ALGEBRAIC QUANTUM FIELD THEORY

by

IVAN FRANCIS WILDE

A thesis presented for the degree of
Doctor of Philosophy of the
University of London,
and the Diploma of Imperial College.

ABSTRACT

Several topics of quantum field theory are discussed within the algebraic context. It is shown that for the charged Bose field there are two natural ways of defining the local field algebras; however, these are relatively antilocal in the sense of Segal and Goodman. We define the charge sectors and show that although they are unitarily inequivalent representations of the observable algebra, they are physically (and, in fact, strongly locally) equivalent. This is a partial justification of the use of abstract algebras.

The converse problem, that of constructing charge carrying fields given the observable algebra in the charge zero sector, is then tackled for the case of a massless boson field in two dimensional space-time. This is achieved by applying the techniques of Doplicher, Haag and Roberts, viz,the use of localised automorphisms. The specific localised automorphisms used are suggested by consideration of Skyrme's model for zero mass.

Finally, we discuss the time evolution corresponding to a bounded interaction density in an arbitrary number of space dimensions. This extends a result of Guenin. A condition on the interaction in order that the resulting time evolution be causal is given.

PREFACE

This thesis results from work carried out as a postgraduate research student in the Mathematics Department of Imperial College from October 1968 to February 1969, and in the Mathematics Department of Bedford College from that time until June 1971, under the supervision of Professor R.F.Streater.

Except as stated in the text, the work in this thesis is original and has not been submitted in this or any other university for any other degree.

I am most grateful to Professor R.F.Streater for his concern and careful guidance during the past three years, for numerous time consuming discussions, and for his constant availability for such.

In addition, I would like to thank Professor T.Gustafson for the hospitality of the Theoretical Physics Institute, in Lund, Sweden, during my stay there for the year 1970, and together with Professor R.F.Streater for making my stay there possible. I would also like to thank Docent Bo Andersson for helping to provide an academically thriving atmosphere.

I am grateful to the Science Research Council for financial support in the form of a Postgraduate Research Studentship held from October 1968 to

September 1971; this being transferred into a NATO award for the year 1970. I am also grateful to the University of Lund for financial support during my time in Lund, in the form of a tredje ammanuensis.

I.F.Wilde

TABLE OF CONTENTS

Introduction

The use of abstract algebras (as opposed to operators on a Hilbert space) was made clear by I.E.Segal, in 1947, in a paper entitled 'Postulates for General Quantum Mechanics'(1). Here, he sets out the postulates in a mathematically cogent form in terms of abstract algebras, and states on these.

In 1957, R.Haag suggested the use of the (unbounded) operator algebras generated by polynomials of Wightman fields (2). These were reformulated in terms of bounded operator algebras and underwent intensive study, notably by H.Araki, H.J.Borchers and R.Haag and B.Schroer (3-9). Meanwhile, Segal was developing his own theory (13).

The formulation as generally accepted today was put forward in 1964 by R.Haag and D.Kastler (10). The axioms set out by Haag and Kastler are sufficiently restrictive so as to allow fruitful investigation, but are more general and less restrictive than the Wightman axioms (11,12). Certainly, the former are more intuitively appealing than the latter.

We shall begin with a brief account of Segal's postulates, and the Haag-Kastler axioms, and shall show how the algebraic approach affords some explanation of superselection rules.

1. The Algebraic Approach to Quantum Field Theory.

§1.1 Quantum Phenomenology (1,13)

We shall take the observables of a system as the basic undefined quantities (in the same sense that a 'line' may be considered as the basic undefined concept in geometry), in terms of which all other physically meaningful objects are to be defined. Originally, an observable was identified with a self-adjoint operator on a Hilbert space, $\underline{\underline{H}}$, and a state was a vector (or, in a more sophisticated formulation, a 'ray', i.e. a family $\{ \lambda\psi \mid \lambda \in \mathbb{C}, |\lambda| = 1;$ fixed $\psi \in \underline{\underline{H}} \}$) in the Hilbert space. The expectation value of an operator A in the state ψ was then taken to be $(\psi, A\psi)$.†

Such a theory is rather unintuitive, and is not sufficiently general. Indeed, in order to have an energy operator for interesting systems, it seems practical to consider at least two Hilbert spaces. This is a consequence of Haag's theorem (14). A recent example of this is the ϕ_2^4 theory of J.Glimm and A.Jaffe (15).

We shall suppose that our observables are bounded. Unbounded objects, such as the energy of an infinite heat bath, are considered to be observable only in that

---

† We shall always use the (Dirac) convention in which an inner product on a complex linear space is linear in the second variable and antilinear in the first.

they are limits of bounded observables; one can measure the energy of any arbitrarily large, but finite, volume of the heat bath, for example.

If α is a real number, we can interpret αA as that bounded observable with values equal to α times those of A. A^2 is the bounded observable obtained by measuring A, and squaring the result. However, A+B and AB can only be similarly defined if A and B are simultaneously observable. We can hope to define A+B as that observable with expectation value, in any state, equal to the sum of those of A and B. This can be done if an observable is determined uniquely by its expectation values in every state. We cannot do the same thing for AB because it is not true that the expectation value of a product is equal to the product of the corresponding expectation values. (The observables will not be 'independent', in general).

It is possible to define a formal product of A and B in terms of A+B and A-B, as was done by Segal (1). However, we shall not do this, but shall accept the following postulate.

POSTULATE. The observables of a physical system are self-adjoint elements of an abstract C*-algebra, $\underline{\underline{A}}$.

Perhaps a few remarks are in order.

1. We have supposed that our observables are bounded. Thus, to each A we associate a non-negative real number $\|A\|$. A is 0 if and only if $\|A\| = 0$. This is the physical interpretation of the norm in the C*-algebra, $\underline{\underline{A}}$.

2. The C*-property, $\| A^*A\| = \|A\|^2$, or, in the case of self-adjoint A, $\|A^2\| = \|A\|^2$, is a natural requirement according to our interpretation of the norm.
3. It is technically convenient to assume that the observables are complete with respect to the norm; if not, we could complete them. It is also convenient to assume that $\underline{\underline{A}}$ has an identity.[†]
4. We have assumed that there is a product AB of any two observables A and B, but as previously noted, the physical interpretation of this is not always clear.
5. We could have taken $\underline{\underline{A}}$ to be a real, rather than a complex, algebra. This was done by Segal (1), but there seems to be no advantage in this. We can always complexify a real algebra.

Having accepted this postulate, we can now apply the beautiful theory of Gelfand and others, on commutative Banach algebras, to recover such concepts as 'exact value of an observable' and 'probability distribution of an observable in a given state'. Let us define the concept of a state.

A state of a system is an assignment of an expected value to each observable, i.e. is an 'expectation' functional on $\underline{\underline{A}}$:

(i) $\omega(\lambda A+B) = \lambda\omega(A)+\omega(B)$, for all $\lambda \in \mathbb{C}$, $A,B \in \underline{\underline{A}}$.

[†] We shall always assume that all C*-algebras that we consider have an identity.

(ii) $\omega(A^*A) \geq 0$ for all $A \in \underline{\underline{A}}$.

(iii) $\omega(\mathbb{1}) = 1$.

In short, a state is an element of the unit sphere of the positive dual, $\underline{\underline{A}}^{*+}$, of $\underline{\underline{A}}$.

A state is called a mixture if it is a convex linear combination of two different states; i.e. if there exist $\omega_1 \neq \omega_2 \in \underline{\underline{A}}^{*+}$ such that

$$\omega = \lambda\omega_1 + (1-\lambda)\omega_2 ,$$

for some $0 < \lambda < 1$.

A state is pure if it is not a mixture.

If $\underline{\underline{A}}$ is realised as operators on a Hilbert space, $\underline{\underline{H}}$, then a vector $\psi \in \underline{\underline{H}}$, with $(\psi,\psi) = 1$, defines a state by

$$\omega(A) = (\psi, A\psi) \quad \text{for } A \in \underline{\underline{A}}.$$

Such a state is called a vector state in the particular realisation. ω is pure if and only if $\underline{\underline{A}}$ leaves no subspace of $\underline{\underline{H}}$ invariant (16). In general, there are more pure states than vector states (1) - another inadequacy of the older formulation.

The variance of an observable A in a state ω is defined to be $\omega(A^2) - \omega(A)^2$. We say that A has an exact value in the state ω if its va-iance therein vanishes; the exact value is then $\omega(A)$. The set of values $\omega(A)$ of A in all such states is the spectrum of the observable.

Now, the commutative C*-algebra generated by an observable A $(= A^*)$ is isomorphic to a subalgebra of $C(\Omega)$, the uniform algebra over a compact Hausdorff space Ω, for some Ω.(1,17) ($C(\Omega)$ is the C*-algebra of contiuous complex-valued functions on Ω , with respect to the sup. norm). Thus any $A = A^* \in \underline{\underline{A}}$ can be considered as a real-valued continuous function on a compact Hausdorff space Ω . Let a be this function. It is natural to say that the exact values of A , i.e. the spectrum of A , is $\{ a(x) \mid x \in \Omega \}$. That this agrees with the above definition is seen as follows. Let ω be a state on $\underline{\underline{A}}$. Then ω clearly defines a state on $C(\Omega)$. By the Riesz-Markov representation theorem we can write

$$\omega(A) = \int_{\Omega} a(x)\, d\mu(x)$$

for some unique probability measure μ on Ω. Suppose ω is pure. Then one can show (1) that μ has total mass in some single point of Ω ;

$$\omega(A) = a(x_0) \quad \text{for some} \quad x_0 \in \Omega.$$

Clearly, in this case, $\omega(A^2) = \omega(A)^2$, and so $a(x_0)$ is an exact value of A. Also, any $x_0 \in \Omega$ defines a state, ω, on the commutative C*-algebra generated by A. This state has the property that $\omega(V^2) = \omega(V)^2$ for all V belonging to this algebra. This implies that ω is pure as a state on this algebra (1). Now, a pure state on a subalgebra of a C*-algebra, $\underline{\underline{A}}$, is the restriction of a pure state on $\underline{\underline{A}}$ to the subalgebra. Hence, given $x_0 \in \Omega$, there is a pure state, ω', on $\underline{\underline{A}}$, such that $\omega'(A) = a(x_0)$, and so $a(x_0)$ is an exact value of A. We conclude that $\{ a(x) \mid x \in \Omega \}$ is a subset of the spectrum of A.

However, if $\omega(A^2) = \omega(A)^2$, then, as above, ω is a pure state on a subalgebra generated by A, and therefore corresponds to a measure on Ω with total mass in some single point. Thus $\{ a(x) \mid x \in \Omega \}$ is equal to spectrum of A. It is the set of values $\{ \omega(A) \mid \omega \in \underline{\underline{A}}^{*+}, \omega \text{ pure} \}$.

The expectation value of an observable in a state ω is the average of its spectral values with respect to a probability distribution uniquely determined by the state;

$$\omega(A) = \int_\Omega a(x)\, d\mu(x) \ .$$

The probability that A has values in a Borel set I in $\mathbb{R}$, in a state ω, is given by $\mu(\Delta)$ where $\omega(A) = \int_\Omega a(x) d\mu(x)$ and $\Delta = \{ x \in \Omega \mid a(x) \in I \}$.

Let us remark, with Segal, that the spectrum of an observable, A, being equal to the values of A in the pure states of $\underline{\underline{A}}$ is representation independent.

The set of pure states is separating for $\underline{\underline{A}}$ (1) - that is, $\omega(A) = 0$ for all pure states ω, implies that $A = 0$. Therefore $A+B$ is uniquely defined in terms of its expectation values in all pure states. This is consistent with our introductory definition of the sum of two observables.

Let us now turn to further requirements on our C*-algebra pertinent to quantum field theory.

§1.2 The Haag-Kastler Axioms (10)

Any particular experiment takes place in a finite region of space-time. That is to say, any experiment can be assigned to a region of Minkowski space, M, namely, the region in which it takes place. (A region is, by definition, a bounded open set). If our apparatus is located in some region in M, we can only expect to measure observables also located within the same space-time region. This is the idea behind the first axiom.

Axiom 1. To each region $\underline{\underline{O}}$ in Minkowski space, M, there corresponds a C*-algebra of observables, $\underline{\underline{A}}(\underline{\underline{O}})$.

The correspondence $\underline{\underline{O}} \to \underline{\underline{A}}(\underline{\underline{O}})$ can be said to determine the theory.

Axiom 2. (Isotony) If $\underline{\underline{O}}_1$, $\underline{\underline{O}}_2$ are regions in M, and $\underline{\underline{O}}_1$ contains $\underline{\underline{O}}_2$, then $\underline{\underline{A}}(\underline{\underline{O}}_2)$ can be identified with a subalgebra of $\underline{\underline{A}}(\underline{\underline{O}}_1)$.

The physical reason for this axiom is obvious. Axioms 1 and 2 allow us to define the inductive limit of the algebras $\underline{\underline{A}}(\underline{\underline{O}})$, indexed by regions in M ;

$$\underline{\underline{A}} = \overline{\overline{\bigcup_{\underline{\underline{O}}} \underline{\underline{A}}(\underline{\underline{O}})}}$$

(the double bar denoting the norm completion).

The algebras $\underline{\underline{A}}(\underline{\underline{O}})$ are called local algebras (hence the label 'Local Quantum Field Theory') and $\underline{\underline{A}}$ is called the quasilocal algebra. The term quasilocal is used to emphasise the fact that $\underline{\underline{A}}$ contains the local algebras, together with their norm limits.

The next axiom is the main one as regards field theory; it corresponds to the fact that no influence can propogate faster than the speed of light (- taken to be unity). Thus we expect two observables associated with space-like separated regions to be simultaneously measurable - this is expressed by requiring them to commute.

<u>Axiom 3</u>. If $\underline{\underline{O}}_1$ and $\underline{\underline{O}}_2$ are space-like separated regions, then $\underline{\underline{A}}(\underline{\underline{O}}_1)$ and $\underline{\underline{A}}(\underline{\underline{O}}_2)$ commute.

This makes sense since, by axiom 2, they can both be identified with subalgebras of $\underline{\underline{A}}(\underline{\underline{O}}_3)$, for any $\underline{\underline{O}}_3$ containing $\underline{\underline{O}}_1$ and $\underline{\underline{O}}_2$. Indeed, any $\underline{\underline{A}}(\underline{\underline{O}})$ is a subalgebra of $\underline{\underline{A}}$.

We would like our theory to be relativistic, so we make the next

<u>Axiom 4</u>. There is a representation α of $\mathbb{P}_+^\uparrow$, the restricted Poincaré group, in Aut$\underline{\underline{A}}$, the automorphism group of $\underline{\underline{A}}$, such that

$$\alpha(\{a,\Lambda\})\underline{\underline{A}}(\underline{\underline{O}}) = \underline{\underline{A}}(\Lambda\underline{\underline{O}}+a)$$

for any region $\underline{\underline{O}}$ in M, and $\{a,\Lambda\} \in \mathbb{P}_+^\uparrow$.

The last axiom is one of technical convenience.

<u>Axiom 5</u>. $\underline{\underline{A}}$ is primitive. (That is, $\underline{\underline{A}}$ possesses a faithful, irreducible representation as operators on a Hilbert space).

It is worth noting that, except for the trivial case, the automorphisms $\alpha(\{a,\Lambda\})$ cannot be inner (10). This reflects the global nature of Poincaré transformations and the exclusion of such from $\underline{\underline{A}}$. To see this, suppose there is a $U(a,\Lambda) \in \underline{\underline{A}}$ such that

$$U(a,\Lambda)AU(a,\Lambda)^{-1} = \alpha(\{a,\Lambda\})A$$

for all $A \in \underline{\underline{A}}$, $\{a,\Lambda\} \in \mathbb{P}_+^\uparrow$, with $U(a,\Lambda)$ unitary. Then, for given $\varepsilon>0$, there is a $V \in \underline{\underline{A}}(\underline{O})$, with $\|V\| = 1$, for some region $\underline{O}$, such that (dropping the $\{a,\Lambda\}$)

$$\| U - V \| < \varepsilon.$$

Let $A \in \underline{\underline{A}}(\underline{O}_1)$, where $\underline{O}_1$ is space-like with respect to $\underline{O}$.

Then $\quad \| \alpha(\{a,\Lambda\})A - A \| = \| UAU^* - A \|$

$$\begin{aligned}
&\leq \| UAU^* - VAV^* \| + \| AVV^* - A \| \quad \text{using } AV = VA,\\
&\leq \|UAU^* - VAU^*\| + \|VAU^* - VAV^*\| + \|A\| \|VV^* - \mathbb{1}\|\\
&\leq 2 \|U - V\| \|A\| + \|A\| 2\varepsilon\\
&\leq 4\varepsilon \|A\| .
\end{aligned}$$

This implies that $\alpha(\{a,\Lambda\})A = A$ which is false except in the trivial case.

§1.3 Physical Equivalence.

Since any abstract C*-algebra is isomorphic to a C*-algebra of bounded operators on a Hilbert space, there appears to be no particular advantage of the former over the latter. However, we claim that the abstract algebra is more fundamental than any particular representation of it. All properties of the system should be inherent in the absract algebra. This belief is justified by consideration of Haag and Kastler's notion of physical equivalence.

Let the system be in a state ω. A given experiment will correspond to the measurement of a finite number of observables $A_1,\ldots,A_n$, with resulting experimental values $p_1,\ldots,p_n$, and with maximum error ε, say. Then

$$|\omega(A_i) - p_i| < \varepsilon \quad \text{for} \quad i = 1,\ldots,n.$$

We cannot determine ω uniquely from this data. Indeed, as far as this particular experiment is concerned, we can only conclude that the system is in some state ω' with

$$|\omega'(A_i) - p_i| < \varepsilon \quad \text{for} \quad i = 1,\ldots,n.$$

Thus

$$|\omega'(A_i) - \omega(A_i)| < 2\varepsilon \quad \text{for} \quad i = 1,\ldots,n.$$

So we see that an experiment will give us a w*-neighbourhood of the state of the system. (The w*-topology in $\underline{\underline{A}}^{*+}$ is that given by the neighbourhood base $\{N(\omega,\Sigma,\varepsilon) \mid \omega \in \underline{\underline{A}}^{*+};\ \Sigma \text{ a finite set of elts. of } \underline{\underline{A}};\ \varepsilon > 0\}$ where $N(\omega,\Sigma,\varepsilon)$ is given by

$$N(\omega,\Sigma,\varepsilon) = \{\ \omega' \in \underline{\underline{A}}^{*+} \mid |\omega'(A) - \omega(A)| < \varepsilon,\ \forall\, A \in \Sigma\ \}).$$

Let π and π' be any two representations of $\underline{\underline{A}}$. That part of $\underline{\underline{A}}^{*+}$ peculiar to a representation is the family of normal states. We consider π and π' as physically equivalent if no experiment can distinguish between them. But an experiment, as we noted above, corresponds to a w*-neighbourhood of the state of the system. This leads us to the

<u>Definition 1.3.1</u> Two representations π and π' of $\underline{\underline{A}}$ are physically equivalent if and only if any w*-neighbourhood of a state of $\underline{\underline{A}}$ which is normal in the representation π contains a state which is normal in the representation π', and vice versa.

<u>Remark</u>. We can replace "normal" by "a finite convex linear combination of vector states" by virtue of the fact that the latter are w*-dense in the set of normal states.

Now, there is a theorem, due to J.Fell (18), which says that π and π' are physically equivalent (Fell's terminology is 'weakly equivalent') if and only if they have the same kernel ; i.e. if and only if

$$\{ A \in \underline{\underline{A}} \mid \pi(A) = 0 \} = \{ A \in \underline{\underline{A}} \mid \pi'(A) = 0 \}.$$

This is the justification for our claim that it is the abstract C*-algebra $\underline{\underline{A}}$ that is basic, rather than any particular representation of it (10). All faithful representations (i.e. those with kernel = $\{0\}$) are physically equivalent.

It should be pointed out that this is rather a matter of opinion. Let us illustrate this for the case of a system of charged particles. We shall see later that the total charge can be used to label the inequivalent representations of the observable algebra, $\underline{\underline{A}}$. These are physically equivalent : any state in the charge 3 sector, for example, can be approximated by a state in the charge 8 sector by adding 5 charges to the original state in a remote region of space.

However, given a state of definite total charge, it is possible to determine this charge, i.e. the sector to which the state belongs, by making a local measurement (- albeit in a very large region). One could therefore argue that the sectors should be considered as physically distinguishable. Of course, we could now add an extra charge to the state, in a remote region, without appreciably changing the value of the above local measurement.

The point is that for any given local measurement, there are states from different sectors between which the measurement cannot distinguish. On the other hand, given any two states belonging to different sectors, there is a local measurement which can distinguish between these.†

We shall illustrate the ideas of Haag and Kastler in the case of a charged Bose field. Although we shall construct $\underline{\underline{A}}$ as a gauge invariant algebra of operators, the charge sectors, as remarked above, will be seen to be inequivalent, but physically equivalent, representations of $\underline{\underline{A}}$.

In further support of these ideas, we shall construct an algebra of observables, in the charge zero sector, corresponding to a two-dimensional massless boson, and from this construct charge carrying fields which behave as fermions. This is contrary to the belief that fermions,

---

† I am grateful to Professor R.F.Streater for discussion of these points.

in principle unobservable, must be basic constituents of a theory (42). Our construction is an explicit example, and a slight variation, of the general theory of S.Doplicher, R.Haag and J.Roberts (19,20). Before doing this, let us first turn to superselection rules.

§1.4 Superselection Rules.

The algebraic formalism affords some explanation of such apparently ad hoc rules. We begin with the Hilbert space approach. As previously remarked, the states are described by unit rays in the underlying Hilbert space $\underline{\underline{H}}$(12). The reason for considering these lies in the fact that ψ and $e^{i\alpha}\psi$ both define the same expectation values,

$$(\psi,A\psi) = (e^{i\alpha}\psi,Ae^{i\alpha}\psi)$$

for all $A \in \underline{\underline{B}}(\underline{\underline{H}})$, the set of all bounded operators on $\underline{\underline{H}}$. The observables are self-adjoint elements of $\underline{\underline{B}}(\underline{\underline{H}})$.

Superselection rules originated in the observation that Dirac's superposition principle does not hold unrestrictedly. For example, one cannot form a state from a sum of two states ϕ and ψ if ϕ and ψ transform under odd and even-dimensional representations of the rotation group. This is because under a rotation of 2π, physically the identity transformation, ϕ is left invariant, but ψ becomes $-\psi$. The state given by $\alpha\phi+\beta\psi$ can only be unchanged if $\alpha=0$ or $\beta=0$. The total electric charge, baryon number and lepton number are also thought to define superselection rules (21).

Let us denote by $\underline{\underline{R}}$ the subset of $\underline{\underline{B}}(\underline{\underline{H}})$ which represents the observables. A state is said to be physically realisable if and only if the projection onto it is an element of $\underline{\underline{R}}$.(12) That the superselection rules are related to $\underline{\underline{R}}'$, the commutant of $\underline{\underline{R}}$ in $\underline{\underline{B}}(\underline{\underline{H}})$, can be seen as follows. If all self-adjoint operators in $\underline{\underline{B}}(\underline{\underline{H}})$ are observable, i.e. belong to $\underline{\underline{R}}$, then in particular all projections are observable, and so all states are physically realisable. Thus there are no superselection rules - all states have a physical meaning. In this case, $\underline{\underline{R}}'$ is trivial, i.e. $\underline{\underline{R}}' = \{ \lambda \mathbb{1} | \lambda \in \mathbb{C} \}$.

On the other hand, if $\underline{\underline{R}}'$ is not trivial, then there exists a non-trivial projection in $\underline{\underline{B}}(\underline{\underline{H}})$ which is not in $\underline{\underline{R}}$, and so not all states are physically-realisable, and we have a superselection rule.

Suppose, following Wightman (21,12), that we make the hypothesis of commuting superselection rules, viz, $\underline{\underline{R}}'$ is abelian. In this case, $\underline{\underline{R}}'$ can be diagoalised, and $\underline{\underline{H}}$ is reduced by a direct sum of orthogonal subspaces; the operators defining superselection rules having definite values on these subspaces, called superselection sectors.

The observables map each superselection sector into itself. Moreover, the restriction of $\underline{\underline{R}}$ to each superselection sector is irreducible. The superpositon principle holds unrestrictedly in each sector.

We can prove that $\underline{\underline{R}}'$ is abelian if it is assumed that the set of physically realisable vectors are total. This assumption, although seemingly innocuous, is, in fact, false in the case of relativistic quantum field theory, if we insist that all observables are local, and that there are no global observables. Nevertheless, it is a nice result, and may be relevant to non-relativistic quantum theory. Let us define the notion of a coherent subset of $\underline{\underline{H}}$.

<u>Definition 1.4.1</u>. A subset K in $\underline{\underline{H}}$ is said to be coherent if and only if it cannot be decomposed as $K = K_1 \cup K_2$ where $K_1 \perp K_2$ and $K_1, K_2 \neq \emptyset$.

For example, if Φ_1 and Φ_2 are any two vectors such that $(\Phi_1, \Phi_2) \neq 0$, then $K = \{\Phi_1, \Phi_2\}$ is coherent.

If $\Phi, \Psi \in S$, some subset of vectors of $\underline{\underline{H}}$, we say that Φ is equivalent to Ψ, denoted $\Phi \sim \Psi$, if there is a coherent subset K in S such that $\Phi, \Psi \in K$. To show that $\sim$ is an equivalence relation :

(i) Clearly, $\Phi \sim \Phi$ (take $K = \{\Phi\}$)

(ii) $\Phi \sim \Psi \Longrightarrow \Psi \sim \Phi$ is obvious.

(iii) Let $\Phi, \Psi \in K_1$, $\Psi, \chi \in K_2$ with K_1 and K_2 coherent. Let $K = K_1 \cup K_2$. We shall show that K is coherent. First we note that $K_1 \not\perp K_2$ since $\Psi \in K_1 \cup K_2$. If K were not coherent, so that $K = K_1' \cup K_2'$ with $K_1' \perp K_2'$, then $K_i'' = (K_i \cap K_1') \cup (K_i \cap K_2')$ $i = 1,2$, would be a non-trivial decomposition of either K_1 or K_2, contradicting their coherence. Thus $\Phi \sim \chi$ and $\sim$ is indeed an equivalence relation.

<u>Theorem 1.4.2</u> (Oksak-Haratian (22))

Let S be the set of physically realisable vectors, $0 \notin S$. If S is total in $\underline{\underline{H}}$, then $\underline{\underline{R}}'$ is commutative.

Proof.

The assumption $O \notin S$ is merely one of convenience; O is orthogonal to all vectors, and is thus never equivalent to a non-zero vector.

We must prove that $A,B \in \underline{\underline{R}}' \Longrightarrow AB = BA$. Let S_α denote the distinct equivalence classes with respect to the equivalence $\sim$ defined above :

$$S = \bigcup_\alpha S_\alpha \ , \ S_\alpha \cap S_\beta = \emptyset \ \text{ if } \alpha \neq \beta .$$

If $\Phi,\Psi \in S$ and $(\Phi,\Psi) \neq 0$, then $\Phi \sim \Psi$. Thus, if $\Phi \in S_\alpha$, $\Psi \in S_\beta$, $\alpha \neq \beta$, then $(\Phi,\Psi) = 0$. We may therefore write

$$\underline{\underline{H}} = \overline{[S]} = \bigoplus_\alpha \underline{\underline{H}}_\alpha$$

where $\underline{\underline{H}}_\alpha = \overline{[S_\alpha]}$, $[\circ]$ denotes the linear span of a set of vectors, and the bar denotes the closure.

<u>Lemma</u> Define $\underline{\underline{R}}'_\alpha = \{ E_\Psi \mid \Psi \in S_\alpha \}'$, where E_Ψ is the projection onto Ψ. Let $A \in \underline{\underline{R}}'_\alpha$, then A maps $\underline{\underline{H}}_\alpha$ into itself, and the restriction of A to $\underline{\underline{H}}_\alpha$ is a multiple of $\mathbb{1}_{\underline{\underline{H}}_\alpha}$.

<u>Proof of lemma</u>

$[A,E_\Psi] = 0$ for all $\Psi \in S_\alpha$, i.e. $AE_\Psi\Phi = E_\Psi A\Phi$ for any $\Phi \in \underline{\underline{H}}$. Take $\Phi = \Psi \Longrightarrow A\Psi = E_\Psi A\Psi = \lambda(\Psi)\Psi$ for some $\lambda(\Psi) \in \mathbb{C}$. Similarly, $A^*\Psi = \overline{\lambda(\Psi)}\Psi$.

Suppose $\lambda(\Psi_1) \neq \lambda(\Psi_2)$, $\Psi_1,\Psi_2 \in S_\alpha$.

$$\text{Then} \quad (\Psi_1,\Psi_2) = \frac{(\overline{\lambda}(\Psi_1)\Psi_1,\Psi_2) - \lambda(\Psi_2)(\Psi_1,\Psi_2)}{\lambda(\Psi_1) - \lambda(\Psi_2)}$$

$$= \frac{(A^*\Psi_1,\Psi_2) - (\Psi_1,A\Psi_2)}{\lambda(\Psi_1) - \lambda(\Psi_2)} = 0 .$$

Now let $S'_\alpha = \{\Psi \in S_\alpha \mid \lambda(\Psi) = \lambda(\Psi_1),\ \text{fixed } \Psi_1 \in S_\alpha \}$.

Then $S_\alpha = S'_\alpha \cup S''_\alpha$, say, and $S' \perp S''$ from the above.

But $S_\alpha = \bigcup_K \{ K \mid K \text{ coherent}, \Psi_1 \in K \text{ in } S \}$, and so S_α is coherent. ($S_\alpha = X \cup Y$, $X \perp Y$, $\Psi_1 \in X \Longrightarrow K = (K \cap X) \cup (K \cap Y)$ $\forall K$ in S_α. K coherent $\Longrightarrow K \cap Y = \emptyset$ for all K in $S_\alpha \Longrightarrow Y = \emptyset$). We conclude that $S''_\alpha = \emptyset$ And so $\lambda(\circ)$ is constant on S_α.

Thus, the restriction of A to $\underline{\underline{H}}_\alpha$ is a constant multiple of $\mathbb{1}_{\underline{\underline{H}}_\alpha}$.

If $A \in \underline{\underline{R}}' \Longrightarrow A \in \bigcap_\alpha \underline{\underline{R}}'_\alpha$, hence $A \restriction \underline{\underline{H}}_\alpha = \lambda_\alpha \mathbb{1}_{\underline{\underline{H}}_\alpha}$ for all α. This proves the lemma.

By the lemma, it is obvious that $\underline{\underline{R}}'$ is commutative, and so the proof of the theorem is complete.

As remarked in (12), $\underline{\underline{R}}'$ is commutative if $\underline{\underline{R}}$ contains a maximal abelian subalgebra $\underline{\underline{R}}_0$ of observables. ($\underline{\underline{R}}_0$ is maximal abelian if and only if $\underline{\underline{R}}_0 = \underline{\underline{R}}'_0$). Then $\underline{\underline{R}}_0$ in $\underline{\underline{R}} \Longrightarrow \underline{\underline{R}}'$ is in $\underline{\underline{R}}'_0 = \underline{\underline{R}}_0$, and so $\underline{\underline{R}}'$ is commutative.

The set-up, then, is the following. We have a C*-algebra of operators representing the observables. The underlying Hilbert space splits as a direct sum of superselection sectors. Each sector corresponds to a definite value of the superselecting operators, and, on each sector, the observables act irreducibly. $\underline{\underline{R}}'$ is commutative.

We can realise such a set-up from the algebraic point of view quite easily. Let $\underline{\underline{A}}$ be the C*-algebra of quasilocal observables. Each state on $\underline{\underline{A}}$ will yield a representation of $\underline{\underline{A}}$ - this is the well-known Gelfand, Neumark, Segal construction (23,24). This representation is irreducible if and only if the state is pure (16). There are many pure states on $\underline{\underline{A}}$, and therefore many irreducible representations to consider. Not all of these will be physically interesting - we must restrict ourselves to a subclass of representations.

In practice, $\underline{\underline{A}}$ is given as an operator algebra, and so there is a natural faithful representation to consider; namely, the representation of $\underline{\underline{A}}$ by itself. If we also require that other interesting irreducible representations be physically equivalent, then they will also be faithful. Let us suppose that they are unitarily inequivalent (- equivalent ones do not provide any further states). Thus, we are concerned with a family $(\pi_\alpha, \underline{\underline{H}}_\alpha)$ of inequivalent, irreducible, faithful representations of $\underline{\underline{A}}$. We can form the direct sum of these, $(\underset{\alpha}{\oplus}\pi_\alpha, \underset{\alpha}{\oplus}\underline{\underline{H}}_\alpha)$, which is also faithful; $\underset{\alpha}{\oplus}\pi_\alpha(A)$ is uniquely determined by $\pi_\beta(A)$ for any $A \in \underline{\underline{A}}$, and β. In particular, we note that $\underset{\alpha}{\oplus}\pi_\alpha(\underline{\underline{A}})$ does not contain the projection E_β onto the subspace $\underline{\underline{H}}_\beta$ of $\underset{\alpha}{\oplus}\underline{\underline{H}}_\alpha$.

Let $\underline{\underline{R}} = \underset{\alpha}{\oplus}\pi_\alpha(\underline{\underline{A}})$, and let $Q \in \underline{\underline{R}}'$; the commutant being taken in $\underline{\underline{B}}(\underset{\alpha}{\oplus}\underline{\underline{H}}_\alpha)$. $\underline{\underline{R}}'$ is determined by its unitary elements, so we may suppose that Q is unitary. (Any element of a C*-algebra is a linear combination of four unitary elements). Suppose there is an $\underline{\underline{H}}_\alpha$ which is not left invariant by Q. Let $Q_\alpha = Q \restriction \underline{\underline{H}}_\alpha$. Then $Q_\alpha : \underline{\underline{H}}_\alpha \rightarrow Q_\alpha\underline{\underline{H}}_\alpha$ is onto and isometric, and intertwines π_α and $\underset{\beta}{\oplus}\pi_\beta \restriction Q_\alpha\underline{\underline{H}}_\alpha$;

$$Q_\alpha\pi_\alpha(\cdot) = \underset{\beta}{\oplus}\pi_\beta(\cdot)Q_\alpha \qquad \text{on } \underline{\underline{H}}_\alpha$$

$$\Longrightarrow Q_\alpha\pi_\alpha = (\underset{\beta}{\oplus}\pi_\beta \restriction Q_\alpha\underline{\underline{H}}_\alpha)Q_\alpha .$$

But π_α is irreducible, and so $Q_\alpha \underline{\underline{H}}_\alpha$ must be contained in $\underline{\underline{H}}_\beta$, some β. (otherwise π_α, being unitarily equivalent to $\oplus_{\beta'} \pi_{\beta'} \upharpoonright Q_\alpha \underline{\underline{H}}_\alpha$ would be reducible).
Hence

$$Q_\alpha \pi_\alpha(\cdot) Q_\alpha^* = \pi_\beta(\cdot) \quad \text{on} \quad Q_\alpha \underline{\underline{H}}_\alpha .$$

But the irreducibility of π_β implies that $Q_\alpha \underline{\underline{H}}_\alpha = \underline{\underline{H}}_\beta$, and so Q_α intertwines π_α and π_β , contradicting their inequivalence. We conclude that any $Q \in \underline{\underline{R}}'$ leaves each subspace $\underline{\underline{H}}_\alpha$ of $\oplus_\alpha \underline{\underline{H}}_\alpha$ invariant, and, by the irreducibility of each π_α, must be a multiple of the identity on each of these subspaces.

Thus, on quite general grounds, we have proved that $\underline{\underline{R}}'$ is commutative. The various subspaces $\underline{\underline{H}}_\alpha$ will correspond to the superselection sectors. Let us note the great difference between the C*-algebra $\underline{\underline{R}}$, and its enveloping von Neumann algebra. From the above, we see that any $Q \in \underline{\underline{R}}'$ can be written as $Q = \Sigma_\alpha \lambda_\alpha E_\alpha$, where E_α is the projection onto $\underline{\underline{H}}_\alpha$, and $\{\lambda_\alpha\}$ is a family of complex numbers, with $\sup_\alpha |\lambda_\alpha| < \infty$. (This last condition ensures that Q is a bounded operator). Thus, the enveloping algebra of $\underline{\underline{R}}$, viz, $\underline{\underline{R}}''$, the double commutant, is equal to the set of bounded operators on $\oplus_\alpha \underline{\underline{H}}_\alpha$ of the form $\oplus_\alpha A_\alpha$, where $A_\alpha \in \underline{\underline{B}}(\underline{\underline{H}}_\alpha)$ and $\sup_\alpha \|A\| < \infty$. The elements of $\underline{\underline{R}}''$ can therefore be quite independent operators on each $\underline{\underline{H}}_\alpha$, whereas $\underline{\underline{R}}$ is determined by its restriction to any one $\underline{\underline{H}}_\alpha$.

2. The Charged Scalar Bose Field.

We have spent some time discussing the general theory - now we shall construct the local observable algebras, $\underline{\underline{A}}(\underline{\underline{O}})$, for the charged field, the charge sectors, and show that these are physically equivalent, but unitarily inequivalent, irreducible representations of the quasilocal algebra $\underline{\underline{A}}$.

This has been discussed by Doplicher, Haag and Roberts (19), as part of a general theory - we feel, however, that an explicit treatment in this case is not without value.

A charged field is a field comprising two independent fields representing the "particle" and "antiparticle", respectively (25). By convention, we choose the "particle" to have charge +1, and its "antiparticle" to have charge -1. (The opposite convention is used for the electron, however).

In mathematical terms, the charged field (i.e. the system comprising a charged field) is the tensor product of two "uncharged", but distinguished, fields. It is therefore described by two neutral fields. It is convenient, for this reason, to develop our notions and notations for the neutral Bose field.

§2.1 The neutral Boson Field.

Let $\underline{\underline{H}}_R$ be a real Hilbert space, and let $\underline{\underline{H}}$ be its complexification.

Definition 2.1.1. The Fock space $\underline{\underline{F}}$ over $\underline{\underline{H}}$ is the Hilbert space completion of the symmetric tensor algebra over $\underline{\underline{H}}$ ·

$$\underline{\underline{F}} = \overline{\circledcirc}\, \underline{\underline{H}}$$

(We shall use the symbol $\circledcirc$ to denote the symmetric tensor product. The bar signifies completion).

Thus the homogeneous components $\underline{\underline{F}}^{(n)}$ of $\underline{\underline{F}}$ are given by $\underline{\underline{F}}^{(o)} = \mathbb{C}$, and for $n \geq 1$

$$\underline{\underline{F}}^{(n)} = \underline{\underline{H}} \overline{\circledcirc} \underline{\underline{F}}^{(n-1)}.$$

$\circledcirc$ is defined for decomposable vectors in $\underline{\underline{F}}^{(n)}$ by

$$\zeta \circledcirc z_n' = \sum_{\pi \varepsilon P_{n+1}} z_{\pi(1)} \otimes \ldots \otimes z_{\pi(n+1)} \Big/ (n+1)!$$

where P_{n+1} is the permutation group on n+1 symbols, $z_{n+1} = \zeta$, and $z_n' = z_1 \circledcirc \ldots \circledcirc z_n$ is a decomposable vector in $\underline{\underline{F}}^{(n)}$. The product is extended to the whole of $\underline{\underline{F}}^{(n)}$ by linearity and continuity.

Define $\underline{\underline{F}}' = \{z \varepsilon \underline{\underline{F}} \mid z = (z_0, z_1, \ldots)$, there is N s.t.

$$\forall\, n \geq N,\ z_n = 0 \}.$$

Definition 2.1.2. Given $\zeta \varepsilon \underline{\underline{H}}$, we define the creation operator, $a^*(\zeta)$, to be the closure of the operator defined on $\underline{\underline{F}}'$ by linear extension of the map given on homogeneous elements by

$$a^*(\zeta) : \underline{\underline{F}}^{(n)} \to \underline{\underline{F}}^{(n+1)}$$

$$a^*(\zeta) : z_n \to \sqrt{n+1}\ \zeta \circledcirc z_n \quad .$$

Definition 2.1.3. The annihilation operator, $a(\zeta)$, is the adjoint of $a^*(\zeta)$, and is given on homogeneous elements of $\underline{\underline{F}}'$ by

$$a(\zeta)z_0 = 0 \quad \forall\, z_0 \in \underline{\underline{F}}^{(o)},$$

$$a(\zeta) : \underline{\underline{F}}^{(n+1)} \to \underline{\underline{F}}^{(n)}$$

$$a(\zeta) : z_{n+1} \to \langle\zeta, z_{n+1}\rangle / \sqrt{n+1}$$

where $\langle\zeta,\circ\rangle$ is defined on $\underline{\underline{F}}^{(m)}$ by

$$\langle\zeta, z_1 \otimes \ldots \otimes z_m\rangle = \sum_{i=1}^{m} (\zeta, z_i)\, z_1 \otimes \ldots \otimes \hat{z}_i \otimes \ldots \otimes z_m$$

(the ^ signifies omission).

Thus, $a^*(\zeta)$ and $a(\zeta)$ are densely defined operators for all $\zeta \in \underline{\underline{H}}$. We note that $a(\zeta) : \underline{\underline{F}}^{(n+1)} \to \underline{\underline{F}}^{(n)}$ is bounded in norm by $\sqrt{n}\|\zeta\|$ and that $a^*(\zeta) : \underline{\underline{F}}^{(n)} \to \underline{\underline{F}}^{(n+1)}$ is bounded in norm by $\sqrt{n+1}\|\zeta\|$.

$a(\zeta_1)$ and $a^*(\zeta_2)$ satisfy the canonical commutation relations on $\underline{\underline{F}}'$:

$$[a(\zeta_1), a^*(\zeta_2)] = (\zeta_1, \zeta_2)\, \mathbb{1}$$

Definition 2.1.4. We define the field $\Phi(\zeta)$ and its canonically conjugate momentum $\Pi(\zeta)$ for $\zeta \in \underline{\underline{H}}$ on $\underline{\underline{F}}'$ by

$$\Phi(\zeta) = 2^{-\frac{1}{2}}(\, a^*(\zeta) + a(\zeta)\,)$$

$$\Pi(\zeta) = 2^{-\frac{1}{2}} i (\, a^*(\zeta) - a(\zeta)\,).$$

Then Φ and Π obey the Heisenberg relations on $\underline{\underline{F}}'$:

$$[\Phi(\zeta_1), \Phi(\zeta_2)] = [\Pi(\zeta_1), \Pi(\zeta_2)] = 0 ,$$

$$[\Pi(\zeta_1), \Phi(\zeta_2)] = -i(\zeta_1, \zeta_2)\mathbb{1}.$$

Using the bounds on a^* and a, one can easily show that $\underline{\underline{F}}'$ is a set of analytic vectors for the symmetric operators $\Phi(\zeta)$ and $\Pi(\zeta)$. (z is an analytic vector for an operator A if $z \in \underline{\text{Dom}A^n}$ for all n, and if the power series $\sum_{n=0}^{\infty} \|A^n z\| \kappa^n / n!$ in κ has a non-zero radius of convergence). By Nelson's theorem (26), it follows that $\Phi(\zeta)$ and $\Pi(\zeta)$ are essentially self-adjoint on $\underline{\underline{F}}'$, and the Weyl relations hold :

$$e^{i\overline{\Pi(\xi)}}e^{i\overline{\Phi(\zeta)}}e^{-i\overline{\Pi(\xi)}}e^{-i\overline{\Phi(\zeta)}} = e^{i(\xi,\zeta)}\,\mathbb{1}$$

where $\xi,\zeta \in \underline{\underline{H}}$. (The bar denotes the closure of the e.s.a. operators).

Definition 2.1.5. Let $\underline{\underline{R}}$ be the C*-algebra generated by the set of unitary operators $\{e^{i\overline{\Pi(\zeta)}}, e^{i\overline{\Phi(\zeta)}} \mid \zeta \in \underline{\underline{H}}_R$. (We recall that $\underline{\underline{H}}_R$ is a real subspace of $\underline{\underline{H}}$).

It is well-known that $\underline{\underline{R}}$ is irreducible, i.e. $\underline{\underline{R}}''$, the double commutant of $\underline{\underline{R}}$ in $\underline{\underline{B}}(\underline{\underline{F}})$, is equal to $\underline{\underline{B}}(\underline{\underline{F}})$. This follows from the fact that the state υ defined on $\underline{\underline{R}}$ by $A \to \upsilon(A) = (\Omega, A\Omega)$, where $\Omega = 1 \in \underline{\underline{F}}^{(o)}$, is pure, and Ω is cyclic. (These facts can be proved via the Stone-von Neumann uniqueness theorem for the case when $\underline{\underline{H}}$ is finite dimensional (27)).

In our applications, we shall only consider $\Phi(h)$, $\Pi(g)$ for h,g in a subset of $\underline{\underline{H}}_R$, and will therefore have a slightly smaller algebra than $\underline{\underline{R}}$. This will not spoil the irreducibility.

Lemma 2.1.6. Let $\underline{D}_1, \underline{D}_2$ be dense in $\underline{\underline{H}}_R$. Let $\underline{\underline{R}}_0$ be the C*-algebra generated by the operators $\{ e^{i\overline{\Phi(h)}}, e^{i\overline{\Pi(g)}} \mid h \in \underline{D}_1, g \in \underline{D}_2 \}$. Then $\underline{\underline{R}}_0'' = \underline{\underline{B}}(\underline{\underline{F}})$.

Proof. We need only show that $\underline{\underline{R}}_0''$ contains $\underline{\underline{R}}$, for then $\underline{\underline{B}}(\underline{\underline{F}}) \supset \underline{\underline{R}}_0'' \supset \underline{\underline{R}} \Longrightarrow \underline{\underline{B}}(\underline{\underline{F}}) \supset \underline{\underline{R}}_0'' \supset \underline{\underline{R}}'' = \underline{\underline{B}}(\underline{\underline{F}})$. This follows if we can show that $e^{i\overline{\Phi(h)}} \in \underline{\underline{R}}_0''$ and $e^{i\overline{\Pi(g)}} \in \underline{\underline{R}}_0''$ for any $h, g \in \underline{\underline{H}}_R$.

It is easy to see that $\Phi(h_n)z$ converges strongly to $\Phi(h)z$ if h_n converges strongly to h in $\underline{\underline{H}}_R$, for $z \in \underline{\underline{F}}'$. Similarly for $\Pi(g_n)$. Since $\underline{\underline{R}}_0''$ is strongly closed, the proof is complete if we can show that $e^{i\overline{\Phi(h_n)}}$ and $e^{i\overline{\Pi(g_n)}}$ converge strongly to $e^{i\overline{\Phi(h)}}$ and $e^{i\overline{\Pi(g)}}$, respectively. Thus, to complete the proof, we shall prove

Lemma 2.1.7. Let $\{A_n\}$, A be a sequence of operators on a Hilbert space, $\underline{\underline{H}}$. Let $\underline{D}$ in $\underline{\underline{H}}$ be a domain of essential self-adjointness (i.e. a core) for A and A_n, $\forall\, n = 1,2,\ldots$. Suppose, further, that $A_n \to A$ strongly on $\underline{D}$. Then $e^{i\overline{A}_n} \to e^{i\overline{A}}$ strongly in $\underline{\underline{H}}$.

Proof. We shall show that the resolvents $R_n(\lambda)$ of $\overline{A}_n$ converge strongly to $R(\lambda)$, the resolvent of $\overline{A}$. Let $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \neq 0$. Let $z \in (A-\lambda)\underline{D}$. Then

$$\begin{aligned} \|(R_n(\lambda)-R(\lambda))z\| &= \|R_n(\lambda)(A-A_n)R(\lambda)z\| \\ &\leq \kappa(\lambda)\, \|(\overline{A}-\overline{A}_n)R(\lambda)z\| \end{aligned}$$

since $\|R_n(\lambda)\| \leq \kappa(\lambda)$ for some constant κ depending on λ (independent of n).

$$= \kappa(\lambda)\ \|(\bar{A}-\bar{A}_n)z'\| \qquad \text{some } z' \in \underline{D},$$

$$= \kappa(\lambda)\ \|(A-A_n)z'\| \to 0 \quad \text{as} \quad n \to \infty.$$

Now, $(A-\lambda)\underline{D}$ is dense in $\underline{\underline{H}}$. To see this, consider first $\lambda = i$. Let $z' \in \underline{D}$. Then

$$\|(A \pm i\mathbb{1})z'\|^2 = \|Az'\|^2 + \|z'\|^2$$

(since A^* is an extension of A, and so $A^*z' = Az'$ for $z' \in \underline{D}$). Let $z \neq 0$ be such that $(z,(A-i\mathbb{1})z') = 0$ for all $z' \in \underline{D}$. It follows that $z \in \underline{\text{Dom}}(A-i\mathbb{1})^*$ and

$$0 = ((A-i\mathbb{1})^*z,z') = ((A^*+i\mathbb{1})z,z').$$

But A is e.s.a. on $\underline{D} \Longrightarrow A^{**} = A^*$. Thus $z \in \underline{\text{Dom}}(A^{**}+i\mathbb{1})$ and

$$0 = ((A^{**}+i\mathbb{1})z,z') \quad \forall\ z' \in \underline{D}.$$

$\underline{D}$ is dense, so $(A^{**}+i\mathbb{1})z = 0$,

$$\Longrightarrow \|A^{**}z\|^2 + \|z\|^2 = 0, \Rightarrow \|z\| = 0,$$

a contradiction. Hence $(A-i\mathbb{1})\underline{D}$ is dense.

Since $A-\lambda\mathbb{1} = \text{Im}\lambda\left\{\dfrac{A-\text{Re}\lambda}{\text{Im}\lambda} - i\mathbb{1}\right\}$, we conclude

that $(A-\lambda\mathbb{1})\underline{D}$ is dense, and therefore $R_n(\lambda) \to R(\lambda)$ strongly. This implies that (28)

$$e^{i\bar{A}_n} \to e^{i\bar{A}} \qquad \text{strongly.} \qquad \text{Q.E.D.}$$

The proof of lemma 2.1.6. is now complete.

Now let us consider the special case when $\underline{\underline{H}} = L^2(\mathbb{R}^3; d\Omega)$, where $d\Omega$ is the relativistic measure on the positive-energy mass-hyperboloid (here identified with $\mathbb{R}^3$); $d\Omega = d^3k/2\sqrt{\underline{k}^2+m^2}$. In this case, $\underline{\underline{F}}^{(n)}$ is the space of all symmetric complex-valued functions of n 3-variables, square-integrable with respect to the indicated measure. $a(\zeta)$ becomes, for $z_n \in \underline{\underline{F}}^{(n)}$,

$$a(\zeta) : z_n(\underline{k}_1,\ldots,\underline{k}_n) \to \sqrt{n} \int \overline{\zeta(\underline{k})} z_n(\underline{k},\underline{k}_2,\ldots,\underline{k}_n)\, d\Omega$$

and $a^*(\zeta)$ becomes

$$a^*(\zeta) : z_n(\underline{k}_1,\ldots,\underline{k}_n) \to \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \zeta(\underline{k}_i) z_n(\underline{k}_1,\ldots,\hat{\underline{k}}_i,\ldots,\underline{k}_n)$$

Definition 2.1.8. Let $f,g \in \underline{\underline{S}}(\mathbb{R}^3)$, the Schwartz space of rapidly decreasing, smooth functions, be real-valued. The neutral relativistic field at time t, and its conjugate momentum, are defined as the operators, with core $\underline{\underline{F}}'$,

$$\phi(f;t) = 2^{-\frac{1}{2}}(a^*(F) + a(F))$$

$$\pi(g;t) = 2^{-\frac{1}{2}} i\,(a^*(G) - a(G))$$

where $F(\underline{k}) = \sqrt{2}\, e^{it\mu}\, \tilde{f}(-\underline{k})$ and $G(\underline{k}) = \sqrt{2}\, e^{it\mu}\, \mu(\underline{k})\tilde{g}(-\underline{k})$,

$\mu(\underline{k}) = \sqrt{(\underline{k}^2 + m^2)}$, and $\tilde{f}(\underline{k}) = (2\pi)^{-3/2} \int e^{i\underline{k}\cdot\underline{x}} f(\underline{x})\, d^3x$,

similarly for $\tilde{g}(\underline{k})$.

Our main concern will be with the time-zero fields which we shall just write as $\phi(f)$ and $\pi(g)$, resp. These are given simply in terms of Φ and Π by

$$\phi(f) = \Phi(F) \quad \text{and} \quad \pi(g) = \Pi(G),$$

where $F(\underline{k})$ and $G(\underline{k})$ are as above, but with $t = 0$.

As before, we have the Heisenberg relations on $\underline{\underline{F}}'$:

$$[\pi(g),\phi(f)] = -i\int \overline{G(\underline{k})}\ F(\underline{k})d\Omega \ = -i \int g(\underline{x})f(\underline{x})d^3x.$$

To each real pair $(f,g) \in \underline{\underline{S}}(\mathbb{R}^3)\times\underline{\underline{S}}(\mathbb{R}^3)$, we can associate a real solution $\xi(\underline{x},t)$ of the Klein-Gordon equation $(\partial_t^2-\nabla^2+m^2)\xi = 0$; namely, the solution with Cauchy-data (f,g) :

$$\dot{\xi}(\underline{x},0) = f(\underline{x}) \ , \quad \xi(\underline{x},0) = g(\underline{x}).$$

(We shall use a dot to denote the time-derivative).

Now, $\phi(f)-\pi(g)$ is e.s.a. on $\underline{\underline{F}}'$, so we can define

$$W(f,g) \equiv e^{i(\overline{\phi(f)-\pi(g)})}.$$

We may write $W(\xi)$ instead of $W(f,g)$ in view of the correspondence $\xi \leftrightarrow (f,g)$. It is not hard to see that the $W(\xi)$ satisfy the Segal-Weyl relations

$$W(\xi_1)\ W(\xi_2) = e^{-\frac{1}{2}i\{\xi_1,\xi_2\}}\ W(\xi_1+\xi_2)$$

where $\{\xi_1,\xi_2\}$ is the Wronskian between the two solutions ξ_1 and ξ_2 :

$$\{\xi_1,\xi_2\} = \int_{t=\text{const.}} (\xi_1(\underline{x},t)\dot{\xi}_2(\underline{x},t) - \dot{\xi}_1(\underline{x},t)\xi_2(\underline{x},t))d^3x.$$

Let us define an action of $P_+^\uparrow$, the restricted Poincaré group, on $\underline{\underline{F}}$. Let $\{a,\Lambda\} \in \mathbb{P}_+^\uparrow$, and define (29) an action $U(a,\Lambda)$ on $\underline{\underline{H}} = L^2(\mathbb{R}^3,d\Omega)$ by

$$U(a,\Lambda)\ :\ h(\underline{k}) \ \rightarrow\ e^{i(a,k)}\ h(\underline{\Lambda^{-1}k})\Big|_{k^0=\mu(\underline{k})}$$

((a,k) is the Lorentz scalar product $a^0k^0-\underline{a}.\underline{k}$)

This is a strongly continuous unitary representation of $\mathbb{P}_+^\uparrow$ in $\underline{\underline{H}}$, which extends to a unitary representation

$\Gamma(U(\cdot,\cdot))$ on $\underline{F}$. $\Gamma(U(\cdot,\cdot))$ is given on decomposable vectors $h_1 \otimes \ldots \otimes h_n$ in $\underline{F}^{(n)}$ by

$$\Gamma(U)\ h_1 \otimes \ldots \otimes h_n = Uh_1 \otimes \ldots \otimes Uh_n$$

$\Gamma(U)$ leaves $\underline{F}'$ invariant, and

$$\Gamma(U) a^{(*)}(h) \Gamma(U)^{-1} = a^{(*)}(Uh)$$

where $a^{(*)}$ denotes a or a^*.

It can be shown that

$$\Gamma(U(a,\Lambda)) W(\xi) \Gamma(U(a,\Lambda))^{-1} = W(\xi_{a,\Lambda})$$

where $\xi_{a,\Lambda}(x) \equiv \xi(\Lambda^{-1}(x-a))$.

We are now in a position to define the local algebras.

Definition 2.1.9. Let $\underline{O}$ be any region in Minkowski space, M. We define $\Xi(\underline{O})$ to be the set of solutions, ξ, of the Klein-Gordon equation, with the following property : there is a flat hyperplane (3-dimensional), J, depending on ξ, such that $J \cap \underline{O} \neq \emptyset$, and the function $\xi \restriction J$ has support in $J \cap \underline{O}$.

Definition 2.1.10. We define $\underline{A}(\underline{O})$, for a region $\underline{O}$ in M, to be the C*-algebra generated by

$$\{ W(\xi) \mid \xi \in \Xi(\underline{O}) \}.$$

$\underline{A}$ is the norm closure in $\underline{B}(\underline{F})$ of $\{ \underline{A}(\underline{O}) \mid \underline{O} \text{ in } M \}$.

Clearly, the $\underline{A}(\underline{O})$ satisfy isotony, and one can show that

$$[\underline{A}(\underline{O}), \underline{A}(\underline{O}_1)] = 0$$

if $\underline{\underline{O}}$ and $\underline{\underline{O}}_1$ are space-like separated, and that

$$\Gamma(U(a,\Lambda))\ \underline{\underline{A}}(\underline{\underline{O}})\ \Gamma(U(a,\Lambda))^{-1} = \underline{\underline{A}}(\underline{\underline{O}}_{a,\Lambda})$$

by using the fact that

$$\Gamma(U(a,\Lambda))\ W(\xi)\ \Gamma(U(a,\Lambda))^{-1} = W(\xi_{a,\Lambda}).$$

Thus $\{\ \underline{\underline{A}}(\underline{\underline{O}})\ \}$ satisfies the axioms 1 - 4 of §1.2. $\underline{\underline{A}}$ is generated by all $W(\xi)$ where ξ has real Cauchy data with compact support: that is, by the operators $e^{i\overline{\phi(f)-\pi(g)}}$ with $f,g \in \underline{\underline{D}}(\mathbb{R}^3)$. Taking f or g to be zero, we see that $\underline{\underline{A}}$ contains all operators of the form $e^{i\overline{\phi(f)}}$, $e^{i\overline{\pi(g)}}$. Now, $\phi(f) = \Phi(F)$ and $\pi(g) = \Pi(G)$, where $F(\underline{k}) = \sqrt{2}\ \tilde{f}(-\underline{k})$, $G(\underline{k}) = \sqrt{2}\ \mu(\underline{k})\tilde{g}(-\underline{k})$. F and G are smooth, rapidly decreasing, and have the property $\overline{F(\underline{k})} = F(-\underline{k})$, $\overline{G(\underline{k})} = G(-\underline{k})$. $\underline{\underline{H}}$ can be written as the complexification of $\underline{\underline{H}}_R \equiv \{\ h \in \underline{\underline{H}}\ |\ \overline{h(\underline{k})} = h(-\underline{k})\ \}$. The F's and G's lie dense in $\underline{\underline{H}}_R$, and so we may apply lamma 2.1.6, to conclude that $\underline{\underline{A}}$ is irreducible. Thus axiom 5 is also satisfied.

§2.2 The Charged Boson Field

As we have said before, the charged field is built from two distinguished neutral fields.

Let $\underline{\underline{F}}^+$ and $\underline{\underline{F}}^-$ be two distinguished Fock spaces over $L^2(\mathbb{R}^3, d\Omega)$. Then the Fock space for the charged field is

$$\underline{\underline{F}} = \underline{\underline{F}}^+ \otimes \underline{\underline{F}}^-.$$

Let $a_{\pm}^{*}$ and $a_{\pm}$ be the creation and annihilation operators in $\underline{\underline{F}}^{\pm}$, respectively. We interpret $a_{+}^{*}\otimes \mathbb{1}$ as the operator creating a particle with charge +1, $a_{+}\otimes \mathbb{1}$ as that destroying a particle with charge +1 ; with analogous interpretations of $\mathbb{1}\otimes a_{-}^{*}$ and $\mathbb{1}\otimes a_{-}$.

Definition 2.2.1. The number operators $N_{\pm}$ are defined on $\underline{\underline{F}}^{\pm\,\prime}$ by

$$N_{\pm} : z_n \to n z_n \quad \forall\, z_n \in \underline{\underline{F}}^{\pm(n)}.$$

The total number operator N in $\underline{\underline{F}}$ is defined as

$$N = N_{+}\otimes \mathbb{1} + \mathbb{1}\otimes N_{-}.$$

The total charge Q in $\underline{\underline{F}}$ is defined as

$$Q = N_{+}\otimes \mathbb{1} - \mathbb{1}\otimes N_{-}.$$

Clearly, N has eigenvalues 0,1,2,.., whilst Q has eigenvalues 0,±1,±2,...

Definition 2.2.2. The charged field, smeared with a real test-function $f \in \underline{S}(\mathbb{R}^3)$ is defined as the closed operator given on $\underline{\underline{F}}' \equiv \underline{\underline{F}}^{+\prime}\otimes\underline{\underline{F}}^{-\prime}$ by

$$\phi(f) = 2^{-\frac{1}{2}}(a_{+}^{*}(F)\otimes \mathbb{1} + \mathbb{1}\otimes a_{-}(F))$$

where, as before, $F(\underline{k}) = \sqrt{2}\tilde{f}(-\underline{k})$.

Its "complex conjugate" is

$$\phi^{*}(f) = 2^{-\frac{1}{2}}(a_{+}(F)\otimes \mathbb{1} + \mathbb{1}\otimes a_{-}^{*}(F)).$$

We see that $\phi(f)$ creates a charge +1, and destroys a charge -1. We say, therefore, that $\phi(f)$ carries a charge +1. Similarly, $\phi^{*}(f)$ is said to carry charge -1.

The momenta are defined, for smooth real g, by

$$\pi(g) = 2^{-\frac{1}{2}} i (a_+^*(G) \otimes \mathbb{1} - \mathbb{1} \otimes a_-(G))$$

$$\pi^*(g) = -2^{-\frac{1}{2}} i (a_+(G) \otimes \mathbb{1} - \mathbb{1} \otimes a_-^*(G))$$

where $G(\underline{k}) = \sqrt{2}\, \mu(\underline{k}) \tilde{g}(\underline{k})$.

The Heisenberg relations hold on $\underline{\underline{F}}'$

$$[\phi(f), \pi^*(g)] = [\phi^*(f), \pi(g)] = i \int \overline{F(\underline{k})} G(\underline{k}) d\Omega$$

$$= i \int f(\underline{x}) g(\underline{x}) d^3x.$$

All other commutators vanish.

As in the neutral case, we would like to construct the local algebras from bounded functions of these fields. However, they are not symmetric on $\underline{\underline{F}}'$. We must take linear combinations. The operators $\phi(f)+\phi^*(f)$, $\pi(g)+\pi^*(g)$, $i(\phi(f)-\phi^*(f))$ and $i(\pi(g)-\pi^*(g))$ are symmetric on $\underline{\underline{F}}'$, and, moreover, $\underline{\underline{F}}'$ is a domain of analytic vectors for them. We can, therefore, define the unitary operators

$$W(f,g) = \exp i(\overline{\phi(f)+\phi^*(f)-\pi(g)-\pi^*(g)})$$

$$W_1(f,g) = \exp i(\overline{i(\phi(f)-\phi^*(f))-i(\pi(g)-\pi^*(g))})$$

– the generators being e.s.a. on $\underline{\underline{F}}'$. In fact, on $\underline{\underline{F}}'$,

$$\phi(f)+\phi^*(f)-\pi(g)-\pi^*(g) = \phi_+(f) \otimes \mathbb{1} + \mathbb{1} \otimes \phi_-(f) - \pi_+(g) \otimes \mathbb{1} - \mathbb{1} \otimes \pi_-(g)$$

and

$$i(\phi(f)-\phi^*(f))-i(\pi(g)-\pi^*(g)) = \phi_+(if) \otimes \mathbb{1} - \pi_+(ig) \otimes \mathbb{1} - \mathbb{1} \otimes \phi_-(if) + \mathbb{1} \otimes \pi_-(ig)$$

where $\phi_{\pm}(f)$ and $\pi_{\pm}(g)$ are defined as in definition 2.1.8 in $\underline{\underline{F}}^{\pm}$, respectively. (We have used the antilinearity of $a(\cdot)$, viz, $a(iF) = -ia(F)$). So if ξ is the solution of the Klein-Gordon equation with f and g as Cauchy data, and if we write $W(\xi)$ for $W(f,g)$, we have that

$$W(\xi) = W_{+}(\xi) \otimes W_{-}(\xi)$$

and

$$W_1(\xi) = W_{+}(i\xi) \otimes W_{-}(-i\xi)$$

where $W_{\pm}(\cdot)$ are the Weyl operators, as defined in the previous section, acting in $\underline{\underline{F}}^{\pm}$, respectively.

§2.3. The Local Field Algebras.

Definition 2.3.1 Let $\underline{\underline{O}}$ be a region in M. $\mathbb{F}(\underline{\underline{O}})$ is defined to be the C*-algebra generated by the $W(\xi)$ and $W_1(\xi)$ with $\xi \in \Xi(\underline{\underline{O}})$.

Let $\mathbb{F}$ be the norm closure in $\underline{\underline{B}}(\underline{\underline{F}})$ of $\{\mathbb{F}(\underline{\underline{O}}) | \underline{\underline{O}} \text{ in } M\}$. $\mathbb{F}$ is called the field algebra of the charged field ; the $\mathbb{F}(\underline{\underline{O}})$ are the local field algebras.

Let $\Gamma_{\pm}(U)$ be the unitary representations of $P_{+}^{\uparrow}$ acting in $\underline{\underline{F}}^{\pm}$ as previously defined. We define the strongly continuous unitary action $\Gamma(U(\cdot,\cdot))$ in $\underline{\underline{F}}$ by

$$\{a,\Lambda\} \to \Gamma_{+}(U(a,\Lambda)) \otimes \Gamma_{-}(U(a,\Lambda)).$$

Thus,

$$\Gamma(U(a,\Lambda))W(\xi)\Gamma(U(a,\Lambda))^{-1} = W(\xi_{a,\Lambda})$$

and

$$\Gamma(U(a,\Lambda))W_1(\xi)\Gamma(U(a,\Lambda))^{-1} = W_1(\xi_{a,\Lambda}).$$

The $\{\mathbb{F}(\underline{\underline{O}})\}$ satisfy our axioms 1 - 4.

There is another rather natural definition of the local field algebras :

Definition 2.3.2. Let $\underline{\underline{O}}$ be a region in M. Let $\hat{\mathbb{F}}(\underline{\underline{O}})$ be the C*-algebra generated by the operators $\{ W_+(\xi)\otimes 1\!\!1,\ 1\!\!1\otimes W_-(\xi) \mid \xi \in \Xi(\underline{\underline{O}}) \}$.

The $\hat{\mathbb{F}}(\underline{\underline{O}})$ define a perfectly satisfactory local relativistic theory, because the $W_\pm(\cdot)$ do. However, $\hat{\mathbb{F}}(\underline{\underline{O}})$ is antilocal with respect to $\mathbb{F}(\underline{\underline{O}})$ in the sense of Segal and Goodman (30). To see this, suppose ξ has Cauchy data $(f,0)$ with suppf compact in $\mathbb{R}^3$. Then $\hat{\mathbb{F}}(\underline{\underline{O}})$ contains the operator $e^{i\overline{\phi_+(f)}}\otimes 1\!\!1$ if $\xi \in \Xi(\underline{\underline{O}})$. But

$$\phi_+(f)\otimes 1 = \tfrac{1}{2}(\phi(f)+\phi^*(f)-i(\pi(f_1)-\pi^*(f_1)))$$

where $\tilde{f}_1(\underline{k}) = \tilde{f}(\underline{k})\ \mu(\underline{k})^{-1}$.

Thus

$$e^{i\overline{\phi_+(f)}}\otimes 1\!\!1 = e^{\frac{1}{2}i\overline{\phi(f)+\phi^*(f)}}\ e^{-\frac{1}{2}i\,\overline{(i\pi(f_1)-i\pi^*(f_1))}}$$

The first term on the right hand side belongs to $\mathbb{F}(\underline{\underline{O}})$, but the second term does not. This is because of the antilocality, in $\underline{x}$-space, of the operator $\mu(\underline{k})^{-1}$ (30). Indeed, $f(\underline{x})$ and $f_1(\underline{x})$ can only both vanish in an open set if they are both identically zero. Thus f_1 cannot be part of compact Cauchy data : f_1 cannot vanish outside suppf.

The algebras $\mathbb{F}(\underline{\underline{O}})$ are preferable to the $\hat{\mathbb{F}}(\underline{\underline{O}})$ because the former associate $\phi(f)$ to the region $\underline{\underline{O}}$ if $(f,0) \leftrightarrow \xi \in \Xi(\underline{\underline{O}})$.

Theorem 2.3.3.

$\mathbb{F}$ is irreducible.

Proof.

Using such equalities as

$$e^{i\overline{\phi_+(f)}}\otimes \mathbb{1} = e^{\frac{1}{2}i\overline{\phi(f)+\phi^*(f)}}e^{-\frac{1}{2}i(\overline{i\pi(f_1)-i\pi^*(f_1)})}$$

and lemma 2.1.6, one shows that $\mathbb{F}''$ contains all the operators

$$\{ W_+(\xi)\otimes \mathbb{1},\ \mathbb{1}\otimes W_-(\xi) \mid \xi \in \Xi = \bigcup_{\underline{\underline{O}}} \Xi(\underline{\underline{O}}) \}.$$

The result now follows from the fact that the $W_\pm$'s form an irreducible set, and the fact that

$$(\underline{\underline{A}}\otimes\underline{\underline{B}})' = \underline{\underline{A}}'\otimes\underline{\underline{B}}'$$

for any two von Neumann algebras $\underline{\underline{A}}$ and $\underline{\underline{B}}$.

§2.4. Gauge Transformations.

We have defined the algebras $\mathbb{F}(\underline{\underline{O}})$, but have not yet specified the observable algebras $\underline{\underline{A}}(\underline{\underline{O}})$. An observable does not change the charge of a state - observables are electrically neutral. This means that we should construct the observables from functions of the fields ϕ,ϕ^*,π,π^* which contain as many starred as unstarred fields. This being so, our observables should remain invariant under the simultaneous so-called gauge transformation $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$, $\pi \to e^{i\alpha}\pi$, $\pi^* \to e^{-i\alpha}\pi^*$.

Thus, we rigorously define such an action on $\underline{\underline{F}}$, and define $\underline{\underline{A}}(\underline{\underline{O}})$ as the gauge invariant part of $\underline{\underline{F}}(\underline{\underline{O}})$.

<u>Definition 2.4.1</u>. On $\underline{\underline{F}}^{\pm(1)}$, define the unitary operator $U_\pm(\alpha)$,

$$U_\pm(\alpha) : h \to e^{\pm i\alpha}h \quad \text{for} \quad 0 \leq \alpha < 2\pi.$$

$U_\pm(\cdot)$ induces a strongly continuous unitary group $\Gamma(U_\pm(\cdot))$ on $\underline{\underline{F}}$:

$$\Gamma(U_\pm(\alpha)) : h_1 \otimes \ldots \otimes h_n \to U_\pm(\alpha)h_1 \otimes \ldots \otimes U_\pm(\alpha)h_n$$

for $h_1 \otimes \ldots \otimes h_n$ a decomposable vector in $\underline{\underline{F}}^{\pm(n)}$.

The tensor product $\Gamma(\alpha) = \Gamma(U_+(\alpha)) \otimes \Gamma(U_-(\alpha))$ defines a representation of $\underline{T}$, the torus, on $\underline{\underline{F}}$. $\underline{T}$ is called the gauge group. Moreover, on $\underline{\underline{F}}'$, we find that

$$\Gamma(\alpha)\phi(f)\Gamma(\alpha)^{-1} = e^{i\alpha}\phi(f)$$

$$\Gamma(\alpha)\phi^*(f)\Gamma(\alpha)^{-1} = e^{-i\alpha}\phi^*(f)$$

(Similarly for π and π^*).

Thus, the indicated gauge transformations correspond to the spatial automorphisms implemented by $\Gamma(\cdot)$. The generator of the strongly continuous, one-parameter group $\Gamma(\cdot)$ is nothing other than $N_+ \otimes \mathbb{1} - \mathbb{1} \otimes N_- = Q$, the total charge operator.

<u>Definition 2.4.2</u>. The local observables are the gauge invariant elements of $\mathbb{F}(\underline{\underline{O}})$;

$$\underline{\underline{A}}(\underline{\underline{O}}) = \mathbb{F}(\underline{\underline{O}}) \cap \Gamma(\underline{T})'$$

where $\Gamma(\underline{T})'$ is the commutant, in $\underline{\underline{B}}(\underline{\underline{F}})$, of the set $\{ \Gamma(\alpha) \mid \alpha \in \underline{T} \}$.

$\underline{\underline{A}}$ is the norm closure, in $\underline{\underline{B}}(\underline{\underline{F}})$, of the set $\{ \underline{\underline{A}}(\underline{\underline{O}}) \mid \underline{\underline{O}} \text{ a region in } M \}$.

As sub-algebras of the $\mathbb{F}(\underline{\underline{O}})$, the $\underline{\underline{A}}(\underline{\underline{O}})$ satisfy axioms 1 - 4. We shall see later that axiom 5 also holds.

§ 2.5. The Charge Sectors.

The charge operator, Q, has eigenvalues $0, \pm 1, \pm 2, ..$; the eigenspace corresponding to the eigenvalue q is denoted $\underline{\underline{F}}_q$. Clearly,

$$\underline{\underline{F}}_q = \bar{\oplus}_n \underline{\underline{F}}^{+(n+q)} \bar{\otimes} \underline{\underline{F}}^{-(n)} \quad \text{if } q \geq 0,$$

or

$$\underline{\underline{F}}_q = \bar{\oplus}_n \underline{\underline{F}}^{+(n)} \bar{\otimes} \underline{\underline{F}}^{-(n-q)} \quad \text{if } q < 0.$$

Also

$$\underline{\underline{F}} = \oplus_{-\infty}^{\infty} \underline{\underline{F}}_q .$$

Since $\underline{\underline{A}}$ commutes with $\Gamma(\cdot)$, we see that $\underline{\underline{A}}$ leaves each $\underline{\underline{F}}_q$ invariant. This is a restatement of the fact that the elements of $\underline{\underline{A}}$ do not carry any charge. Accordingly, the restrictions of $\underline{\underline{A}}$ to the various $\underline{\underline{F}}_q$ define representations, π_q, say, of $\underline{\underline{A}}$. These represen--tations are called the charge sectors.

We expect these representations to be physically equivalent. Indeed, as in our discussion in §1.3, a state in the q-sector can be made arbitrarily close to a state in the q+1-sector by adding a particle with charge +1 in a sufficiently remote region of space. This argument can be made quite rigorous ; we shall need two lemmas.

Lemma 2.5.1. Let $A \in \underline{\underline{A}}(\underline{\underline{O}})$, and let $\underline{\underline{O}}_1$ be a space--like region with respect to $\underline{\underline{O}}$. Suppose (f,0) is the Cauchy data for some solution $\xi \in \Xi(\underline{\underline{O}}_1)$. Then for any $z,z' \in \underline{\underline{F}}'$,

$$(z, A\phi(f)z') = (\phi^*(f)z, Az')$$

i.e. $\phi(f)$ and A weakly commute on a dense set.

Proof.

By construction, $W(\lambda\xi)$, $W_1(\lambda\xi) \in \mathbb{F}(\underline{\underline{O}}_1)$, for all $\lambda \in \mathbb{R}$. Therefore they commute with A, and so, taking $z,z' \in \underline{\underline{F}}'$,

$$(W(-\lambda\xi)z, Az') = (A^*z, W(\lambda\xi)z')$$

and

$$(W_1(-\lambda\xi)z, Az') = (A^*z, W_1(\lambda\xi)z').$$

But $$W(\lambda\xi) = \exp i\lambda(\overline{\phi(f)+\phi^*(f)})$$

and $$W_1(\lambda\xi) = \exp i\lambda(\overline{i\phi(f)-i\phi^*(f)}).$$

The result now follows by taking the derivative, with respect to λ, at $\lambda = 0$, cancelling the i's, and adding the two resulting equations.

Lemma 2.5.2. Let Σ denote the linear span (i.e. finite linear combinations) of decomposable vectors in $\underline{\underline{F}}$ of the form $h = h_1 \odot .. \odot h_n \otimes h_{n+1} \odot ... \odot h_{n+m}$ for some integers n and m, where $h_i \in \underline{\underline{S}}(\mathbb{R}^3)$ for all i.
Let $f \in \underline{\underline{D}}(\mathbb{R}^3)$, be given such that f is normalised to unity in $L^2(\mathbb{R}^3, d\Omega)$. Let $f_{\underline{a}}$ be the translate of f, i.e. $f_{\underline{a}}(\underline{x}) = f(\underline{x}-\underline{a})$. Then $\phi^*(f_{\underline{a}})\phi(f_{\underline{a}})h$ converges

weakly to h as $|\underline{a}| \to \infty$, for $h \in \Sigma$.

Proof.

Let $h \in \Sigma$. Since $\phi^*(f_{\underline{a}})\phi(f_{\underline{a}})h$ is uniformly bounded in norm, we need only show that

$$(\phi^*(f_{\underline{a}})\phi(f_{\underline{a}})h,h') \to (h,h') \quad \text{as } |\underline{a}| \to \infty,$$

for h' in a dense set in $\underline{\underline{F}}$. Now, Σ is such a set, so we choose $h' \in \Sigma$. Writing ϕ and ϕ^* in terms of creation and annihilation operators, we obtain

$$\begin{aligned}(\phi^*(f_{\underline{a}})\phi(f_{\underline{a}})h,h') = \tfrac{1}{2}\Big(&(a_+^*(F)a_+(F)\otimes \mathbb{1}h,h') \\ &+ (a_+(F)\otimes a_-(F)h,h') + (a_+^*(F)\otimes a_-^*(F)h,h') \\ &+ (\mathbb{1}\otimes a_-^*(F)a_-(F)h,h') + (h,h')\int|F(\underline{k})|^2 d\Omega \Big)\end{aligned}$$

where $F(\underline{k}) = \sqrt{2}\,\tilde{f}_{\underline{a}}(-\underline{k}) = \sqrt{2}\,e^{-i\underline{k}\cdot\underline{a}}\,\tilde{f}(-\underline{k})$, and we have made use of the CCR to obtain the last two terms.

The first four terms all contain a factor of the form

$$(F,h'') = \int \sqrt{2}\,e^{i\underline{k}\cdot\underline{a}}\,\tilde{f}(\underline{k})\,h''(\underline{k})\,d\Omega$$

where h" is some function in $\underline{\underline{S}}(\mathbb{R}^3)$. But this converges to zero as $|\underline{a}| \to \infty$, by the Riemann-Lebesgue lemma. The fifth term is just equal to (h,h') because of our normalisation of f. Q.E.D.

We can now prove

Theorem 2.5.3. The representations $\{\pi_q \mid q = 0,\pm1,..\}$ of $\underline{\underline{A}}$ are physically equivalent.

Proof.

It suffices to prove that π_q and π_{q+1} are physically equivalent.

Let ω be a state on $\underline{\underline{A}}$ given by a finite linear convex combination of vector states in the representation π_q. That is, ω has the form

$$\omega(\cdot) = \sum_{i=1}^{N} \lambda_i (z_i, \pi_q(\cdot) z_i)$$

for some integer N, vectors $z_i \in \underline{\underline{F}}_q$, $\|z_i\| = 1$,

and $\lambda_i \in \mathbb{R}$, $\sum_{i=1}^{N} \lambda_i = 1$.

Let $N(\omega, A_1, \ldots, A_p, \varepsilon)$ be a w*-neighbourhood of ω ;

$$N(\omega, A_1, .., A_p, \varepsilon) = \{\omega' \varepsilon \underline{\underline{A}}^{*+} | \ |\omega'(A_\ell) - \omega(A_\ell)| < \varepsilon, \ell = 1, .., p\}.$$

Suppose, first, that $A_1, \ldots, A_p \ \varepsilon \ \underline{\underline{A}}(\underline{\underline{O}})$, for some region $\underline{\underline{O}}$. Let Σ be as in lemma 2.5.2. Then we can choose $h_i \varepsilon \ \Sigma \cap \underline{\underline{F}}_q$, $i = 1, .., N$, $\|h_i\| = 1$, such that the state

$$\omega'(\cdot) = \sum_{i=1}^{N} \lambda_i (h_i, \pi_q(\cdot) h_i)$$

belongs to $N(\omega, A_1, .., A_p, \frac{1}{2}\varepsilon)$. (This is possible because $\Sigma \cap \underline{\underline{F}}_q$ is dense in $\underline{\underline{F}}_q$, and N and p are both <u>finite</u>).

Define a positive linear functional $\rho_{\underline{a}}(\cdot)$ given by

$$\rho_{\underline{a}}(\cdot) = \sum_{i=1}^{N} \lambda_i (\phi(f_{\underline{a}}) h_i, \pi_{q+1}(\cdot) \phi(f_{\underline{a}}) h_i)$$

where suppf is compact in $\mathbb{R}^3$. $\rho_{\underline{a}}$ is not a state, since it is not normalised; $\rho_{\underline{a}}(\cdot)/\rho_{\underline{a}}(\mathbb{1})$ is a state, however.

By lemma 2.5.1, we can write $\rho_{\underline{a}}(A_\ell)$ as

$$\rho_{\underline{a}}(A_\ell) = \sum_{i=1}^{N} \lambda_i (\phi^*(f_{\underline{a}})\phi(f_{\underline{a}})h_i, A_\ell h_i)$$

$\ell = 1,..,p$, for $|\underline{a}|$ sufficiently large.

Now, by lemma 2.5.2, we see that

$$\rho_{\underline{a}}(X) \to \omega'(X) \quad \text{as } |\underline{a}| \to \infty,$$

for $X = 1\!\!1, A_1,..,A_p$; i.e. $\rho_{\underline{a}}(X)/\rho_{\underline{a}}(1\!\!1) \to \omega'(X)$.

Therefore, $\rho_{\underline{a}}(\cdot)/\rho_{\underline{a}}(1\!\!1) \in N(\omega', A_1,..,A_p, \tfrac{1}{2}\varepsilon)$ for $|\underline{a}|$ sufficiently large, in which case $\rho_{\underline{a}}(\cdot)/\rho_{\underline{a}}(1\!\!1)$ is in the neighbourhood $N(\omega, A_1,..,A_p,\varepsilon)$ of ω.

We must remove the condition $A_1,..,A_p \in \underline{\underline{A}}(\underline{\underline{O}})$. Let $A_1,..,A_p \in \underline{\underline{A}}$. We can find $A_1',..,A_p' \in \underline{\underline{A}}(\underline{\underline{O}})$, for some $\underline{\underline{O}}$, such that $\|A_\ell - A_\ell'\| < \varepsilon$, for $\ell = 1,..,p$.

Given ω, we construct $\rho_{\underline{a}}(\cdot)/\rho_{\underline{a}}(1\!\!1)$ as above, and deduce that it belongs to $N(\omega, A_1',..,A_p',\varepsilon)$. But

$$\begin{aligned} |\omega(A_\ell) - \rho_{\underline{a}}(A_\ell)/\rho_{\underline{a}}(1\!\!1)| &\le |\omega(A_\ell') - \rho_{\underline{a}}(A_\ell')/\rho_{\underline{a}}(1\!\!1)| \\ &\quad + 2\|A_\ell' - A_\ell\| \\ &< 3\varepsilon. \end{aligned}$$

Thus $\rho_{\underline{a}}(\cdot)/\rho_{\underline{a}}(1\!\!1) \in N(\omega, A_1,..,A_p, 3\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we conclude that the set of finite convex linear combinations of vector states in π_{q+1} is w*-dense in those in π_q.

Reversing the rôles of q and q+1 and replacing ϕ by ϕ^* in the above argument, we conclude that π_q is physically equivalent to π_{q+1}. QED.

We can now apply Fell's theorem to the various sectors, with the conclusion that all the π_q have the same kernel. It follows that $\oplus_q \pi_q$ has the same kernel as each π_q. But $\oplus \pi_q$ is faithful, and so the same is true of the representations π_q.

Theorem 2.5.4.

Each π_q is a faithful, irreducible representation of $\underline{\underline{A}}$. In particular, $\underline{\underline{A}}$ is primitive. If $q \neq q'$, then π_q and $\pi_{q'}$ are unitarily inequivalent.

Proof.

We have already noted that each π_q is faithful. To prove the irreducibility and inequivalence, we reproduce the proof given by Doplicher, Haag and Roberts (19).

We define the mean of an operator with respect to the unitary representation $\Gamma(\cdot)$ of the gauge group $\underline{T}$,

$$m : \underline{\underline{B}}(\underline{\underline{F}}) \to \underline{\underline{B}}(\underline{\underline{F}})$$

$$m : X \to (2\pi)^{-1}\int_0^{2\pi} \Gamma(\alpha)X\Gamma(\alpha)^{-1}d\alpha$$

The integral is a weak integral. Clearly, m is a map from $\underline{\underline{B}}(\underline{\underline{F}})$ onto $\{ \Gamma(\alpha) \mid \alpha \in \underline{T} \}'$.

Lemma 2.5.5.

(a) $\Gamma(\alpha)m(\cdot)\Gamma(\alpha)^{-1} = m(\Gamma(\alpha)(\cdot)\Gamma(\alpha)^{-1}) = m(\cdot)$.

(b) $m(\cdot)$ is weakly continuous on bounded sets.

Proof.

(a) is obvious.

(b) We shall give an explicit alternative proof to that of (19).

Let X be a weak limit point of the set $\{A\in \underline{\underline{B}}(\underline{\underline{F}}) \mid \|A\| \leq K\}$. Let $\{X_\nu\}$ be a net with $\|X_\nu\| \leq K \;\forall\; \nu$ such that X_ν converges weakly to X. We must show that $m(X_\nu) \to m(X)$ weakly.

Let $z,z' \in \underline{\underline{F}}$. Then

$$(z',(m(X_\nu)-m(X))z) = \int_0^1 (z',\Gamma(\alpha)(X_\nu-X)\Gamma(\alpha)^{-1}d\alpha$$

$$= \int_0^1 (\Gamma(\alpha)^*z',(X_\nu-X)\Gamma(\alpha)^*z)d\alpha$$

where we have identified $\underline{T}$ with $[0,1)$.

Fix $\alpha_0 \in \underline{T}$. Then, given $\varepsilon>0$, there exists $\nu(\alpha_0)$ such that $\forall\; \nu>\nu(\alpha_0)$

$$|(\Gamma(\alpha_0)^*z',(X_\nu-X)\Gamma(\alpha_0)^*z)| < \varepsilon.$$

However, the continuity of $\Gamma(\cdot)$ in α implies that this inequality, with 4ε on the r.h.s., holds for all α in some neighbourhood of α_0.

To see this, put $A_\nu = X_\nu - X$. Let α be fixed. Then, given $\varepsilon>0$, there exists $\nu(\alpha)$ s.t. $\forall\; \nu>\nu(\alpha)$

$$|(\Gamma(\alpha)^*z',A_\nu\Gamma(\alpha)^*z)| < \varepsilon.$$

Let $\nu,\nu'>\nu(\alpha)$. Then

$$|(z',\Gamma(\beta)A_\nu\Gamma(\beta)^*z) - (z',\Gamma(\alpha)A_{\nu'}\Gamma(\alpha)^*z)|$$

$$\leq \ |(z',\Gamma(\beta)A_\nu\Gamma(\beta)^*z) - (z',\Gamma(\alpha)A_\nu\Gamma(\alpha)^*z)|$$
$$+ \ |(z',\Gamma(\alpha)A_\nu\Gamma(\alpha)^*z) - (z',\Gamma(\alpha)A_{\nu'}\Gamma(\alpha)^*z)|$$

$$\leq \ |(z',\Gamma(\beta)A_\nu\Gamma(\beta)^*z) - (z',\Gamma(\beta)A_\nu\Gamma(\alpha)^*z)|$$
$$+ \ |(z',\Gamma(\beta)A_\nu\Gamma(\alpha)^*z) - (z',\Gamma(\alpha)A_\nu\Gamma(\alpha)^*z)|$$
$$+ \ |(z',\Gamma(\alpha)A_\nu\Gamma(\alpha)^*z) - (z',\Gamma(\alpha)A_{\nu'}\Gamma(\alpha)^*z)|$$

$$\leq \|z'\|\|A_\nu\|\,\|\Gamma(\beta)^*z-\Gamma(\alpha)^*z\| + \|z\|\|A_\nu\|\|\Gamma(\beta)^*z'-\Gamma(\alpha)^*z'\|$$
$$+ \ \varepsilon + \varepsilon \quad (\text{since } \|\Gamma(\cdot)\| = 1)$$

$< 3\varepsilon$ provided $|\beta-\alpha| <$ some δ, since $\Gamma(\cdot)$ is strongly continuous, and $\|A_\nu\| \leq 2K \ \forall \ \nu$.

Hence

$$|(z',\Gamma(\beta)A_\nu\Gamma(\beta)^*z)| < 4\varepsilon$$

for $|\beta-\alpha| < \delta$, and $\forall \ \nu>\nu(\alpha)$, as asserted.

Now, by varying α_0 over $\underline{T}$, we get a family of $\nu(\alpha)$'s, and a corresponding family of neighbourhoods, $\{N(\alpha)\}$. The $\{N(\alpha)\}$ cover $\underline{T}$, and so the compactness of $\underline{T}$ implies that there exists $\alpha_1,..,\alpha_N$ s.t. $\underline{T} = \cup_{i=1}^{N} N(\alpha_i)$. Let $\hat{\nu} > \max\{\nu(\alpha_i)\,|\,i=1,..,N\}$. Then, for any $\alpha\varepsilon\underline{T}$, and $\nu>\hat{\nu}$,

$$|(\Gamma(\alpha)^*z',(X_\nu-X)\Gamma(\alpha)^*z)| < 4\varepsilon$$

since $\alpha\varepsilon N(\alpha_i)$, some $i\varepsilon\{1,..,N\}$.

Hence, $\forall \ \nu>\hat{\nu}$,

$$\left|\int_0^1 (\Gamma(\alpha)^*z',(X_\nu-X)\Gamma(\alpha)^*z)d\alpha \ \right| < 4\varepsilon.$$

This completes the proof of part (b) of lemma 2.5.5.

Lemma 2.5.6.

Let $\underline{\underline{B}}$ be a C*-algebra in $\underline{\underline{B}}(\underline{\underline{F}})$ such that $m(\underline{\underline{B}}) \subset \underline{\underline{B}}$, then $(\underline{\underline{B}} \cap \Gamma(\underline{T})')^{-} = \underline{\underline{B}}^{-} \cap \Gamma(\underline{T})'$.

(The bar denoting the weak closure).

Proof.

Since $m(\underline{\underline{B}}) \subset \underline{\underline{B}}$, (a) of lemma 2.5.5 gives $\underline{\underline{B}} \cap \Gamma(\underline{T})' = m(\underline{\underline{B}})$. Thus $(\underline{\underline{B}} \cap \Gamma(\underline{T})')^{-} = m(\underline{\underline{B}})^{-}$.

Now suppose B is a weak limit point of $\underline{\underline{B}}$. Then, by Kaplansky's density theorem (31), B is a weak limit point of a net $\{X_\nu \varepsilon \underline{\underline{B}} | \; \|X_\nu\| \leq \|B\| \; \forall \; \nu\}$. It follows from lemma 2.5.5 (b), that $m(X_\nu) \to m(B)$ weakly, and $m(B) \; \varepsilon \; m(\underline{\underline{B}})^{-}$. Thus $m(\underline{\underline{B}}^{-}) \subset m(\underline{\underline{B}})^{-}$.

Any $B \varepsilon m(\underline{\underline{B}})^{-}$ is the weak limit of elements of $\underline{\underline{B}}$ invariant under m. As above, we can find a net, $\{X_\nu\}$, in $m(\underline{\underline{B}})$ with $m(X_\nu) \to m(B)$ weakly. But $X_\nu \varepsilon \; m(\underline{\underline{B}})$ implies that $m(X_\nu) = X_\nu \to B$ weakly. Thus $m(B) = B$, and so $m(\underline{\underline{B}})^{-} \subset m(\underline{\underline{B}}^{-})$. We have, then, that $m(\underline{\underline{B}}^{-}) = m(\underline{\underline{B}})^{-}$. Therefore $m(\underline{\underline{B}}^{-}) = m(\underline{\underline{B}})^{-} \subset \underline{\underline{B}}^{-}$, and so, by lemma 2.5.5 (a), $\underline{\underline{B}}^{-} \cap \Gamma(\underline{T})' = m(\underline{\underline{B}}^{-})$.

Hence $\underline{\underline{B}}^{-} \cap \Gamma(\underline{T})' = (\underline{\underline{B}} \cap \Gamma(\underline{T})')^{-}$. QED.

We are now in a position to prove theorem 2.5.4.

From the definition of $\underline{\underline{A}}(\underline{\underline{O}})$ and $\underline{\underline{A}}$, we get

$$\underline{\underline{A}}(\underline{\underline{O}}) = \mathbb{F}(\underline{\underline{O}}) \cap \Gamma(\underline{T})' = m(\mathbb{F}(\underline{\underline{O}}))$$

and, by the norm continuity of m,

$$\underline{\underline{A}} = m(\mathbb{F}) = \mathbb{F} \cap \Gamma(\underline{T})'.$$

Now, by theorem 2.3.3, $\mathbb{F}$ is irreducible, and so, by lemma 2.5.6,

$$\underline{\underline{A}}^{-} = \mathbb{F}^{-} \cap \Gamma(\underline{T})' = \Gamma(\underline{T})'.$$

So we see that the sectors $\underline{\underline{F}}_q$ reduce $\underline{\underline{A}}^{-}$ as well as $\underline{\underline{A}}$. Moreover, $\Gamma(\underline{T})' = \oplus_q \underline{\underline{B}}(\underline{\underline{F}}_q)$, and so each $\pi_q(\underline{\underline{A}})$ is irreducible.

Let E_q denote the projection onto $\underline{\underline{F}}_q$. Then $E_q \varepsilon\ \Gamma(\underline{T})' \cap (\underline{T})'' = \underline{\underline{A}}^{-} \cap \underline{\underline{A}}'$. Thus E_q and $E_{q'}$ are the central supports of π_q and $\pi_{q'}$ (31). These are orthogonal for $q \neq q'$, and so π_q and $\pi_{q'}$ are disjoint (see 5.2.1 (iii) of (31)), and are therefore inequivalent. QED.

Thus, regarding $\underline{\underline{A}}$ as an abstract algebra, we see that the axioms 1 - 5 are satisfied.

We proved, in theorem 2.5.3, that the sectors are physically equivalent. In fact, they are strongly locally equivalent in the sense of Borchers (32).

<u>Definition 2.5.7</u>. Let π_1 and π_2 be two representations of the quasilocal algebra $\underline{\underline{A}}$. We say (32) that π_1 and π_2 are strongly locally equivalent if and only if for any region $\underline{\underline{O}}$ in M, the C*-algebras $\pi_1 \restriction \underline{\underline{A}}(\underline{\underline{O}}')$ and $\pi_2 \restriction \underline{\underline{A}}(\underline{\underline{O}}')$, where $\underline{\underline{A}}(\underline{\underline{O}}')$ is the C*-algebra generated by $\{\underline{\underline{A}}(\underline{\underline{O}}_1) \mid \underline{\underline{O}}_1$ a space-like region with respect to $\underline{\underline{O}}\}$

are unitarily equivalent.

π_1 and π_2 are called locally equivalent if and only if $\pi_1 \upharpoonright \underline{\underline{A}}(\underline{\underline{O}}) \approx \pi_2 \upharpoonright \underline{\underline{A}}(\underline{\underline{O}})$ for any region $\underline{\underline{O}}$.

Theorem 2.5.8.

The representations $\{\pi_q \mid q = 0, \pm 1, \pm 2, ..\}$ are strongly locally equivalent.

Proof.

Let $\underline{\underline{O}}$ be an arbitrary region in M. Let $\xi \varepsilon \Xi(\underline{\underline{O}})$ be such that its Cauchy data has the form (h,0).

The unitary operators expis$(\overline{\phi(h)+\phi^*(h)})$ and expis$(\overline{i\phi^*(h)-i\phi(h)})$ commute, and so they can be expressed as complex functions of unit modulus defined on some measure space S; i.e. there is a unitary equivalence U between $\underline{\underline{F}}$ and $L^2(S,dm)$, such that the aforementioned unitary operators are represented as multiplication operators (17).

By taking the strong derivative with respect to s, and using the fact that $|\zeta|^2 = |\text{Re}\zeta|^2 + |\text{Im}\zeta|^2$ for $\zeta \varepsilon \mathbb{C}$, we conclude that

$$\text{Dom } \phi(h) = \text{Dom } \phi^*(h)$$

$$= \text{Dom } (\overline{\phi(h)+\phi^*(h)}) \cap \text{Dom } (\overline{\phi^*(h)-\phi(h)})$$

and that $U\phi(h)U^{-1}$ is multiplication by a complex-valued m-measurable function on S with domain UDom $\phi(h)$. In other words, $\phi(h)$ is a normal operator (17).

$\phi(h)$ can be written as

$$\phi(h) = \sqrt{\phi^*(h)\phi(h)}\ V$$

where $V = \phi(h)/\sqrt{\phi^*(h)\phi(h)}$ is a unitary operator. (This decomposition is obvious by virtue of the fact that $\phi(h)$ is equivalent to a multiplication operator). Now, $\phi^*(h)\phi(h)$ is self-adjoint on its natural domain, and commutes with $\Gamma(\underline{T})$, and so therefore does $\sqrt{\phi^*(h)\phi(h)}$. But $\phi(h)$ is a map from the q-sector into the q+1-sector. It follows that V is a unitary operator mapping $\underline{\underline{F}}_q$ onto $\underline{\underline{F}}_{q+1}$. Moreover, V is a function of operators in $\mathbb{F}(\underline{\underline{O}})$, and so commutes with all operators commuting with $\mathbb{F}(\underline{\underline{O}})$. In particular, V commutes with $\underline{\underline{A}}(\underline{\underline{O}}')$, and since $\underline{\underline{A}}(\underline{\underline{O}}')$ leaves the sectors invariant,

$$V\ \pi_q(\underline{\underline{A}}(\underline{\underline{O}}')) = \pi_{q+1}(\underline{\underline{A}}(\underline{\underline{O}}'))\ V \quad \text{on } \underline{\underline{F}}_q,\ \forall\ q,$$

i.e. $$\pi_q \restriction \underline{\underline{A}}(\underline{\underline{O}}') \simeq \pi_{q+1}\restriction \underline{\underline{A}}(\underline{\underline{O}}').$$

By iteration, we see that

$$\pi_q \restriction \underline{\underline{A}}(\underline{\underline{O}}') \simeq \pi_{q'}\restriction \underline{\underline{A}}(\underline{\underline{O}}')$$

for any $q,q' = 0,\pm1,\pm2,\ldots$ QED.

<u>Remark</u> This proof breaks down if we take the local field algebras $\hat{\mathbb{F}}(\underline{\underline{O}})$ in definition 2.3.2. This is because $\phi(h)$ is equal to

$$\tfrac{1}{2}(\phi(h)+\phi^*(h)) + \tfrac{1}{2}i(i\phi^*(h)-i\phi(h))$$

$$= \tfrac{1}{2}(\phi_+(h)\otimes \mathbb{1} + \mathbb{1}\otimes\phi_-(h)) + \tfrac{1}{2}i(\mathbb{1}\otimes\pi_-(h_1)-\pi_+(h_1)\otimes \mathbb{1})$$

where $\tilde{h}_1(\underline{k}) = \tilde{h}(\underline{k})\mu(\underline{k})^{-1}$.

Consequently, V cannot be associated with $\hat{\mathbb{F}}(\underline{\underline{O}})$, and so we cannot deduce, as before, that V commutes with the commutant of $\hat{\mathbb{F}}(\underline{\underline{O}})$.

Using the ^-localisation, it has been proved by G.Dell'Antonio (57), and independently by J-L.Bonnard (unpublished), that gauge transformations of the second kind (i.e. those in which α is $\underline{x}$-dependent) are not locally implementable. (An automorphism of the global algebra is said to be locally implementable if its restriction to any local algebra is implementable).

This would appear to contradict the work of M.Fitelson and R.Johnson (58), in which they are able to construct the local generators of such transformations (-albeit in two space-time dimensions).

The point is that these transformations may well turn out to be locally implementable with respect to our preferred, and in our opinion, more physical, localisation. This, and related questions, is at present under investigation.

3 The Massless Bose Field, its Sectors and Associated Charged Fields.

In chapter 2, we showed that the inequivalent representations of the observable algebra, $\underline{\underline{A}}$, (defined as the gauge invariant part of the field algebra, $\mathbb{F}$), occuring in an irreducible representation of $\mathbb{F}$ were strongly locally equivalent. Since the observable algebra, $\underline{\underline{A}}$, contains all the physical information, we can regard $\mathbb{F}$ as an auxiliary construct. That is to say, it should be possible to construct charge carrying fields given the algebra $\underline{\underline{A}}$. In other words, given $\underline{\underline{A}}$ in the vacuum sector, we should be able to construct all other sectors.

We shall not do this in complete generality, but rather we shall consider a particular model. We take $\underline{\underline{A}}$ to be the C*-algebra associated with the massless boson field in two space-time dimensions.

We shall construct various inequivalent representations of $\underline{\underline{A}}$ - those given by applying localised automorphisms to the Fock representation. These representations turn out to be strongly locally Fock, and each one contains a strongly continuous representation of the restricted Poincaré group, having energy-momentum spectrum in the closed forward light cone, $\overline{V}^+$.

We find that our "charge" takes doubly-continuous values, i.e. values in $\mathbb{R}\times\mathbb{R}$, and that the charge carrying fields do not obey the Bose-Fermi alternative, except for a discrete countable number of values of the charge.

Finally, we identify an uncountable number of copies of the torus with the gauge group : $G = X_{\alpha\varepsilon I}\, \underline{T}$ where $I = [0,1)\times[0,1)$ and $\underline{T}_\alpha$ is the torus for all $\alpha\varepsilon I$.

Our model is suggested by an early paper of Skyrme (33,34), where there is an explicit formula for the fermion fields in terms of the boson field.

It should be mentioned that the study of the charge sectors given the charge zero sector was initiated by H.J.Borchers in the mid-sixties (32). His results, however, are inconclusive.

§3.1. The Zero-Mass Bose Field in Two Dimensions.

The construction of the local algebras etc. is almost the same as for the case of a massive field - the difference is that the invariant measure on the positive-energy mass-hyperboloid is singular for zero-mass particles. This creates problems if we want to consider the field as a Wightman field, i.e. as an operator-valued distribution. In fact, there is no such field in two dimensions (35,36). This is because there is no Lorentz-invariant tempered, positive distribution defined on the light-cone (35). We avoid this difficulty by restricting our test-function space (37).

We shall give the complete construction of $\underline{\underline{A}}$ for the massless case, as this is simpler, and less messy, than pointing out the differences between the present case and that for the massive field.

Let us take an explicit, and less abstract, formulation of Fock space, and the creation and annihilation operators.

The one-particle space is $K = L^2(\mathbb{R}, d\Omega)$, where $d\Omega = dp/2|p|$. We shall also use ω to denote $|p|$.

Let $\underline{\underline{H}}$ be the Fock space over K ;

$$\underline{\underline{H}} \equiv \overline{\otimes} K = \oplus_{n=0}^{\infty} K_n$$

where $K_0 = \mathbb{C}$, and K_n is the space of symmetric complex-valued functions of n variables, square-integrable with respect to the product measure $\otimes^n d\Omega$.

Definition 3.1.1.(29)

For each $F \in L^2(R, dp)$, we define the annihilation operator as the closed operator, given by its action on homogeneous elements of $\underline{\underline{H}}$ as

$$a(F) : K_0 \rightarrow \{0\},$$

$$a(F) : K_n \rightarrow K_{n-1},$$

$$(a(F)h_n)(p_1, \ldots, p_{n-1}) = \sqrt{n} \int F(p) h_n(p, p_1, \ldots, p_{n-1}) \frac{dp}{\sqrt{2|p|}}$$

$a^*(\overline{F})$ is its adjoint, given explicitly on homogeneous elements of $\underline{\underline{H}}$ as

$$a^*(\overline{F}) \ : \ K_n \to K_{n+1}$$

$$(a^*(\overline{F})h_n)(p_1,\ldots,p_{n+1}) = \frac{1}{\sqrt{n+1}}\sum_{j=1}^{n+1}\sqrt{2|p_j|}\,\overline{F(p_j)}\,h_n(p_1,\ldots,\hat{p}_j,\ldots,p_{n+1})$$

As before, $a(F)$ and $a^*(F)$ define closed, densely-defined unbounded operators in $\underline{\underline{H}}$. If one puts, formally, $F(\cdot) = \delta(\cdot-p)$, $a(F)$ and $a^*(F)$ become the creation and annihilation forms $a(p)$ and $a^*(p)$, as defined in (29), for example.

To define the time-zero fields, ϕ, π, let $\underline{\underline{D}}$ denote the real test-function space of smooth functions on $\mathbb{R}$ with compact support.

Let $\underline{\underline{D}}_0 = \{ f\epsilon\underline{\underline{D}} \mid \tilde{f}(0) = 0 \}$, where, as usual, $\tilde{f}(p) = (2\pi)^{-\frac{1}{2}}\int f(x)e^{ipx}dx$.

Let us denote by $\underline{\underline{M}}$ the set of real-valued pairs $(f,g)\epsilon\underline{\underline{D}}_0\times\underline{\underline{D}}$. Then, for $(f,g)\epsilon\underline{\underline{M}}$, we define

$$\phi(f) = 2^{-\frac{1}{2}}(a^*(F_-)+a(F_+))$$

$$\pi(g) = 2^{-\frac{1}{2}}i(a^*(G_-)-a(G_+))$$

where $F_\pm(p) = |p|^{-\frac{1}{2}}\tilde{f}(\pm p)$; $G_\pm(p) = |p|^{\frac{1}{2}}\tilde{g}(\pm p)$.

Our restriction to $f\epsilon\underline{\underline{D}}_0$ implies that $\tilde{f}(p)$ is analytic in p in a neighbourhood of the origin, and that it behaves like p near the origin. Thus $|p|^{-\frac{1}{2}}\tilde{f}(p)$ is finite at $p = 0$, and so belongs to $L^2(\mathbb{R},dp)$. Accordingly, $\phi(f)$ is a well-defined operator with dense domain. In fact, as in §2.1, $\phi(f)$ and $\pi(g)$ are essentially self-adjoint on $\underline{\underline{H}}'$, the algebraic direct sum of the K_n.

Let ξ be a solution of the two-dimensional wave equation, $\partial_t{}^2\xi-\partial_x{}^2\xi = 0$, with Cauchy data $(f,g)\varepsilon\underline{\underline{M}}$;

$$\dot{\xi}(x,0) = f(x) \quad , \; \xi(x,0) = g(x).$$

We have a correspondence between $\underline{\underline{M}}$ and a subset of real solutions of the wave equation. Let us denote this subset also by $\underline{\underline{M}}$. The restricted Poincare group, in two-dimensions, acts on these solutions, as in §2.1, by

$$\{a,\Lambda\} : \xi \to \xi_{a,\Lambda}$$

$$\xi_{a,\Lambda}(x) = \xi(\Lambda^{-1}(x-a)), \; x \in \mathbb{R}^2.$$

Now, $\underline{\underline{M}}$ is invariant under this action. To see this, we note that the Wronskian between any two solutions is invariant. Taking $\xi\varepsilon\underline{\underline{M}}$, and the constant solution, we have that $\int\dot{\xi}(x,0)dx$ is invariant. But $\tilde{f}(0) = 0$ is equivalent to $\int f(x)dx = 0$, and so this invariant is zero, and implies that $\xi_{a,\Lambda}(x,0)\varepsilon\underline{\underline{D}}_0$.

Let us denote $\overline{\phi(f)-\pi(g)}$ by $\{\phi,\xi\}$, where $\xi \leftrightarrow (f,g) \in \underline{\underline{M}}$. (The notation is meant to indicate that $\phi(f)-\pi(g)$ is the Wronskian between the two solutions $\phi(x,t)$ and $\xi(x,t)$ of the wave-equation). As in §2.1, $\{\phi,\xi\}$ is self-adjoint and has $\underline{\underline{H}}'$ as a core; moreover, we have the Segal-Weyl relations :

$$W(\xi_1)W(\xi_2) = e^{-i\frac{1}{2}\{\xi_1,\xi_2\}}W(\xi_1+\xi_2)$$

$$= e^{-i\{\xi_1,\xi_2\}}W(\xi_2)W(\xi_1),$$

where $W(\xi) = e^{i\{\phi,\xi\}}$, and $\{\xi_1,\xi_2\}$ is the Wronskian.

Just as in §2.1, we can give $\underline{\underline{M}}$ a local structure.

Definition 3.1.2.

We define $\underline{\underline{M}}(\underline{\underline{O}})$ as the set previously denoted by $\Xi(\underline{\underline{O}})$, but we only consider convex regions $\underline{\underline{O}}$ in Minkowski space, M.

Definition 3.1.3.

We define $\underline{\underline{A}}(\underline{\underline{O}})$, for an arbitrary region $\underline{\underline{O}}$ in M, as the von Neumann algebra generated by the set $\{ W(\xi) \mid \xi\epsilon\underline{\underline{M}}(\underline{\underline{O}}_1), \underline{\underline{O}}_1 \text{ in } \underline{\underline{O}}\}$.

So although we have only defined $\underline{\underline{M}}(\underline{\underline{O}})$ for convex regions, $\underline{\underline{O}}$, in M, we have defined $\underline{\underline{A}}(\underline{\underline{O}})$ for arbitrary $\underline{\underline{O}}$.

According to our definition, $W(\xi)\epsilon\underline{\underline{A}}(\underline{\underline{O}})$ if and only if $W(\xi)\epsilon\underline{\underline{A}}(\underline{\underline{O}}_1)$, for some convex (and therefore connected) component $\underline{\underline{O}}_1$ of $\underline{\underline{O}}$. Suppose $\underline{\underline{O}} = \underline{\underline{O}}_1 \cup \underline{\underline{O}}_2$, with $\underline{\underline{O}}_1$, $\underline{\underline{O}}_2$ disjoint convex regions. Then it is natural to require that $\underline{\underline{A}}(\underline{\underline{O}})$ be generated by $\underline{\underline{A}}(\underline{\underline{O}}_1)\cup\underline{\underline{A}}(\underline{\underline{O}}_2)$. If we had defined $\underline{\underline{A}}(\underline{\underline{O}})$ as the algebra generated by the $W(\xi)$ with $\xi\epsilon\underline{\underline{M}}(\underline{\underline{O}})$ (defined as the analogue of $\Xi(\underline{\underline{O}})$), then there would be operators, $W(\xi)$, in $\underline{\underline{A}}(\underline{\underline{O}})$, such that the restrictions of ξ to $\underline{\underline{O}}_1$ and $\underline{\underline{O}}_2$ fail to give elements of $\underline{\underline{A}}(\underline{\underline{O}}_1)$ and $\underline{\underline{A}}(\underline{\underline{O}}_2)$, respectively, because the condition that $\int\dot{\xi}(x,0)dx = 0$ would fail. This will be the case, for example, if $\dot{\xi}(x,t)$ is non-negative in $\underline{\underline{O}}_1$, and non-positive in $\underline{\underline{O}}_2$. $\underline{\underline{A}}(\underline{\underline{O}})$ would not then be equal to the von Neumann algebra generated by $\underline{\underline{A}}(\underline{\underline{O}}_1)$ and $\underline{\underline{A}}(\underline{\underline{O}}_2)$. It is for this reason that we insist that $W(\xi)\varepsilon\underline{\underline{A}}(\underline{\underline{O}})$ only if $\xi\varepsilon\underline{\underline{M}}(\underline{\underline{O}}_1)$ for some convex region $\underline{\underline{O}}_1$ in $\underline{\underline{O}}$.

We have defined $\underline{\underline{A}}(\underline{\underline{O}})$, for each region $\underline{\underline{O}}$, to be a certain von Neumann algebra of operators. This is not essential in that we could have taken them to be C*-algebras, i.e. replaced von Neumann by C*- in definition 3.1.3. However, our results are stronger if they hold for the $\underline{\underline{A}}(\underline{\underline{O}})$ as von Neumann algebras. Since any von Neumann algebra is also a C*-algebra, we can still view $\underline{\underline{A}}(\underline{\underline{O}})$ as an abstract C*-algebra by ignoring the underlying Hilbert space.

$\underline{\underline{A}}$ is defined to be the norm closure of $\{ \underline{\underline{A}}(\underline{\underline{O}}) \mid \underline{\underline{O}} \text{ a region in } M \}$ in $\underline{\underline{B}}(\underline{\underline{H}})$.

The condition $\int\dot{\xi}(x,0)dx = 0$, necessary because of the infra-red problem, i.e. the singularity of the measure $d\Omega$, may be interpreted "physically" by saying that , being a potential, is not observable: only the "field" $-\nabla\phi$ can be observed, or rather, its smeared form $-\int\nabla\phi(x)h(x)dx = \phi(\nabla h) = \phi(f)$, where $f = \nabla h \;\varepsilon\; \underline{\underline{D}}_0$ if $h \;\varepsilon\; \underline{\underline{D}}$. With this interpretation, $\underline{\underline{A}}$ is the algebra of observables of the theory.

We remark that $\underline{\underline{A}}$ is irreducible. This follows from lemma 2.1.6, together with the fact that both $\tilde{\underline{\underline{D}}}$ and $\tilde{\underline{\underline{D}}}_0$ (the Fourier transforms of $\underline{\underline{D}}$ and $\underline{\underline{D}}_0$, resp.) are dense in the set $K_R = \{ f \varepsilon K \mid \overline{f(p)} = f(-p) \}$ in the induced strong topology of K.

Definition 3.1.4.

As in §2.1, we define a strongly continuous unitary representation of $\mathbb{P}_+^\uparrow$ on $\underline{\underline{H}}$ by extension of

$$\{a,\Lambda\} \to U_0(a,\Lambda)$$

$$(U_0(a,\Lambda)h_n)(p_1,\ldots,p_n) = e^{i\sum_{j=1}^n (p_j,a)} h_n(\underline{\Lambda^{-1}p_1},\ldots,\underline{\Lambda^{-1}p_n})\Big|_{p_i^0 = |p_i|}$$

where $(p,a) = p^0 a^0 - p^1 a^1$ and $\underline{\Lambda^{-1}p}$ is the space-component of the two-vector $\Lambda^{-1}p$.

Then

$$U_0(a,\Lambda) W(\xi) U_0(a,\Lambda)^{-1} = W(\xi_{a,\Lambda})$$

and so

$$U_0(a,\Lambda) \underline{\underline{A}}(\underline{\underline{O}}) U_0(a,\Lambda)^{-1} = \underline{\underline{A}}(\Lambda\underline{\underline{O}}+a)$$

with the obvious notation.

In the same way as for the massive case, the vanishing of the Wronskians $\{\underline{\underline{M}}(\underline{\underline{O}}),\underline{\underline{M}}(\underline{\underline{O}}_1)\}$ for space-like regions $\underline{\underline{O}}$ and $\underline{\underline{O}}_1$, and the Segal-Weyl relations imply that $\underline{\underline{A}}(\underline{\underline{O}})$ and $\underline{\underline{A}}(\underline{\underline{O}}_1)$ commute.

Definition 3.1.5.

Let $\alpha : G \to \text{Aut}\,\underline{\underline{A}}$ be a representation of a group G by automorphisms of $\underline{\underline{A}}$. Suppose that the notion $g \to \infty$ in G is meaningful. Then we say that $\underline{\underline{A}}$ is asymptotically abelian with respect to the pair (α, G) if for any $A, B \in \underline{\underline{A}}$,

$$\text{norm} \lim_{g \to \infty} [\alpha(g)A, B] = 0.$$

Lemma 3.1.6.

$\underline{\underline{A}}$ is asymptotically abelian with respect to the space and time translations given in definition 3.1.4.

Proof.

The first part is obvious because of the quasilocal nature of $\underline{\underline{A}}$ and the commutativity of the $\underline{\underline{A}}(\underline{\underline{O}})$ for space-like regions.

For the second part, we must show that

$$\text{norm} \lim_{t \to \infty} [U_0(t) A U_0(t)^{-1}, B] = 0$$

for any $A, B \in \underline{\underline{A}}$, where $U_0(t) = U_0(a, \Lambda)$ with $\{a, \Lambda\}$ a pure time translation.

Suppose that $A = W(\xi) \in \underline{\underline{A}}(\underline{\underline{O}})$, $B = W(\eta) \in \underline{\underline{A}}(\underline{\underline{O}}_1)$. Then

$$[U_0(t) A U_0(t)^{-1}, B]$$

becomes

$$[W(\xi_t), W(\eta)] = \exp i\{\xi_t, \eta\}.$$

Now, any solution ξ of $\partial_t^2 \xi = \partial_x^2 \xi$ can be written in the form

$$\xi(x,t) = f(x+t) + g(x-t)$$

for some functions f and g, which can be chosen to be smooth, and with compact support for the case $\xi \in \underline{\underline{M}}$. Therefore, $\xi_t(x,s)$ has the form

$$\xi_t(x,s) = f(x+s-t) + g(x-s+t)$$

and

$$\{\xi_t,\eta\} = \int\Big((f(x-t)+g(x+t))\dot{\eta}(x,0) - (f'(x-t)-g'(x+t))\eta(x,0)\Big)dx$$

where $f'(y) \equiv df(y)/dy$.

But $\eta(x,0)$ and $\dot{\eta}(x,0)$ have compact support, and so, for sufficiently large t, $\{\xi_t,\eta\} = 0$. Thus

$$[U_0(t)AU_0(t)^{-1},B] = 0$$

for large t, for this choice of A and B. Clearly, the same is true for A chosen to be finite linear combinations of finite products of various $W(\xi)$, where $\xi \in \underline{\underline{M}}(\underline{\underline{O}})$.

It follows that

$$[U_0(t)\underline{\underline{A}}(\underline{\underline{O}})U_0(t)^{-1},B] = 0$$

for large t, and $B = W(\eta)$, $\eta \in \underline{\underline{M}}(\underline{\underline{O}}_1)$, and therefore that

$$[U_0(t)\underline{\underline{A}}(\underline{\underline{O}})U_0(t)^{-1},\underline{\underline{A}}(\underline{\underline{O}}_1)] = 0$$

for large t.

Now let $A,B \in \underline{\underline{A}}$ be arbitrary. The norm density of $\{\underline{\underline{A}}(\underline{\underline{O}})|\underline{\underline{O}}$ in M$\}$ in $\underline{\underline{A}}$ allows us to choose, for given $\varepsilon>0$, A_1 and B_1 ε $\underline{\underline{A}}(\underline{\underline{O}})$ and $\underline{\underline{A}}(\underline{\underline{O}}_1)$, respectively, such that

$$\| [U_0(t)AU_0(t)^{-1},B] - [U_0(t)A_1U_0(t)^{-1},B_1] \| < \varepsilon$$

uniformly in t. The second commutator vanishes for large t, and the proof is complete.

The algebras $\underline{\underline{A}}(\underline{\underline{O}})$, $\underline{\underline{O}}$ in M, and $\underline{\underline{A}}$, satisfy the Haag-Kastler axioms 1-5 of §1.2.

Moreover, U_0 satisfies the spectrum condition, viz, the energy-momentum spectrum lies in the closed forward light-cone, $\overline{V}^+$; there is a non-degenerate eigenvalue 0 of P^μ, $\mu = 0,1$, the generators of $U_0(a,1)$, corresponding to the eigenvector $\Omega = 1$ in $K_0 = \mathbb{C}$. Ω defines a vector state on $\underline{\underline{A}}$, which is called the vacuum state.

§3.2. Localised Automorphisms.

Following Doplicher, Haag and Roberts (20), we make the

Definition 3.2.1.

An automorphism γ of $\underline{\underline{A}}$ is said to be a localised automorphism, localised in a region $\underline{\underline{O}}$, if

$$\gamma(\underline{\underline{A}}(\underline{\underline{O}}_1)) \subset \underline{\underline{A}}(\underline{\underline{O}}_1) \quad \text{for all } \underline{\underline{O}}_1 \supset \underline{\underline{O}}$$

and if

$$\gamma \restriction \underline{\underline{A}}(\underline{\underline{O}}') = \iota \restriction \underline{\underline{A}}(\underline{\underline{O}}'),$$

where ι is the identity automorphism. (We recall that $\underline{\underline{A}}(\underline{\underline{O}}')$ is the C*-algebra generated by all the $\underline{\underline{A}}(\underline{\underline{O}}_1)$, with $\underline{\underline{O}}_1$ space-like with respect to $\underline{\underline{O}}$.)

In other words, γ is localised in a region $\underline{\underline{O}}$ if it has no effect on observables outside $\underline{\underline{O}}$, but maps observables in $\underline{\underline{O}}$ into observables again located within $\underline{\underline{O}}$.

If duality holds, viz, $\underline{\underline{A}}(\underline{\underline{O}}) = \underline{\underline{A}}(\underline{\underline{O}}')'$, for all $\underline{\underline{O}}$, then $\gamma \restriction \underline{\underline{A}}(\underline{\underline{O}}') = \iota \restriction \underline{\underline{A}}(\underline{\underline{O}}')$ implies that if $\underline{\underline{O}}_1 \supset \underline{\underline{O}}$ then $\gamma(\underline{\underline{A}}(\underline{\underline{O}}_1)) \subset \underline{\underline{A}}(\underline{\underline{O}}_1)$. This is because, for $A \in \underline{\underline{A}}(\underline{\underline{O}}_1)$, $B \in \underline{\underline{A}}(\underline{\underline{O}}_1')$,

$$[A,B] = 0 \implies \gamma[A,B] = 0$$

$$\implies [\gamma(A),\gamma(B)] = 0 \implies [\gamma(A),B] = 0$$

since $\gamma \restriction \underline{\underline{A}}(\underline{\underline{O}}') = \iota \restriction \underline{\underline{A}}(\underline{\underline{O}}')$

$$\implies \gamma(A) \in \underline{\underline{A}}(\underline{\underline{O}}_1')' = \underline{\underline{A}}(\underline{\underline{O}}_1).$$

However, in general, duality will not hold unless the regions $\underline{\underline{O}}$ are suitably shaped (3). It is for this reason that the $\underline{\underline{O}}$ are taken to be "double-cones" in (20).

It is the purpose of this section to construct localised automorphisms with the above properties.

The effect of our automorphisms will be to add c-numbers to the fields ϕ and π. The operators $W(\xi)$, therefore, just pick up a phase. We could treat ϕ and π separately; but to show that Poincaré transformations are implemented in the various sectors we must exploit the properties of solutions of the wave equation, and the treatment is unified and "compactified" if we consider $\{\phi,\xi\}$ rather than ϕ and π separately.

Let $\theta: \mathbb{R} \to \mathbb{R}$ be such that $d\theta(x)/dx \in \underline{\underline{D}}$ and $\theta(-\infty) = 0$. Thus θ is a smooth step function, vanishing for large negative argument.

Each such θ defines a pair of solutions of the wave equation $(\partial_t^2-\partial_x^2)\Theta(x,t) = 0$ by setting

$\Theta(x,t) = \theta(x+t)$ or $= \theta(x-t)$.

<u>Definition 3.2.1</u>.

Let $\underline{\underline{N}}^{\pm}$ denote the set of such solutions, and let $\underline{\underline{N}}$ be the real linear span of $\underline{\underline{N}}^{+}$ and $\underline{\underline{N}}^{-}$.

Clearly, $\Theta\epsilon\underline{\underline{N}}$ if and only if $\dot{\Theta}\epsilon\underline{\underline{M}}$ and $\Theta(-\infty,0) = 0$. We want to give $\underline{\underline{N}}$ a local structure - we can do this via $\underline{\underline{M}}$.

<u>Definition 3.2.2</u>.

Let $\underline{\underline{O}}$ be a region in M. We define $\underline{\underline{N}}(\underline{\underline{O}})$ to be the real linear span of the set

$$\{\Theta\epsilon\underline{\underline{N}}\,|\,\dot{\Theta}\epsilon\underline{\underline{M}}(\underline{\underline{O}}_1),\ \underline{\underline{O}}_1 \text{ convex, } \underline{\underline{O}}_1 \text{ in } \underline{\underline{O}}\}.$$

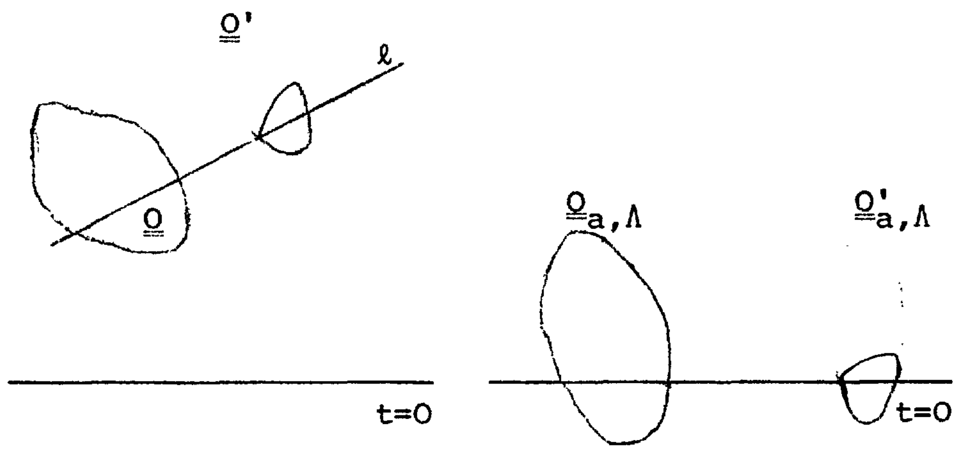
<u>Lemma 3.2.3</u>.

If $\Theta\epsilon\underline{\underline{N}}(\underline{\underline{O}})$, and if $\underline{\underline{O}}'$ is space-like with respect to $\underline{\underline{O}}$, then $\dot{\Theta}(x,t)$ and $\partial_x\Theta(x,t)$ are zero on $\underline{\underline{O}}'$; i.e.

$\Theta(x,t)$ is constant on connected components of $\underline{\underline{O}}'$.

Proof.

Let ℓ be a space-like line passing through $\underline{\underline{O}}$ and $\underline{\underline{O}}'$. By a Poincaré transformation, $\{a,\Lambda\}$, we can transform ℓ into the line $\{(x,t)\,|\,t=0\}$.



We may suppose that $\underline{\underline{O}}$ is simply connected because of our definition of $\underline{\underline{N}}(\underline{\underline{O}})$ in terms of the $\underline{\underline{M}}(\underline{\underline{O}}_1)$.

Consider $\Theta_{a,\Lambda}(x,t)$. This can be written as

$$\Theta_{a,\Lambda}(x,t) = \theta_1(x+t) + \theta_2(x-t)$$

with $\theta_1 \varepsilon\ \underline{\underline{N}}^+$, $\theta_2 \varepsilon\ \underline{\underline{N}}^-$. Now, $\dot{\Theta}_{a,\Lambda}(x,t)\ \varepsilon\ \underline{\underline{M}}(\underline{\underline{O}}_{a,\Lambda})$ implies that $\dot{\Theta}_{a,\Lambda}(x,0)$ and $\ddot{\Theta}_{a,\Lambda}(x,0)$ have support in an interval (a,b), space-like with respect to $\underline{\underline{O}}'_{a,\Lambda}$. Thus $\theta_1'(x)-\theta_2'(x)$ and $\theta_1''(x)+\theta_2''(x)$ have support in (a,b). (The dash denotes ∂_x).

It follows that $\theta_1'(x)+\theta_2'(x)$ is constant for x<a and for x>b (with different constants resp.). This, together with supp($\theta_1'(x)-\theta_2'(x)$) in (a,b) implies that $\theta_1'(x)$ and $\theta_2'(x)$ are both constant outside the interval (a,b). But θ_1 and θ_2 ε $\underline{\underline{N}}^{\pm}$ and so θ_1' and θ_2' are zero outside (a,b). Thus

$$\Theta'_{a,\Lambda}(x,0) = \theta_1'(x) + \theta_2'(x)$$

and

$$\dot{\Theta}_{a,\Lambda}(x,0) = \theta_1'(x) - \theta_2'(x)$$

vanish on any interval disjoint from (a,b). It follows that $\dot{\Theta}(x,t)$ and $\partial_x\Theta(x,t)$ vanish on $\underline{\underline{O}}'$, as asserted.

We shall realise the additive groups, $\underline{\underline{N}}$, $\underline{\underline{N}}(\underline{\underline{O}})$, by automorphisms of $\underline{\underline{A}}$. To this end, we make:

<u>Definition 3.2.4</u>.

For any Θ ε $\underline{\underline{N}}$, the transformation γ is defined on elements of $\underline{\underline{A}}$ of the form $W(\xi)$ by

$$\gamma : W(\xi) \to e^{i\{\Theta,\xi\}}W(\xi)$$

<u>Lemma 3.2.5</u>.

For each region $\underline{\underline{O}}$ in M, there is a unitary operator, V, (non-unique) which effects the transformation γ ;

$$\gamma(W(\xi)) = VW(\xi)V^{-1} \qquad \text{for all } W(\xi)\ \varepsilon\ \underline{\underline{A}}(\underline{\underline{O}}).$$

<u>Proof</u>.

Let $\Theta_1 \varepsilon$ $\underline{\underline{M}}$ be such that $\Theta_1(x,t) = \Theta(x,t)$ if

$(x,t) \in \underline{\underline{O}}$. Put $V = W(-\Theta_1)$. Then, according to the Weyl relations :

$$VW(\xi)V^{-1} = W(\xi)W(-\Theta_1)W(\Theta_1)e^{i\{\Theta_1,\xi\}}$$

$$= e^{i\{\Theta,\xi\}}W(\xi)$$

since $\{\Theta,\xi\}$ only depends on the values of Θ in $\underline{\underline{O}}$. QED.

Corollary 3.2.6.

To each $\Theta \in \underline{\underline{N}}$ there exists a unique automorphism of $\underline{\underline{A}}$ which reduces to γ on elements of the form $W(\xi)$.

Proof.

Since the $W(\xi)$ generate $\underline{\underline{A}}(\underline{\underline{O}})$, for $\xi \in \underline{\underline{M}}(\underline{\underline{O}}_1)$, $\underline{\underline{O}}_1$ convex in $\underline{\underline{O}}$, γ can be extended uniquely to an automorphism $\gamma_{\underline{\underline{O}}}$ of $\underline{\underline{A}}(\underline{\underline{O}})$, implemented by V. Clearly, if $\underline{\underline{O}}_2$ contains $\underline{\underline{O}}_1$, then $\gamma_{\underline{\underline{O}}_2} \upharpoonright \underline{\underline{A}}(\underline{\underline{O}}_1) = \gamma_{\underline{\underline{O}}_1}$. By taking the inductive limit, we obtain the required automorphism which we shall also denote by γ.

Remark.

$\Theta(x,t)+c$ defines the same automorphism as $\Theta(x,t)$ for any constant, c. Thus the requirement that $\Theta(-\infty,0) = 0$ is one of convenience. The important property of $\Theta(x,t)$, as far as we are concerned, is the value of the difference $\Theta(+\infty,0)-\Theta(-\infty,0)$.

The next lemma expresses the fact that γ is a localised automorphism.

Lemma 3.2.7.

Let γ be defined as in corollary 3.2.6, with $\theta\epsilon\underline{\underline{N}}(\underline{\underline{O}})$. Let $\underline{\underline{O}}_1$ be space-like with respect to $\underline{\underline{O}}$. Then $\gamma \upharpoonright \underline{\underline{A}}(\underline{\underline{O}}_1) = \iota \upharpoonright \underline{\underline{A}}(\underline{\underline{O}}_1)$.

Proof.

We need only show that $\{\theta,\xi\} = 0$ for all $\xi\epsilon\underline{\underline{M}}(\underline{\underline{O}}_2)$, with $\underline{\underline{O}}_2$ a convex region in $\underline{\underline{O}}_1$. The Wronskian $\{\theta,\xi\}$ is given by

$$\{\theta,\xi\} = \int_{\ell}\left(\theta(x,t)\dot{\xi}(x,t) - \dot{\theta}(x,t)\xi(x,t)\right)d\ell$$

where ℓ is any space-like line. Choosing ℓ to run through $\underline{\underline{O}}$ and $\underline{\underline{O}}_2$, and using lemma 3.2.3, we find that

$$\{\theta,\xi\} = \text{const.}\int_{\ell}\dot{\xi}(x,t)d\ell = \text{const.}\{1,\xi\} = 0,$$

as required. QED.

Suppose that $\theta(x,t) = \theta(x+t)$. Then the corresponding automorphism γ corresponds to a displacement of the fields :

$$\phi(f) \to \phi(f) + \int f(x)\theta(x)dx$$

$$\pi(g) \to \pi(g) + \int g(x)\frac{d\theta}{dx}(x)dx$$

or, if we avail ourselves the distribution-theoretic notation $\phi(x)$, $\pi(x)$, where $\phi(f)$ and $\pi(g)$ are written symbolically as $\phi(f) = \int\phi(x)f(x)dx$, and $\pi(g) = \int\pi(x)g(x)dx$, then we find that γ corresponds to the displacements ;

$$\phi(x) \to \phi(x) + \theta(x)$$

and

$$\pi(x) \to \pi(x) + \theta'(x).$$

If $\Theta(x,t) = \theta(x-t)$, then γ would correspond to the displacements

$$\phi(x) \to \phi(x) + \theta(x)$$

and $\quad \pi(x) \to \pi(x) - \theta'(x).$

Since π is the time-derivative of ϕ, we see that γ is given by the displacement

$$\phi(x,t) \to \phi(x,t) + \Theta(x,t).$$

We call γ a gauge transformation of the second kind, since it is an addition of a space-time dependent function, $\Theta(x,t)$. We notice that if supp $f \cap$ supp θ' = supp $g \cap$ supp $\theta' = \emptyset$, then $\phi(f)$ and $\pi(g)$ remain unaltered (-this because $f \in \underline{\underline{D}}_0$). This is why we localise Θ in terms of its derivative.

If we admit the limiting procedure $\theta_1 \to H(x-x_0)$, where H is the Heavyside step-function, and the function Θ_1 of lemma 3.2.5 is $\Theta_1(x,t) = \theta_1(x+t)$, then the unitary operator of lemma 3.2.5 becomes essentially the fermion operator of Skyrme (33,34). These limits are not rigorous, however. It is essential to consider inequivalent representations of $\underline{\underline{A}}$.

§3.3 The New Representations.

We consider $\underline{\underline{A}}$ and the various $\underline{\underline{A}}(\underline{\underline{O}})$ as abstract C*-algebras - we shall denote by π_0 the representation of $\underline{\underline{A}}$ by itself on $\underline{\underline{H}}$, which we now write as $\underline{\underline{H}}_0$.

For any of our localised automorphisms, γ, we can define a new representation π_γ, say, obtained by composing π_0 with γ, viz,

$$\pi_\gamma \equiv \pi_0 \cdot \gamma$$

acting on $\underline{\underline{H}}_0$.

Thus $\pi_\gamma(A) = \pi_0(\gamma(A))$ for $A \varepsilon \underline{\underline{A}}$.

Definition 3.3.1.

Define Γ, $\Gamma(\underline{\underline{O}})$, $\Gamma^{\pm}$ as the automorphism groups of $\underline{\underline{A}}$, obtained earlier, by choosing Θ in $\underline{\underline{N}}$, $\underline{\underline{N}}(\underline{\underline{O}})$, and $\underline{\underline{N}}^{\pm}$, respectively.

These give various inequivalent representations of $\underline{\underline{A}}$. To prove this, we must first identify $\underline{\underline{M}}$ with a subset of K, the one-particle space of $\underline{\underline{H}}_0$, and then prove a lemma on the implementability of certain gauge transformations.

For each $\xi \leftrightarrow (f,g) \varepsilon \underline{\underline{M}}$, we can associate the function

$$h(p) = 2\omega\tilde{g}(p) - i\tilde{f}(p) \;\varepsilon\; K = L^2(\mathbb{R}, d\Omega),$$

where, as usual, $\omega = |p|$, and $d\Omega = dp/2\omega$.

As (f,g) run over $\underline{\underline{D}}_0\times\underline{\underline{D}}$, the corresponding h's run over a dense subset of K. Moreover, $\{\xi_1,\xi_2\}$ is nothing but minus the imaginary part of the scalar product (h_1,h_2) in K :

$$\begin{aligned}\operatorname{Im}\,(h_1,h_2) &= \operatorname{Im}\int\overline{h_1(p)}\,h_2(p)\,d\Omega \\ &= \operatorname{Im}\int\left(\overline{2\omega\tilde{g}_1(p)-i\tilde{f}_1(p)}\right)\left(2\omega\tilde{g}_2(p)-i\tilde{f}_2(p)\right)d\Omega \\ &= \int\left(\overline{\tilde{f}_1(p)}\,2\omega\tilde{g}_2(p) - 2\omega\overline{\tilde{g}_1(p)}\,\tilde{f}_2(p)\right)d\Omega \\ &= \int\left(\overline{\tilde{f}_1(p)}\,\tilde{g}_2(p) - \overline{\tilde{g}_1(p)}\,\tilde{f}_2(p)\right)dp \\ &= \int\left(f_1(x)g_2(x) - g_1(x)f_2(x)\right)dx \\ &= -\{\xi_1,\xi_2\}\end{aligned}$$

as asserted.

Now we need the lemma ;

<u>Lemma 3.3.2</u>.

Let $\chi:\underline{\underline{M}}\to\mathbb{R}$ be a real linear functional on $\underline{\underline{M}}$. Suppose that χ is unbounded (where $\underline{\underline{M}}$ is identified, as above, with a subset of K). Then the transformation

$$W(\xi)\to e^{i\chi(\xi)}W(\xi)$$

$\xi\in\underline{\underline{M}}$, is not unitarily implementable.

<u>Proof</u>. (This is an adaption of Manuceau (38,39), after Roepstorff (40)).

χ unbounded implies that there exists a sequence $\{\xi_n\}$, $\xi_n\in\underline{\underline{M}}$ $n=1,2,..,$ with $\xi_n\to 0$ in K, such that $\chi(\xi_n)\to\pi$ as $n\to\infty$.

Now, by lemma 2.1.7, $\xi_n \to 0$ implies that $W(\xi_n) \to \mathbb{1}$ strongly in $\underline{\underline{H}}_0$.

Suppose the transformation $W(\cdot) \to e^{i\chi(\cdot)}W(\cdot)$ were implementable, by U, say. Then

$$UW(\xi_n)U^{-1} = e^{i\chi(\xi_n)}W(\xi_n), \quad \forall\ n.$$

But as $n \to \infty$, the l.h.s. converges strongly to $\mathbb{1}$, whereas the r.h.s. converges strongly to $-\mathbb{1}$; a contradiction.

<u>Theorem 3.3.3</u>.

(i) If $\gamma_1 \varepsilon \Gamma^+$ and $\gamma_2 \varepsilon \Gamma^+$, then $\pi_0 \cdot \gamma_1 \simeq \pi_0 \cdot \gamma_2$ if and only if $\theta_1(\infty) = \theta_2(\infty)$, where $\theta_1, \theta_2\ \varepsilon\ \underline{\underline{N}}^+$ define γ_1 and γ_2.

(ii) If $\gamma^{\pm} \varepsilon \Gamma^{\pm}$ are defined by $\theta(x \pm t)$, respectively, then the two representations $\pi^{\pm} = \pi_0 \cdot \gamma^{\pm}$ are unitarily inequivalent unless $\theta(\infty) = 0$.

<u>Proof</u>.

(i) Suppose, first, that $\theta_1(\infty) = \theta_2(\infty)$. Let $\theta(x) = \theta_1(x) - \theta_2(x)$. Then $\Theta(x,t) = \theta(x+t)\ \varepsilon\ \underline{\underline{M}}$. Hence, if $\Theta_j(x,t) = \theta_j(x+t)$, $j=1,2$,

$$W(\Theta)\gamma_1\big(W(\xi)\big)W(\Theta)^{-1} = W(\Theta)W(\xi)W(\Theta)^{-1}e^{i\{\Theta_1,\xi\}}$$

$$= W(\xi)e^{i\{\Theta_1-\Theta,\xi\}} = e^{i\{\Theta_2,\xi\}}W(\xi)$$

$$= \gamma_2\big(W(\xi)\big).$$

Since $\{W(\xi)|\xi\varepsilon\underline{\underline{M}}\}$ generates $\underline{\underline{A}}$, we have

$$W(\Theta)\gamma_1(A)W(\Theta)^{-1} = \gamma_2(A)$$

for all $A \varepsilon \underline{\underline{A}}$. Thus

$$\pi_0\cdot\gamma_2(A) = \pi_0\left(W(\Theta)\gamma_1(A)W(\Theta)^{-1}\right)$$

$$= \pi_0\left(W(\Theta)\right)\pi_0\cdot\gamma_1(A)\pi_0\left(W(\Theta)\right)^{-1}$$

and so $\pi_0\left(W(\Theta)\right) \varepsilon \pi_0(\underline{\underline{A}})$ provides the required unitary equivalence.

To prove the inequivalence for $\theta_1(\infty) \neq \theta_2(\infty)$, we shall show that $\{\Theta,\cdot\}$ is an unbounded functional on $\underline{\underline{M}}$, and use lemma 3.3.2.

More precisely, let $\gamma = \gamma_1\cdot\gamma_2^{-1}$. γ is defined by $\Theta(x,t) = \theta(x+t)$, where $\theta(\infty) \neq 0$; i.e. $\theta'\varepsilon \underline{\underline{D}}$ but $\theta'\not\varepsilon \underline{\underline{D}}_0$. (We write $\theta'(x)$ for $d\theta(x)/dx$, and $\tilde{\theta}'(p)$ is the Fourier transform of $\theta'(x)$, <u>not</u> the derivative of $\tilde{\theta}(p)$). Therefore $\tilde{\theta}'(0) \neq 0$. It follows, by the continuity of $\tilde{\theta}'(p)$, that there exists $\delta>0$ and $b>0$ such that, for $|p| < \delta$, we have that $|\mathrm{Re}\tilde{\theta}'(p)| > b$. (Since $\theta'(x)$ is real, $\tilde{\theta}'(0)$ is also real).

Now suppose $\{\Theta,\cdot\}$ were a bounded linear functional on $\underline{\underline{M}}$ - identified with a dense subset of K. Then the same is true for $\{\Theta,\cdot\}$ on the subset of elements of M of the form $\xi = (0,g)$. On such elements, $\{\Theta,\cdot\}$ is given by

$$\{\Theta,(0,g)\} = -\int \tilde{g}(p)\tilde{\theta}'(-p)\ dp$$

$$= -\int 2\omega\tilde{g}(p)\tilde{\theta}'(-p)\ d\Omega.$$

But, as g runs over $\underline{\underline{D}}$, $2\omega\tilde{g}(p)$ runs over a dense subset of a real subspace, K_R, of K, - the real subspace of hermitian functions, i.e. those with $\overline{h(p)} = h(-p)$. $\{\Theta,\cdot\}$ can be extended, therefore, by continuity, to the whole of this subspace. This implies (by Riesz' lemma) that $\tilde{\theta}'(\cdot) \in K_R$.

This is a contradiction, however, because, as remarked above, $|\mathrm{Re}\tilde{\theta}'(\cdot)| > b > 0$ in a neighbourhood of p=0, and so

$$\int|\tilde{\theta}'(p)|^2 d\Omega = \infty.$$

Hence, $\{\Theta,\cdot\}$ is unbounded on $\underline{\underline{M}}$. Thus, by lemma 3.3.2, γ is not implementable. But then it follows that $\pi_0\cdot\gamma_1$ and $\pi_0\cdot\gamma_2$ cannot be unitarily equivalent. This completes the proof of (i).

(ii) We consider $\gamma^+\cdot(\gamma^-)^{-1} = \gamma$, say. Then γ is defined by $\Theta(x,t) = \theta(x+t)-\theta(x-t)$.

Now,

$$\{\Theta,\xi\} = -\int \dot{\Theta}(x,0)g(x)dx$$

for ξ of the form (0,g),

$$= -2\int g(x)\theta'(x)dx$$

$$= -2\int 2\omega\tilde{g}(p)\tilde{\theta}'(-p)\ d\Omega$$

The proof now proceeds exactly as in (i). The unitary operator giving equivalence if $\theta(+\infty) = 0$ is $W(\Theta)$. QED.

Theorem 3.3.4.

Let $\gamma \varepsilon \Gamma^+$. Then restricted Poincaré transformations are implemented in the representation $\pi_0 \cdot \gamma$.

Proof.

Suppose γ is given by $\Theta(x,t) = \theta(x+t)$. Under $\{a,\Lambda\} \varepsilon \mathbb{P}_+^\uparrow$, $W(\xi)$ becomes $W(\xi_{a,\Lambda})$. Now, in the representation $\pi_0 \cdot \gamma$, these are represented by $e^{i\{\Theta,\xi\}}W(\xi)$ and $e^{i\{\Theta,\xi_{a,\Lambda}\}}W(\xi_{a,\Lambda})$, respectively; so we must prove the implementability of the map

$$e^{i\{\Theta,\xi\}}W(\xi) \rightarrow e^{i\{\Theta,\xi_{a,\Lambda}\}}W(\xi_{a,\Lambda})$$

for $\xi \varepsilon \underline{\underline{M}}$, by operators in $\underline{\underline{H}}_0$:

i.e. the map

$$W(\xi) \rightarrow e^{i\{\Theta,\xi_{a,\Lambda}-\xi\}}W(\xi_{a,\Lambda}).$$

Now, $U_0(a,\Lambda)$ implements $W(\xi) \rightarrow W(\xi_{a,\Lambda})$, so we must implement

$$W(\xi_{a,\Lambda}) \rightarrow W(\xi_{a,\Lambda})e^{i\{\Theta,\xi_{a,\Lambda}-\xi\}}$$

$$= W(\xi_{a,\Lambda})e^{i\{\Theta,\xi_{a,\Lambda}\}}e^{-i\{\Theta_{a,\Lambda},\xi_{a,\Lambda}\}}$$

since $\{\cdot,\cdot\}$ is $\mathbb{P}_+^\uparrow$ invariant,

$$= W(\xi_{a,\Lambda})e^{i\{\Theta-\Theta_{a,\Lambda},\xi_{a,\Lambda}\}}.$$

But ξ is arbitrary, so we must prove that

$$W(\xi) \to W(\xi)e^{i\{\Theta-\Theta_{a,\Lambda},\xi\}}$$

is implemented for any $\{a,\Lambda\} \in \mathbb{P}_{+}^{\uparrow}$.

By theorem 3.3.3 (i), this is true if $\Theta(\infty,0) = \Theta_{a,\Lambda}(\infty,0)$, and $\Theta(-\infty,0) = \Theta_{a,\Lambda}(-\infty,0) = 0$. Any Lorentz transformation Λ takes the form $x+t \to \alpha(x+t)$, $x-t \to \alpha^{-1}(x-t)$ for some $\alpha>0$. Therefore, if $\{a,\Lambda\} = \{(a^0,a^1),\Lambda(\alpha)\}$, then $\Theta_{a,\Lambda}(x,t) = \theta(\alpha(x+t)+a^0+a^1)$. Clearly, $\Theta_{a,\Lambda}(\infty,0) = \theta(\infty) = \Theta(\infty,0)$ etc. QED.

Remark 1.

Space and time inversions are not implemented except in π_0. This follows because, according to theorem 3.3.3, $\Theta(-x,t)$ and $\Theta(x,-t)$ define representations inequivalent to that defined by $\Theta(x,t)$.

Remark 2.

Since, for any region $\underline{O}$, there are functions in $\underline{N}(\underline{O})$, we see that $\Gamma(\underline{O})$ is non-trivial however small $\underline{O}$ is.

Theorem 3.3.5.

Let $\gamma\in\Gamma^{+}$. Then π_0 and $\pi_0\cdot\gamma$ are strongly locally equivalent.

Proof.

Let $\underline{O}$ be a given region. If $\gamma\in\Gamma^{+}(\underline{O})$, then, by lemma 3.2.7, $\gamma \upharpoonright \underline{\underline{A}}(\underline{\underline{O}}') = \iota$ and there is nothing to prove. If $\gamma \notin \Gamma^+(\underline{\underline{O}})$, the idea of the proof is to move γ into $\underline{\underline{O}}$ by a unitarily implementable automorphism. Let $\gamma \varepsilon \Gamma^+(\underline{\underline{O}}_1)$, and suppose γ is defined by $\Theta_1 \varepsilon\ \underline{\underline{N}}^+(\underline{\underline{O}}_1)$. Let $\Theta_2 \varepsilon\ \underline{\underline{N}}^+(\underline{\underline{O}})$ with $\Theta_2(\infty,0) = \Theta_1(\infty,0)$, and let γ_2 be the automorphism corresponding to $\Theta_2(x,t)$.

According to theorem 3.3.3 (i), $\gamma \cdot \gamma_2^{-1}$ is implemented; so $\pi_0 \simeq \pi_0 \cdot \gamma \cdot \gamma_2^{-1}$.

But $\gamma_2^{-1} \varepsilon \Gamma^+(\underline{\underline{O}})$, (it corresponds to $-\Theta_2(x,t)$), and so $\gamma_2^{-1} \upharpoonright \underline{\underline{A}}(\underline{\underline{O}}') = \iota$.

Therefore $\pi_0 \upharpoonright \underline{\underline{A}}(\underline{\underline{O}}') \simeq \pi_0 \cdot \gamma \upharpoonright \underline{\underline{A}}(\underline{\underline{O}}')$. QED.

Theorem 3.3.6.

If $\gamma \varepsilon \Gamma^+$, then the operators in $\pi_0 \cdot \gamma$ implementing $\mathbb{P}_+^\uparrow$ may be chosen so as to give a strongly continuous unitary representation of $\mathbb{P}_+^\uparrow$ with energy-momentum spectrum in the closed forward light-cone.

Proof.

Suppose that γ is given by $\Theta(x,t) = \theta(x+t)$. $\pi(\underline{A}) = \pi_0 \cdot \gamma(\underline{A})$ is an algebra of operators acting in $\underline{\underline{H}}_0$. The action of $\mathbb{P}_+^\uparrow$ in π is to map the operator $e^{i\{\Theta,\xi\}} W(\xi)$ into the operator $e^{i\{\Theta,\xi_g\}} W(\xi_g)$, $g \varepsilon\ \mathbb{P}_+^\uparrow$. We saw in the proof of theorem 3.3.4, that this action is implemented by $V(g) = U_0(g) W(\Theta - \Theta_{g^{-1}})$, where U_0 is the representation of $\mathbb{P}_+^\uparrow$ in π_0. The phase of $V(g)$ is arbitrary. The Weyl relations imply that

$$V(g)V(h) = U_0(g)W(\theta-\theta_{g^{-1}})U_0(h)W(\theta-\theta_{h^{-1}})$$

$$= U_0(gh)W(\theta_{h^{-1}}-\theta_{h^{-1}g^{-1}})W(\theta-\theta_{h^{-1}})$$

$$= U_0(gh)W(\theta-\theta_{h^{-1}g^{-1}})\exp-\tfrac{1}{2}i\{\theta_{h^{-1}}-\theta_{h^{-1}g^{-1}},\theta-\theta_{h^{-1}}\}$$

$$= V(gh)\exp-\tfrac{1}{2}i\{\theta,\theta_h\}\exp\tfrac{1}{2}i\{\theta,\theta_{gh}\}\exp-\tfrac{1}{2}i\{\theta,\theta_g\}$$

using the invariance of $\{\cdot,\cdot\}$ under $\mathbb{P}_+^\uparrow$.

Thus

$$U_1(g) \equiv U_0(g)W(\theta-\theta_{g^{-1}})\exp\tfrac{1}{2}i\{\theta,\theta_g\}$$

is a strongly continuous unitary representation of $\mathbb{P}_+^\uparrow$ in π, implementing the Poincaré transformations.

To show that the spectrum condition holds, we shall compute the generators of time and space translations.

Let s be a time translation. Then

$$\theta_s(x,t) = \theta(x,t-s) = \theta(x+t-s),$$

and so

$$U_1(s) = U_0(s)\exp i(\overline{\phi(\theta'-\theta'_{-s}) - \pi(\theta-\theta_{-s})})\exp\tfrac{1}{2}i\{\theta,\theta_s\}.$$

The generator is the sum of the strong derivatives w.r.t. s, at $s=0$, of the three unitary operators appearing in the expression (divided by i) :-

there being a common dense domain for these operators, viz, those elements of $\underline{\underline{H}}_0'$ with rapid decrease for large arguments, or the one-dimensional analogue of Σ in lemma 2.5.2.

Now,

$$\phi(\theta'-\theta'_{-s}) - \pi(\theta-\theta_{-s})$$

$$= 2^{\frac{1}{2}}(a^*(F_-)+a(F_+)) - 2^{\frac{1}{2}}i(a^*(G_-)-a(G_+))$$

where $F_\pm(p) = |p|^{-\frac{1}{2}}(\theta'-\theta'_{-s})^\sim(\pm p)$,

and $G_\pm(p) = |p|^{\frac{1}{2}}(\theta-\theta_{-s})^\sim(\pm p)$.

Thus, the strong derivative, at s=0, (divided by i), of the middle term in the expression for U_1 is, on Σ,

$$2^{-\frac{1}{2}}\int\left(-a^*(p)\tilde{\theta}'(-p)i\frac{\sqrt{\omega}}{p}(\omega-p) + a(p)\tilde{\theta}'(p)i\frac{\sqrt{\omega}}{p}(\omega-p)\right)dp$$

where $\omega=|p|$, and we have used the forms $a^*(p)$ and $a(p)$ together with a symbolic integration.

The strong derivatives of $U_0(s)$ and $\exp\frac{1}{2}i\{\Theta,\Theta_s\}$ at s=0, are given on Σ respectively by

$$H_0 = \int\omega a^*(p)a(p)dp,$$

and $\int \tilde{\theta}'(-p)\tilde{\theta}'(p)dp$.

Hence $U_1(s)$ has generator

$$\int\Big\{\omega a^*(p)a(p) - 2^{-\frac{1}{2}}ia^*(p)\tilde{\theta}'(-p)\sqrt{\omega}\left(\frac{\omega-p}{p}\right)$$

$$+ 2^{-\frac{1}{2}}ia(p)\tilde{\theta}'(p)\sqrt{\omega}\left(\frac{\omega-p}{p}\right) + \tilde{\theta}'(p)\tilde{\theta}'(-p)\Big\}dp$$

$$= \int \omega b^*(p) b(p) dp$$

where

$$b^*(p) \equiv a^*(p) + 2^{-\frac{1}{2}} i \frac{\tilde{\theta}'(p)}{\sqrt{\omega}} \frac{(\omega - p)}{p}$$

$$b(p) \equiv a(p) - 2^{-\frac{1}{2}} i \frac{\tilde{\theta}'(-p)}{\sqrt{\omega}} \frac{(\omega - p)}{p} .$$

In the same way, we find (using the fact that the generator for space-translations $U_0(a)$ is $-\int pa^*(p)a(p)dp$) that the generator of space-translations is given by

$$- \int pb^*(p) b(p) dp.$$

Evidently,

$$\int \omega b^*(p) b(p) dp \geq \left| \int pb^*(p) b(p) dp \right|$$

i.e.

$$(h, \int \omega b^*(p) b(p) dp \; h) \geq \left| (h, \int pb^*(p) b(p) dp \; h) \right|$$

for h in the domain of the operators in question. This is just a statement of the fact that the energy-momentum spectrum lies in the closed forward light-cone. QED.

Remark 1.

This result is not surprising. Indeed, the transformation γ has the effect of taking

$$\phi(x) \to \phi(x) + \theta(x)$$

and

$$\pi(x) \to \pi(x) + \theta'(x).$$

In terms of $a^*(p)$ and $a(p)$, this is effected by the transformation

$$a^*(p) \to b^*(p) \text{ and } a(p) \to b(p),$$

where b and b^* are as in the previous proof. Thus, in as much as γ can be extended to unbounded operators, we expect that $\int \omega a^*(p)a(p)dp$ and $-\int pa^*(p)a(p)dp$ transform into $\int \omega b^*(p)b(p)dp$ and $-\int pb^*(p)b(p)dp$, respectively, and we would expect these latter to implement time and space translations in the representation $\pi_0 \cdot \gamma$, as is, in fact, the case.

Remark 2.

The vacuum in the representation π_0, viz, $\Omega = (1,0,0,..) \in \underline{\underline{H}}_0$, defines a vector state in the representation $\pi = \pi_0 \cdot \gamma$. This state no longer has zero energy;

$$(\Omega, H_1\Omega) = \tfrac{1}{2}\int\left(\dot{\Theta}(x,0)^2 + \left(\frac{d\Theta}{dx}(x,0)\right)^2\right)dx$$

where H_1 is the generator of time translations $U_1(t)$. This means that the phase of U_1 is chosen such that the average energy of the vector state Ω is equal to the classical energy of the solution $\Theta(x,t)$ of the wave equation.

Remark 3.

It is of interest to note that the representation

$U_1(a,\Lambda)$, restricted to the cyclic subspace generated by Ω, is infinitely-divisible (41). This follows by applying the criterion of Streater (41).

§3.4. The Sectors and Charged Fields.

We now turn to the definition of the sectors, and the charge carrying fields. As we saw in the last section, two representations π_1 and π_2, given by Θ_1 and $\Theta_2 \varepsilon \underline{\underline{N}}^+$, are unitarily equivalent if and only if $\theta_1(\infty) = \theta_2(\infty)$. An equivalence class of such representations will be called a sector, labelled by $\theta(\infty)$, called the charge of the sector, taking values in IR. For example, π_0 has charge zero.

The charged fields will be defined as unitary transformations from one sector to another. We shall see that these will anticommute if they have a certain charge, or multiples of that charge. There will be fields that are neither Bose nor Fermi, as we expect from our form of lemma 2.3 of (20); viz, if $\gamma_1,\gamma_2 \varepsilon \Gamma^+$ are localised in space-like separated convex regions, and correspond to the same sector, of charge α, say, and if U is such that

$$\gamma_2(\cdot) = U\gamma_1(\cdot)U^{-1},$$

then $\gamma_1(U) = e^{\pm i\alpha^2}U$. The sign of the phase depends on whether γ_1 is localised to the right or left of γ_2. That $\gamma(U) \neq \pm U$ is possible, is due to the fact that we have only one space dimension. Indeed, in three-

-dimensional space, only $\gamma(U) = \pm U$ is possible (20).

We select an arbitrary, but fixed, $\Theta \in \underline{\underline{N}}^+$, with $\Theta(\infty,0) = \theta(\infty) = 1$. For each $\alpha \in \mathbb{R}$, we shall write Θ_α for $\alpha\Theta \in \underline{\underline{N}}^+$. Such a Θ_α, and its corresponding automorphism, γ_α, will be called standard.

For each $\alpha \in \mathbb{R}-\{0\}$, let $\underline{\underline{H}}_\alpha$ be a Hilbert space isomorphic to $\underline{\underline{H}}_0$. Thus we have a family of copies of $\underline{\underline{H}}_0$, indexed by $\mathbb{R}$.

Let ψ^*_α be an isometric operator from $\underline{\underline{H}}_0$ onto $\underline{\underline{H}}_\alpha$, with $\psi^*_0 = \mathbb{1}_{\underline{\underline{H}}_0}$. Define the representation π_α of $\underline{\underline{A}}$ on $\underline{\underline{H}}_\alpha$ by

$$\pi_\alpha(A) = \psi^*_\alpha \pi_0 \cdot \gamma_\alpha(A)\psi_\alpha \qquad A \in \underline{\underline{A}},$$

where $\psi_\alpha : \underline{\underline{H}}_\alpha \to \underline{\underline{H}}_0$ is the inverse of ψ^*_α.
By theorem 3.3.3(i), $(\pi_\alpha,\underline{\underline{H}}_\alpha)$ and $(\pi_\beta,\underline{\underline{H}}_\beta)$ are unitarily inequivalent if $\alpha\neq\beta$. Define

$$\underline{\underline{H}} \equiv \bigoplus_\alpha \underline{\underline{H}}, \text{ and } \pi \equiv \bigoplus_\alpha \pi_\alpha .$$

$\underline{\underline{H}}$ will be the Hilbert space in which the field algebra will be defined. If $U_\alpha(a,\Lambda)$ represents $\mathbb{P}^\uparrow_+$ in $(\pi_\alpha,\underline{\underline{H}}_\alpha)$, then $\underline{\underline{H}}$ carries the representation $\bigoplus_\alpha U_\alpha$, which satisfies the spectrum condition because the U_α do.

We define ψ^*_α on $\underline{\underline{H}}$ by linear extension and continuity of the map

$$\psi^*_\alpha \restriction \underline{\underline{H}}_\beta = \psi_{\alpha+\beta}{}^*\psi_\beta \restriction \underline{\underline{H}}_\beta , \quad \forall\ \beta \in \mathbb{R}.$$

Clearly, ψ_α^* is a unitary operator in $\underline{\underline{H}}$; it is the charged field operator corresesponding to the standard automorphism γ_α. We can now extend the definition to any $\gamma_\mu \varepsilon\ \Gamma^+$, corresponding to $M\ \varepsilon\ \underline{\underline{N}}^+$, with $M(x,t) = \mu(x+t)$ and $\mu(\infty) = \alpha$.

Definition 3.4.1.

We define the representation π_μ of $\underline{\underline{A}}$ on $\underline{\underline{H}}$ by

$$\pi_\mu(A) \equiv \psi_\alpha^* \pi_0 \cdot \gamma_\mu(A) \psi_\alpha \quad \text{for } A\ \varepsilon\ \underline{\underline{A}}$$

and the field $\psi^*(\mu)$ with charge $\alpha = \mu(\infty)$ by extension of

$$\psi^*(\mu) \upharpoonright \underline{\underline{H}}_\beta \equiv \pi_{\alpha+\beta}(W(M-\Theta_\alpha))\psi_\alpha^* \upharpoonright \underline{\underline{H}}_\beta$$

for each $\beta\ \varepsilon\ \mathbb{R}$, where $\pi_{\alpha+\beta}$ is a standard representation.

We understand this definition as follows : ψ_α^* acting on $\underline{\underline{H}}_\beta$ creates a standard charge in $\underline{\underline{H}}_{\alpha+\beta}$. We can change this standard state to the required state, determined by M, by an element of $\underline{\underline{A}}$, namely $W(M-\Theta_\alpha)$; and this must be done in the representation $\pi_{\alpha+\beta}$.

$\psi^*(\mu)$ is a unitary operator in $\underline{\underline{H}}$.

Lemma 3.4.2.

Let $\underline{\underline{O}}$ be any given space-time region, and let $M\ \varepsilon\ \Gamma^+(\underline{\underline{O}})$. Then $\psi^*(\mu)$ commutes with $\pi(\underline{\underline{A}}(\underline{\underline{O}}'))$.

Proof.

It suffices to prove this on each $\underline{\underline{H}}_\beta$, since, by construction, π is reduced by $\underset{\beta}{\oplus} \underline{\underline{H}}_\beta$.

Let $A \in \underline{\underline{A}}(\underline{\underline{O}}')$, and suppose that $\mu(\infty) = \alpha$.

Then

$$\psi^*(\mu)\pi(A) \restriction \underline{\underline{H}}_\beta = \psi^*(\mu)\pi_\beta(A)$$

$$= \pi_{\alpha+\beta}\big(W(M-\Theta_\alpha)\big)\psi^*_\alpha\psi^*_\beta\pi_0\big(\gamma_\beta(A)\big)\psi_\beta$$

$$= \psi^*_{\alpha+\beta}\ \pi_0\big(\gamma_{\alpha+\beta}(W)\big)\ \psi_{\alpha+\beta}\psi^*_{\alpha+\beta}\ \pi_0\big(\gamma_\beta(A)\big)\psi_\beta$$

(writing W for $W(M-\Theta_\alpha)$),

$$= \psi^*_{\alpha+\beta}\ \pi_0\big(\gamma_\beta\big(\gamma_\alpha(W)A\big)\big)\psi_\beta$$

$$= \psi^*_{\alpha+\beta}\ \pi_0\big(\gamma_\beta\big(\gamma_\alpha(W)\gamma_\mu(A)\big)\big)\psi_\beta$$

since $\gamma_\mu \restriction \underline{\underline{A}}(\underline{\underline{O}}') = \iota$ by lemma 3.2.7,

$$= \psi^*_{\alpha+\beta}\ \pi_0\{\gamma_\beta\big(\gamma_\alpha(W)\gamma_\mu(A)\gamma_\alpha(W)^{-1}\gamma_\alpha(W)\big)\}\psi_\beta.$$

But $\gamma_\alpha(W) = e^{i\nu}W$ for some $\nu \in \mathbb{R}$, and so

$$\gamma_\alpha(W)\gamma_\mu(A)\gamma_\alpha(W)^{-1} = W\gamma_\mu(A)W^{-1}$$

$$= W(M-\Theta_\alpha)\gamma_\mu(A)W(M-\Theta_\alpha)^{-1}$$

$$= \gamma_\alpha(A).$$

Hence

$$\psi^*(\mu)\pi(A) \restriction \underline{\underline{H}}_\beta = \psi^*_{\alpha+\beta}\ \pi_0\{\gamma_\beta\big(\gamma_\alpha(A)\gamma_\alpha(W)\big)\}\ \psi_\beta$$

$$= \psi^*_{\alpha+\beta}\ \pi_0\big(\gamma_{\alpha+\beta}(AW)\big)\ \psi_\beta$$

$$= \psi^*_{\alpha+\beta}\ \big(\pi_0\cdot\gamma_{\alpha+\beta}(A)\big)\big(\pi_0\cdot\gamma_{\alpha+\beta}(W)\big)\ \psi_\beta$$

$$= \pi_{\alpha+\beta}(A)\ \pi_{\alpha+\beta}(W)\ \psi^*_\alpha \restriction \underline{\underline{H}}_\beta$$

$$= \pi(A)\ \psi^*(\mu) \restriction \underline{\underline{H}}_\beta \qquad \text{QED.}$$

Definition 3.4.3.

We define $\mathbb{F}^+(\underline{\underline{O}})$ to be the von Neumann algebra generated by the set

$$\{\psi^*(\mu)\,|\,\mu \leftrightarrow M \epsilon \underline{\underline{N}}^+(\underline{\underline{O}})\} \cup \{\pi(A)\,|\,A \epsilon \underline{\underline{A}}(\underline{\underline{O}})\}.$$

By lemma 3.4.2, we see that

$$[\mathbb{F}^+(\underline{\underline{O}}_1), \pi(A(\underline{\underline{O}}_2))] = 0$$

if $\underline{\underline{O}}_1$ and $\underline{\underline{O}}_2$ are space-like separated regions.

Lemma 3.4.4.

Let $\underline{\underline{O}}_1$ and $\underline{\underline{O}}_2$ be space-like separated convex regions in Minkowski space. If $M_1 \epsilon\ \underline{\underline{N}}^+(\underline{\underline{O}}_1)$ and $M_2 \epsilon\ \underline{\underline{N}}^+(\underline{\underline{O}}_2)$, then

$$\psi^*(\mu_1)\psi^*(\mu_2) = \psi^*(\mu_2)\psi^*(\mu_1)e^{i\nu}$$

where $M_j(x,t) = \mu_j(x,t)$ and $\nu = \pm\mu_1(\infty)\mu_2(\infty)$ according as to whether $\underline{\underline{O}}_1$ is to the left or the right of $\underline{\underline{O}}_2$.

Proof.

Let $\mu_1(\infty) = \alpha$, and $\mu_2(\infty) = \beta$, and consider $\psi^*(\mu_1)\psi^*(\mu_2)$ on any subspace $\underline{\underline{H}}_\varepsilon$ of $\underline{\underline{H}}$.

$$\psi^*(\mu_1)\ \psi^*(\mu_2) \restriction \underline{\underline{H}}_\varepsilon$$

$$= \pi_{\alpha+\beta+\varepsilon}(W(M_1-\Theta_\alpha))\ \psi^*_\alpha\ \pi_{\varepsilon+\beta}(W(M_2-\Theta_\beta))\ \psi^*_\beta$$

$$= \psi^*_{\alpha+\beta+\varepsilon}\ \pi_0\{\gamma_{\alpha+\beta+\varepsilon}(W(M_1-\Theta_\alpha))\gamma_{\varepsilon+\beta}(W(M_2-\Theta_\beta))\}\ \psi_\varepsilon$$

$$= \psi^*_{\alpha+\beta+\varepsilon}\ \pi_0\{W(M_1-\Theta_\alpha)W(M_2-\Theta_\beta)e^{iX}\}\ \psi_\varepsilon$$

where

$$X = \{\Theta_{\alpha+\beta+\varepsilon}, M_1-\Theta_\alpha\} + \{\Theta_{\beta+\varepsilon}, M_2-\Theta_\beta\}.$$

Now, $\{\Theta_i, \Theta_j\} = 0$ for any standards Θ_i and Θ_j, as they are proportional. Thus

$$X = \{\Theta_{\alpha+\beta+\varepsilon}, M_1\} + \{\Theta_{\beta+\varepsilon}, M_2\}$$

and so

$$W(M_1-\Theta_\alpha)W(M_2-\Theta_\beta)e^{iX} = W(M_1+M_2-\Theta_\alpha-\Theta_\beta)e^{iY}$$

where $Y = \frac{1}{2}\{M_2, M_1\} + \frac{1}{2}\{M_1, \Theta_\beta\} + \frac{1}{2}\{\Theta_\alpha, M_2\}$

$$+ \{\Theta_{\alpha+\beta+\varepsilon}, M_1\} + \{\Theta_{\beta+\varepsilon}, M_2\}$$

$$= \{\Theta_{\alpha+\beta+\varepsilon}, M_1+M_2\} - \tfrac{1}{2}\{\Theta_\alpha, M_2\}$$

$$- \tfrac{1}{2}\{M_1, M_2\} - \tfrac{1}{2}\{\Theta_\beta, M_1\}.$$

Interchanging M_1 and M_2 and α and β, we deduce that

$$\psi^*(\mu_1)\ \psi^*(\mu_2) = \psi^*(\mu_2)\ \psi^*(\mu_1)e^{i\nu}$$

where $\nu = \{M_1, M_2\}$

$$= \int\left(M_1(x,0)\dot{M}_2(x,0) - \dot{M}_1(x,0)M_2(x,0)\right)dx$$

$$= \int\left(\mu_1(x)\mu_2'(x) - \mu_1'(x)\mu_2(x)\right)dx$$

$$= \pm\ \mu_1(\infty)\ \mu_2(\infty) \qquad \text{as required.}$$

Remark.

If $\mu_1(\infty)$ and $\mu_2(\infty)$ are of the form $\sqrt{(2n+1)\pi}$ for some integer n, then $\psi^*(\mu_1)$ and $\psi^*(\mu_2)$ anticommute at space-like separation.

So far we have only considered Γ^+, for convenience. In the same way, the group Γ leads to a family of inequivalent representations π_{α_1,α_2}, corresponding to $M \in \underline{\underline{N}}$, where $M(x,t) = M^+(x,t) + M^-(x,t)$, and $M^+(\infty,0) = \alpha_1$, $M^-(\infty,0) = \alpha_2$.

The charged field, acting on $\underline{\underline{H}}_{\beta_1,\beta_2}$, is

$$\psi^*(\mu_1,\mu_2) = \pi_{\alpha_1+\beta_1,\alpha_2+\beta_2}\big(W(M-\Theta_{\alpha_1,\alpha_2})\big)\ \psi^*_{\alpha_1,\alpha_2},$$

where

$$M^+(x,t) = \mu_1(x+t)\ , \quad M^-(x,t) = \mu_2(x-t)\ ,$$

$$\Theta_{\alpha_1,\alpha_2}(x,t) = \alpha_1\theta_1(x+t) + \alpha_2\theta_2(x-t),$$

for some "standards" θ_j; and $\psi^*_{\alpha_1,\alpha_2}$ is the standard charged field.

There is an analogue of lemma 3.4.4, but the expression for ν is not quite so simple.

Just as in definition 3.4.3, the field algebra, $\mathbb{F}$, is defined to be the C*-algebra generated by $\pi(A)$ and the ψ^*'s. (π being the direct sum of the various inequivalent representations).

We have called the ψ^*'s charge carrying fields -

- we cannot think of them as particles :e.g. the "two-particle" state $\psi^*_\alpha \psi^*_\beta \Omega$ is the same as the "one-particle" state $\psi^*_{\alpha+\beta}\Omega$.

We can define a gauge group on $\oplus_{\iota,\kappa} \underline{\underline{H}}_{\iota,\kappa}$ as $G = X_{\alpha,\beta} \underline{T}_{\alpha,\beta}$, where $(\alpha,\beta) \in [0,1) \times [0,1)$ and $\underline{T}_{\alpha,\beta}$ is the torus, for each (α,β). ($\underline{\underline{H}}_{\iota,\kappa}$ is the representation space for the irreducible representation $\pi_{\iota,\kappa}$ of $\underline{\underline{A}}$). G acts on $\oplus_{\iota,\kappa} \underline{\underline{H}}_{\iota,\kappa}$ as follows;

$$g_{\alpha,\beta} \restriction \underline{\underline{H}}_{\iota,\kappa} \equiv e^{img_{a,0}} e^{ing_{0,b}}$$

where $m,n \in \mathbb{Z}$, $a,b \in [0,1)$ are given by

$$\iota = m + a \ , \qquad \kappa = n + b \ .$$

$g_{a,0}$ and $g_{0,b}$ are the $(a,0)$ and $(0,b)$ components of $g_{\alpha,\beta} \in G$, respectively.

This ties in with the ideas of Doplicher, Haag and Roberts (19), where it is shown that, without loss of generality, the gauge group may be chosen to be compact, and that the physical spectrum of $\underline{\underline{A}}$, i.e. the representations of $\underline{\underline{A}}$ occuring in $\mathbb{F}$, is in one-one correspondence with the irreducible representations occuring in the gauge group. Indeed, we have defined G so that this is the case.

Let us also remark that, contrary to the philosophy that fermions must be involved in the basic formalism (42), we have constructed such from the observable algebra, $\underline{\underline{A}}$, given in the charge zero sector.

4. The Time Evolution of Quantised Fields.

So far we have considered only free systems. The purpose of this last chapter is to discuss the time evolution corresponding to a certain class of interactions. These will be rather "mild", and, in general, non-local. Nevertheless, it is felt that this may be a "half-way" step to more realistic theories. Indeed, we can define, quite rigorously, a time evolution corresponding to a four-fermion interaction, provided the interaction is given by smeared fermions. We can also define a time evolution corresponding to an interaction of the form

$$\frac{1}{1 + \int f(x):\phi^{2n}(x):dx}$$

a "smeared" version of interaction densities recently considered by Efimov and Salam and co-workers(43,44).

However, it still remains to investigate the "unsmeared" version of these theories; i.e. the limit as all test-functions become δ-functions.

It may also be possible to treat two space-time dimensional models such as $:\phi^{2n}:$ by putting in a spectral cut-off on the self-adjoint operator $\int g(x):\phi^{2n}(x):dx$, obtaining a time evolution automorphism by our rather elementary methods, and then taking the limit (in some sense) as this spectral cut-off is removed. This could be an alternative to the methods of Glimm and Jaffe, or those of Segal, and

Simon and Høegh-Krohn (45). The idea of putting in a spectral cut-off appears in Guenin's lectures at Colorado in 1966 (46).

§4.1. The Guenin-Interaction Picture.

The conventional approach to study interactions is to use the so-called interaction (or Dirac) picture of the time-evolution (see e.g. (25)). This is related to the Heisenberg picture as follows :

$$\psi_I(t) = e^{iH_0 t} e^{-iHt} \psi_H$$

$$A_I(t) = e^{iH_0 t} A_H(O) e^{-iH_0 t}$$

where the subscripts refer to the "picture", and H_0 and H are the "free" and "interacting" Hamiltonians respectively. H is given as a sum of H_0 and V, where V represents the interaction. It is then usual to "solve" the equation

$$\frac{d}{dt}\psi_I(t) = -i\, e^{iH_0 t} V e^{-iH_0 t}\ \psi_I(t)$$

for $\psi_I(t)$, by iteration - the perturbation expansion. Unfortunately, in many cases of interest, the formal object V has little or no mathematical meaning (47), the operator H having even less!

Even if we can give a meaning to H as an operator, there is no reason to expect the perturbation series to converge; in fact, in some cases it has been shown to diverge (48). Segal has suggested that, in order to avoid the divergences of quantum field theory, one might take the time-evolution as an automorphism of the algebra of observables, not necessarily spatial (49). The operative question, however, is how can one define these automorphisms? We shall follow Guenin in considering the "inverse" interaction picture. Guenin's idea (46) was to let the states evolve trivially, rather than the operators as in the usual interaction picture. Thus;

$$\psi_G(t) = e^{-iH_0 t}\psi_H$$

and therefore

$$A_G(t) = e^{-iH_0 t} e^{iHt}\, A\, e^{-iHt} e^{iH_0 t}$$

The Heisenberg picture operators are thus

$$A_H(t) = e^{iH_0 t}\, A_G(t)\, e^{-iH_0 t}.$$

The point is that the map

$$A \equiv A_G(0) \to A_G(t)$$

may have a well-defined meaning, even though the "unitary" operators e^{iHt} may not exist.

We will show this to be the case for our "mild" interactions; this extends to arbitrary dimension Guenin's result (46). (Actually, we shall only consider four dimensions, but the extension is trivial).

We should point out that the time-evolution, as an automorphism group, has been successfully illustrated for the Heisenberg ferromagnet (50), certain fermion systems (51), and quantum spin systems (52,53).

We shall take a slight variation of the axioms 1-5 of §1.2. Explicitly, we shall adopt axioms 1, 2, and 3 of §1.2, together with

Axiom 4'

We are given a continuous homomorphism τ_0, from $\mathbb{R}^4$ into Aut $\underline{\underline{A}}$, the automorphism group of $\underline{\underline{A}}$ (furnished with ths strong topology of operators on a Banach space), such that

$$\tau_0(a)\underline{\underline{A}}(\underline{\underline{O}}) = \underline{\underline{A}}(\underline{\underline{O}}_a)$$

where $\underline{\underline{O}}_a = \{x \in \mathbb{R}^4 \mid x-a \in \underline{\underline{O}}\}$.

Thus, the continuity means that

$$\|\tau_0(a)A - A\| \to 0$$

as $a \to 0$ in $\mathbb{R}^4$, for each $A \in \underline{\underline{A}}$.

We shall call τ_0 the free field dynamics.

Examples. 1.

The $\underline{\underline{A}}(\underline{\underline{O}})$, $\underline{\underline{A}}$ defined in §2.1 for the massive neutral boson field, together with

$$\tau_0(a)(\cdot) = U(a)(\cdot)U(a)^{-1}$$

$a \varepsilon \mathbb{R}^4$, do not satisfy the continuity requirement, axiom 4'. We must "resmear" the algebras.

If $A \varepsilon \underline{\underline{A}}$, and $f \varepsilon \underline{\underline{D}}(\mathbb{R}^4)$, define

$$A(f) = \int f(x)\tau_0(x)A\, d^4x,$$

where the integral is a strong integral on Fock space. Then

$$\begin{aligned}
\|\tau_0(a)A(f)-A(f)\| &= \left\| \int f(x)\tau_0(a+x)A d^4x - \int f(x)\tau_0(x)A d^4x \right\| \\
&= \left\| \int \big(f(x-a) - f(x)\big)\, \tau_0(x)A\, d^4x \right\| \\
&\leq \int |f(x-a) - f(x)|\, \|\tau_0(x)A\|\, dx \\
&= \|A\| \int |f(x-a) - f(x)|dx \\
&\rightarrow 0 \text{ as } a \rightarrow 0, \text{ as required.}
\end{aligned}$$

The algebras $\underline{\underline{A}}(\underline{\underline{O}})$ must now be adjusted to take account of the finite size of the support of the various f's. (This corrects an error in (54)).

2. In two space-time dimensions, we could do the above modification for the case when $\tau_0(\cdot)$ is the space-time automorphism group of Glimm and Jaffe corresponding to a $:\phi^{2n}:$ interaction.

3. The usual algebra generated by even powers of a

free Dirac field satisfies our requirement. (The continuity condition of axiom 4' holds because of the bounded nature of fermion fields.)

§4.2. The Cut-Off Interaction.

Let $V \in \underline{\underline{A}}(\underline{\underline{O}}_1)$ be such that $V = V^*$, and write $V_{\underline{a}}$ for $\tau_0(0,\underline{a})V$, and $V(t)$ for $\tau_0(-t,0)$. The minus sign appears here, rather than in many places in the sequel.

We would like to consider the translationally invariant interaction given by $\int V_{\underline{a}} d^3a$. However, this will not generally exist as an element of $\underline{\underline{A}}$. We must first introduce a space cut-off, and remove it, in some sense, later on.

Let us define the cut-off interaction

$$V_r = \int_{|\underline{a}| \leq r} V_{\underline{a}} d^3a$$

as a Riemann-Bochner integral (55), and, so defined, it lies in $\underline{\underline{A}}$. We have that

$$\tau_0(-t,0)V_r \equiv V_r(t) = \int_{|\underline{a}| \leq r} V_{\underline{a}}(t) d^3a.$$

This perturbs the dynamics $\tau_0(t)$ and defines a new one-parameter group of automorphisms. We shall show that the limit as $r \to \infty$ exists, and defines a space-translation invariant time-evolution.

As we have already said, Guenin defines

$$A_G(t) = e^{-iH_0t}e^{iHt} A e^{-iHt}e^{iH_0t},$$

which, when "differentiated" and "solved" by iteration, yields

$$A_G(t) = A + i\int_0^t ds \left[e^{-iH_0s}Ve^{iH_0s}, A\right] + \ldots .$$

where $H = H_0 + V$. This suggests the following definition of the (Guenin) interaction picture evolution corresponding to the interaction V_r ;

$t \in \mathbb{R} \; : t \to \tau_r^I(t)$

$$\tau_r^I(t)A \equiv A + i\int_0^t dt_1[V_r(t_1), A]$$

$$+ i^2\int_0^t dt_1\int_0^{t_1} dt_2[V_r(t_1),[V_r(t_2), A]]$$

$$+ \ldots\ldots$$

for any $A \in \underline{\underline{A}}$.

Now, τ_0 is an automorphism, and so

$$\|V_r(t)\| = \|V_r\| \leq \int_{|\underline{a}| \leq r} \|V_{\underline{a}}\| d^3a = \frac{4}{3}\pi r^3\|V\|$$

for all t. The $(n+1)^{th}$ term of $\tau_r^I(t)A$ is a multiple commutator which can be expanded to give 2^n various permutations of $V_r(t_1)\ldots V_r(t_n)A$.

Its norm is therefore bounded above by

$$2^n \, \|V_r\|^n \, \|A\| \, \frac{|t|^n}{n!}$$

It follows that the series converges in $\underline{\underline{A}}$ in norm, for all $t \in \mathbb{R}$. $\tau_r^I(t)$ is thus a well-defined map : $\underline{\underline{A}} \to \underline{\underline{A}}$ for all $t \in \mathbb{R}$.

Lemma 4.2.1.

(i) $\tau_r^I(t)(\lambda A+B) = \lambda\tau_r^I(t)A + \tau_r^I(t)B$

(ii) $\left(\tau_r^I(t)A\right)^* = \tau_r^I(t)A^*$

(iii) $\tau_r^I(t)(AB) = \tau_r^I(t)A \; \tau_r^I(t)B$

for all $t\in \mathbb{R}$, $\lambda\in\mathbb{C}$, $A,B\in\underline{\underline{A}}$.

Proof.

(i) is obvious, and so is (ii) since $V = V^*$.

(iii) Denote by S_n the sum of the first n terms of the series for $\tau_r^I(t)A$, $t\in \mathbb{R}$, $A\in\underline{\underline{A}}$. Then using the fact that

$$\frac{dS}{dt}{}_{n+1}(t) = i\left[V_r(t), S_n(t)\right]$$

it is not hard to show that

$$\frac{d}{dt}\tau_r^I(t)A = i\left[V_r(t), \tau_r^I(t)A\right] .$$

Similarly,

$$\frac{d}{dt}\tau_r^I(t)AB = i\left[V_r(t), \tau_r^I(t)AB\right] .$$

Now,

$$\frac{d}{dt}\left(\tau_r^I(t)A\ \tau_r^I(t)B\right) = i\left[V_r(t),\tau_r^I(t)A\right]\tau_r^I(t)B$$

$$+\ i\ \tau_r^I(t)A\left[V_r(t),\tau_r^I(t)B\right]$$

$$= i\left[V_r(t),\ \tau_r^I(t)A\ \tau_r^I(t)B\right].$$

Thus we see that $\tau_r^I(t)(AB)$ and $\tau_r^I(t)A\ \tau_r^I(t)B$ satisfy the same differential equation and the same initial condition, viz, $\tau_r^I(0)AB = AB$. It follows that

$$\tau_r^I(t)(AB) = \tau_r^I(t)A\ \tau_r^I(t)B$$

for all t. QED.

Remark.

Lemma 4.2.1 says that $t \to \tau_r^I(t)$ is a map $: \mathbb{R} \to \text{End}\ \underline{\underline{A}}$ (- the endomorphism algebra of $\underline{\underline{A}}$).

Theorem 4.2.2.

Let τ_0 be given as in axiom 4', and let τ_r^I be defined as above. Then, writing $\tau_0(t)$ for $\tau_0(t,\underline{O})$, we have that $\tau_0(\cdot)\tau_r^I(\cdot)$ is a one-parameter group of endomorphisms of $\underline{\underline{A}}$.

Proof.

We must show that

$$\tau_0(s)\tau_r^I(s)\tau_0(t)\tau_r^I(t)A = \tau_0(s+t)\tau_r^I(s+t)A$$

for all $A \in \underline{\underline{A}}$, and $s,t \in \mathbb{R}$.

Or, equivalently, since $\tau_0(\cdot)$ is a one-parameter group of automorphisms,

$$\tau_r^I(s)\tau_0(t)\tau_r^I(t)A = \tau_0(t)\tau_r^I(s+t)A.$$

Fix t ; then as in lemma 4.2.1, the l.h.s. satisfies the differential equation

$$\frac{d}{ds}(\cdot) = i\,[V_r(s),(\cdot)]\ .$$

The r.h.s. satisfies

$$\frac{d}{ds}\{\tau_0(t)\tau_r^I(s+t)A\} = \tau_0(t)\,i[V_r(s+t),\tau_r^I(s+t)A]$$

$$= i[V_r(s),\tau_0(t)\tau_r^I(s+t)A]\ .$$

Since they both satisfy the same initial condition, we conclude that they are equal for all s. QED.

Corollary 4.2.3.

The endomorphisms $\tau_r^I(\cdot)$ are automorphisms of $\underline{\underline{A}}$.

Proof.

Let $B \in \underline{\underline{A}}$. Put $A = \tau_0(-t)\tau_r^I(-t)\tau_0(t)B$.
Then $A \in \underline{\underline{A}}$, and $\tau_0(t)\tau_r^I(t)A = \tau_0(t)B$, by the theorem.
Since $\tau_0(t)$ is an automorphism, we conclude that $\tau_r^I(t)A = B$, and so $\tau_r^I(t)$ is a map from $\underline{\underline{A}}$ onto $\underline{\underline{A}}, \forall t \in \mathbb{R}$.

Suppose $\tau_r^I(t)A = 0$. Then $\tau_0(t)\tau_r^I(t)A = 0$, and acting on this with $\tau_0(-t)\tau_r^I(-t)$ yields, by the theorem, that $A = 0$. Therefore $\tau_r^I(t)$ is one-one for all $t \in \mathbb{R}$. QED.

§4.3. Removal of the Cut-Off.

Let us now consider the limit of $\tau_r^I(\cdot)$ as $r\to\infty$.

Definition 4.3.1.

Let r_0 be the smallest real number such that

$$[V_{\underline{a}}(t), V_{\underline{b}}(s)] = 0$$

for all $\underline{a},\underline{b}$ with $|\underline{a}-\underline{b}| > r_0$, where $|t|<1$, and $|s|<1$.

Definition 4.3.2.

Let $A \in \underline{\underline{A}}(\underline{\underline{O}})$. Let r_A be the smallest real number such that

$$[V_{\underline{a}}(t), A] = 0$$

for all $|\underline{a}| > r_A$, where $|t| < 1$.

Both r_0 and r_A are well-defined on account of axiom 4' and axiom 3.

Theorem 4.3.3.

There exists $\delta>0$ such that the limit $\tau_r^I(t)A$ exists as $r\to\infty$, for all $A\in\underline{\underline{A}}$, uniformly in $|t|\leq\delta$.

Proof.

Let $A \in \underline{\underline{A}}(\underline{\underline{O}})$ be given, and suppose $|t| < 1$. Then, according to definitions 4.3.1 and 4.3.2, if $|t_j| \leq |t|$ and $|t_k| \leq |t|$, we have

$$[V_{\underline{a}_j}(t_j), V_{\underline{a}_k}(t_k)] = 0 \quad \text{if } |\underline{a}_j - \underline{a}_k| > r_0$$

and

$$[V_{\underline{a}_j}(t_j), A] = 0 \quad \text{if } |\underline{a}_j| > r_A .$$

Consider the general term, u_n, in the series for $\tau_r^I(t)A$, viz,

$$i^n \int_0^t dt_n \ldots \int_0^{t_2} dt_1 \int_R d^3 a_n \ldots \int_R d^3 a_1 [V_{\underline{a}_n}(t_n), [\ldots, [V_{\underline{a}_1}(t_1), A] \ldots]$$

where $R = \{ \underline{a} \mid |\underline{a}| \leq r \}$.

Working from the inside bracket, we see that $[V_{\underline{a}_1}(t_1), A]$ is zero if $|\underline{a}_1| > r_A$; similarly for the double commutator,

$$[V_{\underline{a}_2}(t_2), V_{\underline{a}_1}(t_1)A] = [V_{\underline{a}_2}(t_2), V_{\underline{a}_1}(t_1)]A + V_{\underline{a}_1}(t_1)[V_{\underline{a}_2}(t_2), A]$$

vanishes unless $|\underline{a}_2 - \underline{a}_1| \leq r_0$ or $|\underline{a}_2| \leq r_A$. The same goes for the other term $-[V_{\underline{a}_2}(t_2), AV_{\underline{a}_1}(t_1)]$.

In general, the j-fold commutator will vanish unless $|\underline{a}_j| \leq r_A$ or $|\underline{a}_j - \underline{a}_1| \leq r_0$ or ... or $|\underline{a}_j - \underline{a}_{j-1}| \leq r_0$.

It follows that the integrand of u_n is zero outside the 3n-dimensional region S(n) given by

$$S(n) = \cup_{j\le n} S_j \;,$$

$$S_1 = \{a\varepsilon\, \mathbb{R}^{3n} \,|\, |\underline{a}_1|\le r_A\}$$

$$S_2 = \{a\varepsilon\, \mathbb{R}^{3n} \,|\, |\underline{a}_2|\le r_A\}\cup\{a\,|\,|\underline{a}_2-\underline{a}_1|\le r_0\}$$

$$\cdots\cdots$$

$$S_j = \{a\,|\,|\underline{a}_j|\le r_A\}\cup\{a\,|\,|\underline{a}_j-\underline{a}_{j-1}|\le r_0\}\cup\ldots\cup\{a\,|\,|\underline{a}_j-\underline{a}_1|\le r_0\}.$$

Thus $\|u_n\|$ is bounded by

$$\int_0^{|t|} dt_n \ldots \int_0^{t_2} dt_1 \int_{S(n)} d^3a_n \ldots d^3a_1 \,\|\, [V_{\underline{a}_n}(t_n),[\ldots,[V_{\underline{a}_1}(t_1),A]\ldots]\,\|.$$

Expanding the commutator gives the bound

$$\frac{|t|^n}{n!}\, 2^n \,\|V\|^n\, \|A\| \int_{S(n)} d^3a_n \ldots d^3a_1 .$$

The integral over S(n) can be split up into n! (overlapping) parts, each of the form of a polysphere in suitable co-ordinates, namely,

$|\underline{a}_1|\le r_A$, $|\underline{a}_2'|\le$const., $|\underline{a}_3'|\le$const., ...

where $\underline{a}_j'$ is either $\underline{a}_j$, when the constant is r_A, or is one of $\underline{a}_j-\underline{a}_{k-1}$ $(2\le k\le j)$ when the constant is r_0.

For each $j \in \{1,\ldots,n\}$ let us put

$$\alpha_{j,1} = \int_{|\underline{a}_j|\le r_A} d^3 a_j = \tfrac{4}{3}\pi r_A^3 = D, \text{ say}$$

and

$$\alpha_{j,k} = \int_{|\underline{a}_j-\underline{a}_{k-1}|\le r_0} d^3 (a_j-a_{k-1}) = \tfrac{4}{3}\pi r_0^3 = d, \text{ say,}$$

for $k = 2,\ldots,j$. Then the typical term in the integral over $S(n)$ is $\alpha_{1,1}\alpha_{2,i_2}\ldots\alpha_{n,i_n}$

where $i_j \in \{1,\ldots,j\}$. Hence, the integral over $S(n)$ is bounded by the sum

$$\Sigma\, \alpha_{1,1}\alpha_{2,i_2} \ldots \alpha_{n,i_n}$$

(where the sum extends over all possible values of $i_2, i_3, \ldots, i_n$).

$$= D(D+d)(D+2d) \ldots (D+(n-1)d).$$

So a final estimate for $\|u_n\|$ is

$$\|u_n\| \le \frac{|t|^n}{n!} 2^n \|V\|^n \|A\| D(D+d) \ldots (D+(n-1)d).$$

Since each $S(n)$ is a bounded $3n$-dimensional region, each term of the series for $\tau_r^I(t)A$ becomes independent of r for r sufficiently large.

But $\|\tau_r^I(t)A\|$ is bounded by the series

$$\sum_{n=0}^{\infty} \|A\| \frac{|t|^n}{n!} 2^n \|V\|^n D(D+d) \ldots (D+(n-1)d),$$

which is uniformly convergent for $|t| \leq \delta$, where $\delta < (2\,||V||\,d)^{-1}$.

Therefore $\tau_r^I(t)A$ converges in norm, as $r \to \infty$, uniformly in $|t| \leq \delta$. The convergence depends on d, which depends only on r_0, and is independent of A. Thus $\tau_r^I(t)A$ converges uniformly in $|t| \leq \delta$ for all $A \in \{ \underline{\underline{A}}(\underline{\underline{O}}) \mid \underline{\underline{O}}$ a region in M$\}$.

Now let $B \in \underline{\underline{A}}$ be arbitrary, and let $\varepsilon > 0$ be given. Then there is an $A \in \underline{\underline{A}}(\underline{\underline{O}})$, some $\underline{\underline{O}}$, such that

$$||A - B|| < \varepsilon/3.$$

Therefore

$$|| \tau_r^I(t)B - \tau_s^I(t)B|| \leq ||\tau_r^I(t)A - \tau_s^I(t)A|| + 2\,||A-B||$$
$$< \varepsilon$$

for $|t| \leq \delta$, and r,s sufficiently large. The $\tau_r^I(t)B$ therefore converge in norm as $r \to \infty$. QED.

Remark.

If we had taken s dimensions, rather than 3, the only difference would be that D and d would be replaced by the volumes of s-dimensional hyperspheres of radii r_A and r_0, respectively. The conclusion of the theorem remains unaltered.

Corollary 4.3.4.

The family $\{ \tau_0(t)\tau_\infty^I(t) \mid t \in \mathbb{R} \}$, where $\tau_\infty^I(t)$

denotes the limit of $\tau_r^I(t)$ as $r \to \infty$, define a strongly continuous one-parameter group in Aut $\underline{\underline{A}}$.

Proof.

The relation $\tau_0(s)\tau_r^I(s)\tau_0(t)\tau_r^I(t) = \tau_0(s+t)\tau_r^I(s+t)$ implies, by the theorem, that

$$\tau_0(s)\tau_\infty^I(s)\tau_0(t)\tau_\infty^I(t) = \tau_0(s+t)\tau_\infty^I(s+t)$$

provided $|s|,|t|$ and $|s+t|$ are all $\leq \delta$.

The fact that each $\tau_\infty^I(t)$ is an automorphism follows just as in corollary 4.2.3.

We shall utilise this group property to extend $\tau_0(\cdot)\tau_\infty^I(\cdot)$ to $|t| \leq 2\delta$. Let us write $\tau(\cdot)$ for $\tau_0(\cdot)\tau_\infty^I(\cdot)$.

For $|\theta| \leq 2\delta$, define $\tau(\theta) = \tau(x)\tau(y)$ where $|x|,|y| \leq \delta$, and $x+y = \theta$. The r.h.s. is a well-defined automorphism of $\underline{\underline{A}}$. For $|\theta| \leq \delta$, this definition is no more than an identity.

Suppose that $\delta < |\theta| \leq 2\delta$, $x+y = \theta = x'+y'$, $|x|,|x'|,|y|,|y'| \leq \delta$, and $x \neq x'$. Then

$$\tau(x)\tau(y) = \tau(x')\tau(x-x')\tau(y)$$

$$= \tau(x')\tau(y-y')\tau(y)$$

(using $x-x' = y'-y$)

$$= \tau(x')\tau(y') \quad ;$$

all the automorphisms occuring are well-defined since their arguments have modulus not greater than δ.

This simply means that $\tau(\cdot)$ is independent of how we write $\theta = x+y$, i.e. is well-defined.

We have extended the range of t from $|t| \leq \delta$ to $|t| \leq 2\delta$. In the same way, we can extend this to $|t| \leq 4\delta$, and so on.

Thus $\tau(t)$ is a well-defined automorphism for each $t \in \mathbb{R}$, and satisfies the group property. It remains to show that $\tau(t)$ is strongly continuous in t;i.e. $\tau(\cdot)A$ is norm continuous in t for each $A \in \underline{\underline{A}}$.

Let $A \in \underline{\underline{A}}(\underline{\underline{O}})$, and let $\varepsilon>0$ be given.

Then

$$\| \tau(s+t)A - \tau(s)A \| = \| \tau(t)A - A\|$$

$$= \| \tau_\infty^I(t)A - \tau_0(-t)A\|$$

since τ_0 has the group property,

$$\leq \| \tau_\infty^I(t)A-A\| + \| A-\tau_0(-t)A\| .$$

But

$$\| \tau_\infty^I(t)A-A\| \leq \|A\| \sum_{n=1}^{\infty} \frac{|t|^n}{n!} 2^n \|V\|^n D \ldots (D+(n-1)d)$$

$$< \varepsilon/2 \quad \text{for } |t| \text{ sufficiently small.}$$

Also

$$\| A - \tau_0(-t)A\| < \varepsilon/2 \text{ for small } |t| \text{ by axiom 4'.}$$

Therefore

$$\| \tau(t')A - \tau(t)A\| < \varepsilon$$

for $|t'-t|$ sufficiently small.

Now let $A \in \underline{\underline{A}}$, and let $B \in \underline{\underline{A}}(\underline{\underline{O}})$, some $\underline{\underline{A}}(\underline{\underline{O}})$, with $\| A - B\| < 2\varepsilon/3$.

Then

$$\| \tau(t')A-\tau(t)A \| \leq \| \tau(t')B-\tau(t)B\| + 2\| A-B\|$$

$$< \varepsilon$$

for sufficiently small $|t'-t|$. QED.

Remark.

The conclusion of corollary 4.3.4 depends on the continuity assumption (axiom 4') of τ_0.

Suppose that $\underline{\underline{A}}$ is given as a C*-algebra of operators on a Hilbert space, $\underline{\underline{H}}$. A weaker continuity assumption on τ_0 is then that, for each $A \varepsilon \underline{\underline{A}}$ and $h \varepsilon \underline{\underline{H}}$,

$$\| \tau_0(a)Ah - Ah\| \to 0 \text{ in } \underline{\underline{H}}, \text{ as } a \to 0 \text{ in } \mathbb{R}^4.$$

$\tau_r^I(t)A$ can be defined as a series of strong integrals on $\underline{\underline{H}}$, and theorem 4.2.2, corollary 4.2.3, and theorem 4.3.3 remain unaltered. However, corollary 4.3.4 must be modified :

Corollary 4.3.4'.

If τ_0 satisfies the weaker continuity condition above, then the family $\{\tau_0(t)\tau_\infty^I(t) \mid t \varepsilon \mathbb{R}\}$ define a one-parameter group of automorphisms satisfying the same continuity condition as $\tau_0(t)$.

(Note that it remains true that $\tau_\infty^I(t)A$ is norm continuous in t, at t=0, for fixed A).

Theorem 4.3.5.

$\tau(t)$ commutes with space translations.

Proof.

Let $A \in \underline{\underline{A}}(\underline{O})$, some $\underline{O}$, and let $|t| \leq \delta$.

Then we have

$$\tau_0(0,\underline{a})\tau(t)A = \tau_0(t,\underline{a})\left(A + i\int_0^t dt_1 \int d^3a_1 [V_{\underline{a}_1}(t_1),A] + \ldots\right)$$

$$= \tau_0(t)\left(\tau_0(0,\underline{a})A + i\int_0^t dt_1 \int d^3a_1 [V_{\underline{a}+\underline{a}_1}(t_1),\tau_0(0,\underline{a})A] + \ldots\right)$$

$$= \tau_0(t)\tau_\infty^I(t)\left(\tau_0(0,\underline{a})A\right)$$

$$= \tau(t)\tau_0(0,\underline{a})A.$$

By continuity in A, we obtain the result for arbitrary $A \in \underline{\underline{A}}$. To remove the restriction $|t| \leq \delta$, we use the group property of τ.

Let s be given. Then there is t, with $|t| \leq \delta$, and an integer n such that nt = s. Thus, for $A \in \underline{\underline{A}}$,

$$\tau_0(0,\underline{a})\tau(s)A = \tau_0(0,\underline{a})\tau(s/n)^n A = \tau(s/n)^n \tau_0(0,\underline{a})A$$

by repeated application of the result for $|t| \leq \delta$,

$$= \tau(s)\tau_0(0,\underline{a})A \qquad \text{QED.}$$

We have obtained a translation-invariant automorphism group $\{\tau(t) \mid t \in \mathbb{R}\}$ by removing the space cut-off on V_r, despite the fact that V_r itself has no limit. If, in addition, we have a continuous representation of O(3) in Aut $\underline{\underline{A}}$, and if we choose V to be invariant under this action, the theory will be Euclidean invariant. (For example, an invariant V can be obtained by averaging an arbitrary V over the group O(3) with respect to its Haar measure).

§4.4. The Heisenberg Fields.

Suppose now that $\underline{\underline{A}}$ is the quasi-local algebra for the free field, as discussed in §2.1. $\tau_0(t)$ and $\tau(t)$ are then continuous in the weaker sense that $\|\tau(t)Ah - Ah\| \to 0$ as $t \to 0$ for each $A \in \underline{\underline{A}}$ and each $h \in \underline{\underline{H}}$, the Fock space.

In this case, V_r is a bounded operator in $\underline{\underline{H}}$, and therefore $H_0 + V_r$, where H implements the time translations of §2.1, is a well-defined, unbounded self-adjoint operator with domain equal to Dom H_0. It is not hard to see that $\tau_r(t) \equiv \tau_0(t)\tau_r^I(t)$ is implemented by the unitary group with generator $H_0 + V_r$.

From our estimates in theorem 4.3.3, it is clear that the convergence of $\tau_r^I(t)A$ to $\tau_\infty^I(t)A$ for $|t| \le \delta$ depends only on D and d and $\|A\|$. Now, if $A \in \underline{\underline{A}}(\underline{O})$, D depends only on $\underline{O}$. Thus $\tau_r^I(t)A$ converges to $\tau_\infty^I(t)A$ uniformly in $\|A\| \le 1$ and $A \in \underline{\underline{A}}(\underline{O})$, for fixed $\underline{O}$, provided $|t| \le \delta$.

Now let $W(\alpha\xi) \in \underline{\underline{A}}(\underline{O})$, where

$$W(\alpha\xi) = \exp i\alpha(\overline{\phi(f)-\pi(g)}) \ ,\alpha \in \mathbb{R}, \text{ as in §2.1.}$$

This is strongly continuous in α, and the implementability of $\tau_r(t)$ implies that the same is true of $\tau_r(t)W(\alpha\xi)$. By the above comments, this converges in norm, for $|t| \leq \delta$, as $r \to \infty$, uniformly in α, and so we conclude that $\tau(t)W(\alpha\xi)$ is strongly continuous in α, for $|t| \leq \delta$.

We can therefore define the sharp-time Heisenberg fields $\phi(f,t)$, $\pi(g,t)$ for $|t| \leq \delta$, as the self-adjoint generators of $\tau(t)W(\alpha\xi)$ for ξ given by Cauchy data $(f,0)$ and $(0,-g)$, respectively. Clearly, the domains of $\phi(f,t)$ and $\pi(g,t)$ will depend on t.

<u>Remark</u>.

We do not expect $\tau(t)$ to be implemented if $\tau_0(t)$ is - this because of Haag's theorem (14). To obtain a representation of $\underline{\underline{A}}$ in which $\tau(t)$ is implemented, one method would be to find an invariant state on $\underline{\underline{A}}$ and then employ the G.N.S. construction (23,24) to obtain a unitary operator $U(t)$ implementing $\tau(t)$ on a new Hilbert space. The difficulty is in showing that $U(t)$ is strongly continuous in t, as is necessary if we are to define the Hamiltonian as its generator. This has been done successfully by J.Glimm and A.Jaffe (45), and can also be done in our case in two-dimensional space-time using their techniques (56).

Another undesirable property of $\tau(t)$ is its violation of causality. The various terms in $\tau^I_\infty(t)A$ spread out further and further, so that although A may be located within a bounded region, $\tau(t)A$ is spread over all space for any arbitrarily small time, t.

§4.5. Almost-factorisable Interactions.

We can prove that $\tau(t)$ is causal under an extra condition on V. Unfortunately, we do not know if there are any such V satisfying this condition.

To avoid the complications of the time-dependence of the $\underline{\underline{O}}$, let us suppose that we are given the local algebras, $\underline{\underline{A}}(\underline{\underline{O}})$, for a fixed time, say t=0, so that now $\underline{\underline{O}}$ is a region in $\mathbb{R}^3$.

For example, the time-zero algebras for the free boson field are generated by the time-zero fields $\phi(f)$, $\pi(g)$ with $f,g \in \underline{\underline{D}}(\mathbb{R}^3)$. (Note that $\underline{\underline{A}}$ is the same here as in the 4-dimensional case).

Axioms 1-3, for the three dimensional regions will still be meaningful, and we shall suppose that they hold. Axiom 4' is the same except that we suppose $\tau_0(t)\underline{\underline{A}}(\underline{\underline{O}}) = \underline{\underline{A}}(\underline{\underline{O}}_{|t|})$, where $\underline{\underline{O}}_{|t|}$ is the region $\underline{\underline{O}}$ "spread out" by a distance $|t|$: $\underline{\underline{O}}_{|t|} = \bigcup_{B \text{ in } \underline{\underline{O}}} B_{|t|}$, where B is a ball, and $B_{|t|}$ is the ball obtained by increasing the radius of B by $|t|$.

We would like $\tau(t)$ to have this property, viz, $\tau(t)\underline{\underline{A}}(\underline{\underline{O}}) = \underline{\underline{A}}(\underline{\underline{O}}_{|t|})$.

Definition 4.5.1.

Let $V \in \underline{\underline{A}}(\underline{\underline{O}})$, for some region $\underline{\underline{O}}$. We say that V is almost-factorisable if, for any given $\varepsilon>0$, $\delta>0$, there is a finite cover of $\underline{\underline{O}}$ by open balls $\{B_i | i=1,..,n\}$, with radii equal to δ, and elements $V_i \in \underline{\underline{A}}(B_i)$ such that

$$\| V - \sum_{i=1}^{n} V_i \| < \varepsilon.$$

In other words, V can be approximated by a sum of elements in arbitrarily small regions. The following theorem is, therefore, not so surprising. (An example of such a V is given by any element in the intersection of all the $\underline{\underline{A}}(\underline{\underline{O}})$. In this case, V can be written as $V = \Sigma_{i=1}^{n} V_i$, where $V_i = \frac{1}{n}V$, which clearly will satisfy the conditions of the definition. However, in cases of interest, the intersection of all the $\underline{\underline{A}}(\underline{\underline{O}})$ is trivial, or at least is contained in the centre of $\underline{\underline{A}}$, in which case, τ_{∞}^{I} is equal to the identity automorphism).

Theorem 4.5.2.

Let $V = V^* \in \underline{\underline{A}}(R)$, where R is a ball of radius ρ, be almost-factorisable. Suppose that τ_0 is causal, i.e. satisfies our modification of axiom 4', above. Then τ is causal.

Proof.

Let $A \in \underline{\underline{A}}(\underline{\underline{O}})$, $\underline{\underline{O}}$ a region in $\mathbb{R}^3$. We note that if $W \in \underline{\underline{A}}(S)$, where S is a ball of radius σ, then $\int d^3a[W_{\underline{a}}, A] \in \underline{\underline{A}}(\underline{\underline{O}}_{2\sigma})$.

The causality of τ_0 implies, therefore, that

$$\int d^3a_1 \ldots \int d^3a_m\ \tau_0(t)[W_{\underline{a}_1}(t_1), [\ \ldots, [W_{\underline{a}_m}(t_m), A] \ldots]$$

(where $W_{\underline{a}}(t) = \tau_0(-t) W_{\underline{a}}$)

$$= \int d^3a_1 \ldots d^3a_m \tau_0(t-t_1)[W_{\underline{a}_1}, \tau_0(t_1-t_2)[W_{\underline{a}_2}, \ldots$$

$$\ldots, \tau_0(t_{m-1}-t_m)[W_{\underline{a}_m}, \tau_0(t_m)A]\ldots]$$

$$\in \underline{\underline{A}}(\underline{\underline{O}}_{2\sigma m+|t|})$$

Let $|\theta|<|t|<(2\|V\|d)^{-1}$, with d as in theorem 4.3.3, and let $\varepsilon>0$ be given. Let N be an integer, and $\varepsilon'>0$. Then, since V is almost-factorisable, there is a finite cover of R, by open balls $\{B_i |\ i=1,\ldots,n\}$ with radii equal to $\delta = \dfrac{|t| - |\theta|}{2N}$, and elements $V_i \in \underline{\underline{A}}(B_i)$ such that

$$\|V - \Sigma_{i=1}^n V_i\| < \varepsilon'.$$

Now, it follows from our previous remark that

$$S_N(\theta)A = \tau_0(\theta)A + \sum_{j=1}^{N} i^j \int_0^\theta dt_1 \ldots \int_0^{t_{j-1}} dt_j \int d^3a_1 \ldots \int d^3a_j$$

$$\tau_0(\theta)[W_{\underline{a}_1}(t_1), \ldots [W_{\underline{a}_j}(t_j), A]\ldots]$$

where $W = \Sigma_{i=1}^{n} V_i$, belongs to $\underline{\underline{A}}(\underline{\underline{O}}_{2N\delta+|\theta|}) = \underline{\underline{A}}(\underline{\underline{O}}_{|t|})$.

But $||V - W|| < \varepsilon'$ implies that

$$||S_N(\theta)A - \tau(\theta)|_N A|| < \varepsilon,$$

provided ε' is sufficiently small; where $\tau(\theta)|_N A$ is the sum of the first N+1 terms of the series for $\tau(\theta)A$. Now, $\tau(\theta)|_N A \to \tau(\theta)A$ in norm, as $N \to \infty$ (by our estimates in theorem 4.3.3), and so the same is true for $S_N(\theta)A$.

The norm completeness of $\underline{\underline{A}}(\underline{\underline{O}}_{|t|})$ implies, therefore, that $\tau(\theta)A \in \underline{\underline{A}}(\underline{\underline{O}}_{|t|})$. Now, $\tau(\theta)A$ is norm continuous in θ if $\tau_0(\theta)A$ is. Under this assumption on τ_0, we conclude that $\tau(t)A$, as a norm limit of $\tau(\theta)A$ as $\theta \to t$, belongs to $\underline{\underline{A}}(\underline{\underline{O}}_{|t|})$.

The group property of τ ensures that this property holds for all t, and the proof is complete.

<u>Remark 1</u>.

If we assume that the $\underline{\underline{A}}(\underline{\underline{O}})$ are given as operator algebras, and as such are weakly, or equivalently, strongly closed, then we need only assume the weaker condition that $\tau_0(t)A$ is strongly continuous in t. Indeed, this implies that the same is true of $\tau(t)A$, and so $\tau(t)A$ is the strong limit of $\tau(\theta)A$ as $\theta \to t$.

But each $\tau(\theta)A \in \underline{\underline{A}}(\underline{\underline{O}}_{|t|})$ and so $\tau(t)A \in \underline{\underline{A}}(\underline{\underline{O}}_{|t|})$ if this is strongly closed.

Remark 2.

If V is such that $\tau(t)$ is causal, we can exploit the uniformity in $\|A\| \leq 1$, $A \in \underline{\underline{A}}(\underline{\underline{O}})$, of the convergence of the series for $\tau_{\infty}^{I}(t)A$, and define the Heisenberg fields $\phi(f,t)$ and $\pi(g,t)$ for all time, t. Indeed, $\tau_r(t)A \to \tau(t)A$ uniformly in $\|A\| \leq 1$, $A \in \underline{\underline{A}}(\underline{\underline{O}}_{|t|})$ for any fixed t, and so the restriction $|t| \leq \delta$ can be removed.

References.

1. Segal,I.E.: Postulates for general quantum mechanics. Ann.Math. 48, 930–948 (1947).

2. Haag,R.: Colloques sur les problèmes mathématiques de la théorie quantique des champs, Lille, 1957. Paris: C.N.R.S. 1959.

3. Araki,H.: A lattice of von Neumann algebras associated with the quantum theory of a free Bose field. J.Math.Phys. 4, 1343–1362 (1963).

—— : On the algebra of all local observables. Prog.Theor.Phys. 32, 844–854 (1964).

4. —— : von Neumann algebras of local observables for the free scalar field. J.Math.Phys. 5, 1–13 (1964).

—— : Type of von Neumann algebra associated with the free field. Prog.Theor.Phys. 32, 956–965 (1964).

5. Borchers,H.J.: On the structure of the algebra of field operators. Nuov.Cim. 24, 214–236 (1962).

—— : On the structure of the algebra of field operators II. Commun.math.Phys. 1, 49–56 (1965).

6. —— : Energy and momentum as observables in quantum field theory. Commun.math.Phys. 2, 49–54 (1966).

7. —— ,Haag,R.,Schroer,B.: The vacuum state in quantum field theory. Nuov.Cim. 29, 148–162 (1963).

8. Haag,R.,Schroer,B.: Postulates of quantum field theory. J.Math.Phys. 3, 248–256 (1962).

9. Borchers,H.J.,Zimmerman,W.: On the self-adjointness of field operators. Nuov.Cim. 31, 1047–1059 (1964).

10. Haag,R.,Kastler,D.: An algebraic approach to quantum field theory. J.Math.Phys. 5, 848–861 (1964).

11. Wightman,A.S.: Quantum field theory in terms of vacuum expectation values. Phys.Rev. 101, 860–866(1956).

12. Streater,R.F.,Wightman,A.S.: PCT, spin and statistics, and all that. New York: Benjamin 1964.

13. Segal,I.E.: Mathematical problems of relativistic physics. Amer.Math.Soc. 1963. See also references herein to Segal's work.

14. Haag,R.: On quantum field theory. Dan.Mat.Fys. Medd. 29, 12 (1955).

15. Glimm,J.,Jaffe,A.: The $\lambda(\phi^4)_2$ quantum field theory without cutoffs III.The physical vacuum. Acta Math. 125, 203–267 (1970).

16. Dixmier,J: Les C*-algèbres et leurs répresentations, 2nd. éd. Paris: Gauthier-Villars 1969.

17. Segal,I.E.,Kunze,R.: Integrals and operators. New York: McGraw-Hill 1968.

18. Fell,J.M.G.: The dual spaces of C*-algebras. Trans.Amer.Math.Soc. 94, 365–403 (1960).

19. Doplicher,S.,Haag,R.,Roberts,J.E.: Fields, observables and gauge transformations I. Commun.Math.Phys. 13, 1–23 (1969).

20. ——— : Fields, observables and gauge transformations II. Commun.Math.Phys. 15, 173–200 (1969).

21. Wightman,A.S.: Dispersion relations and elementary particles; lectures at Les Houches 1960, ed.C.de Witt, R.Omnes. Paris: Hermann 1960.

22. Streater,R.F.: Lectures at Imperial College, London, 1968–1969, unpublished.

23. Gelfand,I.,Neumark,M.: On the imbedding of normed rings into the ring of operators in Hilbert space. Rec.Math.n.s. 12, 197–213 (1943).

24. Segal,I.E.; Irreducible representations of operator algebras. Amer.Math.Soc.,Bull 53, 73–88 (1947).

25. Schweber,S.S.: An introduction to relativistic quantum field theory. Harper and Row 1961.

26. Nelson,E.: Analytic vectors. Ann.Math.70,572–615(1959).

27. Guichardet,A.: Algebrès d'observables associées aux relations de commutation. Paris: A.Colin, Collection Interscience 1968.

28. Kato,T.: Perturbation theory of linear operators. Berlin-Heidelberg-New York: Springer 1966. (See page 502).
Kallman,R.R.: Groups of inner automorphisms of von Neumann algebras. J.Funct.An. 7,43–60(1971)(Theorem 1.1)

29. Hepp,K.: Théorie de la renormalisation. Lecture Notes in Physics,2. Berlin-Heidelberg-New York :Springer 1969.

30. Segal,I.E.,Goodman,R.W.: Anti-locality of certain Lorentz invariant operators. J.Math.Mech. 14, 629–638 (1965).

31. Dixmier,J.: Les algèbres d'opérateurs dans l'espace hilbertien, 2nd éd. Paris: Gauthier Villars 1969.

32. Borchers,H.J.: Applications of mathematics to problems in theoretical physics. Cargèse lectures 1965, ed. F.Lurcat. New York: Gordon and Breach 1967.

33. Skyrme,T.H.R.: A non-linear theory of strong interactions. Proc.Roy.Soc. 247A, 260–278 (1958).

34. Williams,J.G.: The topological analysis of a non-linear field theory, Ph.D. thesis. Birmingham University, England, 1969, unpublished.

35. Wightman,A.S.: High energy electromagnetic interactions and field theory. Cargèse lectures, 1964, ed. M.Levy. New York: Gordon and Breach 1967.

36. Tarski,J.: Representation of fields in a two-dimensional model theory.J.Math.Phys.5,1713–1722(1964).

37. Schroer,B.: Infrateilchen in der quantenfeldtheorie. Fortschr.der Phys. 11,1–31 (1963).

38. Manuceau,J.: Étude de quelques automorphismes de la C*-algèbre du champ de bosons libres. Ann.Inst.H.P. 8, 117–138 (1968).

39. —— : C*-algèbres de relations de commutation. Ann.Inst.H.P. 8, 139–161 (1968).

40. Roepstorff,G.: Coherent photon states and spectral condition. Commun.math.Phys. 19, 301–314 (1970).

41. Streater,R.F.: Local quantum theory, ed. R.Jost, Varrenna lectures,1968. London:Academic Press 1969.

42. Heisenberg,W.: Introduction to the unified field theory of elementary particles. London:Interscience 1966.

43. Salam,A.: Lectures at Imperial College, autumn 1969, unpublished.

44. —— ,Delbourgo,R.,Strathdee,J.: Phys.Rev. 187, 1999–2007 (1969) ; and references therein.

45. Glimm,J.,Jaffe,A.: Field theory models. Les Houches lectures, ed. C. de Witt, R.Stora. New York: Gordon and Breach 1971.
Segal,I.E.; Construction of non-linear quantum processes I. Ann.Math. 92, 463–481 (1970).
Simon,B.,Høegh-Krohn,R.: Hypercontractive semigroups and two-dimensional self-coupled Bose fields. J.Funct.Anal., to appear.

46. Guenin,M.: Mathematical methods of theoretical physics. Lectures at Colorado, 1966. Ed. W.E.Britten A.O.Barut, M.Guenin. New York: Gordon & Breach 1967.

—— : On the interaction picture. Commun.math. Phys. 3, 120–132 (1966).

47. Segal,I.E.: Transformations in Wiener space and squares of quantum fields. Adv.in Math.4,91–108 (1970).

48. Jaffe,A.: Divergence of perturbation theory for bosons. Commun.math.Phys. 1, 127–149 (1965).

49. Segal,I.E.: Colloques sur les problèmes mathématiques de la théorie quantique des champs, Lille, 1957. Paris: C.N.R.S. 1959.

—— : Quasi-finiteness of the interaction Hamiltonian of certain quantum fields. Ann.Math. 72, 594–602 (1960).

50. Streater,R.F.: The Heisenberg ferromagnet as a quantum field theory. Commun.math.Phys.6,233–247(1967).

51. —— : On certain non-relativistic quantised fields. Commun.math.Phys. 7, 93–98 (1968).

52. Robinson,D.W.: Statistical mechanics of quantum spin systems II. Commun.math.Phys. 7, 337–348 (1968).

53. Ruelle,D.: Statistical mechanics. New York: Benjamin 1969.

54. Streater,R.F.,Wilde,I.F.: The time evolution of quantised fields with bounded quasi-local interaction density. Commun.math.Phys. 17, 21–32 (1970).

55. See e.g. Yosida,K.: Functional analysis, p.132. Berlin-Heidelberg-New York: Springer 1965.

56. Høegh-Krohn,R.: Boson fields with bounded interaction densities. Commun.math.Phys. 17, 179–193 (1970).

57. Dell'Antonio,G.: preprint, communicated to the author orally via R.F.Streater.

58. Fitelson,M.,Johnson,R.: On the construction of a vector current with time component obeying the algebra of currents I. Nucl.Phys. B18,151–161 (1970).