OUTPUT FEEDBACK: A GEOMETRIC APPROACH

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ABSTRACT

This thesis presents an original extension of the geometric theory of linear multivariable systems. The theory is developed for the case when feedback control for the system concerned is restricted to be derived from only the observable outputs of the system.

The main results obtained are in the field of non-interaction for linear multivariable systems, the chief application of the geometric theory introduced by Wonham and Morse, and described in the thesis. Necessary and sufficient conditions are obtained for the existence of decoupling controls, and a method of constructing a dynamic controller of low order is given for the case when it is not possible to obtain non-interaction by non-dynamic output feedback.

A further extension is made in the case of pole assignment by output feedback, and for the less stringent condition of system stabilisation, a theorem providing a sufficient condition for stabilising a system by output feedback is proved using the geometric theory. Additional topics concerned with output feedback control are discussed including the geometric theory of observers.

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For Mark and Lucy

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NOTATION

Capital letters denote matrices; underlined capital letters denote linear vector spaces. The same symbol, e.g. A, is used to denote both a matrix A and its map, e.g. $A: \underline{X} \to \underline{X}$. The superscript T, e.g. A^T, is used to denote the transpose of a matrix. The zero space is denoted by 0 and the empty space by $\emptyset$. The dimension of a space $\underline{V}$ is denoted by $\dim(\underline{V})$, the rank of a matrix A by rank(A), and the range of A by $\underline{A}$ or $\{A\}$.

The sum of two spaces, i.e. the space spanned by the union of their bases, is denoted by +, that of several spaces by Σ. Their direct sum is denoted by $\oplus$. The orthogonal complement of a space $\underline{V}$ is denoted $\underline{V}^{\perp}$. $A^{-1}\underline{V}$ denotes the inverse image of $\underline{V}$ under A, or the set $\{x: Ax \in \underline{V}\}$ of vectors x of appropriate dimension. The null space of a map A is denoted by $\underline{N}(A)$. The restriction of a map A to a subspace $\underline{V}$ is denoted by $A|\underline{V}$, and $\{A|\underline{V}\}$ denotes the subspace defined by $\underline{V} + A\underline{V} + \ldots + A^{n-1}\underline{V}$, where A is a matrix with n columns and $\dim(\underline{V}) \leq n$.

For k a fixed positive integer, $\underline{k}$ denotes the set $\{1,2,\ldots,k\}$, and a set of k subspaces $\{\underline{R}_i\}$, $i \in \underline{k}$, is denoted by $\{\underline{R}_i\}_{\underline{k}}$. Also, $k_o = \{0,1,\ldots,k\}$.

If a(s) is a polynomial in s, $a^+(s)$, $a^-(s)$ are used to denote its factors with roots in the open right half or closed left half of the complex plane, respectively. For b(s) another polynomial, $a(s)|b(s)$ is read as "a(s) divides b(s)".

CHAPTER ONE

Introduction

1.1. Introduction

The problems of analysis and design of linear multivariable control systems have received considerable attention over a period of at least the past twenty years. This is not surprising since many physical systems can be approximated closely by systems of this type, though, for ease of analysis, a single predominant input only was often considered and utilised for control, with only a single output chosen to be controlled. Much work has been devoted recently however to extending results which have been obtained for single input, single output systems to the multivariable case, and ways have been found for using classical design techniques to deal with multivariable system design in the frequency domain (R1, M1), where the system is expressed usually in the form of a transfer function matrix. The technique of optimal control theory (AF1)(BH1) has been shown to be directly applicable to multivariable systems, the representation of the system in this case being in state variable form. However, this approach does not seem to solve many practical engineering problems. It appears therefore that no entirely satisfactory technique exists for overcoming the design problems inherent in multivariable systems.

In the last few years some attention has been devoted to the inherent structure which exists in a multivariable system. It is perhaps because insufficient regard is paid to this structure that the previously mentioned methods have not been entirely satisfactory. The module-theoretic treatment (K2) of the minimal realisation problem by Kalman demonstrated how a fundamental algebraic treatment of the structure of dynamical systems leads to a clear understanding of the problems of realisation and the development of successful algorithms (HK1). This theory has not however been shown to have an extension to the problems of control. The possibility of a useful geometric treatment of the structure of multivariable systems with direct application to control problems has been introduced by Wonham and Morse (WM1). This has been shown to give a precise interpretation of the problems of disturbance localization and decoupling, leading to a transparent explanation of the difficulties involved in achieving these design constraints. Further applications of this theory are being discovered in multivariable tracking (W2), (BhP1), observer theory (W3) and canonical representations of linear multivariable systems (WM2).

At present the geometric approach to such problems has been concerned with the case where no limitation is imposed on which states of the system are employed for feedback control. In this thesis, the restriction is made that feedback control may only be implemented from the system outputs, which are linear combinations of the states, and from which the states themselves cannot be directly measured. From an engineering viewpoint, output feedback is often the only case considered for practical reasons.

1.2. Outline of the thesis and original contributions

The multivariable control problem with particular reference to output feedback is introduced in Chapter Two. Also in this chapter, the basic geometric concepts of invariant subspaces and controllability subspaces of (A,B) are defined and their properties examined.

In Chapter Three the general problem of decoupling a linear time-invariant multivariable system, without the use of additional dynamics in the form of compensators, is considered using the geometric concepts of Chapter One. The whole of Section 3.3. is an original contribution to this problem. In particular, Theorem 3.3.1. establishes new necessary and sufficient conditions for a set of controllability subspaces to form a solution to the output feedback decoupling problem in its geometric interpretation. In the particular case when the number of inputs equals the number of output blocks to be decoupled, Theorem 3.3.2. provides new necessary and sufficient conditions for the existence of a solution to the output feedback decoupling problem.

The introduction of additional dynamics, and the solution of the resulting decoupling problem is considered in Chapter Four. The output feedback decoupling problem in this case is the subject of Sections 4.2 - 4.5, which represent an original contribution to the subject. Theorem 4.2.1. provides a necessary and sufficient condition for the existence of a dynamic output feedback decoupling control, and Theorem 4.3.1. establishes necessary and sufficient conditions for the existence of such a control using additional dynamics of low order, and such that complete controllability of the augmented state space is preserved in the decoupled system. In addition the proof of this theorem contains a new and original procedure for constructing a solution to the output feedback decoupling problem. The controllability subspaces so constructed moreover form a solution for which the augmented state space is completely spanned by the direct summation of these subspaces. This procedure is compared with existing methods using alternative approaches in Section 4.6.

The problem of pole assignment in multivariable systems is considered in Chapter Five. The output feedback pole assignment problem is approached using geometric ideas in Section 5.2., and Theorem 5.2.2. presents a new necessary condition for complete pole assignment in this case. Section 5.3. considers existing work on output feedback pole assignment and a new, but unproven, method is put forward for determining in a simple manner an output feedback matrix such that the closed loop system has a cyclic state space. Section 5.4. is concerned with the problem of stabilization of a multivariable system by output feedback. A geometric approach is used in Theorem 5.4.2. to provide a new concise proof of a recently established result, Theorem 5.4.1.. The remainder of this chapter is concerned with the addition of dynamics to achieve complete pole assignment by output feedback. In particular, Section 5.6. describes the geometric approach to observer theory, from which the concept of a dual observer follows clearly, the precise demonstration of this being original.

CHAPTER TWO

Feedback control and basic geometric concepts

2.1. Mathematical description of linear time-invariant multivariable systems

There exist in common usage three main ways in which the dynamical behaviour of a linear time-invariant multivariable, or multiple input, multiple output, system is described.

(i) state space description:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1.1}$$

$$y(t) = Cx(t) \tag{2.1.2}$$

where $u(t)$ is an m-vector of controlled inputs, $y(t)$ is a p-vector of measured outputs, $x(t)$ is an n-vector of state variables and A,B and C are matrices with real, constant elements.

(ii) weighting function matrix description:

$$y(t) = \int_0^t W(t,s)u(s)ds \tag{2.1.3}$$

where $W(t,s)$ is a matrix of weighting functions.

(iii) Laplace or Fourier transform description:

$$y(s) = G(s)u(s) \tag{2.1.4}$$

or

$$L(S)y(s) = M(s)u(s) \tag{2.1.5}$$

where G(s) is a matrix of rational transfer functions, L(s) and M(s) are matrices of polynomials in s.

Of course, explicit relationships exist between these three descriptions, (ii) and (iii) in particular being equivalent descriptions in the time and frequency domains respectively. The equations of (i) differ in their use of the concept of state variables x(t), where $x(t) \in \underline{X}$, the state (vector) space. In general, $\underline{X} = \underline{R}^n$, and this will be true in the following. A large proportion of the study of linear, time-invariant multivariable systems has made use of the state space formulation (i), and its mathematical properties are only now beginning to be fully understood (K1), (Po1). The results described in this thesis will be predominantly concerned with this formulation.

Let $\underline{U}$ denote the space of m-vector valued functions that are defined and continuous on $\underline{T} = (0,T)$. Consider the state evolution map F: $\underline{T} \times \underline{X} \times \underline{U} \rightarrow \underline{X}$. This is given by the equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds \qquad (2.1.6)$$

which defines the trajectory of the state $x(t)$, $0 \leq t \leq T$, under the influence of the control $u(t)$, $0 \leq t \leq T$, starting from an initial condition $x(0)$.

2.2. Feedback control

From a practical viewpoint, the most useful form of control is that which is generated by feedback from information related to the behaviour of the system. Owing initially to the popularity of the linear quadratic optimal regulator problem (AF1), the most widely studied form of feedback is that described by

$$u(t) = Fx(t) \qquad (2.2.1)$$

Since, in general, $C \neq I$ in most practical situations, a control of this form cannot be directly implemented. This may be overcome in one way, by implementing (2.2.1) in the form

$$u(t) = F\hat{x}(t) \qquad (2.2.2)$$

where $\hat{x}(t)$ denotes an estimate of the state vector $x(t)$, the determination of which may be based upon available information from the input $u(t_1)$ and output $y(t_1)$, $0 \leq t_1 \leq t$. The determination of $\hat{x}(t)$ takes the form of a dynamical system, termed as either an "observer" (L1),(BG1) in the deterministic case, or a "filter" (KB1),(B1) in the stochastic case.

In many cases it may be possible to choose F in (2.2.1) such that

$$F = KC \tag{2.2.3}$$

for some matrix K. The following lemma provides a necessary and sufficient condition on F for this to be possible.

Lemma 2.2.1.

A solution K to the matrix equation

$$F = KC \tag{2.2.3}$$

where K is $m \times p$, C and F are $p \times n$ and $m \times n$, and p and m, respectively, are their ranks, exists if and only if

$$\underline{N}(C) \subset \underline{N}(F) \tag{2.2.4}$$

Proof: Assume $\underline{N}(C) \subset \underline{N}(F)$. Taking orthogonal complements

$$\underline{F}^T \subset \underline{C}^T \tag{2.2.5}$$

Since $\underline{F}^T$ is spanned by the columns f_i, $i \in \underline{m}$, of F^T, thus

$$f_i = C^T k_i \tag{2.2.6}$$

for some $k_i \in \underline{R}^p$, $i \in \underline{m}$. Writing

$$K^T = (k_1 \ldots k_m) \tag{2.2.7}$$

yields the required solution.

Assuming K exists, transposing (2.2.3)

$$F^T = C^T K^T \tag{2.2.8}$$

Thus

$$\underline{F}^T \subset \underline{C}^T \tag{2.2.9}$$

or

$$\underline{N}(C) \subset \underline{N}(F) \tag{2.2.10}$$

If it is not possible to obtain a closed loop system with the required properties using a control where F satisfies (2.2.3), then it may be possible to define, by a suitable extension of the state space, a higher order system with the required closed loop properties and for which F satisfies (2.2.3). This approach has been termed "dynamic compensation" (P1),(PD1),(BP1). A specific form of state space extension developed by Morse and Wonham (MW1) will be used extensively in the following, and is described here.

2.3. State space extension

Let $\tilde{\underline{X}}$ denote an $\tilde{n}$-dimensional extension of the state space $\underline{X}$. Denoting the extended state space by $\hat{\underline{X}}$, then

$$\hat{\underline{X}} = \underline{X} \oplus \tilde{\underline{X}} \qquad (2.3.1)$$

Denoting the input and output spaces by $\underline{U}$ and $\underline{Y}$ respectively, the extended input and output spaces are defined by

$$\hat{\underline{U}} = \underline{U} \oplus \tilde{\underline{U}} \qquad (2.3.2)$$

and

$$\hat{\underline{Y}} = \underline{Y} \oplus \tilde{\underline{Y}} \tag{2.3.3}$$

The system (2.1.1),(2.1.2), and the additional dynamic elements

$$\dot{\tilde{x}}(t) = \tilde{B}\tilde{u}(t) \tag{2.3.4}$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) \tag{2.3.5}$$

where $\tilde{u} \in \tilde{\underline{U}}$, $\tilde{y} \in \tilde{\underline{Y}}$, can be described in the extended space by

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t) \tag{2.3.6}$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) \tag{2.3.7}$$

where $\hat{x}(t) = \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix}$, $\hat{y}(t) = \begin{bmatrix} y(t) \\ \tilde{y}(t) \end{bmatrix}$, $\hat{u}(t) = \begin{bmatrix} u(t) \\ \tilde{u}(t) \end{bmatrix}$,

$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} B & 0 \\ 0 & \tilde{B} \end{bmatrix}$, $\hat{C} = \begin{bmatrix} C & 0 \\ 0 & \tilde{C} \end{bmatrix}$.

In general, $\tilde{B}$ and $\tilde{C}$ will be chosen so that

$$\tilde{B} = \tilde{C} = I_{\tilde{n}} \tag{2.3.8}$$

the $\tilde{n} \times \tilde{n}$ identity matrix.

Consider now a feedback control described by

$$\hat{u}(t) = \hat{F}\hat{y}(t) + \hat{G}v(t) \tag{2.3.9}$$

where $\hat{F}$ can be expressed as

$$\hat{F} = \hat{K}\hat{C} \tag{2.3.10}$$

and $\hat{G}$, $\hat{K}$ have the partitioned forms

$$\hat{G} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{2.3.11}$$

Applying this control to the system (2.3.6),(2.3.7), and forming the closed loop transfer function between y(s) and v(s), yields

$$y(s) = (I - G(s)H(s))^{-1}G(s)K(s)v(s) \tag{2.3.12}$$

where $H(s) = K_{11} + K_{12}(sI - K_{22})^{-1}K_{21}$

$K(s) = G_1 + K_{12}(sI - K_{22})^{-1}G_2$

$G(s) = C(sI - A)^{-1}B$

This is therefore a more restrictive form of dynamic compensation than that considered by Rosenbrock (R1), where

H(s) and K(s) are not constrained to have the same characteristic polynomial, i.e. the determinant of $(sI - K_{22})$. However, its consideration does conveniently restrict the choice of compensators K(s), H(s) to be both physically realizable, and, if K_{22} is chosen accordingly, asymptotically stable.

2.4. Invariant and controllability subspaces of (A,B)

The development in this section is due to Wonham and Morse (WM1), who introduced the concept of an invariant subspace of (A,B). This is a natural extension to the feedback situation of the concept of simple invariance, i.e. a subspace $\underline{U} \subset \underline{X}$ is said to be A-invariant, if $A\underline{U} \subset \underline{U}$ for some map $A: \underline{X} \rightarrow \underline{X}$.

Definition 2.4.1.

A subspace $\underline{V} \subset \underline{X}$ is called an invariant subspace of (A,B), if it is (A + BF)-invariant for some F, i.e. if the set of matrices

$$F(\underline{V}) = \{ F: (A + BF)\underline{V} \subset \underline{V} \}$$

is non-empty.

Lemma 2.4.1.

The set $F(\underline{V})$ defined above is non-empty if and only if

$$A\underline{V} \subset \underline{V} + \underline{B} \qquad (2.4.1)$$

Proof: Necessity follows simply, since by definition

$$\underline{B} + (A + BF)\underline{V} \supset A\underline{V}$$

$$(A + BF)\underline{V} \subset \underline{V}$$

which implies (2.4.1). For sufficiency, let $\underline{X}_1$ be any subspace of $\underline{X}$ such that

$$\underline{V} \oplus \underline{X}_1 = \underline{X} \qquad (2.4.2)$$

and define Q: $\underline{X} \rightarrow \underline{X}_1$ as the projection of $\underline{X}$ onto $\underline{X}_1$ along $\underline{V}$. Then, from (2.4.1)

$$QA\underline{V} \subset Q\underline{B} \qquad (2.4.3)$$

or, writing V and B as basis matrices for $\underline{V}$ and $\underline{B}$,

$$QAV = QBZ \qquad (2.4.4)$$

But, by Lemma 2.2.1, Z can be written as

$$Z = -FV \qquad (2.4.5)$$

since $\underline{N}(V) = 0$ by choice of V as a basis matrix. Thus, (2.4.4) becomes

$$Q(A + BF)V = 0 \tag{2.4.6}$$

or, since $\underline{N}(Q) = \underline{V}$ by choice of Q

$$(A + BF)\underline{V} \subset \underline{V} \tag{2.4.7}$$

The second important geometric concept introduced by Wonham and Morse is that of a controllability subspace.

Definition 2.4.2.

A subspace $\underline{R} \subset \underline{X}$ is called a controllability subspace of (A,B) if for some F

$$\{A + BF \mid \underline{R} \cap \underline{B}\} = \underline{R} \tag{2.4.8}$$

Lemma 2.4.2.

Given A,B, and a subspace $\underline{R} \subset \underline{X}$, $\underline{R}$ is a controllability subspace of (A,B), if and only if

$$A\underline{R} \subset \underline{R} + \underline{B} \tag{2.4.9}$$

and

$$\underline{R} = \lim \underline{S}^j \qquad j = 0,1,\ldots \tag{2.4.10}$$

where $\underline{S}^o = 0$, $\underline{S}^{j+1} = (A\underline{S}^j + \underline{B}) \cap \underline{R}$.

Proof: From (2.4.8), $\underline{R}$ is an invariant subspace of (A,B) since

$$(A + BF)\underline{R} = (A + BF)\{A + BF \mid \underline{B} \cap \underline{R}\}$$

$$= (A + BF)(\underline{R} \cap \underline{B}) + \ldots + (A + BF)^n(\underline{R} \cap \underline{B})$$

$$\subset \underline{R}$$

Therefore, by Lemma 2.4.1., $F(\underline{R})$ is non-empty if and only if (2.4.9) is true. Let

$$\{A + BF \mid \underline{B} \cap \underline{R}\} = \underline{R}$$

Then $F \in F(\underline{R})$. Define, for $j \in \underline{n}$

$$\underline{P}^j = \sum_{i=0}^{j-1} (A + BF)^i(\underline{B} \cap \underline{R})$$

Then

$$\underline{P}^j = (A\underline{P}^{j-1} + \underline{B}) \cap \underline{R} \tag{2.4.11}$$

For the proof of equation (2.4.11), it is easily seen that $\underline{P}^j$ can be written, from its definition, as

$$\underline{P}^j = (A + BF)\underline{P}^{j-1} + \underline{B} \cap \underline{R}$$

$$= ((A + BF)\underline{P}^{j-1} + \underline{B}) \cap \underline{R} \qquad j \in \underline{n} \qquad (2.4.12)$$

for, since $\underline{P}^{j-1} \subset \underline{R}$, $j \in \underline{n}$, it follows that $(A + BF)\underline{P}^{j-1} \subset \underline{R}$
But

$$\underline{B} + (A + BF)\underline{P}^{j-1} = \underline{B} + A\underline{P}^{j-1}$$

which, used in (2.4.12), yields (2.4.11).

Thus

$$\underline{R} = \sum_{i=1}^{n-1} (A + BF)^i(\underline{B} \cap \underline{R})$$

$$= \underline{P}^n$$

$$= \lim \underline{S}^j \qquad j = 0,1,\ldots \qquad (2.4.13)$$

Conversely, if $\underline{R} = \lim \underline{S}^j$, $j = 0,1,\ldots$, then

$$\underline{R} = \underline{S}^n$$

$$= \underline{P}^n$$

$$= \{A + BF \mid \underline{B} \cap \underline{R}\} \tag{2.4.14}$$

The following fact follows immediately from the proof of this lemma.

Corrollary to Lemma 2.4.2.

If $\underline{R}$ is a controllability subspace of (A,B), then

$$\{A + BF \mid \underline{B} \cap \underline{R}\} = \underline{R}$$

for all $F \in F(\underline{R})$, i.e. such that $(A + BF)\underline{R} \subset \underline{R}$.

For a more intuitive viewpoint, it is worth considering the concepts of invariant and controllability subspaces of (A,B) in the following way. Consider the equation describing the state trajectory for the system (2.1.1),(2.1.2), i.e.

$$x(t) = e^{At}x(o) + \int_0^t e^{A(t-s)}Bu(s)ds \tag{2.4.15}$$

where x(o) is the initial state at time t = 0. An invariant subspace $\underline{V}$ of (A,B), by using the result of Lemma 2.4.1., can then be viewed as a set of x(0), such that for some u(t) for each x(0), t > 0, then $x(t) \in \underline{V}$, for all $t \geq 0$. Note that the state trajectory can start at any vector x(0) contained in $\underline{V}$, the criterion being that x(t) will range only over $\underline{V}$.

The concept of a controllability subspace of (A,B) can then be regarded as a natural step from this, by forming the controllable subspace $\underline{R} \subset \underline{V}$, i.e.

$$\underline{R} = \underline{B} \cap \underline{V} + (A + BF)\underline{B} \cap \underline{V} + \dots + (A + BF)^{n-1}\underline{B} \cap \underline{V}$$

$$= \left\{ A + BF \;\middle|\; \underline{B} \cap \underline{V} \right\} \tag{2.4.16}$$

where F is chosen such that that $(A + BF)\underline{V} \subset \underline{V}$. Equation (2.4.16) will later be shown to be equivalent to (2.4.8) for $\underline{R} \subset \underline{V}$ (Theorem 3.2.1.).

Alternatively, $\underline{R}$ can be considered in a reachability context as the set of x(t) which are reachable from the origin, x(0) = 0, by trajectories entirely contained in $\underline{V}$, i.e.

$$x(t) = \int_0^t e^{A(t-s)} Bu(s)ds$$

$$\in \underline{V} \tag{2.4.17}$$

for some control u(t), $t \geq 0$. Note here that, in contrast to the case of invariant subspaces of (A,B), the system state is constrained to start from the origin x(0) = 0, and range entirely in connected invariant subspaces of (A,B) to form a controllability subspace of (A,B).

2.5. Algebraic properties of invariant subspaces of (A,B)

In this section, the special algebraic properties of invariant subspaces of (A,B), which were noted (WM1) by Wonham and Morse, will be presented in greater detail. The majority of the definitions following have been taken from Birkhoff and Bartee (BB1).

Definition 2.5.1.

A partially ordered set is any set S with a binary relation $<$ which is reflexive, antisymmetric, and transitive, i.e. which satisfies

(i) $X \leq X$, for all $X \in S$,

(ii) $X \leq Y$ and $Y \leq X$ imply $X = Y$, and

(iii) $X < Y$ and $Y < Z$ imply $X < Y$, for X,Y, and $Z \in S$.

Definition 2.5.2.

A lower bound of a partially ordered set S is an element X of S satisfying $X < Y$, for all $Y \in S$. A greatest lower bound P is any lower bound P such that $Q < P$ for any other lower bound Q of S.

Clearly, greatest lower bounds are unique by (ii) of Definition 2.5.1. A similar definition obviously exists for upper bounds, and least upper bounds, which are similarly unique.

Definition 2.5.3.

A lattice is a partially ordered set in which any two elements X, Y have a greatest lower bound, $X \wedge Y$, and a least upper bound, $X \vee Y$, where the binary operations $\wedge$ and $\vee$ satisfy the idempotent, commutive, and associative identities:

L1. $X \wedge X = X \qquad X \vee X = X$

L2. $X \wedge Y = Y \wedge X \qquad X \vee Y = Y \vee X$

L3. $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$

$X \vee (Y \vee Z) = (X \vee Y) \vee Z$

for all X, Y, and Z contained in the set.

From these definitions, it is clear that the set of all subspaces of a linear vector space $\underline{X}$ is a lattice, and in this case, the operations $\vee$, and $\wedge$ above are +, and $\cap$ respectively.

In any lattice, the semi-distributive laws hold:

$$(X \wedge Y) \vee (X \wedge Z) < X \wedge (Y \vee Z) \qquad (2.5.1a)$$

$$X \vee (Y \wedge Z) < (X \vee Y) \wedge (X \vee Z) \qquad (2.5.1b)$$

In the case of linear vector subspaces, therefore, the relation $<$ becomes $\subset$ and (2.5.1a,b) can be written

$$(X \cap Y) + (X \cap Z) \subset X \cap (Y + Z) \qquad (2.5.2a)$$

$$X + (Y \cap Z) \subset (X + Y) \cap (X + Z) \qquad (2.5.2b)$$

The following important fact holds in particular for invariant subspaces of (A,B).

Assertion 2.5.1.

The sum of two invariant subspaces of (A,B), $\underline{V}_1 + \underline{V}_2$, is also an invariant subspace of (A,B). The intersection of two invariant subspaces of (A,B), $\underline{V}_1 \cap \underline{V}_2$, is not in general an invariant subspace of (A,B).

Proof: For the first statement, this follows trivially from (2.4.1) and the associative property of the operation +. The second statement is a consequence of (2.5.2b) since

$$\begin{aligned} \underline{B} + (\underline{V}_1 \cap \underline{V}_2) &\subset (\underline{V}_1 + \underline{B}) \cap (\underline{V}_2 + \underline{B}) \\ &\supset A\underline{V}_1 \cap A\underline{V}_2 \\ &\supset A(\underline{V}_1 \cap \underline{V}_2) \end{aligned} \tag{2.5.3}$$

with equality in the first relationship not holding in general.

Definition 2.5.4.

A semilattice is a set with an idempotent, commutative, and associative binary operation.

Clearly, any lattice is a semilattice under $\wedge$ and under $\vee$.

From the foregoing Definitions and Assertion, it is now a simple matter to identify the set of subspaces defined by

$$T = \{ \underline{V} : A\underline{V} \subset \underline{V} + \underline{B},\ \underline{V} \subset \underline{W} \subset \underline{X} \}$$

as a semilattice, partially ordered by $\subset$, with binary operation +, under which it is closed (WM1).

The following assertion can now be seen to hold for the set T.

<u>Assertion 2.5.2.</u>

The set T defined by

$$T = \{ \underline{V} : A\underline{V} \subset \underline{V} + \underline{B},\ \underline{V} \subset \underline{W} \subset \underline{X} \}$$

has a least upper bound contained in T, which is unique.

<u>Proof:</u> This follows directly from the above definitions and Assertion 2.5.1.

The following lemma relates to the construction of the least upper bound $\underline{V}^M$ of the set T, a subspace which will prove to be of importance in the following.

<u>Lemma 2.5.1.</u>

The least upper bound $\underline{V}^M$ of the set T of subspaces $\underline{V}$ defined in Assertion 2.5.2., is given by $\underline{V}^p$, where $p = \dim(\underline{W})$ and

$$\underline{V}^0 = \underline{W} \tag{2.5.4}$$

$$\underline{V}^{j+1} = \underline{W} \cap A^{-1}(\underline{V}^j + \underline{B}) \qquad j = 0,1,\ldots \tag{2.5.5}$$

<u>Proof:</u> The equivalence of the sequence

$$\underline{V}^0 = \underline{W}, \quad \underline{V}^{j+1} = \underline{V}^j \cap A^{-1}(\underline{V}^j + \underline{B}) \qquad (2.5.6)$$

to that defined by (2.5.4), (2.5.5) is first established. From (2.5.6)

$$A^{-1}(\underline{V}^{j+1} + \underline{B}) = A^{-1}\left((\underline{V}^j \cap A^{-1}(\underline{V}^j + \underline{B})) + \underline{B} \right)$$

$$\subset A^{-1}(\underline{V}^j + \underline{B}) \qquad (2.5.7)$$

Thus

$$A^{-1}(\underline{V}^{j+1} + \underline{B}) \subset A^{-1}(\underline{V}^j + \underline{B}) \subset \ldots \subset A^{-1}(\underline{W} + \underline{B}) \qquad (2.5.8)$$

The equivalence of the sequences can now be made clear by expanding the first few terms of (2.5.6). The equivalence is trivial for $\underline{V}^0$ and $\underline{V}^1$. For $\underline{V}^2$, from (2.5.6),

$$\underline{V}^2 = \underline{V}^1 \cap A^{-1}(\underline{V}^1 + \underline{B})$$

$$= \underline{V}^0 \cap A^{-1}(\underline{V}^0 + \underline{B}) \cap A^{-1}(\underline{V}^1 + \underline{B}) \qquad (2.5.9)$$

Since, by (2.5.8), $A^{-1}(\underline{V}^1 + \underline{B}) \subset A^{-1}(\underline{V}^0 + \underline{B})$, it follows that

$$\underline{V}^0 \cap A^{-1}(\underline{V}^0 + \underline{B}) \cap A^{-1}(\underline{V}^1 + \underline{B}) = \underline{V}^0 \cap A^{-1}(\underline{V}^1 + \underline{B}) \tag{2.5.10}$$

which equals $\underline{V}^2$ by (2.5.5). This process can be repeated for $\underline{V}^j$, $j \geq 2$.

Now, let $\underline{V}^M$ be the least upper bound contained in T. Since $A\underline{V}^M \subset \underline{V}^M + \underline{B}$, it follows that

$$\underline{V}^M = \underline{V}^M \cap A^{-1}(\underline{V}^M + \underline{B}) \tag{2.5.11}$$

Assume $\underline{V}^M \subset \underline{V}^j$. Then

$$\begin{aligned} \underline{V}^M &\subset \underline{V}^j \cap A^{-1}(\underline{V}^j + \underline{B}) \\ &= \underline{V}^{j+1} \end{aligned} \tag{2.5.12}$$

Hence, since $\underline{V}^M \subset \underline{V}^0$, $\underline{V}^M \subset \underline{V}^j$, for all j. Therefore, since if, by (2.5.6), $\underline{V}^{j+1} \subset \underline{V}^j$, $\dim(\underline{V}^{j+1}) < \dim(\underline{V}^j)$ if $\underline{V}^j \neq \underline{V}^M$ then the known existence of $\underline{V}^M$ and the fact that $\underline{V}^M \subset \underline{V}^j$, for all j, implies that there exists a finite integer k such that $\underline{V}^j = \underline{V}^k$, for all $j \geq k$.

Since $\underline{V}^M \subset \underline{V}^k$, and

$$\underline{V}^k = \underline{V}^k \cap A^{-1}(\underline{V}^k + \underline{B}) \tag{2.5.13}$$

or

$$A\underline{V}^k \subset \underline{V}^k + \underline{B} \tag{2.5.14}$$

by uniqueness of the least upper bound in T, it follows that $\underline{V}^k = \underline{V}^M$. Clearly, since $p = \dim(\underline{W}) = \dim(\underline{V}^0)$, and the sequence $\underline{V}^j$, $j = 0,1,\ldots,$ is monotonically decreasing, $k \leq p$. Hence

$$\underline{V}^p = \underline{V}^k = \underline{V}^M \tag{2.5.15}$$

and the lemma is proved.

In the following, $\underline{V}^M \subset \underline{W} \subset \underline{X}$, will be termed the maximal invariant subspace of (A,B) contained in $\underline{W}$, a subspace of $\underline{X}$.

2.6. Properties of controllability subspaces of (A,B).

The following theorem (WM1) presents an important property of controllability subspaces.

Theorem 2.6.1.

Given a controllability subspace $\underline{R} \subset \underline{X}$ of (A,B), $\dim(\underline{R}) = p$, let a(s) be an arbitrary monic polynomial of degree p. Then $F \in F(\underline{R})$ can be chosen so that the characteristic polynomial of $(A + BF)|\underline{R}$ is a(s). Further, for any non-zero vector $b \in \underline{R} \cap \underline{B}$, F can be chosen such that

$$\{A + BF \mid \underline{b}\} = \underline{R} \qquad (2.6.1)$$

Proof: Choose $F_1 \in F(\underline{R})$ arbitrarily, and denote $(A + BF_1)$ by A_1. Let $b_1 \in \underline{R} \cap \underline{B}$, and p_1 be the largest integer such that

$$b_1, A_1 b_1, \ldots, A_1^{p_1-1} b_1$$

are linearly independent. Set $r_1 = b_1$ and $r_j = A_1 r_{j-1} + b_1$, $j = 2, \ldots, p_1$. Then the r_i, $i \in \underline{p_1}$ are independent and $r_i \in \underline{R}$, $i \in \underline{p_1}$. If $p_1 < p$, choose $b_2 \in \underline{R} \cap \underline{B}$, independent of the set of r_i, $i \in \underline{p_1}$. Repeating the procedure for b_2, setting $r_{p_1+i} = A_1 r_{p_1+i-1} + b_2$, $i \in \underline{p_2}$, and repeating for b_3, etc., a set of r_i, $i = 1, \ldots, p$ is obtained, which are independent and in $\underline{R}$, with the property

$$r_{i+1} = A_1 r_i + \tilde{b}_i \qquad i \in \underline{p-1} \qquad (2.6.2)$$

where $\tilde{b}_i \in \underline{R} \cap \underline{B}$. Choosing $F_2 \in F(\underline{R})$ so that

$$BF_2 r_i = \tilde{b}_i \qquad i \in \underline{p} \qquad (2.6.3)$$

with $\tilde{b}_p$ arbitrary, then

$$r_{i=1} = (A_1 + BF_2) r_i \qquad i \in \underline{p-1} \qquad (2.6.4)$$

A solution F_2 to (2.6.3) is guaranteed by Lemma 2.2.1. By independence of the r_i, therefore

$$\{ A_1 + BF_2 \mid \underline{r}_1 \} = \underline{R} \qquad (2.6.5)$$

Thus, an m-vector f can be found so that the characteristic polynomial of $(A_1 + BF_2 + r_1 f^T) \mid \underline{R}$ is a(s).(W1)

An alternative viewpoint for showing the pole assignment properties of controllability subspaces will assume the following result.

Lemma 2.6.1. (W1)

A pair (A,B) is controllable if and only if, for every choice of a symmetric† set of complex numbers Λ, there is a matrix F such that (A + BF) has Λ for its set of eigenvalues.

Proof: For this proof, reference should be made to Wonham (W1).

Assuming this, write $\underline{X} = \underline{R} \oplus \underline{L}$ for some subspace $\underline{L} \subset \underline{X}$, and let P be the projection of $\underline{X}$ onto $\underline{R}$ along $\underline{L}$. Choose $F \in F(\underline{R})$, and let $\overline{A}$ be the restriction of (A + BF) to $\underline{R}$. Applying the projection P to the equations

$$\dot{x}(t) = (A + BF)x(t) + BGv(t) \qquad (2.6.6)$$

where G is defined by $B\underline{G} = \underline{R} \cap \underline{B}$

† i.e. such that complex numbers occur in conjugate pairs

yields

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}Gv(t) \tag{2.6.7}$$

where $\bar{x} = Px \in \underline{R}$, $P(A + BF) = \bar{A}P$, and $PB = \bar{B}$. The controllability matrix H for this system is then

$$H = [\bar{B}G \quad \bar{A}\bar{B}G \ \dots \ \bar{A}^{n-1}\bar{B}G]$$

$$= P\,[B \ (A + BF)B \ \dots \ (A + BF)^{n-1}B\,]\,G \tag{2.6.8}$$

Assuming G is chosen to be of full rank, it is easily seen that if (A,B) is a controllable pair, then rank (H) = dim($\underline{R}$) and ($\bar{A}$,$\bar{B}$G) is a controllable pair. Complete eigenvalue assignability then follows immediately from Lemma 2.6.1.

The concepts and properties of invariant and controllability subspaces of (A,B) which have been established in this chapter will now be applied to some problems in multivariable system theory, with particular emphasis being placed on the output feedback solution, i.e. when the state feedback matrix F can be written, for some matrix K, as

$$F = KC$$

It will be shown that many of these problems have clear and elegant solutions when formulated in the geometric terms developed in this chapter, and useful insight is gained into the problems.

CHAPTER THREE

Decoupling by output feedback: restricted case

3.1. The state feedback decoupling problem

Consider a general partitioning of the output equation (2.1.2) given by

$$\begin{bmatrix} y_1(t) \\ \cdot \\ \cdot \\ \cdot \\ y_k(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ \cdot \\ \cdot \\ \cdot \\ C_k \end{bmatrix} x(t) \qquad (3.1.1)$$

where $y_i(t)$ is a p_i-vector. To avoid trivial cases, it is assumed that the subspaces $\underline{C}_i^T$, $i \in \underline{k}$, are mutually independent, i.e.

$$\underline{C}_i^T \cap \sum_{j \neq i} \underline{C}_j^T = 0 \qquad i \in \underline{k} \qquad (3.1.2)$$

and that $\underline{C}_i^T \neq 0$, $\underline{C}_i^T \neq \underline{R}^n$, $i \in \underline{k}$. Consider the state feedback control

$$u(t) = Fx(t) + \sum_{i \in \underline{k}} G_i v_i(t) \qquad (3.1.3)$$

The state feedback decoupling problem is then to chose a control of the form (3.1.3) such that $v_i(t)$ completely controls $y_i(t)$, $i \in \underline{k}$, without affecting $y_j(t)$, $j \neq i$, $j \in \underline{k}$. Here, "completely" is used in the same sense as complete controllability, and in the following, it will be assumed that in (2.1.1) the pair (A,B) is completely controllable, i.e. that $\{ A \mid \underline{B} \} = \underline{X} = \underline{R}^n$.

The decoupling condition can be stated more formally as follows. For $v_i(t)$ to control $y_i(t)$ completely requires that

$$C_i \{ A + BF \mid \{ BG_i \} \} = \underline{C}_i \qquad i \in \underline{k} \tag{3.1.4}$$

and for $v_i(t)$ to leave $y_j(t)$ unaffected requires that

$$C_j \{ A + BF \mid \{ BG_i \} \} = 0, \; j \neq i, \; j \in \underline{k} \tag{3.1.5}$$

To obtain the geometric formulation of the problem, consider a set $\{ \underline{R}_i \}_{\underline{k}}$ of controllability subspaces such that

$$\{ A + BF \mid \{ BG_i \} \} = \underline{R}_i \qquad i \in \underline{k} \tag{3.1.6}$$

and where G_i, $i \in \underline{k}$ is chosen such that

$$B\underline{G}_i = \underline{B} \cap \underline{R}_i \tag{3.1.7}$$

yielding for (3.1.6)

$$\{ A + BF \mid \underline{R}_i \cap \underline{B} \} = \underline{R}_i \qquad i \in \underline{k} \qquad (3.1.8)$$

Restricting these subspaces to be controllability subspaces ensures that the dynamic effect of each $v_i(t)$ is contained within the subspace, and that the whole subspace is reachable. From (3.1.4) it now follows that

$$C_i\underline{R}_i = \underline{C}_i \qquad i \in \underline{k} \qquad (3.1.9)$$

or, denoting $\underline{N}(C_i)$ by $\underline{N}_i$,

$$\underline{R}_i + \underline{N}_i = \underline{X} \qquad i \in \underline{k} \qquad (3.1.10)$$

must hold. From (3.1.5) it follows trivially that

$$\underline{R}_i \subset \bigcap_{j \neq i} \underline{N}_j \qquad i \in \underline{k} \qquad (3.1.11)$$

is necessary to restrict the influence of $v_i(t)$ on $y_j(t)$, $j \neq i$, to be zero.

In its geometric formulation, the problem thus becomes one of determining a set $\{\underline{R}_i\}_{\underline{k}}$ satisfying (3.1.8), (3.1.10), and (3.1.11) with the additional requirement that

$$\bigcap_{i \in \underline{k}} F(\underline{R}_i) \neq \emptyset \qquad (3.1.12)$$

Here, $F(\underline{R}_i)$ is defined as the set of matrices F such that $(A + BF)\ \underline{R}_i \subset \underline{R}_i$, as in Chapter Two. This requirement ensures that there exists an F which will "work" for all the $\underline{R}_i$, $i \in \underline{k}$.

This problem has been termed (MW2) the "restricted decoupling problem", or RDP, and will be referred to thus in the following.

3.2. Existence of state feedback solutions to RDP

The requirements of (3.1.10) and (3.1.11) indicate that $\underline{R}_i$ must be large enough to satisfy (3.1.10), whilst small enough such that (3.1.11) holds. For this reason, a reasonable approach is to consider the set of maximal (least upper bound) controllability subspaces $\{\underline{R}_i^M\}_{\underline{k}}$ of (A,B), such that $\underline{R}_i^M \subset \bigcap_{j \neq i} \underline{N}_j$, $i \in \underline{k}$. The following theorem (WM1) defines the maximal controllability subspace $\underline{R}^M$ of (A,B) contained in a given subspace $\underline{W}$ of the state space $\underline{X}$. From now on, $\underline{R}_i^M$ will always be defined such that $\underline{R}_i^M \subset \bigcap_{j \neq i} \underline{N}_j$.

Theorem 3.2.1.

Let $\underline{V}^M$ denote the maximal invariant subspace of (A,B) contained in $\underline{W} \subset \underline{X}$. Then the maximal controllability subspace $\underline{R}^M$ of (A,B) contained in $\underline{W}$ is given by

$$\underline{R}^M = \{ A + BF \mid \underline{B} \cap \underline{V}^M \} \qquad (3.2.1)$$

for any $F \in F(\underline{V}^M)$.

__Proof:__ Let $\underline{R} \subset \underline{W}$, and

$$\{A + BF_1 \mid \underline{R} \cap \underline{B}\} = \underline{R} \qquad (3.2.2)$$

Since $\underline{R}$ is an invariant subspace of $\underline{W}$, then $\underline{R} \subset \underline{V}^M$. Let $\underline{V}^M = \underline{R} \oplus \underline{V}$, for some $\underline{V} \subset \underline{X}$. Choose F_2 such that $F_2\underline{R} = F_1\underline{R}$ and

$$(A + BF_2)\underline{V} \subset \underline{V}^M \qquad (3.2.3)$$

Then $F_2 \in F(\underline{V}^M)$ and

$$\underline{R} = \{A + BF_2 \mid \underline{B} \cap \underline{R}\}$$

$$\subset \{A + BF_2 \mid \underline{B} \cap \underline{V}^M\} = \{A + BF_2 \mid \underline{B} \cap \underline{R}^M\} \text{ since } \underline{B} \cap \underline{V}^M = \underline{B} \cap \underline{R}^M \; ^{\dagger}$$

$$= \underline{R}^M \qquad (3.2.4)$$

Thus $\underline{R}^M$ is a controllability subspace by Defn. 2.4.2. and maximal in $\underline{W}$.
Existence of F_2 such that $F_2\underline{R} = F_1\underline{R}$ and (3.2.3) holds, follows from the construction procedure of Lemma 2.4.1.

The following lemma now provides a simple necessary condition for the existence of a solution to RDP.

__Lemma 3.2.1.__

A solution to RDP exists only if

† note that since $\underline{B} \cap \underline{V}^M \subset \underline{R}^M$ by (3.2.1), then $\underline{B} \cap \underline{V}^M \subset \underline{B} \cap \underline{R}^M$. Also, since $\underline{R}^M \subset \underline{V}^M$, then $\underline{B} \cap \underline{R}^M \subset \underline{B} \cap \underline{V}^M$. Thus $\underline{B} \cap \underline{R}^M = \underline{B} \cap \underline{V}^M$.

$$\underline{R}_i^M + \underline{N}_i = \underline{X}, \qquad i \in \underline{k} \tag{3.2.5}$$

Proof: It is clear that if $\{\underline{R}_i\}_{\underline{k}}$ is a solution to RDP, then $\underline{R}_i$ satisfies (3.1.11). Then by the maximality of $\underline{R}_i^M$, $\underline{R}_i \subset \underline{R}_i^M$ and (3.2.5) follows from (3.1.10).

Therefore, if (3.2.5) holds and

$$\bigcap_{i \in \underline{k}} F(\underline{R}_i^M) \neq \emptyset \tag{3.2.6}$$

then $\{\underline{R}_i^M\}_{\underline{k}}$ is certainly a solution to RDP. However (3.2.6) is not a necessary condition for the existence of a solution, and there may in fact exist a set of smaller $\underline{R}_i \subset \underline{R}_i^M$ such that

$$\bigcap_{i \in \underline{k}} F(\underline{R}_i) \neq \emptyset \tag{3.2.7}$$

although (3.2.6) does not hold. In the following two cases however, under certain restrictions, $\{\underline{R}_i^M\}_{\underline{k}}$ can be shown to provide a solution to RDP.

(i) rank (G) = rank $(G_1 \ldots G_k) = m$. Under this condition, the following lemma (MW2) is applicable.

Lemma 3.2.2.

A solution to RDP, such that rank (G) = m, exists if and only if

$$\underline{B} = \sum_{i \in \underline{k}} \underline{B} \cap \underline{R}_i^M \tag{3.2.8}$$

in which case $\{\underline{R}_i^M\}_{\underline{k}}$ is a solution.

__Proof__: If rank (G) = m, then if $\{\underline{R}_i\}_{\underline{k}}$ is a solution

$$\underline{B} = B\underline{G} = B\sum_{i \in \underline{k}} \underline{G}_i \subset \sum_{i \in \underline{k}} \underline{B} \cap \underline{R}_i \subset \underline{B} \tag{3.2.9}$$

since G_i is chosen such that $B\underline{G}_i = \underline{B} \cap \underline{R}_i$, $i \in \underline{k}$. Therefore

$$\underline{B} = \sum_{i \in \underline{k}} \underline{B} \cap \underline{R}_i \tag{3.2.10}$$

and by the maximality of $\underline{R}_i^M$,

$$\underline{B} = \sum_{i \in \underline{k}} \underline{B} \cap \underline{R}_i^M \tag{3.2.11}$$

Proof of sufficiency requires showing that (3.2.8) implies that $\bigcap_{i \in \underline{k}} F(\underline{R}_i^M) \neq \emptyset$, and

$$\underline{R}_i^M + \underline{N}_i = \underline{X}, \qquad i \in \underline{k} \tag{3.2.12}$$

Since the proof of this is fairly involved and requires additional concepts which will not be needed elsewhere, it will be omitted here.

__(ii) rank (B) = k__. Under this condition the following lemma (WM1) applies.

Lemma 3.2.3.

A solution $\{\underline{R}_i\}_{\underline{k}}$ to RDP where rank (B) = k, exists if and only if

$$\underline{B} = \sum_{i\in\underline{k}} \underline{B} \cap \underline{R}_i^M \tag{3.2.13}$$

Moreover, $\{\underline{R}_i^M\}_{\underline{k}}$ is, in this case, the only solution.

Proof: For necessity, define $\underline{B}_i$, $i \in \underline{k}$, by

$$\underline{B} \cap \underline{R}_i = (\underline{B} \cap \underline{R}_i \cap \sum_{j\neq i} \underline{R}_j) \oplus \underline{B}_i \tag{3.2.14}$$

The $\underline{B}_i$ will now be shown to be mutually independent. By definition of $\underline{B}_j$, $\underline{B}_j \subset \underline{B} \cap \underline{R}_j$, and hence

$$\underline{B}_i \cap \sum_{j\neq i} \underline{B}_j \subset \underline{B}_i \cap \sum_{j\neq i} (\underline{B} \cap \underline{R}_j) \tag{3.2.15}$$

But, $\sum_{j\neq i} (\underline{B} \cap \underline{R}_j) \subset \underline{B} \cap \sum_{j\neq i} \underline{R}_j$, and $\underline{B}_i = \underline{B}_i \cap \underline{R}_i$ since $\underline{B}_i \subset \underline{R}_i$ by (3.2.14). But this implies that

$$\underline{B}_i \cap \sum_{j\neq i} (\underline{B} \cap \underline{R}_j) \subset \underline{B}_i \cap \underline{R}_i \cap \underline{B} \cap \sum_{j\neq i} \underline{R}_j \tag{3.2.16}$$

which intersection is zero, by definition of $\underline{B}_i$. Hence, it follows from (3.2.15) that the $\underline{B}_i$ are mutually independent. Now, for $F \in F(\underline{R}_i)$,

$$\underline{R}_i = \{A + BF \mid \underline{B} \cap \underline{R}_i\}$$

$$= \{A + BF \mid \underline{B}_i\} + \underline{\tilde{R}}_i \qquad (3.2.17)$$

from (3.2.14), where, since $\underline{B} \cap \underline{R}_i \subset \underline{B}_i \oplus \sum_{j \neq i} \underline{R}_j$,

$$\underline{\tilde{R}}_i \subset \{A + BF \mid \sum_{j \neq i} \underline{R}_j\}$$

$$\subset \sum_{j \neq i} \underline{R}_j$$

$$\subset \underline{N}_i \qquad (3.2.18)$$

using the fact that $F \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$, and $\underline{R}_i \subset \bigcap_{j \neq i} \underline{N}_j$. Therefore,

$$\underline{X} = \underline{R}_i + \underline{N}_i$$

$$= \{A + BF \mid \underline{B}_i\} + \underline{\tilde{R}}_i + \underline{N}_i$$

$$= \{A + BF \mid \underline{B}_i\} + \underline{N}_i \qquad (3.2.19)$$

by (3.2.18). Hence, since $\underline{N}_i \neq \underline{X}$, then $\underline{B}_i \neq 0$, $i \in \underline{k}$. Furthermore, consideration of (3.2.14) yields that $\underline{B} \cap \underline{B}_i$ equals $\underline{B}_i$, $i \in \underline{k}$. This, together with the existence of the k non-zero mutually independent subspaces $\underline{B}_i$, yields that

$$\underline{B} = \bigoplus_{i \in \underline{k}} \underline{B}_i \qquad (3.2.20)$$

Since $\underline{B}_i \subset \underline{B} \cap \underline{R}_i \subset \underline{B} \cap \underline{R}_i^M$, (3.2.13) follows immediately.

For uniqueness of the solution $\{\underline{R}_i^M\}_{\underline{k}}$, observe that (3.2.20) and the fact that $\underline{B}_i \neq 0$, $i \in \underline{k}$, imply that $\dim(\underline{B}_i) = 1$, $i \in \underline{k}$. Assume, as can be proved (WM1), that the subspaces $\underline{B} \cap \underline{R}_i^M$, $i \in \underline{k}$, are independent. Then, by (3.2.13), it follows that

$$\dim(\underline{B} \cap \underline{R}_i^M) = 1 \qquad i \in \underline{k} \qquad (3.2.21)$$

Thus, since $\underline{B} \cap \underline{R}_i \subset \underline{B} \cap \underline{R}_i^M$, and $\dim(\underline{B} \cap \underline{R}_i) = 1$,

$$\underline{B} \cap \underline{R}_i = \underline{B} \cap \underline{R}_i^M \qquad i \in \underline{k} \qquad (3.2.22)$$

Let $\underline{R}_i^M = \underline{R}_i \oplus \tilde{\underline{R}}_i$, for some $\tilde{\underline{R}}_i$, $i \in \underline{k}$, and choose F_i such that $F_i \underline{R}_i = F \underline{R}_i$, for $F \in F(\underline{R}_i)$, and

$$(A + BF_i)\tilde{\underline{R}}_i \subset \underline{R}_i^M \qquad (3.2.23)$$

This is always possible by Lemma 2.4.1. Thus $F_i \in F(\underline{R}_i) \cap F(\underline{R}_i^M)$. Then

$$\underline{R}_i = \{A + BF_i \mid \underline{B} \cap \underline{R}_i\}$$

$$= \{ A + BF_i \mid \underline{B} \cap \underline{R}_i^M \}$$

$$= \underline{R}_i^M \qquad (3.2.24)$$

proving uniqueness of the solution $\{\underline{R}_i^M\}_{\underline{k}}$. The proof that the subspaces $\underline{B} \cap \underline{R}_i^M$, $i \in \underline{k}$, are independent, will be omitted here for the sake of brevity, and since it will be of no further use.

The proof of sufficiency follows similar lines to that for Lemma 3.2.2. and will also be omitted here.

In these special cases therefore a solution to RDP can be readily obtained using the procedure of Lemma 2.5.1. to obtain $\{\underline{V}_i^M\}_{\underline{k}}$, $\underline{V}_i^M \subset \bigcap_{j \neq i} \underline{N}_j$, and then determining $\{\underline{R}_i^M\}_{\underline{k}}$ using the result of Theorem 3.2.1. Systematic procedures for determining general solutions to RDP, if they exist, when

$$\bigcap_{i \in \underline{k}} F(\underline{R}_i^M) = \emptyset \qquad (3.2.25)$$

are as yet unknown.

3.3. Existence of output feedback solutions to RDP.

Consider that, in addition to the decoupling requirements (3.1.6),(3.1.10) - (3.1.12), it is required that there exists a solution K to the equation

$$F = KC \tag{3.3.1}$$

for some $F \in \bigcap_{i\in \underline{k}} F(\underline{R}_i)$. Such an F will not in general exist. The following theorem provides necessary and sufficient conditions for the existence of some $F \in \bigcap_{i\in \underline{k}} F(\underline{R}_i)$ such that (3.3.1) holds. The case is considered when the pair (C,A) is observable. (D1)

Theorem 3.3.1.

For the system described by (2.1.1), (2.1.2), given (C,A) observable and $\{\underline{R}_i\}_{\underline{k}}$, a solution to RDP, there exists an $F \in \bigcap_{i\in \underline{k}} F(\underline{R}_i)$ such that F = KC for some K, if and only if

(i) $$\underline{R}^* = \bigcap_{i\in \underline{k}} (\sum_{j\neq i} \underline{R}_j) = 0 \tag{3.3.2}$$

(ii) $$A(\underline{R}_i \cap \underline{N}(C)) \subset \underline{R}_i \qquad i \in \underline{k} \tag{3.3.3}$$

Proof: Necessity of (i) is first established. For any two linear vector subspaces $\underline{V}$ and $\underline{W}$, and a map A, it is easy to show that $A(\underline{V} \cap \underline{W}) \subset A\underline{V} \cap A\underline{W}$, and $A(\underline{V} + \underline{W}) = A\underline{V} + A\underline{W}$. Using these results it can be seen that, for all $F \in \bigcap_{i\in \underline{k}} F(\underline{R}_i)$

$$(A + BF)\underline{R}^* = (A + BF) \bigcap_{i\in \underline{k}} (\sum_{j\neq i} \underline{R}_j)$$

$$\subset \bigcap_{i\in \underline{k}} (A + BF) \sum_{j\neq i} \underline{R}_j$$

$$= \bigcap_{i \in \underline{k}} \left(\sum_{j \neq i} (A + BF)\underline{R}_j \right)$$

$$\subset \bigcap_{i \in \underline{k}} \left(\sum_{j \neq i} \underline{R}_j \right)$$

$$= \underline{R}^* \tag{3.3.4}$$

Thus, $\underline{R}^*$ is an invariant subspace of (A,B). Also, from (3.1.11)

$$\underline{R}^* = \bigcap_{i \in \underline{k}} \left(\sum_{j \neq i} \underline{R}_j \right)$$

$$\subset \bigcap_{i \in \underline{k}} \left(\sum_{j \neq i} \left(\bigcap_{s \neq j} \underline{N}_s \right) \right)$$

$$\subset \bigcap_{i \in \underline{k}} \underline{N}_i$$

$$= \underline{N}(C) \tag{3.3.5}$$

Assume F = KC for some $F \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$. Then, from (3.3.5), since $C\underline{R}^* = 0$,

$$(A + BKC)\underline{R}^* = A\underline{R}^* \tag{3.3.6}$$

and by (3.3.4) it follows that

$$A\underline{R}^* \subset \underline{R}^* \tag{3.3.7}$$

Thus, from (3.3.5),

$$C\left\{A \mid \underline{R}^*\right\} = 0 \tag{3.3.8}$$

contradicting, if $\underline{R}^* \neq 0$, the assumption that (C,A) is an observable pair. To establish necessity of (ii), assume $F = KC$ for some $F \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$. Then

$$(A + BKC)\underline{R}_i \subset \underline{R}_i \qquad i \in \underline{k} \tag{3.3.9}$$

or

$$\underline{R}_i \subset (A + BKC)^{-1}\underline{R}_i \qquad i \in \underline{k} \tag{3.3.10}$$

Using the easily proved result that for any subspace $\underline{V}$ and a map A, $(A^{-1}\underline{V})^{\perp} = A^T\underline{V}^{\perp}$, taking orthogonal complements of both sides of (3.3.10) yields

$$(A^T + C^TK^TB^T)\underline{R}_i^{\perp} \subset \underline{R}_i^{\perp} \qquad i \in \underline{k} \tag{3.3.11}$$

By Lemma 2.4.1, it then follows that

$$A^T\underline{R}_i^{\perp} \subset \underline{R}_i^{\perp} + \underline{C}^T \qquad i \in \underline{k} \tag{3.3.12}$$

Taking orthogonal complements again yields (ii).

For sufficiency, assume (i) and (ii) are satisfied for $\{\underline{R}_i\}_{\underline{k}}$, a solution to RDP. For each $i \in \underline{k}$, define $\underline{\tilde{R}}_i \subset \underline{X}$ by

$$\underline{R}_i = (\underline{R}_i \cap \underline{N}(C)) \oplus \underline{\tilde{R}}_i \tag{3.3.13}$$

Then, by (ii) and the construction of Lemma 2.4.1, it follows that there exists an $F_i \in F(\underline{R}_i)$ such that

$$(A + BF_i)\underline{R}_i \subset \underline{R}_i \tag{3.3.14}$$

and

$$F_i(\underline{R}_i \cap \underline{N}(C)) = 0 \tag{3.3.15}$$

This implies that

$$(\underline{R}_i \cap \underline{N}(C)) \subset \underline{N}(F_i) \tag{3.3.16}$$

Writing R_i as a basis matrix for $\underline{R}_i$, it follows therefore from (3.3.16) that

$$\begin{aligned}\underline{N}(CR_i) &= R_i^{-1}(\underline{R}_i \cap \underline{N}(C)) \\ &\subset R_i^{-1}\underline{N}(F_i)\end{aligned}$$

$$= R_i^{-1}(\underline{R}_i \cap \underline{N}(F_i))$$

$$= \underline{N}(F_i R_i) \tag{3.3.17}$$

By Lemma 2.2.1, therefore, there exists a solution K_i to the equation

$$K_i C R_i = F_i R_i \tag{3.3.18}$$

Hence

$$(A + BK_i C)\underline{R}_i \subset \underline{R}_i \tag{3.3.19}$$

or $K_i C \in F(\underline{R}_i)$, $i \in \underline{k}$. Now $\underline{R}^* = 0$ implies that the $\{\underline{R}_i\}_{\underline{k}}$ are mutually independent. Define P_i as the projection of $\underline{X}$ onto $\underline{R}_i$ along $\bigoplus_{j \neq i} \underline{R}_j$, and let

$$KC = \sum_{i \in \underline{k}} K_i C P_i \tag{3.3.20}$$

Since $\underline{R}_i \subset \bigcap_{j \neq i} \underline{N}_j$, (3.3.20) becomes

$$KC = \sum_{i \in \underline{k}} K_{ii} C_i P_i$$

$$= \sum_{i \in \underline{k}} K_{ii} C_i \tag{3.3.21}$$

where $K_i = [K_{i1}\ K_{i2}\ \dots\ K_{ik}]$. Hence, K defined by

$$K = [K_{11}\ K_{22}\ \dots\ K_{kk}] \tag{3.3.22}$$

is such that

$$\begin{aligned}(A + BKC)\underline{R}_i &= (A + \sum_{i\in \underline{k}} BK_i CP_i)\underline{R}_i \\ &= (A + BK_i C)\underline{R}_i \\ &\subset \underline{R}_i \qquad\qquad i \in \underline{k}\end{aligned} \tag{3.3.23}$$

or $KC \in \bigcap_{i\in \underline{k}} F(\underline{R}_i)$, proving sufficiency.

It is interesting to note that for the case k=2, condition (ii) of Theorem 3.3.1. implies condition (i).

Recalling the second special case of the previous section, i.e. when rank(B) = k, the following theorem establishes necessary and sufficient conditions for the existence of an output feedback solution to RDP in this case.

Theorem 3.3.2.

An output feedback solution to RDP, where rank(B) = k, and when (C,A) is an observable pair, exists if and only if $\{\underline{R}_i^M\}_{\underline{k}}$ is a solution to RDP and

(i) $$(\underline{R}^M)^* = \bigcap_{i\in \underline{k}} (\sum_{j\neq i} \underline{R}_j^M) = 0 \tag{3.3.24}$$

(ii) $A(\underline{R}_i^M \cap \underline{N}(C)) \subset \underline{R}_i^M \qquad i \in \underline{k}$ (3.3.25)

Moreover, $\{\underline{R}_i^M\}_{\underline{k}}$ is the only such solution.

Proof: This follows directly from Theorem 3.3.1., and Lemma 3.2.3. which establishes uniqueness of the solution to RDP in this special case.

For the first special case of the previous section, i.e. when rank(G) = m, since the solution $\{\underline{R}_i^M\}_{\underline{k}}$ is not unique, it may be possible to find a solution $\{\underline{R}_i\}_{\underline{k}}$ which satisfies (3.3.2), (3.3.3), although $(\underline{R}^M)^* \neq 0$. In this special case therefore, (3.3.24) and (3.3.25) are only sufficient conditions for the existence of an output feedback solution to RDP when rank(G) = m. If they are fulfilled, then $\{\underline{R}_i^M\}_{\underline{k}}$ is certainly a solution.

As a result of the following lemma, however, it is possible to say something further about this special case.

Lemma 3.3.1.

For the special cases (i) and (ii) of section 3.2., if $\{\underline{R}_i\}_{\underline{k}}$ is a solution to RDP, then

$$\sum_{i \in \underline{k}} \underline{R}_i = \underline{X} \tag{3.3.26}$$

Proof: From the initial assumption that

$$\{A \mid \underline{B}\} = \underline{X} \tag{3.3.27}$$

it follows that

$$\sum_{i \in \underline{k}} \{A \mid \underline{B} \cap \underline{R}_i\} = \underline{X} \tag{3.3.28}$$

since, from the proofs of Lemmas 3.2.2. and 3.2.3.,

$$\underline{B} = \sum_{i \in \underline{k}} \underline{B} \cap \underline{R}_i \tag{3.3.29}$$

if rank(G) = m, or rank(B) = k. From a result due to Wonham (W1) that controllability is unaltered by state feedback, then

$$\sum_{i \in \underline{k}} \{A + BF \mid \underline{B} \cap \underline{R}_i\} = \underline{X} \tag{3.3.30}$$

for any F. The result follows if $F \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$.

The following theorem can now be stated.

<u>Theorem 3.3.3.</u>

If the $\{\underline{R}_i^M\}_{\underline{k}}$ are independent, i.e. $(\underline{R}^M)^* = 0$, an output feedback solution to RDP, where rank(G) = m, exists if and only if

$$A(\underline{R}_i^M \cap N(C)) \subset \underline{R}_i^M \qquad i \in \underline{k} \tag{3.3.31}$$

Moreover, $\{\underline{R}_i^M\}_{\underline{k}}$ is the only such solution.

<u>Proof</u>: Sufficiency follows directly from Theorem 3.3.1.

For necessity and uniqueness, assume (3.3.31) does not hold, but that there exists a solution $\{\underline{R}_i\}_{\underline{k}}$ to RDP such that $\underline{R}_i \neq \underline{R}^M$, $i \in \underline{k}$, and

$$A(\underline{R}_i \cap \underline{N}(C)) \subset \underline{R}_i \qquad i \in \underline{k} \qquad (3.3.32)$$

Since the $\underline{R}_i^M$, $i \in \underline{k}$, are independent however,

$$\sum_{i \in \underline{k}} \underline{R}_i \neq \sum_{i \in \underline{k}} \underline{R}_i^M = \underline{X} \qquad (3.3.33)$$

which, by Lemma 3.3.1., implies that $\{\underline{R}_i\}_{\underline{k}}$ is not a solution to RDP. This contradiction proves necessity and uniqueness.

For these special cases therefore, the set $\{\underline{R}_i^M\}_{\underline{k}}$ can be constructed using the result of Theorem 3.2.1. and tested as to whether or not it provides an output feedback solution to RDP by the results of the foregoing theorems of this section. In the general case, however, and where $\{\underline{R}_i^M\}_{\underline{k}}$ does not satisfy (3.3.2) and (3.3.3), no systematic method is known for generating $\{\underline{R}_i\}_{\underline{k}}$, if it exists, which satisfies (3.3.2) and (3.3.3) and is a solution to RDP. This restriction is shared by state feedback solutions to RDP, as was pointed out at the end of section 3.2., since there also in the general case, if $\{\underline{R}_i^M\}_{\underline{k}}$ does not provide a solution, only intuition can derive a set of non-maximal controllability subspaces which will be a solution.

If it is the case that the $\{\underline{R}_i^M\}_{\underline{k}}$ does not provide a solution, the decoupling problem can be solved if an extension of the state space is permitted, i.e. dynamic compensation is used, This subject will be dealt with in detail in the next chapter. Before this, however, it is of interest to consider the problem of pole assignment in the decoupled system.

3.4. Pole assignment in the decoupled system.

The problem of the extent to which it is possible to arbitrarily assign the poles of the decoupled system is now considered. This problem has been thoroughly investigated by Morse and Wonham (WM1),(MW2), in the case of state feedback solutions to RDP, and the theory will be reviewed here.

The structure of the decoupled system is first considered. Let $\{\underline{R}_i\}_{\underline{k}}$ be a solution to RDP, and assume that $\sum_{i\in\underline{k}} \underline{R}_i = \underline{X}$. Define

$$\underline{E}_0 = \underline{R}^* = \bigcap_{i\in\underline{k}} (\sum_{j\neq i} \underline{R}_j) \tag{3.4.1}$$

and let $\underline{E}_i$ be chosen such that

$$\underline{R}_i = \underline{E}_i \oplus (\underline{R}_i \cap \underline{R}^*) \qquad i \in \underline{k} \tag{3.4.2}$$

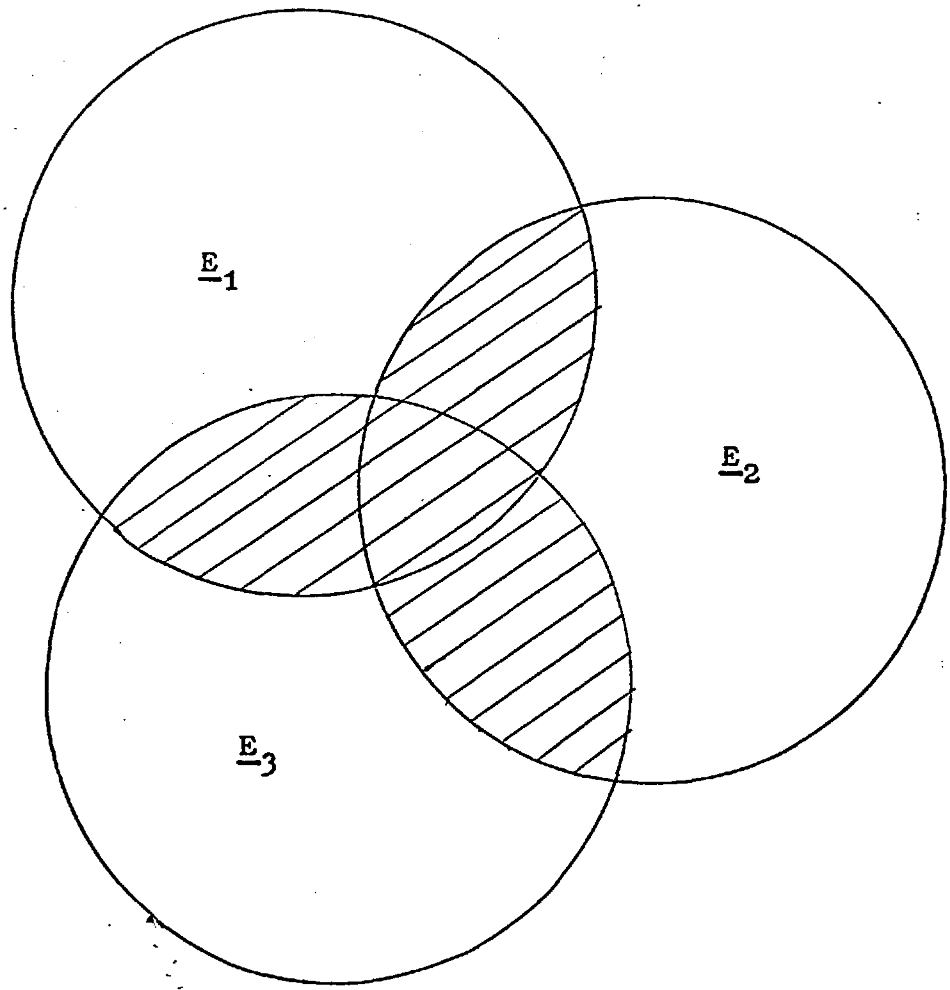

Figure 3.4.1. The three circles represent controllability subspaces $\underline{R}_i$, i = 1, 2, 3. The subspaces $\underline{E}_i \subset \underline{R}_i$ are indicated, $\underline{E}_0 = \underline{R}^*$ being the shaded area, and can be seen to be independent.

For the case k = 3, these subspaces are represented diagramatically in Figure 3.4.1. to assist in clarifying the arguements used in proving the following results. The next lemma exhibits the properties of the subspaces $\underline{E}_i$, $i \in \underline{k}_o$.

Lemma 3.4.1.

The subspaces $\underline{E}_i$, $i \in \underline{k}_o$, have the properties if $\sum_{i \in \underline{k}} \underline{R}_i = \underline{X}$ that

(i) $$\bigoplus_{i \in \underline{k}_o} \underline{E}_i = \underline{X} \qquad (3.4.3)$$

(ii) $$(A + BF)\underline{E}_i \subset \underline{E}_i + \underline{E}_0 \qquad i \in \underline{k}_o \qquad (3.4.4)$$

for all $F \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$.

Proof: Clearly, from the choice of the $\underline{E}_i$, $i \in \underline{k}_o$,

$$\underline{E}_0 \cap \sum_{i \in \underline{k}} \underline{E}_i = 0 \qquad (3.4.5)$$

For $i \in \underline{k}$, by the definition of the $\underline{E}_i$,

$$\underline{R}_i \cap \sum_{j \neq i} \underline{R}_j = (\underline{E}_i \oplus (\underline{R}_i \cap \underline{R}^*)) \cap (\sum_{j \neq i} (\underline{E}_j \oplus (\underline{R}_j \cap \underline{R}^*)))$$

(3.4.6)

But, by linearity

$$\sum_{j\neq i}(\underline{E}_j + (\underline{R}_j \cap \underline{R}^*)) = \sum_{j\neq i}\underline{E}_j + \sum_{j\neq i}(\underline{R}_j \cap \underline{R}^*) \qquad (3.4.7)$$

Hence, by (3.4.6),

$$\underline{E}_i \cap \sum_{j\neq i}\underline{E}_j \subset (\underline{E}_i + (\underline{R}_i \cap \underline{R}^*) \cap (\sum_{j\neq i}\underline{E}_j + \sum_{j\neq i}(\underline{R}_j \cap \underline{R}^*))$$

$$= (\underline{E}_i + (\underline{R}_i \cap \underline{R}^*) \cap (\sum_{j\neq i}(\underline{E}_j + (\underline{R}_j \cap \underline{R}^*)))$$

$$= \underline{R}_i \cap \sum_{j\neq i}\underline{R}_j \qquad (3.4.8)$$

But, $\underline{R}_i \cap \sum_{j\neq i}\underline{R}_j \subset \underline{R}^*$, and $\underline{E}_i \cap \sum_{j\neq i}\underline{E}_j \subset \underline{E}_i$, implying, by (3.4.8) that

$$\underline{E}_i \cap \sum_{j\neq i}\underline{E}_j \subset \underline{E}_i \cap \underline{R}^*$$

$$= 0 \qquad (3.4.9)$$

which follows from the construction of the $\underline{E}_i$, $i \in \underline{k}$. Thus the $\underline{E}_i$, $i \in \underline{k}_o$, are independent. Also,

$$\underline{X} = \sum_{i\in\underline{k}}\underline{R}_i = \sum_{i\in\underline{k}}\underline{E}_i + \sum_{i\in\underline{k}}(\underline{R}_i \cap \underline{R}^*)$$

$$\subset \sum_{i\in\underline{k}}\underline{E}_i + \underline{R}^* = \sum_{i\in\underline{k}_o}\underline{E}_i \qquad (3.4.10)$$

proving (3.4.3).

It has already been shown in Theorem 3.3.1. that

$$(A + BF)\underline{R}^* \subset \underline{R}^* \qquad (3.4.11)$$

for all $F \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$. Hence, (3.4.4) for i = 0, follows from (3.4.1). Since, from (3.4.2), $\underline{E}_i \subset \underline{R}_i$, $i \in \underline{k}$, then

$$\begin{aligned}(A + BF)\underline{E}_i &\subset (A + BF)\underline{R}_i \\ &\subset \underline{R}_i \\ &\subset \underline{E}_i + \underline{E}_0 \qquad i \in \underline{k} \qquad (3.4.12)\end{aligned}$$

for all $F \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$, proving (3.4.4) for $i \in \underline{k}$, and completing the proof of the lemma.

In order to describe the structure of the decoupled system in terms of the subspaces $\underline{E}_i$, $i \in \underline{k}_o$, the following lemma is necessary.

Lemma 3.4.2.

Define P_i, $i \in \underline{k}_o$, as the projection map of $\underline{X}$ onto $\underline{E}_i$ along $\bigoplus_{j \neq i} \underline{E}_j$. For all $F \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$, there exist maps A_i such that

$$A_i P_i = P_i(A + BF) \qquad i \in \underline{k} \qquad (3.4.13)$$

In addition,

$$C_i = C_i P_i \qquad i \in \underline{k} \qquad (3.4.14)$$

and, if G_i is chosen so that $B\underline{G}_i = \underline{B} \cap (\underline{E}_i + \underline{E}_0)$,

$$P_i B\underline{G}_j = 0 \qquad j \neq i,\ j \in \underline{k}_0,\ i \in \underline{k} \qquad (3.4.15)$$

and

$$\{P_i(A + BF) \mid P_i B\underline{G}_i\} = \underline{E}_i \qquad i \in \underline{k} \qquad (3.4.16)$$

Proof: Since $\underline{N}(P_i) = \sum_{j \neq i} \underline{E}_j$, and by Lemma 3.4.1.,

$$P_i(A + BF)\sum_{j\neq i}\underline{E}_j \subset P_i \sum_{j\neq i}\underline{E}_j$$

$$= 0 \qquad i \in \underline{k} \qquad (3.4.17)$$

therefore

$$\underline{N}(P_i) = \sum_{j\neq i}\underline{E}_j \subset \underline{N}(P_i(A + BF)) \quad i \in \underline{k} \qquad (3.4.18)$$

implying the existence of the A_i, $i \in \underline{k}$, in (3.4.13).

From (3.4.2), $\underline{E}_i + \underline{E}_0 = \underline{R}_i + \underline{R}^*$, and since, as has been shown previously, $\underline{R}^* \subset \underline{N}(C) \subset \underline{N}(C_i)$, $i \in \underline{k}$, it follows that

$$C_i(\underline{E}_j + \underline{E}_0) = C_i(\underline{R}_j + \underline{R}^*)$$

$$= 0 \qquad j \neq i,\ j,i \in \underline{k} \qquad (3.4.19)$$

Thus, since by Lemma 3.4.1., $\sum_{i\in \underline{k}_o} \underline{E}_i = \underline{X}$, and $\underline{P}_i = \underline{E}_i$,

$$C_i = C_i\left(\sum_{i\in \underline{k}_o} \underline{P}_i\right)$$

$$= C_i P_i \qquad i \in \underline{k} \qquad (3.4.20)$$

proving (3.4.14). Equation (3.4.15) follows simply from the choice of G_i, since $\underline{BG}_j \subset \underline{E}_j + \underline{E}_0 \subset \underline{N}(P_i)$, $j \neq i$.

For (3.4.16), since

$$\underline{R}_j \subset \underline{E}_j + \underline{E}_0 \subset \underline{N}(P_i) \qquad j\neq i \qquad i \in \underline{k} \qquad (3.4.21)$$

it follows that

$$\underline{E}_i = P_i\underline{X} = P_i\left(\sum_{j\in \underline{k}} \underline{R}_j\right) = P_i\underline{R}_i \qquad i \in \underline{k} \qquad (3.4.22)$$

Also, $P_i(A + BF)\underline{E}_j \subset P_i(\underline{E}_j + \underline{E}_0)$, $j \neq i$, $j \in \underline{k}_o$, $i \in \underline{k}$, which implies that $P_i(A + BF) = P_i(A + BF) \sum_{j\in \underline{k}_o} P_j = P_i(A + BF)P_i$. Thus, from (3.4.22), for $i \in \underline{k}$,

$$\underline{E}_i = P_i\{A + BF \mid BG_i\} = \{P_i(A + BF) \mid P_iBG_i\} \qquad (3.4.23)$$

This lemma shows that the decoupled system can be represented as $(k + 1)$, with k independent, subsystems. This is best demonstrated in matrix form. For $i \in \underline{k}_o$, write P_i in some basis as the product of two full rank matrices, $P_i = L_i M_i$, and define $A_i = M_i(A + BF)L_i$, $B_i = M_i BG_i$, and $C_i = C_i L_i$. For $i \in \underline{k}$, define $A_{oi} = M_o(A + BF)L_i$ and $B_{oi} = M_o BG_i$. The following properties now follow directly from the results of Lemma 3.4.2.

$$M_i(A + BF) = A_i M_i \qquad i \in \underline{k} \qquad (3.4.24)$$

$$C_i = C_i M_i \qquad i \in \underline{k} \qquad (3.4.25)$$

$$M_i BG_j = 0 \qquad j \neq i,\ j \in \underline{k}_o,\ i \in \underline{k} \qquad (3.4.26)$$

and (A_i, B_i), $i \in \underline{k}$, are completely controllable. It is now possible to describe the decoupled system by the set of equations

$$\dot{x}_i(t) = A_i x_i(t) + B_i v_i(t) \qquad i \in \underline{k} \qquad (3.4.27)$$

$$y_i(t) = C_i x_i(t) \qquad i \in \underline{k} \qquad (3.4.28)$$

$$\dot{x}_o(t) = \sum_{i \in \underline{k}} A_{oi} x_i(t) + A_o x_o(t) + \sum_{i \in \underline{k}} B_{oi} v_i(t) \qquad (3.4.29)$$

where $x_i(t) = M_i x(t)$, $i \in \underline{k}_0$. The $(k + 1)$th. subsystem, described by (3.4.29), has state space $\underline{E}_0$ and therefore has no effect on the output $y(t)$ since $\underline{E}_0 \subset \underline{N}(C)$, and for all $F \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$, $(A + BF)\underline{E}_0 \subset \underline{E}_0$. Furthermore, by (3.4.27), the i th. completely controllable subsystem is driven only from the i th. input v_i, with output y_i, $i \in \underline{k}$, by (3.4.28), whilst the $(k + 1)$th. subsystem may be driven by all the inputs v_i, $i \in \underline{k}$.

From this structure the solution of the pole assignment problem is clear. From Theorem 2.6.1., it follows that the eigenvalues of $(A + BF)$ restricted to $\underline{E}_i$, $i \in \underline{k}$, can be arbitrarily assigned, for F restricted to being contained in $\bigcap_{i \in \underline{k}} F(\underline{R}_i)$. Furthermore, $\underline{R}^*$ is an invariant subspace of $(A,\underline{B})$ and it is possible, by Theorem 3.2.1., to find the maximal controllability subspace $\underline{R}$ of (A,B) contained in $\underline{R}^*$, i.e.

$$\underline{R} = \{ A + BF \mid \underline{B} \cap \underline{R}^* \} \qquad (3.4.30)$$

where $F \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$. Since $\underline{R}$ is a controllability subspace, the eigenvalues of $(A + BF)$ restricted to $\underline{R}$ can also be arbitrarily assigned. Thus, the only fixed eigenvalues are those of the restriction of $(A + BF)$ to the subspace of $\underline{R}^*$ which remains after removal of $\underline{R}$, i.e. $\tilde{\underline{R}}$ where

$$\underline{R}^* = \underline{R} \oplus \underline{\tilde{R}} \qquad (3.4.31)$$

Consider now the output feedback case. Of necessity, if an output feedback solution to RDP exists, and (C,A) is observable, by Theorem 3.3.1., $\underline{R}^* = 0$. Thus, in the foregoing development, $\underline{E}_0 = 0$, and $\underline{E}_i = \underline{R}_i$, $i \in \underline{k}$. Let $x_i(t) = M_i x(t)$, $i \in \underline{k}$, where now P_o, and hence M_o equals zero. Then, by the results of Lemma 3.4.2.,

$$\dot{x}_i(t) = A_i x_i(t) + B_i v_i(t) \qquad i \in \underline{k} \qquad (3.4.32)$$

$$y_i(t) = C_i x_i(t) \qquad i \in \underline{k} \qquad (3.4.33)$$

The eigenvalue assignment problem becomes therefore the problem of assigning the eigenvalues of $(A_i + B_i K_i C_i)$ by choice of K_i, for $i \in \underline{k}$. For each of the k subsystems the problem is then simply the general output feedback eigenvalue assignment problem. This problem will be considered in a later chapter. It may be noted however, that (C_i, A_i) are observable pairs with respect to each $\underline{R}_i$, $i \in \underline{k}$. This follows because the decoupled system is completely observable, as observability cannot be destroyed by output feedback and therefore, since the system is entirely composed of the k independent subsystems, each subsystem must be completely observable.

CHAPTER FOUR

Decoupling by output feedback: extended case

4.1. The extended decoupling problem

The situation now considered is when it is necessary to introduce some form of dynamic compensation in order to achieve a required decoupling control. The method considered here for introducing dynamics into the system will be that described in section 2.3., whereby the state space $\underline{X}$ is extended to form a larger state space $\hat{\underline{X}}$. In the extended space, the system is described by the equations

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t) \qquad (4.1.1)$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) \qquad (4.1.2)$$

where $\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}$, $\hat{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$, and $\hat{x}(t) = \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix}$, $\hat{u}(t) = \begin{bmatrix} u(t) \\ \tilde{u}(t) \end{bmatrix}$, $\hat{y}(t) = \begin{bmatrix} y(t) \\ \tilde{x}(t) \end{bmatrix}$. The extension is denoted by $\tilde{\underline{X}}$, and thus

$$\hat{\underline{X}} = \underline{X} \oplus \tilde{\underline{X}} \qquad (4.1.3)$$

The dimension of $\tilde{\underline{X}}$ is denoted by $\tilde{n}$.

Consider now the decoupling problem in the extended state space. The outputs to be decoupled remain unaltered, i.e.,

$$y_i(t) = [C_i \quad 0]\hat{x}(t) = \overline{C}_i\hat{x}(t) \qquad i \in \underline{k} \tag{4.1.4}$$

The feedback control is described by the equation

$$\hat{u}(t) = \hat{F}\hat{x}(t) + \sum_{i\in\underline{k}} \hat{G}_i v_i(t) \tag{4.1.5}$$

Note that, if $\overline{\underline{N}}_i = \underline{N}(\overline{C}_i)$, for $i \in \underline{k}$, then

$$\overline{\underline{N}}_i = \underline{N}_i + \tilde{\underline{X}} \tag{4.1.6}$$

from the structure of $\overline{C}_i$ given in (4.1.4). Hence the decoupling problem can now be seen to be that of finding a set $\{\hat{\underline{R}}_i\}_{\underline{k}}$ of controllability subspaces of $(\hat{A},\hat{B})$ such that

$$\hat{\underline{R}}_i + \underline{N}_i + \tilde{\underline{X}} = \hat{\underline{X}} \qquad i \in \underline{k} \tag{4.1.7}$$

$$\hat{\underline{R}}_i \subset (\bigcap_{j\neq i} \underline{N}_j) + \tilde{\underline{X}} \qquad i \in \underline{k} \tag{4.1.8}$$

and

$$\bigcap_{i\in\underline{k}} F(\hat{\underline{R}}_i) \neq \emptyset \tag{4.1.9}$$

corresponding to the conditions (3.1.10)-(3.1.12) for RDP. This problem has been called the "extended decoupling problem", or EDP, (MW1)(MW2). The additional freedom introduced in EDP over RDP is the dimension $\tilde{n}$ of the extension $\tilde{\underline{X}}$. In all other respects, the problem formulated here is identical to RDP.

Let P be defined as the projection map of $\hat{\underline{X}}$ onto $\underline{X}$ along $\tilde{\underline{X}}$. Then for the map $\hat{A}: \hat{\underline{X}} \to \hat{\underline{X}}$, corresponding to the matrix $\hat{A}$ in (4.1.1), and for the map $\hat{B}: \hat{\underline{U}} \to \hat{\underline{X}}$, corresponding to the matrix $\hat{B}$ in (4.1.1), the following can be seen to be true for the maps $P\hat{A}$, $\hat{A}P$, $P\hat{B}$,

$$P\hat{A} = \hat{A}P = A, \ P\hat{B} = B \tag{4.1.10}$$

where A and B are now regarded as maps from $\hat{\underline{X}}$ into $\hat{\underline{X}}$. This is consistent since clearly $\underline{X} \subset \hat{\underline{X}}$. In matrix form P clearly can be written as $P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, an $\hat{n} \times \hat{n}$ matrix in which I is n x n. It follows that the null space of P is $\tilde{\underline{X}}$.

The following lemma will be useful in proving necessary and sufficient conditions for the existence of a solution to EDP.

Lemma 4.1.1.

Let $\underline{U}$ and $\underline{V}$ be any two subspaces such that $\tilde{\underline{X}} \subset \underline{U} \subset \hat{\underline{X}}$, and $\tilde{\underline{X}} \subset \underline{V} \subset \hat{\underline{X}}$. Then

$$P(\underline{U} \cap \underline{V}) = P\underline{U} \cap P\underline{V} \qquad (4.1.11)$$

Proof: It is known (MW1) that for any $\underline{U}$, $\underline{V}$ and map P,

$$P(\underline{U} \cap \underline{V}) \subset P\underline{U} \cap P\underline{V} \qquad (4.1.12)$$

with equality if and only if

$$(\underline{U} + \underline{V}) \cap \underline{N}(P) = \underline{U} \cap \underline{N}(P) + \underline{V} \cap \underline{N}(P) \qquad (4.1.13)$$

It has already been seen that in this case $\underline{N}(P) = \tilde{\underline{X}}$, and, by assumption $\tilde{\underline{X}} \subset \underline{U}$ and $\tilde{\underline{X}} \subset \underline{V}$. Hence for these subspaces and projection P, (4.1.13) holds, and the lemma is proved.

The following theorem now establishes a necessary and sufficient condition for the existence of a solution to EDP (MW1).

Theorem 4.1.1.

A solution to EDP exists if and only if

$$\underline{R}_i^M + \underline{N}_i = \underline{X} \qquad i \in \underline{k} \qquad (4.1.14)$$

Proof: Define the projection map P as above, and let $\{\hat{\underline{R}}_i\}_{\underline{k}}$ be a solution to EDP, for some $\tilde{n} \geq 0$. Then, for $i \in \underline{k}$, since $\hat{\underline{R}}_i$ is a controllability subspace of $(\hat{A},\hat{B})$,

$$\hat{A}\underline{\hat{R}}_i \subset \underline{\hat{R}}_i + \underline{\hat{B}} \tag{4.1.15}$$

Then, by linearity,

$$P\hat{A}\underline{\hat{R}}_i \subset P\underline{\hat{R}}_i + P\underline{\hat{B}} \tag{4.1.16}$$

or, defining $\underline{R}_i = P\underline{\hat{R}}_i$, and from (4.1.10),

$$A\underline{R}_i \subset \underline{R}_i + \underline{B} \tag{4.1.17}$$

showing $\underline{R}_i, i \in \underline{k}$, so defined, to be an invariant subspace of (A,B).

To show that $\underline{R}_i$ is, in fact, a controllability subspace of (A,B), recall that, by Lemma 2.4.2., since $\underline{\hat{R}}_i$ is a controllability subspace of $(\hat{A},\hat{B})$,

$$\underline{\hat{R}}_i = \lim \underline{\hat{S}}_i^j, \qquad j = 0,1,\ldots \tag{4.1.18}$$

where $\underline{\hat{S}}_i^0 = 0$, $\underline{\hat{S}}_i^j = (\hat{A}\underline{\hat{S}}_i^{j-1} + \underline{\hat{B}}) \cap \underline{\hat{R}}_i$. Define $\underline{S}_i^j = P\underline{\hat{S}}_i^j$, $j = 0,1,\ldots$. Then, using Lemma 4.1.1. and (4.1.10),

$$\begin{aligned}\underline{S}_i^j &= P((\hat{A}\underline{\hat{S}}_i^{j-1} + \underline{\hat{B}}) \cap \underline{\hat{R}}_i) \\ &= P(\hat{A}\underline{\hat{S}}_i^{j-1} + \underline{\hat{B}}) \cap P\underline{\hat{R}}_i\end{aligned}$$

$$= (\hat{A}P\hat{\underline{S}}_i^{j-1} + \underline{B}) \cap \underline{R}_i$$

$$= (A\underline{S}_i^{j-1} + \underline{B}) \cap \underline{R}_i \tag{4.1.19}$$

Also, by definition,

$$\underline{R}_i = P\hat{\underline{R}}_i = P(\lim \hat{\underline{S}}_i^j) = \lim P\hat{\underline{S}}_i^j = \lim \underline{S}_i^j \tag{4.1.20}$$

where $\underline{S}_i^0 = P\hat{\underline{S}}_i^0 = 0$, and $\underline{S}_i^j = (A\underline{S}_i^{j-1} + \underline{B}) \cap \underline{R}_i$, by (4.1.19). This, together with (4.1.17), ensures, by Lemma 2.4.2., that $\underline{R}_i$ is a controllability subspace of (A,B). This is true for all $i \in \underline{k}$.

To show necessity of (4.4.14) it remains only to show that

$$\underline{R}_i = P\hat{\underline{R}}_i \subset P(\bigcap_{j \neq i} \underline{N}_j + \tilde{\underline{X}}) = \bigcap_{j \neq i} \underline{N}_j \tag{4.1.21}$$

by (4.1.8), and that

$$\underline{R}_i + \underline{N}_i = P(\hat{\underline{R}}_i + \underline{N}_i + \tilde{\underline{X}}) = P\hat{\underline{X}} = \underline{X} \tag{4.1.22}$$

by (4.1.7). Then, since $\underline{R}_i^M$ is the maximal controllability of (A,B) contained in $\bigcap_{j \neq i} \underline{N}_j$, by (4.1.21),

$$\underline{R}_i \subset \underline{R}_i^M \tag{4.1.23}$$

and therefore, by (4.1.22),

$$\underline{R}_i^M + \underline{N}_i = \underline{X} \qquad i \in \underline{k}$$

proving necessity.

For sufficiency, assume that (4.1.14) holds. Let $\underline{\tilde{X}}$ be defined with dimension $\tilde{n} = \sum_{i \in \underline{k}} \dim(\underline{R}_i^M)$. Define the maps $M_i : \underline{\hat{X}} \rightarrow \underline{\tilde{X}} \subset \underline{\hat{X}}$, $i \in \underline{k}$, such that

$$\underline{R}_i^M \cap \underline{N}(M_i) = 0 \qquad i \in \underline{k} \qquad (4.1.24)$$

$$\underline{M}_i = M_i \underline{R}_i^M \qquad i \in \underline{k} \qquad (4.1.25)$$

and such that the $\underline{M}_i$, $i \in \underline{k}$, are mutually independent. For $\underline{\tilde{X}}$ of such a dimension, the M_i clearly exist. For example, let $k = 2$. Then in $\underline{\hat{X}}$, $\underline{R}_1^M$ and $\underline{R}_2^M$ have basis matrices of the form $\begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} R_2 \\ 0 \end{bmatrix}$. Now, let the $\tilde{n} \times \hat{n}$ matrices corresponding to the maps M_1 and M_2 be given by

$$M_1 = \begin{bmatrix} R_1^T & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ R_2^T & 0 \end{bmatrix} \qquad (4.1.26)$$

Noting that $\underline{N}(M_i) = \underline{N}(R_i^T) + \underline{\tilde{X}} = (\underline{R}_i^M)^{\perp} + \underline{\tilde{X}}$, (4.1.24) clearly follows; (4.1.25) can also be seen to hold.

Returning to the proof of sufficiency of (4.1.14), define

$$\hat{\underline{R}}_i = (P + M_i)\underline{R}_i^M \qquad i \in \underline{k} \tag{4.1.27}$$

For the example above, the map $(P + M_1)$ has the matrix form $(\hat{n} \times \hat{n})$

$$P + M_1 = \begin{bmatrix} I & 0 \\ R_1^T & 0 \\ 0 & 0 \end{bmatrix} \tag{4.1.28}$$

and similarly for i = 2. Then, using (4.1.10) and noting that $\underline{M}_i \subset \tilde{\underline{X}}$, it follows that, in $\hat{\underline{X}}$,

$$\begin{aligned} \hat{A}\hat{\underline{R}}_i &= \hat{A}(P + M_i)\underline{R}_i^M \\ &= A\underline{R}_i^M \\ &\subset \underline{R}_i^M + \underline{B} \\ &\subset \hat{\underline{R}}_i + \underline{B} + \tilde{\underline{X}} \\ &= \hat{\underline{R}}_i + \hat{\underline{B}} \end{aligned} \tag{4.1.29}$$

Thus, the $\hat{\underline{R}}_i$ are invariant subspaces of $(\hat{A},\hat{B})$, and, since the M_i were chosen independent, the $\hat{\underline{R}}_i$ are also independent.

It remains to show that $\{\hat{\underline{R}}_i\}_{\underline{k}}$ so constructed, is a solution to EDP, i.e. that they are controllability subspaces and satisfy (4.1.7)-(4.1.9). To show that they are controllability subspaces of $(\hat{A},\hat{B})$, set, for $i \in \underline{k}$,

$$\hat{\underline{S}}_i^0 = 0, \; \hat{\underline{S}}_i^j = (\hat{A}\hat{\underline{S}}_i^{j-1} + \hat{\underline{B}}) \cap \hat{\underline{R}}_i \tag{4.1.30}$$

and

$$\underline{S}_i^0 = 0, \; \underline{S}_i^j = (A\underline{S}_i^{j-1} + \underline{B}) \cap \underline{R}_i^M \tag{4.1.31}$$

Clearly, $(P + M_i)\underline{S}_i^0 \subset \hat{\underline{S}}_i^0$. It will now be shown by induction, that $(P + M_i)\underline{S}_i^j \subset \hat{\underline{S}}_i^j$, $j = 1,2,\ldots$; assume $(P + M_i)\underline{S}_i^{j-1} \subset \hat{\underline{S}}_i^{j-1}$. Then

$$\begin{aligned}
\hat{\underline{S}}_i^j &\supset (\hat{A}(P + M_i)\underline{S}_i^{j-1} + \underline{B} + \tilde{\underline{X}}) \cap (P + M_i)\underline{R}_i^M \\
&= (A\underline{S}_i^{j-1} + \underline{B} + \tilde{\underline{X}}) \cap (P + M_i)\underline{R}_i^M \\
&\supset (P + M_i)((A\underline{S}_i^{j-1} + \underline{B} + \tilde{\underline{X}}) \cap \underline{R}_i^M) \\
&= (P + M_i)\underline{S}_i^j
\end{aligned} \tag{4.1.32}$$

as required.

Since $\underline{R}_i^M = \lim \underline{S}_i^j$, $\hat{\underline{R}}_i \supset \lim \hat{\underline{S}}_i^j \supset \lim (P+M_i)\underline{S}_i^j = (P+M_i)\underline{R}_i^M = \hat{\underline{R}}_i$, therefore, $\hat{\underline{R}}_i = \lim \hat{\underline{S}}_i^j$, $j = 0,1,\ldots$, $i \in \underline{k}$, proving that the $\hat{\underline{R}}_i$, $i \in \underline{k}$, are controllability subspaces of $(\hat{A},\hat{B})$.

To show that $\bigcap_{i\in\underline{k}} F(\hat{\underline{R}}_i) \neq \emptyset$, define $\hat{P}_i$ as the projection of $\hat{\underline{X}}$ onto $\hat{\underline{R}}_i$ along $\bigoplus_{j\neq i} \hat{\underline{R}}_j$. Then for $\hat{F}_i \in F(\hat{\underline{R}}_i)$, $i \in \underline{k}$, it is easy to see that

$$\hat{F} = \sum_{i\in\underline{k}} \hat{F}_i \hat{P}_i \in \bigcap_{i\in\underline{k}} F(\hat{\underline{R}}_i) \tag{4.1.33}$$

Finally, to show that (4.1.7) and (4.1.8) hold, it follows from the structure of the $\hat{\underline{R}}_i$ that

$$\hat{\underline{R}}_i \subset \underline{R}_i^M + \tilde{\underline{X}} \subset \bigcap_{j\neq i} \underline{N}_j + \tilde{\underline{X}} \tag{4.1.34}$$

and, by (4.1.14),

$$(P + M_i)\underline{R}_i^M + (P + M_i)\underline{N}_i = (P + M_i)\underline{X} \tag{4.1.35}$$

yielding

$$\hat{\underline{R}}_i + \underline{N}_i + \tilde{\underline{X}} = \underline{X} + \tilde{\underline{X}} = \hat{\underline{X}} \tag{4.1.36}$$

This completes the proof of Theorem 4.1.1.

This theorem shows that it is always possible to decouple a linear multivariable system by state feedback, provided (4.1.14) holds, if dynamics of sufficiently high order are adjoined to the system. Moreover, since the $\{\hat{\underline{R}}_i\}_{\underline{k}}$ which form a solution to EDP can be made independent, by the results of Section 3.4., arbitrary assignment of all the eigenvalues of $(\hat{A} + \hat{B}\hat{F})$ is always possible. In fact, to satisfy these properties it is not necessary to use an extension of as large an order as $\sum_{i \in \underline{k}} \dim(\underline{R}_i^M)$. A full development of this may be found in Morse and Wonham (MW1), (MW2).

The possibility of the existence of output feedback solutions to EDP is now considered.

4.2. Existence of an output feedback solution to EDP.

An important application of the technique of extending the state space will now be shown to be that of obtaining an output feedback decoupling control, for a system for which a solution $\{\underline{R}_i\}_{\underline{k}}$ to RDP cannot be found to satisfy the conditions derived in Section 3.3., i.e.

$$\underline{R}^* = 0 \tag{4.2.1}$$

$$A(\underline{R}_i \cap \underline{N}(C)) \subset \underline{R}_i \qquad i \in \underline{k} \tag{4.2.2}$$

The following lemma provides a basis for the development of a procedure for constructing a set of controllability subspaces $\{\hat{\underline{R}}_i\}_{\underline{k}}$ of $(\hat{A},\hat{B})$ which are an output feedback solution to EDP.

Lemma 4.2.1.

Given a set of controllability subspaces $\{\hat{\underline{R}}_i\}_{\underline{k}}$ of $(\hat{A},\hat{B})$, such that

$$\bigoplus_{i\in\underline{k}} \hat{\underline{R}}_i = \hat{\underline{X}} \qquad (4.2.3)$$

suppose there exist $\hat{F}_i \in F(\hat{\underline{R}}_i)$, $i \in \underline{k}$ such that

$$\underline{N}(\hat{C}\hat{R}_i) \subset \underline{N}(\hat{F}_i\hat{R}_i) \qquad i \in \underline{k} \qquad (4.2.4)$$

Then, there exists $\hat{F} \in \bigcap_{i\in\underline{k}} F(R_i)$ such that

$$\underline{N}(\hat{C}) \subset \underline{N}(\hat{F}) \qquad (4.2.5)$$

Here $\hat{R}_i$, $i \in \underline{k}$ denotes basis matrices for $\hat{\underline{R}}_i$, that is matrices whose columns span the subspace.

Proof: Define $\hat{P}_i$, $i \in \underline{k}$ as the projection of $\hat{\underline{X}}$ onto $\hat{\underline{R}}_i$ along $\bigoplus_{j\neq i} \hat{\underline{R}}_j$. For $\hat{F}_i \in F(\hat{\underline{R}}_i)$, $i \in \underline{k}$, such that (4.2.4) holds, let

$$\hat{F} = \sum_{i\in\underline{k}} \hat{F}_i\hat{P}_i \qquad (4.2.6)$$

Then

$$(\hat{A} + \hat{B}\hat{F})\hat{\underline{R}}_i = (\hat{A} + \hat{B}\sum_{i\in \underline{k}}\hat{F}_i\hat{P}_i)\hat{\underline{R}}_i$$

$$= (\hat{A} + \hat{B}\hat{F}_i)\hat{\underline{R}}_i$$

$$\subset \hat{\underline{R}}_i \qquad i \in \underline{k} \qquad (4.2.7)$$

Now, (4.2.4) implies that

$$\underline{N}(\hat{C}) \cap \hat{\underline{R}}_i \subset \underline{N}(\hat{F}_i) \qquad i \in \underline{k} \qquad (4.2.8)$$

or, taking orthogonal complements

$$\hat{\underline{F}}_i^T \subset \hat{\underline{R}}_i^{\perp} + \underline{C}^T \qquad i \in \underline{k} \qquad (4.2.9)$$

Taking orthogonal complements of (4.2.6) also yields

$$\hat{\underline{F}}^T = \sum_{i\in \underline{k}} \hat{P}_i^T \hat{\underline{F}}_i^T \qquad (4.2.10)$$

and since $\hat{P}_i^T \hat{\underline{R}}_i^{\perp} = (\hat{P}_i^{-1}\hat{\underline{R}}_i)^{\perp} = (\hat{\underline{X}})^{\perp} = 0$, it follows from (4.2.9) that

$$\hat{\underline{F}}^T \subset \sum_{i\in \underline{k}} \hat{P}_i^T \hat{\underline{C}}^T = \hat{\underline{C}}^T \qquad (4.2.11)$$

since $\sum_{i \in \underline{k}} \hat{P}_i = I$. Taking orthogonal complements of (4.2.11) yields (4.2.5) and proves the lemma.

From this lemma it is clear that in order to obtain an output feedback solution to EDP it is sufficient to provide a construction procedure which will yield a set of controllability subspaces $\{\hat{\underline{R}}_i\}_{\underline{k}}$ for which (4.2.3) and (4.2.4) are true. Before establishing such a construction procedure, it is necessary to include the following lemma.

Lemma 4.2.2.

Given $\bar{\underline{R}} \subset \underline{X}$, a controllability subspace of $(\hat{A},\hat{B})$ with $\hat{\underline{X}} = \underline{X} \oplus \tilde{\underline{X}}$, then

$$\hat{\underline{R}} = \bar{\underline{R}} + \underline{U} \tag{4.2.12}$$

where $\underline{U} \subset \tilde{\underline{X}}$, is also a controllability subspace of $(\hat{A},\hat{B})$.

Proof: Since $\bar{\underline{R}}$ is a controllability subspace of $(\hat{A},\hat{B})$, by Lemma 2.4.2.,

$$\bar{\underline{R}} = \lim \bar{\underline{S}}^j \qquad j = 0,1,\ldots \tag{4.2.13}$$

where $\bar{\underline{S}}^0 = 0$, $\bar{\underline{S}}^j = (\hat{A}\bar{\underline{S}}^{j-1} + \hat{\underline{B}}) \cap \bar{\underline{R}}$. Let

$$\hat{\underline{S}}^0 = 0,\ \hat{\underline{S}}^j = (\hat{A}\hat{\underline{S}}^{j-1} + \hat{\underline{B}}) \cap \hat{\underline{R}} \tag{4.2.14}$$

Then

$$\hat{\underline{S}}^1 = \hat{\underline{B}} \cap \hat{\underline{R}}$$

$$= (\underline{B} + \tilde{\underline{X}}) \cap (\bar{\underline{R}} + \underline{U})$$

$$= (\hat{\underline{B}} \cap \bar{\underline{R}}) + \underline{U} \qquad (4.2.15)$$

Further, if $\hat{\underline{S}}^j = \bar{\underline{S}}^j + \underline{U}$,

$$\hat{\underline{S}}^{j+1} = (\hat{A}\hat{\underline{S}}^j + \hat{\underline{B}}) \cap \hat{\underline{R}}$$

$$= (\ (\hat{A}\hat{\underline{S}}^j + \hat{\underline{B}}) \cap \bar{\underline{R}}) \ + \underline{U}$$

$$= (\ (\hat{A}\bar{\underline{S}}^j + \hat{\underline{B}}) \cap \bar{\underline{R}}) \ + \underline{U}$$

$$= \bar{\underline{S}}^{j+1} + \underline{U} \qquad (4.2.16)$$

By induction therefore,

$$\hat{\underline{R}} = \bar{\underline{R}} + \underline{U} = \lim \bar{\underline{S}}^j + \underline{U} = \lim \hat{\underline{S}}^j \qquad (4.2.17)$$

Also

$$\hat{A}\hat{\underline{R}} = \hat{A}\bar{\underline{R}} \subset \bar{\underline{R}} + \hat{\underline{B}} \subset \hat{\underline{R}} + \hat{\underline{B}} \qquad (4.2.18)$$

which, by Lemma 2.4.2., completes the proof.

Using this lemma and Lemma 4.2.1. a necessary and sufficient condition for the existence of an output feedback solution to EDP can be established.

Theorem 4.2.1.

Given A, B and $\underline{N}_i$, $i \in \underline{k}$, let $\underline{R}_i^M$ be the maximal controllability subspace of (A,B) contained in $\bigcap_{j \neq i} \underline{N}_j$, $i \in \underline{k}$. Then, there exists a solution $\{\hat{\underline{R}}_i\}_{\underline{k}}$ to EDP such that

$$\hat{F} = \hat{K}\hat{C} \qquad (4.2.19)$$

for some $\hat{F} \in \bigcap_{i \in \underline{k}} F(\underline{R}_i)$, if and only if

$$\underline{R}_i^M + \underline{N}_i = \underline{X} \qquad i \in \underline{k} \qquad (4.2.20)$$

Proof: Necessity follows immediately from Theorem 4.1.1., which establishes (4.2.20) as a necessary condition for $\bigcap_{i \in \underline{k}} F(\hat{\underline{R}}_i) \neq \emptyset$, where $\{\hat{\underline{R}}_i\}_{\underline{k}}$ is any solution to EDP.

For sufficiency, if $\sum_{i \in \underline{k}} \dim(\underline{R}_i^M) \leq n$, choose $\tilde{n} = n$, and if $\sum_{i \in \underline{k}} \dim(\underline{R}_i^M) > n$, choose $\tilde{n} = \sum_{i \in \underline{k}} \dim(\underline{R}_i^M)$. It is clear that for such a large extension it is possible to choose maps $M_i : \hat{\underline{X}} \rightarrow \tilde{\underline{X}}$, $i \in \underline{k}$, as in Theorem 4.1.1., such that the subspaces

$$\underline{L}_i = (P + M_i)\underline{R}_i^M \qquad i \in \underline{k} \qquad (4.2.21)$$

are independent, and moreover

$$\underline{N}(\hat{C}L_i) = 0 \qquad i \in \underline{k} \qquad (4.2.22)$$

where L_i is a basis matrix for $\underline{L}_i$, $i \in \underline{k}$. To clarify this consider the following example.

Example 4.2.1.

For

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

the maximal controllability subspaces $\underline{R}_i^M$, $i = 1,2$, can be easily determined as

$$\underline{R}_1^M = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad (4.2.23)$$

$$\underline{R}_2^M = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad (4.2.24)$$

Since $\sum_{i=1}^{2} \dim(\underline{R}_i^M) = 6 > n = 5$, choose $\tilde{n} = 6$. It is then possible to choose M_1 such that

$$\underline{L}_1 = (P + M_1)\underline{R}_1^M = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad (4.2.25)$$

and M_2 such that

$$\underline{L}_2 = (P + M_2)\underline{R}_2^M = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad (4.2.26)$$

Operating $\hat{C} = \begin{bmatrix} C & 0 \\ 0 & I_{\tilde{n}} \end{bmatrix}$ on $\hat{\underline{R}}_1$ and $\hat{\underline{R}}_2$ will illustrate that (4.2.22) holds.

Returning to the proof of Theorem 4.2.1., it is clear that, by the same argument as in the proof of Theorem 4.1.1., the $\{\underline{L}_i\}_{\underline{k}}$ so constructed are controllability subspaces of $(\hat{A},\hat{B})$ and form a solution to EDP. However $\bigoplus_{i\in\underline{k}} \underline{L}_i \neq \hat{\underline{X}}$, which is required for Lemma 4.2.1. to be applied. Since the extension of the state space is so large, however, it is possible to choose subspaces $\underline{U}_i \subset \tilde{\underline{X}}$, $i \in \underline{k}$, such that

$$\underline{M}_i \cap \underline{U}_i = 0 \qquad\qquad i \in \underline{k} \qquad\qquad (4.2.27)$$

and to be independent, and of sufficient dimension that, setting $\hat{\underline{R}}_i = \underline{L}_i + \underline{U}_i$, $i \in \underline{k}$,

$$\bigoplus_{i\in\underline{k}} \hat{\underline{R}}_i = \hat{\underline{X}} \qquad\qquad (4.2.28)$$

To see this, for Example 4.2.1., choose

$$\underline{U}_1 = \left\{ \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix} \right\} \qquad\qquad (4.2.29)$$

$$\underline{U}_2 = \left\{ \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix} \right\} \qquad (4.2.30)$$

It is then simple to verify (4.2.27).

Using Lemma 4.2.2. shows that the $\{\hat{\underline{R}}_i\}_{\underline{k}}$ so constructed are controllability subspaces of $(\hat{A},\hat{B})$. Also

$$\begin{aligned} \hat{\underline{R}}_i + \overline{\underline{N}}_i &= \underline{L}_i + \underline{U}_i + \underline{N}_i + \tilde{\underline{X}} \\ &= \underline{L}_i + \overline{\underline{N}}_i \\ &= \hat{\underline{X}} \end{aligned} \qquad (4.2.31)$$

and

$$\begin{aligned} \hat{\underline{R}}_i &= \underline{L}_i + \underline{U}_i \\ &\subset \bigcap_{j\neq i} \underline{N}_j + \tilde{\underline{X}} \\ &= \bigcap_{j\neq i} \overline{\underline{N}}_j \end{aligned} \qquad (4.2.32)$$

Therefore, the $\{\underline{\hat{R}}_i\}_{\underline{k}}$ forms a solution to EDP. From (4.2.27) and (4.2.22), it follows that

$$\underline{N}(\hat{C}\hat{R}_i) = 0 \qquad i \in \underline{k} \tag{4.2.33}$$

Using this together with (4.2.28), by Lemma 4.2.1. the $\{\underline{\hat{R}}_i\}_{\underline{k}}$ so constructed form an output feedback solution to EDP, completing the proof of sufficiency for Theorem 4.2.1.

From Example 4.2.1., it can be seen that the dimension of the extension $\underline{\tilde{X}}$ is, in this case, much larger than necessary. Consider the following controllability subspaces in place of those constructed in the foregoing:

$$\underline{\hat{R}}_1 = \left\{ \begin{bmatrix}1\\0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\\1\end{bmatrix} \right\} \tag{4.2.34}$$

$$\underline{\hat{R}}_2 = \left\{ \begin{bmatrix}0\\0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\-1\\0\end{bmatrix} \right\} \tag{4.2.35}$$

Operating $\hat{C}$ on these subspaces clearly shows that they satisfy (4.2.6), and in fact form an output feedback solution to EDP.

4.3. Extension of low order for output feedback decoupling

As in the case of the general extended decoupling problem of Section 4.1., it is not necessary to use an extension of such a large dimension as $\max(\sum_{i\in \underline{k}} \dim(\underline{R}_i^M), n)$ in order to obtain an output feedback solution to EDP (D3). The following theorem provides a sufficient dimension of low order, and the proof of the existence of $\{\hat{\underline{R}}_i\}_{\underline{k}}$ which solves the output feedback EDP will be by a refined version of the construction used in the proof of Theorem 4.2.1.

Theorem 4.3.1.

Given A, B, and $\underline{N}_i$, $i \in \underline{k}$, let $\{\underline{R}_i^M\}_{\underline{k}}$ be the set of maximal controllability subspaces of (A,B) such that

$$\sum_{i\in \underline{k}} \underline{R}_i^M = \underline{X} \qquad (4.3.1)$$

$$\underline{R}_i^M \subset \bigcap_{j\neq i} \underline{N}_j \qquad i \in \underline{k} \qquad (4.3.2)$$

$$\underline{R}_i^M + \underline{N}_i = \underline{X} \qquad i \in \underline{k} \qquad (4.3.3)$$

If $\tilde{n} = n_o + v$, where

$$n_o = (\sum_{i\in \underline{k}} \dim(\underline{R}_i^M)) - n \qquad (4.3.4)$$

and $v = \dim(\underline{V})$, for any subspace $\underline{V}$ such that

$$\underline{N}(C) = \underline{V} \oplus (\underline{R}^M)^*, \tag{4.3.5}$$

then a solution $\{\hat{\underline{R}}_i\}_{\underline{k}}$ to EDP exists such that $\hat{F} = \hat{K}\hat{C}$, for some $\hat{F} \in \bigcap_{i \in \underline{k}} F(\hat{\underline{R}}_i)$.

<u>Proof</u>: The proof of existence of an output feedback solution $\{\hat{\underline{R}}_i\}_{\underline{k}}$ to EDP under these conditions will use the following construction. This derives from the set $\{\underline{R}_i^M\}_{\underline{k}}$, a set $\{\hat{\underline{R}}_i\}_{\underline{k}}$ of independent controllability subspaces of $(\hat{A}, \hat{B})$, each member $\hat{\underline{R}}_i$ of the set having the property that

$$\hat{\underline{R}}_i \cap \underline{N}(\hat{C}) = 0 \tag{4.3.6}$$

or $\underline{N}(\hat{C}\hat{R}_i) = 0$, and also that $\bigoplus_{i \in \underline{k}} \hat{\underline{R}}_i = \hat{\underline{X}}$. Then, by Lemma 4.2.1., there exists a matrix $\hat{K}$ such that $\hat{K}\hat{C} \in \bigcap_{i \in \underline{k}} F(\hat{\underline{R}}_i)$. The method of construction moreover ensures that $\{\hat{\underline{R}}_i\}_{\underline{k}}$ forms a solution to EDP.

Intuitively, it can be seen that the extension of the state space by dimension n_o defined in (4.3.4) ensures that the $\hat{\underline{R}}_i$ can be made independent, and the extension by dimension $v = \dim(\underline{V})$, where $\underline{V}$ is defined in (4.3.5), ensures that (4.3.6) can be made to hold. Obviously, if $(\underline{R}^M)^* = \underline{N}(C)$, then the further extension dimension v equals 0, but both requirements can still be fulfilled.

Construction procedure: Defining $\underline{V}$ as in (4.3.5), where $(\underline{R}^M)^* = \bigcap_{i \in k} (\sum_{j \neq i} \underline{R}_j^M)$, let R_i, V be basis matrices for $\underline{R}_i^M$, $\underline{V}$, $i \in \underline{k}$, respectively; that is, the columns of R_i span $\underline{R}_i^M$, $i \in \underline{k}$, and are independent, and similarly for $\underline{V}$. Construct the partitioned $n \times (n + \tilde{n})$ matrix R given by

$$R = [R_1 \ \ldots \ R_k \ V] \qquad (4.3.7)$$

and define the $\tilde{n} \times (n + \tilde{n})$ matrix M such that

$$\underline{M}^T = \underline{N}(R) \qquad (4.3.8)$$

where $M = [M_1 \ \ldots \ M_k \ M_{k+1}]$, the partitioning being consistent with that of R in (4.3.7). Set the $(n + \tilde{n}) \times (n + \tilde{n})$ matrix $\bar{R}$ as

$$\bar{R} = \begin{bmatrix} R \\ M \end{bmatrix}$$

$$= \begin{bmatrix} R_1 & \ldots & R_k & V \\ M_1 & \ldots & M_k & M_{k+1} \end{bmatrix} \qquad (4.3.9)$$

Then (4.3.8) ensures that $\bar{R}$ has full rank equal to $(n + \tilde{n})$

Define

$$\underline{\bar{R}}_i = \left\{ \begin{bmatrix} R_i \\ M_i \end{bmatrix} \right\} \qquad i \in \underline{k} \qquad (4.3.10)$$

Then,

(a) if $\underline{V} = 0$, set

$$\underline{\hat{R}}_i = \underline{\bar{R}}_i \qquad i \in \underline{k} \qquad (4.3.11)$$

(b) if $\underline{V} \neq 0$, find linearly independent subspaces $\underline{\tilde{R}}_i \subset \underline{M}_{k+1}$, for $i \in \underline{\tilde{k}} \subset \underline{k}$, such that

$$\underline{N}(C) \cap (\underline{\bar{R}}_i + \underline{\tilde{R}}_i) = 0 \qquad i \in \underline{\tilde{k}} \qquad (4.3.12)$$

and

$$\bigoplus_{i \in \underline{k}} \underline{\tilde{R}}_i = \underline{M}_{k+1} \qquad (4.3.13)$$

The existence of $\{\underline{\tilde{R}}_i\}_{\underline{\tilde{k}}}$ will be established in the proof of the theorem. Now, set

$$\underline{\hat{R}}_i = \underline{\bar{R}}_i + \underline{\tilde{R}}_i \qquad i \in \underline{\tilde{k}} \qquad (4.3.14)$$

$$\underline{\hat{R}}_i = \underline{\bar{R}}_i \qquad i \in \underline{k} - \underline{\tilde{k}} \qquad (4.3.15)$$

In order to prove Theorem 4.3.1., it will now be shown that the set of $\underline{\hat{R}}_i$, $i \in \underline{k}$, as defined by (4.3.11) or

(4.3.14) and (4.3.15), is a solution to EDP such that $\hat{F} = \hat{K}\hat{C}$ for some $\hat{F} \in \bigcap_{i\in \underline{k}} F(\underline{\hat{R}}_i)$. For simplicity, $\underline{R}_i$ is written for $\underline{R}_i^M$, $i \in \underline{k}$, and $\underline{R}^*$ for $(\underline{R}^M)^*$ in the following.

For $i \in \underline{k}$, by construction

$$R_i^{-1}(\underline{R}^* + \underline{V}) \subset \underline{M}_i^T \tag{4.3.16}$$

for this not being true implies that, for $i \in \underline{k}$ there exists some vector $x_i \neq 0$, of appropriate dimension, such that

$$R_i x_i \in \underline{R}^* + \underline{V},\ x_i \notin \underline{M}_i^T \tag{4.3.17}$$

Since $\underline{R}^* = \bigcap_{i\in \underline{k}} (\sum_{j\neq i} \underline{R}_j)$, this implies that

$$R_i x_i \in \sum_{j\neq i} \underline{R}_j + \underline{V} \tag{4.3.18}$$

Thus there exists vectors x_j, $j \neq i$ and $x_{\overline{v}}$ such that

$$R_i x_i + \sum_{j\neq i} R_j x_j + V x_{\overline{v}} = 0 \tag{4.3.19}$$

which implies that for $x = (x_1 \ldots x_k x_{\overline{v}})^T$, the elements of which are defined above,

$$x \in \underline{N}(R) \tag{4.3.20}$$

But $\underline{M}^T = \underline{N}(R)$ and $x_i \notin \underline{M}_i^T$ contradicts this; hence (4.3.16) is true. Then, by (4.3.5),

$$R_i^{-1}(\underline{N}(C)) \subset \underline{M}_i^T \qquad i \in \underline{k} \qquad (4.3.21)$$

Now,

$$\underline{N}(\widehat{C\overline{R}}_i) = \underline{N}\begin{bmatrix} CR_i \\ M_i \end{bmatrix} \qquad i \in \underline{k} \qquad (4.3.22)$$

and $\underline{N}(\widehat{C\overline{R}}_i) \neq 0$ implies that there exists a vector $x \neq 0$, such that $x \in R_i^{-1}\underline{N}(C)$, and $x \in \underline{N}(M_i)$. But (4.3.21) implies that $x \in \underline{M}_i^T = \underline{N}^{\perp}(M_i)$. Hence $x = 0$, and

$$\underline{N}(\widehat{C\overline{R}}_i) = 0 \qquad i \in \underline{k} \qquad (4.3.23)$$

Noting that $\overline{\underline{R}}_i$ has the form $(P + L_i)\underline{R}_i$, $\underline{L}_i \subset \tilde{\underline{X}}$, $i \in \underline{k}$, and using the same arguments as the proofs of Theorem 4.1.1., and Theorem 4.2.1., it can be shown that the set $\{\overline{\underline{R}}_i\}_{\underline{k}}$ forms a solution to EDP. Also, if $\underline{V} = 0$, $\bigoplus_{i \in \underline{k}} \overline{\underline{R}}_i = \hat{\underline{X}}$, which, together with (4.3.23) and Lemma 4.2.1., proves that $\hat{F} = \hat{K}\hat{C}$ can be satisfied for some $\hat{F} \in \bigcap_{i \in \underline{k}} F(\overline{\underline{R}}_i)$. Thus the theorem is proved in this case.

If $\underline{V} \neq 0$,

$$V^{-1}\sum_{i\in \underline{k}} \underline{R}_i \subset \underline{M}^T_{k+1} \tag{4.3.24}$$

for, if not, assume there exists a vector $x_{\bar{v}}$ such that

$$Vx_{\bar{v}} \in \sum_{i\in \underline{k}} \underline{R}_i, \quad x_{\bar{v}} \notin \underline{M}^T_{k+1} \tag{4.3.25}$$

This implies that there exists x_i, $i \in \underline{k}$, such that

$$\sum_{i\in \underline{k}} R_i x_i + Vx_{\bar{v}} = 0 \tag{4.3.26}$$

and hence $x = (x_1 \ldots x_k x_{\bar{v}})^T$ is contained in $\underline{N}(R) = \underline{M}^T$. But $x_{\bar{v}} \notin \underline{M}^T_{k+1}$ contradicts this, proving (4.3.24). Now, by (4.3.1) and (4.3.24)

$$\underline{R}^v = V^{-1}\underline{X} \subset \underline{M}^T_{k+1} \tag{4.3.27}$$

where $\underline{R}^v$ is the entire linear vector space of dimension v. Hence, since $\underline{M}^T_{k+1} = \underline{N}^{\perp}(M_{k+1})$

$$\underline{N}^{\perp}(M_{k+1}) \supset \underline{R}^v \tag{4.3.28}$$

But, since the number of columns of M_{k+1} equals v, then $\underline{N}(M_{k+1}) \subset \underline{R}^v$, implying by (4.3.28), that

$$\underline{N}(M_{k+1}) = 0 \tag{4.3.29}$$

Therefore sufficient non-zero independent subspaces $\underline{\tilde{R}}_i \subset \underline{M}_{k+1}$ can be chosen such that, for some $\underline{\tilde{k}} \subset \underline{k}$,

$$\bigoplus_{i\in\underline{\tilde{k}}} \underline{\tilde{R}}_i = \underline{M}_{k+1} \tag{4.3.30}$$

and $\sum_{i\in\underline{k}} \dim(\underline{\tilde{R}}_i) = v.$

It will now be shown how the subspaces $\underline{\tilde{R}}_i$, $i \in \underline{\tilde{k}}$, can be chosen so that (4.3.12) is satisfied, by partitioning M_{k+1} according to the following scheme. Write, for some $s \leq k$,

$$\begin{bmatrix} V \\ M_{k+1} \end{bmatrix} = \begin{bmatrix} V_1 & \cdots & V_s \\ \tilde{M}_1 & \cdots & \tilde{M}_s \end{bmatrix} \tag{4.3.31}$$

such that for each $j \in \underline{s}$, there exists $\underline{R}_i$, $i \in \underline{k}$, for which

$$\underline{V}_j \cap \underline{R}_i = 0 \tag{4.3.32}$$

This partitioning is always possible, subject to possible rearrangement of the columns of $\begin{bmatrix} V \\ M_{k+1} \end{bmatrix}$, since for some V_j, let

$$\underline{Z}_i = \underline{V}_j \cap \underline{R}_i \qquad i \in \underline{k} \tag{4.3.33}$$

If $\underline{Z}_i \neq 0$, for all $i \in \underline{k}$, and the $\underline{Z}_i$ are independent, V_j can be further partitioned

$$V_j = \left[Z_1 \; \ldots \; Z_k \right] \tag{4.3.34}$$

Then, for $m \neq i$, $i \in \underline{k}$, $m \in \underline{k}$

$$\begin{aligned} \underline{Z}_i \cap \underline{R}_m &= \underline{V}_j \cap \underline{R}_i \cap \underline{R}_m \\ &= (\underline{V}_j \cap \underline{R}_i) \cap (\underline{V}_j \cap \underline{R}_m) \\ &= \underline{Z}_i \cap \underline{Z}_m \\ &= 0 \end{aligned} \tag{4.3.35}$$

If the $\underline{Z}_i$, $i \in \underline{k}$, are assumed to be not independent, then

$$\begin{aligned} \underline{Z}^* &= \bigcap_{i \in \underline{k}} \left(\sum_{j \neq i} \underline{Z}_j \right) \\ &\neq 0 \end{aligned} \tag{4.3.36}$$

and by (4.3.33), $\underline{Z}^* \subset \underline{R}^* \cap \underline{V}$, which is a contradiction, by construction of $\underline{R}^*$ and $\underline{V}$, defined in (4.3.5).

Now, for i, j such that (4.3.32) holds, set

$$\underline{\tilde{R}}_i = \underline{\tilde{M}}_j \qquad i \in \underline{\tilde{k}} \qquad (4.3.37)$$

and

$$\underline{\hat{R}}_i = \underline{\bar{R}}_i + \underline{\tilde{R}}_i \qquad i \in \underline{\tilde{k}} \qquad (4.3.38)$$

$$\underline{\hat{R}}_i = \underline{\bar{R}}_i \qquad i \in \underline{k} - \underline{\tilde{k}} \qquad (4.3.39)$$

Following the proof of Theorem 4.2.1., it is clear that $\{\underline{\hat{R}}_i\}_{\underline{k}}$ so defined forms a solution to EDP. It must now be proved that

$$\underline{N}(\hat{C}\hat{R}_i) = 0 \qquad i \in \underline{\tilde{k}} \qquad (4.3.40)$$

Writing, for $i \in \underline{\tilde{k}}$

$$\hat{C}\hat{R}_i = \begin{bmatrix} CR_i & 0 \\ M_i & \tilde{M}_j \end{bmatrix} \qquad (4.3.41)$$

it can be seen that $\underline{N}(\hat{C}\hat{R}_i) \neq 0$ implies the existence of a vector $x \neq 0$, such that $x \in [R_i \ 0]^{-1}\underline{N}(C)$ and $x \in \underline{N}[M_i \ \tilde{M}_j]$. However, similarly to (4.3.16), by construction, for $i \in \underline{\tilde{k}}$,

$$\begin{bmatrix} M_i^T \\ \tilde{M}_j^T \end{bmatrix} \supset [R_i \ V_j]^{-1}(\underline{R}^* + \sum_{l \neq j} \underline{V}_l) \qquad (4.3.42)$$

But, by (4.3.32),

$$[R_i \ V_j]^{-1}(\underline{R}^* + \sum_{l \neq j} \underline{V}_l) = R_i^{-1}(\underline{R}^* + \sum_{l \neq j} \underline{V}_l)$$

$$+ V_j^{-1}(\underline{R}^* + \sum_{l \neq j} \underline{V}_l)$$

$$\supset R_i^{-1}(\underline{R}^* + \sum_{l \neq j} \underline{V}_l)$$

$$= R_i^{-1}(\underline{R}^* + \underline{V}) \qquad (4.3.43)$$

Thus,

$$[R_i \ 0]^{-1}\underline{N}(C) \subset \begin{bmatrix} M_i^T \\ \tilde{M}_j^T \end{bmatrix}$$

$$= \underline{N}^{\perp}[M_i \ \tilde{M}_j] \qquad (4.3.44)$$

which implies that x = 0, or (4.3.40) holds.

Since $\underline{N}(\hat{R}_i) \subset \underline{N}(\hat{C}\hat{R}_i)$, it follows that

$$\bigoplus_{i \in \underline{k}} \hat{\underline{R}}_i = \hat{\underline{X}} \qquad (4.3.45)$$

and application of Lemma 4.2.1., completes the proof of Theorem 4.3.1.

4.4. Example of construction procedure

The following example will be used to illustrate the construction procedure described in the proof of Theorem 4.3.1.

Consider the system described by (2.1.1), (2.1.2), where

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 1 & 0 & 1 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (4.4.1)$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad (4.4.2)$$

Using the procedure of Lemma 2.5.1. and the result of Theorem 3.2.1., the set $\{\underline{R}_i^M\}_{\underline{k}}$, $\underline{k} = \{1, 2\}$, can be constructed as

$$\underline{R}_1^M = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \underline{R}_2^M = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad (4.4.3)$$

It can then be seen that

$$(\underline{R}^M)^* = \{[\,0 \quad 1 \quad -1 \quad 0 \quad -1\,]^T\} \tag{4.4.4}$$

and since (C,A) is an observable pair, the system does not fulfill condition (i) of Theorem 3.3.2. for existence of an output feedback solution to the restricted decoupling problem. It can be easily verified that condition (ii) of that theorem is also not satisfied.

However, since $\underline{R}_i^M + \underline{N}_i = \underline{X}$, i = 1, 2, for this example an output feedback solution to the extended problem does exist. Moreover, since $\underline{R}_1^M + \underline{R}_2^M = \underline{X}$, the conditions of Theorem 4.3.1. are fulfilled, and an extension of dimension given by (4.3.4), i.e. $\tilde{n} = 2$, is sufficient.

To proceed with the construction of the solution $\{\hat{\underline{R}}_i\}_{\underline{k}}$ to the problem, let

$$\underline{V} = \{[\,0 \quad 0 \quad 0 \quad 0 \quad 1\,]^T\} \tag{4.4.5}$$

Then $(\underline{R}^M)^* \oplus \underline{V} = \underline{N}(C)$. Constructing R yields

$$R = [\, R_1 \quad R_2 \quad V \,]$$

$$= \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad (4.4.6)$$

for which

$$\underline{N}(R) = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\} \qquad (4.4.7)$$

Thus

$$\overline{R} = \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$= \begin{bmatrix} R_1 & R_2 & V \\ M_1 & M_2 & M_3 \end{bmatrix} \tag{4.4.8}$$

Since $\underline{V} \neq 0$, and $\underline{R}_1 \cap \underline{V} = 0$, choose

$$\hat{\underline{R}}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}, \quad \hat{\underline{R}}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \tag{4.4.9}$$

Using the procedure contained in the proof of Lemma 2.4.1., the feedback matrices $\hat{F}_i \in F(\hat{\underline{R}}_i)$, i = 1, 2, can be determined. It can be easily verified that $\underline{N}(\hat{C}\hat{R}_i) = 0$, i = 1, 2, where

$$\hat{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.4.10}$$

and that $\hat{\underline{R}}_1 \oplus \hat{\underline{R}}_2 = \hat{\underline{X}}$. Therefore, the output feedback decoupling matrix $\hat{K}$ can be determined, and found to have the general form

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & 0 & 1 \\ k_{21} & k_{22} & k_{23} & -\frac{1}{2} & 1 \\ -1-k_{11}-k_{21} & -\frac{1}{2}-k_{12}-k_{22} & -\frac{1}{2}-\frac{1}{2}k_{13} & 1 & -2 \\ k_{11} & \frac{1}{2}+k_{12} & \frac{1}{2}+k_{23}-\frac{1}{2}k_{13} & -\frac{3}{2} & 1 \\ k_{51} & k_{52} & k_{23} & \frac{1}{2} & -1 \end{bmatrix} \qquad (4.4.11)$$

where k_{11}, k_{12}, k_{13}, k_{21}, k_{22}, k_{23}, k_{51}, and k_{52} are arbitrary. Also, it is easy to verify that

$$\underline{\hat{R}}_i + \underline{\bar{N}}_i = \underline{\hat{X}} \qquad i = 1, 2 \qquad (4.4.12)$$

$$\underline{\hat{R}}_i \subset \underline{\bar{N}}_j \qquad j \neq i, \ \ i = 1, 2 \qquad (4.4.13)$$

where

$$\bar{C}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (4.4.14)$$

$$\bar{C}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad (4.4.15)$$

4.5. Pole assignment in the extended decoupled system

The decoupled system in the extended state space can be described by

$$\dot{\hat{x}}(t) = (\hat{A} + \hat{B}\hat{F})\hat{x}(t) + \hat{B}\sum_{i\in \underline{k}}\hat{G}_i v_i(t) \qquad (4.5.1)$$

$$y_i(t) = \overline{C}_i\hat{x}(t) \qquad i \in \underline{k} \qquad (4.5.2)$$

It has been seen in the foregoing that the state space of the original system (2.1.1), (2.1.2), can be extended so that, for $\{\hat{\underline{R}}_i\}_{\underline{k}}$ a solution to EDP, $\hat{\underline{R}}^* = 0$. Sufficient dimension for such an extension is given by (MW2) $\tilde{n}$, where $\tilde{n} = \sum_{i\in\underline{k}} \dim(\underline{R}_i^M) - \dim(\sum_{i\in\underline{k}} \underline{R}_i^M)$. In this case, similar arguments to those used in Section 3.4. lead to a representation of (4.5.1), (4.5.2), in the form

$$\dot{\hat{x}}_i(t) = \hat{A}_i\hat{x}_i(t) + \hat{B}_i v_i(t) \qquad i \in \underline{k} \qquad (4.5.3)$$

$$y_i(t) = \overline{C}_i\hat{x}_i(t) \qquad i \in \underline{k} \qquad (4.5.4)$$

In the case of state feedback, therefore, the pole assignment problem has an easy solution, since $(\hat{A}_i, \hat{B}_i)$, $i \in \underline{k}$, are controllable pairs. That is, full pole assignment is possible in the extended decoupled system. In fact, a smaller extension than that mentioned above can be shown to be sufficient (MW1).

In the output feedback case, the construction procedure of the previous section will also lead to a representation of the form (4.5.3), (4.5.4), for the decoupled system.

However, feedback in each of the k subsystems is restricted to be from the output $y_i(t)$, and controllability of (4.5.3) is not sufficient in this case for complete pole assignment. Also, a further extension of the state space of dimension equal to $\dim(\underline{V})$, where $\underline{V}$ was defined in Section 4.3., over that for the state feedback case, is not in general sufficient to allow complete pole assignment. Therefore, this problem falls inside the general problem of obtaining pole assignment by output feedback, a problem to which the next chapter will be devoted.

4.6. Alternative techniques for output feedback decoupling

It was pointed out in Section 2.2. that techniques had already been determined for the implementation of state feedback using output information, that is, the Luenberger observer (L1), and the technique of dynamic compensation due to Pearson et al. (P1), (PD1),(BP1). In the following, it will be shown how these techniques can be applied in the decoupling problem.

Consider first the possibility of using a Luenberger observer to implement a state feedback decoupling control (D2). The control has the form

$$u(t) = Fx(t) + \sum_{i \in \underline{k}} G_i v_i(t) \qquad (4.6.1)$$

and the Luenberger observer is described by the equations

$$\dot{e}(t) = \bar{F}e(t) + Hx(t) + Lu(t) \tag{4.6.2}$$

$$\hat{x}(t) = Nx(t) + Me(t) \tag{4.6.3}$$

where e(t) is an r-vector, and $\hat{x}(t)$ is an n-vector estimate of x(t). The conditions relating the system and observer parameters are

$$L = TB \tag{4.6.4}$$

$$TA - \bar{F}T = H = \bar{H}C \tag{4.6.5}$$

$$I - MT = N = \bar{N}C \tag{4.6.6}$$

for some T, $\bar{H}$ and $\bar{N}$. If (C,A) is observable, then solutions to (4.6.4) - (4.6.6) exist, for $r = n - p$. Introducing $\hat{x}(t)$ for x(t) in (4.6.1) yields the composite closed loop system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + BFN & BFM \\ H + LFN & \bar{F} + LFM \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \sum_{i\in \underline{k}} \begin{bmatrix} BG_i \\ LG_i \end{bmatrix} v_i(t) \tag{4.6.7}$$

$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \qquad (4.6.8)$$

These equations will be denoted by

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}v(t) \qquad (4.6.9)$$

$$y(t) = \bar{C}\bar{x}(t) \qquad (4.6.10)$$

Forming the controllability matrix $\bar{P}$ for this system

$$\bar{P} = \left[\bar{B} \quad \bar{A}\,\bar{B} \ \ldots \ \bar{A}^{n+r-1}\bar{B}\right], \qquad (4.6.11)$$

using the relations (4.6.4) - (4.6.6), $\bar{P}$ can be shown to have the form

$$\bar{P} = \begin{bmatrix} I \\ T \end{bmatrix} \begin{bmatrix} P & P_1 \end{bmatrix} \qquad (4.6.12)$$

where $P_1 = \left[(A + BF)^n BG \ \ldots \ (A + BF)^{n+r-1}BG\right]$, and P is the controllability matrix for the state feedback decoupled system

$$P = \left[BG \quad (A + BF)BG \ \ldots \ (A + BF)^{n-1}BG\right] \qquad (4.6.13)$$

It is evident from (4.6.12) that the system (4.6.9) is not completely controllable, because of the block dyadic nature of $\overline{P}$. It can also be shown that if $\{\underline{R}_i^M\}_{\underline{k}}$ is the set of maximal controllability subspaces of (A,B), then $\{\overline{\underline{R}}_i\}_{\underline{k}}$, where

$$\overline{\underline{R}}_i = \left\{ \begin{bmatrix} R_i^M \\ TR_i^M \end{bmatrix} \right\} \qquad i \in \underline{k} \qquad (4.6.14)$$

is the set of controllability subspaces for the composite decoupled system (4.6.9), (4.6.10), i.e.

$$\{ \overline{A} \mid \underline{\tilde{B}} \cap \overline{\underline{R}}_i \} = \overline{\underline{R}}_i \qquad i \in \underline{k} \qquad (4.6.15)$$

where $B = \begin{bmatrix} B \\ L \end{bmatrix}$. Consideration of $\{\overline{\underline{R}}_i\}_{\underline{k}}$ shows that it forms a solution to the decoupling problem.

It is not required that the $\overline{\underline{R}}_i$, $i \in \underline{k}$, be independent. However, for full pole assignment in the decoupled system, additional dynamic compensation may be required as described in Section 4.5. Extra dynamics will also be necessary if $\bigcap_{i \in \underline{k}} F(\underline{R}_i^M) = \emptyset$, in which case a solution to EDP must be found, and then implemented using the observer.

The essential difference between using an observer, and the technique developed in Section 4.3., is that, in the latter case, the whole extended state space $\hat{\underline{X}}$ is decoupled into independent controllability subspaces, and the output performance of the system is entirely predicted by its transfer function matrix. This is not true for an observer, for which the uncontrollable modes may appear at the output.

The second alternative to be considered is that proposed by Howze and Pearson (HP1). Their main theorem states: "Assume (A,B) controllable, (C,A) observable, and (the system) (A,B,C) can be decoupled by state feedback. It is possible to compensate the system by means of a compensator of order $(mq + n_o)$ such that decoupling and arbitrary placement of $(n + mq + n_o)$ poles can be achieved with output feedback". Here, q is the smallest non-negative integer such that

$$\text{rank}\begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^q \end{bmatrix} = n \qquad (4.6.16)$$

that is, (q - 1) is the system observability index. The method used is based upon earlier developments of Pearson et al. concerning the use of dynamic compensators for optimal control and pole assignment (P1), (PD1), (BP1). The results in (HP1) are obtained only for the special case of RDP where rank(B) = k = p, i.e. rank(C_i) = 1, $i \in \underline{k}$, however further generalisation would not appear to present additional problems.

Obviously, $mq \geq n - p \geq v$, which implies that in general the dimension of extension using this method will be larger than that for the method of Section 4.3. Also, the compensator presented by Howze and Pearson requires that $\bigcap_{i \in \underline{k}} F(\underline{R}_i^M) \neq \emptyset$, or, by Lemma 3.2.3., $\underline{B} = \sum_{i \in \underline{k}} \underline{B} \cap \underline{R}_i^M$. This is not necessary for the method of Section 4.3. The latter method, however, does not in general result in full pole assignment being possible as previously remarked.

CHAPTER FIVE

Pole assignment by output feedback

5.1. The state feedback pole assignment problem

Consider the system described by equations (2.1.1), (2.1.2), which are repeated here for convenience

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5.1.1}$$

$$y(t) = Cx(t) \tag{5.1.2}$$

A problem which has been widely studied is that of determining a feedback matrix F such that the eigenvalues of (A + BF) correspond to a set Λ of predetermined values, the only restriction on which is that complex members of the set should occur in conjugate pairs.

The existence of some F corresponding to any set was first established by Wonham (W1) in the following theorem.

Theorem 5.1.1.

The pair (A,B) is controllable if and only if, for every choice of the set Λ , there is a matrix F such that (A + BF) has Λ for its set of eigenvalues.

Proof: The result obtained in Theorem 2.6.1., concerning the eigenvalue assignment properties in controllability subspaces provides a simple proof of Theorem 5.1.1.. For, if $\underline{R} = \underline{X}$

$$A\underline{R} \subset \underline{R} + \underline{B} \tag{5.1.3}$$

holds automatically, and

$$\{A \mid \underline{B} \cap \underline{R}\} = \underline{R} \tag{5.1.4}$$

is equivalent to complete controllability of (A,B).

A less restrictive condition than the ability to assign arbitrarily the eigenvalues of (A + BF) was introduced by Wonham (W1). This is the requirement that the eigenvalues of (A + BF) should have negative real parts, or that (A + BF) should be stable. A weaker condition than controllability can then be defined as follows.

Definition 5.1.1.

The pair (A,B) is said to be stabilizable if there exists an $m \times n$ matrix F such that (A + BF) is stable.

Let a(s) be the minimal polynomial of $\underline{X}$ with respect to A. Then a(s) can be factorized into its stable ($a^-(s)$) and unstable ($a^+(s)$) parts, that is

$$a(s) = a^+(s)\, a^-(s) \tag{5.1.5}$$

Define the subspace $\underline{X}^+(A) \subset \underline{X}$ by

$$\underline{X}^{+}(A) = \left\{ x : a^{+}(A)x = 0, \; x \in \underline{X} \right\} \tag{5.1.6}$$

The following theorem can then be stated.

Theorem 5.1.2.

The pair (A,B) is stabilizable if and only if

$$\underline{X}^{+}(A) \subset \{ A \mid \underline{B} \} \tag{5.1.7}$$

Proof: Intuitively, (5.1.7) implies that the unstable modes of A are controllable, and the result follows. The complete proof may be found in Wonham (W1).

Neither of these theorems consider any restriction on the state feedback matrix F. In the following, the concepts of pole assignment and stabilization will be considered in the context of output feedback, where a solution K to the equation

$$F = KC \tag{5.1.8}$$

must exist.

5.2. Pole assignment by output feedback: existence of a feedback matrix

In certain cases it may be possible to obtain arbitrary eigenvalues by output feedback. For example, it is possible for the system described by equations (5.1.1), (5.1.2) where

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -2 & -2 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (5.2.1)$$

as was pointed out by Pearson and Ding (PD1). As yet no necessary and sufficient existence conditions are known which a system has to satisfy for this property to hold although such a result would clearly be of interest. A possible approach to this problem would appear to be through the study of the canonical forms of equations (5.1.1), (5.1.2), (L2), (K1), (P1), (WM2). In the following, the next theorem due to Wonham and Morse (WM2) is used in considering the problem.

Theorem 5.2.1.

Let $\{A \mid \underline{B}\} = \underline{X}$, $\dim(\underline{B}) = m$. There exist controllability subspaces $\underline{R}_i$, $i \in \underline{m}$ of (A,B) such that

$$\dim(\underline{B} \cap \underline{R}_i) = 1 \qquad i \in \underline{m} \qquad (5.2.2)$$

and

$$\underline{R}_1 \oplus \underline{R}_2 \oplus \ldots \oplus \underline{R}_m = \underline{X} \qquad (5.2.3)$$

If the $\underline{R}_i$ are ordered so that $\dim(\underline{R}_i) \geq \dim(\underline{R}_{i+1})$, $i \in \underline{m-1}$, then the list of integers $\{\dim(\underline{R}_i),\ i \in \underline{m}\}$ uniquely characterizes the orbit of (A,B) under Υ where Υ is the group of transformations

$$(A,B) \longrightarrow (T(A + BF)T^{-1},\ TBG) \tag{5.2.4}$$

<u>Proof</u>: The proof of this theorem (WM2) is long and will not be included here.

It is of interest however to consider the algorithm for constructing the controllability subspaces $\underline{R}_i$, $i \in \underline{m}$, which is as follows. Write

$$\underline{X}_o = 0,\ \underline{X}_j = \underline{B} + A\underline{B} + \ldots + A^{j-1}\underline{B} \quad j \in \underline{n} \tag{5.2.5}$$

and define

$$\mu_1 = \min\{j : j \in \underline{n},\ A^j\underline{B} \subset \underline{X}_j\} \tag{5.2.6}$$

Let x_1 be any vector such that

$$x_1 \in \underline{B},\ A^j x_1 \notin \underline{X}_j, \quad j \in \underline{\mu_1 - 1} \tag{5.2.7}$$

For $x_r \in \underline{B}$, $r \in \underline{i-1}$, write

$$\underline{B}_{i-1} = \{ x_1, \ldots, x_{i-1} \} \qquad (5.2.8)$$

and define

$$\mu_i = \min \{ j : j \in \underline{n}, A^j \underline{B} \subset \underline{X}_j + A^j \underline{B}_{i-1} \} \qquad (5.2.9)$$

Choose $d_i \in \underline{B}$ and $e_i \in \underline{B}_{i-1}$ such that

$$d_i \notin \underline{B}_{i-1}, \; A^{\mu_i}(d_i - e_i) \in \underline{X}_{\mu_i} \qquad (5.2.10)$$

Set $x_i = d_i - e_i$, $\underline{B}_i = \{ B_{i-1}, x_i \}$, to complete the recursive determination of the μ_i and x_i, $i \in \underline{m}$. To construct the $\underline{R}_i$, $i \in \underline{m}$, choose $z_{ij} \in \underline{B}$, $i \in \underline{m}$, $j \in \mu_i$ so that

$$A^{\mu_i} x_i = z_{i1} + A z_{i2} + \ldots + A^{\mu_i - 1} z_{i\mu_i} \qquad (5.2.11)$$

Write

$$e_{ij} = A^{j-1} x_i - A^{j-2} z_{i\mu_i} - \ldots - z_{i(\mu_i - j+2)} \qquad (5.2.12)$$

and then cyclic $\underline{R}_i$, $i \in \underline{m}$ are defined by

$$\underline{R}_i = \{ e_{i1}, \ldots, e_{i\mu_i} \} \qquad (5.2.13)$$

Consider constructing $\{\underline{R}_i\}_{\underline{m}}$ for the example above, where A,B and C are given in (5.2.1). In this case m = 2, and

$$\underline{R}_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -12 \\ 17 \end{bmatrix} \right\}, \quad \underline{R}_2 = \left\{ \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\} \qquad (5.2.14)$$

can be constructed using the algorithm. For $\underline{N}(C) = \{(0\ 0\ 1)^T\}$, it is immediately apparent that

$$\underline{R}_i \cap \underline{N}(C) = 0 \qquad i = 1,2 \qquad (5.2.15)$$

Based on this observation, the following theorem is put forward as a necessary condition for pole assignment by output feedback, in the case when Λ can be decomposed into m subsets Λ_i of size equal to $\dim(\underline{R}_i)$, $i \in \underline{m}$, and in each Λ_i, complex elements occur in conjugate pairs, otherwise denoted as being symmetric.

Theorem 5.2.2.

Given $\{A \mid \underline{B}\} = \underline{X}$, $\dim(\underline{B}) = m$ and any set Λ capable of being decomposed as described above, there exists a matrix K such that the eigenvalues of $(A + BKC) = \Lambda = \bigcup_{i \in \underline{m}} \Lambda_i$, only if

$$A(\underline{R}_i \cap \underline{N}(C)) \subset \underline{R}_i \qquad (5.2.16)$$

<u>Proof</u>: For $F = KC$ such that spectrum $(A + BF) = \Lambda$ let $F \in \bigcap_{i \in \underline{m}} F(\underline{R}_i)$ and

$$\Lambda_i = \text{spectrum } (A + BF \mid \underline{R}_i) \qquad i \in \underline{m} \qquad (5.2.17)$$

since $\underline{R}_i$, $i \in \underline{m}$, is a controllability subspace of (A,B). Also, since F is such that the $\underline{R}_i$ $i \in \underline{m}$ are cyclic, and, from (4.2.2), $\dim(\underline{B} \cap \underline{R}_i) = 1$, F is uniquely determined by $\{\Lambda_i\}_{\underline{m}}$. But by assumption $F = KC$. Hence $KC \in \bigcap_{i \in \underline{m}} F(\underline{R}_i)$ and (5.2.16) follows since

$$(A + BKC)\underline{R}_i \subset \underline{R}_i \qquad i \in \underline{m} \qquad (5.2.18)$$

As a further example, consider the system described by (5.1.1),(5.1.2) where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (5.2.19)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad (5.2.20)$$

In this example, the $\underline{R}_i$, $i \in \underline{m}$ may be constructed as

$$\underline{R}_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \underline{R}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

$$\underline{R}_3 = \left\{ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \right\} \qquad (5.2.21)$$

It is evident that

$$A(\underline{R}_i \cap \underline{N}(C)) \not\subset \underline{R}_i \qquad i = 1,2,3 \qquad (5.2.22)$$

and, in fact, complete pole assignment cannot be achieved for this system using output feedback.

5.3. Pole assignment by output feedback: a lower bound.

Jameson (J1) and Davison (Da1)(DaC1) have considered the problem of pole assignment by output feedback, and a lower bound on the number of eigenvalues of (A + BKC) that can be arbitrarily chosen, for some K, has been established. The original theorem (Da1) considered the case when the state space $\underline{X}$ is cyclic with respect to A.

Theorem 5.3.1.

If (A,B) is a controllable pair, C has rank $p \leq n$, and the state space $\underline{X}$ is cyclic with respect to A, then a matrix K can be found such that p eigenvalues of (A + BKC) are arbitrarily close (but not necessarily equal) to p predetermined values, chosen such that complex values occur in conjugate pairs.

Proof: The proof (Da1) of this theorem, which is algebraic in nature, will not be included here.

This theorem has been extended (DaC1) to the case where $\underline{X}$ is not cyclic, by using the result of the following theorem, which was first established by Brash and Pearson (BP1).

Theorem 5.3.2.

Given a system described by (4.1.1), (4.1.2), where (A,B) is a controllable pair, there exists a matrix K such that the state space $\underline{X}$ is cyclic with respect to (A + BKC).

Proof: The proof of this theorem (BP1) is of a fairly complicated algebraic nature and will not be included here.

The following is put forward as a possible simple method of constructing the matrix K of Theorem 5.3.2. Consider the decomposition of the state space $\underline{X}$ studied by Kalman (K1). Write $B = (b_1\ b_2 \ldots b_m)$, where b_i is the ith. column of B, $i \in \underline{m}$. Construct the following m columns of n vectors:

$$\begin{array}{llll} b_1 & b_2 & \cdots & b_m \\ Ab_1 & Ab_2 & \cdots & Ab_m \\ A^2b_1 & A^2b_2 & \cdots & A^2b_m \\ \text{etc.} & & & \end{array} \tag{5.3.1}$$

such that a vector in a particular row and column is included only if it is linearly independent of the other vectors in that row, and all the proceding rows, and every vector above it in the column is included. Now write the subspace $\underline{U}_i \subset \underline{X}$ as the span of the vectors in the ith. column of the vector array (5.3.1), $i \in \underline{m}$. It can be shown that

$$\bigoplus_{i \in \underline{m}} \underline{U}_i = \underline{X} \tag{5.3.2}$$

$$\underline{U}_i \cap \underline{B} = \underline{b}_i \qquad i \in \underline{m} \tag{5.3.3}$$

if (A,B) is controllable, and rank (B) = m. Let $p_i = \dim(\underline{U}_i)$. For some ordering of the $\underline{U}_i$ set

$$r_1 = b_1,\ r_{1+i} = Ar_i \qquad i = 1, \ldots, p_1-1$$

$$\begin{aligned} r_{p_1+i+1} &= Ar_{p_1+i} \text{ if } r_{p_1+i} \in \underline{N}(C) \\ &= Ar_{p_1+i} + b_2 \text{ if } r_{p_1+i} \notin \underline{N}(C) \\ & \qquad\qquad i = p_1, \ldots, p_2-1 \end{aligned}$$

$$r_{p_2+i+1} = Ar_{p_2+i} \text{ if } r_{p_2+i} \in \underline{N}(C)$$
$$= Ar_{p_2+i} + b_3 \text{ if } r_{p_2+i} \notin \underline{N}(C)$$
$$i = p_2, \ldots, p_3-1$$

and so on, until $\sum_{i \in \underline{m}} p_i = n$ vectors r_i are obtained such that, for $\tilde{b}_i \in \underline{B}$

$$r_{i+1} = Ar_i + \tilde{b}_i \qquad i \in \underline{n-1} \qquad (5.3.4)$$

Choose K such that

$$BKCr_i = \tilde{b}_i \qquad (5.3.5)$$

where $\tilde{b}_i = 0$ for all $r_i \in \underline{N}(C)$ by construction. Hence K will always exist. Substituting (5.3.5) in (5.3.4) yields

$$r_{i+1} = (A + BKC)r_i \qquad i \in \underline{n-1} \qquad (5.3.6)$$

showing that $\underline{X}$ is cyclic with respect to (A + BKC), or

$$\{A + BKC \mid \underline{r}_1\} = \underline{X} \qquad (5.3.7)$$

Whether this construction of K works depends heavily on the choice of the order in which the $\underline{U}_i$ are taken. In particular, it will fail if for some choice of order, any of r_{p_i}, $i \in \underline{m-1}$, are contained in $\underline{N}(C)$. It has not been possible to find any criterion for the choice of the order of the $\underline{U}_i$ which will guarantee that the construction will work. If this were possible it would provide a simple proof for Theorem 5.3.2. The example of Section 5.2, where A,B and C are given by (5.2.19), (5.2.20) is used here to demonstrate the construction. The $\underline{U}_i$ can be simply determined as

$$\underline{U}_i = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \underline{U}_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \underline{U}_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \tag{5.3.8}$$

Choose $r_1 = b_3 \in \underline{N}(C)$. Hence $r_2 = Ar_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \notin \underline{N}(C)$

Choosing therefore $r_3 = Ar_2 + b_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \underline{N}(C)$, then

r_4 can be chosen such that $r_4 = Ar_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin \underline{N}(C)$, and

$r_5 = Ar_4 + b_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. The matrix K is then determined

as satisfying

$$BKCr_2 = b_1, \; BKCr_4 = b_2 \tag{5.3.9}$$

Such a K is given by

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.3.10}$$

It is simple to check that (5.3.7) holds. It is worth noting that the procedure will also work if $r_1 = b_1$, and $r_3 = Ar_2 + b_3$, but not if $r_1 = b_2$ is chosen.

The approach to the output feedback pole assignment problem described in this section gives little indication as to which (n - p) eigenvalues cannot be altered by output feedback. If any of these eigenvalues are in the right half complex plane, then this approach cannot yield a stable closed loop system. The following section goes some way towards answering this question.

5.4. Stabilization by output feedback.

In many cases where it is not possible to achieve arbitrary pole placement by output feedback, it may be possible to arbitrarily alter those eigenvalues which are unstable and hence stabilize the closed loop system.

Consider the system described by equations (5.1.1), (5.1.2) which is assumed to be controllable and observable, and define the subspace $\underline{V} \subset \underline{X}$, as the smallest A-invariant subspace containing $\underline{N}(C)$, that is, $\underline{V}$ is the smallest subspace such that

$$A\underline{V} \subset \underline{V}, \quad \underline{N}(C) \subset \underline{V} \qquad (5.4.1)$$

From this, it can be seen that

$$A^T\underline{V}^{\perp} \subset \underline{V}^{\perp}, \quad \underline{V}^{\perp} \subset \underline{C}^T \qquad (5.4.2)$$

and $\underline{V}^{\perp}$ is maximal with these properties. Recalling the construction of maximal invariant subspaces in Chapter Two, it is easy to see that $\underline{V}^{\perp}$ is given by $\underline{V}^{\perp} = \underline{V}^n$, where

$$\underline{V}^o = \underline{C}^T, \ \underline{V}^{j+1} = \underline{C}^T \cap (A^T)^{-1}\underline{V}^j \qquad (5.4.3)$$

Taking orthogonal complements it is easy to see that

$$\underline{V} = \underline{N}(C) + A\underline{N}(C) + \ldots + A^{n-1}\underline{N}(C) \qquad (5.4.4)$$

The following theorem was recently presented by Li (Li1).

Theorem 5.4.1.

There exists a matrix K such that the eigenvalues of $(A + BKC)$ are stable if the eigenvalues of $A \mid \underline{V}$ (A restricted to $\underline{V}$) are stable.

Proof: The proof presented by Li (Li1) is complex, and will not be included here. The result however can be shown to be a special case of a result obtained by Wonham (W2) in relation to the multivariable tracking problem, and will be proved as a direct result of the following theorem (D4).

Define $a_F(s)$ as the minimal polynomial of $\underline{X}$ with respect to $(A + BF)$, and $a_F^+(s)$, $a_F^-(s)$ as its unstable and stable factors respectively. For $F = 0$, define $a(s) = a^+(s)a^-(s)$ similarly. Denote

$$\underline{X}^+(A + BF) = \underline{N}(a_F^+(A + BF)) \qquad (5.4.5)$$

$$\underline{X}^+(A) = \underline{N}(a^+(A)) \tag{5.4.6}$$

$\underline{V}$ is defined as above.

Theorem 5.4.2.

There exists F with

$$\underline{V} \subset \underline{N}(F) \tag{5.4.7}$$

such that (A + BF) has stable eigenvalues, if and only if

$$\underline{X}^+(A) \cap \underline{V} = 0 \tag{5.4.8}$$

Proof: F can always be chosen so that

$$\underline{V} \subset \underline{N}(F) \tag{5.4.9}$$

Let x be any vector such that $x \in \underline{X}^+(A + BF) \cap \underline{V}$. Then

$$a_F^+(A + BF)x = 0 \tag{5.4.10}$$

and, since $x \in \underline{V}$, and $A\underline{V} \subset \underline{V}$,

$$a_F^+(A + BF)x = a_F^+(A)x \tag{5.4.11}$$

Let b(s) be the minimal polynomial of x with respect to A. Then (5.4.10) and (5.4.11) imply that $b(s) \mid a_F^+(s)$

($b(s)$ divides $a_F^+(s)$). Hence $b(s)$ must have only unstable roots. By definition of $a(s)$ as the minimal polynomial of $\underline{X}$ with respect to A, it follows that $b(s) \mid a(s)$, and hence $b(s) \mid a^+(s)$. Therefore

$$a^+(A)x = 0 \tag{5.4.12}$$

or $x \in \underline{X}^+(A) \cap \underline{V}$. Hence

$$\underline{X}^+(A + BF) \cap \underline{V} \subset \underline{X}^+(A) \cap \underline{V} \tag{5.4.13}$$

By a similar argument, the reverse inclusion holds, and therefore

$$\underline{X}^+(A + BF) \cap \underline{V} = \underline{X}^+(A) \cap \underline{V} \tag{5.4.14}$$

For sufficiency therefore, assume (5.4.8) holds. Then, from (5.4.14)

$$\underline{X}^+(A + BF) \cap \underline{V} = 0 \tag{5.4.15}$$

and since $\underline{X} = \underline{X}^+(A + BF) \oplus \underline{X}^-(A + BF)$, it follows that

$$\underline{V} \subset \underline{X}^-(A + BF) \tag{5.4.16}$$

Since the system is assumed to be controllable, F can therefore be chosen to satisfy (5.4.7) such that (A + BF) has stable eigenvalues.

Necessity is proved by noting that (A + BF) having all stable eigenvalues implies that

$$\underline{X}^{+}(A + BF) = 0 \qquad (5.4.17)$$

and hence (5.4.8) follows from (5.4.14).

The result of Theorem 5.4.1 can now be obtained directly from Theorem 5.4.2 by noting that

$$\underline{N}(C) \subset \underline{V} \subset \underline{N}(F) \qquad (5.4.18)$$

implies that a solution K exists to F = KC.

These results obviously hold also for the dual system described by the equations

$$\dot{z} = A^{T}z + C^{T}w \qquad (5.4.19)$$

$$v = B^{T}z \qquad (5.4.20)$$

where, in place of $\underline{V}$, the subspace of the dual state space $\underline{Z}$ considered is $\underline{W}$, where $\underline{W}$ is the smallest A^{T}-invariant subspace of $\underline{Z}$ containing $\underline{N}(B^{T})$. Since the eigenvalues of

$(A + BKC)$ and $(A + BKC)^T$ are the same, the following theorem can be stated (Li1).

Theorem 5.4.3.

The controllable and observable system described by (5.1.1), (5.1.2) is stabilizable by output feedback if $\{\text{eigenvalues of } A \,|\, \underline{V}\} \cap \{\text{eigenvalues of } A^T \,|\, \underline{W}\}$ contains only stable eigenvalues.

Proof: This follows simply from Theorem 5.4.1 and its dual.

5.5. Pole assignment by output feedback: extension of the state space

Two ways exist of achieving complete pole assignment by output feedback using an extension of the state space, these being the dynamic compensator of Brasch and Pearson (BP1) and an observer (L1), (W3).

Considering first the dynamic compensator, this involves the determination of a sufficient dimension for an extension of the form described in Section 2.3. Brasch and Pearson (BP1) have shown that for a controllable and observable system, as described by (5.1.1), (5.1.2), a sufficient dimension for such an extension is equal to $\min(p_c, p_o)$, where p_c and p_o are the smallest integers such that

$$\text{rank } (B \; AB \; \dots \; A^{p_c}B) = n \tag{5.5.1}$$

and

$$\text{rank } (C^T, A^T C^T, \ldots, (A^T)^{p_o} C^T) = n \qquad (5.5.2)$$

Their approach is a complex algebraic one, which does not readily lend itself to the determination of the minimal order for such an extension, an unsolved problem. However, the geometric approach also has not been fruitful in respect of this problem in the absence of any sufficient condition for pole assignment by output feedback. A possible approach to this problem may be by consideration of $\{\underline{R}_i\}_{\underline{m}}$ constructed in Section 5.2.

The pole assignment properties using observers (L1) are well known and will not be pursued here. It is of interest however, to consider the geometric approach to dynamic observers which is described in the following section.

5.6. Geometric theory of observers

This theory (W3),(WM2) is based upon the geometric concept of a cover. Let $\underline{Z}$ be any subspace of $\underline{X}$ such that $\underline{Z} \cap \underline{B} = 0$.

Definition 5.6.1.

A subspace $\underline{V} \subset \underline{X}$ is a cover for $\underline{Z}$, relative to the pair (A,B) if

$$\underline{AV} \subset \underline{V} + \underline{B}, \quad \underline{V} + \underline{B} \supset \underline{Z} \qquad (5.6.1)$$

The cover index of $\underline{Z}$ is the smallest integer v with the property that for every symmetric set Δ of v numbers, there exists a cover $\underline{V}$ of $\underline{Z}$ with $\dim(\underline{V}) = v$, and an F such that

$$(A + BF)\underline{V} \subset \underline{V}, \quad \text{spectrum}(A + BF \,|\, \underline{V}) = \Delta \qquad (5.6.2)$$

Consider now the $\{\underline{R}_i\}_{\underline{m}}$, the construction and properties of which were described in Section 5.2. Define

$$\gamma = \max\{i : \underline{Z} \subset \underline{B} + \underline{R}_i + \underline{R}_{i+1} + \dots + \underline{R}_m\} \qquad (5.6.3)$$

In the general case for $\underline{Z} \subset \underline{X}$, the determination of v and corresponding $\underline{V}$ for various Δ is an unsolved problem. For the case when $\dim(\underline{Z}) = 1$, however, it can be shown (WM2), assuming the pair (A,B) is controllable, that $v = \dim(\underline{R}_\gamma) - 1$.

An observer can now be demonstrated as the dual structure of a cover. Assume that the pair (C,A) is observable, or dually, (A^T, C^T) is controllable. Let $\underline{X}'$ denote the dual state space.

<u>Definition 5.6.2.</u>

A subspace $\underline{V}' \subset \underline{X}'$ is an observer for $\underline{Z}' \subset \underline{X}'$, relative to (C,A), if

$$A^T \underline{V}' \subset \underline{V}' + \underline{C}^T, \quad \underline{V}' + \underline{C}^T \supset \underline{Z}' \tag{5.6.4}$$

The observer index of $\underline{Z}'$ is the smallest integer w having the property that for every symetric set Γ of w numbers, there exists an observer $\underline{V}'$ for $\underline{Z}'$ with $\dim(\underline{V}') = w$, and an F such that

$$(A^T + C^T F^T)\underline{V}' \subset \underline{V}' , \quad \text{spectrum}(A^T + C^T F^T \mid \underline{V}') = \Gamma \tag{5.6.5}$$

The relationship of this definition of the geometric concept of an observer to that described by Luenberger (L1), is immediately made obvious by considering (5.6.4) in matrix terms. In particular, let $\dim(\underline{Z}') = 1$, and let z^T be a basis vector for $\underline{Z}'$. Similarly, let V^T be a basis matrix for $\underline{V}'$. Then (5.6.5) implies that a $w \times w$ matrix T exists such that

$$(A^T + C^T F^T)V^T = V^T T^T \tag{5.6.6}$$

or

$$V(A + FC) = TV \tag{5.6.7}$$

Also, (5.6.4) implies that

$$\underline{N}\begin{bmatrix} C \\ V \end{bmatrix} \subset \underline{N}(z^T) \qquad (5.6.8)$$

or

$$z^T = MC + NV \qquad (5.6.9)$$

for some M and N, and spectrum(T) = Γ, by (5.6.5). The relationship to the form of a Luenberger observer described by

$$\dot{e}(t) = Te(t) - VFy(t) + VBu(t) \qquad (5.6.10)$$

$$\hat{x}(t) = Ne(t) + My(t) \qquad (5.6.11)$$

which estimates the single linear functional

$$u(t) = z^T x(t) \qquad (5.6.12)$$

by

$$u(t) = z^T \hat{x}(t) = My(t) + Ne(t) \qquad (5.6.13)$$

is now clear. The error $(e(t) - Vx(t))$ is then governed by the dynamics

$$(\dot{e}(t) - V\dot{x}(t)) = T(e(t) - Vx(t)) \tag{5.6.14}$$

and, as has been shown, the eigenvalues of T are arbitrary.

The concept of a dual observer, introduced by Brasch (Br1), and described by Luenberger (L3), is now straightforward, and leads to an interesting new approach. In a similar fashion as for the observer, from the definition of a cover, it follows that there exists a v $\times$ v matrix S such that

$$(A + BF)V = VS \tag{5.6.15}$$

from (5.6.2), or

$$AV - VS = BFV \tag{5.6.16}$$

Also, from (5.6.1),

$$\underline{N}\begin{bmatrix} V^T \\ B^T \end{bmatrix} \subset \underline{N}(Z^T) \tag{5.6.17}$$

or

$$Z = VM + BN \tag{5.6.18}$$

and spectrum(S) = Δ. The dual observer can be described as a special form of controller which permits an approximation to complete freedom as to how input is made to the system. For example, if the output y(t) were to be introduced into the system in the form

$$x(t) = Ax(t) + Ky(t) \tag{5.6.19}$$

then the eigenvalues of the system, i.e. of (A + KC), could be arbitrarily assigned by choice of K, if the system were observable.

The dual observer takes the form described by

$$\dot{e}(t) = Se(t) + Mw(t) \tag{5.6.20}$$

$$w(t) = y(t) + CVe(t) \tag{5.6.21}$$

$$u(t) = FVe(t) + Nw(t) \tag{5.6.22}$$

where AV - VS = BFV, K = VM + BN, from (5.6.16) and (5.6.18). A straightforward development then yields that

$$\dot{x}(t) + V\dot{e}(t) = A(x(t) + Ve(t)) + VMw(t) + BNw(t)$$

$$= A(x(t) + Ve(t)) + Kw(t)$$

$$= (A + KC)(x(t) + Ve(t)) \tag{5.6.23}$$

from (5.6.20)-(5.6.22). Thus the eigenvalues of (A + KC) can be chosen arbitrarily, if the system is observable, as can those of S if $\underline{V}$ is chosen such that $\dim(\underline{V}) \geq v$, where v is the cover index of $\underline{K}$.

The problem of determining the minimal order for an observer or its dual is therefore that of determining the cover index for a subspace $\underline{Z} \subset \underline{X}$, or the observer index for $\underline{Z}' \subset \underline{X}'$, which in the general case is an unsolved problem.

CHAPTER SIX

Conclusions and areas of further research

6.1. Conclusions

The intention of this thesis has been to extend the theory relating to linear multivariable systems using the geometric approach, by considering the related concepts of invariant and controllability subspaces. This extension has been obtained in the field of output feedback control, an area of obvious practical significance.

The theory relating to the main application of this geometric approach, that is, decoupling, has been extended in the non-dynamic feedback case to provide a useful necessary and sufficient condition for a solution to the decoupling problem to be implemented by output feedback. In the special case, when the number of system inputs equals the number of sets of system outputs to be decoupled, the case which is most generally treated in the literature, a necessary and sufficient condition has been established for the existence of any output feedback solution to the decoupling problem.

Consideration of the case when the control is permitted to contain dynamic elements has led to a necessary and sufficient condition for the existence of an output feedback decoupling control of this form. For economic reasons, it is of importance to minimise the order of the additional dynamics required. As a contribution towards achieving this, a constructive method has been presented for obtaining an output feedback solution to the decoupling problem which requires in general a small order of dynamic compensation relative to the order of the system dynamics. It has not been possible, however, to show if this order is minimal, or if not to discover other ways of achieving a minimal order.

Also of interest in multivariable design is the problem of pole assignment by output feedback. This is an entirely new area for application of the geometric concepts which have been considered here, and this application has proved as yet to be of limited success. The discovery of a necessary condition for pole assignment by output feedback has perhaps pointed the way to the existence of more useful results in the case of non-dynamic feedback. Certainly the geometric ideas utilised here have provided a simple proof concerning the stabilizability of multivariable systems by output feedback. In the dynamic control case, a geometric theory of pole assignment compensators, though intuitively its existence is apparent, has proved difficult to formulate. In contrast, the geometric theory of dynamic observers is well developed and has been shown to exhibit clearly the concept of a "dual observer", and open an interesting new approach to the problem of minimizing observer order.

An assessment of the value of the geometric theory presented here in relation to the continuously expanding store of information concerning the analysis and design of multivariable systems is difficult to make. That it is most suited to problems involving "hard" design constraints is emphasised by its success in dealing with the decoupling problem. However, it has been shown that some extension is possible into the field of "soft" constraints, such as pole assignment, stabilization and observer theory. Additional application has been found in the field of multivariable tracking systems.

6.2. Suggested areas of further research

As has been pointed out elsewhere, the general decoupling problem is as yet unsolved. The solution to this lies in the ability to systematically generate non-maximal invariant or controllability subspaces. This fact is equally applicable to the case of output feedback decoupling, where it would be of interest to establish a systematic procedure for determining invariant subspaces of (A,B), maximal or not, which can be generated by output feedback, that is, which are (A + BKC)-invariant for some K. A search over all such subspaces, together with Theorem 2.3.1 would then yield a complete solution to the existence of a non-dynamic output feedback control for decoupling.

In the field of dynamic decoupling control, the remaining interest lies in determining minimal order dynamics.

In the general field of linear multivariable design theory, the opportunities for further research are vast. The problem of pole assignment by output feedback alone requires intimate insight into the structure of a multivariable system. This may or may not be provided by the geometric approach, perhaps requiring more powerful tools such as multilinear algebra or affine geometry. A possible approach to the general multivariable design problem may be through a "softening" of the constraints in the decoupling problem. Basically, this would require an effective method of measuring the "distance" between two vectors, or vector subspaces, a concept which is open to development.

Finally, it would be of interest to explore the computational aspects of the geometric approach described here. Some aspects of this have been considered (MW2), and the basic vector space operations expressed in terms of matrix computations.

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