\eta'\right)^2 + R^4\right\} \\ & + \left[\frac{\tilde{\Phi}R'R}{4} - \left(\frac{R'}{2\sigma^3R^5}\right)'\left[1 + O(\sigma^{-1/2})\right]\right]\left\{\left(R - \sqrt{\tfrac{2}{\sigma}}\eta'\right)^2 - R^2\right\} \\ & + \frac{1}{\sigma^{7/2}}\left(\frac{R'}{\sqrt{2}R^4}\right)'\left\{\eta'\sin\eta' e^{-\eta'} + 4\cos\eta' e^{-\eta'} + 3\sin\eta' e^{-\eta'} + \frac{e^{-2\eta'}}{2} - \frac{9}{2}\right\} \\ & + \frac{R'^2}{2\sigma^3R^5}\left[R - \sqrt{\tfrac{2}{\sigma}}\eta'\right]\left\{\eta'\left[\cos\eta' - \sin\eta'\right]e^{-\eta'} - 6\sin\eta' e^{-\eta'} \right. \\ & \qquad \left. - \cos\eta' e^{-\eta'} - e^{-2\eta'} + 2\right\} \\ & + O(\sigma^{-7/2})\end{aligned}$$

(2.3.19)a

where we have used the boundary condition (2.3.13)b to show that $\left(R - \sqrt{\tfrac{2}{\sigma}}\eta'\right)g_s^i$ is zero at $\eta' = 0$. If we substitute for f_s^o from (2.3.17)b into (2.3.18)b and integrate from $\eta = 0$

to a point $\eta = \eta$ still in the outer layer we obtain

$$\eta g_s^o = \frac{-\mathfrak{P}'}{16}\left\{\eta^4 - 2\eta^2 R^2\right\} + \left\{\frac{\mathfrak{P} R' R}{4} - \left(\frac{R'}{2\sigma^3 R^5}\right)'\left(1 + O(\sigma^{-1/2})\right)\right\}\eta^2 \qquad (2.3.19)\text{b}$$

where we have used the fact that g_s^o is regular at $\eta = 0$ to show that ηg_s^o is zero there. We now give an explanation why we have evaluated only some of the terms of order $\sigma^{-7/2}$ in (2.3.19)a,b. The terms of this order which are given explicitly are those which arise from the order σ^{-3} terms in f_s^i, f_s^o through the equation of continuity. However, terms of similar order will arise from the order $\sigma^{-7/2}$ terms in f_s^i, f_s^o again through the equation of continuity, and these are the terms which we have not given explicitly.
The essential 'physical' difference between the terms is that the latter, when combined with the order $\sigma^{-7/2}$ terms in f_s^i, give a resultant velocity parallel to the pipe wall whilst the other terms lead to a component of velocity normal to the pipe wall. We shall in fact see that in the evaluation of the stream function in the Stokes layer up to order $\sigma^{-7/2}$ the terms not shown explicitly are not required. If we use the condition that (2.3.19)a,b must match at the edge of the Stokes layer we obtain

$$0 = \frac{-\mathfrak{P}' R^4}{16} - \frac{\mathfrak{P} R^3 R'}{4} + \left(\frac{R'}{2\sigma^3 R^5}\right)' R^2 + \frac{R'^2}{\sigma^3 R^4} - \frac{9}{2\sqrt{2}\,\sigma^{7/2}}\left(\frac{R'}{R^4}\right)' + O(\sigma^{-7/2}) \qquad (2.3.20)$$

which we integrate once to give

$$Q = \frac{\mathfrak{P} R^4}{16} - \left(\frac{R'}{2\sigma^3 R^3}\right)\left[1 - \frac{9}{\sqrt{2\sigma}\,R}\right] + O(\sigma^{-7/2}) \qquad (2.3.21)$$

where Q is an arbitrary constant which we can obtain by substituting for $\mathfrak{P}$ from (2.3.12) into (2.3.21) and replacing

p'_{00} by its asymptotic form given by (2.3.3). If we then integrate from $\zeta = 0$ to $\zeta = K$ and use (2.3.14) we find that

$$Q = \left\{\frac{R^{-4}(K) - R^{-4}(0)}{64\sigma^2}\right\} \Big/ \int_0^K \frac{d\zeta}{R^4} + O(\sigma^{-5/2}) \qquad (2.3.22)$$

Thus if the ends of the pipe have the same mean radius the term of order σ^{-2} in (2.3.22) vanishes. In fact it can be shown that all the higher order terms also vanish. For Q non-zero we can see that there is no need to distinguish between the Stokes and outer layers as far as the dominant term in the velocity field is concerned. If we take Φ, Q as given by (2.3.21), (2.3.22) we can use (2.3.17)a,b, (2.3.19)a,b to show that we can write f_s, g_s as follows

$$\left.\begin{aligned} f_s &= A_0\left[\frac{\eta^2 - R^2}{\sigma^2 R^4} + O(\sigma^{-5/2})\right] \\ g_s &= A_0\left[\frac{R'\eta(\eta^2 - R^2)}{R^5} + O(\sigma^{-5/2})\right] \end{aligned}\right\} \qquad (2.3.23)\ \text{a,b}$$

where $$A_0 = \left[R^{-4}(K) - R^{-4}(0)\right] \Big/ 4\int_0^K \frac{d\zeta}{R^4} \qquad (2.3.23)\text{c}$$

and if we introduce a stream function ψ_s by

$$\eta f_s = \frac{\partial \psi_s}{\partial \eta}, \quad \eta g_s = -\frac{\partial \psi_s}{\partial \zeta} \qquad (2.3.24)\ \text{a,b}$$

then we can show that

$$\psi_s = \frac{A_0}{4}\left\{\frac{\eta^4 - 2\eta^2 R^2}{\sigma^2 R^4}\right\} + O(\sigma^{-5/2}) \qquad (2.3.25)$$

and this represents a steady flow which is always directed towards the widest end of the pipe.

When Q is zero we can use (2.3.20), (2.3.21) to show that f_s^i, g_s^i given by (2.3.17)a, (2.3.19)a may be written in the form

$$f_s^i = \frac{R'}{2\sigma^3 R^5}\left\{\eta'(\cos\eta' - \sin\eta')e^{-\eta'} - 6\sin\eta' e^{-\eta'} - \cos\eta' e^{-\eta'} - e^{-2\eta'} + 2 + O(\sigma^{-1/2})\right\}$$

$$g_s^i = \frac{R'}{2\sigma^3 R^5}\left\{\eta'(\cos\eta' - \sin\eta')e^{-\eta'} - 6\sin\eta' e^{-\eta'} - \cos\eta' e^{-\eta'} - e^{-2\eta'} + 2 + O(\sigma^{-1/2})\right\}$$
$$+ \frac{1}{R}\left(\frac{R'}{\sqrt{2}\sigma^{7/2}R^4}\right)\left\{\eta'\sin\eta' e^{-\eta'} + 4\cos\eta' e^{-\eta'} + 3\sin\eta' e^{-\eta'} + \frac{e^{-2\eta'}}{2} - \frac{9}{2}\right\}$$
$$+ O(\sigma^{-7/2}) \qquad (2.3.26)\ a,b$$

We can see from (2.3.26)a,b that the order σ^{-3} term in (2.3.26)b is just R' times the order σ^{-3} term in (2.3.26)a. Thus at any point in the Stokes layer the dominant velocity is parallel to the wall of the pipe. We can also show that the order $\sigma^{-7/2}$ terms not shown explicitly in (2.3.26)a,b similarly represent a velocity parallel to the pipe wall. We define a stream function ψ_s^i in the Stokes layer by

$$f_s^i = \frac{-\sqrt{\frac{\sigma}{2}}}{R - \sqrt{\frac{2}{\sigma}}\eta'}\frac{\partial\psi_s^i}{\partial\eta'}, \quad g_s^i = \frac{-1}{R - \sqrt{\frac{2}{\sigma}}\eta'}\frac{\partial\psi_s^i}{\partial\zeta} + R' f_s^i \qquad (2.3.27)\ a,b$$

and we can then show that ψ_s^i is given by

$$\psi_s^i = \frac{-R'}{2\sqrt{2}\sigma^{7/2}R^4}\left\{2\eta'\sin\eta' e^{-\eta'} + 8\cos\eta' e^{-\eta'} + 6\sin\eta' e^{-\eta'} + e^{-2\eta'} + 4\eta' - 9 + O(\sigma^{-1/2})\right\} \qquad (2.3.28)$$

and with Q equal to zero we can show that the stream function ψ_s^o in the outer layer is given by

$$\psi_s^o = \frac{R'}{2\sigma^3 R^7}\left\{\eta^4 - 2\eta^2 R^2\right\}\left\{1 + O(\sigma^{-1/2})\right\} - \frac{9R'}{2\sqrt{2}\sigma^{7/2}R^8}\left\{\eta^4 - 2\eta^2 R^2\right\} + O(\sigma^{-4}) \qquad (2.3.29)$$

Thus ψ_s^o has only been evaluated explicitly up to order σ^{-3}. However this expression matches up to order $\sigma^{-7/2}$ with (2.3.28) at the edge of the Stokes layer, the term of order $\sigma^{-7/2}$ not shown explicitly giving a contribution of order σ^{-4} there. The flux through the pipe associated with this steady streaming with Q zero is clearly zero (at least up to order $\sigma^{-7/2}$) since

$$\psi_s^o\Big|_{\eta=0} - \psi_s^i\Big|_{\eta'=0} = O(\sigma^{-4}) \qquad (2.3.30)$$

We now determine the order ϵ correction to the axisymmetric solution. The order ϵR_M term in the expansion of the velocity was found to be determined by (2.3.7), (2.3.9). Suppose that we denote the steady parts of G_{11}, H_{11}, F_{11}, P_{11} by G_s, H_s, F_s, P_s respectively, then we can use (2.2.10), (2.2.19) to rewrite (2.3.7), (2.3.9) in the form

$$\nabla^2 F_s = P_s' + \frac{1}{4}\left\{ f_{00}\frac{\partial \tilde{f}_{01}}{\partial \zeta} + f_{01}\frac{\partial \tilde{f}_{00}}{\partial \zeta} + g_{00}\frac{\partial \tilde{f}_{01}}{\partial \eta} + g_{01}\frac{\partial \tilde{f}_{00}}{\partial \eta} + g_{00}\frac{\partial \tilde{f}_{01}}{\partial \eta} + g_{01}\frac{\partial \tilde{f}_{00}}{\partial \eta} + \frac{h_{00}}{\eta}\frac{\partial \tilde{f}_{01}}{\partial \theta} + \frac{h_{01}}{\eta}\frac{\partial \tilde{f}_{00}}{\partial \theta} \right\} + \text{COMPLEX CONJUGATE}$$

$$\frac{\partial}{\partial \eta}(\eta G_s) + \frac{\partial H_s}{\partial \theta} + \eta \frac{\partial F_s}{\partial \zeta} = 0 \qquad \text{(2.3.31) a,b}$$

$$f_s + \epsilon F_s = g_s + \epsilon G_s = \epsilon H_s = O(\epsilon^2), \quad \eta = R + \epsilon S \cos M\theta \qquad \text{(2.3.32) a,b,c}$$

where f_s, g_s are now assumed to be known. From (2.2.7), (2.2.9) we can show that

$$P_s(K) - P_s(0) = 0 \qquad \text{(2.3.33)}$$

We also require that P_s, F_s are independent of θ at $\eta = 0$, and that G_s, H_s vary with $\cos\theta$, $\sin\theta$ there. If we eliminate p^+ from (2.2.5)a,b and substitute for g,h,f from (2.2.9)a,b,c into the resulting equation and equate terms of order $\epsilon R_M \delta$ we obtain

$$\left\{ \frac{\partial^2}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial}{\partial \eta} + \frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{\eta^2} \right\}\left[\frac{\partial G_s}{\partial \theta} - \frac{\partial}{\partial \eta}(\eta H_s) \right] = \gamma(\eta, \zeta, \sigma) \qquad \text{(2.3.34)}$$

where
$$\gamma(\eta,\zeta,\sigma) = \frac{1}{4}\left[\frac{\partial}{\partial \theta}\left\{ g_{00}\frac{\partial \tilde{g}_{01}}{\partial \eta} + g_{01}\frac{\partial \tilde{g}_{00}}{\partial \eta} + f_{00}\frac{\partial \tilde{g}_{01}}{\partial \zeta} + f_{01}\frac{\partial \tilde{g}_{00}}{\partial \zeta} + \frac{h_{00}}{\eta}\frac{\partial \tilde{g}_{01}}{\partial \theta} + \frac{h_{01}}{\eta}\frac{\partial \tilde{g}_{00}}{\partial \theta} \right\} - \frac{\partial}{\partial \eta}\left\{ \eta g_{00}\frac{\partial \tilde{h}_{01}}{\partial \eta} + \eta g_{01}\frac{\partial \tilde{h}_{00}}{\partial \eta} + \eta f_{00}\frac{\partial \tilde{h}_{01}}{\partial \zeta} + \eta f_{01}\frac{\partial \tilde{h}_{00}}{\partial \zeta} + h_{00}\frac{\partial \tilde{h}_{01}}{\partial \theta} + h_{01}\frac{\partial \tilde{h}_{00}}{\partial \theta} + h_{00}\tilde{g}_{01} + h_{01}\tilde{g}_{00} \right\} \right]$$

$$+\frac{1}{4}\Big[\frac{\partial}{\partial\theta}\Big\{\tilde{g}_{00}\frac{\partial g_{01}}{\partial\eta}+\tilde{g}_{01}\frac{\partial g_{00}}{\partial\eta}+\tilde{f}_{00}\frac{\partial g_{01}}{\partial\zeta}+\tilde{f}_{01}\frac{\partial g_{00}}{\partial\zeta}+\frac{\tilde{h}_{00}}{\eta}\frac{\partial g_{01}}{\partial\theta}+\frac{\tilde{h}_{01}}{\eta}\frac{\partial g_{00}}{\partial\theta}\Big\}$$
$$-\frac{\partial}{\partial\eta}\Big\{\eta\tilde{g}_{00}\frac{\partial h_{01}}{\partial\eta}+\eta\tilde{g}_{01}\frac{\partial h_{00}}{\partial\eta}+\eta\tilde{f}_{00}\frac{\partial h_{01}}{\partial\zeta}+\eta\tilde{f}_{01}\frac{\partial h_{00}}{\partial\zeta}+\tilde{h}_{00}\frac{\partial h_{01}}{\partial\theta}+$$
$$\tilde{h}_{01}\frac{\partial h_{00}}{\partial\theta}+\tilde{h}_{00}g_{01}+\tilde{h}_{01}g_{00}\Big\}\Big] \qquad (2.3.35)$$

Suppose that we denote (G_s, H_s, F_s) by (G_s^i, H_s^i, F_s^i) and (G_s^o, H_s^o, F_s^o) in the Stokes and outer layers respectively. We can use (2.2.22), (2.2.28), (2.3.3), (2.3.4), (2.3.31), (2.3.34) to show that these functions are determined by

$$\Big\{\frac{\partial^2}{\partial\eta'^2}-\frac{\sqrt{2\sigma}}{R-\sqrt{\frac{2}{\sigma}}\eta'}\frac{\partial}{\partial\eta'}+\frac{2}{\sigma(R-\sqrt{\frac{2}{\sigma}}\eta')^2}\frac{\partial^2}{\partial\theta^2}\Big\}F_s^i=\frac{2P_s'}{\sigma}+\frac{\sqrt{2}SR'}{\sigma^{5/2}R^5}\Big\{2e^{-2\eta'}-2\cos\eta' e^{-\eta'}-\sin\eta' e^{-\eta'}$$
$$-\eta'\sin\eta' e^{-\eta'}+O(\sigma^{-3})\Big\}\cos 2\theta$$

$$\nabla^2 F_s^o = P_s'$$

$$-\frac{\partial}{\partial\eta'}\Big[(R-\sqrt{\tfrac{2}{\sigma}}\eta')G_s^i\Big]+\sqrt{\frac{2}{\sigma}}\frac{\partial H_s^i}{\partial\theta}+(R-\sqrt{\tfrac{2}{\sigma}}\eta')\Big[\sqrt{\frac{2}{\sigma}}\frac{\partial F_s^i}{\partial\zeta}+R'\frac{\partial F_s^i}{\partial\eta'}\Big]=0$$

$$\frac{\partial}{\partial\eta}(\eta G_s^o)+\frac{\partial H_s^o}{\partial\theta}+\eta\frac{\partial F_s^o}{\partial\zeta}=0$$

$$\Big\{\frac{\partial^2}{\partial\eta'^2}+\frac{\sqrt{\frac{2}{\sigma}}}{R-\sqrt{\frac{2}{\sigma}}\eta'}\frac{\partial}{\partial\eta'}+\frac{2}{\sigma(R-\sqrt{\frac{2}{\sigma}}\eta')^2}\frac{\partial^2}{\partial\theta^2}+\frac{2}{\sigma(R-\sqrt{\frac{2}{\sigma}}\eta')^2}\Big\}\Big[\frac{\partial G_s^i}{\partial\theta}+\frac{\partial}{\partial\eta'}(R-\sqrt{\tfrac{2}{\sigma}}\eta')H_s^i\Big]$$
$$=O(\sigma^{-5/2})$$

$$\Big\{\frac{\partial^2}{\partial\eta^2}-\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{\eta^2}\Big\}\Big[\frac{\partial G_s^o}{\partial\theta}-\frac{\partial}{\partial\eta}(\eta H_s^o)\Big]$$
$$=0$$

(2.3.36)
a,b,c,d,e,f

The non-linear terms on the right hand sides of (2.3.31)a, (2.3.34) are all exponentially small away from the Stokes layer.

The boundary conditions (2.3.32) become

$$f_s^i + \epsilon f_s^i = g_s^i + \epsilon G_s^i = \epsilon H_s^i = O(\epsilon^2), \quad \eta' = -\sqrt{\frac{\sigma}{2}}\,\epsilon S \cos M\theta \qquad \text{(2.3.37) a,b,c}$$

We can easily show that the solutions of (2.3.36)a,b which match at the edge of the Stokes layer and satisfy the above boundary condition are

$$F_s^i = \frac{P_s'}{2\sigma}\left\{\eta'^2 - \sqrt{2\sigma}\,R\eta'\right\} + \frac{SR'}{\sigma^{5/2}\sqrt{2}}\left\{e^{-2\eta'} + 3\sin\eta' e^{-\eta'} - 2\cos\eta' e^{-\eta'} - \eta'\cos\eta' e^{-\eta'}\right\}\cos M\theta$$
$$- \frac{\tilde{\mathcal{L}} RS}{2}\left\{1 - \sqrt{\frac{2}{\sigma}}\frac{\eta'}{R}\right\}^M \cos M\theta + O(\sigma^{-3})$$

$$F_s^o = P_s'\left\{\eta^2 - R^2\right\} - \frac{\tilde{\mathcal{L}} RS}{2}\left(\frac{\eta}{R}\right)^M \cos M\theta + O(\sigma^{-3}) \qquad \text{(2.3.38)a,b}$$

where $\tilde{\mathcal{L}}$ is given by (2.3.21), and solutions of (2.3.36)b which are singular at $\eta=0$, have been rejected. Similarly the solutions of (2.3.36)e,f which match at the edge of the Stokes layer are

$$\frac{\partial}{\partial\eta'}\left[\left(R\sqrt{\frac{\sigma}{2}} - \eta'\right)H_s^i\right] + \frac{\partial G_s^i}{\partial\theta} = -B(\zeta,\sigma)\left(1 - \sqrt{\frac{2}{\sigma}}\frac{\eta'}{R}\right)^{M+1}\sin M\theta + O(\sigma^{-5/2})$$

$$\frac{\partial}{\partial\eta}\left(\eta H_s^o\right) - \frac{\partial G_s^o}{\partial\theta} = B(\zeta,\sigma)\left(\frac{\eta}{R}\right)^{M+1}\sin M\theta \qquad \text{(2.3.39) a,b}$$

where $B(\zeta, \sigma)$ is for the moment an arbitrary function of ζ, σ and solutions which are singular at $\eta=0$ have been rejected, since these solutions would lead to G_s^o, H_s^o being singular at $\eta=0$. We have assumed for the sake of convenience that only terms proportional to $\sin M\theta$ need be retained in (2.3.39)a,b. (The vanishing of the other θ dependent solutions would otherwise be found when the boundary conditions were applied). If we substitute for F_s^i from (2.3.38)a into the equation of continuity (2.3.36)c, and eliminate G_s^i from the resulting equation and (2.3.39)a we obtain

$$\frac{\partial}{\partial\eta'}\left\{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)\frac{\partial}{\partial\eta'}\left[\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)H_s^i\right]+\frac{2}{\sigma}\frac{\partial^2 H_s^i}{\partial\theta^2}\right\}=\left[-M\left(\frac{S\Phi}{R^{M-1}}\right)'+\frac{2(M+2)B}{R^{M+1}}\right]\frac{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^{M+1}\sin M\theta}{\sigma}+O(\sigma^{-7/2})$$

and we can show that the corresponding equation in the outer layer is

$$\frac{\partial}{\partial\eta}\left\{\eta\frac{\partial}{\partial\eta}\left[\eta H_s^o\right]\right\}+\frac{\partial^2 H_s^o}{\partial\theta^2}=\left[-\frac{M}{\sigma}\left(\frac{S\Phi}{R^{M-1}}\right)'+\frac{(M+2)B}{R^{M+1}}\right]\eta^{M+1}\sin M\theta+O(\sigma^{-3})$$

and the solutions of these equations which match at the edge of the Stokes layer are

$$H_s^i=\frac{\left[-M\left(\frac{S\Phi}{R^{M-1}}\right)'+\frac{2(M+2)B}{R^{M+1}}\right]}{4(M+1)}\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^{M+1}\sin M\theta+C\left(1-\sqrt{\tfrac{2}{\sigma R}}\eta'\right)^{M-1}\sin M\theta+O(\sigma^{-3})$$

$$H_s^o=\frac{\left[-M\left(\frac{S\Phi}{R^{M-1}}\right)'+2(M+2)B\right]}{4(M+1)}\eta^{M+1}\sin M\theta+C\left(\eta/R\right)^{M-1}\sin M\theta+O(\sigma^{-3})$$

(2.3.40) a,b

where C is for the moment an arbitrary function of ζ, σ and solutions which are singular at $\eta=0$ have been rejected. We have again assumed for convenience that any θ dependence of H_s^i, H_s^o is with $\sin M\theta$.

Having determined F_s^i, H_s^i, we can substitute for these functions into (2.3.36)c and integrate from $\eta'=-\sqrt{\frac{\sigma}{2}}\epsilon S\cos M\theta$ to a point $\eta'=\eta'$ still in the Stokes layer. We thus obtain

$$\left[\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)G_s^i\right]_{-\sqrt{\frac{\sigma}{2}}\epsilon S\cos M\theta}^{\eta'}$$

$$=\frac{-P_s''}{16}\left\{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^4-2R^2\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^2+R^4\right\}+\frac{P_s''R'R}{4}\left\{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^2-R^2\right\}$$

$$+\left\{M+2\left(\frac{S\Phi}{R^{M-1}}\right)'-\frac{2MB}{R^{M+1}}\right\}\left\{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^{M+2}-R^{M+2}\right\}\cos M\theta$$

$$+\frac{SR'^2}{\sqrt{2}R^5\sigma^{5/2}}\left\{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)\left[e^{-2\eta'}+3\sin\eta' e^{-\eta'}-2\cos\eta' e^{-\eta'}-\eta'\cos\eta' e^{-\eta'}\right]+2R\right\}\cos M\theta$$

$$+O(\sigma^{-3},\epsilon)$$

However, using (2.3.37), we have that

$$\left[\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)\left(g_s^i + \epsilon G_s^i\right)\right]_{\eta' = -\sqrt{\frac{\sigma}{2}}\epsilon S\cos M\theta} = O(\epsilon^2)$$

and so using (2.3.19)a, (2.3.20) we obtain

$$\begin{aligned}\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)G_s^i = {} & \frac{-P_s''}{16}\left\{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^4 - 2R^2\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^2 + R^4\right\} + \frac{P_s'R'R}{4}\left\{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^2 - R^2\right\} \\ & + \left\{\frac{M+2\left(\frac{S\hat{\Phi}}{R^{M-1}}\right)' - \frac{2MB}{R^{M+1}}}{8(M+1)}\right\}\left\{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^{M+2} - R^{M+2}\right\}\cos M\theta \\ & - \frac{C}{R^{M-1}}\left\{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)^M - R^M\right\}\cos M\theta - \tilde{\Phi}\frac{R^2SR'}{2}\cos M\theta \\ & + \frac{\delta R'^2}{\sqrt{2}R^6\sigma^{5/2}}\left\{\left(R-\sqrt{\tfrac{2}{\sigma}}\eta'\right)\left[e^{-2\eta'} + 3\sin\eta' e^{-\eta'} - 2\cos\eta' e^{-\eta'} - \eta'\cos\eta' e^{-\eta'}\right]\right\}\cos M\theta \\ & + O(\epsilon, \sigma^{-3})\end{aligned}$$

(2.3.41)a

and if we substitute for F_s^o, H_s^o from (2.3.38)b, (2.3.40)b into (2.3.36)c, integrate from $\eta = 0$ to $\eta = \eta$ still in the outer layer, and use the fact that G_s^o is regular at $\eta = 0$ we obtain

$$\begin{aligned}\eta G_s^o = {} & \frac{-P_s''}{16}\left\{\eta^4 - 2\eta^2R^2\right\} + \frac{P_s'R'R\eta^2}{4} + \left\{\frac{M+2\left(\frac{S\Phi}{R^{M-1}}\right)' - \frac{2MB}{R^{M+1}}}{8(M+1)}\right\}\eta^{M+2}\cos M\theta \\ & - \frac{C}{R^{M-1}}\eta^M\cos M\theta + O(\sigma^{-3})\end{aligned}$$

(2.3.41)b

and if (2.3.41)a,b are to match at the edge of the Stokes layer we require that

$$O(\sigma^{-3}) = \frac{P_s''R^4}{4} + \frac{P_s'R'R^3}{4}$$

$$\text{and} \quad O(\sigma^{-3}) = \tilde{\Phi}\frac{R^2SR'}{2} + RC + \left\{\frac{2MBR - (M+2)\left(\frac{S\hat{\Phi}}{R^{M-1}}\right)'R^{M+1}}{8(M+1)}\right\}$$

(2.3.42) a,b

We can integrate (2.3.42)a once to give

$$P_s' = \frac{Z}{R^4} + O(\sigma^{-3})$$

where Z is a constant, which after integrating from $\xi = 0$ to

ζ =K and using (2.3.33), is found to be zero. If we put $\eta' = -\sqrt{\frac{\sigma}{2}}\,\epsilon S \cos M\theta$ in (2.3.40)a and use (2.3.37), then equate terms proportional to $\epsilon \sin M\theta$ we obtain an equation for B,C which we solve together with (2.3.42)b to give

$$\frac{B}{R^{M+1}} = -\frac{1}{2}\left(\frac{S\zeta}{R^{M-1}}\right)' - \frac{(M+1)SR'}{R^{M}} + O(\sigma^{-3})$$

$$\frac{C}{R^{M-1}} = \frac{(M+2)\tilde{\zeta} SR'}{4R^{M-2}} + \frac{R^2}{4}\left(\frac{S\tilde{\zeta}}{R^{M-1}}\right)' + O(\sigma^{-3})$$

and then we can show that G_s^i, H_s^i, F_s^i will be given by

$$G_s^i = \Big\{4Q\Big[(\eta/R)^{M+1} - (\eta/R)^{M-1}\Big]\Big[\frac{S'}{R^2} - \frac{SR'}{R^3}\Big] - 8Q(\eta/R)^{M+1}\frac{SR'}{R^3}$$
$$+ \frac{SR'^2}{\sqrt{2}\,\sigma^{5/2}R^5}\Big[e^{-2\eta'} + 3\sin\eta' e^{-\eta'} - 2\cos\eta' e^{-\eta'} - \eta'\cos\eta' e^{-\eta'}\Big]\Big\}\cos M\theta + O(\sigma^{-3})$$

$$H_s^i = \Big\{4Q\Big[(\eta/R)^{M+1} - (\eta/R)^{M-1}\Big]\Big[\frac{SR'}{R^3} - \frac{S'R}{R^2}\Big]\Big\}\sin M\theta + O(\sigma^{-3})$$

$$F_s^i = \Big\{-\frac{8Q}{R^3}(\eta/R)^{M} + \frac{SR'}{\sqrt{2}\,\sigma^{5/2}R^5}\Big[e^{-2\eta'} + 3\sin\eta' e^{-2\eta'} - 2\cos\eta' e^{-\eta'} - \eta'\cos\eta' e^{-\eta'}\Big]\Big\}\cos M\theta$$
$$+ O(\sigma^{-3})$$

(2.3.43)
a,b,c

and G_s^o, H_s^o, F_s^o will be given by

$$G_s^o = \Big\{4Q\Big[(\eta/R)^{M+1} - (\eta/R)^{M-1}\Big]\Big[\frac{S'}{R^2} - \frac{SR'}{R^3}\Big] - 8Q(\eta/R)^{M+1}\frac{SR'}{R^3}\Big\}\cos M\theta + O(\sigma^{-3})$$

$$H_s^o = \Big\{4Q\Big[(\eta/R)^{M+1} - (\eta/R)^{M-1}\Big]\Big[\frac{SR'}{R^3} - \frac{S'}{R^2}\Big]\Big\}\sin M\theta + O(\sigma^{-3})$$

$$F_s^o = \Big\{-\frac{8Q}{R^3}(\eta/R)^{M}\Big\}\cos M\theta + O(\sigma^{-3})$$

(2.3.44)
a,b,c

where we have replaced $\tilde{\zeta}$ by (2.3.21). We have deliberately written the terms in (2.3.43)a,b,c which are porportional to Q in terms of η since they do not have a Stokes layer type of behaviour. Moreover, if the pipe is such that R(K) and R(O) are not equal, we can see from (2.3.22) that Q is of order σ^{-2} and so the dominant steady streaming of order ϵ is given by the terms proportional to $\mathcal{Q}$. In this case there is no need to distinguish between the two layers and we write

$$G_s = \frac{A_o}{4\sigma^2}\left\{\left[\left(\eta/R\right)^{M+1} - \left(\eta/R\right)^{M-1}\right]\left[\frac{S'}{R^2} - \frac{SR'}{R^3}\right] - 2\left(\eta/R\right)^{M+1}\frac{SR'}{R^3}\right\}\cos M\theta + O(\sigma^{-\frac{5}{2}})$$

$$H_s = \frac{A_o}{4\sigma^2}\left\{\left[\left(\eta/R\right)^{M+1} - \left(\eta/R\right)^{M-1}\right]\left[\frac{SR'}{R^3} - \frac{S'}{R^2}\right]\right\}\sin M\theta + O(\sigma^{-5/2})$$

$$F_s = \frac{-2A_oS}{\sigma^2 R^3}\left(\eta/R\right)^{M}\cos M\theta + O(\sigma^{-5/2})$$

(2.3.45) a,b,c

where A_o is given by (2.3.23)c. However, if R(K) and R(0) are equal, then $\mathcal{Q}$ is zero and we can write (2.3.43)a,b,c, (2.3.44)a,b,c as follows

$$G_s^i = \frac{SR'^2}{\sqrt{2}\,\sigma^{5/2}R^5}\left\{e^{-2\eta'} + 3\sin\eta' e^{-\eta'} - 2\cos\eta' e^{-\eta'} - \eta'\cos\eta' e^{-\eta'}\right\}\cos M\theta + O(\sigma^{-3})$$

$$F_s^i = \frac{SR'}{\sqrt{2}\,\sigma^{5/2}R^5}\left\{e^{-2\eta'} + 3\sin\eta' e^{-\eta'} - 2\cos\eta' e^{-\eta'} - \eta'\cos\eta' e^{-\eta'}\right\}\cos M\theta + O(\sigma^{-3})$$

$$H_s^i = G_s^o = F_s^o = H_s^o = O(\sigma^{-3})$$

and so in this case the dominant steady streaming of order ϵ is confined to the Stokes layer and has no swirling component of velocity. Similarly, if we choose $S = \gamma R$, where γ is a constant, then the dominant steady streaming of order ϵ given by (2.3.45) will also have no swirling component of velocity. This particular choice of S, R corresponds to the pipe having a uniform cross-sectional shape.

2.4 Calculation of the steady streaming for small σ

When σ is small the Stokes layer completely fills the pipe and there is no need to split the flow field into separate regions. We again solve for the order R_M axisymmetric steady streaming velocity, $(g_s, 0, f_s)$, and then for the order ϵR_M non-axisymmetric correction to this solution. We first consider the form of the Stokes flow for small σ. If we expand the Bessel functions appearing in (2.2.25), (2.2.26) using the series form for Bessel functions of small argument we obtain

$$p_{00}' = \frac{-i\alpha}{R^4}\left\{ \gamma_0 + \frac{\gamma_0 i\sigma R^2}{6} + \gamma_1 i\sigma + O(\sigma^2) \right\}$$

where $\gamma_0 = \left\{ \int_0^K \frac{d\xi}{R^4} \right\}^{-1}$, $\gamma_1 = -\frac{\gamma_0^2}{6}\int_0^K \frac{d\xi}{R^2}$

and for convenience we now choose $\alpha = \gamma_0^{-1}$ in which case we can write

$$p_{00}' = \frac{-i}{R^4}\left\{ 1 + \frac{i\sigma R^2}{6} + \frac{\gamma_1}{\gamma_0} i\sigma + O(\sigma^2) \right\} \tag{2.4.1}$$

We can then use (2.2.22), (2.2.28) to show that

$$f_{00} \sim -i\left\{ \frac{\eta^2 - R^2}{4R^4} \right\} + O(\sigma)$$

$$g_{00} \sim -i\eta\left\{ \frac{\eta^2 - R^2}{4R^5} \right\} R' + O(\sigma)$$

$$f_{01} \sim \frac{iS}{2R^3}\left(\eta/R\right)^M \cos M\theta + O(\sigma)$$

$$g_{01} \sim \frac{iSR'}{2R^3}\left(\eta/R\right)^{M+1}\cos M\theta \pm \frac{i}{4}\left\{ \left(\eta/R\right)^{M+1} - \left(\eta/R\right)^{M-1} \right\}\left\{ \frac{SR'}{R^3} - \frac{S'}{R^2} \right\}\cos M\theta + O(\sigma)$$

$$h_{01} \sim \frac{i}{4}\left\{ \left(\eta/R\right)^{M+1} - \left(\eta/R\right)^{M-1} \right\}\left\{ \frac{S'}{R^2} - \frac{SR'}{R^3} \right\}\sin M\theta + O(\sigma)$$

Using the notation of §2.3 we can see that g_s, f_s, p_s are determined by the equations (2.3.10) together with the boundary conditions

$$\left.\begin{aligned} &f_s = g_s = 0, \quad \eta = R \\ &p_s(K) - p_s(0) = 0 \end{aligned}\right\} \tag{2.4.2 a,b,c}$$

We also require that p_s, f_s, g_s are regular at $\eta = 0$. If we now let σ tend to zero in (2.3.10)a,b and use (2.2.22)a, (2.2.28)a, (2.4.1) and the series expansions of Bessel functions, we obtain

$$\left.\begin{aligned} &\left\{\frac{\partial^2}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial}{\partial\eta}\right\} f_s = p_s' + \frac{R'}{16R^9}\left\{-\eta^4 + 2\eta^2R^2 - R^4\right\} + O(\sigma) \\ &\frac{\partial}{\partial\eta}(\eta g_s) + \eta\frac{\partial f_s}{\partial s} = 0 \end{aligned}\right\} \qquad \text{(2.4.3) a,b}$$

The solution of (2.4.3)a which satisfies (2.4.2)a is

$$f_s = p_s'\left\{\frac{\eta^2 - R^2}{4}\right\} - \frac{R'}{1152R^9}\left\{2\eta^6 - 9\eta^4R^2 + 18R^4\eta^2 - 11R^6\right\} + O(\sigma) \qquad \text{(2.4.4)a}$$

We can then substitute for f_s from (2.4.4)a into (2.4.3)b and integrate from $\eta = 0$ to $\eta = \eta$ to give

$$\begin{aligned} \eta g_s = &-\frac{p_s''}{16}\left\{\eta^4 - 2\eta^2R^2\right\} + \frac{p_s'}{4}R'R\eta^2 \\ &+ \left(\frac{R'}{4608R^9}\right)'\left\{\eta^8 - 6\eta^6R^2 + 18\eta^4R^4 - 22\eta^2R^6\right\} \\ &+ \frac{R'^2}{384R^8}\left\{-\eta^6 + 6\eta^4R^2 - 11\eta^2R^4\right\} + O(\sigma) \end{aligned} \qquad \text{(2.4.4)b}$$

where we have used the fact that g_s is regular at $\eta = 0$ to show that ηg_s is zero there. If we put $\eta = R$ in (2.4.4)b and use (2.4.2)b we obtain the Reynolds equation for the pressure which we integrate once to give

$$\frac{16C}{R^4} = p_s' - \frac{R'}{32R^5} + O(\sigma) \qquad \text{(2.4.4)c}$$

where C is a constant which after integrating both sides of the above equation from $\zeta = 0$ to $\zeta = K$ and using (2.4.2)c is found to be given by

$$C = 2^{-11}\left\{R^{-4}(K) - R^{-4}(0)\right\} \Big/ \int_0^K \frac{d\zeta}{R^4} + O(\sigma) \qquad \text{(2.4.5)}$$

and C is zero if the ends of the pipe have the same mean radius. (It can be shown that all the higher order terms in (2.4.5) also vanish in this case). If we introduce the stream function ψ_s defined by (2.3.24) we can show that

$$\begin{aligned}\psi_s = \; & C\{\eta^4 - 2\eta^2 R^2\}/4R^4 \\ & \frac{-R'}{4608R}\left\{\left(\eta/R\right)^8 - 6\left(\eta/R\right)^6 + 9\left(\eta/R\right)^4 - 4\left(\eta/R\right)^2\right\}\end{aligned} \qquad (2.4.6)$$

Thus we see that, as in §2.3, there is no net flux through the pipe if the ends have the same radius.

We now determine the order ϵR_M correction to the axisymmetric solution. If we again denote the steady velocity of this order by (G_s, H_s, F_s) and the corresponding pressure by P_s then these functions are determined by (2.3.31), (2.3.32), (2.3.33), (2.3.34). If we now let σ tend to zero in these equations we can show that

$$\left.\begin{aligned} & \nabla^2 F_s = P_s' + \frac{1}{4}\left\{\left(\eta/R\right)^{M+2} - \left(\eta/R\right)^{M}\right\}\frac{SR'}{R^6}\cos M\theta + O(\sigma) \\ & \frac{\partial}{\partial\eta}(\eta G_s) + \frac{\partial H_s}{\partial\theta} + \eta\frac{\partial F_s}{\partial s} = 0 \\ & \left\{\frac{\partial^2}{\partial\eta^2} - \frac{1}{\eta}\frac{\partial}{\partial\eta} + \frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\eta^2}\right\}\left[\frac{\partial}{\partial\eta}(\eta H_s) - \frac{\partial G_s}{\partial\theta}\right] \\ & \qquad = \frac{1}{8}\mu(\zeta)\left\{\left(\eta/R\right)^{M+3} - \left(\eta/R\right)^{M+1}\right\}\sin M\theta + O(\sigma) \end{aligned}\right\} \qquad \begin{matrix}(2.4.7)\\ a,b,c\end{matrix}$$

where $$\mu(\zeta) = \frac{2(M-1)SR'^2}{R^6} - \frac{S''}{R^4} - \frac{(M-1)SR''}{R^5} + \frac{2S'R'}{R^5} \qquad (2.4.8)$$

and the boundary conditions are given by (2.3.32), (2.3.3) where f_s, g_s are taken as given by (2.4.4)a,b. The method of solution now follows the methods used in §2.2, 2.3 and so we briefly explain the various steps used and give the final results.

We first integrate the equation (2.4.7)a and after using the boundary condition on F_s at $\eta = R$ the only unknown function in F_s is P_s. We can then substitute for F_s into (2.4.7)b and after integrating (2.4.7)c we can eliminate G_s from the solution of this equation and the equation of continuity. This gives a differential equation for H_s which is easily integrated. We can then substitute for F_s, H_s into (2.4.7)b and integrate to give G_s. The boundary conditions on G_s, H_s then give all the unknown functions in G_s, H_s and the Reynolds equation for P_s. After integrating the Reynolds equation twice we find that P_s is just a constant. The expressions for F_s, G_s, H_s are found to be as follows.

$$
\begin{aligned}
G_s = \Bigg\{ & -\left[\left(\frac{\eta}{R}\right)^{M+5} - \left(\frac{\eta}{R}\right)^{M-1}\right]\left[(M+6)R^{M+5}\left(\frac{SR'}{R^{M+8}}\right)' + \frac{\mu R^2 M}{2}\right] \Big/ 384(M+2)(M+3) \\
& + \left[\left(\frac{\eta}{R}\right)^{M+3} - \left(\frac{\eta}{R}\right)^{M-1}\right]\left[(M+4)R^{M+3}\left(\frac{SR'}{R^{M+6}}\right)' + \frac{\mu R^2 M}{2}\right] \Big/ 128(M+1)(M+2) \\
& + \left[\left(\frac{\eta}{R}\right)^{M+1} - \left(\frac{\eta}{R}\right)^{M-1}\right]\Bigg[\frac{-MSR'^2}{192R^4} + \frac{MSRR'p_s'}{4} - \frac{M(M+5)\mu R^2}{256(M+1)(M+2)(M+3)} \\
& \qquad - \frac{R^{M+1}}{2}\left(\frac{F^+}{R^M}\right)' - \frac{M}{64(M+2)}\left(\frac{R^{M+5}\left(\frac{SR'}{R^{M+8}}\right)'}{2(M+3)} - \frac{R^{M+5}\left(\frac{SR'}{R^{M+6}}\right)'}{M+1}\right)\Bigg] \\
& + \left(\frac{\eta}{R}\right)^{M-1}\left[\frac{SR'^2}{96R^4} - \frac{RR'Sp_s'}{2}\right]\Bigg\} \cos M\theta + O(\sigma)
\end{aligned}
$$

$$
F_s = \left\{\frac{SR'}{16R^4}\left[\frac{(\eta/R)^{M+4}}{2(M+2)} - \frac{(\eta/R)^{M+2}}{(M+1)}\right] + F^+\left(\frac{\eta}{R}\right)^M\right\} \cos M\theta
$$

$$H_s = \Bigg\{ \Big[\left(\tfrac{\eta}{R}\right)^{M+5} - \left(\tfrac{\eta}{R}\right)^{M-1} \Big] \Big[M R^{M+5} \left(\frac{\delta R'}{R^{M+8}} \right)' + \frac{\mu R^2 (M+6)}{2} \Big] \Big/ 384(M+2)(M+3)$$

$$- \Big[\left(\tfrac{\eta}{R}\right)^{M+3} - \left(\tfrac{\eta}{R}\right)^{M-1} \Big] \Big[M R^{M+3} \left(\frac{\delta R'}{R^{M+6}} \right)' + \frac{\mu R^2 (M+4)}{2} \Big] \Big/ 128(M+1)(M+2)$$

$$+ \Big[\left(\tfrac{\eta}{R}\right)^{M+1} - \left(\tfrac{\eta}{R}\right)^{M-1} \Big] \Big[\frac{R^{M+1}}{2} \left(\frac{F^+}{R^M} \right)' + \frac{M+2\,\delta R'^2}{192 R^4} - \frac{(M+2) p_s' \delta R R'}{4} + \mu R^2 (M+5) \Big] \Big/ 256(M+1)(M+3)$$

$$+ \Big[\frac{R^{M+5} \left(\frac{\delta R'}{R^{M+8}} \right)'}{2(M+3)} - \frac{R^{M+3} \left(\frac{\delta R'}{R^{M+6}} \right)'}{(M+1)} \Big] \Big/ 64 \Bigg] \Bigg\} \sin M\theta + O(\sigma)$$

(2.4.9)
a,b,c

where p_s', μ are as given by (2.4.4)c, (2.4.8) and F^+ is given by

$$F^+ = -\frac{\delta R p_s'}{2} + \frac{\delta R'}{96 R^4} \left\{ \frac{M^2 + 6M + 11}{(M+1)(M+2)} \right\}$$

2.5. Discussion of Results

We have again seen that in both the low and high frequency limits the geometry of the fluid container is crucial in determining the nature of the steady streaming. In particular the difference between the mean radii of the pipe ends has an important role. If this difference is zero then the steady streaming is confined between the nodes of the pipe (i.e. where R' is zero) and this steady streaming is produced by the Reynolds stresses associated with the oscillatory Stokes flow. If the ends of the pipe do not have the same mean radius these Reynolds stresses induce a steady pressure difference between the ends of the pipe which must be balanced by the pressure difference associated with the axisymmetric flow

$$(u, v, w) = \frac{[\eta^2 - R^2]}{R^5} (R'\eta, 0, R) \qquad (2.5.1)$$

which is merely Poiseuille flow in a pipe with radius equal to the local radius of the pipe. This effect is particularly important in the high frequency limit where the steady streaming velocity field is dominated by the component of velocity given by (2.5.1). In contrast to this we find that in the low frequency limit the steady velocity field is affected at the same order in σ by the component of velocity given by (2.5.1), and that induced directly by the Reynolds stresses. We can easily show that the stream surfaces associated with (2.5.1) are given by

$$\eta = \lambda R, \quad 0 \leqslant \lambda \leqslant 1 \tag{2.5.2}$$

In Fig. 3 we have shown the steady streaming in a wavy axisymmetric pipe whose ends have the same mean radius. In Fig. 4 we have shown the steady streaming in the Stokes layer at the pipe wall in more detail. The steady streaming shown in Fig. 4 is qualitatively similar to that found by Lyne (1971) who considered oscillatory viscous flow adjacent to a wavy wall. Our results correspond to the wavelength of the wall being much greater than both the thickness of the Stokes layer at the wall and the amplitude of oscillation of a fluid particle far from the wall. We recall that in Chapter 1 we considered oscillatory viscous flow in a two-dimensional channel of slowly varying depth. If one of the walls of the channel was taken to be wavy then the steady streaming in the Stokes layer at the wall was found to be identical to that found by Lyne.

In Fig. 5 we have sketched the steady streaming given by (2.4.6) for a pipe defined by

$$\eta = 1 - \tfrac{1}{2}\exp\left(-[\zeta-4]^2\right), \quad 0 \leqslant \zeta \leqslant 8$$

The ends of this pipe have the same radius and so C in (2.4.6) is zero. In contrast to the high frequency limit we see that there is no region of recirculation near the pipe wall. The flow in such a pipe might be of some interest as a model for oscillatory flow in a narrow constricted blood vessel. However, in such a flow the condition that the pipe radius changes slowly would be violated and so δ, defined by (2.1.2), would not be small.

Finally we compare the order of magnitude of the high frequency steady streaming given by (2.3.28) and that found by Lyne (1970) for oscillatory flow in a curved pipe. A calculation shows that in the Stokes layers of these flows the ratio of typical axial steady velocities for flows with similar order basic velocities and pipes of similar radius is $\frac{L}{R_0}$ where R_0 is the radius of curvature of the curved pipe. Thus we might expect that for flow in a curved pipe of varying radius the effects of curvature and narrowing are equally important as far as the Stokes layer type of steady streaming is concerned. The steady streaming of the form given by (2.5.1) would clearly be more important than both the latter contributions since, as shown by (2.3.23), (2.3.26) this effect appears at lower order in σ . Lyne (1970) has discussed the relevance of his work to the flow in the human aorta. The parameters δ and σ for such a flow are typically of order, 10^{-3}, 10.0 but the parameter R_M is of order 10^2. Thus our theory is not strictly applicable but it is likely that the effect of narrowing of the aorta is at least as important as the effect of curvature as far as the steady streaming for such a flow is concerned.

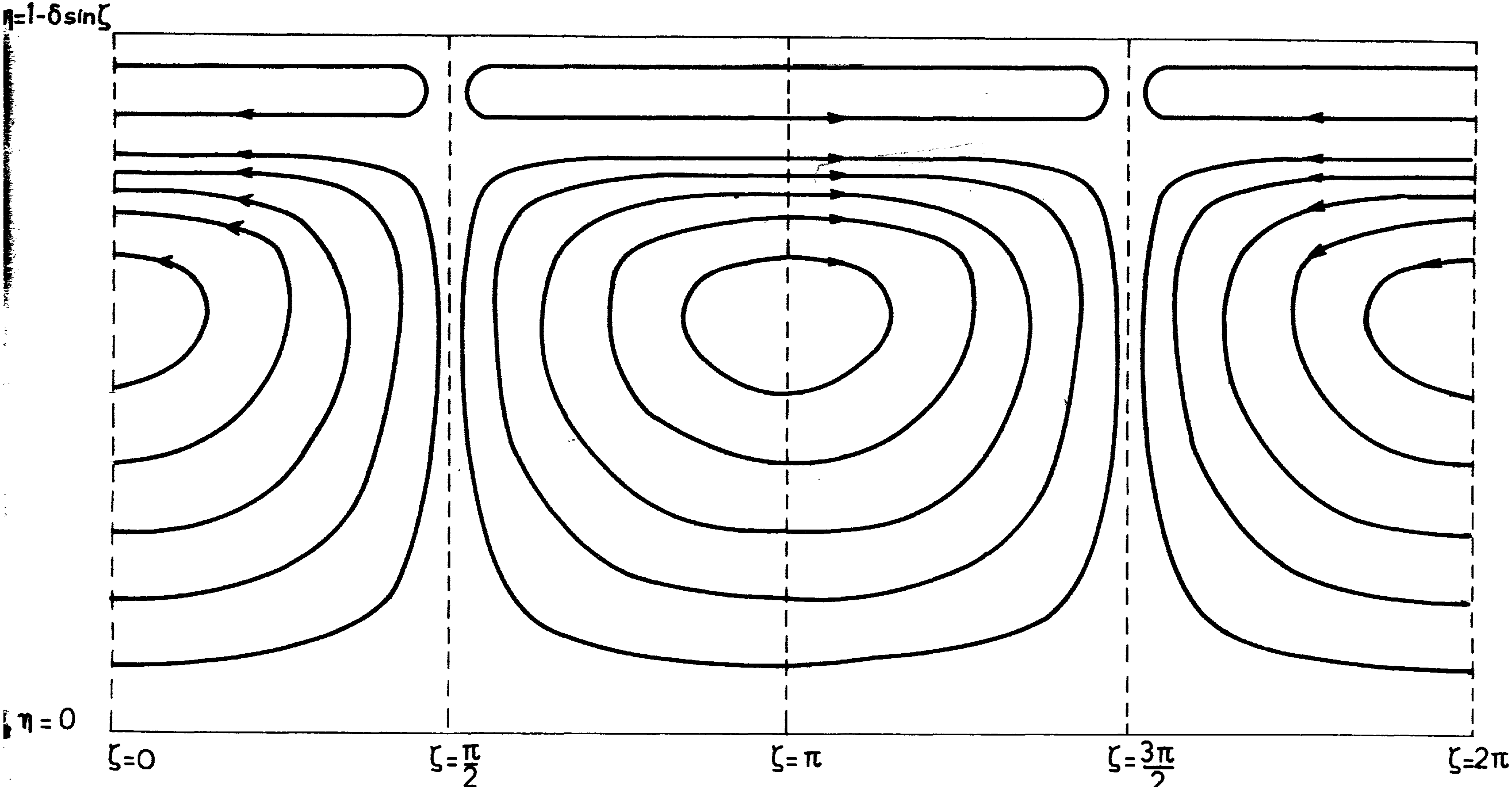

Fig. 3 Steady streaming in a pipe defined by $\eta=1-\delta\sin\zeta$, $0\leqslant\zeta\leqslant2\pi$ with $\sigma^{1/2}\delta<<1$ and $\sigma>>1$

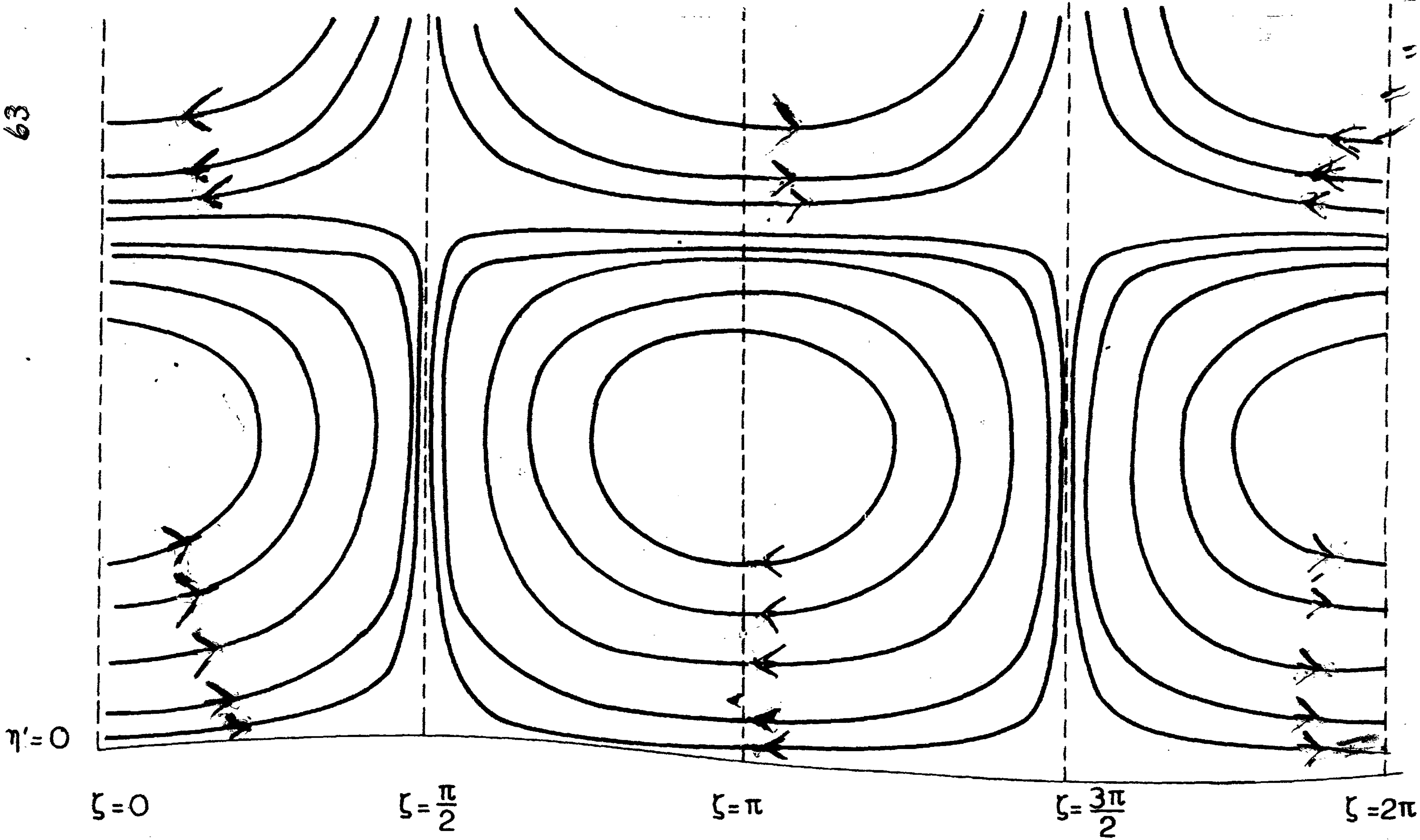

Fig. 4 Steady streaming in the Stokes layer in a pipe defined by $\eta = 1 - \delta \sin\zeta$, $0 \leq \zeta \leq 2\pi$ with $\sigma^{1/2}\delta \ll 1$ and $\sigma \gg 1$.

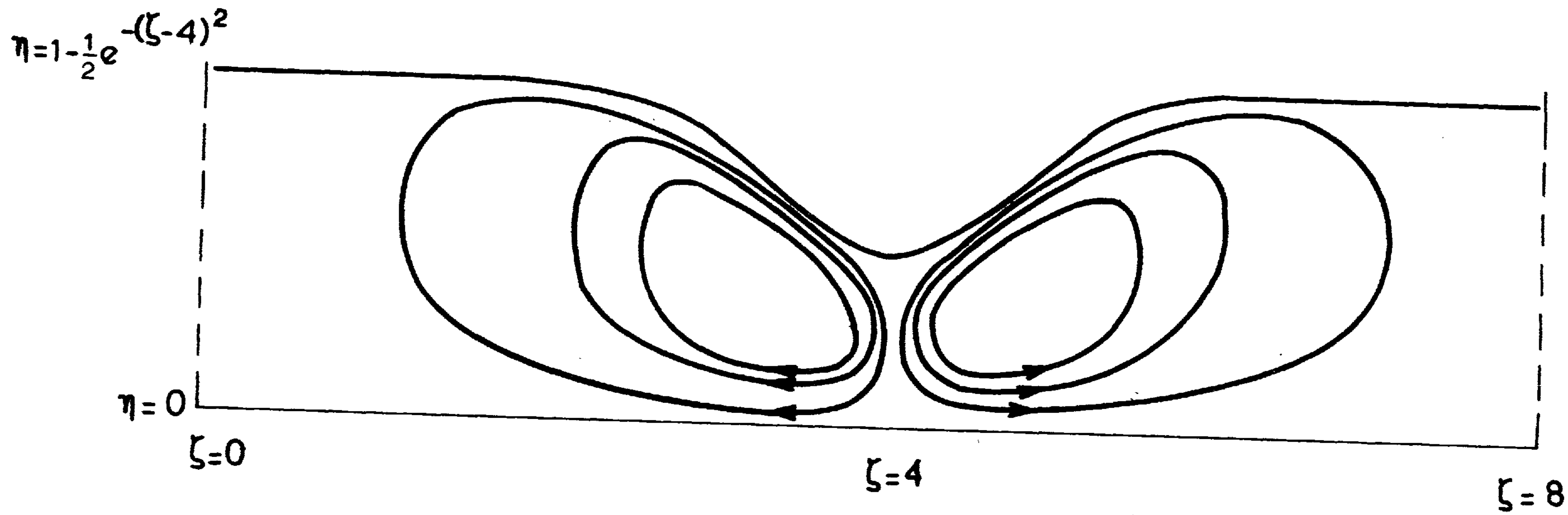

Fig.5 Steady streaming in the pipe defined by $\eta=1-\frac{1}{2}e^{-(\zeta-4)^2}$, $0\leqslant\zeta\leqslant 8$ with $\sigma<<1$.

PART 2

THE STABILITY OF SOME UNSTEADY VISCOUS FLOWS

CHAPTER 111

THE LINEAR STABILITY OF UNSTEADY CYLINDER FLOWS

3.1 Introduction

We consider the stability of the flow between concentric cylinders when the outer cylinder is at rest and the inner one has angular velocity $\Omega_1\{1+\epsilon\cos\omega t\}$. This problem has been considered experimentally by Donnelly (1964) who found that modulation enhanced the stability of the flow. Moreover, he found that for all ϵ this enhancement was a maximum for a certain value, .27, of a frequency parameter, σ, defined to be the square of the ratio of the separation of the cylinders to the thickness of the Stokes layer associated with the oscillatory motion of the inner cylinder.

In this chapter we examine the stability of this flow to disturbances which are small enough for linearization to be a valid approximation. The procedure adopted is as follows.

In §3.2 we determine the nature of the basic flow and derive the partial differential equations governing the linear stability of this flow. These equations are solved subject to there being no relative velocity at the walls of the cylinders. We follow Venezian (1969) and Rosenblat & Herbert (1970) and use the so-called 'periodicity' criterion to define a boundary between stability and instability. The above authors considered the linear stability of the thermal analogue of this problem. The results of the former and latter authors corresponded to taking parameters corresponding to ϵ and σ respectively to be small.

In §3.3 we obtain an asymptotic expansion of the Taylor number in terms of ϵ and σ when the latter are both small. In actual fact we seek a solution to the partial differential system by letting ϵ tend to zero with σ/ϵ fixed and equal to α say.

This is done so that the dominant time dependences of the partial differential equations balance in some sense. A similar idea was used by Di Prima & Stuart (1972) who considered the global stability of the flow between eccentric rotating cylinders when the cylinders move with constant angular velocity. We expand the perturbation velocities and the Taylor number in powers of ϵ and replace σ by $\alpha\epsilon$ everywhere in the partial differential system. We then equate like powers of ϵ and obtain ordinary differential systems which contain the time variable only as a parameter. We find that the order ϵ^0 system gives the ordinary steady velocity field multiplied by an arbitrary function of the time variable. This function is determined by solvability conditions on the order ϵ ordinary differential system. The order ϵ term in the expansion of T in powers of ϵ is then specified by insisting that this function of ωt is in fact periodic in ωt. Higher order terms in the expansion of T are determined by considering the higher order systems.

In §3.4 we consider the limit of σ tending to infinity with ϵ arbitrary. The time dependence of the basic flow is then confined to a thin layer near the inner cylinder. We shall refer to this layer as being the 'inner' layer. However, the interaction of the basic flow with the disturbance in this layer causes the disturbance velocity field to have a time dependence which persists throughout the fluid. Hence a second Stokes layer is required at the outer cylinder to satisfy the relevant boundary conditions there. We shall refer to this layer as the 'outer' layer and the region between the Stokes layers will be called the 'central' region. In each region we expand the disturbance velocity in Fourier series in time and then expand the coefficients in the series in powers of $\sigma^{-1/2}$. The Taylor number is also expanded in powers of $\sigma^{-1/2}$. The disturbance velocity is then determined in each region by equating like powers of $\sigma^{-1/2}$ in the relevant differential systems . We then 'match' velocities where different regions meet and the terms in the expansion of T are essentially determined by matching the steady parts of the perturbation velocity

In §3.5 we describe the numerical work required to solve the ordinary differential systems appearing in §3.3 , §3.4 , and in §3.6 we discuss the results of our work and their relevance to the work of Donnelly (1964) .

3.2 The basic flow and the disturbance equations

We consider viscous incompressible flow between concentric cylinders of infinite length and radii R_1 , R_2 ($R_2 > R_1$) . We assume that the separation of the cylinders is small compared to the mean radius of the cylinders . Thus we have that

$$d = R_2 - R_1 \ll R_1$$

and we shall therefore neglect terms of order d/R_1 throughout . We take cylindrical polar coordinates (r , θ , z) with the z-axis along the axis of the cylinders . We take (u , v , w) to be the corresponding velocity vector . We also take p , ρ , ν , and t to be the pressure , density , kinematic viscosity , and time respectively . We can easily show that the basic velocity field will be given by (0 , V(r,t) , 0) where

$$\left.\begin{aligned} &\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{\nu}\frac{\partial}{\partial t}\right] V = 0 \\ &V = \Omega_1 R_1 \{1 + \epsilon \cos\omega t\} \ , \ r = R_1 \\ &V = 0 \ , \ r = R_2 \end{aligned}\right\} \qquad (3.2.1)$$

and the pressure distribution associated with this velocity field is then determined by

$$\frac{\partial}{\partial r}\left(\frac{P}{\rho}\right) = \frac{V^2}{r} \tag{3.2.2}$$

We now introduce the dimensionless variables ζ , τ by

$$\left.\begin{aligned} \zeta &= \frac{r - R_1}{d} \\ \tau &= \omega t \end{aligned}\right\} \tag{3.2.3)a,b}$$

and a dimensionless velocity by

$$\bar{V} = \frac{V}{\Omega_1 R_1} \tag{3.2.4}$$

and the frequency parameter σ , mentioned in §3.1 , is defined by

$$\sigma = \frac{\omega d^2}{\nu} \tag{3.2.5}$$

If we now write (3.2.1) in terms of the dimensionless quantities introduced above we obtain a differential system which we can solve to give

$$\bar{V} = \left\{1 - \zeta + \frac{\epsilon}{2}\left[\frac{\sinh(\sqrt{i\sigma}[1-\zeta])e^{i\tau}}{\sinh\sqrt{i\sigma}} + \text{COMPLEX CONJUGATE}\right]\right\} \tag{3.2.6}$$

We do not require any knowledge of p/ρ in the following work and so we do not solve for it here .

Suppose now that the basic flow is perturbed in such a way that the disturbed state may be charecterized by u , $v + V$, w , δp . We make the usual assumption that the flow is periodic along the z-axis with wavelength $2\pi/k$. Thus we write

$$\left.\begin{aligned} u &= \frac{-\nu}{2d} u^*(\zeta,\tau)\cos kz \\ v &= \frac{\Omega_1 R_1}{2} v^*(\zeta,\tau)\cos kz \\ w &= \frac{-\nu}{2d} v^*(\zeta,\tau)\sin kz \end{aligned}\right\} \tag{3.2.7)a,b,c}$$

This is the usual scaling for the problem and the minus signs have merely been introduced for the sake of convenience. We can show from the momentum and continuity equations by linearizing and the usual manipulations that u^*, v^* are determined by

$$\left.\begin{aligned} \left[M - \sigma\frac{\partial}{\partial\tau}\right] M u^* &= -a^2 T v^* \bar{V} \\ \left[M - \sigma\frac{\partial}{\partial\tau}\right] v^* &= -u^* \frac{\partial\bar{V}}{\partial\zeta} \\ u^* = v^* = \frac{\partial u^*}{\partial\zeta} &= 0 \;, \quad \zeta = 0, 1 \end{aligned}\right\} \qquad (3.2.8)$$

These equations are valid in the limit d/R_1 tending to zero with ζ, τ, etc. held fixed. (For details of the derivation of the above system the reader should see for example Chandrasekhar (1961).) The quantities appearing above which are as yet undefined are given by

$$\left.\begin{aligned} a &= kd \\ M &\equiv \frac{\partial^2}{\partial\zeta^2} - a^2 \\ T &= \frac{2\Omega_1^2 d^3 R_1}{\nu^2} \end{aligned}\right\} \qquad (3.2.9)\text{a, b, c}$$

Thus a is a non-dimensional wavenumber and T is the usual Taylor number. Clearly (3.2.8) specifies an eigenvalue problem for u^*, v^* which leads to an eigenrelation of the form

$$F(T, \sigma, \epsilon, a) = 0 \qquad (3.2.10)$$

We now stipulate that u^*, v^* must be periodic functions of τ. This serves to define a boundary between stability and instability and the corresponding smallest value of the Taylor number will be called the critical Taylor number and we shall denote this value of T by T_0.

3.3 The low frequency limit

We now determine the nature of T as ϵ and σ tend to zero , in which case the Stokes layer associated with the oscillatory motion of the inner cylinder completely fills the gap between the cylinders . If we expand $\overline{V}$ given by (3.2.6) for small σ and substitute into (3.2.8) we obtain (after dropping the star notation

$$\left.\begin{aligned} \left[M-\sigma\frac{\partial}{\partial\tau}\right]Mu &= -a^2 Tv\left\{\chi_0+\epsilon\chi_1\cos\tau+\epsilon\sigma\chi_2\sin\tau+\epsilon\sigma^2\chi_3\cos\tau+\dots\right\} \\ \left[M-\sigma\frac{\partial}{\partial\tau}\right]v &= u\left\{1+\epsilon\cos\tau+\epsilon\sigma\phi_2\sin\tau+\epsilon\sigma^2\phi_3\cos\tau+\dots\right\} \\ u = v = \frac{\partial u}{\partial\zeta} &= 0, \quad \zeta = 0,1 \end{aligned}\right\} \qquad (3.3.1)\text{a}$$

where the first five χ_i are given by

$$\left.\begin{aligned} \chi_0 &= \chi_1 = 1-\zeta \\ \chi_2 &= \frac{\{\zeta^3-3\zeta^2+2\zeta\}}{6} \\ \chi_3 &= \frac{\{3\zeta^5-15\zeta^4+20\zeta^3-8\zeta\}}{360} \\ \chi_4 &= \frac{\{21\zeta^6-3\zeta^7-42\zeta^5+56\zeta^3-32\zeta\}}{15{,}120} \end{aligned}\right\} \qquad (3.3.2)\text{b}$$

and for convenience we have defined $\phi_i = -\dfrac{d\chi_i}{d\zeta}$.

We now constrain ϵ and σ to tend to zero in such a way that the τ dependences of the right and left hand sides of the two differential equations appearing above 'balance' in some sense . If we assume that T varies little from its unmodulated value we can see that the responses of the $\frac{\partial}{\partial\tau}$ terms of the left hand sides of these equations are proportional to the τ dependences imposed by the $\epsilon\cos\tau$ terms of the right hand sides if we have

$$\sigma \sim \epsilon$$

Hence we write

$$\sigma = \alpha\epsilon \tag{3.3.2}$$

and let ϵ tend to zero with α fixed. This corresponds physically to letting the frequency and velocity amplitude of the inner cylinder tend to zero in such a way that the oscillatory displacement of the cylinder, $\alpha^{-1}(Td/2R_1)^{\frac{1}{2}}$, remains constant if d, R_1 are held fixed.

We now expand the perturbation velocities in the form

$$\left.\begin{aligned} u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \\ v &= v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots \end{aligned}\right\} \tag{3.3.3)a, b}$$

and the Taylor number is expanded in the form

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots \tag{3.3.3)c}$$

We shall in fact see that $T_i = 0$ for i odd which is only to be expected since changing ϵ to $-\epsilon$ does not essentially change the physical problem under consideration. If we substitute for u, v, T from above into (3.3.1) we have

$$\left.\begin{aligned} &\left[M - \alpha\epsilon\frac{\partial}{\partial\tau}\right] M\left[u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots\right] \\ &\qquad = -a^2\left\{T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots\right\}\left\{\chi_0 + \epsilon\chi_1 \cos\tau \cdots\right\}\left[v_0 + \epsilon v_1 + \cdots\right] \\ &\left[M - \alpha\epsilon\frac{\partial}{\partial\tau}\right]\left[v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots\right] \\ &\qquad = \left\{1 + \epsilon\cos\tau + \cdots\right\}\left[u_0 + \epsilon u_1 + \cdots\right] \\ &\left[u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots\right] = \left[v_0 + \epsilon v_1 + \epsilon^2 v_2 \cdots\right] = \frac{\partial}{\partial\zeta}\left[u_0 + \epsilon u_1 + \epsilon^2 u_2 \cdots\right] = 0,\ \zeta = 0, 1 \end{aligned}\right\} \tag{3.3.4}$$

where we have replaced σ by $\alpha\epsilon$ everywhere. If we equate terms of order ϵ^0 in (3.3.4) we have

$$\left.\begin{aligned} &M^2 u_0 + a^2 T_0 \chi_0 v_0 = 0 \\ &u_0 - M v_0 = 0 \\ &u_0 = v_0 = \frac{\partial u_0}{\partial\zeta} = 0,\ \zeta = 0, 1 \end{aligned}\right\} \tag{3.3.5}$$

We notice that τ does not appear in (3.3.5) and so we have an ordinary differential system whose solution may be written in the form

$$\left.\begin{aligned} u_0 &= B_0(\tau) f_0(\zeta) \\ v_0 &= B_0(\tau) g_0(\zeta) \end{aligned}\right\} \qquad (3.3.6)\text{a,b}$$

where $f_0(\zeta)$, $g_0(\zeta)$ are determined by

$$\left.\begin{aligned} \left[\frac{d^2}{d\zeta^2} - a^2\right] f_0 + a^2 T_0 X_0 g_0 = 0 \\ f_0 - \left[\frac{d^2}{d\zeta^2} - a^2\right] g_0 = 0 \\ f_0 = g_0 = \frac{df_0}{d\zeta} = 0, \ \zeta = 0,1 \end{aligned}\right\} \qquad (3.3.7)$$

and $B_0(\tau)$ will be determined by the order ϵ system . In the following work we shall find it useful to define $\mathcal{L}_1$, $\mathcal{L}_2$, N as follows

$$\left.\begin{aligned} \mathcal{L}_1(u,v) &= M^2 u + a^2 T_0 X_0 v \\ \mathcal{L}_2(u,v) &= u - Mv \\ N &\equiv \frac{d^2}{d\zeta^2} - a^2 \end{aligned}\right\} \qquad (3.3.8)\text{a,b,c}$$

If we now equate terms of order ϵ in (3.3.4) and use (3.3.6) we have

$$\left.\begin{aligned} \mathcal{L}_1(u,v) &= \alpha \frac{dB_0}{d\tau} N f_0 - B_0 \cos\tau \, a^2 T_0 X_1 g_0 \\ &\quad - B_0 a^2 T_1 X_0 g_0 \\ \mathcal{L}_2(u,v) &= -\alpha \frac{dB_0}{d\tau} g_0 - B_0 \cos\tau \, f_0 \\ u_1 = v_1 &= \frac{\partial u_1}{\partial \zeta} = 0 \ , \ \zeta = 0,1 \end{aligned}\right\} \qquad (3.3.9)$$

After solving the order ϵ system the only unknown quantities on the right hand sides of the above differential equations are

B_0 and T_1 . We now introduce the adjoint function pair , (f_0^+, g_0^+) , which we define by

$$\left.\begin{array}{l} N^2 f_0^+ + g_0^+ = 0 \\ a^2 T_0 \chi_0 f_0^+ - N g_0^+ = 0 \\ f_0^+ = g_0^+ = \dfrac{df_0}{d\zeta} = 0 \; , \; \zeta = 0,1 \end{array}\right\} \qquad (3.3.10)$$

The eigenvalues a , T_0 of (3.3.7) , (3.3.10) are the same but as the form of the equations shows the function pairs (f_0, g_0) , (f_0^+, g_0^+) are not the same . Having defined the adjoint function pair we can show, that the condition that the system (3.3.10) has a solution is that the integral from ζ = 0 to ζ = 1 of the sum of f_0^+ times the right hand side of the first equation in (3.3.9) and g_0^+ times the right hand side of the second is zero . (See for example Ince (1927) .) Thus we have that

$$\alpha \frac{dB_0}{d\tau} \left\{ \int_0^1 \left[f_0^+ N f_0 - g_0^+ g_0 \right] d\zeta \right\} - B_0 \cos\tau \left\{ \int_0^1 \left[a^2 T_0 \chi_1 f_0^+ g_0 + g_0^+ f_0 \right] d\zeta \right\} = B_0 a^2 T_1 \int_0^1 \chi_1 f_0^+ g_0 \, d\zeta$$

which is an ordinary differential equation for B_0 and has a periodic solution if T_1 = 0, and B_0 is then given by

$$B_0(\tau) = A \exp\left\{ \frac{-\Upsilon}{\alpha} \sin\tau \right\} \qquad (3.3.11)$$

where

$$\Upsilon = \frac{\int_0^1 \left[a^2 T_0 \chi_1 f_0^+ g_0 + g_0^+ f_0 \right] d\zeta}{\int_0^1 \left[g_0^+ g_0 - f_0^+ N f_0 \right] d\zeta} \qquad (3.3.12)$$

and A is a constant dependent on the parameters of the problem which can only be determined by a consideration of the corresponding non-linear problem . Having determined B_0 we can substitute back into (3.3.9) to show that

$$(u_1, v_1) = B_0(\tau)\cos\tau\,(f_1, g_1) + B_1(\tau)(f_0, g_0) \tag{3.3.13}$$

where (f_1, g_1) is determined by

$$\left.\begin{aligned} N^2 f_1 + a^2 T_0 \chi_0 g_1 &= -\Gamma N f_0 - a^2 T_0 \chi_1 g_0 \\ f_1 - N g_1 &= \Gamma g_0 - f_0 \\ f_1 = g_1 = \frac{df_1}{d\zeta} &= 0\,, \quad \zeta = 0, 1 \end{aligned}\right\} \tag{3.3.14}$$

The solution of (3.3.9) is clearly unique only upto an arbitrary multiple of the basic eigenfunction pair (f_0, g_0) . Hence the inclusion of the as yet arbitrary function of τ , $B_1(\tau)$, times this eigenfunction pair in (3.3.13) . The determination of B_1 requires the consideration of the order ϵ^2 differential system , which from (3.3.4) , (3.3.6) , (3.3.13) is found to be

$$\left.\begin{aligned} \mathcal{L}_1(u_2, v_2) &= \alpha \frac{dB_1}{d\tau} N f_0 - B_1 \cos\tau\, a^2 T_0 \chi_1 g_0 - \alpha B_0 \sin\tau\, F_{11}(\zeta) \\ &\quad - \tfrac{1}{2} B_0 \left\{ 2a^2 T_2 \chi_0 g_0 + F_{12}\left[1 + \cos 2\tau\right] \right\} \\ \mathcal{L}_2(u_2, v_2) &= -\alpha \frac{dB_1}{d\tau} g_0 - B_1 \cos\tau\, f_0 - \alpha B_0 \sin\tau\, G_{11}(\zeta) \\ &\quad - \tfrac{1}{2} B_0 \left\{ 1 + \cos 2\tau \right\} G_{12}(\zeta) \\ u_2 = v_2 = \frac{\partial u_2}{\partial \zeta} &= 0\,, \quad \zeta = 0, 1 \end{aligned}\right\} \tag{3.3.15}$$

where the functions F_{11} , F_{12} , G_{11} , G_{12} are given by

$$\left.\begin{aligned} F_{11}(\zeta) &= N f_1 + a^2 T_0 \chi_2 g_0 \\ F_{12}(\zeta) &= \Gamma N f_1 + a^2 T_0 \chi_1 g_1 \\ G_{11}(\zeta) &= \phi_2 f_0 - g_1 \\ G_{12}(\zeta) &= f_1 - \Gamma g_1 \end{aligned}\right\} \quad \text{(3.3.16) a,b,c,d}$$

Having solved the order ϵ^0 , ϵ^1 systems the right hand sides of the equations in (3.3.15) are known except for $B_1(\tau)$ and T_2 . If we now invoke the solvability condition on this system we obtain an ordinary differential equation for $B_1(\tau)$, which has a solution periodic in τ if

$$a^2 T_2 = \frac{-\int_0^1 \{ f_0^+ F_{12} + g_0^+ G_{12} \} d\zeta}{\int_0^1 \{ \chi_0 f_0^+ g_0 \} d\zeta} \tag{3.3.17}$$

and $B_1(\tau)$ is then given by

$$B_1 = B_0 \left\{ \Pi_1 \cos\tau + \frac{\Pi_2}{\alpha} \sin 2\tau \right\} \tag{3.3.18}$$

where

$$\left.\begin{aligned} \Pi_1 &= \frac{\int_0^1 \{ f_0^+ F_{11} + g_0^+ G_{11} \} d\zeta}{\int_0^1 \{ g_0^+ g_0 - f_0^+ N f_0 \} d\zeta} \\ \Pi_2 &= \frac{-\int_0^1 \{ f_0^+ F_{12} + g_0^+ G_{12} \} d\zeta}{4\int_0^1 \{ g_0^+ g_0 - f_0^+ N f_0 \} d\zeta} \end{aligned}\right\} \tag{3.3.19)a,b$$

and the functions F_{11} , etc. which appear in the above integrals are as defined by (3.3.16)a,b,c,d respectively . If we now substitute for $B_1(\tau)$ from above into (3.3.13) we obtain

$$(u_1, v_1) = B_0 \cos\tau \, (f_1 + \Pi_1 f_0, g_1 + \Pi_1 g_0) + \frac{B_0}{\alpha} \sin 2\tau \, \Pi_2 \, (f_0, g_0) \tag{3.3.20}$$

Having determined $B_1(\tau)$ we can substitute back into (3.3.15) to show that (u_2, v_2) is of the form

$$\begin{aligned} (u_2, v_2) = B_0 \Big[& \alpha \sin\tau \, (f_2, g_2) + \cos 2\tau \, (f_3, g_3) + (f_4, g_4) + \frac{\sin 3\tau}{\alpha} (f_5, g_5) \\ & + \frac{\sin\tau}{\alpha} (f_5, g_5) \Big] + B_2 \, (f_0, g_0) \end{aligned} \tag{3.3.21}$$

where the function pairs (f_2, g_2) , (f_3, g_3) , (f_4, g_4) , (f_5, g_5) are solutions of (3.3.22) with H_2 , etc. as given by (3.3.23) .

$$\left.\begin{aligned} N^2 f_i + a^2 T_0 \chi_0 g_i &= H_i \\ f_i - N g_i &= K_i \\ f_i = \frac{df_i}{d\zeta} = g_i &= 0, \quad \zeta = 0, 1 \end{aligned}\right\} \tag{3.3.22}$$

$$\left.\begin{aligned}
H_1 &= -\left\{\Gamma_1 N f_0 + F_{11}\right\} \\
H_2 &= \left\{2\Gamma_2 N f_0 - \frac{F_{12}}{2} - \frac{\Gamma_1}{2}\left(\Gamma N f_0 + a^2 T_0 \chi_1 g_0\right)\right\} \\
H_4 &= -\left\{a^2 T_2 \chi_0 g_0 + \frac{F_{12}}{2} + \frac{\Gamma_1}{2}\left(\Gamma N f_0 + a^2 T_0 \chi_1 g_0\right)\right\} \\
H_5 &= -\frac{1}{2}\left\{a^2 T_0 \Gamma_2 \chi_1 g_0 + \Gamma \Gamma_2 N f_0\right\} \\
K_2 &= \left\{\Gamma_1 g_0 - G_{11}\right\} \\
K_3 &= -\left\{2\Gamma_2 g_0 + \frac{G_{12}}{2} + \frac{\Gamma_1}{2}\left(\phi_1 f_0 - \Gamma g_0\right)\right\} \\
K_4 &= -\frac{1}{2}\left\{G_{12} + \Gamma_1\left(\phi_1 f_0 - \Gamma g_0\right)\right\} \\
K_5 &= \frac{1}{2}\left\{\Gamma \Gamma_2 g_0 - \Gamma_2 \phi_1 f_0\right\}
\end{aligned}\right\} \qquad \begin{matrix}(3.3.23)\\ a,b,c,d,e,f,g,h\end{matrix}$$

It follows from an inspection of (3.3.14) , (3.3.23) that the function pairs (f_5, g_5) ,$\frac{\Gamma}{2}$(f_1 , g_1) differ only by an arbitary multiple of the basic eigenfunction pair , (f_0 , g_0) , and this fact will be used later . The arbitary function of τ , $B_2(\tau)$, appearing in (3.3.21) is now calculated using the order ϵ^3 system of equations .

Equating terms of order ϵ^3 in (3.3.4) and using the expressions for u_0 , v_0 , etc. already calculated we can show that

$$\left.\begin{aligned}
\mathcal{L}_1(u_3, v_3) &= \alpha \frac{dB_2}{d\tau} N f_0 - B_2 \cos\tau \, a^2 T_0 \chi_1 g_0 - B_0 a^2 T_3 \chi_0 g_0 \\
&\quad + B_0\left\{\alpha^2 \cos\tau \, F_{21} + \alpha \sin 2\tau \, F_{22} + \cos 3\tau \, F_{23} \right. \\
&\quad \left. + \cos\tau \, F_{24} + \frac{\sin 4\tau}{\alpha} F_{25} + \frac{\sin 2\tau}{\alpha} F_{26}\right\} \\
\mathcal{L}_2(u_3, v_3) &= -\alpha \frac{dB_2}{d\tau} g_0 - B_2 \cos\tau \, f_0 \\
&\quad + B_0\left\{\alpha^2 \cos\tau \, G_{21} + \alpha \sin 2\tau \, G_{22} + \cos 3\tau \, G_{23} \right. \\
&\quad \left. + \cos\tau \, G_{24} + \frac{\sin 4\tau}{\alpha} G_{25} + \frac{\sin 2\tau}{\alpha} G_{26}\right\} \\
u_3 &= v_3 = \frac{\partial u_3}{\partial \zeta} = 0 \; , \quad \zeta = 0, 1
\end{aligned}\right\} \qquad (3.3.24)$$

where F_{21} , F_{22} , etc. are given by

$$
\left.
\begin{aligned}
F_{21} &= -a^2T_0 g_0 \chi_3 + N f_2 \\
F_{22} &= \tfrac{1}{2}a^2T_0\chi_2(g_1+\Pi_1 g_0) - \tfrac{1}{2}a^2T_0\chi_1 g_2 - N\left\{2f_3 + \tfrac{\Pi}{2}f_2\right\} \\
F_{23} &= -\tfrac{1}{2}a^2T_0\Pi_2 g_0 - \tfrac{1}{2}a^2T_0\chi_1 g_3 + N\left\{3f_5 - \tfrac{\Pi}{2}f_3\right\} \\
F_{24} &= -\tfrac{1}{2}a^2T_0\chi_2\Pi_2 g_0 - \tfrac{1}{2}a^2T_0\chi_1 g_3 - a^2T_2\left\{g_0\chi_1 + g_1\chi_0 + \Pi_1\chi_0 g_0\right\} \\
&\quad - a^2T_0\chi_1 g_4 + N\left\{f_5 - \tfrac{\Pi}{2}f_3 - \Pi f_4\right\} \\
F_{25} &= -\tfrac{\Pi}{2}Nf_5 - \tfrac{1}{2}a^2T_0\chi_1 g_5 \\
F_{26} &= 2F_{25} - a^2T_2\Pi_2\chi_0 g_0 \\
\\
G_{21} &= -g_2 - \phi_3 f_0 \\
G_{22} &= -\tfrac{1}{2}f_2 - \tfrac{1}{2}\phi_2(f_1+\Pi_1 f_0) + 2g_3 + \tfrac{1}{2}\Pi g_3 \\
G_{23} &= -\tfrac{1}{2}f_3 + \tfrac{1}{2}\phi_2\Pi_2 f_0 - 3g_5 + \tfrac{1}{2}\Pi g_3 \\
G_{24} &= -\left(f_4 + \tfrac{1}{2}f_3\right) - \tfrac{1}{2}\phi_2\Pi_2 f_0 - g_5 + \tfrac{1}{2}\Pi g_3 + \Pi g_4 \\
G_{25} &= -\tfrac{1}{2}f_5 + \tfrac{1}{2}\Pi g_5 \\
G_{26} &= 2G_{25}
\end{aligned}
\right\}
\quad
\begin{aligned}
&(3.3.25)\\
&a, b, \ldots\ldots. l
\end{aligned}
$$

Having solved the order ϵ^0, ϵ^1, ϵ^2 systems the only unknown quantities on the right hand sides of (3.3.24) are B_2 and T_3 which may be determined by invoking the solvability condition for the system in question .We find that $T_3 = 0$ if $B_2(\tau)$ is to be a periodic function of τ . The function $B_2(\tau)$ is then given by

$$B_2 = B_0\left\{\alpha\Pi_3 \sin\tau + \Pi_4\cos\tau + \frac{\Pi_5}{\alpha}\sin 3\tau + \frac{\Pi_6}{\alpha}\sin\tau + \frac{\Pi_7}{\alpha^2}\cos 4\tau + \frac{\Pi_8}{\alpha^2}\cos 2\tau\right\} \qquad (3.3.26)$$

where Π_3, Π_4 , etc. are determined by

$$\Pi_3 = \mu_3, \; \Pi_4 = -\frac{\mu_4}{2}, \; \Pi_5 = \frac{\mu_5}{3}, \; \Pi_6 = \mu_6, \; \Pi_7 = -\frac{\mu_7}{4}, \; \Pi_8 = -\frac{\mu_8}{2}$$

and $$\mu_i = \frac{\int_0^1 \{ f_0^+ F_{2\,i-2} + g_0^+ G_{2\,i-2} \} d\rho}{\int_0^1 \{ g_0^+ g_0 - f_0^+ N f_0 \} d\rho} \qquad i = 3,4,5,6,7,8$$

However we recall that (f_5, g_5) and $\frac{\Pi_1}{2}$(f_1, g_1) differ only by an arbitrary multiple of the function pair (f_0, g_0), and so by using (3.3.17) with μ_8 as above we can show that Π_8 is in fact zero . Having determined B_2 we can write (u_2 , v_2) as follows

$$\begin{aligned}(u_2, v_2) = B_0 \Big[& \alpha \sin\tau (f_2 + \Pi_3 f_0, g_2 + \Pi_3 g_0) + \cos 2\tau (f_3 + \Pi_4 f_0, g_3 + \Pi_4 g_0) \\ & + (f_4, g_4) + \frac{\sin 3\tau}{\alpha} (f_5 + \Pi_5 f_0, g_5 + \Pi_5 g_0) \\ & + \frac{\sin\tau}{\alpha} (f_5 + \Pi_6 f_0, g_5 + \Pi_6 g_0) + \frac{\cos 4\tau}{\alpha^2} (\Pi_7 f_0, \Pi_7 g_0) \Big]\end{aligned} \tag{3.3.27}$$

If we substitute for B_2 from (3.3.26) into (3.3.24) we can show that (u_3 , v_3) is of the form

$$\begin{aligned}(u_3, v_3) = B_0 \Big[& \alpha^2 \cos\tau (f_6, g_6) + \alpha \sin 2\tau (f_7, g_7) + \cos 3\tau (f_8, g_8) \\ & + \cos\tau (f_9, g_9) + \frac{\sin k\tau}{\alpha} (f_{10}, g_{10}) + \frac{\sin 4\tau}{\alpha} (f_{10}, g_{10}) \\ & + \frac{\sin 2\tau}{\alpha} (f_{11}, g_{11}) + \frac{\cos 5\tau}{\alpha^2} (f_{12}, g_{12}) + \frac{\cos 3\tau}{\alpha^2} (f_{13}, g_{13}) \Big] \\ & + B_3(\tau) (f_0, g_0)\end{aligned} \tag{3.3.28}$$

where B_3 is an arbitrary function of τ and (f_6, g_6) , (f_7, g_7) , etc. are solutions of (3.3.22) with

$$H_6 = \Gamma_3 N f_0 + F_{21}$$
$$H_7 = -\left(2\Gamma_4 + \frac{\pi\Gamma_3}{2}\right) N f_0 - \frac{1}{2} a^2 T_0 \Gamma_3 \chi_1 g_0 + F_{22}$$
$$H_8 = \left(3\Gamma_5 - \frac{\pi\Gamma_4}{2}\right) N f_0 - \frac{1}{2} a^2 T_0 \Gamma_4 \chi_1 g_0 + F_{23}$$
$$H_9 = \left(\Gamma_6 - \frac{\pi\Gamma_4}{2}\right) N f_0 - \frac{1}{2} a^2 T_0 \Gamma_4 \chi_1 g_0 + F_{24}$$
$$H_{10} = -\left(4\Gamma_7 + \frac{\pi\Gamma_5}{2}\right) N f_0 - \frac{1}{2} a^2 T_0 \Gamma_5 \chi_1 g_0 + F_{25}$$
$$H_{11} = -\left(\frac{\pi\Gamma_5}{2} + \frac{\pi\Gamma_6}{2}\right) N f_0 - \frac{1}{2} a^2 T_0 (\Gamma_5 + \Gamma_6) \chi_1 g_0 + F_{26}$$
$$H_{12} = -\frac{\pi\Gamma_7}{2} N f_0 - \frac{1}{2} a^2 T_0 \Gamma_2 \chi_1 g_0$$
$$K_6 = -\Gamma_3 g_0 + G_{21}$$
$$K_7 = \left(2\Gamma_4 + \frac{\pi\Gamma_3}{2}\right) g_0 - \frac{\Gamma_3}{2} f_0 + G_{22}$$
$$K_8 = \left(\frac{\pi\Gamma_4}{2} - 3\Gamma_5\right) g_0 - \frac{\Gamma_4}{2} f_0 + G_{23}$$
$$K_9 = \left(\frac{\pi\Gamma_4}{2} - \Gamma_6\right) g_0 - \frac{\Gamma_4}{2} f_0 + G_{24}$$
$$K_{10} = \left(4\Gamma_7 + \frac{\pi\Gamma_5}{2}\right) g_0 - \frac{\Gamma_5}{2} f_0 + G_{25}$$
$$K_{11} = \left(\frac{\pi\Gamma_5}{2} + \frac{\pi\Gamma_6}{2}\right) g_0 - \left(\frac{\Gamma_5 + \Gamma_6}{2}\right) f_0 + G_{26}$$
$$K_{12} = \frac{\pi\Gamma_7}{2} g_0 - \frac{\Gamma_7}{2} f_0$$

We can obtain $B_3(\tau)$ by equating terms of order ϵ^4 in (3.3.4) and invoking the solvability condition on the resulting differential system. If $B_3(\tau)$ is to be a periodic function of τ we find that T_4 is of the form

$$T_4 = \alpha^2 T_{40} + T_{42} \tag{3.3.29}$$

where
$$a^2 T_{40} = \frac{\displaystyle\int_0^1 \Big\{ -f_0^+\big[a^2 T_0(\chi_1 g_6 + \chi_2[g_2 + \Gamma_3 g_0] + \chi_3[g_1 + \Gamma_1 g_0]) + \Gamma N f_6\big] - g_0^+\big[f_6 + \phi_2[f_2 + \Gamma_3 f_0] + \phi_3[f_1 + \Gamma_1 f_0] - \Gamma g_6\big]\Big\} d\rho}{\displaystyle 2\int_0^1 \chi_0 f_0^+ g_0 \, d\rho}$$

and
$$a^2 T_{42} = \frac{\displaystyle\int_0^1 \Big\{ -f_0^+\big[a^2 T_0(\chi_1 g_9 + \chi_2[g_5 + \Gamma_6 g_0]) + \Gamma N f_9 + a^2 T_2 (2\chi_0 g_4 + \chi_1[g_1 + \Gamma_1 g_0])\big] - g_0^+\big[f_9 + \phi_2[f_5 + \Gamma_6 f_0] - \Gamma g_9\big] d\rho}{\displaystyle 2\int_0^1 \chi_0 f_0^+ g_0 \, d\rho}$$

(3.3.30)a,b

If the order ϵ^5, ϵ^6 differential systems are considered we find that T_5 is zero and T_6 is of the form

$$T_6 = \alpha^4 T_{60} + \alpha^2 T_{62} + T_{64} \tag{3.3.31}$$

and so it follows that we can write T in the form

$$\begin{aligned} T = T_0 + \epsilon^2 T_2 + \epsilon^4 [\alpha^2 T_{40} + T_{42}] \\ + \epsilon^6 [\alpha^4 T_{60} + \alpha^2 T_{62} + T_{64}] + O(\epsilon^8) \end{aligned} \tag{3.3.32}$$

3.4 The high frquency solution

We now consider the limit of σ tending to infinity with ϵ arbitrary . In this case the Stokes layer associated with the oscillatory motion of the inner cylinder is thin compared to the separation of the cylinders . If we let σ tend to infinity in (3.2.6) we can show that

$$V \sim 1 - \zeta + \frac{\epsilon}{2}\left\{ e^{-\sqrt{i\sigma}\,\zeta + i\tau} + e^{-\sqrt{-i\sigma}\,\zeta - i\tau} \right\} \tag{3.4.1}$$

We can see from above that the time dependent part of the basic flow is confined to a thin region near the inner cylinder . In contrast to this behaviour we shall seethat the disturbance velocity field has a time dependence throughout the fluid . Hence the disturbance velocity field must have a Stokes layer at the outer cylinder in order to satisfy the no-slip boundary condition there . As stated earlier we shall refer to these layers as the 'inner' and the 'outer' layers respectively . The region between these layers will be referred to as the 'central' region .

We first define the following new variables

$$\left.\begin{aligned} \zeta^* &= 1 - \zeta \\ \eta^* &= \zeta^*\sqrt{\tfrac{\sigma}{2}} \\ \eta &= \zeta\sqrt{\tfrac{\sigma}{2}} \end{aligned}\right\} \tag{3.4.2)a,b,c}$$

Thus η^*, η are Stokes layer variables for the inner and outer layers respectively . We now define (u ,v) , (U , V) , (u^*, v^*) to be the disturbance velocities in each region beginning with the inner layer . We can use (3.2.8) , (3.4.1) , (3.4.2) to show that the relevant differential equations to determine these function pairs in each region are given by

$$\left\{\frac{\partial^2}{\partial\eta^2} - \frac{2a^2}{\sigma} - 2\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^2} - \frac{2a^2}{\sigma}\right\}u = -\frac{4a^2T}{\sigma^2}\left\{1 - \eta\sqrt{\tfrac{2}{\sigma}} + \frac{\epsilon}{2}\left[e^{-\eta(1+i)+i\tau} + c.c.\right]\right\}v$$

$$\left\{\frac{\partial^2}{\partial\eta^2} - \frac{2a^2}{\sigma} - 2\frac{\partial}{\partial\tau}\right\}v = \frac{2}{\sigma}\left\{1 + \frac{\epsilon\sigma^{1/2}}{2\sqrt{2}}\left[(1+i)e^{-\eta(1+i)+i\tau} + c.c.\right]\right\}u \qquad (3.4.3)\text{a,b}$$

$$\mathcal{L}_1(U,V) = \sigma M\frac{\partial U}{\partial\tau}$$

$$\mathcal{L}_2(U,V) = -\sigma\frac{\partial V}{\partial\tau} \qquad (3.4.4)\text{a,b}$$

$$\left\{\frac{\partial^2}{\partial\eta^{*2}} - \frac{2a^2}{\sigma} - \frac{2\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^{*2}} - \frac{2a^2}{\sigma}\right\}u^* = -\frac{4a^2T\sqrt{2}\,\eta^*}{\sigma^{5/2}}V^*$$

$$\left\{\frac{\partial^2}{\partial\eta^{*2}} - \frac{2a^2}{\sigma} - 2\frac{\partial}{\partial\tau}\right\}V^* = \frac{2u^*}{\sigma} \qquad (3.4.5)\text{a,b}$$

where C. C denotes 'complex conjugate' and M , $\mathcal{L}_1$, $\mathcal{L}_2$ are as determined by (3.2.9)b , (3.3.8)a,b with T_0 replaced by T. We can see from (3.2.8) that the required boundary conditions are

$$u = v = \frac{\partial u}{\partial\eta} = 0 \quad , \eta = 0$$

$$u^* = v^* = \frac{\partial u^*}{\partial\eta^*} = 0 \quad , \eta^* = 0 \qquad (3.4.6)\text{a,b}$$

and we also stipulate that the perturbation velocities must ' match' where different regions meet .

We now expand the perturbation velocities in each region in Fourier series in time . This is possible since we are seeking solutions which are periodic in τ .Thus we write

$$\left.\begin{aligned} u &= u_s + \frac{1}{2}\sum_{n=1}^{\infty}\left(u_n e^{in\tau} + \tilde{u}_n e^{-in\tau}\right) \\ v &= v_s + \frac{1}{2}\sum_{n=1}^{\infty}\left(v_n e^{in\tau} + \tilde{v}_n e^{-in\tau}\right) \end{aligned}\right\} \qquad (3.4.7)a,b$$

$$\left.\begin{aligned} U &= U_s + \frac{1}{2}\sum_{n=1}^{\infty}\left(U_n e^{in\tau} + \tilde{U}_n e^{-in\tau}\right) \\ V &= V_s + \frac{1}{2}\sum_{n=1}^{\infty}\left(V_n e^{in\tau} + \tilde{V}_n e^{-in\tau}\right) \end{aligned}\right\} \qquad (3.4.8)a,b$$

$$\left.\begin{aligned} u^* &= U_s + \frac{1}{2}\sum_{n=1}^{\infty}\left(u_n^* e^{in\tau} + \tilde{u}_n^* e^{-in\tau}\right) \\ v^* &= V_s + \frac{1}{2}\sum_{n=1}^{\infty}\left(v_n^* e^{in\tau} + \tilde{v}_n^* e^{-in\tau}\right) \end{aligned}\right\} \qquad (3.4.9)a,b$$

where $\sim$ denotes 'complex conjugate' . The expression (u_s , v_s) represents the steady part of the disturbance velocity in the inner layer . In the other two layers we can denote the steady part of the perturbation velocity by the same expression (U_s , V_s) This is because there is no Stokes layer type of behaviour for the steady part of the perturbation velocity in the outer layer . However , in the inner layer this is not the case since the interaction of the basic flow and the disturbance causes the steady part of the disturbance velocity to have terms proportional to decaying exponentials . Thus it is necessary to distinguish between the steady parts of the perturbation velocity in the inner layer and away from it .

We consider first the outer layer and if we substitute for u , v from (3.4.7)a,b into (3.4.5)a,b then equating terms proportional to $e^{in\tau}$ we obtain

$$\left.\begin{aligned} \left\{D_*^2 - \frac{2a^2}{\sigma} - 2in\right\}\left\{D_*^2 - \frac{2a^2}{\sigma}\right\} u_n^* &= -\frac{4a^2 T_1^* \sqrt{2}\, v_n^*}{\sigma^{5/2}} \\ \left\{D_*^2 - \frac{2a^2}{\sigma} - 2in\right\} v_n^* &= \frac{2\, u_n^*}{\sigma} \end{aligned}\right\} \qquad (3.4.10)a,b$$

where $D_* \equiv \dfrac{d}{d\eta}$

and from (3.4.6)b it follows that the relevant boundary conditions are

$$u_n^* = v_n^* = \mathcal{D}_* u_n^* = 0 \quad , \eta^* = 0 \qquad (3.4.11)$$

We seek a solution of the system specified by (3.4.10) , (3.4.11) by expanding u_n^*, v_n^*, T in the form

$$\left.\begin{aligned} u_n^* &= \mathcal{V}_n(\sigma)\left\{u_n^{0*} + \frac{u_n^{1*}}{\sigma^{1/2}} + \frac{u_n^{2*}}{\sigma} + \cdots\right\} \\ v_n^* &= \mathcal{V}_n(\sigma)\left\{v_n^{0*} + \frac{v_n^{1*}}{\sigma^{1/2}} + \frac{v_n^{2*}}{\sigma} + \cdots\right\} \\ T &= T_0 + \frac{T_1}{\sigma^{1/2}} + \frac{T_2}{\sigma} + \cdots \end{aligned}\right\} \qquad (3.4.12)a,b,c$$

where $\mathcal{V}_n(\sigma)$ is for the moment an arbitrary function of σ . If we substitute the above expansions into (3.4.10) , (3.4.11) and equate terms of order σ^0 after dividing throughout by $\mathcal{V}_n(\sigma)$ we obtain

$$\{\mathcal{D}_*^2 - 2in\}\mathcal{D}_*^2 u_n^{0*} = \{\mathcal{D}_*^2 - 2in\} v_n^{0*} = 0$$

$$u_n^{0*} = v_n^{0*} = \mathcal{D}_* u_n^{0*} = 0 \, , \eta^* = 0$$

which we can solve to give

$$\left.\begin{aligned} u_n^{0*} &= C_n^0\left\{e^{-\eta^* n^{1/2}(1+i)} + n^{1/2}(1+i)\eta^* - 1\right\} \\ v_n^{0*} &= 0 \end{aligned}\right\} \qquad (3.4.13)a,b$$

where C_n^0 is an arbitrary constant and exponentially increasing functions of η^* have been rejected . If we equate terms of order $\sigma^{-1/2}$ after performing the substitutions described above we obtain a differential system for u_n^{1*}, v_n^{1*} which we solve to give

$$\left.\begin{aligned} u_n^{1*} &= C_n^1\left\{e^{-\eta^* n^{1/2}(1+i)} + n^{1/2}(1+i)\eta^* - 1\right\} \\ v_n^{1*} &= 0 \end{aligned}\right\} \qquad (3.4.14)a,b$$

where C_n^1 is another arbitrary constant . Similarly we can equate terms of order σ^{-1} and use (3.4.13) to obtain a differential system for u_n^{2*} , v_n^{2*} which we can solve to give

$$\left.\begin{aligned} u_n^{2*} &= C_n^2\left\{e^{-\eta^* n^{1/2}(1+i)} + n^{1/2}(1+i)\eta^* - 1\right\} \\ &\quad + a^2 C_n^0\left\{\frac{-\eta^* e^{-\eta^* n^{1/2}(1+i)}}{n^{1/2}(1+i)} + \frac{\eta^*}{n^{1/2}(1+i)} - \eta^{*2} + \frac{n^{1/2}(1+i)\eta^{*3}}{3}\right\} \\ v_n^{2*} &= C_n^0\left\{\frac{-\eta^* e^{-\eta^* n^{1/2}(1+i)}}{n^{1/2}(1+i)} - \frac{e^{-\eta^* n^{1/2}(1+i)}}{in} + \frac{1}{in} - \frac{(1-i)\eta^*}{n^{1/2}}\right\} \end{aligned}\right\} \qquad (3.4.15)\text{a,b}$$

where C_n^2 is yet another arbitrary constant. We can continue in this way to determine any number of terms in the expansions of u_n^*, v_n^*. If we then write $\eta^* = \zeta^*(\sigma/2)^{\frac{1}{2}}$ we can show that u_n^*, v_n^* have the following asymptotic forms at the edge of the outer layer

$$\left.\begin{aligned} u_n^* &\sim \mathcal{V}_n \sigma^{1/2}\left\{\frac{C_n^0}{a}\sqrt{\frac{n}{2}}(1+i)\sinh a\zeta^* + \sigma^{-1/2}\left[-C_n^0\cosh a\zeta^* + \frac{C_n^1}{a}\sqrt{\frac{n}{2}}(1+i)\sinh a\zeta^*\right] + O(\sigma^{-1})\right\} \\ v_n^* &\sim \frac{-u_n^*}{in\sigma}\left\{1 + O(\sigma^{-1})\right\} \end{aligned}\right\} \qquad (3.4.16)\text{a,b}$$

(To help understand why the hyperbolic functions appear above it is helpful to know that we can show inductively that the terms $-C_n^0$, $C_n^0 n^{1/2}(1+i)\eta^*$ in u_n^{0*} lead to the terms $\dfrac{-C_n^0(a^2\eta^{*2}2)^m}{2m!}$, $\dfrac{C_n^0(a^2\eta^{*2}2)^m a\eta^*(1+i)n^{1/2}}{a\,(2m+1)!}$ in u_n^{2m*} and the terms $+\dfrac{C_n^0(a^2\eta^{*2}2)^m}{i\sigma n 2m!}$, $\dfrac{-C_n^0(a^2\eta^{*2}2)^m a\eta^*(1-i)}{a\sigma n^{1/2}(2m+1)!}$ in v_n^{2m*} respectively .)

We next consider the 'central' region and if we substitute for U, V from (3.4.8)a,b into (3.4.4)a,b and equate terms proportional to $e^{in\tau}$ we obtain

$$\left.\begin{aligned} N^2U_n - in\sigma N U_n &= -a^2 T X_0 V_n \\ U_n - NV_n &= -in\sigma V_n \end{aligned}\right\} \qquad (3.4.17)\text{a,b}$$

where N is as defined by (3.3.8)c . We now expand U_n, V_n in the form

$$\left.\begin{aligned} U_n &= \mu_n(\sigma)\left\{U_n^0 + \frac{U_n^1}{\sigma^{1/2}} + \cdots\right\} \\ V_n &= \mu_n(\sigma)\left\{V_n^0 + \frac{V_n^1}{\sigma^{1/2}} + \cdots\right\} \end{aligned}\right\} \qquad (3.4.18)\text{a,b}$$

where $\mu_n(\sigma)$ is for the moment an arbitary function of σ . Taking T as given by (3.4.12)c and substituting for U_n , V_n from above into (3.4.17) we can equate terms of equal order in $\sigma^{-1/2}$ to give differential equations which are easily solved to give

$$\left.\begin{aligned} U_n &= \mu_n\left\{A_n^0\cosh a\rho^* + B_n^0\sinh a\rho^* + \sigma^{-1/2}\left[A_n^1\cosh a\rho^* + B_n^1\sinh a\rho^*\right] + O(\sigma^{-1/2})\right\} \\ V_n &= \frac{-\mu_n}{in\sigma}U_n\left\{1+O(\sigma^{-1})\right\} \end{aligned}\right\} \quad (3.4.19)a,b$$

where A_n^0 , B_n^0 , etc. are arbitary constants . It follows immediately from (3.4.16) , (3.4.19) that if the perturbation velocities in the outer layer and the central region are to match where these regions meet then we require that

$$\mu_n = \sigma^{1/2}\mathcal{V}_n$$

$$A_n^0 = 0$$

$$B_n^0 = \frac{C_n^0(1+i)n^{1/2}}{2^{1/2}a}$$

$$A_n^1 = -C_n^0$$

$$B_n^1 = \frac{C_n^1(1+i)n^{1/2}}{2^{1/2}a}$$

Hence U_n , V_n in the central region may be written in the form

$$\left.\begin{aligned} U_n &= \mu_n\left\{B_n^0\sinh a\rho^* + \sigma^{-1/2}\left[B_n^1\sinh a\rho^* - \frac{B_n^0 a\sqrt{2}\cosh a\rho^*}{(1+i)n^{1/2}}\right]\right\} \\ V_n &= \frac{-\mu_n}{in\sigma}U_n\left\{1+O(\sigma^{-1})\right\} \end{aligned}\right\} \quad (3.4.20)$$

It now remains for us to consider the steady part of the perturbation velocity away from the inner layer . We can show from (3.4.4) , (3.4.5) , (3.4.6) , (3.4.8) , (3.4.9) that (U_s , V_s) is in fact determined by

$$\left.\begin{aligned} &N^2 U_s + a^2 T \chi_0 V_s = 0 \\ &U_s - N V_s = 0 \\ &U_s = V_s = \frac{dU_s}{d\zeta} = 0 \ , \ \zeta = 1 \end{aligned}\right\} \qquad (3.4.21)$$

which we solve by taking T as in (3.4.12)c and expanding U_s , V_s in the form

$$\left.\begin{aligned} U_s &= \gamma(\sigma)\left\{ U_s^0 + \frac{U_s^1}{\sigma^{1/2}} + \cdots \right\} \\ V_s &= \gamma(\sigma)\left\{ V_s^0 + \frac{V_s^1}{\sigma^{1/2}} + \cdots \right\} \end{aligned}\right\} \qquad (3.4.22)\text{a,}$$

where $\gamma(\sigma)$ is for the moment another arbitary function of σ . Substituting for T and U_s , V_s from above into (3.4.21) and equating terms of order σ^0 , $\sigma^{-1/2}$, etc. after dividing by $\gamma(\sigma)$ throughout we can show that

$$\left.\begin{aligned} &N^2 U_s^0 + a^2 T_0 \chi_0 V_s^0 = 0 \\ &U_s^0 - N V_s^0 = 0 \\ &U_s^0 = V_s^0 = \frac{dU_s^0}{d\zeta} = 0 \ , \ \zeta = 1 \end{aligned}\right\} \qquad (3.4.23)$$

and for $k \geq 1$

$$\left.\begin{aligned} &N^2 U_s^k + a^2 T_0 \chi_0 V_s^k = -\sum_{r=0}^{k-1} a^2 \chi_0 V_s^r T_{k-r} \\ &U_s^k - N V_s^k = 0 \\ &U_s^k = V_s^k = \frac{dU_s^k}{d\zeta} = 0 \ , \ \zeta = 1 \end{aligned}\right\} \qquad (3.4.24)$$

We must now solve for the perturbation velocity in the inner layer and ensure that it matches with the perturbation velocity in the central region where these regions meet . Substituting for u , v from (3.4.7) into (3.4.3) and then equating terms proportional to $e^{in\tau}$, n =0,1,2,3,4,5........., we obtain

$$\left.\begin{aligned}\left\{D^2-\frac{2a^2}{\sigma}\right\}^2 u_s &= -\frac{4a^2T}{\sigma^2}\left\{v_s\left(1-\eta\sqrt{\tfrac{2}{\sigma}}\right)+\frac{\epsilon}{4}\left(\tilde{v}_1 e^{-\eta(1+i)}+c.c.\right)\right\}\\ \left\{D^2-\frac{2a^2}{\sigma}\right\}v_s &= \frac{2}{\sigma}\left\{u_s+\frac{\epsilon}{4}\sqrt{\frac{\sigma}{2}}\left(\tilde{u}_1(1+i)e^{-\eta(1+i)}+c.c.\right)\right\}\end{aligned}\right\} \qquad (3.4.25)\text{a,b}$$

$$\left.\begin{aligned}&\left\{D^2-\frac{2a^2}{\sigma}-2i\right\}\left\{D^2-\frac{2a^2}{\sigma}\right\}u_1\\ &\qquad= -\frac{4a^2T}{\sigma^2}\left\{v_1\left(1-\eta\sqrt{\tfrac{2}{\sigma}}\right)+\frac{\epsilon}{4}\left(v_s e^{-\eta(1+i)}+\frac{v_2}{2}e^{-\eta(1-i)}\right)\right\}\\ &\left\{D^2-\frac{2a^2}{\sigma}-2i\right\}v_1 =\\ &\qquad \frac{2}{\sigma}\left\{u_1+\epsilon\sqrt{\frac{\sigma}{2}}\left(u_s e^{-\eta(1+i)}+\frac{u_2}{2}(1-i)e^{-\eta(1-i)}\right)\right\}\end{aligned}\right\} \qquad (3.4.26)\text{a,b}$$

and for n$\geqslant$2

$$\left.\begin{aligned}&\left\{D^2-\frac{2a^2}{\sigma}-2in\right\}\left\{D^2-\frac{2a^2}{\sigma}\right\}u_n = -\frac{4a^2T}{\sigma^2}\left\{v_n\left(1-\eta\sqrt{\tfrac{2}{\sigma}}\right)\right.\\ &\qquad\qquad\left.+\epsilon\left(v_s e^{-\eta(1+i)}+\frac{v_2}{2}e^{-\eta(1-i)}\right)\right\}\\ &\left\{D^2-\frac{2a^2}{\sigma}-2in\right\}v_n = \frac{2}{\sigma}\left\{u_n+\frac{\epsilon}{2}\sqrt{\frac{\sigma}{2}}\left(u_{n-1}(1+i)e^{-\eta(1+i)}\right.\right.\\ &\qquad\qquad\left.\left.+u_{n+1}(1-i)e^{-\eta(1-i)}\right)\right\}\end{aligned}\right\} \qquad (3.4.27)\text{a,b}$$

where $D = d/d\eta$

In order to find the relevant scaling for (u_s , v_s) in the above equations we consider the basic eigenfunction pair (f_0, g_0) , determined by (3.3.7) , which represents the steady perturbation velocity for the problem with ϵ = 0 . We can show from (3.3.7) that near ζ = 0

$$\left.\begin{aligned} f_0 &\sim \zeta^2 \sim \sigma^{-1}\\ g_0 &\sim \zeta \sim \sigma^{-1/2}\end{aligned}\right\} \qquad (3.4.28)\text{a,b}$$

Since we are seeking a solution which is in some sense a perturbation from the problem with zero ϵ , we assume that the correct scaling for u_s , v_s follows from (3.4.28) . Hence we have that

$$u_s \sim \sigma^{-1}$$

$$v_s \sim \sigma^{-1/2}$$

and the above scaling for (u_s , v_s) , together with (3.4.25) , (3.4.26) , (3.4.27) , suggests the following scalings for u_1 , v_1 , u_2 , etc.

$$u_{2n-1} \sim \sigma^{-5n/2} \quad , u_{2n} \sim \sigma^{-(5n+2)/2}$$

$$v_{2n-1} \sim \sigma^{-(6n-2)/2} \quad , v_{2n} \sim \sigma^{-(5n+1)/2}$$

$$n = 1, 2, 3, \ldots$$

Hence we expand the above functions as follows

$$\left.\begin{aligned} u_s &= \sigma^{-1}\left\{u_s^0 + \frac{u_s^1}{\sigma^{1/2}} + \cdots\right\} \\ v_s &= \sigma^{-1/2}\left\{v_s^0 + \frac{v_s^1}{\sigma^{1/2}} + \cdots\right\} \\ u_1 &= \sigma^{-5/2}\left\{u_1^0 + \frac{u_1^1}{\sigma^{1/2}} + \cdots\right\} \\ v_1 &= \sigma^{-3/2}\left\{v_1^0 + \frac{v_1^1}{\sigma^{1/2}} + \cdots\right\} \\ &\text{etc} \end{aligned}\right\} \qquad (3.4.29)$$

If we now substitute the above expansions into (3.4.25) , (3.4.26) and take T as in (3.4.12)c we can show that

$$\left.\begin{aligned} &\left\{D^2 - \frac{2a^2}{\sigma}\right\}^2\left[u_s^0 + \frac{u_s^1}{\sigma^{1/2}} + \cdots\right] = -\frac{4a^2}{\sigma^{3/2}}\left\{T_0 + \frac{T_1}{\sigma^{1/2}} + \cdots\right\}\left\{\left(1 - \eta\sqrt{\tfrac{2}{\sigma}}\right)\left[v_s^0 + \cdots\right]\right. \\ &\qquad\qquad \left. + \frac{\epsilon}{4\sigma}\left(\left[\tilde{v}_1^0 + \frac{\tilde{v}_1^1}{\sigma^{1/2}} + \cdots\right]e^{-\eta(1+i)} + c.c.\right)\right\} \\ &\left\{D^2 - \frac{2a^2}{\sigma}\right\}\left[v_s^0 + \frac{v_s^1}{\sigma^{1/2}} + \cdots\right] = \frac{2}{\sigma^{3/2}}\left\{\left[u_s^0 + \frac{u_s^1}{\sigma^{1/2}} + \cdots\right] + \frac{\epsilon}{4\sigma\sqrt{2}}\left(\left[\tilde{u}_1^0 + \cdots\right](1+i)e^{-\eta(1+i)}\right.\right. \\ &\qquad\qquad \left.\left. + c.c\right)\right\} \end{aligned}\right\} \begin{matrix}(3.4.30)\\ \text{a, b}\end{matrix}$$

$$\left.\begin{aligned} &\left\{D^2 - \frac{2a^2}{\sigma} - 2i\right\}\left\{D^2 - 2i\right\}\left[u_1^0 + \frac{u_1^1}{\sigma^{1/2}} + \cdots\right] \\ &\quad = -4a^2\left\{T_0 + \frac{T_1}{\sigma^{1/2}} + \cdots\right\}\left\{\frac{\left(1 - \eta\sqrt{\frac{2}{\sigma}}\right)}{\sigma}\left[v_1^0 + \frac{v_1^1}{\sigma^{1/2}} + \cdots\right] + \epsilon\left[v_s^0 + \cdots\right]e^{-\eta(1+i)}\right. \\ &\qquad\qquad \left. + \frac{\epsilon}{2\sigma^{3/2}}\left[v_2^0 + \frac{v_2^1}{\sigma^{1/2}} + \cdots\right]e^{-\eta(1-i)}\right\} \\ &\left\{D^2 - \frac{2a^2}{\sigma} - 2i\right\}\left[v_1^0 + \frac{v_1^1}{\sigma^{1/2}} + \cdots\right] \\ &\quad = 2\left\{\frac{\left[u_1^0 + \cdots\right]}{\sigma^2} + \frac{\epsilon}{\sqrt{2}}(1+i)\left[u_s^0 + \cdots\right]e^{-\eta(1+i)} + \frac{\epsilon}{\sqrt{2}\sigma^3}\left[u_2^0 + \cdots\right](1-i)e^{-\eta(1-i)}\right\} \end{aligned}\right\} \begin{matrix}(3.4.31)\\ \text{a, b}\end{matrix}$$

It follows from (3.4.6)a , (3.4.7) , (3.4.29) that the relevant boundary conditions are

$$u_s^i = u_n^i = v_s^i = v_n^i = \mathcal{D}u_s^i = \mathcal{D}u_n^i = 0,\ \zeta = 0,\ n = 1,2,\ldots,\ i = 0,1,2\ldots \qquad (3.4.32)$$

We can see from (3.4.30) , (3.4.32) that the first five (u_s^i, v_s^i) can be determined without any knowledge of the expansion of (u_1 , v_1) Having calculated these terms we can see from (3.4.31) , (3.4.32) that the first five terms in the expansion of (u_1, v_1) can then be determined . We can then return to (3.4.30) and calculate the next five (u_s^i , v_s^i) and , if we write down the equations for (u_2 , v_2) , we find that we can also calculate the first five (u_2^i , v_2^i) . Continuing in this way we can calculate any number of terms in the expansions of (u_s , v_s) , (u_1, v_1) , (u_2, v_2) , etc. We first equate terms of order σ^0, $\sigma^{-1/2}$, σ^{-1}, $\sigma^{-3/2}$, σ^{-2} in (3.4.30) and solve the resulting differential equations subject to (3.4.32) . We obtain

$$u_s^0 = B_0\eta^2$$
$$u_s^1 = B_1\eta^2 + A_0\eta^3$$
$$u_s^2 = B_2\eta^2 + A_1\eta^3 + \frac{a^2B_0}{3}\eta^4$$
$$u_s^3 = B_3\eta^2 + A_2\eta^3 + \frac{a^2B_1}{3}\eta^4 + \frac{a^2}{5}\left(A_0 - \frac{C_0T_0}{6}\right)\eta^5$$
$$u_s^4 = B_4\eta^2 + A_3\eta^3 + \frac{a^2B_2}{3}\eta^4 + \frac{a^2}{5}\left(A_1 - \frac{C_0T_1}{6} - \frac{C_1T_0}{6}\right)\eta^5 + \frac{a^2}{30}\left(a^2B_0 + \frac{\sqrt{2}C_0T_0}{3}\right)\eta^6$$

$$v_s^0 = C_0\eta$$
$$v_s^1 = C_1\eta$$
$$v_s^2 = C_2\eta + \frac{a^2C_0}{3}\eta^3$$
$$v_s^3 = C_3\eta + \frac{a^2C_1}{3}\eta^3 + \frac{B_0}{6}\eta^4$$
$$v_s^4 = C_4\eta + \frac{a^2C_2}{3}\eta^3 + \frac{B_1}{6}\eta^4 + \frac{1}{10}\left(A_0 + \frac{a^4C_0}{3}\right)\eta^5$$

where A_0, B_0, etc. are arbitrary constants and an arbitrary constant in u_s^0 has been set equal to zero . This is not strictly neccessary at this stage and has been done to save a great deal of tedious algebra . The reason for this choice will become apparent later .

Having determined the first five (u_s^i , v_s^i) we can equate terms of order σ^0, $\sigma^{-1/2}$, σ^{-1}, $\sigma^{-3/2}$, σ^{-2} in (3.4.31) and solve the resulting differential equations successively subject to (3.4.32) . The first few terms in the expansions of u_1, v_1 are found to be as follows

$$u_1^0 = P_0\left\{e^{-\eta(1+i)} - 1 + \eta(1+i)\right\} - \frac{a^2\epsilon C_0 T_0}{8}\left\{2(1+i)\eta^2 e^{-\eta(1+i)} + 10\eta e^{-\eta(1+i)} + 5[1-i](e^{-\eta(1+i)} - 1)\right\}$$

$$u_1^1 = P_1\left\{e^{-\eta(1+i)} - 1 + \eta(1+i)\right\} - \frac{a^2\epsilon (C_0 T_1 + C_1 T_0)}{8}\left\{2(1+i)\eta^2 e^{-\eta(1+i)} + 10\eta e^{-\eta(1+i)} + 5[1-i](e^{-\eta(1+i)} - 1)\right\}$$

$$v_1^0 = \frac{-\sqrt{2}\epsilon B_0 e^{-\eta(1+i)}}{24}\left\{4\eta^3 + 3(1-i)\eta^2 - 3i\eta\right\}$$

$$v_1^1 = \frac{-\sqrt{2}\epsilon B_1 e^{-\eta(1+i)}}{24}\left\{4\eta^3 + 3(1-i)\eta^2 - 3\eta\right\} - \frac{\sqrt{2}\epsilon A_0 e^{-\eta(1+i)}}{32}\left\{4\eta^4 + 4(1-i)\eta^3 - 6\eta^2 + 3(1+i)\eta\right\}$$

where P_0 , P_1 , P_2 are arbitrary constants and exponentially increasing functions of η have been rejected . We notice immediately from above that , in contrast to u_1^0 , u_1^1 , the terms v_1^0 , v_1^1 consist solely of exponentially decaying terms , thus giving no contribution to the velocity field at the edge of the inner layer . However the appearance of the terms u_1^0, u_1^1 etc. on the right hand side of (3.4.31)b ensures that this not always the case for higher order terms in the expansion of v_1 . Having calculated the first five (u_1^i , v_1^i) we can calculate more of the (u_s^i , v_s^i) and the first few of the (u_2^i, v_2^i) . This enables us to calculate more of the (u_1^i , v_1^i) and so on . If we then write $\eta = \zeta(\sigma/2)^{\frac{1}{2}}$ we can show that u_1 and v_1 have the following asymptotic forms at the edge of the inner layer

$$\mu_1 \sim \sigma^{-2}\left\{\frac{P_0(1+i)\sinh a\zeta}{a\sqrt{2}} + \sigma^{-1/2}\left[\frac{5a^2\epsilon C_0T_0\cosh a\zeta}{4(1+i)} - P_0\cosh a\zeta + \frac{P_1(1+i)\sinh a\zeta}{a\sqrt{2}}\right] + \sigma^{-1}\left[\frac{P_2(1+i)\sinh a\zeta}{a\sqrt{2}} - P_1\cosh a\zeta + 5a^2\epsilon(C_0T_1 + C_1T_0\cosh a\zeta)\right) + O(\sigma^{-3/2})\right\}$$

$$V_1 \sim \frac{-\mu_1}{i\sigma}\left\{1 + O(\sigma^{-3/2})\right\} \qquad (3.4.33)\text{a,b}$$

We must now match (3.4.33)a,b with the expressions (3.4.20)a,b (with n = 1) at theedge of the inner layer . Clearly we must choose $\mu_1 = \sigma^{-5/2}$ and $P_0 = 0$. The first non-zero terms in the these asymptotic forms will then match if we choose B_1^0, P_1 as follows

$$\left.\begin{aligned} B_1^0 &= \frac{5a^2\epsilon C_0T_0}{4(1+i)\sinh a} \\ P_1 &= \frac{-\sqrt{2}\,5a^3\epsilon C_0T_0\coth a}{8i} \end{aligned}\right\} \qquad (3.4.34)\text{a,b}$$

Before matching the next terms we redefine C_1 as follows

$$C_1 = \frac{8P_1(1+i)}{5a^2\epsilon T_0} + C_1^*$$

C_1^* now being an arbitrary constant . The next terms in the expansions will then match if we choose P_2 , B_1^1 as follows

$$\left.\begin{aligned} B_1^1 &= \frac{5a^2\epsilon(C_0T_1 + C_1T_0)}{4(1+i)\sinh a} \\ P_2 &= \frac{-\sqrt{2}\,5a^3\epsilon[C_0T_1 + C_1^*T_0]\coth a}{8i} - \frac{a\sqrt{2}P_1\tanh a}{(1+i)} \end{aligned}\right\} \qquad (3.4.35)$$

and higher order terms can be matched by a similar procedure . Since we are primarily interested in calculating the first few terms of the expansion of the Taylor number , and the information already calculated will suffice for that purpose , we do not pursue the determination of (u_2, v_2) , etc. here. Taking μ_1 , B_1^0 as above we see that the dominant unsteady velocity in the central region is given by

$$\frac{5a^2\epsilon C_0T_0 \sinh a(1-\zeta)}{8\sinh a\ \sigma^{5/2}}\left(\cos\tau+\sin\tau,\ \frac{\cos\tau-\sin\tau}{\sigma}\right)$$

Having calculated $u_1^0, u_1^1, v_1^0, v_1^1$ we can equate terms of order $\sigma^{-5/2}$, σ^{-3} in (3.4.25)a,b and solve the resulting differential equations subject to (3.4.28) to give

$$u_s^5 = B_5\eta^2 + A_4\eta^3 + \frac{a^2B_3}{3}\eta^4 + \frac{a^2}{5}\left\{A_2 - \frac{[C_0T_2+C_1T_1+C_2T_0]}{6}\right\}\eta^5 + \frac{a^2}{30}\left\{a^2B_1 + \frac{\sqrt{2}[C_0T_1+C_1T_0]}{3}\right\}\eta^6$$
$$+\frac{a^4}{70}\left[A_0 - \frac{C_0T_0}{3}\right]\eta^7 + \frac{a^2\epsilon^2B_0T_0\sqrt{2}}{192}\left\{e^{-2\eta}\left[4\eta^3+27\eta^2+72\eta+75\right]+78\eta-75\right\}$$

$$u_s^6 = B_6\eta^2 + A_5\eta^3 + \frac{a^2B_4}{3}\eta^4 + \frac{a^2}{5}\left\{A_3 - \frac{[C_0T_3+C_1T_2+C_2T_1+C_3T_0]}{6}\right\}\eta^5 + \frac{a^2}{30}\left\{a^2B_2 + \frac{\sqrt{2}[C_0T_2+C_1T_1+C_2T_0]}{3}\right\}$$
$$+\frac{a^4}{70}\left\{A_1 - \frac{[C_0T_1+C_1T_0]}{3}\right\}\eta^7 + \frac{a^2B_0}{630}\left\{a^4 - \frac{T_0}{4}\right\}\eta^8$$
$$+\frac{a^2\epsilon^2A_0T_0\sqrt{2}}{256}\left\{e^{-2\eta}\left[4\eta^4+36\eta^3+144\eta^2+297\eta+264\right]+231\eta-264\right\}$$
$$+\frac{a^2\epsilon^2[B_0T_1+B_1T_0]\sqrt{2}}{192}\left\{e^{-2\eta}\left[4\eta^3+27\eta^2+72\eta+75\right]+78\eta-75\right\}$$

$$v_s^5 = C_5\eta + \frac{a^2C_3}{3}\eta^3 + \frac{B_2\eta^4}{6} + \frac{1}{10}\left\{A_1 + \frac{a^4C_1}{3}\right\}\eta^5 + \frac{a^2B_0}{30}\eta^6$$
$$-\frac{a^2\epsilon^2C_0T_0}{32}\left\{e^{-2\eta}\left[2\eta^2+9\eta+8\right]-10\cos\eta\, e^{-\eta}+2\right\}$$

$$v_s^6 = C_6\eta + \frac{a^2C_4}{3}\eta^3 + \frac{B_3}{6}\eta^4 + \frac{1}{10}\left\{A_2 + \frac{a^4C_0}{9} - \frac{C_0T_0}{9}\right\}\eta^7$$
$$-\frac{a^2\epsilon^2[C_0T_1+C_1T_0]}{32}\left\{e^{-2\eta}\left[2\eta^2+9\eta+8\right]-10\cos\eta\, e^{-\eta}+2\right\}$$
$$+\frac{5a^3\epsilon^2C_0T_0\coth a}{32}\left\{e^{-2\eta}+2e^{-\eta}\left[3\sin\eta-\cos\eta\right]e^{-\eta}-4\eta e^{-\eta}\cos\eta+1\right\}$$

If we calculate a few more terms in the expansion of (u_s , v_s) we can show that u_s and v_s have the following asymptotic forms at the edge of the inner layer

$$u_s \sim \sum_{i=0}^{7}\sigma^{-i/2}\left\{S_1(A_i,B_i,C_i,a,T_0,\zeta) + \text{terms proportional to } T_k,\ 0<k\leq i\right\}$$
$$+\sigma^{-3}\left\{\frac{\alpha_1}{\sqrt{2}}\left[\zeta - \frac{a^4\zeta^5}{120}\right] - \frac{a^2T_0\gamma_1\zeta^4}{24} + \frac{a^2\gamma_1\zeta^5}{120} + O(\zeta^6)\right\}$$
$$+\sigma^{-7/2}\left\{\frac{\alpha_2}{\sqrt{2}}\left[\zeta - \frac{a^4\zeta^5}{120}\right] - \frac{a^2T_0\gamma_2\zeta^4}{24} + \frac{a^2\gamma_2\zeta^5}{120} + \beta_1(1-\frac{a^2\zeta^4}{24}) + O(\zeta^6)\right\} + O(\sigma^{-4})$$

$$v_s \sim \sum_{i=0}^{7} \sigma^{-i/2}\left\{S_2(A_i,B_i,C_i,a,T_0,\zeta) + \text{terms prop. to } T_k,\ 0<k\le i\right\}$$
$$+\ \sigma^{-3}\left\{\frac{\alpha_1}{\sqrt{2}}\left[\zeta - \frac{a^4\zeta^5}{120}\right] - \frac{a^2 T_0 \gamma_1 \zeta^4}{120} + O(\zeta^6)\right\} + \sigma^{-7/2}\left\{\frac{\alpha_2}{\sqrt{2}}\left[\zeta - \frac{a^4\zeta^5}{120}\right]\right.$$
$$\left. + \frac{a^2\gamma_2\zeta^5}{120} - \frac{a^2 T_0\gamma_2\zeta^4}{24} + \beta_1\left(1-\frac{a^4\zeta^4}{24}\right) + O(\zeta^6)\right\} + O(\sigma^{-4}) \qquad (3.4.36)\text{a,b}$$

where

$$\left.\begin{aligned}
&\alpha_1 = \frac{a^2\epsilon^2 B_0 T_0\sqrt{2}\,13}{32}, \quad \alpha_2 = \frac{B_1\alpha_1}{B_0} + \frac{a^2\epsilon^2 A_0 T_0\sqrt{2}\,231}{256}\\
&\beta_1 = -\frac{a^2\epsilon^2 B_0 T_0\sqrt{2}\,26}{64}, \quad \gamma_1 = -\frac{a^2\epsilon^2 C_0 T_0\sqrt{2}}{16}\\
&\gamma_2 = \frac{C_1\mu_1}{C_0} + \frac{a^3\epsilon^2 C_0 T_0 \coth a\delta}{32}
\end{aligned}\right\} \qquad (3.4.37)\text{a,b,c,d,e}$$

and S_1, S_2 represent the following series

$$\left.\begin{aligned}
S_1 &= B_i\left\{\zeta^2 + \frac{a^2\zeta^4}{6} + \frac{a^4\zeta^6}{120} + \frac{\zeta^8}{5040}\left[a^6 - a^2 T_0\right]\right\} + \frac{a^2 C_i T_0}{120\sqrt{2}}\left[-\zeta^5 - \frac{a^2\zeta^7}{14}\right]\\
&\quad + \frac{A_i}{2\sqrt{2}}\left\{\zeta^3 + \frac{a^2\zeta^5}{10} + \frac{a^4\zeta^7}{280}\right\} + O(\zeta^9)\\
S_2 &= \frac{B_i}{24}\left\{\zeta^4 + \frac{a^2\zeta^6}{10}\right\} + \frac{C_i}{\sqrt{2}}\left\{\zeta + \frac{a^2\zeta^3}{6} + \frac{a^4\zeta^5}{120} + \frac{\zeta^7}{5040}\left[a^6 - a^2 T_0\right]\right\}\\
&\quad + \frac{A_i}{40\sqrt{2}}\left\{\zeta^5 + \frac{a^2\zeta^7}{14}\right\} + O(\zeta^8)
\end{aligned}\right\} \qquad (3.4.38)$$

It now remains for us to determine the constants A_0, B_0, etc. such that the above asymptotic forms match with (3.4.22)a,b where the central region and the inner layer overlap. We first note that (f_0 , g_0), defined by (3.3.7), have the following form for small ζ

$$(f_0, g_0) = \left(S_1(A,B,C,a,T_0,\zeta),\ S_2(A,B,C,a,T_0,\zeta)\right) \qquad (3.4.39)$$

where A , B , C are given by

$$A = \frac{\sqrt{2}}{3} f_0'''(0), \quad B = f_0''(0), \quad C = \sqrt{2}\, g_0'(0) \qquad (3.4.40)\text{a,b,c}$$

where a dash denotes a derivative with respect to ζ and S_1 , S_2 are as defined above . Hence if we choose $\gamma = 1$ and (U_s^0, V_s^0) = (f_0 , g_0) then (3.4.23) is automatically satisfied and the first terms of (3.4.22)a,b and (3.4.36)a,b match if we choose

$A_0 = A$, $B_0 = B$, $C_0 = C$. Similarly if we put

$$T_i = 0$$
$$(U_s^i, V_s^i) = \frac{C_i}{C_0}(f_0, g_0) \ , i = 1, \ldots, 5$$

then the next five terms of (3.4.22)a,b and (3.4.36)a,b match at the edge of the inner layer . With the above choices for T_1, T_2, etc. we can see from (3.4.24) that (U_s^6 , V_s^6) is determined by

$$\left.\begin{array}{l} N^2 U_s^6 + a^2 T_0 \chi_0 V_s^6 = -a^2 T_6 \chi_0 g_0 \\ U_s^6 - N V_s^6 = 0 \end{array}\right\} \qquad (3.4.41)a,b$$

with boundary conditions

$$U_s^6 = V_s^6 = \frac{dU_s^6}{d\zeta} = 0, \zeta = 1 \qquad (3.4.42)$$

and if the order σ^{-3} terms in (3.4.22)a,b and (3.4.36)a,b are to match at the edge of the inner layer we require that

$$\left.\begin{array}{l} U_s^6 \sim S_1(A_6, B_6, C_6, a, T_0, \zeta) + \frac{\alpha_1}{\sqrt{2}}\left(\zeta - \frac{a^4\zeta^5}{120}\right) - \frac{a^2 T_0 \gamma_1 \zeta^4}{24} \\ \qquad + \frac{a^2 \gamma_1 \zeta^5}{120} + O(\zeta^6) \\ V_s^6 \sim S_2(A_6, B_6, C_6, a, T_0, \zeta) + \frac{\alpha_1}{6\sqrt{2}}\left(\zeta^3 + \frac{a^2\zeta^5}{20}\right) + \gamma\Big(1 \\ \qquad + \frac{a^2\zeta^2}{2} + \frac{a^4\zeta^4}{24}\Big) + O(\zeta^6) \end{array}\right\} \qquad (3.4.43)a,b$$

at the edge of the inner layer . However the above series are the small ζ series solutions of (3.4.41) with boundary conditions

$$U_s = 0 , \ \frac{dU_s^6}{d\zeta} = \frac{\alpha_1}{\sqrt{2}} \ , \ V_s^6 = \gamma_1 , \ \zeta = 0 \qquad (3.4.44)$$

Therefore if we consider (3.4.41) with boundary conditions (3.4.42) , (3.4.44) then the solution will automatically satisfy the requirements on (U_s^6 , V_s^6) away from the inner layer and for some A_6, B_6, C_6 will be of the form given by

(3.4.43) at the edge of this layer . Thus the problem reduces to solving the system specified by (3.4.41) , (3.4.42) , (3.4.44) . In fact , since we are only interested in finding T_6 , we merely use the condition that this system has a solution which gives

$$a^2 T_6 = \frac{\left\{ -\gamma_1 g_0'^{+}(0) - \frac{\alpha_1}{\sqrt{2}} f_0''^{+}(0) \right\}}{\int_0^1 \chi_0 f_0^+ g_0 \, d\xi}$$

where (f_0^+, g_0^+) is the adjoint function pair defined by (3.3.10) . Using (3.4.37) , (3.4.40) we can show that the above expression can be written in the form

$$T_6 = \epsilon^2 T_0 \frac{\left\{ 13 f_0''(0) f_0''^{+}(0) - 4 g_0'(0) g_0'^{+}(0) \right\}}{32 \int_0^1 \chi_0 f_0^+ g_0 \, d\xi} \tag{3.4.45}$$

and a similar procedure for the order $\sigma^{-7/2}$ terms in the expressions (3.4.22)a,b , (3.4.36)a,b shows that

$$T_7 = \epsilon^2 T_0 \frac{\left\{ 40 a \cot a \, g_0'(0) g_0'^{+}(0) + 100 f_0''(0) f_0'''^{+}(0) + 77 f_0'''(0) f_0''^{+}(0) \right\}}{256 \int_0^1 \chi_0 f_0^+ g_0 \, d\xi} \tag{3.4.46}$$

and T_8 , T_9 , etc. can be obtained by a similar procedure if higher order terms in the expansion of the perturbation velocity are considered . However we have seen that to order σ^{-4} T may be written in the form

$$T = T_0 + T_6 \sigma^{-3} + T_7 \sigma^{-7/2} + O(\sigma^{-4}) \tag{3.4.47}$$

where T_6 and T_7 are determined by (3.4.45) and (3.4.46) respectively and T_0 is the Taylor number for the steady problem with $\epsilon = 0$.

3.5 The numerical work

If we wish to obtain the critical Taylor number , T_c, associated with (3.3.25) we must take into account the variation of a with ϵ near its critical value for the problem with zero ϵ . A calculation similar to the one given by Venezian (1969) shows that if this effect is taken into account then T_c is given by

$$T_c = T_0^c + \epsilon^2 T_2^c + \epsilon^4\left[\alpha^2 T_{40}^c + T_{42}^c - \frac{\left(\frac{\partial T_2}{\partial a}\right)^2}{\left(2\frac{\partial^2 T_0}{\partial a^2}\right)}\right] + O(\epsilon^6) \tag{3.5.1}$$

where T_0^c is the critical value of T_0 for the problem with zero ϵ , and T_2^c, T_4^c, etc. , which are functions of a , T_0 denote T_2 , T_4 , etc. evaluated with $T_0 = T_0^c$ and a equal to its critical value , a^c, corresponding to $T_0 = T_0^c$.

Similarly if the variation of a with σ is taken into account we find that the critical Taylor number associated with (3.4.47) is given by

$$T_c = T_0^c + \frac{T_6^c}{\sigma^3} + \frac{T_7^c}{\sigma^{7/2}} + O(\sigma^{-4}) \tag{3.5.2}$$

where T_0^c is as defined above and T_6^c , T_7^c represent T_6 , T_7 evaluated with $T_0 = T_0^c$ and $a = a^c$.

All the computations were for the critical case and as a starting point we assumed the following well-known values for a , T_0^c ..

$$a = 3.1266, \quad T_0^c = 3389.9 \tag{3.5.3)a,b}$$

We then solved the ordinary differential systems (3.3.7),(3.3.10) by a fourth order Runge-Kutta scheme with 40 steps . The method of solution followed that described in detail by Eagles (1971) and so we do not describe it again here . The solutions obtained were in good agreement with the corresponding solutions obtained by Di Prima&Stuart (1972) when normalized in the same way . Having determined the function pairs(f_o, g_o) , (f_o^{+}, g_o^{+}) we used Simpson's rule to evaluate the integrals in (3.3.12) and we were able to show that

$$\gamma = -26.18 \tag{3.5.4}$$

We then solved the system (3.3.14) again by using the method described in detail for such systems by Eagles (1971) . We then evaluated (3.3.17) and obtained

$$T_2^c = -208.6 \tag{3.5.5}$$

We then solved the systems (3.3.22) for i = 2 , 6 and used the results to show that

$$T_{40} = 1.7 \tag{3.5.6}$$

This value is correct only to two significant figures wheras T_o^c is correct to four significant figures . If we now substitute for T_o^c , T_2^c , T_{40}^c from above into (3.5.1) and eliminate α using (3.3.2) we obtain

$$T_c = 3389.9 - 208.6\,\epsilon^2 + 1.7\,\epsilon^2\sigma^2 + O(\epsilon^4, \epsilon^2\sigma^4) \tag{3.5.7}$$

For details of the function pairs (f_o , g_o) , (f_o^{+} , g_o^{+}) and some of the more important higher order function pairs see the Tables at the end of this thesis .

The high frequency critical Taylor number requires only the knowledge of the basic eigenfunction pair together with the adjoint function pair . After evaluating the integrals appearing in (3.4.45) , (3.4.46) using Simpson's rule once again we found that T_c was expressible in the form

$$T_c = 3389.9\left\{1 - \frac{\epsilon^2}{\sigma^3}\left[4.898 - \frac{84.81}{\sigma^{1/2}}\right]10^4 + O(\sigma^{-4})\right\} \tag{3.5.8}$$

3.6 Discussion of results

We first discuss the nature of the velocity field associated with the disturbance imposed on the flow . In the low frequency limit the order ϵ^0 velocity is just the usual steady velocity multiplied by a function of τ , $B_0(\tau)$, this function being determined by (3.3.11) . Since Γ is in fact negative it follows that the order ϵ velocity has maximum value when τ has the following values

$$\tau_m = (4m+1)\pi/2 \quad , \quad m = 0,1,2 \ldots \tag{3.6.1}$$

This is surprising since we would expect that the maximum velocity would occur when the inner cylinder was moving with its maximum velocity rather than when it had its maximum deceleration as is suggested by (3.6.1) . A related result was found by Di Prima & Stuart (1972) when they investigated the global stability of the flow between eccentric rotating cylinders . Suppose we wish to determine the position in a cycle when the the Taylor vortex activity is a maximum when terms of order ϵ are taken into account . The above authors have suggested that a relevant physical quantity to consider might be the axial velocity component near the outer cylinder . However , since the latter is in fact zero at the point in question , a Taylor series expansion of the axial component near $\zeta = 1$ shows that the axial velocity near $\zeta = 1$ is proportional to its derivative with respect to ζ evaluated at $\zeta = 1$. It follows by considering the equation of continuity that this is equal to $-\left.\frac{\partial^2 u}{\partial \zeta^2}\right|_{\zeta=1}$. A calculation using (3.3.11) , (3.3.20) shows that , if terms upto order ϵ are taken into account, then this quantity has its maximum value when

$$\tau = \tau_m + \frac{\epsilon}{\Gamma}\left\{\alpha\left[f_1''(1)/f_0''(1) + \Gamma_1\right] + 2\Gamma_2\right\} + O(\epsilon^2)$$

and the computations show that this becomes

$$\tau = \tau_m + \epsilon\left[0.44\alpha - 0.03\right] + O(\epsilon^2) \qquad (3.6.2)$$

Thus we see that the position of maximum vortex activity in a cycle is before or after the position of maximum deceleration of the inner cylinder depending on whether or not , α , which we recall is proportional to the angular displacement of the inner cylinder , is greater than 14.7 . With $T \sim 3390$ this value of α in fact corresponds to an angular diplacement of the order $600(d/R_1)^{\frac{1}{2}}$

In the high frequency limit it is possible to isolate the steady and unsteady parts of the disturbance velocity by Fourier analysis . The unsteady flow is dominated by components with the same frequency as that of the basic flow . The presence of the Stoke layer at the outer cylinder means that the relevant measure of vortex activity is not nec~~c~~essarily the same as the one used above . However , if we use the one described above and just take the first term of the expansion of u into account together with the dominant steady component , f_0 , we find that $\left.\frac{\partial^2 u}{\partial \zeta^2}\right|_{\zeta=1}$ has its maximum value when the inner cylinder has its maximum velocity .

It is of interest to know that the quantities γ , T_2 which we derived in §3.3 are related to the coefficients in the expansion of T in powers of a small growth rate σ_0 for the problem with zero ϵ .Essentially the latter problem is to solve the following system

$$\left.\begin{aligned} &\mathcal{L}_1(u,v) = \sigma_0 N u \\ &\mathcal{L}_2(u,v) = \sigma_0 v \\ &u = u' = v = 0, \ \zeta = 0, 1 \end{aligned}\right\} \qquad (3.6.3)$$

where $\mathcal{L}_1, \mathcal{L}_2$, N are as defined (3.3.8)a,b,c . The above system can be solved by expanding u , v , T in the form

$u = u_0 + \sigma_0 u_1 + \cdots,$

$$T = T_0 + \sigma_0 T_1' + \dots$$

and the system can then be solved by using the method of §3.3 . The results of such a calculation show that

$$(u_0, v_0) = (f_0, g_0) , \quad (u_1, v_1) = -\frac{1}{\gamma}(\delta_1 + f_0, g_1) \qquad (3.6.4)\text{a,b}$$

$$T_1' = -2T_0/\gamma , \quad T_2'/(T_1')^2 = 2T_2/T_0^2 + 1/T_0 \qquad (3.6.5)\text{a,b}$$

where (f_0, g_0) , (f_1, g_1) , T_2, γ are as defined in §3.3 . The ratio T_1/T_0 was computed by Davey (1962) who found that

$$T_1'/T_0 = 1/13.1$$

wheras , taking γ as being -26.18 , we can infer from above that

$$T_1'/T_c = 1/13.09$$

Thus we have reasonably good agreement with Davey's work , the slight error possibly being due to the fact that he used a slightly different value for the Taylor number T_0. Unfortunately Davey did not compute T_2' in his work but a rough estimate of T_2' can be obtained by interpolation from Table 3 of Davey , Di Prima & Stuart (1968) . The resulting value agrees within 30 % of the value which we can obtain using (3.6.5)b .

We have seen that the critical Taylor number at which instability first occurs is given by (3.5.7) in the limit of ϵ and σ tending to zero . Thus we see that the dominant correction to T_c from its unmodulated value is negative . For fixed ϵ the critical Taylor number increases as σ increases from zero , but , unless σ is taken to be greater than ~ 12 , T_c will always be less than its unmodulated value . In Fig. 6 we have shown the variation of T_c with σ for fixed values of ϵ . We have also calculated the order ϵ^4, $\sigma^4\epsilon^2$ correction terms in (3.5.7) . These terms include corrections to T_c through a being dependent on ϵ .

We found that the order ϵ^4 term was -1303 and the order $\epsilon^2\sigma^4$ term was zero to two decimal places .

In the limit of σ tending to infinity with ϵ arbitrary we found that the critical Taylor number was given by (3.5.8) . Again it seems that the dominant effect of modulation is to destabilize the flow since the first correction term of T from its unmodulated value is negative . For fixed ϵ we see that T increases as σ decreases but unless σ is greater than ~ 20 T is less than its unmodulated value , assuming of course that the next term in the expansion is negligible for σ of this order of magnitude . We have shown the variation of T with σ for fixed ϵ in Fig.7 .

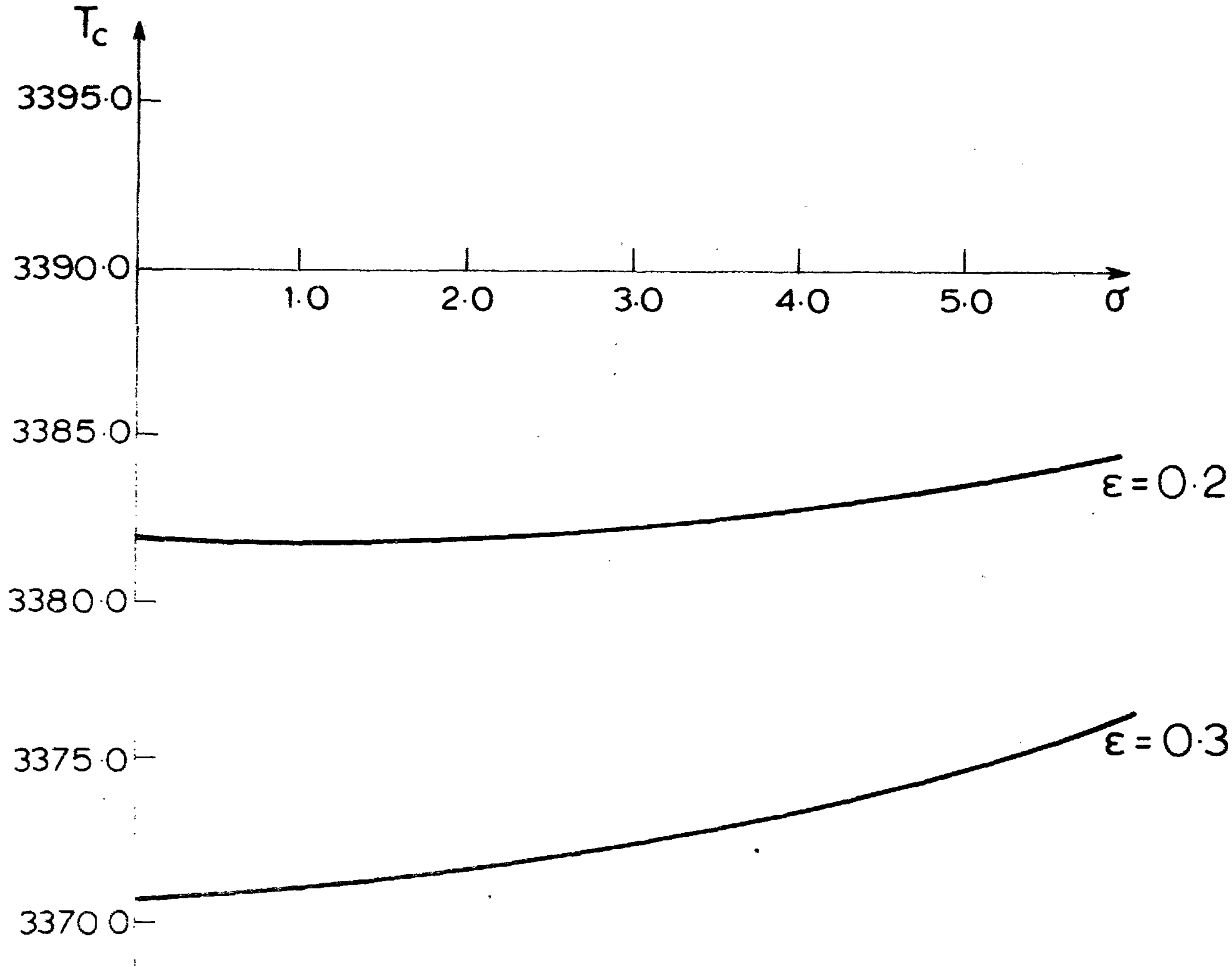

Fig **6**: The critical Taylor number as a function of σ in the low frequency limit.

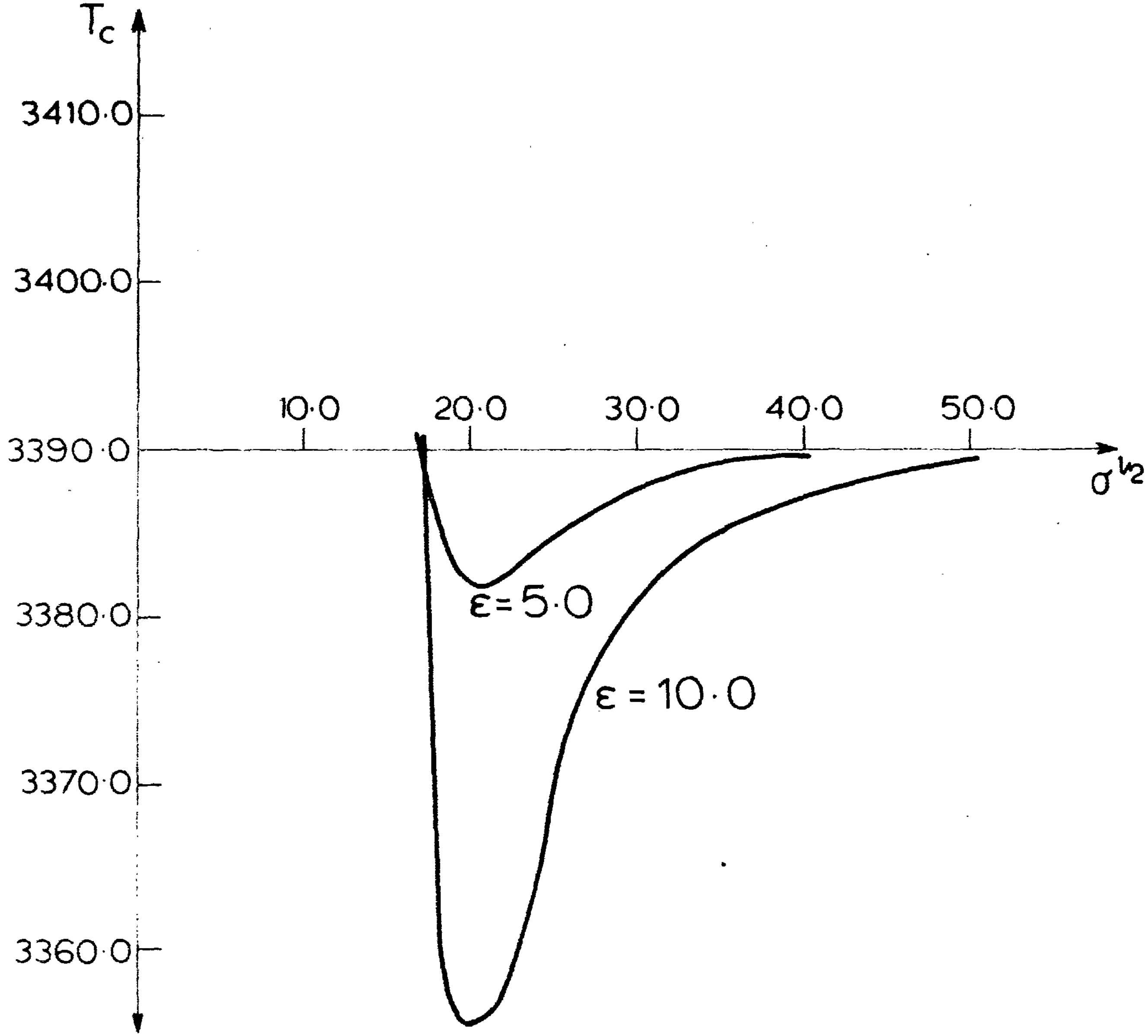

Fig 7. The critical Taylor number as a function of $\sigma^{1/2}$ in the high frequency limit.

CHAPTER 1V

THE NON-LINEAR STABILITY OF UNSTEADY CYLINDER FLOWS

4. Introduction

In view of the results of the previous chapter with regard to the work of Donnelly (1964) we now see if better agreement between theory and experiment can be obtained by taking non-linear effects into account . The procedure adopted in this chapter is as follows.

In §4.2 we formulate the non-linear differential system governing the stability of the flow. This system is just the system (3.2.8) of the previous chapter together with the non-linear terms neglected in the derivation of the latter.

In §4.3 we consider the low frequency limit and obtain a solution to the differential system by the method of multiple scales. We again balance the dominant time-dependences of the system by letting ϵ tend to zero with σ/ϵ fixed and equal to α say. The Taylor number is perturbed by an ammount of order ϵ from its critical value for the problem with zero ϵ. The perturbation velocity is then expanded in powers of ϵ and we find that the time-dependent amplitude of the leading Fourier mode satisfies the following differential equation.

$$\alpha \frac{dA}{d(\omega t)} = \frac{-\gamma}{2T_0}\left\{T_1 + 2T_0 \cos\omega t\right\}A + a_1 A^3 \qquad (4.1.1)$$

Here T_1 is the order ϵ alteration to the perturbed Taylor number and a_1, γ are negative constants. The non-linear effects are represented by the A^3 term above ,and, since a_1 is negative, it follows that the effect of the non-linear terms is to stabilize the flow. If we suppose that the speed of the inner cylinder is now given by $\Omega_1\{1+\epsilon f(\omega t)\}$,where $f(\omega t)$ is a slowly varying function of ωt, we can show that the corresponding amplitude equation is given by

$$\alpha \frac{dA}{d(\omega t)} = \frac{-\gamma}{2T_0}\left\{T_1 + 2T_0 f(\omega t)\right\}A + a_1 A^3 \qquad (4.1.2)$$

If we put f identically equal to zero above and rescale certain quantities appearing in this equation we can obtain the third-order truncated amplitude equation given by Davey (1962) ; , and T_1 is then proportional to the growth rate of linear theory. Moreover, if we define an instantaneous Taylor number , T_i , by

$$T_i = T\{1 + \epsilon f(\omega t)\}^2$$

then it follows that, since $T_1 + 2T_0 f$ is the order ϵ alteration of T_i from T_0 , Davey's amplitude equation may be regarded as being valid for this problem if we replace the Taylor number T used by Davey by T_i defined above.

In §4.4 we consider the nature of the solutions of (4.1.2) for various functions f(ωt). In particular we consider the case of f(ωt) being cos(ωt) and examine the possibility of A(ωt) being a periodic function of ωt . We find that such solutions exist if T_1 is positive.

In §4.5 we examine the limit of σ tending to infinity with ϵ arbitrary. We assume that the Taylor number is given by

$$T = T_0 + \sigma^{-3} T_6' + O(\sigma^{-7/2}) \qquad (4.1.3)$$

and it follows from the high frequency linear theory of the previous chapter that the flow is unstable to infinitesimally small disturbances if

$$T_6' > -48980\, T_0 \epsilon^2$$

when $f(\omega t) = \cos \omega t$.

In fact we consider only the case when f(ωt) = cos(ωt) so that we can use the method of §3.4 which requires that the perturbations are periodic in ωt. We find that to the order of magnitude in σ which we work the non-linear effects are only important through their effect on the steady part of the perturbation velocity. The latter is assumed to be of order $\sigma^{-3/2}$ in the central region and we find that the amplitude , A_0^0, of the leading steady Fourier mode is given by

$$\left(A_s^0\right)^2 = \frac{r}{2a_1}\left\{\frac{-T_6}{T_0} + \frac{T_6'}{T_0}\right\} \tag{4.1.4}$$

where T_6 is given by (3.4.45). Putting ϵ equal to zero in the above equation we find that , after eliminating T_6' using (4.1.3) , that $\overset{0}{A_s}$ is then the equilibrium amplitude solution of Davey's truncated third-order amplitude equation.

In §4.6 we discuss the relevance of this work to the experimental observations of Donnelly (1964) . We find that the low frequency calculation explains some of his results but our theory does not predict an optimum value σ for the enhancement of stability.

4.2 The basic flow and the disturbance equations

Using the notation of the previous chapter we recall that the basic dimensionless velocity, $(0, \overline{V}, 0)$, for the flow between concentric cylinders when the outer one is at rest and the inner one has angular velocity $\Omega_1\{1+\epsilon\cos\omega t\}$ is

$$\overline{V} = \left\{1 - \xi + \frac{\epsilon}{2}\left[\frac{\sinh\sqrt{i\sigma}(1-\xi)\, e^{i\tau}}{\sinh\sqrt{i\sigma}} + \text{COMPLEX CONJUGATE}\right]\right\} \tag{4.2.1}$$

Suppose that we again perturb the flow such that the disturbed state may be characterized by u , v + V , w , p/ρ + δp , where p is the basic pressure distribution , then if we rescale u , v , w as follows

$$u = -\nu u^*/2d \;, \quad v = \Omega_1 R_1 v^*/2 \;, \quad w = -\nu w^*/2d$$

and define the variable ϕ by

$$\phi = z/d \tag{4.2.2}$$

we can show that u^*, v^*, w^* satisfy the following equations

$$\left.\begin{aligned}
&\left\{\mathcal{L} - \sigma\frac{\partial}{\partial\tau}\right\}\mathcal{L}u^* = T\overline{V}\frac{\partial^2 v^*}{\partial\phi^2} - \frac{1}{2}\frac{\partial^2 Q_1}{\partial\phi^2} + \frac{1}{2}\frac{\partial^2 Q_2}{\partial\xi\,\partial\phi} \\
&\left\{\mathcal{L} - \sigma\frac{\partial}{\partial\tau}\right\}v^* = -\frac{\partial\overline{V}}{\partial\xi}u^* - \frac{1}{2}Q_3 \\
&\frac{\partial u^*}{\partial\xi} + \frac{\partial w^*}{\partial\phi} = 0
\end{aligned}\right\} \tag{4.2.3)a,b,c}$$

where $$\mathcal{L} \equiv \frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial \phi^2} \tag{4.2.4}$$

and
$$\left.\begin{aligned} Q_1 &= u^*\frac{\partial u^*}{\partial \zeta} + w^*\frac{\partial u^*}{\partial \phi} - \frac{T}{2}V^{*2} \\ Q_2 &= u^*\frac{\partial w^*}{\partial \zeta} + w^*\frac{\partial w^*}{\partial \phi} \\ Q_3 &= u^*\frac{\partial v^*}{\partial \zeta} + w^*\frac{\partial v^*}{\partial \phi} \end{aligned}\right\} \text{(4.2.5)a,b,c}$$

and T is the Taylor number defined by

$$T = 2\Omega_1^2 R_1 d^3/\nu^2$$

The boundary conditions are determined by there being no relative velocity at the boundaries. Thus we require that

$$u^* = v^* = w^* = 0, \quad \zeta = 0, 1 \tag{4.2.6}$$

Finally we note that the equations (4.2.3)a,b,c are the 'small gap' equations obtained from the full equations by letting d/R_1 tend to zero with the variables $\zeta, \tau, \phi, u^*, v^*, w^*$ and the parameters T, σ, ϵ held fixed.

4.3 The low frequency limit

If we replace $\bar{V}$ in (4.2.3)a,b,c by its asymptotic form for small σ and drop the star notation we obtain

$$\left.\begin{aligned} \left\{\mathcal{L} - \sigma\frac{\partial}{\partial \tau}\right\}\mathcal{L}u &= T\left\{\chi_0 + \epsilon\chi_1\cos\tau + \epsilon\sigma\chi_2\sin\tau \ldots\right\}\frac{\partial^2 v}{\partial \phi^2} \\ &\quad -\frac{1}{2}\frac{\partial^2 Q_1}{\partial \phi^2} + \frac{1}{2}\frac{\partial^2 Q_2}{\partial \phi \partial \zeta} \\ \left\{\mathcal{L} - \sigma\frac{\partial}{\partial \tau}\right\}v &= \left\{1 + \epsilon\cos\tau + \epsilon\sigma\phi_2\sin\tau + \ldots\right\}u - \frac{1}{2}Q_3 \\ \frac{\partial u}{\partial \zeta} + \frac{\partial w}{\partial \phi} &= 0 \end{aligned}\right\} \text{(4.3.1)a,b,c}$$

where the functions χ_i, ϕ_i are as defined by (3.3.1). The relevant boundary conditions are

$$u = v = w = 0, \quad \zeta = 0, 1 \tag{4.3.2}$$

Following the method of the previous chapter we seek a solution of the above partial differential system by letting ϵ tend to zero with σ/ϵ fixed and equal to α. We then expand u, v, w, T in the form

$$\left.\begin{aligned}
u &= \epsilon^{1/2}\{u_{01}\cos a\phi + \epsilon^{1/2}[u_{10}+u_{12}\cos 2a\phi] + \epsilon[u_{21}\cos a\phi \\
&\qquad + u_{23}\cos 3a\phi] + O(\epsilon^{3/2})\} \\
v &= \epsilon^{1/2}\{v_{01}\cos a\phi + \epsilon^{1/2}[v_{10}+v_{12}\cos 2a\phi] + \epsilon[v_{21}\cos a\phi \\
&\qquad + v_{23}\cos 3a\phi] + O(\epsilon^{3/2})\} \\
w &= \epsilon^{1/2}\{w_{01}\sin a\phi + \epsilon^{1/2}[w_{10}+w_{12}\sin 2a\phi] + \epsilon[w_{21}\sin a\phi \\
&\qquad + w_{23}\sin 3a\phi] + O(\epsilon^{3/2})\} \\
T &= T_0 + \epsilon T_1 + O(\epsilon^2)
\end{aligned}\right\} \quad (4.3.3)\text{a,b,c,d}$$

where a is again a wavenumber. This expansion procedure is similar to that used by Di Prima and Stuart (1973) who were considering the stability of the flow between eccentric rotating cylinders . The $\epsilon^{\frac{1}{2}}$ scaling factor in (4.3.3)a,b,c follows from (4.3.3)d and that if T is slightly greater than T_0 then the amplitude of the disturbance is proportional to $(T - T_0)^{\frac{1}{2}}$. If we replace σ by $\alpha\epsilon$ in (4.3.1) and substitute for u , v , etc. from above into (4.3.1) , (4.3.2) and equate terms of order $\epsilon^{\frac{1}{2}}$ we obtain

$$\left.\begin{aligned}
&M^2 u_{01} + a^2 T_0 \chi_0 v_{01} = 0 \\
&u_{01} - M v_{01} = 0 \\
&\frac{\partial u_{01}}{\partial \zeta} + a w_{01} = 0 \\
&u_{01} = v_{01} = w_{01} = 0, \ \zeta = 0,1
\end{aligned}\right\} \quad (4.3.4)$$

where M is as defined by (3.2.9)b. The solution of the above system is given by

$$\left.\begin{aligned}
u_{01} &= A(\tau) f_0(\zeta) \\
v_{01} &= A(\tau) g_0(\zeta) \\
w_{01} &= -A(\tau)\frac{df_0(\zeta)}{d\zeta}\Big/ a
\end{aligned}\right\} \quad (4.3.5)\text{a,b,c}$$

where $f_0(\zeta)$, $g_0(\zeta)$ are as defined by (3.3.7). If we now substitute for u , v , etc. from (4.3.3) into (4.3.1) , (4.3.2) , equate terms of order ϵ and then use (4.3.5) we can show that

$$\left.\begin{aligned}&\frac{\partial^4 u_{10}}{\partial \zeta^4} = \frac{\partial u_{10}}{\partial \zeta} = 0\\&\frac{\partial^2 v_{10}}{\partial \zeta^2} = \frac{-A^2}{4}\frac{d}{d\zeta}\left(f_0 g_0\right)\\&u_{10} = v_{10} = \frac{\partial u_{10}}{\partial \zeta} = 0\ ,\ \zeta = 0,1\end{aligned}\right\}\tag{4.3.6}$$

and
$$\left.\begin{aligned}&\left\{\frac{\partial^2}{\partial \zeta^2} - 4a^2\right\}u_{12} + 4a^2 T_0 \chi_0 v_{12} = -\frac{A^2}{2}a^2 T_0 g_0^2 + \frac{A^2}{2}\left\{\frac{df_0}{d\zeta}\frac{d^2 f_0}{d\zeta^2} - f_0\frac{d^3 f_0}{d\zeta^3}\right\}\\&u_{12} - \left\{\frac{\partial^2}{\partial \zeta^2} - 4a^2\right\}v_{12} = -\frac{A^2}{4}\left\{g_0\frac{df_0}{d\zeta} - f_0\frac{dg_0}{d\zeta}\right\}\\&\frac{\partial u_{12}}{\partial \zeta} + 2a w_{12} = 0\\&u_{12} = v_{12} = w_{12} = 0\ ,\ \zeta = 0,1\end{aligned}\right\}\tag{4.3.7}$$

and so we can write

$$u_{10} = w_{10} = 0,\ v_{10} = A^2 g_2(\zeta) \tag{4.3.8)a,b,c (}$$

where g_2 is the solution of

$$\left.\begin{aligned}&\frac{d^2 g_2}{d\zeta^2} = -\frac{1}{4}\frac{d}{d\zeta}\left(f_0 g_0\right)\\&g_2 = 0\ ,\ \zeta = 0,1\end{aligned}\right\}\tag{4.3.9}$$

We can also write u_{12} , v_{12} , w_{12} in the form

$$\left.\begin{aligned}&u_{12} = A^2 f_3(\zeta)\\&v_{12} = A^2 g_3(\zeta)\\&w_{12} = A^2 h_3(\zeta)\end{aligned}\right\}\tag{4.3.10)a,b,c (}$$

where f_3 , g_3 , h_3 are defined by

$$\left.\begin{aligned}&\left\{\frac{d^2}{d\zeta^2} - 4a^2\right\}f_3 + 4a^2 T_0 \chi_0 g_3 = \frac{1}{2}\left\{\frac{df_0}{d\zeta}\frac{d^2 f_0}{d\zeta^2} - f_0\frac{d^3 f_0}{d\zeta^3}\right\} - \frac{a^2 T_0}{2}g_0^2\\&f_3 - \left\{\frac{d^2}{d\zeta^2} - 4a^2\right\}g_3 = \frac{1}{4}\left\{f_0\frac{dg_0}{d\zeta} - g_0\frac{df_0}{d\zeta}\right\}\\&\frac{df_3}{d\zeta} + 2a h_3 = 0\\&f_3 = g_3 = h_3 = 0\ ,\ \zeta = 0,1\end{aligned}\right\}\tag{4.3.11}$$

If we now substitute for u , v , etc. from (4.3.3) into (4.3.1) , (4.3.2) and equate terms of order ϵ and then use (4.3.5) , (4.3.8) , (4.3.10) we can show that u_{21} , v_{21} , w_{21} are determined by

$$\left.\begin{aligned}&M^2 u_{21} + a^2 T_0 \chi_0 v_{21} = \alpha\frac{dA}{d\tau}Nf_0 - Aa^2 T_0 \chi_1 g_0 \cos\tau - Aa^2 T_1 \chi_0 g_0 + A^3 F_1(\zeta)\\&u_{21} - M v_{21} = -\alpha\frac{dA}{d\tau}g_0 - Af_0\cos\tau + A^3 G_1(\zeta)\\&\frac{\partial u_{21}}{\partial \zeta} + a w_{21} = 0\\&u_{21} = v_{21} = w_{21} = 0\ ,\ \zeta = 0,1\end{aligned}\right\}\tag{4.3.12}$$

where $F_1(\zeta)$, $G_1(\zeta)$ are given by

$$\left.\begin{aligned} F_1(\zeta) &= \frac{1}{8}\left\{f_0\frac{d^3f_3}{d\zeta^3} + 2\frac{df_0}{d\zeta}\frac{d^2f_3}{d\zeta^2} - \frac{d^2f_0}{d\zeta^2}\frac{df_3}{d\zeta} - 2\frac{d^3f_0}{d\zeta^3}f_3\right\} \\ &\quad + \frac{3a^2}{8}\left\{f_0\frac{df_3}{d\zeta} + 2\frac{df_0}{d\zeta}f_3\right\} - \frac{a^2T_0}{4}\left\{g_0g_3 + 2g_0g_2\right\} \\ G_1(\zeta) &= \frac{1}{4}\left\{f_0\frac{dg_3}{d\zeta} + f_3\frac{dg_0}{d\zeta} + 2g_3\frac{df_0}{d\zeta} + \frac{g_0}{2}\frac{df_3}{d\zeta} + 2f_0\frac{dg_2}{d\zeta}\right\} \end{aligned}\right\} \qquad (4.3.13)\text{a,b}$$

and the operator N is as defined by (3.3.8)c. If we now use the condition that the above system should have a solution we obtain the following ordinary differential equation for $A(\tau)$

$$\alpha\frac{dA}{d\tau} = -\Upsilon\left\{\cos\tau + \frac{T_1}{2T_0}\right\}A + a_1A^3 \qquad (4.3.14)$$

where Υ is given by (3.3.12) and has the numerical value -26.18. The constant a_1 is defined by

$$a_1 = \frac{\int_0^1\{f_0^+F_1 + g_0^+G_1\}d\zeta}{\int_0^1\{g_0^+g_0 - f_0^+Nf_0\}d\zeta} \qquad (4.3.15)$$

where (f_0^+, g_0^+) is the adjoint function pair defined by (3.3.10), and F_1, G_1 are as defined by (4.3.13)a,b respectively. The constant a_1 is in fact related to the constant $\overline{a_1}$ introduced by Davey . We can easily show that

$$a_1 = \overline{a_1}/8$$

if we choose the function pair (f_0, g_0) equal to the function pair $(\overline{u_1}, v_1)$ of Daveys work. The functions F_1, $\overline{u_2}$, v_2 introduced by him are then given by

$$\begin{aligned} F_1 &= -4g_2 \\ \overline{v_2} &= -4g_3 \\ \overline{u_2} &= -4f_3 \end{aligned}$$

The numerical calculations performed by Davey in fact showed that with $g(\frac{1}{2}) = 1$, a_1 has the value -10.05.

Finally suppose now that the speed of the inner cylinder is given by $\Omega_1(1+\epsilon f(\omega t))$. The dimensionless velocity $\overline{V}$ is then as shown below for small σ

$$\overline{V} = \left\{ \chi_0 + \sum_{i=0}^{\infty} \epsilon \sigma^i \chi_i(\xi) \frac{d^i f}{d\tau^i} \right\}$$

and the method used above leads to the following equation for the corresponding amplitude function , $A(\tau)$.

$$\alpha \frac{dA}{d\tau} = -\Upsilon \left\{ f(\tau) + \frac{T_1}{2T_0} \right\} A + a_1 A^3 \tag{4.3.16}$$

where a_1, Υ are as defined earlier.

4.4 The solution of the amplitude equation

The equation (4.3.16) is a 'Bernouilli' type of equation and if we use the usual substitutions for such equations and write

$$A^2 = B^{-1} \tag{4.4.1}$$

we can substitute for A from above into (4.3.16) to obtain a first order linear differential equation for B whose solution is given by

$$\left[B(\tau') \phi(\tau') \right]_0^{\tau} = -\frac{2a_1}{\alpha} \int_0^{\tau} \phi(\tau')\, d\tau' \tag{4.4.2}$$

where $$\phi = \exp\left\{ -\frac{\Upsilon}{\alpha} \left[2 \int^{\tau'} f(x)\,dx + \frac{T_1 \tau'}{T_0} \right] \right\}$$

We now consider the nature of the solution of (4.3.16) when $f(\tau) = \tanh(\tau)$. In this case the speed of the inner cylinder changes slowly from $\Omega_1(1-\epsilon)$ at $\tau = -\infty$ to $\Omega_1(1+\epsilon)$ at $\tau = +\infty$. We can then write the equation (4.3.16) in the form

$$\alpha \frac{dA}{d\tau} = -\Upsilon \left\{ \tanh \tau + \frac{T_1}{2T_0} \right\} A + a_1 A^3 \tag{4.4.3}$$

and so as τ tends to infinity A tends to the following equilibrium amplitude solution

$$A_E = \left[\gamma (2T_0 + T_1) / 2a_1 T_0 \right]^{1/2} \tag{4.4.4}$$

which is no more than the usual equilibrium amplitude solution for the steady problem with the Taylor number based on the final speed of the inner cylinder.

Suppose now that f is identically zero. We can show that the time-dependent amplitude , A_D , of Davey's work is related to $A(\tau)$ by

$$A_D = \epsilon^{1/4} A/2 \tag{4.4.5}$$

and that Davey's time variable , t_D, is related to τ by

$$t_D = \tau / \sigma \tag{4.4.6}$$

and so with f identically zero we can write (4.3.16) as follows

$$\frac{dA_D}{d\tau} = 13.09 \left\{ \frac{T}{T_0} - 1 \right\} A_D - 80.2\, A_D^3 \tag{4.4.7}$$

where we have replaced T by $\dfrac{T - T_0}{\epsilon}$ and α by σ/ϵ. This amplitude equation is identical to the truncated third-order equation found by Davey.

<u>The special case $f(\tau) = \cos(\tau)$</u>

Before examining the special case $f(\tau) = \cos(\tau)$ we wish to point out that the following analysis is similar to that given by Di Prima and Stuart (1973) who solved an amplitude equation similar to (4.3.14) which arises in the non-linear study of the eccentric rotating cylinders stability problem .

Taking $f(\tau)$ as being $\cos(\tau)$ in (4.4.2) we have

$$\left[B(\tau')\Psi(\tau')\right]_0^{\tau} = -\frac{2a_1}{\alpha}\int_0^{\tau}\Psi(\tau')\,d\tau' \qquad (4.4.8)$$

where $\Psi(\tau') = \exp\left\{-\frac{\gamma}{\alpha}\left(2\sin\tau' + \frac{T_1\tau'}{T_0}\right)\right\}$

The equation (4.4.8) contains an unknown constant , $B(0)$, which is specified by insisting that $B(\tau)$ is a periodic function of τ . If we solve for this constant and substitute into (4.4.8) we find that

$$B(\tau) = \frac{-2a_1}{\alpha}\left\{\frac{\int_0^{2\pi}\Psi(\tau')d\tau' + \left[\Psi(2\pi)-1\right]\int_0^{\tau}\Psi(\tau')\,d\tau'}{\left[\Psi(2\pi)-1\right]\Psi(\tau)}\right\} \qquad (4.4.9)$$

and in general this form can not be simplified further. However , in the limit of T_1/T_0 tending to infinity with α fixed, we can use (4.4.1) , (4.4.9) to show that

$$A(\tau) \sim \left(\frac{\gamma T_1}{2a_1T_0}\right)^{1/2}\left\{1 + O\left(\frac{T_1}{T_0}\right)^{-1}\right\} \qquad (4.4.10)$$

and the dominant term on the right hand side of (4.4.10) is just the equilibrium amplitude solution for the corresponding problem without modulation and the same Taylor number. Similarly if we let α^{-1} tend to zero with T_1/T_0 fixed we can use (4.4.1) , (4.4.9) to show that

$$A(\tau) \sim \left(\frac{\gamma T_1}{2a_1T_0}\right)^{1/2}\left\{1 + O(\alpha^{-1})\right\} \qquad (4.4.11)$$

which shows that for σ small but large compared to ϵ modulation has negligible effect on the stability of the flow. This is consistent with the experimental observations of Donnelly (1964) which showed that , as the period of oscillation of the inner cylinder tended to zero , the critical angular velocity at which instability first appeared decreased to its unmodulated value

We recall that we obtained (4.4.9) by letting ϵ tend to zero with α , T_1/T_0 fixed. Suppose now that we let T_1/T_0 tend to zero with α held fixed (4.4.9). We find that

$$B(\tau) \sim \frac{a_1 T_0}{\pi \Gamma T_1} \exp\left(\frac{2\gamma \sin \tau}{\alpha}\right) I_0\left(\frac{-2\gamma}{\alpha}\right)\left\{1 + O\left(\frac{T_1}{T_0}\right)\right\} \tag{4.4.12}$$

where I_0 is the modified Bessel function of zero order. We now let α tend to zero in (4.4.12) and it then follows using (4.4.6) that $A(\tau)$ is given by

$$A(\tau) \sim \left[\frac{-T_1 \pi^{3/2}(-2\gamma)^{1/2}}{a_1 T_0}\right] \alpha^{-1/4} \exp\left\{\frac{\pi}{\alpha}(1-\sin\tau)\right\}\left[1 + O\left(\frac{T_1}{T_0}, \alpha\right)\right] \tag{4.4.13}$$

and so for small $\alpha, T_1/T_0$ it follows that $A(\tau)$ behaves somewhat like a δ-function , being exponentially small away from the regions near $\tau = (2n + \frac{1}{2})\pi$, n = 0, 1, 2, 3, 4, 5, 6, ... However the factor $(T_1/T_0)^{\frac{1}{2}}$ multiplying the exponential above ensures that $A(\tau)$ does not become infinite at these points. A sketch of $A(\tau)$ in this limit is shown in Fig. 8.

4.5 The high frequency solution

We now investigate the possibility of the existence of equilibrium perturbations of small but finite size in the limit of σ tending to infinity with ϵ arbitrary. We consider only the case when the inner cylinder has angular velocity $\Omega_1(1+\epsilon\cos\omega t)$. Thus we seek periodic solutions from the outset and can therefore use the method of §3.4.

We recall that in the latter section the effect of modulation in the bulk of the fluid first appeared in the order σ^{-3} term in the expansion of the steady component of the perturbation velocity in powers of $\sigma^{-1/2}$ when the dominant term was of order σ^0 . Hence we perturb the Taylor number in the form

$$T = T_0 + \frac{T_6'}{\sigma^3} + O(\sigma^{-7/2}) \tag{4.5.1}$$

in order that the effects of modulation and non-linearities appear at the same order when we expand in powers of $\sigma^{-1/2}$. It is important at this stage to distinguish between T_6' given above and T_6 introduced in §3.3. We recall that the T_6 is the order σ^{-3} correction term in the expansion of the critical Taylor number in powers of $\sigma^{-1/2}$ for the linear stability problem. On the other hand T_6' is determined by (4.5.1) for any given value of T and the flow is stable or unstable according to linear theory depending on whether or not $T_6' > T$.

We saw in §3.4 that the effect of modulation in the central region was to cause the perturbation velocity to have radial and azimuthal components proportional to $e^{i\tau}$ of order $\sigma^{-5/2}$, $\sigma^{-7/2}$ when the dominant steady velocity was of order σ^{0}. We could also show that the corresponding terms proportional to $e^{2i\tau}$ were of order $\sigma^{-7/2}$, $\sigma^{-9/2}$ respectively. Suppose that we choose the dominant steady velocity in the central region to be of order $\sigma^{-3/2}$ then the linear theory of the previous chapter suggests that we expand the perturbation velocity in the form

$$\left.\begin{aligned}
u &= \sigma^{-3/2}\Big\{U_s^0 + \frac{U_s^1}{\sigma^{1/2}} + \cdots\cdots + \frac{e^{i\tau}}{\sigma^{3/2}}\Big[U_1^0 + \frac{U_1^1}{\sigma^{1/2}} + \cdots\Big] + C.C \\
&\qquad + \frac{e^{2i\tau}}{\sigma^{7/2}}\Big[U_2^0 + \frac{U_2^1}{\sigma^{1/2}} + \cdots\Big] + C.C.\Big\}\cos a\phi \\
v &= \sigma^{-3/2}\Big\{V_s^0 + \frac{V_s^1}{\sigma^{1/2}} + \cdots\cdots + \frac{e^{i\tau}}{\sigma^{9/2}}\Big[V_1^0 + \frac{V_1^1}{\sigma^{1/2}} + \cdots\Big] + C.C. \\
&\qquad + \frac{e^{2i\tau}}{\sigma^{9/2}}\Big[V_2^0 + \frac{V_2^1}{\sigma^{1/2}} + \cdots\Big] + C.C\Big\}\cos a\phi \\
w &= \sigma^{-3/2}\Big\{W_s^0 + \frac{W_s^1}{\sigma^{1/2}} + \cdots + \frac{e^{i\tau}}{\sigma^{5/2}}\Big[W_1^0 + \frac{W_1^1}{\sigma^{1/2}} + \cdots\Big] + C.C \\
&\qquad + \frac{e^{2i\tau}}{\sigma^{7/2}}\Big[W_2^0 + \frac{W_2^1}{\sigma^{1/2}} + \cdots\Big] + C.C\Big\}\sin a\phi
\end{aligned}\right\} \quad (4.5.2)\text{a,b.}$$

where C.C. denotes 'complex conjugate' and the choice of scaling for the steady component of velocity was made so as to be proportional to $(T - T_o)^{\frac{1}{2}}$, which from (4.5.1) is of order $\sigma^{-3/2}$.

However if we let σ tend to infinity in (4.2.1) we can use (4.2.3) to show that in the central region the perturbation velocity is determined by

$$\left.\begin{aligned}
&\left\{\mathcal{L}-\sigma\frac{\partial}{\partial\tau}\right\}\mathcal{L}u = \chi_o T\frac{\partial^2 v}{\partial\phi^2} - \frac{1}{2}\frac{\partial^2 Q_1}{\partial\phi^2} + \frac{1}{2}\frac{\partial^2 Q_2}{\partial\zeta\partial\phi}\\
&\left\{\mathcal{L}-\sigma\frac{\partial}{\partial\tau}\right\}v = u - \frac{1}{2}Q_3\\
&\frac{\partial u}{\partial\zeta} + \frac{\partial w}{\partial\phi} = 0
\end{aligned}\right\}\qquad (4.5.3)\text{a,b,c}$$

where $\mathcal{L}$, Q_1, Q_2, Q_3 are defined by (4.2.4) , (4.2.5). The expansions (4.5.2)a,b,c are clearly no longer suitable if we wish to retain the non-linear terms above . In order to take the non-linear effects into account we modify the expansions (4.5.2)a,b,c to give

$$\left.\begin{aligned}
u = {}&\sigma^{-3/2}\left\{U_s^0+\frac{U_s^1}{\sigma^{1/2}}+\dots+\frac{e^{i\tau}}{\sigma^{5/2}}\left[U_1^0+\frac{U_1^1}{\sigma^{1/2}}+\dots\right]+c.c.+\frac{e^{2i\tau}}{\sigma^{7/2}}\left[U_2^0+\frac{U_2^1}{\sigma^{1/2}}+\dots\right]+c.c.\right\}\cos a\phi\\
&+\sigma^{-3}\left\{U_s^{00}+\frac{U_s^{10}}{\sigma^{1/2}}+\dots+\left(U_s^{02}+\frac{U_s^{12}}{\sigma^{1/2}}+\dots\right)\cos 2a\phi\right\}+O(\sigma^{-5})\\
v = {}&\sigma^{-3/2}\left\{V_s^0+\frac{V_s^1}{\sigma^{1/2}}+\dots+\frac{e^{i\tau}}{\sigma^{3/2}}\left[U_1^0+\frac{U_1^1}{\sigma^{1/2}}+\dots\right]+c.c.+\frac{e^{2i\tau}}{\sigma^{9/2}}\left[V_2^0+\frac{V_2^1}{\sigma^{1/2}}+\dots\right]+c.c.\right\}\cos a\phi\\
&+\sigma^{-3}\left\{V_s^{00}+\frac{V_s^{10}}{\sigma^{1/2}}+\dots+\left(V_s^{02}+\frac{V_s^{12}}{\sigma^{1/2}}+\dots\right)\cos 2a\phi\right\}+O(\sigma^{-6})\\
w = {}&\sigma^{-3/2}\left\{W_s^0+\frac{W_s^1}{\sigma^{1/2}}+\dots+\frac{e^{i\tau}}{\sigma^{3/2}}\left[W_1^0+\frac{W_1^1}{\sigma^{1/2}}+\dots\right]+c.c.+\frac{e^{2i\tau}}{\sigma^{7/2}}\left[W_2^0+\frac{W_2^1}{\sigma^{1/2}}+\dots\right]+c.c.\right\}\sin a\phi\\
&+\sigma^{-3}\left\{W_s^{00}+\frac{W_s^{10}}{\sigma^{1/2}}+\dots+\left(W_s^{02}+\frac{W_s^{12}}{\sigma^{1/2}}+\dots\right)\sin 2a\phi\right\}+O(\sigma^{-5})
\end{aligned}\right\}\qquad (4.5.4)\text{a,b,c}$$

where the terms in the expansions with the triple index notation are produced by non-linear interactions.

From now on we shall use the words fundamental , mean , first harmonic ,etc. with reference to the ϕ-dependence only. The non-linear interaction of the steady fundamental components of velocity with themselves leads to the steady mean and first harmonic terms of order σ^{-3} in the above expansions. The non-linear interactions involving the unsteady fundamental terms produces steady and unsteady mean and first harmonic terms which are at most of order $\sigma^{-13/2}$. Since we shall consider terms only upto order $\sigma^{-9/2}$ these terms are negligible for our purposes. The dominant steady mean and first harmonic terms produced by the interaction described above interact non-linearly with the dominant steady fundamental terms to produce steady fundamental terms of order $\sigma^{-9/2}$. Similar terms are produced by the non-linear interaction of the order $\sigma^{-13/2}$ terms with themselves and the other terms in the aboveexpansions , but these terms will be at most of order σ^{-8} and so negligible for our purposes. Thus we see that in the central region the steady fundamental terms upto order $\sigma^{-9/2}$ are unaffected by any non-linear interactions involving unsteady terms.

We recall that in the high frequency linear theory of §3.4 the steady part of the perturbation velocity in the outer layer exhibited no Stokes layer type of behaviour. However , in this section the steady fundamental component of the perturbation velocity "inherits" such a behaviour through ~~the~~ the non-linear interaction of the unsteady components of the perturbation velocity which of course does have this type of behaviour. Hence we must distinguish between the steady fundamental components in the central region and the outer layer. However we can show that in both the inner and outer layers, the residual steady fundamental components of velocity at the edges of these layers is first affected by non-linearities of order $\sigma^{-9/2}$ (when the dominant steady fundamental component is of order $\sigma^{3/2}$) , and the effect is independent of any non-linear interaction of the unsteady parts of the perturbation velocity . Thus the first order non-linear correction to the linear theory of §3.4 is independent of the time dependence of the basic flow .

Having said this we find that the solution of the problem is trivial , since all the information which we require is embedded in §3.4 and §4.3 . If we substitute for u , v , w in the central region from (4.5.4) into (4.5.3) and take T as in (4.5.1) we obtain the following after equating steady fundamental terms of order $\sigma^{-3/2}$.

$$\left.\begin{aligned} N^2 U_s^0 + a^2 T_0 \chi_0 V_s^0 &= 0 \\ U_s^0 - N V_s^0 &= 0 \end{aligned}\right\} \qquad (4.5.5)\text{a,b}$$

Here N is as defined by (3.3.8)c . The method of §3.4 shows that at the edge of the inner layer U_s^0 , V_s^0 must match onto the small ζ series solution of (4.5.5) with boundary conditions

$$U_s^0 = V_s^0 = \frac{dU_s^0}{d\zeta} = 0 \; , \; \zeta = 0 \qquad (4.5.6)$$

and at the edge of the outer layer we require that U_s , V_s match onto the series solution of (4.5.5) with boundary conditions

$$U_s^0 = V_s^0 = \frac{dU_s^0}{d\zeta} = 0 \; , \; \zeta = 1 \qquad (4.5.7)$$

and so we write

$$(U_s^0, V_s^0) = A_s^0 (f_0(\zeta), g_0(\zeta)) \qquad (4.5.8)\text{a,b}$$

where A_s is an amplitude constant to be determined and f , g are as defined by (3.3.7) . Similarly for n=1,2,3,4,5 we can show that

$$(U_s^n, V_s^n) = A_s^n (f_0, g_0)$$

where A_s^n are unknown amplitude constants to be determined. If we substitute for u, v, w from (4.5.4) into (4.5.3) and equate steady mean and first harmonic terms of order σ^{-3} we can show that (U_s^{00}, V_s^{00}, W_s^{00}) and (U_s^{02}, V_s^{02}, W_s^{02}) satisfy the differential equations in (4.3.6), (4.3.7) with these vectors replacing the vectors (u_{10}, v_{10}, w_{10}) and (u_{11}, v_{11}, w_{11}) respectively, and A replaced by A_s^0. The matching conditions at the edge of the inner and outer layers have no dependence on the time dependence of the basic flow and merely require that at the edge of the inner layer (U_s^{00}, V_s^{00}, W_s^{00}) matches onto the small ζ series solution of the differential equations determining this vector in the central region with boundary conditions

$$U_s^{00} = V_s^{00} = W_s^{00} = 0, \; \zeta = 0$$

and at the edge of the outer layer we require that the vector matches onto the series solution of the differential equations for small (1 − ζ) with boundary conditions

$$U_s^{0} = V_s^{00} = W_s^{00} = 0, \; \zeta = 1$$

and so we have

$$U_s^{00} = W_s^{00} = 0, \quad V_s^{00} = (A_s^0)^2 g_2(\zeta) \qquad (4.5.9)\text{a,b,c}$$

where $g_2(\zeta)$ is defined by (4.3.9). Similarly we obtain

$$(U_s^{02}, V_s^{02}, W_s^{02}) = (A_s^0)^2 (f_3, g_3, h_3) \qquad (4.5.10)\text{a,b,c}$$

where $f_3(\zeta)$, $g_3(\zeta)$, $h_3(\zeta)$ are defined by (4.3.11).

If we now substitute for u, v, w from (4.5.4) into (4.5.3) and take T as in (4.5.1) and equate steady fundamental terms of order $\sigma^{-9/2}$ we obtain

$$\left.\begin{aligned} N^2 U_s^q + a^2 \tau_0 \chi_0 V_s^q &= -(A_s^0)^2 a^2 \tau_6' \chi_0 g_0 + (A_s^0)^3 F_1(\zeta) \\ U_s^q - N V_s^q &= (A_s^0)^3 G_1(\zeta) \end{aligned}\right\} \qquad (4.5.11)\text{a,b}$$

where F_1, G_1 are as defined by (4.3.13)a,b. The matching condition at the edge of the outer layer requires that U_s^q, V_s^q there match onto the series solution of (4.5.11) for small ($1 - \zeta$) with boundary conditions

$$U_s^q = V_s^q = \frac{dU_s^q}{d\zeta} = 0, \ \zeta = 1 \qquad (4.5.12)$$

Without modulation the corresponding conditions at the edge of the inner layer would be that U_s^q, V_s^q there match onto the small ζ series solution of (4.5.11) with boundary conditions

$$U_s^q = V_s^q = \frac{dU_s^q}{d\zeta} = 0, \ \zeta = 0 \qquad (4.5.13)$$

However, with modulation the non-linear interaction of the basic flow and the disturbance in the inner layer affects this matching in an identical way to that observed in §3.4. An analysis similar to that given in the latter section shows that, if modulation is to be taken into account, then the conditions (4.5.13) should be replaced by

$$U_s^q = 0, \ V_s^q = \frac{-a^2\epsilon^2 \tau_0 A_s^0}{8}\frac{dg_0}{d\zeta}, \ \frac{dU_s^q}{d\zeta} = \frac{13a^2\epsilon^2 \tau_0 A_s^0}{32}\frac{d^2 f_0}{d\zeta^2}, \ \zeta = 0 \qquad (4.5.14)$$

and so U_s^q, V_s^q are given by the solution (4.5.11) with boundary conditions (4.5.12), (4.5.14). The condition that this system should have a solution can be shown to be given by

$$a^2 \tau_6' \int_0^1 \chi_0 f_0^+ g_0 \, d\zeta - (A_s^0)^2 \int_0^1 \{ f_0^+ F_1 + g_0^+ G_1 \} d\zeta$$
$$= \frac{a^2\epsilon^2 \tau_0}{32} \left\{ 13\frac{d^2 f_0^+}{d\zeta^2}\frac{d^2 f_0}{d\zeta^2} - 4\frac{dg_0^+}{d\zeta}\frac{dg_0}{d\zeta} \right\}_{\zeta=0}$$

where (f_0^+, g_0^+) is the adjoint function pair defined by (3.3.10) and $\chi_0 = 1 - \zeta$. The terms in this equation are more recognizable after a few substitutions. We can use (3.3.12) , (3.4.45) , (4.3.15) to show that the above equation can be written in the form

$$\left(A_5^0\right)^2 = \frac{\pi}{2a_1 T_0}\left\{T_6' - T_6\right\} \tag{4.5.15}$$

We recall that T_6 was in fact the order σ^{-3} correction to the Taylor number ,T, expanded in powers of $\sigma^{-1/2}$ for the high frequency linear theory. Thus if (4.5.15) is to have a real solution we require that $T_6' > T_6$,and so finite amplitude perturbations can exist only when T is greater than its critical value of linear stability theory.

4.6 Discussion of results

We first discuss the experimental work of Donnelly in more detail. As stated earlier he considered the flow between concentric cylinders when the outer one is at rest and the inner one moving with angular velocity $\Omega_1(1+\epsilon\cos\omega t)$. Before saying how Donnelly defined the critical Taylor number for the flow we first discuss the important features of the stability of the unmodulated flow.

When the outer cylinder is at rest and the inner one moving wit angular velocity Ω_1 it can be shown by linear stability theory that the flow first becomes unstable when the Taylor number reaches the value 3389.9 . For T slightly greater than this value the non-linear theory of Davey (1962) shows that equilibrium perturbations to the flow can exist. The amplitude of the Taylor vortex flow is then proportional to ($T - T_0$ $)^{\frac{1}{2}}$. It can also be shown that equilibrium amplitude flows can not exist for T less than T_0 . Thus for T less than T_0 the amplitude of the Taylor vortex flow is zero , and then when T reaches the value T_0 the amplitude begins to grow like $(T - T_0)^{\frac{1}{2}}$.

With this in mind Donnelly defined the critical Taylor number to be that value for which a slight increase in $\mathcal{T}$ caused the amplitude of the Taylor vortex flow to increase rapidly. With this definition of the critical Taylor number he found that the flow was stabilized for all ϵ , σ in the sense that the critical Taylor number was always greater than T_0 . The maximum enhancement for all vaues of ϵ was when σ took the value .27 .

Low frequency results with $f(\tau) = \cos(\tau)$

The first difficulty which we must overcome is to decide what property of the time dependent amplitude , $A(\tau)$, Donnelly actually measured in his experiments. We feel that the most relevant property of $A(\tau)$ is its mean value , $\bar{A}$, defined by

$$\bar{A} = \frac{1}{2\pi}\int_0^{2\pi} A\, d\tau \qquad (4.6.1)$$

We saw in §4.4 that A was in general only known in integral form . Thus A must be evaluated numerically using an integration routine and then $\bar{A}$ can be evaluated using the same routine. The result of such acalculation for various values of T_1 / T_0 , α is shown in Fig. 9 . We have also shown the corresponding equilibrium amplitude solution for the unmodulated flow at the same Taylor number. We see that , as suggested by (4.4.11), the effect of modulation vanishes as α tends to infinity , the curves tending to the equilibrium amplitude solution for the unmodulated flow . The results of Fig. 9 suggest that modulation stabilizes the flow in the sense that the value of $\bar{A}$ for any given values of

T_1/T_0, α is always less than its unmodulated value. However, unlike the results of Donnelly, our results show no optimum value of α, and hence σ for given value of ϵ, at which the enhancement of stability is most pronounced. The enhancement of stability shown in Fig. 9 decreases as α increases. In Fig. 10 we have shown the results of our low frequency theory in a form more suitable for comparison with Donnelly's results. We see that there is poor agreement between theory and experiment. This is perhaps due to $\bar{A}$ not being the relevant property of $A(\tau)$ as far as the latter's results are concerned.

Finally we would like to suggest that a more promising method of experimentally checking our theory would be to try and obtain the behaviour of A as a function of τ. This could be perhaps done by measuring the difference of the torque on the inner cylinder from its laminar value, a quantity which is proportional to A^2. We have shown A^2 as a function of τ for various values of α and $T_1/T_0 = \frac{1}{2}$ in Fig. 11.

<u>High frequency results with $f(\tau) = \cos(\tau)$</u>

In the limit of σ tending to infinity with ϵ arbitary we found that the amplitude of the dominant steady fundamental component of the perturbation velocity was given by (4.5.15). Suppose that we write

$$A = \sigma^{-3/4} A_s^0$$

and eliminate T_6' from (4.5.15) using (4.5.1). We obtain

$$A^2 = \frac{4}{2a_1 T_0}\left\{T - T_0 - \frac{T_6}{\sigma^3}\right\}$$

and if we put ϵ, and hence T_6, equal to zero we can show that A is then the equilibrium amplitude solution for the problem without modulation and the same Taylor number. If we denote this equilibrium amplitude by A_E we can show that

$$A - A_E = -\pi T_6 \big/ 2a_1 (A + A_E)\,\sigma^3$$

and so as T becomes appreciably greater than T_0 , A tends to A_E . We have shown A as a function of T for different values of ϵ in Fig. 12 .

In contrast to the low frequency results we see that the amplitude grows quite quickly as soon as the critical Taylor number of linear theory is reached . Since T was in actual fact negative , we conclude that in the high frequency limit the effect of modulation is to destabilize the flow .

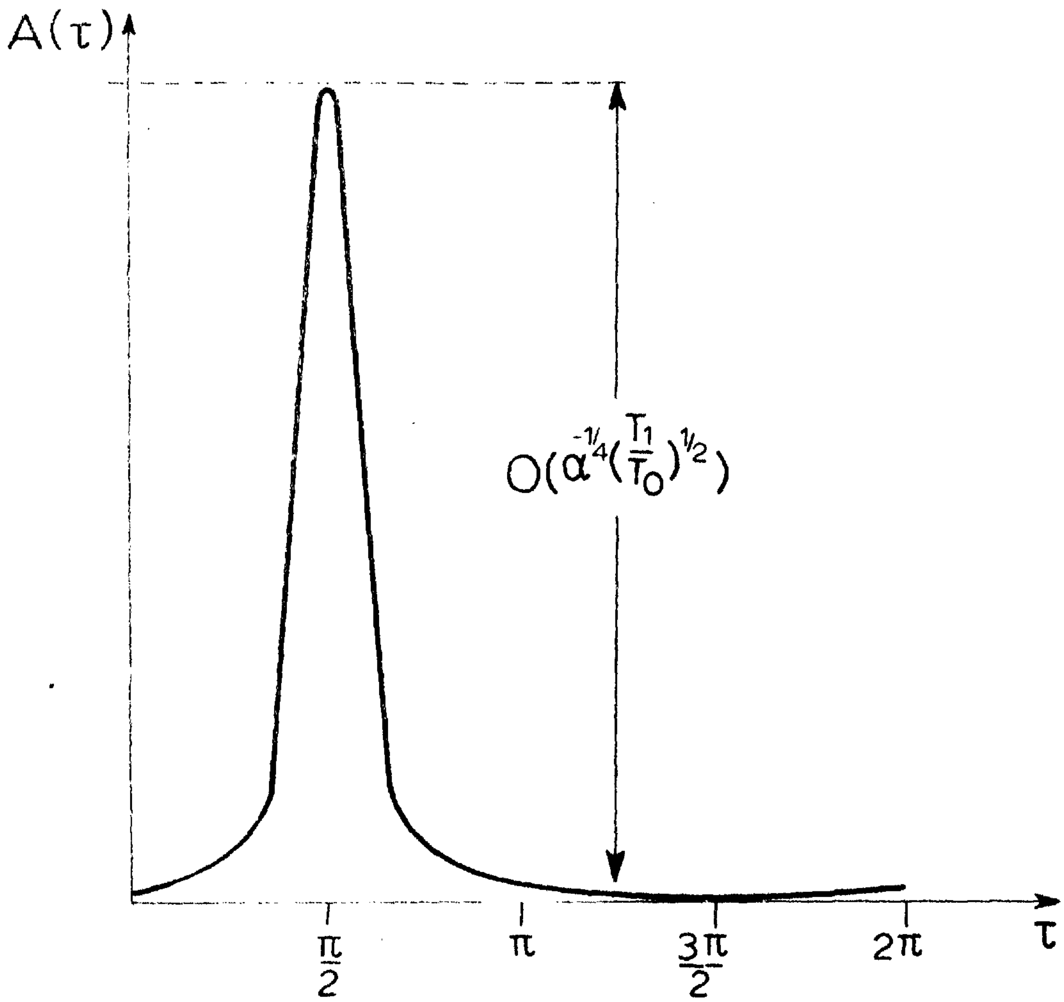

Fig 8: Amplitude as a function of τ for small $\alpha, \frac{T_1}{T_0}$

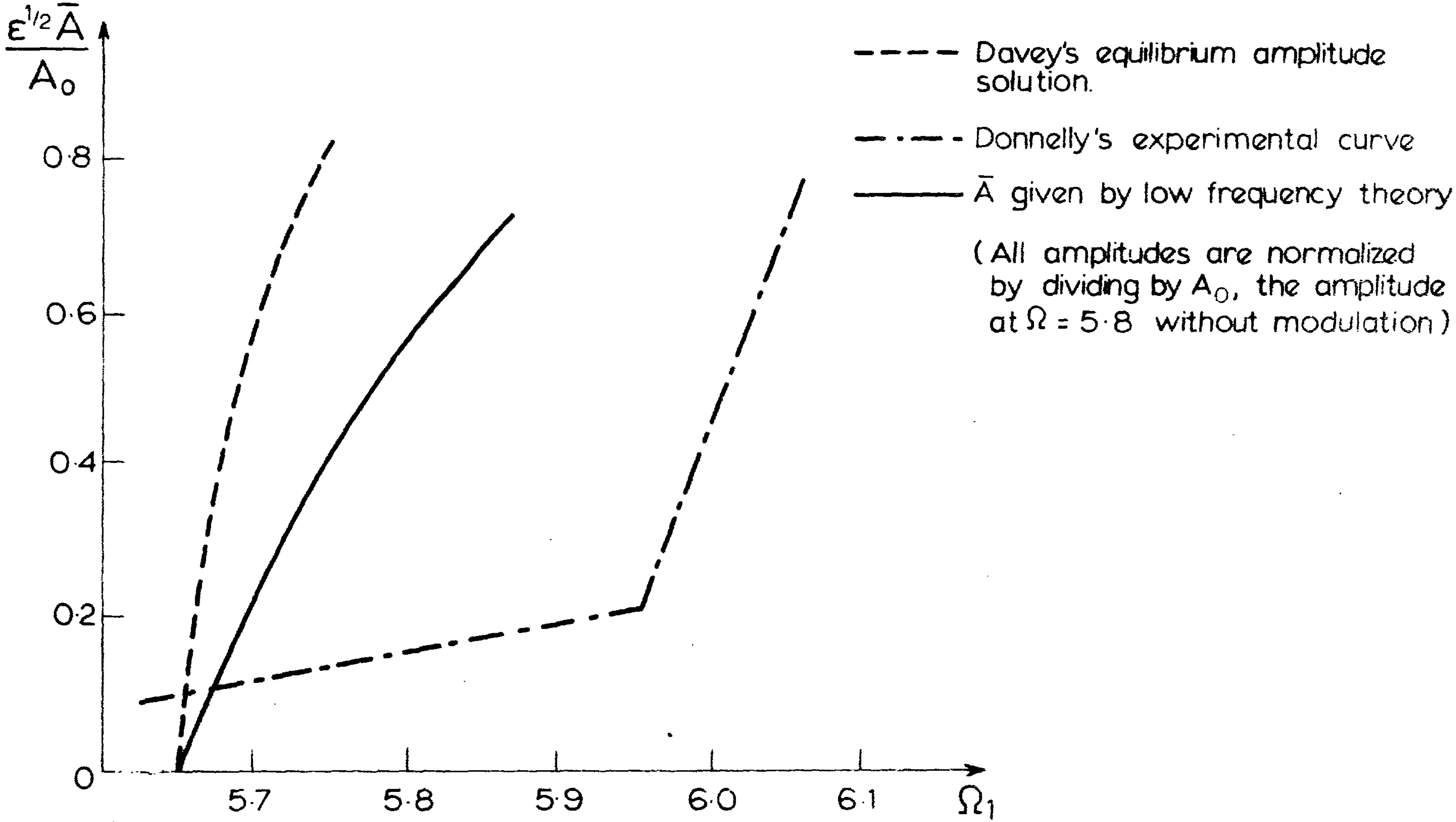

Fig **9**: Comparison with Donnelly's results for $\varepsilon = 0{\cdot}08$, period $= 46{\cdot}1$.

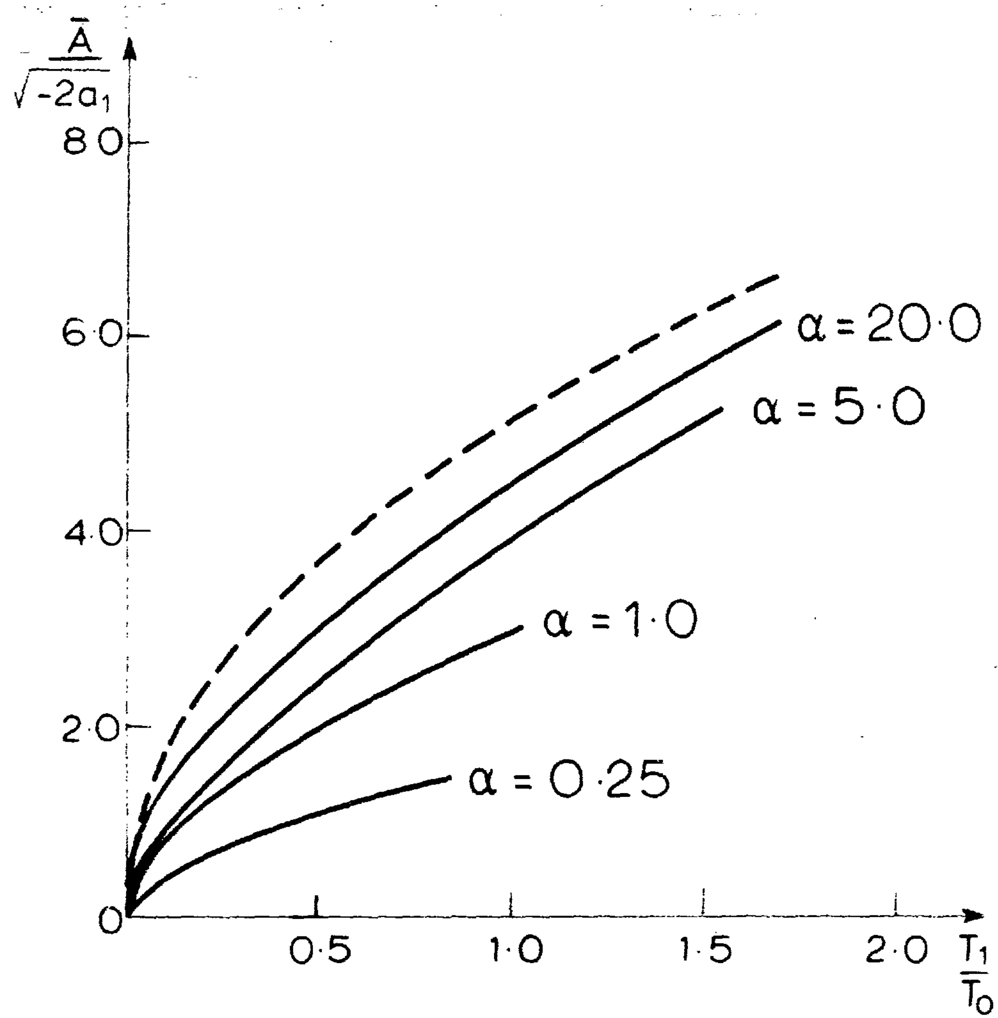

Fig **10**: $\bar{A}$ as a function of $\frac{T_1}{T_0}$. The dotted line represents the value of $\bar{A}$ for the unmodulated problem with the same Taylor number.

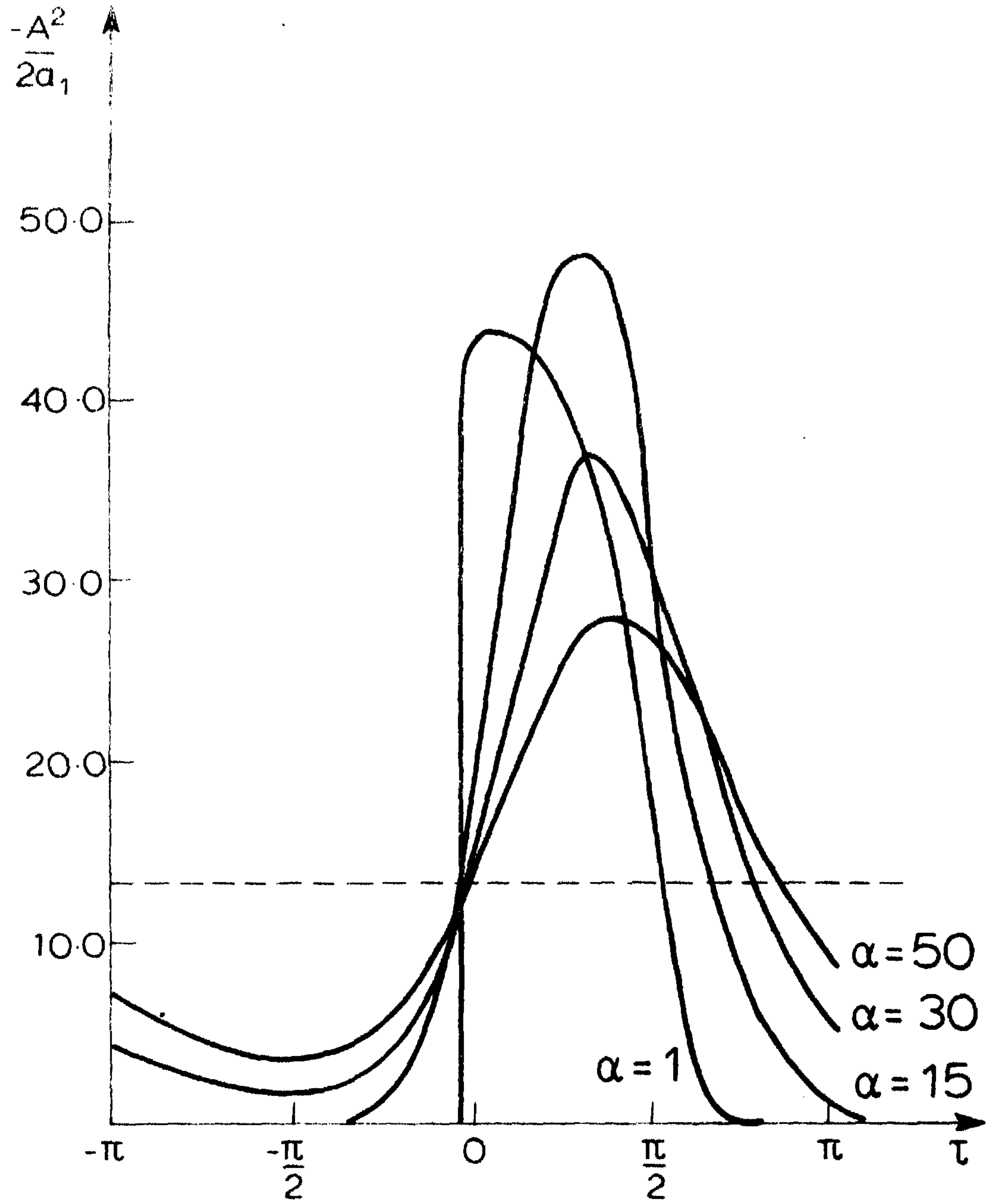

Fig 11: A^2 as a function of α for $\frac{T_1}{T_0} = 0 \cdot 5$. The dotted line represents the constant value of A^2 for the unmodulated problem with the same Taylor number. The difference G of the torque on the inner cylinder from its laminar value, for this value of $\frac{T_1}{T_0}$, can be shown to be given by

$$G = \left\{ \frac{2\pi \Omega_1 R_1^3 \mu}{R_2 - R_1} \right\} \left\{ 0{\cdot}037 \, A^2 \varepsilon \right\} .$$

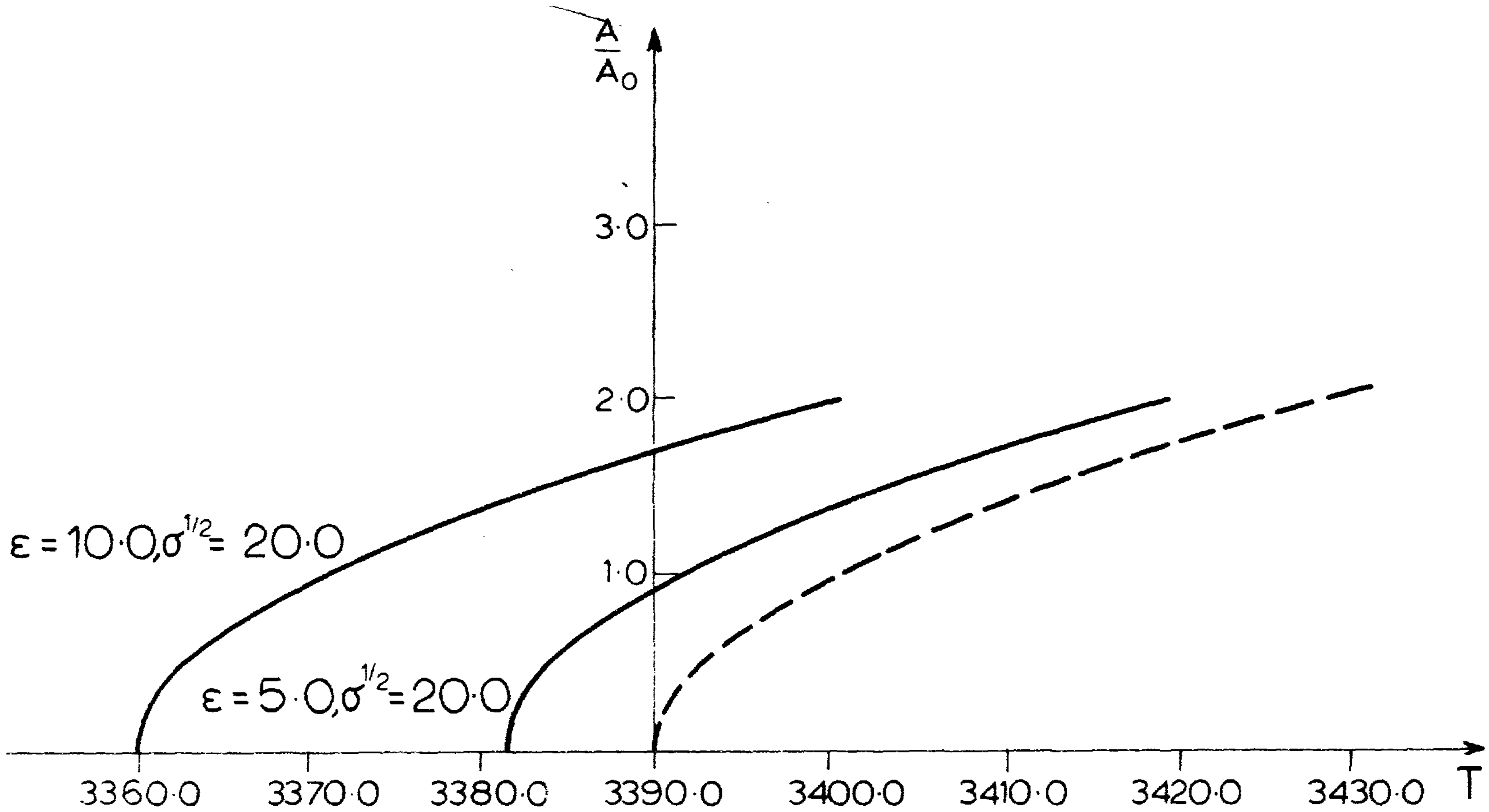

Fig 12 The amplitude A as a function of T in the high frequency limit. The dotted line represents the amplitude for the unmodulated problem with the same Taylor number. (All amplitudes are normalized by dividing by A_0, the amplitude at $T = 3400$ without modulation.)

CHAPTER V

THE MODULATION OF THERMAL INSTABILITY-LINEAR THEORY

5.1 Introduction

In this chapter we investigate the thermal analogue of the cylinder problem considered in Chapter 111 . We consider the stability of the fluid confined between parallel planes which are separated by a distance d . The temperature of the upper plane is maintained at zero and the lower one has temperature $\beta d(1 + \epsilon \cos \omega t)$. Thus ϵ will again be an amplitude parameter and we again introduce a frequency parameter σ defined to be the square of the ratio of d to the thickness of the oscillatory layer associated with the basic temperature field . We assume that the boundaries are stress free surfaces and we again consider the limits of low and high frequencies separately . Venezian (1969) has considered this problem for small ϵ and arbitary σ . If σ is allowed to tend to zero and infinity in his work we should expect to recover the results of our work . However this is not the case and since our low frequency results have also been obtained by Herbert in some unpublished work using a Galerkin type method we believe that the error lies in Venezian's work . The procedure adopted in this chapter is as follows .

In §5.2 we determine the temperature distribution of the basic state and we then obtain the equations governing the linear stability of this state . We again insist that the boundary between instability and stability is determined by the disturbance velocity and temperature fields being periodic in ωt.

In §5.3 we seek a solution to these equations by letting ϵ and σ tend to zero with σ/ϵ fixed and equal to α say . We then obtain a solution of the equations by expanding in powers of ϵ . In contrast to §3.3 we find that the ordinary differential systems which arise from equating like powers of ϵ can be solved exactly without using any numerical techniques .

In §5.4 we consider the limit of σ tending to infinity with ϵ arbitrary . We again find that , in contrast to the basic state , the disturbance has a time dependence throughout the fluid . However , since we are using the free surface conditions , there is no need for an oscillatory layer near the upper boundary .

In §5.5 we give a brief discussion of our results .

5.2 Formation of the equations for the stability of the flow

We consider the flow between the planes $z = 0$, $z = d$ with respect to a Cartesian coordinate system (x , y , z). These planes are taken to be free surfaces . We take $\underset{\sim}{u}$, T , p , t to be the velocity, temperature , pressure and time respectively . We also define the constants χ, α , ν , T_0, ρ to be the thermal conductivity , coefficient of volumetric expansion , kinematic viscosity , and averages of temperature and density respectively . The governing equations in the Boussinesq approximation are

$$\left.\begin{aligned} &\nabla\cdot\underset{\sim}{u} = 0 \\ &\frac{\partial \underset{\sim}{u}}{\partial t} = -(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} - \frac{1}{\rho}\nabla p + \{1-\alpha(T-T_0)\}(0,0,-g) + \nu\nabla^2\underset{\sim}{u} \\ &\frac{\partial T}{\partial t} = -(\underset{\sim}{u}\cdot\nabla)T + \chi\nabla^2 T \end{aligned}\right\} \qquad (5.2.1)\ a,b,c$$

where $\underset{\sim}{u}\cdot\nabla \equiv u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

and $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

We seek an equilibrium state in which the fluid is at rest and the temperature is zero at $z = d$ and equal to $\beta d(1 + \epsilon\cos\omega t)$ at $z = 0$. The required solution of these equations is

$$\underset{\sim}{u} = \underset{\sim}{0} \quad , \quad T = \overline{T}(z,t) \quad , \quad p = \overline{p}(z,t)$$

and $\overline{T}$ is given by

$$\overline{T} = \beta d \left\{ 1 - \frac{z}{d} + \frac{\epsilon}{2}\left[\frac{\sinh\left(\frac{i\omega}{K}\right)^{1/2}(d-z)\, e^{i\omega t}}{\sinh\left(\frac{i\omega}{K}\right)^{1/2} d} + \text{COMPLEX CONJUGATE} \right] \right\} \qquad (5.2.2)$$

We now perturb this equilibrium state and Fourier analyse in the xy plane . Thus we write

$$\underset{\sim}{u} = (u, v, w)\exp i[a_x x + a_y y]$$
$$T = \overline{T} + \Theta(z,t)\exp i[a_x x + a_y y]$$
$$p = \overline{p} + p(z,t)\exp i[a_x x + a_y y]$$

If we substitute these expressions into (5.2.1) and neglect non-linear terms then after eliminating u , v , p , θ from the resulting equations we obtain the following equation for w

$$\left\{\frac{\partial^2}{\partial z^2} - a^2\right\}\frac{\partial^2 w}{\partial t^2} - (K+\nu)\left\{\frac{\partial^2}{\partial z^2} - a^2\right\}^2\frac{\partial w}{\partial t} + K\nu\left\{\frac{\partial^2}{\partial z^2} - a^2\right\}^3 w - g\alpha a^2\frac{\partial \overline{T}}{\partial z} w = 0 \qquad (5.2.3)$$

where $a = \left[a_x^2 + a_y^2\right]^{1/2}$

Following the method of Chapter 111 we seek a solution which has a time dependence only in terms of ωt.Hence we define a new time variable τ by

$$\tau = \omega t \qquad (5.2.4)$$

and we seek a solution periodic in τ , thus defining a boundary between stability and instability .We now introduce the following dim -ensionless quantities

$$z' = z/d, \quad a' = ad, \quad T' = \overline{T}/\beta d, \quad w' = \nu w/\alpha a^2 g d^4 \qquad (5.2.5)$$
a,b,c,d

and we define the frequency parameter σ by

$$\sigma = \omega d^2/\chi \tag{5.2.6}$$

and the Prandtl number p_a by

$$p_a = \nu/\chi \tag{5.2.7}$$

and the Rayleigh number R by

$$R = \alpha g \beta d^4/\chi\nu \tag{5.2.8}$$

It then follows that (5.2.3) may be written in the form shown below after dropping the dash notation

$$M\left\{M - \sigma\frac{\partial}{\partial\tau}\right\}\left\{M - \frac{\sigma}{p_a}\frac{\partial}{\partial\tau}\right\}w = a^2 R\frac{\partial T}{\partial z}w \tag{5.2.9}$$

where $M \equiv \dfrac{\partial^2}{\partial z^2} - a^2$

and $\dfrac{\partial T}{\partial z}$ is given from (5.2.2) , (5.2.5) by

$$\frac{\partial T}{\partial z} = -1 - \frac{\epsilon}{2}\left\{\frac{(i\sigma)^{1/2}\cosh(i\sigma)^{1/2}(1-z)\,e^{i\tau}}{\sinh(i\sigma)^{1/2}} + \text{COMPLEX CONJUGATE}\right\} \tag{5.2.10}$$

The relevant boundary conditions for the problem are given by

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0 \quad , z = 0,1 \tag{5.2.11}$$

(See Chandrasekhar (1961)) In contrast to Chapter 111 we have a differential system in terms of only one perturbation quantity , w . However this does not change our approach to the problem and we see that this in actual fact makes the problem easier to solve .

5.3 The low frequency limit

We now consider the nature of the solution of (5.2.9) , (5.2.11) as ϵ and σ tend to zero . Using (5.2.10) we see that we must solve the following differential system

$$M\left[M-\sigma\frac{\partial}{\partial\tau}\right]\left[M-\frac{\sigma}{P_r}\frac{\partial}{\partial\tau}\right]w$$
$$= -a^2R\left[1+\epsilon\phi_1\cos\tau+\epsilon\sigma\phi_2\sin\tau+\ldots\ldots\right]w \qquad (5.3.1)$$

with boundary conditions

$$w=\frac{\partial^2 w}{\partial z^2}=\frac{\partial^4 w}{\partial z^4}=0\ ,\quad z=0,1 \qquad (5.3.2)$$

where ϕ_1, ϕ_2 , etc. are as defined in §3.3 . If we let σ tend to zero with σ/ϵ fixed and equal to α we see that the τ dependences of the right and left hand sides of (5.3.1) will balance in some sense . Thus we write

$$\sigma = \alpha\epsilon \qquad (5.3.3)$$

and let ϵ tend to zero with α fixed . We expand w, R in the form

$$w = w_0+\epsilon w_1+\epsilon^2 w_2+\ldots\ldots$$
$$R = R_0+\epsilon R_1+\epsilon^2 R_2+\ldots\ldots$$

in which case (5.3.1)can be written in the form

$$M\left\{M-\alpha\epsilon\frac{\partial}{\partial\tau}\right\}\left\{M-\frac{\alpha\epsilon}{P_a}\frac{\partial}{\partial\tau}\right\}\left[w_0+\epsilon w_1+\ldots\right]$$
$$= -a^2\left[R_0+\epsilon R_1+\ldots\ldots\right]\left[1+\epsilon\phi_1\cos\tau\ldots\right]\left[w_0+\epsilon w_1\ldots\right] \qquad (5.3.4)$$

and it follows from (5.3.2) that the relevant boundary conditions are

$$w_i=\frac{\partial^2 w_i}{\partial z^2}=\frac{\partial^4 w_i}{\partial z^4}=0\ ,\ z=0,1\ ,\ i=0,1,2.. \qquad (5.3.5)$$

If we equate terms of order ϵ^0 in (5.3.4) we obtain

$$M^3 w_0 + a^2 R_0 w_0 = 0 \tag{5.3.6}$$

with boundary conditions given by (5.3.5) with i = 0 . Thus τ does not appear in either the differential equation or the boundary conditions and so we have an ordinary differential system for w_0 whose solution will contain τ only as a parameter . The appropriate solution is given by

$$w_0 = B_0(\tau) \sin\pi z \tag{5.3.7}$$

and the corresponding values of a and R_0 are

$$a = \pi/2^{1/2}$$
$$R_0 = 27\pi^4/4 \tag{5.3.8}$$

and $B_0(\tau)$ is an as yet arbitrary function of τ . Thus we see that the order ϵ^0 velocity is just the usual velocity for the problem with zero ϵ multiplied by a function of τ . If we equate terms of order ϵ in (5.3.4) and use (5.3.7) , (5.3.8) we can show that

$$M^2 w_1 + \frac{27\pi^6}{8} w_1 = \frac{9\pi^4}{4}\left\{\frac{1+p_A}{p_A}\right\} \alpha \frac{dB_0}{d\tau} \sin\pi z - a^2 R_1 \sin\pi z\, B_0$$
$$- \frac{27\pi^6}{8} B_0 \cos\tau \sin\pi z \tag{5.3.9}$$

with boundary conditions given by (5.3.5) with i = 1 . It is an easy matter to show that the required solvability condition for such a system is that the integral from z = 0 to z = 1 of the right hand side of (5.3.9) multiplied by $\sin\pi z$ is zero . Hence we have

$$\frac{9\pi^4}{4}\left\{\frac{1+p_A}{p_A}\right\} \alpha \frac{dB_0}{d\tau} - \frac{27\pi^6}{8} B_0 \cos\tau$$
$$= a^2 R_1 B_0$$

which is an ordinary differential equation for $B_0(\tau)$ which has a periodic solution if $R_1 = 0$ and $B_0(\tau)$ is then given by

$$B_0 = A \exp\{3\pi^2 p_A \sin\tau / 2(1+p_A)\} \tag{5.3.10}$$

where A is a constant , dependent on the parameters of the problem , which can only be determined by considering the corresponding non-linear problem . Having determined B_0 we can substitute back into (5.3.9) and solve the resulting differential equation subject to the appropriate boundary conditions to give

$$w_1 = B_1(\tau)\sin\pi z \tag{5.3.11}$$

In order to find the unknown function of τ , $B_1(\tau)$, we equate terms of order ϵ in (5.3.4) and if we then use (5.3.8) , (5.3.10) , (5.3.11) we can show that

$$\begin{aligned} M^2 w_2 + \frac{27\pi^6}{8} w_2 = {} & \frac{9\pi^4}{4}\frac{(1+\rho_A)}{\rho_A}\alpha\frac{dB_1}{d\tau}\sin\pi z - \frac{27\pi^6}{8}B_1\cos\tau\sin\pi z \\ & - a^2 R_2 B_0 \sin\pi z + \frac{27\pi^6\rho_A}{16(1+\rho_A)^2} B_0(1+\cos 2\tau)\sin\pi z \\ & - \frac{9\pi^4}{4}\left\{\frac{1}{1+\rho_A} + \frac{3\pi^2}{2}\rho_A\right\}\alpha\, B_0 \sin t \sin\pi z \end{aligned} \tag{5.3.12}$$

with boundary conditions given by (5.3.5) with i = 2 . If we impose the solvability condition on this system we obtain an ordinary differential equation for B_1 , which has a solution periodic in τ if

$$R_2 = \frac{27\pi^4\rho_A}{8(1+\rho_A)^2} \tag{5.3.13}$$

and B_1 is then given by

$$B_1 = -B_0\left\{\frac{\rho_A(11+3\rho_A)\cos\tau}{8(1+\rho_A)^2} + \frac{3\pi^2\rho_A^2\sin 2\tau}{8(1+\rho_A)^3\alpha}\right\} \tag{5.3.14}$$

and so it follows that we can write

$$w_1 = -B_0\left\{\frac{\rho_A(11+3\rho_A)\cos\tau}{8(1+\rho_A)^2} + \frac{3\pi^2\rho_A^2\sin 2\tau}{8(1+\rho_A)^3\alpha}\right\} \tag{5.3.15}$$

Having determined B_1 we can substitute back into (5.3.12) and solve the resulting differential equation subject to the appropriate boundary conditions to give

$$w_2 = \alpha B_0 \sin\tau \, w_{21} + B_2 \sin\pi z \qquad (5.3.16)$$

where w_{21} is given by

$$w_{21} = \frac{\pi}{4}\left\{ \cos\pi z \left\{ \frac{3Qz}{27\pi^2} + \frac{(3z^2 - z^3 - 2z)}{6} \right\} - \sin\pi z \left\{ \frac{11}{6\pi}\left(z - \frac{z^2}{2}\right) \right\} \right.$$
$$\left. \frac{-3Q}{27\pi^2}\left\{ \cos\pi z - \cosh cz \cos dz + \frac{(\sin 2d \cosh cz \sin dz + \sinh 2c \sinh cz \cos dz)}{2(\sin^2 d + \sinh^2 c)} \right\} \right\} \qquad (5.3.17)$$

where $\pm(c \pm id)$ are the complex roots of the auxil~~l~~iary equation

$$\left(m^2 - \frac{\pi^2}{2}\right)^3 + \frac{27\pi^6}{8} = 0$$

and B_2 is an unknown function of τ which can only be determined by considering the order ϵ^3 system . If we equate terms of order ϵ^3 in (5.3.4) and use the expressions for w_0 , B_0 etc. already calculate we can show that

$$M^2 w_3 + \frac{27\pi^6}{8} w_3 = \frac{9\pi^4}{4}\left(\frac{1+\rho_A}{\rho_A}\right)\alpha \frac{dB_2}{d\tau}\sin\pi z - \frac{27\pi^6}{8} B_2 \cos\tau \sin\pi z$$
$$- a^2 R_3 B_0 \sin\pi z + B_0 \left\{ \alpha^2 \cos\tau \, H_1 + \alpha \sin 2\tau \, H_2 \right.$$
$$\left. + \cos 3\tau \, H_3 + \cos\tau \, H_4 + \frac{\sin 4\tau}{\alpha} H_5 \right\} \qquad (5.3.18)$$

with boundary conditions given by (5.3.5) with i = 3 . The functions H_1 , H_2 ,etc. appearing above are defined by

$$H_1 = \frac{(1+\rho_A)}{\rho_A} M^2 w_{21} - \frac{27\pi^6}{8}\phi_3 \sin\pi z + \frac{3\pi^2(11+3\rho_A)}{16\,(1+\rho_A)^2}\sin\pi z$$

$$H_2 = \frac{3\pi^2}{4} M^2 w_{21} + \frac{27\pi^6}{128}\phi_2 \frac{\rho_A(11+3\rho_A)}{(1+\rho_A)^2}\sin\pi z + \frac{9\pi^4}{64}\frac{\rho_A(49+9\rho_A)}{(1+\rho_A)^2}\sin\pi z$$

$$H_3 = -\frac{81\pi^8\phi_2\rho_A^2 \sin\pi z}{128\,(1+\rho_A)^2} - \frac{27\pi^6}{256}\frac{\rho_A^2(22+3\rho_A)}{(1+\rho_A)^4}\sin\pi z$$

$$H_4 = \frac{81\pi^8}{128}\phi_2\frac{\rho_A^2\sin\pi z}{(1+\rho_A)^2} - \frac{27\pi^6\rho_A}{256}\frac{(19\rho_A^2+55\rho_A+16)}{(1+\rho_A)^4}\sin\pi z$$

$$H_5 = \frac{-81\,\rho_A^3\sin\pi z}{256^{6}(1+\rho_A)^5}$$

If we now invoke the solvability condition on the system defined by (5.3.5) ,. (5.3.18) we obtain an ordinary differential equation for B_2 which has a solution periodic in τ if $R_3 = 0$ and B_2 is then given by

$$B_2 = B_0\left\{\alpha\Gamma_1\sin\tau + \Gamma_2\cos 2\tau + \frac{\Gamma_3}{\alpha}\sin 3\tau + \frac{\Gamma_4}{\alpha}\sin\tau + \frac{\Gamma_5}{\alpha^2}\cos 4\tau\right\} \qquad (5.3.19)$$

where Γ_1, Γ_2 etc. are defined by

$$\Gamma_1 = \frac{-8\rho_A}{9\pi^4(1+\rho_A)}\int_0^1 H_1\sin\pi z\,dz\,, \quad \Gamma_2 = \frac{4\rho_A}{9\pi^4(1+\rho_A)}\int_0^1 H_2\sin\pi z\,dz$$

$$\Gamma_3 = \frac{-8\rho_A}{27\pi^4(1+\rho_A)}\int_0^1 H_3\sin\pi z\,dz, \quad \Gamma_4 = \frac{-8\rho_A}{9\pi^4(1+\rho_A)}\int_0^1 H_4\sin\pi z\,dz$$

$$\Gamma_5 = \frac{2\rho_A}{9\pi^4(1+\rho_A)}\int_0^1 H_5\sin\pi z\,dz$$

Hence w_2 may be written in the form

$$w_2 = \alpha B_0\sin\tau\,(w_{21} + \Gamma_1\sin\pi z) + B_0\Big[\Gamma_2\cos 2\tau + \frac{\Gamma_3}{\alpha}\sin 3\tau + \frac{\Gamma_4}{\alpha}\sin\tau + \frac{\Gamma_5}{\alpha^2}\cos 4\tau\Big]\sin\pi z \qquad (5.3.20)$$

Substituting for B_2 from (5.3.19) into (5.3.18) and solving the resulting differential equation subject to the appropriate boundary conditions we find that the solution is of the form

$$w_3 = B_0\left\{\alpha^2\cos\tau\, w_{31} + \alpha\sin 2\tau\, w_{32} + \cos 3\tau\, w_{33} + \cos\tau\, w_{34}\right\} + B_3(\tau)\sin\pi z \qquad (5.3.21)$$

where B_3 is an unknown function of τ and w_{3i} are solutions of

$$\left(M^3 + \frac{27\pi^6}{8}\right) w_{3i} = K_i$$

$$w_{3i} = \frac{d^2 w_{3i}}{dz^2} = \frac{d^4 w_{3i}}{dz^4} = 0 \ , \ z = 0, 1$$

with

$$K_1 = H_1 + \frac{9\pi^4}{4} \Pi_1 \left(1 + \frac{1}{\rho_A}\right) \sin \pi z$$

$$K_2 = H_2 - \frac{9\pi^4}{2} \Pi_2 \left(1 + \frac{1}{\rho_A}\right) \sin \pi z$$

$$K_3 = H_3 + \frac{27\pi^4}{4} \Pi_3 \left(1 + \frac{1}{\rho_A}\right) \sin \pi z$$

$$K_4 = H_4 + \frac{9\pi^4}{4} \left(1 + \frac{1}{\rho_A}\right) \sin \pi z$$

If we equate terms of order ϵ^4 in (5.3.4) and use the solvability condition on the resulting differential equation with boundary conditions given by (5.3.5) with i = 4 we obtain an ordinary differential equation for B_3 which has a solution periodic in τ if we choose R_4 as follows

$$R_4 = \alpha^2 R_{40} + R_{42} \tag{5.3.22}$$

where R_{40} is given by

$$\begin{aligned} a^2 R_{40} \int_0^1 \sin^2 \pi z \, dz = {} & \frac{3\pi^2}{4} \int_0^1 M^2 w_{31} \sin \pi z \, dz - \frac{27\pi^6}{8} \int_0^1 w_{31} \sin \pi z \, dz \ \frac{27\pi^6}{8} \int_0^1 \phi_2 (w_{21} + \Pi_1 \sin \pi z) \sin \pi z \, dz \\ & - \frac{3\pi^2}{4(1+\rho_A)} \int_0^1 M (w_{21} + \Pi_1 \sin \pi z) \sin \pi z \, dz \\ & + \frac{27\pi^6 \rho_A (11 + 3\rho_A)}{128 \ (1+\rho_A)^2} \int_0^1 \phi_3 \sin^2 \pi z \, dz \end{aligned} \tag{5.3.23}$$

If the terms in the above integral are evaluated we find that

$$R_{40} = -0.015\pi^{-4} - \frac{3\rho_A}{2(1+\rho_A)^4} \left\{ 1 + \frac{9}{64} (1+\rho_A)^2 \right\}$$

(The contributions to R_{40} from the terms involving w_{31} in fact cancel and so there is no need to calculate w_{31} if we do not wish to calculate higher order terms in the expansion of R)

If we now write $\alpha = \sigma/\epsilon$ in (5.3.22) we see that the Rayleigh number is expressible in the form

$$\begin{aligned} R = {} & \frac{27\pi^4}{4} \left\{ 1 + \frac{\epsilon^2 \rho_A}{2(1+\rho_A)^2} \right\} - \left\{ 0.015\pi^{-4} + \frac{3\rho_A}{2(1+\rho_A)^2} \left(1 + \frac{9}{64}(1+\rho_A)^2\right) \right\} \epsilon^2 \sigma^2 \\ & + O(\epsilon^4, \epsilon^2 \sigma^4) \end{aligned} \tag{5.3.24}$$

and this expression is identical to one obtained by Dr. Herbert at Imperial College in some unpublished work using a Galerkin type method .

5.4 The high frequency limit

Suppose now that we let σ tend to infinity in (5.2.10) . We obtain

$$\frac{\partial T}{\partial z} \sim -1 - \frac{\epsilon}{2}\left\{\sqrt{i\sigma}\, e^{-\eta(1+i)+i\tau} + \text{COMPLEX CONJUGATE}\right\} \tag{5.4.1}$$

Thus for large σ the time dependence of the basic state is confined to a thin layer at the lower boundary . We shall refer to this layer as the 'inner' layer and the region away from this layer will be called the 'outer' layer . The interaction of the basic temperature distribution with the disturbance in the inner layer causes the latter to have a time dependence which persists throughout the flow . However , in contrast to the cylinder problem , there is no need for a Stokes layer at the upper boundary in order to satisfy the required boundary conditions . This is because we are not using the rigid boundary conditions but the somewhat artificial 'free surface' conditions .

We define an inner layer variable η by

$$\eta = z\sqrt{\frac{\sigma}{2}} \tag{5.4.2}$$

and we take w , W to be the velocities normal to the boundaries in the inner and outer layers respectively . It follows from (5.2.9) , (5.4.1) that the relevant differential equations to determine these functions are

$$\left\{\frac{\partial^2}{\partial\eta^2} - \frac{2a^2}{\sigma}\right\}\left\{\frac{\partial^2}{\partial\eta^2} - \frac{2a^2}{\sigma} - 2\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^2} - \frac{2a^2}{\sigma} - \frac{2}{P_a}\frac{\partial}{\partial\tau}\right\}w$$
$$= -8a^2Rw\sigma^{-3}\left\{1 + \frac{\epsilon}{2}\left(\sqrt{i\sigma}\, e^{-\eta(1+i)+i\tau} + \text{COMPLEX CONJUGATE}\right)\right\} \tag{5.4.3}$$

$$\left\{\frac{\partial^2}{\partial z^2} - a^2\right\}\left\{\frac{\partial^2}{\partial z^2} - a^2 - \sigma\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial z^2} - a^2 - \frac{\sigma}{P_a}\frac{\partial}{\partial\tau}\right\}W = -a^2RW \tag{5.4.4}$$

and it follows from (5.2.11) that the required boundary conditions ar

$$\left.\begin{aligned} w = \frac{\partial^2 w}{\partial \eta^2} = \frac{\partial^4 w}{\partial \eta^4} = 0, \ \eta = 0 \\ W = \frac{\partial^2 W}{\partial z^2} = \frac{\partial^4 W}{\partial z^4} = 0, \ z = 1 \end{aligned}\right\} \quad (5.4.5)\text{a,b}$$

and we stipulate that w and W must match where the different regions overlap . Since we are seeking solutions which are periodic in τ we expand w , W in Fourier series as follows

$$\left.\begin{aligned} w = w_s + \frac{1}{2}\sum_{n=1}^{\infty}\left(w_n e^{in\tau} + \tilde{w}_n e^{-in\tau}\right) \\ W = W_s + \frac{1}{2}\sum_{n=1}^{\infty}\left(W_n e^{in\tau} + \tilde{W}_n e^{-in\tau}\right) \end{aligned}\right\} \quad (5.4.6)\text{a,b}$$

Hence w_s , W_s represent the steady parts of the velocities normal to the boundaries in each layer . If we substitute for W from above into(5.4.4) and equate terms proportional to $e^{in\tau}$ we obtain

$$\{\mathcal{D}^2 - a^2\}\{\mathcal{D}^2 - a^2 - i\sigma n\}\{\mathcal{D}^2 - a^2 - \frac{i\sigma n}{p_a}\}W_n = -a^2 R W_n \quad (5.4.7)$$

where $\mathcal{D} \equiv d/dz$

We solve the above differential equation by expanding W_n , R in the form

$$\left.\begin{aligned} R = R_0 + \frac{R_1}{\sigma^{-1/2}} + \frac{R_2}{\sigma} + \ldots \ldots \ldots \\ W_n = \mu_n(\sigma)\left\{W_n^0 + \frac{W_n^1}{\sigma^{1/2}} + \frac{W_n^2}{\sigma} + \ldots \ldots \ldots\right\} \end{aligned}\right\} \quad (5.4.8)\text{a,b}$$

where $\mu_n(\sigma)$ is for the moment an arbitrary function of σ . It follows from (5.4.5) , (5.4.6) , (5.4.8) that the required boundary conditions are

$$W_n = \mathcal{D}^2 W_n = \mathcal{D}^4 W_n = 0, \ z = 1, \ n = 1, 2, \ldots, i = 0, 1, \ldots \quad (5.4.9)$$

If we substitute for W_n , R from above into (5.4.7) , equate like powers of $\sigma^{1/2}$, and solve the resulting differential equations subject to (5.4.9) we can show that

$$W_n = \mu_n \left\{ A_0 \sinh a(1-z) + \frac{A_1 \sinh a(1-z)}{\sigma^{1/2}} + O(\sigma^{-1}) \right\} \tag{5.4.10}$$

where A_0, A_1 are arbitrary constants . If we expand W_s in the form

$$W_s = \beta(\sigma) \left\{ W_s^0 + \frac{W_s^1}{\sigma^{1/2}} + \frac{W_s^2}{\sigma} + \ldots\ldots \right\}$$

we can similarly show that W_s^i are determined by

$$\left.\begin{aligned} &\{(D^2 - a^2)^3 + a^2 R_0\} W_s^0 = 0 \\ &\{(D^2 - a^2)^3 + a^2 R_0\} W_s^i = -a^2 \sum_{k=0}^{i-1} W_s^k R_{i-1-k}, \; i = 1, 2 \ldots \end{aligned}\right\} \tag{5.4.11}$$

with boundary conditions

$$W_s^i = D^2 W_s^i = D^4 W_s^i = 0, \; z = 1, \; i = 0, 1, 2, \ldots \tag{5.4.12}$$

We now consider the flow in the inner layer .If we substitute for w from (5.4.6) into (5.4.3) and equate terms proportional to e^{int} for n = 0 , 1 , ... we can show that

$$\left\{D^2 - \frac{2a^2}{\sigma}\right\}^3 w_s = -\frac{8a^2 R}{\sigma^3} \left\{ w_s + \frac{\epsilon}{4} \left(\sqrt{i\sigma}\, \tilde{w}_1 e^{-\eta(1+i)} + \text{COMPLEX CONJUGATE} \right) \right\} \tag{5.4.13}$$

$$\left\{D_\eta^2 - \frac{2a^2}{\sigma}\right\} \left\{D_\eta^2 - \frac{2a^2}{\sigma} - \frac{2i}{Pa}\right\} \left\{D_\eta^2 - \frac{2a^2}{\sigma} - 2i\right\} w_1$$

$$= -\frac{8a^2 R}{\sigma^3} \left\{ w_1 + \epsilon \sqrt{i\sigma}\, w_s e^{-\eta(1+i)} + \frac{\epsilon \sqrt{-i\sigma}\, w_2}{2} e^{-\eta(1-i)} \right\} \tag{5.4.14}$$

and for n = 2 , 3 , 4 ,.........

$$\left\{D_\eta^2 - \frac{2a^2}{\sigma}\right\} \left\{D_\eta^2 - \frac{2a^2}{\sigma} - \frac{2in}{Pa}\right\} \left\{D_\eta^2 - \frac{2a^2}{\sigma} - 2in\right\} w_n \tag{5.4.15}$$

$$= -\frac{8a^2 R}{\sigma^3} \left\{ w_n + \frac{\epsilon \sqrt{i\sigma}}{2} e^{-\eta(1+i)} w_{n-1} + \frac{\epsilon \sqrt{-i\sigma}}{2} e^{-\eta(1-i)} w_{n+1} \right\}$$

where $D_\eta \equiv d/d\eta$

A consideration of the problem with $\epsilon = 0$ shows that near $z = 0$

$$w_s \sim z \sim \sigma^{-1/2}$$

and it follows that , by using (5.4.14) , (5.4.15) , that if w_s has this scaling then

$$w_n \sim \sigma^{-(5n+1)/2} \quad , n = 1, 2, \ldots \ldots \ldots$$

Hence we expand w_s , w_n in the form

$$w_s = \sigma^{-1/2}\left\{w_s^0 + \frac{w_s^1}{\sigma^{1/2}} + \cdots\cdots\right\}$$

$$w_n = \sigma^{-(5n+1)/2}\left\{w_n^0 + \frac{w_n^1}{\sigma^{1/2}} + \cdots\cdots\right\}$$

for n = 1,2,3,4,5,.........

If we take R as given by (5.4.8) we obtain the following equtions to determine w_s , w_n

$$\left\{D_\eta^2 - \frac{2a^2}{\sigma}\right\}\left\{w_s^0 + \frac{w_s^1}{\sigma^{1/2}} + \cdots\right\} = -8a^2\left\{R_0 + \frac{R_1}{\sigma^{1/2}} + \cdots\right\}\left\{\sigma^{-3}\left(w_s^0 + \frac{w_s^1}{\sigma^{1/2}} + \cdots\right) + \frac{\epsilon}{4\sqrt{2}}\sigma^{-5}\left[(1+i)e^{-\eta(1+i)}(w_1^0 + \cdots) + \text{COMPLEX CONJUGATE}\right]\right\} \tag{5.4.16}$$

$$\left\{D_\eta^2 - \frac{2a^2}{\sigma}\right\}\left\{D_\eta^2 - \frac{2a^2}{\sigma} - \frac{2in}{Pa}\right\}\left\{D_\eta^2 - \frac{2a^2}{\sigma} - 2in\right\}\left\{w_1^0 + \cdots\right\}$$
$$= -8a^2\left\{R_0 + \frac{R_1}{\sigma^{1/2}} + \cdots\right\}\left\{\sigma^{-3}(w_1^0 + \cdots\cdots) + \frac{\epsilon}{\sqrt{2}}(1+i)e^{-\eta(1+i)}(w_s^0 + \cdots\cdots) + \frac{\epsilon}{2\sqrt{2}}\sigma^{-5}(1-i)e^{-\eta(1-i)}(w_2^0 + \cdots)\right\} \tag{5.4.17}$$

and for n = 2,3,4,5,6........

$$\left\{D_\eta^2 - \frac{2a^2}{\sigma}\right\}\left\{D_\eta^2 - \frac{2a^2}{\sigma} - \frac{2in}{Pa}\right\}\left\{D_\eta^2 - \frac{2a^2}{\sigma} - 2in\right\}\left\{w_n^0 + \cdots\right\}$$
$$= -8a^2\left\{R_0 + \cdots\right\}\left\{\sigma^{-3}(w_n^0 + \cdots) + \frac{\epsilon}{2\sqrt{2}}(1+i)e^{-\eta(1+i)}(w_{n-1}^0 + \cdots) + \frac{\epsilon\sigma^{-5}}{2\sqrt{2}}(1-i)e^{-\eta(1-i)}(w_{n+1}^0 + \cdots)\right\} \tag{5.4.18}$$

and it follows from (5.4.5)a that the relevant boundary conditions ar

$$w_s^i = w_n^i = D_\eta^2 w_s^i = D_\eta^2 w_n^i = D_\eta^4 w_s^i = D^4 w_n^i = 0, \quad \eta = 0 \tag{5.4.19}$$

$$i = 0, 1, \ldots \qquad n = 1, 2, 3, 4, \ldots$$

It follows from (5.4.16) , (5.4.19) that the first ten w_s^i can be obtained without any knowledge of w_1^i, w_2^i, etc. Thus equating terms of order σ^0, $\sigma^{-1/2}$, ... $\sigma^{-9/2}$ in (5.4.16) and solving the resulting differential equations subject to (5.4.19) we can show that

$$w_s^0 = P_0 \eta$$
$$w_s^1 = P_1 \eta$$
$$w_s^2 = P_2 \eta + Q_0 \eta^3$$
$$w_s^3 = P_3 \eta + Q_1 \eta^3$$
$$w_s^4 = P_4 \eta + Q_2 \eta^3 + S_0 \eta^5$$
$$w_s^5 = P_5 \eta + Q_3 \eta^3 + S_1 \eta^5$$
$$w_s^6 = P_6 \eta + Q_4 \eta^3 + S_2 \eta^5 + \left\{ \frac{a^2 P_0}{630}(a^4 - R_0) - \frac{a^4 Q_0}{70} + \frac{a^2 S_0}{7} \right\} \eta^7$$
$$w_s^7 = P_7 \eta + Q_5 \eta^3 + S_3 \eta^5 + \left\{ \frac{a^2}{630}(a^4 - P_0 R_1 - P_1 R_0) - \frac{a^4 Q_1}{70} + \frac{a^4 S_1}{7} \right\} \eta^7$$
$$w_s^8 = P_8 \eta + Q_6 \eta^3 + S_4 \eta^5 + \left\{ \frac{a^2}{630}(a^4 - P_0 R_2 - P_1 R_1 - P_2 R_0) - \frac{a^4 Q_2}{70} + \frac{a^2 S_2}{7} \right\} \eta^7$$
$$+ \left\{ \frac{a^4 P_0}{9.8.7.5.3}(a^4 - R_0) - \frac{a^2 Q_0}{9.7.5.3}\left(\frac{R_0}{8} + a^4\right) - \frac{a^4 S_0}{9.7.4} \right\} \eta^9$$
$$w_s^9 = P_0 \eta + Q_7 \eta^3 + S_5 \eta^5 + \left\{ \frac{a^2}{630}(a^4 - P_0 R_3 - P_1 R_2 - P_2 R_1 - P_3 R_0) - \frac{a^4 Q_3}{70} + \frac{a^2 S_3}{7} \right\} \eta^7$$
$$+ \left\{ \frac{a^4 (a^4 P_1 - P_1 R_0 - P_0 R_1)}{9.8.7.5.3} - \frac{a^2}{9.7.5.3}\left[\frac{R_0 Q_1}{8} + \frac{R_1 Q_0}{8} + a^4 Q_1 \right] - \frac{a^4 S_1}{9.7.4} \right\} \eta^9$$

where P_0, Q_0, etc. are arbitrary constants . We have choosen certain arbitrary constants in w_s^0, w_s^1, w_s^2, w_s^3, w_s^4 to be zero for the sake of convenience . The reason for this choice will become clearer later .

We now calculate w_1^0 which will enable us to calculate w_s^{10}. Equating terms of order σ^0 in (5.4.17) and using the expression for w_s^6 shown above we obtain

$$\mathcal{D}_\eta^2\left\{\mathcal{D}_\eta^2 - \frac{2i}{p_a}\right\}\left\{\mathcal{D}_\eta^2 - 2i\right\} w_{10} = -4\sqrt{2}\,a^2 R_o P_o \epsilon (1+i)\,\eta\, e^{-\eta(1+i)}$$

with boundary conditions given by (5.4.19) with i = 0 , n = 1 . The above system has the following solution if $p_a \neq 1$.

$$w_1^0 = -\frac{\sqrt{2}\,a^2 R_o P_o \epsilon}{4}\left\{\frac{p_a}{p_a-1}\left[\eta^2 + \frac{(9p_a-5)(1-i)\eta}{2(p_A-1)}\right]e^{-\eta(1+i)} + A^*\left[e^{-\eta(1+i)}-1\right] + B^*\left[e^{-\eta(1+i)/p_a^{1/2}}-1\right]\right\} + G_o^*\eta \qquad (5.4.20)\text{a}$$

If $p_a = 1$ then the solution is of the form

$$w_1^0 = \frac{\sqrt{2}\,a^2 R_o P_o \epsilon(1+i)}{48}\left\{\left[2\eta^3 + 9\eta^2(1-i)\right]e^{-\eta(1+i)} + A^*\left[e^{-\eta(1+i)}-1\right] + B^*\eta\, e^{-\eta(1+i)}\right\} + G_o^*\eta \qquad (5.4.20)\text{b}$$

where A^* , B^* when $p_a \neq 1$ are given by

$$A^* = \frac{4i p_a (3p_a - 3p_a^2 - 1)}{(p_A-1)^3}$$
$$B^* = \frac{4i p_a^2}{(p_a-1)^3} \qquad (5.4.21)\text{a, b}$$

and otherwise we have

$$A^* = -24(1+i)$$
$$B^* = -33i \qquad (5.4.22)\text{a, b}$$

and the unknown constant G_o^* will be determined by the matching conditions . We can similarly show that

$$\text{if } p_a \neq 1 \quad w_1^0 = \frac{-\sqrt{2}\,a^2 R_o P_o \epsilon}{4}\left\{\frac{p_a}{p_a-1}\left[\eta^2 + \frac{(9p_a-5)(1-i)\eta}{2(p_a-1)}\right]e^{-\eta(1+i)} + A^*\left[e^{-\eta(1+i)}-1\right] + B^*\left[e^{-\eta(1+i)/p_a^{1/4}}-1\right]\right\} + G_1^*\eta$$

$$\text{if } p_A = 1 \quad w_1^1 = \frac{\sqrt{2}\,a^2 R_o P_o \epsilon(1+i)}{48}\left\{\left[2\eta^3 + 9\eta^2(1-i)\right]e^{-\eta(1+i)} + A^*\left[e^{-\eta(1+i)}-1\right] + B^*\eta\, e^{-\eta(1+i)}\right\} + G_1^*\eta \qquad (5.4.23)\text{a, b}$$

where A^*, B^* are as above and G will be determined later. It follows from (5.4.17) that we can continue in this way to determine $w_1^2, w_1^3, \ldots\ldots\ldots\ldots w_1^9$ and then we can calculate more of the w_s^i and the first few terms in the expansion of w_2. We are then in a position to calculate more terms in the expansion of w_1 and so on. It can be shown inductively that the terms independent of η in (5.4.20), namely $\frac{\sqrt{2}a^2R_0P_0(A^*+B^*)\epsilon}{4}$, $\frac{-\sqrt{2}a^2R_0P_0\epsilon(1+i)A^*}{48}$ lead to terms $\frac{\sqrt{2}a^2R_0P_0\epsilon(A^*+B^*)\eta^{2n}2^n}{4.2n!}$, $\frac{-\sqrt{2}a^2R_0P_0\epsilon(1+i)A^*\eta^{2n}2^n}{48\;2n!}$ respectively in w_1^{2n} for n = 1,2,3,4...... Similarly the term $G_0^*\eta$ in (5.4.20)a, leads to a term $\frac{G_0^*\eta^{2n+1}2^n}{2n+1!}$ in w_1^{2n} for n = 1,2,3,4........... If we consider the asymptotic form of w_1 at the edge of the inner layer and write $\eta = z(\sigma/2)^{\frac{1}{2}}$ we can show that at the edge of this layer

$$w_1 \sim \sigma^{-5/2}\left\{\frac{G_0^* \sinh az}{a\sqrt{2}} + \sigma^{-1/2}\left\{\frac{G_1^* \sinh az}{a\sqrt{2}} + H_0^*\cosh az\right\} + O(\sigma^{-1})\right\} \qquad (5.4.24)$$

where $$H_0^* = \frac{\sqrt{2}a^2R_0P_0\epsilon(A^*+B^*)}{4} \quad \text{if } pa \neq 1$$

and $$H_0^* = \frac{-\sqrt{2}a^2R_0P_0\epsilon(1+i)A^*}{48} \quad \text{if } pa = 1$$

We must match (5.4.24) with W_1 given by (5.4.10) with n = 1 at the edge of the inner layer. Clearly we must take $\mu_1 = \sigma^{-5/2}, G_0^* = 0$ and the first terms of the series will then match if we choose

$$\left.\begin{aligned} G_1^* &= -a\sqrt{2}\coth a\, H_0^* \\ A_0 &= \frac{H_0^*}{\sinh a} \end{aligned}\right\} \qquad \begin{matrix}(5.4.25)\text{a}\\ \text{b}\end{matrix}$$

Higher order terms in the expansions of w_1, W_1 can also be matched by a similar procedure as can also the expansions of w_2, W_2, etc. However we have enough information already about the expansions of w_1, W_1 for the determination of the first non-zero correction term to R_0 and so we do not pursue this here .

Having determined w_1^0 we can use (5.4.16) to obtain a differential equation for w_s^{10} which when solved subject to (5.4.19) gives

$$
\begin{aligned}
w_s^{10} = {} & P_{10}\eta + Q_8\eta^3 + S_6\eta^5 + \Big\{\frac{a^2}{630}\big(a^4P_4 - R_0P_4 - R_1P_3 - R_2P_1 - R_3P_0 - R_4P_0\big) \\
& \qquad - \frac{a^4Q_4}{70} + \frac{a^2S_4}{7}\Big\}\eta^7 \\
& + \Big\{\frac{a^4}{9.8.7.5.3}\big(a^4P_2 - R_0P_2 - R_1P_1 - R_2P_0\big) \\
& \quad - \frac{a^4S_2}{9.74} - \frac{a^2}{9.7.5.3}\Big(\frac{R_0Q_2 + R_1Q_1 + R_2Q_0}{8} + a^4Q_2\Big)\Big\}\eta^9 \\
& + \Big\{\frac{a^6P_0(a^4 - R_0)}{11.10.9.7.6.5} - \frac{a^4Q_0(5a^4 + R_0)}{11.10.9.7.5.4} - \frac{a^2S_0\,(8a^4R_0)}{11.9.7.65}\Big\}\eta^{11} \\
& + \tilde{\phi}
\end{aligned}
\tag{5.4.26}
$$

where for $p_a \neq 1$, $\tilde{\phi}$ is given by

$$
\begin{aligned}
\tilde{\phi} = \frac{a^4R_0^2\epsilon^2P_0}{128}\Big\{ & \frac{p_a}{p_a - 1}\big[\eta^2 + 6\eta + \tfrac{21}{2}\big]e^{-2\eta} + \frac{(9p_a - 5)p_0 i}{(p_a - 1)^2}\big[\eta + 3\big]e^{-2\eta} \\
& + (1+i)\tilde{A}^* e^{-2\eta} + 8(1-i)(\hat{A}^* + \hat{B}^*)e^{-\eta(1+i)} \\
& + \frac{64(1+i)\tilde{B}^* e^{-\eta\left[1 + \frac{1}{\sqrt{p_a}} - \frac{i}{\sqrt{p_a}} + i\right]}}{\Big\{\Big(1 + \frac{1}{\sqrt{p_0}}\Big) + i\Big(1 - \frac{1}{\sqrt{p_a}}\Big)\Big\}^6} \\
& + X_0 + Y_0\eta^2 + Z_0\eta^4\Big\}^{6} + \text{COMPLEX CONJUGATE}
\end{aligned}
\tag{5.4.27a}
$$

and if $p_a = 1$, $\tilde{\phi}$ is given by

$$
\begin{aligned}
\tilde{\phi} = \frac{-a^4R_0^2\epsilon^2P_0}{1536}\Big\{ & \big[2\eta^3 + 18\eta^2 + 63\eta + 84\big]e^{-2\eta} + 9(1+i)\big[\eta^2 + 6\eta + \tfrac{21}{2}\big]e^{-2\eta} \\
& + \tilde{A}^* e^{-2\eta} - 8i\,\hat{A}^* e^{-\eta(1+i)} + \tilde{B}^*\big[\eta + 3\big]e^{-2\eta} \\
6.128 \quad -6 \quad & \Big\{+X_0 + Y_0\eta^2 + Z_0\eta^4\Big\}\Big\} + \text{COMPLEX CONJUGATE}
\end{aligned}
\tag{5.4.27b}
$$

where X_0, Y_0, Z_0 are choosen so as to satisfy $w_s^{10} = D_1^2 w_s^{10} = D_1^4 w_s^{10} = 0$ at $\eta = 0$

In particular we note that Z_0 is given by

$$\left.\begin{aligned} \chi_0 &= -\frac{p_a(1+i)}{p_a-1} + \frac{2\tilde{\hat{A}}^*}{3}(1-3i) + \frac{4}{3}(1-i)\tilde{\hat{B}}^* , \quad p_a \neq 1 \\ -6\,\chi_0 &= \frac{-13\ \cancel{93+145i}}{\cancel{2}} - \frac{2\tilde{\hat{A}}^*}{3}(1+2i) - \frac{2\tilde{\hat{B}}^*}{3} , \quad p_a = 1 \end{aligned}\right\} \qquad (5.4.28)\ \text{a, b}$$

If we consider the asymptotic form of w_s at the edge of the inner layer and write $\eta = z(\sigma/2)^{\frac{1}{2}}$ we can show that

$$\begin{aligned} w_s \sim \sum_{i=0}^{7} \sigma^{-i/2}\Big\{ S(P_i, Q_i, S_i, a, R_0, z) + \text{terms proportional to } R_K \ 0<k\leq i \Big\} \\ + \sigma^{-7/2}\Big\{ \frac{a^4 R_0^2 e^2 P_0 (Z_0+\tilde{Z}_0) z^4}{512} + O(z^6) \Big\} + O(\sigma^{-4}) \end{aligned} \qquad (5.4.29)$$

where S is defined by

$$S = \frac{P_0}{\sqrt{2}}\Big\{ z + \frac{a^2(a^4-R_0)z^7}{5040} \Big\} + \frac{Q_0}{2\sqrt{2}}\Big\{ z^3 - \frac{a^4 z^7}{280} \Big\} + \frac{S_0}{4\sqrt{2}}\Big\{ z^5 + \frac{a^2 z^7}{14} \Big\} + O(z^9)$$

We recall that in the outer layer

$$W_s = \beta(\sigma)\Big\{ W_s^0 + \frac{W_s^1}{\sigma^{1/2}} + \cdots \Big\} \qquad (5.4.30)$$

where the W_s^i are determined by (5.4.11) , (5.4.12) . It follows from (5.4.29), (5.4.30) that we must choose $\beta = 1$ if w_s , W_s are to match at the edge of the inner layer . We must then choose a , R_0, P_0 , Q_0, S_0 such that W_s as determined by (5.4.11)a , (5.4.12) satisfie

$$W_s = S(P_0, Q_0, S_0, a, R_0, z) \qquad (5.4.31)$$

at the edge of the inner layer . However it can easily be shown that S is just the small z series solution of (5.4.11)a with the following boundary conditions

$$W_s^o = \mathcal{D}^2 W_s^o = \mathcal{D}^4 W_s^o = 0 \quad , \ z = 0 \tag{5.4.32}$$

Hence if we determine a solution of (5.4.11)a satisfying the conditions (5.4.12) with i = 0 and (5.4.32) then this solution will nec~~c~~essarily satisfy (5.4.31) at the edge of the inner layer . Thus we write

$$\left.\begin{aligned} W_s^o &= \sin \pi z \\ a &= \pi/\sqrt{2} \\ R_o &= 27\pi^4/4 \end{aligned}\right\} \qquad \begin{matrix}(5.4.33)\\ a,b,c\end{matrix}$$

and the corresponding values of P_o, Q_o, S_o are found to be

$$\begin{aligned} P_o &= \pi\sqrt{2} \\ Q_o &= -\pi^3\sqrt{2}/3 \\ S_o &= \pi^5\sqrt{2}/30 \end{aligned} \qquad \begin{matrix}(5.4.34)\\ a,b,c\end{matrix}$$

Similarly if we put $R_1 = R_2 \ldots\ldots = R_6 = 0$, $W_s^i = P_i W_s^o / P_o$, i=1,2.....6 then (5.4.29) will match upto order σ^{-3} . With the above choices of a, R_o, etc. it follows that W_s^7 is determined by

$$\left[\{\mathcal{D}^2 - a^2\}^3 + \frac{27\pi^6}{8}\right] W_s^7 = -\frac{\pi^2}{2} R_7 \sin \pi z \tag{5.4.35}$$

with boundary conditions

$$W_s^7 = \mathcal{D}^2 W_s^7 = \mathcal{D}^4 W_s^7 = 0 \quad , \ z = 1 \tag{5.4.36}$$

and at the edge of the inner layer we require that

$$W_s^7 = S(P_7, Q_7, S_7, a, R_o, z) + \frac{a^4 R_o^2 \epsilon^2 P_o}{512} (Z_o + \bar{Z}_o) z^4 + O(z^6) \tag{5.4.37}$$

However it can easily be shown that (5.4.37) is the small z series solution of (5.4.35) with the following boundary conditions

$$W_s^7 = \mathcal{D}^2 W_s^7 = 0, \quad \mathcal{D}^4 W_s^7 = \frac{3a^4\epsilon^2 R_o^2 P_o}{64}\left\{Z_o + \tilde{Z}_o\right\}, \; z=0 \qquad (5.4.38)$$

Thus the problem reduces to solving (5.4.35) with boundary conditions (5.4.36) , (5.4.38) since the solution will ~~nec~~necessarily satisfy (5.4.37) at the edge of the inner layer for some P_7, Q_7, S_7 .In actual fact we do not determine W_s but merely use the condition that the system specified above has a solution . This reduces to

$$\frac{R_7}{R_o} = \frac{-81\sqrt{2}\pi^8}{256}\left\{Z_o + \tilde{Z}_o\right\}$$

and using the expression for Z_o already calculated we can show that

$$\frac{R_7}{R_o} = \frac{27\sqrt{2}\,\pi^8 \cdot 2}{256\,(p_a-1)^3}\left\{16p_a^4 - 69p_a^3 + 66p_a^2 - 21p_a + \frac{16p_a^5(2\sqrt{p_a+1}-p_a)}{(1+p_a)^2} \cancel{- \frac{16p_a^2(p_a-1)^2}{(p_a+1)^2} + \frac{32p_a}{(p_a+1)^2}}^{1/2}\right\}, \; p_a \neq 1$$

$$\frac{R_7}{R_o} = \frac{27.35\,\cancel{1709}\sqrt{2}\,\pi^8}{256}, \; p_a = 1 \qquad (5.4.39)$$

a,b

5.5 Discussion of results

We have seen that the method of Chapter 111 may be applied to the thermal convection problem when the temperature of the lower plane is modulated about a fixed value . The critical Rayleigh number associated with (5.3.24) is given by

$$R_c = \frac{27\pi^4}{4}\left\{1 + \frac{p_a\epsilon^2}{2(1+p_a)^2}\right\} - \left\{\frac{3p_a}{2(1+p_a)^4}\left(1+\frac{9}{64}(1+p_a)^2\right) + 0.015\pi^{-4}\right\}\epsilon^2\sigma^2 + O(\epsilon^4, \epsilon^2\sigma^4) \qquad (5.5.1)$$

The dependence of a on ϵ first affects the critical Rayleigh number at order ϵ^4. This result has also been obtained by Herbert by a Galerkin type of method .

The work of Herbert was in fact done without relating the parameters ϵ , σ and in view of the well known similarity between the thermal convection problem and the cylinder problem discussed earlier we have some justification for assuming that the parameters ϵ , σ in (3.5.7) are independent .

The result of Venezian (1969) corresponding to (5.5.1) differs in the order $\epsilon^2\sigma^2$ term . Venezian obtained a solution to the problem by letting ϵ tend to zero with σ arbitrary . He expanded the perturbation velocity in powers of ϵ . We feel that the discrepancy between our results and his may be due to the fact that the latter does not allow for any time dependence in the order ϵ^0 velocity field . It follows from (5.5.1) that the flow is destabilized as σ increases with ϵ held fixed . In contrast to the cylinder problem the order ϵ^2 term is positive thus stabilizing the flow .

For large σ with ϵ arbitrary we found that

$$R = R_0 + \frac{R_7}{\sigma^{7/2}} + O(\sigma^{-4}) \tag{5.5.2}$$

where R_0 and R_7 are given by (5.4.33) ,(5.4.39) respectively . This is in fact the critical Taylor number at which instability first occurs since taking the variation of a with σ into account shows that the critical Rayleigh number is first affected at order σ^{-7} by this effect .

Venezian says that the first correction term in (5.5.2) is of order σ^{-2} but the reason for this statement is not clear . A calculation for the rigid boundary problem shows that the corresponding correction term there would be of order σ^{-5} if we were to repeat the method of solution desribed in §5.4 for that problem . Thus it seems that the nature of the boundary conditions is important in determining the order of magnitude of the correction term .

TABLE 1

ζ	$g_0^+(\zeta)$	$f_0^+(\zeta)$
0.00000	0.00000	0.00000
0.02500	-0.02502	0.00000
0.05000	-0.05010	0.00001
0.07500	-0.07519	0.00003
0.10000	-0.10015	0.00005
0.12500	-0.12474	0.00008
0.15000	-0.14871	0.00010
0.17500	-0.17178	0.00013
0.20000	-0.19364	0.00016
0.22500	-0.21402	0.00019
0.25000	-0.23265	0.00022
0.27500	-0.24929	0.00025
0.30000	-0.26376	0.00027
0.32500	-0.27589	0.00030
0.35000	-0.28556	0.00032
0.37500	-0.29271	0.00034
0.40000	-0.29731	0.00035
0.42500	-0.29937	0.00036
0.45000	-0.29896	0.00037
0.47500	-0.29617	0.00037
0.50000	-0.29113	0.00037
0.52500	-0.28399	0.00037
0.55000	-0.27494	0.00036
0.57500	-0.26419	0.00035
0.60000	-0.25194	0.00034
0.62500	-0.23841	0.00032
0.65000	-0.22383	0.00030
0.67500	-0.20842	0.00028
0.70000	-0.19239	0.00026
0.72500	-0.17594	0.00023
0.75000	-0.15925	0.00020
0.77500	-0.14247	0.00018
0.80000	-0.15274	0.00015
0.82500	-0.10916	0.00012
0.85000	-0.09281	0.00009
0.87500	-0.07673	0.00007
0.90000	-0.06094	0.00005
0.92500	-0.04543	0.00003
0.95000	-0.03014	0.00001
0.97500	-0.01503	0.00000
1.00000	-0.00000	0.00000

TABLE 2

ζ	$g_0(\zeta)$	$f_0(\zeta)$
0.00000	0.00000	0.00000
0.02500	-0.01231	0.04215
0.05000	-0.02467	0.15874
0.07500	-0.03708	0.33572
0.10000	-0.04950	0.56013
0.12500	-0.06187	0.81997
0.15000	-0.07411	1.10426
0.17500	-0.08611	1.42098
0.20000	-0.09775	1.70706
0.22500	-0.10893	2.00834
0.25000	-0.11952	2.29961
0.27500	-0.12940	2.57454
0.30000	-0.13846	2.82766
0.32500	-0.14660	3.05436
0.35000	-0.15373	3.25087
0.37500	-0.15977	3.41419
0.40000	-0.16465	3.54210
0.42500	-0.16833	3.63311
0.45000	-0.17077	3.68643
0.47500	-0.17194	3.70191
0.50000	-0.17186	3.68003
0.52500	-0.17052	3.62183
0.55000	-0.16797	3.52890
0.57500	-0.16423	3.40331
0.60000	-0.15938	3.42756
0.62500	-0.15347	3.06458
0.65000	-0.14658	2.85766
0.67500	-0.13880	2.63004
0.70000	-0.13023	2.38686
0.72500	-0.12096	2.13114
0.75000	-0.11109	1.86778
0.77500	-0.10074	1.60155
0.80000	-0.09001	1.33745
0.82500	-0.07898	1.08076
0.85000	-0.06776	0.83702
0.87500	-0.05644	0.61209
0.90000	-0.04507	0.41213
0.92500	-0.03372	0.24370
0.95000	-0.02242	0.11379
0.97500	-0.01119	0.02987
1.00000	-0.00000	0.00000

TABLE 3

ζ	$g_1(\zeta)$	$f_1(\zeta)$
0.00000	0.00000	0.00000
0.02500	-0.09814	0.35985
0.05000	-0.19680	1.35263
0.07500	-0.29609	2.85589
0.10000	-0.39578	4.75703
0.12500	-0.49536	6.95296
0.15000	-0.59410	9.34970
0.17500	-0.69115	11.86212
0.20000	-0.78553	14.41359
0.22500	-0.87623	16.93577
0.25000	-0.96224	19.36837
0.27500	-1.04254	21.65883
0.30000	-1.11619	23.76218
0.32500	-1.18231	25.64069
0.35000	-1.24013	27.26365
0.37500	-1.28898	28.60708
0.40000	-1.32832	29.65339
0.42500	-1.35773	30.39113
0.45000	-1.37694	30.81454
0.47500	-1.38580	30.92330
0.50000	-1.38431	30.72211
0.52500	-1.37261	30.22032
0.55000	-1.35093	29.43155
0.57500	-1.31967	28.37338
0.60000	-1.27931	27.06694
0.62500	-1.23043	25.53658
0.65000	-1.17372	23.80961
0.67500	-1.10991	21.91599
0.70000	-1.03981	19.88815
0.72500	-0.96428	17.76077
0.75000	-0.88418	15.57067
0.77500	-0.80041	13.35683
0.80000	-0.71382	11.16029
0.82500	-0.62525	9.02437
0.85000	-0.53547	6.99475
0.87500	-0.44517	5.11982
0.90000	-0.35492	3.45099
0.92500	-0.26514	2.04316
0.95000	-0.17608	0.95531
0.97500	-0.08778	0.25118
1.00000	0.00000	0.00000

TABLE 4

ζ	$g_2(\zeta)$	$f_2(\zeta)$
0.00000	0.00000	0.00000
0.02500	0.02502	-0.08037
0.05000	0.05011	-0.30259
0.07500	0.07525	-0.63979
0.10000	0.10036	-1.06705
0.12500	0.12527	-1.56140
0.15000	0.14978	-2.10174
0.17500	0.17368	-2.66886
0.20000	0.19673	-3.24540
0.22500	0.21868	-3.81581
0.25000	0.23930	-4.36632
0.27500	0.25837	-4.88492
0.30000	0.27566	-5.36131
0.32500	0.29100	-5.78682
0.35000	0.30424	-6.15442
0.37500	0.31524	-6.45859
0.40000	0.32389	-6.69530
0.42500	0.33014	-6.86192
0.45000	0.33394	-6.95712
0.47500	0.33529	-6.98085
0.50000	0.33422	-6.93418
0.52500	0.33076	-6.81928
0.55000	0.32500	-6.63927
0.57500	0.31705	-6.39819
0.60000	0.30702	-6.10091
0.62500	0.29505	-5.75303
0.65000	0.28131	-5.36083
0.67500	0.26596	-4.93117
0.70000	0.24919	-4.47152
0.72500	0.23119	-3.98983
0.75000	0.21214	-3.49452
0.77500	0.19223	-2.99450
0.80000	0.17166	-2.49912
0.82500	0.15058	-2.01821
0.85000	0.12918	-1.56208
0.87500	0.10759	-1.14158
0.90000	0.08593	-0.76817
0.92500	0.06431	-0.45395
0.95000	0.04277	-0.21182
1.00000	0.00000	0.00000

TABLE 5

ζ	$g_3(\zeta)$	$f_3(\zeta)$
0.00000	0.00000	0.00000
0.02500	0.02505	-0.11128
0.05000	0.05032	-0.41738
0.07500	0.07593	-0.87939
0.10000	0.10182	-1.46175
0.12500	0.12789	-2.13216
0.15000	0.15394	-2.86142
0.17500	0.17971	-3.62323
0.20000	0.20492	-4.39418
0.22500	0.22927	-5.15354
0.25000	0.25244	-5.88319
0.27500	0.27414	-6.56749
0.30000	0.29406	-7.19321
0.32500	0.31194	-7.74940
0.35000	0.32754	-8.22726
0.37500	0.34067	-8.62007
0.40000	0.35116	-8.92305
0.42500	0.35888	-9.13324
0.45000	0.36376	-9.24939
0.47500	0.36575	-9.27183
0.50000	0.36487	-9.20233
0.52500	0.36117	-9.04401
0.55000	0.35471	-8.80120
0.57500	0.34564	-8.47931
0.60000	0.33411	-8.08475
0.62500	0.32031	-7.62480
0.65000	0.30444	-7.10754
0.67500	0.28675	-6.54179
0.70000	0.26749	-5.93700
0.72500	0.24692	-5.30326
0.75000	0.22530	-4.65126
0.77500	0.20290	-3.99231
0.80000	0.17999	-3.33835
0.82500	0.15680	-2.70200
0.85000	0.13355	-2.09668
0.87500	0.11044	-1.53667
0.90000	0.08761	-1.03732
0.92500	0.06516	-0.61515
0.95000	0.04312	-0.28814
0.97500	0.02145	-0.07589
1.00000	0.00001	0.00002

TABLE 6

ζ	$g_4(\zeta)$	$f_4(\zeta)$
0.00000	0.00000	0.00000
0.02500	0.02505	-0.11331
0.05000	0.05033	-0.42497
0.07500	0.07595	-0.89528
0.10000	0.10187	-1.48802
0.12500	0.12797	-2.17029
0.15000	0.15405	-2.91233
0.17500	0.17987	-3.68741
0.20000	0.20513	-4.47167
0.22500	0.22953	-5.24403
0.25000	0.25275	-5.98608
0.27500	0.27449	-6.68191
0.30000	0.29445	-7.31808
0.32500	0.31237	-7.88346
0.35000	0.32801	-8.36913
0.37500	0.34116	-8.76826
0.40000	0.35166	-9.07601
0.42500	0.35939	-9.28938
0.45000	0.36426	-9.40713
0.47500	0.36624	-9.42959
0.50000	0.36533	-9.35859
0.52500	0.36159	-9.19731
0.55000	0.35510	-8.95015
0.57500	0.34598	-8.62264
0.60000	0.33440	-8.22127
0.62500	0.32054	-7.75346
0.65000	0.30462	-7.22744
0.67500	0.28688	-6.65213
0.70000	0.26756	-6.03717
0.72500	0.24693	-5.39280
0.75000	0.22527	-4.72989
0.77500	0.20283	-4.05990
0.80000	0.17988	-3.39498
0.82500	0.15667	-2.74795
0.85000	0.13341	-2.13243
0.87500	0.11030	-1.56297
0.90000	0.08748	-1.05514
0.92500	0.06505	-0.62577
0.95000	0.04304	-0.29313
0.97500	0.02141	-0.07721
1.00000	0.00001	0.00002

TABLE 7

ζ	$g_5(\zeta)$	$f_5(\zeta)$
0.00000	0.00000	0.00000
0.02500	0.02501	-0.08083
0.05000	0.05009	-0.30487
0.07500	0.07521	-0.64577
0.10000	0.10031	-1.07899
0.12500	0.12525	-1.58172
0.15000	0.14986	-2.13292
0.17500	0.17395	-2.71328
0.20000	0.19729	-3.30526
0.22500	0.21967	-3.89299
0.25000	0.24086	-4.46235
0.27500	0.26062	-5.00089
0.30000	0.27874	-5.49781
0.32500	0.29504	-5.94394
0.35000	0.30933	-6.33172
0.37500	0.32145	-6.65510
0.40000	0.33128	-6.90955
0.42500	0.33873	-7.09196
0.45000	0.34372	-7.20057
0.47500	0.34621	-7.23496
0.50000	0.34620	-7.19591
0.52500	0.34370	-7.08533
0.55000	0.33877	-6.90623
0.57500	0.33149	-6.66258
0.60000	0.32196	-6.35928
0.62500	0.31030	-6.00203
0.65000	0.29668	-5.59734
0.67500	0.28124	-5.15236
0.70000	0.26418	-4.67491
0.72500	0.24568	-4.17339
0.75000	0.22594	-3.65672
0.77500	0.20517	-3.13437
0.80000	0.18356	-2.61628
0.82500	0.16131	-2.11293
0.85000	0.13860	-1.63528
0.87500	0.11559	-1.19486
0.90000	0.09243	-0.80376
0.92500	0.06924	-0.47478
0.95000	0.04609	-0.22142
0.97500	0.02301	-0.05805
1.00000	0.00000	0.00000

TABLE 8

ζ	$g_6(\zeta)$	$f_6(\zeta)$
0.00000	0.00000	0.00000
0.02500	0.02502	-0.08646
0.05000	0.05014	-0.32555
0.07500	0.07535	-0.68847
0.10000	0.10060	-1.14857
0.12500	0.12576	-1.68125
0.15000	0.15066	-2.26396
0.17500	0.17508	-2.87615
0.20000	0.19880	-3.49921
0.22500	0.22157	-4.11644
0.25000	0.24317	-4.71303
0.27500	0.26334	-5.27600
0.30000	0.28186	-5.79418
0.32500	0.29852	-6.25811
0.35000	0.31313	-6.66008
0.37500	0.32552	-6.99395
0.40000	0.33557	-7.25521
0.42500	0.34317	-7.44081
0.45000	0.34824	-7.54915
0.47500	0.35075	-7.57996
0.50000	0.35068	-7.53424
0.52500	0.34806	-7.41416
0.55000	0.34295	-7.22299
0.57500	0.33542	-6.96498
0.60000	0.32559	-6.64531
0.62500	0.31360	-6.26999
0.65000	0.29960	-5.84579
0.67500	0.28378	-5.38017
0.70000	0.26632	-4.88121
0.72500	0.24743	-4.35758
0.75000	0.22732	-3.81850
0.77500	0.20620	-3.27371
0.80000	0.18428	-2.73344
0.82500	0.16178	-2.20848
0.85000	0.13887	-1.71015
0.87500	0.11574	-1.25039
0.90000	0.09253	-0.84179
0.92500	0.06935	-0.49770
0.95000	0.04628	-0.23236
0.97500	0.02334	-0.06099
1.00000	0.00051	0.00000

TABLE 9

ζ	$g_9(\zeta)$	$f_9(\zeta)$
0.00000	0.00000	0.00000
0.02500	0.02507	-0.10589
0.05000	0.05048	-0.39273
0.07500	0.07637	-0.81866
0.10000	0.10270	-1.34705
0.12500	0.12930	-1.94599
0.15000	0.15594	-2.58774
0.17500	0.18229	-3.24829
0.20000	0.20803	-3.90702
0.22500	0.23278	-4.54636
0.25000	0.25622	-5.15150
0.27500	0.27801	-5.71019
0.30000	0.29785	-6.21249
0.32500	0.31548	-6.65059
0.35000	0.33066	-7.01863
0.37500	0.34322	-7.31252
0.40000	0.35301	-7.52982
0.42500	0.35995	-7.66954
0.45000	0.36399	-7.73202
0.47500	0.36512	-7.71880
0.50000	0.36338	-7.63244
0.52500	0.35885	-7.47642
0.55000	0.35163	-7.25503
0.57500	0.34187	-6.97321
0.60000	0.32974	-6.63647
0.62500	0.31544	-6.25083
0.65000	0.29919	-5.82269
0.67500	0.28121	-5.35883
0.70000	0.26177	-4.86632
0.72500	0.24112	-4.35260
0.75000	0.21952	-3.82541
0.77500	0.19723	-3.29292
0.80000	0.17452	-3.76379
0.82500	0.15162	-2.24730
0.85000	0.12877	-1.75352
0.87500	0.10616	-1.29358
0.90000	0.08395	-0.87985
0.92500	0.06223	-0.52637
0.95000	0.04105	-0.24915
0.97500	0.02035	-0.06671
1.00000	- 0.00005	-0.00053

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APPENDIX

STEADY FLOW IN AN AXISYMMETRIC PIPE OF SLOWLY VARYING RADIUS

We now use the method of the first two chapters to consider steady flow in an axisymmetric pipe of slowly varying radius . If D_0 is a characteristic radius of the pipe and and L a characteristic length along the pipe then we define a parameter δ by

$$\delta = \mathcal{D}_0^2 / L^2 \qquad \text{A1}$$

Since the pipe radius varies slowly we assume that δ is small . If U_0 is a characteristic velocity along the pipe we define a modified Reynolds number , R_M , by

$$R_M = U_0 \mathcal{D}_0^2 / L\nu \qquad \text{A2}$$

where ν is the kinematic viscosity . We assume that R_M is small and we shall seek a solution by expanding the velocity and pressure in powers of R_M and δ . The pressure difference which we maintain between the ends of the pipe is steady . If the pipe is defined in cylindrical polar coordinates (r, θ, z) by

$$0 \leqslant z \leqslant KL \;, \quad 0 \leqslant r \leqslant \mathcal{D}_0 R\left(\tfrac{z}{L}\right) \qquad \text{A3}$$

then we impose a pressure difference such that

$$p(R(K), K) - p(R(0), 0) = G_0$$

The order δ pressure term is found to be a function of both the radial and axial variables in contrast to the pressures evaluated in the first two chapters which were only functions of one variable . It is for this reason that we prescribe the above pressure difference evaluated at the pipe wall .

Manton (1971) has considered the steady problem for an infinitely long pipe . However he based his Reynolds number on the flux through the pipe and hence insisted that there was no net flux through the pipe associated with the order R_M, δ, etc. velocity components . Having done this the pressure difference between any two points is fixed . This is in contrast to our work where we first impose a pressure difference , on which we base our Reynolds number , and the flux through the pipe is then fixed . We shall see that the methods are in fact equivalent .

Using the notation of Chapter 11 we can write the momentum equations in the form

$$\left.\begin{aligned} R_M\delta\left(g\frac{\partial g}{\partial \eta}+f\frac{\partial g}{\partial \zeta}\right) &= -\frac{\partial P}{\partial \eta}+\delta\left(\frac{\partial^2}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial}{\partial \eta}-\frac{1}{\eta^2}\right)g+\delta^2\frac{\partial^2 g}{\partial \zeta^2} \\ R_M\left(g\frac{\partial f}{\partial \eta}+f\frac{\partial f}{\partial \zeta}\right) &= -\frac{\partial P}{\partial \zeta}+\left(\frac{\partial^2}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial}{\partial \eta}\right)f+\delta\frac{\partial^2 f}{\partial \zeta^2} \end{aligned}\right\} \qquad \text{A4}$$

and the equation of continuity is

$$\frac{\partial}{\partial \eta}(\eta g)+\eta\frac{\partial f}{\partial \zeta}=0 \qquad \text{A5}$$

The required boundary conditions are

a. There is no velocity at the surface of the pipe , ie f and g are zero at $\eta = R(\zeta)$.

b. The pressure difference , $P(R(K),K) - P(R(0),0)$, is independent of R_M , δ and is equal to α say .

c. The velocity is regular at $\eta = 0$. A6 a,b,c

Suppose that we expand f , g , P in the form

$$\left.\begin{aligned} f = f_0+\delta f_1+R_M f_M+R_M^2 f_{MM}+O(R_M^3,\delta^2,R_M\delta) \end{aligned}\right\} \qquad \text{A7 a,b,c}$$

etc.

If we substitute these expressions into A4 we obtain the following system of equations after equating terms of order 1, δ, R_M, R_M^2

$$\left.\begin{aligned} D^2 f_0 &= \frac{\partial P_0}{\partial \zeta}, \quad \frac{\partial P_0}{\partial \eta}=0 \\ D^2 f_1 &= \frac{\partial P_1}{\partial \zeta}-\frac{\partial^2 f_0}{\partial \zeta^2}, \quad \frac{\partial P_1}{\partial \eta}=(D^2-1/\eta^2)g_0 \\ D^2 f_M &= \frac{\partial P_M}{\partial \zeta}+g_0\frac{\partial f_0}{\partial \eta}+f_0\frac{\partial f_0}{\partial \zeta}, \quad \frac{\partial P_M}{\partial \eta}=0 \\ D^2 f_{MM} &= \frac{\partial P_{MM}}{\partial \zeta}+g_0\frac{\partial f_M}{\partial \eta}+g_M\frac{\partial f_0}{\partial \eta}+f_0\frac{\partial f_M}{\partial \zeta}+f_M\frac{\partial f_0}{\partial \zeta}, \quad \frac{\partial P_{MM}}{\partial \eta}=0 \end{aligned}\right\} \qquad \text{A8 a,b,c,d}$$

where $D^2 = \partial^2/\partial\eta^2 + \eta^{-1}\partial/\partial\eta$

and the boundary conditions from A6 are

$$\left.\begin{aligned} &f_0 = f_1 = \cdots \quad g_0 = g_1 = \cdots\cdots = 0\,,\ \eta = R \\ &P_0(K) - P_0(0) = \alpha, P_1(R(K),K) - P_1(R(0),0) = P_M(K) - P_M(0) = P_{MM}(K) - P_{MM}(0) = 0 \end{aligned}\right\} \quad \text{A9 a,b}$$

where we have used the fact that from A8 , P_1 alone is a function of both η and ζ . If we substitute for f and g from A7 into the equation of continuity and equate terms of order $1, \delta, R_M, R_M^2$ we can show that

$$\frac{\partial}{\partial\eta}(\eta g_K) + \eta\frac{\partial f_K}{\partial\zeta} = 0 \qquad \text{A10}$$

for K = 0,1,M,MM

If w is the axial velocity along the pipe the flux through any cross-section of the pipe is

$$Q = 2\pi\int_0^{D_0 R} r w\, dr$$

But $w = U_0 f$, $\eta = r/D_0$ and so Q is given by

$$Q = 2\pi\int_0^R U_0 D_0^2 \eta f\, d\eta$$

Suppose that we define a stream function ,ψ, by

$$\eta f = \frac{\partial\psi}{\partial\eta}\ ,\ \eta g = -\frac{\partial\psi}{\partial\zeta}$$

then it follows that Q is given by

$$Q = 2\pi D_0^2 U_0\, \psi\Big|_0^R$$

If we choose ψ to be zero along the axis of the pipe and then use A7 we have that

$$Q = 2\pi U_0 D_0^2 \left\{\psi_0 + \psi_1\delta + \psi_M R_M + \cdots\right\}_{\eta=R}$$

where ψ_0 corresponds to f_0, g_0 etc.

<u>Evaluation of f_0, g_0, P_0, etc.</u>

Using A8 , A9 we can show that

$$f_o = \frac{1}{4}\frac{dP_o}{d\zeta}\left\{\eta^2 - R^2\right\} \qquad \text{A11}$$

and from A10 with K = 0 we can show that

$$\frac{\partial}{\partial\eta}(\eta g_o) = \frac{-1}{4}P_o''\left\{\eta^3 - \eta R^2\right\} + \frac{1}{2}P_o'\eta R'R \qquad \text{A12}$$

where a dash denotes a derivative with respect to ζ . We now integrate both sides of the above equation from $\eta = 0$ to $\eta = \eta$ and use the fact that g_o is regular at $\eta = 0$ to show that

$$\eta g_o = \frac{-1}{16}P_o''\left\{\eta^4 - 2\eta^2R^2\right\} + \frac{1}{4}R'RP_o'\eta^2 \qquad \text{A13}$$

and if we now put $\eta = R$ and use A9a we have that

$$0 = R^4P_o'' + 4R^3R'P_o' \qquad \text{A14}$$

which is the Reynolds equation for the pressure which we can integrate once to give

$$P_o' = \frac{A_o}{R^4} \qquad \text{A15}$$

where A_o is an arbitrary constant , which may be determined by integrating both sides of A15 from $\zeta = 0$ to $\zeta = K$ and then using A7b, thus giving

$$A_o = \alpha \Big/ \int_0^K \frac{d\zeta}{R^4} \qquad \text{A16}$$

and for convenience we choose $\alpha = 16\int_0^K d\zeta/R^4$. Using A15 we can then show that

$$f_o = -4\left\{\frac{\eta^2 - R^2}{R^4}\right\}, \quad g_o = -4R'\left\{\frac{\eta^3 - \eta R^2}{R^5}\right\} \qquad \text{A17 a,b}$$

from which we can deduce that the corresponding streamfunction ψ_o is given by

$$\psi_o = -\left\{(\eta/R)^4 - 2(\eta/R)^2\right\} \qquad \text{A18}$$

and from A15 , A16 , with $\alpha = -16\int_0^K d\zeta/R^4$ we can show that

$$P_0(\zeta) - P_0(0) = -16\int_0^{\zeta}\frac{d\zeta}{R^4} \qquad \text{A19}$$

We now evaluate the order δ pressure and velocity . From A8b , with f_0, g_0 as given by A17 , we can show that

$$\frac{\partial P_1}{\partial \eta} = -32\eta\frac{R'}{R^5} \qquad \text{A20}$$

and

$$D^2 f_1 = \frac{\partial P_1}{\partial \zeta} - 16\eta^2\left(\frac{R'}{R^5}\right)' + 8\left(\frac{R'}{R^3}\right)' \qquad \text{A21}$$

Integrating A20 once we obtain

$$P_1 = Q_1 - 16\eta^2\frac{R'}{R^5} \qquad \text{A22}$$

where Q_1 is an arbitary function of ζ . If we substitute for P_1 from A22 into A21 we have

$$D^2 f_1 = Q_1' + 8\left(\frac{R'}{R^3}\right)' - 32\eta^2\left(\frac{R'}{R^5}\right)'$$

which we integrate to give

$$f_1 = \frac{1}{4}\left\{Q_1' + 8\left(\frac{R'}{R^3}\right)'\right\}\left\{\eta^2 - R^2\right\} - 2\left(\frac{R'}{R^5}\right)'\left\{\eta^4 - R^4\right\} \qquad \text{A23}$$

and if we now substitute into A11, with K = 1 , integrate from $\eta = 0$ to $\eta = \eta$, and use the fact that g is regular at $\eta = 0$ we can show that

$$\eta g_1 = -\frac{1}{16}\left[Q_1' + 8\left(\frac{R'}{R^3}\right)'\right]'\left[\eta^4 - 2\eta^2R^2\right] + \frac{1}{4}\left[Q_1' + 8\left(\frac{R'}{R^3}\right)'\right]R'R\eta^2$$
$$+ 2\left(\frac{R'}{R^5}\right)''\left\{\frac{\eta^6 - 3\eta^2R^4}{6}\right\} - 4\left(\frac{R'}{R^5}\right)'R^3R'\eta^2 \qquad \text{A24}$$

If we put $\eta = R$ in A24 and use A6a we obtain the Reynolds equation for Q .

$$0 = \left\{\left[Q_1' + 8\left(\frac{R'}{R^3}\right)'\right]R^4\right\}' - \frac{8}{3}\left\{\left(\frac{R'}{R^5}\right)'R^6\right\}' \qquad \text{A25}$$

We integrate this equation once to give

$$\frac{16 B_1}{R^4} = Q_1' + 8\left(\frac{R'}{R^3}\right)' - \frac{32}{3}\left\{R^2\left(\frac{R'}{R^5}\right)'\right\} \qquad \text{A26}$$

where B_1 is an arbitary constant . Using A22 , A26 we can show that

$$P_1(\eta,\zeta) = -16\eta^2\left(\frac{R'}{R^5}\right) + \int_0^\zeta \frac{16B_1}{R^4} - \frac{8R'}{R^3} + \frac{32}{3}\int_0^\zeta R^2\left(\frac{R'}{R^5}\right)' d\zeta + \frac{24R'(0)}{R^3(0)} + P_1(R(0),0) \qquad \text{A27}$$

from which we deduce that , after using A9b , B_1 is determined by

$$B_1\int_0^K \frac{d\zeta}{R^4} = 24\left(\frac{R'(K)}{R^3(K)} - \frac{R'(0)}{R^3(0)}\right) - \frac{32}{3}\int_0^K R^2\left(\frac{R'}{R^5}\right)' d\zeta \qquad \text{A28}$$

The associated stream function , ψ_1, is seen from A23 , A24 , A26 to be given by

$$\psi_1 = B_1\left\{(\eta/R)^4 - 2(\eta/R)^2\right\} + \left\{5R'^2 - RR''\right\}\left\{\frac{(\eta/R)^6 - 2(\eta/R)^4 + (\eta/R)^2}{3}\right\} \qquad \text{A29}$$

We shall now calculate the order R_M velocity and pressure . If we substitute for f_0 and g_0 from A17 into A8c we can show that

$$D^2 f_M = \frac{\partial P_M}{\partial \zeta} - \frac{32R'}{R^9}\left\{\eta^4 - 2\eta^2R^2 + R^4\right\}$$

and $\qquad$ A30 a,b

$$\frac{\partial P_M}{\partial \eta} = 0$$

The solution of A30 which satisfies the boundary condition A9a is

$$f_M = \frac{1}{4}\left\{P_M' - \frac{32R'}{R^5}\right\}\left\{\eta^2 - R^2\right\} - \frac{8R'}{9R^9}\left\{\eta^6 - R^6\right\} + \frac{4R'}{R^7}\left\{\eta^4 - R^4\right\} \qquad \text{A31}$$

Substituting for f_M in A10 with K = M and integrating from $\eta = 0$ to $\eta = \eta$ and using the fact that g_M is regular at $\eta = 0$ we can show that

$$\begin{aligned}\eta g_M = {} & \frac{1}{16}\left\{\eta^4 - 2\eta^2R^2\right\}\left\{P_M' - \frac{32R'}{R^5}\right\}' + \frac{1}{4}\eta^2R'R\left\{P_M' - \frac{32R'}{R^5}\right\} \\ & + \frac{1}{9}\left(\frac{R'}{R^9}\right)'\left\{\eta^8 - 4\eta^2R^6\right\} - \frac{8}{3}R'R^5\eta^2\left(\frac{R'}{R^9}\right) - \frac{2}{3}\left(\frac{R'}{R^7}\right)'\left\{\eta^6 - 3\eta^2R^4\right\} \\ & + 8\left(\frac{R'}{R^7}\right)\eta^2R'R^3\end{aligned} \qquad \text{A32}$$

If we now put $\eta = R$ and use A10 we obtain the Reynolds equation for P_M

$$0 = \left\{P_M'R^4\right\}' - 16\left\{\frac{R'}{R}\right\}' \qquad \text{A33}$$

which we integrate once to give

$$P_M' = \frac{16C_M}{R^4} + \frac{16R'}{R^5}. \qquad \text{A34}$$

and now if we integrate both sides of A34 from $\zeta = 0$ to $\zeta = K$ and use A9b we find that the constant C_M is given by

$$C_M = \frac{\{R^{-4}(K) - R^{-4}(0)\}}{4\int_0^K \frac{d\xi}{R^4}} \qquad \text{A35}$$

It is an easy matter to show that the streamfunction associated with the order R_M velocity field is

$$\psi_M = C_M\left\{(\eta/R)^4 - 2(\eta/R)^2\right\} - \frac{R'}{R^9}\left\{\frac{1}{9}(\eta/R)^8 - \frac{2}{3}(\eta/R)^6 + (\eta/R)^4 - \frac{4}{9}(\eta/R)^2\right\} \qquad \text{A36}$$

The order R_M^2 velocity and pressure can now be calculated in the same way as that just described for the order R_M quantities. After a great deal of algebra we find that

$$P_{MM} = \frac{16 D_{MM}}{R^4} - \frac{1}{45R^4}\left(\frac{11P}{3} - 11Q\right) - 32 C_M \frac{R'}{R^5}$$

$$\psi_{MM} = \sum_{n=1}^{6} (-1)^{n+1} a_n (\eta/R)^{2n} - \frac{4}{3} C_M \frac{R'}{R}\left\{(\eta/R)^6 - 3(\eta/R)^2\right\}$$
$$+ \frac{2}{9} C_M \frac{R'}{R}\left\{(\eta/R)^8 - 4(\eta/R)^2\right\} + \left\{D_{MM} + 2C_M\frac{R'}{R}\right\}\left\{(\eta/R)^4 - 2(\eta/R)^2\right\}$$

A37 a,b

where Q and P are defined by

$$Q = -\frac{4R''}{R} + \frac{20R'^2}{R^2}, \quad P = \frac{16R'^2}{R^2}$$

and

$$a_1 = \frac{1}{900}\left(\frac{19P}{24} + 13Q\right), \quad a_2 = \frac{1}{720}\left(29Q + \frac{11P}{3}\right)$$

$$a_3 = \frac{1}{24}\left(\frac{2P}{9} + Q\right), \quad a_4 = \frac{1}{144}(P - 13Q)$$

$$a_5 = \frac{1}{480}\left(P + \frac{8Q}{3}\right), \quad a_6 = \frac{1}{1800}\left(\frac{P}{3} + Q\right)$$

The constant D_{MM} is determined by the following equation

$$16 D_{MM}\int_0^K \frac{d\xi}{R^4} = \frac{1}{45}\int_0^K \left(\frac{11P}{3} - 11Q\right) R^{-4}\, d\xi + 8C_M\left(R^{-4}(0) - R^{-4}(K)\right) \qquad \text{A38}$$

The recovery of Manton's results

We have shown that the streamfunction may be written in the form

$$\psi = \left\{ \left(\tfrac{\eta}{R}\right)^4 - 2\left(\tfrac{\eta}{R}\right)^2 \right\} \left\{ -1 + B_1\delta + C_M R_M + D_{MM} R_M^2 \right\}$$
$$+ \delta\left\{5R'^2 - RR''\right\}\left\{ \tfrac{1}{3}\left(\tfrac{\eta}{R}\right)^6 - \tfrac{2}{3}\left(\tfrac{\eta}{R}\right)^4 + \tfrac{1}{3}\left(\tfrac{\eta}{R}\right)^2 \right\}$$
$$+ \frac{R_M R'}{R}\left\{ -\tfrac{1}{9}\left(\tfrac{\eta}{R}\right)^8 + \tfrac{2}{3}\left(\tfrac{\eta}{R}\right)^6 - \left(\tfrac{\eta}{R}\right)^4 + \tfrac{4}{9}\left(\tfrac{\eta}{R}\right)^2 \right\}\left\{1 - 2C_M^2 R_M\right\}$$
$$+ R_M^2 \sum_{n=1}^{6} (-1)^{n+1} a_n \left(\tfrac{\eta}{R}\right)^{2n} + O(R_M^3, R_M\delta, \delta^2) \qquad \text{A39}$$

This is exactly the same form as that found by Manton if we put $B_1 = C_M = D_{MM} = 0$. If we redefine the Reynolds number in an equivalent way to that used by Manton we see that our results are the same .

Using A39 we see that the flux through the tube is

$$Q = 2\pi U_0 D_0^2 \left\{ 1 - B_1\delta - C_M R_M - D_{MM} R_M^2 + O(R_M^3, \delta^2, R_M\delta) \right\}$$

and using A2 we have that

$$Q = 2\pi U_0 D_0^2 \left\{ 1 - B_1\delta - \frac{C_M U_0 D_0^2}{L\nu} - \frac{D_{MM} U_0^2 D_0^4}{L^2\nu^2} + O(R_M^3, R_M\delta, \delta^2) \right\} \qquad \text{A40}$$

If we define a modified Reynolds number , R_M^* , based on the flux by

$$R_M^* = \frac{Q}{2\pi L\nu}$$

and invert A40 to give

$$D_0^2 U_0 = \frac{Q}{2\pi}\left\{ 1 + B_1\delta + C_M R_M^* + (D_{MM} + 2C_M^2) R_M^{*2} + O(\delta^2, R_M^{*3}, R_M^*\delta) \right)$$

then

$$R_M = R_M^* \left\{ 1 + B_1\delta + C_M R_M^* + (D_{MM} + 2C_M^2) R_M^{*2} + O(\delta^2, R_M^{*3}, R_M^*\delta) \right\} \qquad \text{A41}$$

Suppose we define a streamfunction $\tilde{\psi}$ by

$$r u = \frac{\partial \tilde{\psi}}{\partial r} \quad , \quad r w = \frac{-\partial \tilde{\psi}}{\partial z}$$

where u , v are the velocity components correspondingto the coordinates r , z , then ψ and $\tilde{\psi}$ are related by

$$\tilde{\psi} = U_0 D_0^2 \psi$$

If we now use A39 , A40 ; A41 we can show that

$$\begin{aligned}\tilde{\psi} = \frac{Q}{2\pi}\Big[& -(\eta/R)^4 + 2(\eta/R)^2 + \delta\left(5R'^2 - R''R\right)\left(\tfrac{1}{3}(\eta/R)^6 - \tfrac{2}{3}(\eta/R)^4 + \tfrac{1}{3}(\eta/R)^2\right) \\ & + \frac{R_M^* R'}{R^9}\left\{-\tfrac{1}{9}(\eta/R)^8 + \tfrac{2}{3}(\eta/R)^6 - (\eta/R)^4 + \tfrac{4}{9}(\eta/R)^2\right\} \\ & + R_M^{*2}\sum_{n=1}^{6}(-1)^{n+1} a_n (\eta/R)^{2n} + O\left(\delta^2, R_M^*\delta, R_M^{*3}\right)\Big] \end{aligned} \qquad \text{A42}$$

which is the result obtained by Manton . We notice that A42 was derived without any knowledge of B_1 , etc. Thus A42 is valid for a finite or an infinite pipe . We have merely considered a pipe of finite length and based a Reynolds number on the pressure difference between the ends . Hence we were able to find explicit forms for $B_{1,1}$ etc. by stipulating that the pressure difference between the ends had no contribution of order δ , etc.