

COUNTING CURVES ON SURFACES IN CALABI-YAU 3-FOLDS

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ABSTRACT. Motivated by S-duality modularity conjectures in string theory, we define new invariants counting a restricted class of 2-dimensional torsion sheaves, enumerating pairs $Z \subset H$ in a Calabi-Yau threefold X . Here H is a member of a sufficiently positive linear system and Z is a 1-dimensional subscheme of it. The associated sheaf is the ideal sheaf of $Z \subset H$, pushed forward to X and considered as a certain Joyce-Song pair in the derived category of X . We express these invariants in terms of the MNOP invariants of X .

1. INTRODUCTION

Throughout this paper we fix a Calabi-Yau 3-fold X with $H^1(\mathcal{O}_X) = 0$. String theorists conjecture the modularity of certain generating functions of counts of supersymmetric D4-D2-D0 BPS states of X [GSY07, GY07, OSV01]. Motivated by this we define invariants of X which count 0- and 1-dimensional subschemes (i.e. D2-D0 branes) of hyperplanes (D4 branes) in X . We briefly discuss the relationship to the physicists' S-duality conjecture in Section 4.

We wish to count pairs of the form

$$(1) \quad Z \subset H \quad \text{in } X,$$

where H is a suitably ample hypersurface and Z is a 1-dimensional subscheme of H . While we demand that H be pure of dimension 2 (i.e. a divisor in X), the subscheme Z need not be pure, so is in general the union of a curve and a 0-dimensional subscheme.

Since $H^1(X, \mathcal{O}_X) = 0$, all deformations of H are in the same linear system

$$|H| := \mathbb{P}(H^0(\mathcal{O}_X(H))).$$

This carries a universal hyperplane

$$(2) \quad \begin{array}{c} \mathcal{H} \subset X \times |H| \\ \downarrow \\ |H|. \end{array}$$

The data (1) are naturally parameterized by the relative Hilbert scheme of 1-dimensional subschemes of the fibers of the family (2). That is, if we fix

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$\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, then the moduli space of pairs (1) is the relative Hilbert scheme

$$(3) \quad \text{Hilb}_{\beta, n}(\mathcal{H}/|H|)$$

of 1-dimensional subschemes Z of the universal hyperplane whose pushforward to X has fundamental class $\beta = [Z] = \text{ch}_2(i_* \mathcal{O}_Z)$ and holomorphic Euler characteristic $n = \chi(\mathcal{O}_Z)$.

We will produce a symmetric perfect obstruction theory on (3) by considering it as a moduli space of torsion sheaves of the form

$$(4) \quad i_* I_Z \in \text{Coh}(X).$$

Here $i: H \hookrightarrow X$ is the inclusion, and I_Z denotes the ideal sheaf of Z considered as a subscheme of H . So (4) is a torsion sheaf on X with rank 1 on its 2-dimensional support H . When $Z \subset H$ is a Cartier divisor, it is the pushforward of a line bundle on H .

Theorem 1. *Suppose that H is sufficiently positive with respect to $\beta \in H_2(X; \mathbb{Z})$ and $n \in \mathbb{Z}$, in the sense of Definition 2.2. Then*

$$\text{Hilb}_{\beta, n}(\mathcal{H}/|H|)$$

is a locally complete moduli space of torsion sheaves (4) on X . It admits a symmetric perfect obstruction theory and so a virtual cycle of virtual dimension 0.

There are two obvious problems to overcome in proving Theorem 1:

- The sheaves (4) need not be (Gieseker) semistable when H is reducible or nonreduced.
- Deformations of (4) need not be sheaves of the same form. They could be arbitrary torsion sheaves of the same topological type, like the pushforward $i_* L$ of a general line bundle on a hyperplane H (rather than one which is a subsheaf of \mathcal{O}_H).

We circumvent these by using certain Joyce-Song pairs [JS11]. These come with a different notion of stability which gets round the first problem. And for $H \gg 0$, they allow us to see that deformations of (4) are also push forwards of ideal sheaves, so that $\text{Hilb}_{\beta, n}(\mathcal{H}/|H|)$ is indeed an open and closed subscheme of the stack of all coherent sheaves on X .

Our Joyce-Song pairs are of the form

$$\left(\mathcal{I}_Z, s \in H^0(\mathcal{I}_Z(H)) \right),$$

where Z and H are as before, but \mathcal{I}_Z denotes the ideal sheaf of Z when considered as a subscheme of X (rather than H).¹ Since the torsion-free rank-1 sheaf \mathcal{I}_Z is automatically Gieseker stable, the only further stability condition we need is that s should be nonzero. Therefore we get an injection

$$(5) \quad \mathcal{O}(-H) \xrightarrow{s} \mathcal{I}_Z$$

¹We use straight I s to denote ideal sheaves on H and curly \mathcal{I} s for ideal sheaves on X .

whose cokernel $i_* I_Z$ is the torsion sheaf (4). That is, if we let $I^\bullet \in D(X)$ denote the two term complex formed by (5), then

the Joyce-Song complexes I^\bullet (5) are quasi-isomorphic to the torsion sheaves (4).

More globally we show in Proposition 2.1 that this observation gives an isomorphism between

- an open and closed subscheme of the moduli space of Joyce-Song pairs (5), and
- the relative Hilbert scheme (3) parameterizing the sheaves (4).

Finally, for $H \gg 0$, Joyce and Song show that the deformation theory of the complexes $I^\bullet \in D(X)$ gives this space a symmetric perfect obstruction theory. Since I^\bullet is quasi-isomorphic to $i_* I_Z \in D(X)$, this shows that the deformation theory of the sheaf $i_* I_Z$ indeed endows $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ with a symmetric perfect obstruction theory.

Using Theorem 1 we can define an invariant

$$N_{\beta,n}^H(X) := \int_{[\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)]^{\text{vir}}} 1.$$

Since the obstruction theory is symmetric, by [Beh09] this can also be written as a weighted Euler characteristic of $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$. The invariants $N_{\beta,n}^H$ are closely related to the MNOP invariants $I_{\beta,n}(X)$ counting the ideal sheaves \mathcal{I}_Z of X .

Theorem 2. *Under the conditions of Theorem 1 we have*

$$N_{\beta,n}^H = (-1)^{c-1} c \cdot I_{\beta,n},$$

where $c = c(\beta, n, H, X)$ is the topological number

$$\int_X \left(\frac{1}{6} H^3 + H^2 \text{td}_2(X) - H \cdot \beta \right) - n.$$

Example 1.1. Most of the examples worked out explicitly in [GY07, Sections 2.1 to 2.6] fit into the setting of this section and hence, the corresponding invariants are captured by the invariants $N_{\beta,n}^H$. As an illustration, we work out the following two simple cases for linear hyperplane sections H of the quintic Calabi-Yau threefold X .

- \mathcal{I}_p is the ideal sheaf of a point $p \in X$, then $\beta = 0$, and $n = 1$, and hence

$$c = \text{td}_2(X) \cdot H + H^3/6 - 1 = 25/6 + 5/6 - 1 = 4.$$

Also $I_{0,1} = -\chi(X) = 200$, and hence by $N_{0,1}^H = (-1)^3 \times 4 \times 200 = -800$.

- \mathcal{I}_C is the ideal sheaf of a line $C \subset X$, so $\beta \cdot H = 1$, $n = 1$, and hence

$$c = 25/6 + 5/6 - 1 = 3.$$

Also $I_{\beta,1} = 2875$, the number of rational degree 1 curves in a generic quintic, and hence $N_{\beta,1}^H = (-1)^2 \times 3 \times 2875 = 8625$. \square

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2. PROOF OF THEOREM 1

A Joyce-Song stable pair [JS11] is a pair

$$(E, s \in H^0(E(H))),$$

such that

- E is coherent sheaf on X , Gieseker semistable with respect to H , and
- s is a section which does not factor through any destabilizing subsheaf of E .

A family of such pairs over a base scheme B is a sheaf \mathcal{E} over $X \times B$, flat over B , and a section of $\mathcal{E} \otimes \pi_X^* \mathcal{O}(H)$ such that the restriction of (\mathcal{E}, s) to any fiber $X \times \{b\}$ is a stable pair in the above sense. Fixing a Chern character $c \in H^{\text{ev}}(X, \mathbb{Q})$, there is a projective scheme

$$J_c(X)$$

representing the moduli functor which assigns to a scheme B the set of isomorphism classes² of Joyce-Song stable pairs over B .

Forgetting the section s and passing to the S-equivalence class of the sheaf E gives a morphism

$$J_c(X) \longrightarrow M_c(X)$$

to the moduli space of Gieseker semistable sheaves E of Chern character c .

We take

$$c = (1, 0, -\beta, -n).$$

Then the Hilbert scheme $I_n(X, \beta)$ of ideal sheaves \mathcal{I}_Z defines an open and closed subscheme³ of $M_c(X)$ (see for instance the proof of Theorem 2.7 in [PT09] for a careful proof of this fact). Thus its inverse image

$$J_n(X, \beta) := J_c(X) \times_{M_c(X)} I_n(X, \beta)$$

is both open and closed in $J_c(X)$. It therefore has the same deformation theory as $J_c(X)$, so we can use the results of Joyce and Song.

Proposition 2.1. $J_n(X, \beta) \cong \text{Hilb}_{\beta, n}(\mathcal{H}/|H|)$.

²Isomorphism is meant in the strict sense: two families of stable pairs (\mathcal{E}_i, s_i) are isomorphic if there is an isomorphism $\mathcal{E}_1 \rightarrow \mathcal{E}_2$ taking s_1 to s_2 . Stable pairs have no automorphisms so there is no need to tensor by line bundles pulled back from the base (as one does to define isomorphism for families of stable sheaves, for example: stable sheaves have $\text{Aut} = \mathbb{C}^*$).

³In fact it is all of $M_c(X)$ when $H^2(X, \mathbb{Z})$ is torsion-free.

Proof. $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ carries a pair of universal subschemes

$$(6) \quad \mathcal{Z} \subset \mathcal{H} \subset X \times \text{Hilb}_{\beta,n}(\mathcal{H}/|H|).$$

Letting i denote the second inclusion, we get the right hand column and central row of the following commutative diagram of short exact sequences.

$$(7) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{O}_{X \times \text{Hilb}}(-\mathcal{H}) & \xrightarrow{s} & \mathcal{I}_{\mathcal{Z}} & \longrightarrow & i_* \mathcal{I}_{\mathcal{Z}} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{X \times \text{Hilb}}(-\mathcal{H}) & \longrightarrow & \mathcal{O}_{X \times \text{Hilb}} & \longrightarrow & i_* \mathcal{O}_{\mathcal{H}} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & i_* \mathcal{O}_{\mathcal{Z}} & = & i_* \mathcal{O}_{\mathcal{Z}} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0. \end{array}$$

Filling in the diagram gives the top row, producing the flat family of stable pairs

$$(8) \quad \mathcal{O} \xrightarrow{s} \mathcal{I}_{\mathcal{Z}}(\mathcal{H} - \pi_X^* H) \otimes \pi_X^* \mathcal{O}(H)$$

over the base $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$. This is classified by a map from the base to the moduli space of stable pairs:

$$(9) \quad \text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \longrightarrow J_n(X, \beta).$$

Similarly, $X \times J_n(X, \beta)$ carries a universal stable pair

$$(10) \quad \mathcal{O} \longrightarrow \mathcal{E} \otimes \pi_X^* \mathcal{O}(H),$$

flat over $J_n(X, \beta)$. Since the restriction of \mathcal{E} to each X -fiber is torsion-free of rank 1, its double dual $\mathcal{E}^{\vee\vee}$ is locally free by [Kol90, Lemma 6.13]. Therefore it defines a map from $J_n(X, \beta)$ to $\text{Pic}(X)$ which takes closed points to the trivial line bundle \mathcal{O}_X . But $H^1(X, \mathcal{O}_X) = 0$ so $\text{Pic}(X)$ is a union of discrete reduced points and the map is constant. Pulling back a Poincaré line bundle shows that $\mathcal{E}^{\vee\vee}$ is the pullback $\pi_J^* L$ of some line bundle L on $J_n(X, \beta)$. Therefore $\mathcal{E} \subset \mathcal{E}^{\vee\vee}$ must take the form

$$(11) \quad \mathcal{E} \cong \pi_J^* L \otimes \mathcal{I}_{\mathcal{Z}}$$

for some subscheme $\mathcal{Z} \subset X \times J_n(X, \beta)$. Since \mathcal{E} is flat over $J_n(X, \beta)$, so \mathcal{Z} must be too.

Composing the section (10) with $\mathcal{E} \rightarrow \mathcal{E}^{\vee\vee}$ gives a section

$$(12) \quad \mathcal{O}_{X \times J_n(X, \beta)} \longrightarrow \mathcal{O}(H) \boxtimes L,$$

nonzero on each X -fiber by stability. Its zero locus is a divisor

$$\mathcal{H} \in |\mathcal{O}(H) \boxtimes L|,$$

giving a classifying map $J_n(X, \beta) \rightarrow |H|$ such that the pullback of $\mathcal{O}_{X \times |H|}(\mathcal{H})$ is $\mathcal{O}(H) \boxtimes L$. Since the section (12) factors through (11) it follows that

$$\mathcal{Z} \subset \mathcal{H},$$

giving us a classifying map from our base $J_n(X, \beta)$ to $\text{Hilb}_{\beta, n}(\mathcal{H}/|H|)$. By inspection it is the inverse of the map (9). \square

Definition 2.2. Assume that $\beta \in H_2(X; \mathbb{Z})$ and $n \in \mathbb{Z}$ are given. We say H is sufficiently positive with respect to β, n if

$$(13) \quad H^i(X, \mathcal{I}_{\mathcal{Z}}(H)) = 0$$

for $i > 0$ for any ideal sheaf $\mathcal{I}_{\mathcal{Z}} \in I_n(X, \beta)$. For fixed (β, n) the ideal sheaves $\mathcal{I}_{\mathcal{Z}} \in I_n(X, \beta)$ form a bounded family, so (13) is satisfied for all H sufficiently positive. \circlearrowright

Corollary 2.3. Suppose that $H^1(X, \mathcal{O}_X) = 0$ and β, n, H satisfy (13). Then, using the same notation as in (6), the deformation-obstruction theory of the sheaves $i_* I_{\mathcal{Z}}$ (4) given by [HT10, Section 4.4],

$$(14) \quad \tau^{[1,2]} R\pi_{H*} R\mathcal{H}om(i_* I_{\mathcal{Z}}, i_* I_{\mathcal{Z}})[2] \longrightarrow \mathbb{L}_{\text{Hilb}_{\beta, n}(\mathcal{H}/|H|)},$$

defines a symmetric perfect obstruction theory on $\text{Hilb}_{\beta, n}(\mathcal{H}/|H|)$.

In (14), π_H denotes the projection from $X \times \text{Hilb}_{\beta, n}(\mathcal{H}/|H|)$ to its second factor. We recall the definition of the arrow [HT10, Section 4]. Project the Atiyah class

$$\text{At}(i_* I_{\mathcal{Z}}) \in \text{Ext}^1(i_* I_{\mathcal{Z}}, i_* I_{\mathcal{Z}} \otimes \mathbb{L}_{X \times \text{Hilb}})$$

to

$$\text{Ext}^1(i_* I_{\mathcal{Z}}, i_* I_{\mathcal{Z}} \otimes \pi_H^* \mathbb{L}_{\text{Hilb}})$$

and consider the resulting element as a map

$$(15) \quad \pi_H^* \mathbb{L}_{\text{Hilb}}^{\vee} \longrightarrow R\mathcal{H}om(i_* I_{\mathcal{Z}}, i_* I_{\mathcal{Z}})[1].$$

By adjunction this induces

$$(16) \quad \mathbb{L}_{\text{Hilb}}^{\vee} \longrightarrow R\pi_{H*} R\mathcal{H}om(i_* I_{\mathcal{Z}}, i_* I_{\mathcal{Z}})[1].$$

Next we project to the $\tau^{\geq 0}$ truncation of the right hand side. In turn this receives a map from the truncation $\tau^{[0,1]}$, and in [HT10, Section 4.4] it is shown that the map from $\mathbb{L}_{\text{Hilb}}^{\vee}$ lifts uniquely to it. Dualizing and using Serre duality down the fibers of π_H gives (14).

Proof of Corollary 2.3. For $H \gg 0$, Joyce and Song [JS11, Theorem 12.20] give a symmetric perfect obstruction theory on $J_c(X)$:

$$(17) \quad \tau^{[1,2]} R\pi_{J*} R\mathcal{H}om(\mathbb{I}^{\bullet}, \mathbb{I}^{\bullet})[2] \longrightarrow \mathbb{L}_{J_c(X)}.$$

Here π_J is the projection $X \times J_c(X) \rightarrow J_c(X)$ and $\mathbb{I}^{\bullet} \in D(X \times J_c(X))$ is the 2-term complex

$$\mathcal{O}_{X \times J_c(X)}(-\pi_X^* H) \longrightarrow \mathcal{E}$$

defined by the universal stable pair $\mathcal{O} \rightarrow \mathcal{E} \otimes \pi_X^* \mathcal{O}(H)$. (From now on we suppress the π_X^* .) The arrow in (17) is the composition of Serre duality for $R\pi_{J*} R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)$ and the Atiyah class of \mathbb{I}^\bullet , just as in (16).

We restrict their result to $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ using Proposition 2.1. Their precise $H \gg 0$ condition is that the sheaves in their stable pairs are H -regular; this becomes the cohomology vanishing condition (13). The first row of the diagram (7) gives the quasi-isomorphism

$$\mathbb{I}^\bullet \otimes \mathcal{O}(H - \mathcal{H}) = \{ \mathcal{O}_{X \times \text{Hilb}_{\beta,n}(\mathcal{H}/|H|)}(-\mathcal{H}) \rightarrow \mathcal{I}_{\mathcal{Z}} \} \simeq i_* I_{\mathcal{Z}}.$$

Using the line bundle $\mathcal{O}(1)$ pulled back from the projective space $|H|$, we have the isomorphism

$$\mathcal{O}(\mathcal{H}) \cong \mathcal{O}(H) \boxtimes \mathcal{O}(1).$$

In particular, $\mathbb{I}^\bullet \simeq i_* I_{\mathcal{Z}}(\mathcal{H} - H) \cong i_* I_{\mathcal{Z}}(1)$, so (17) can be written

$$\tau^{[1,2]} R\pi_{H*} R\mathcal{H}om(i_* I_{\mathcal{Z}}(1), i_* I_{\mathcal{Z}}(1)) [2] \longrightarrow \mathbb{L}_{\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)}.$$

Again the arrow is given by the composition of Serre duality and the Atiyah class of $i_* I_{\mathcal{Z}}(1)$.

This is almost identical to (14, 16) except for the twist by $\mathcal{O}(1)$. Now

$$\text{At}(i_* I_{\mathcal{Z}}(1)) = \text{At}(i_* I_{\mathcal{Z}}) + \text{id}_{i_* I_{\mathcal{Z}}} \otimes \text{At}(\mathcal{O}(1))$$

in $\text{Ext}^1(i_* I_{\mathcal{Z}}(1), i_* I_{\mathcal{Z}}(1) \otimes \pi_H^* \mathbb{L}_{\text{Hilb}}) \cong \text{Ext}^1(i_* I_{\mathcal{Z}}, i_* I_{\mathcal{Z}} \otimes \pi_H^* \mathbb{L}_{\text{Hilb}})$. That is, tensoring with $\mathcal{O}(1)$ changes the map (15) by addition of the following composition

$$\pi_H^* \mathbb{L}_{\text{Hilb}}^\vee \xrightarrow{\text{At}(\mathcal{O}(1))} \mathcal{O}_{X \times \text{Hilb}}[1] \xrightarrow{\text{id}_{i_* I_{\mathcal{Z}}}} R\mathcal{H}om(i_* I_{\mathcal{Z}}, i_* I_{\mathcal{Z}})[1].$$

Therefore (16) changes by addition of the composition

$$\mathbb{L}_{\text{Hilb}}^\vee \xrightarrow{\text{At}(\mathcal{O}(1))} R\pi_{H*} \mathcal{O}_{X \times \text{Hilb}}[1] \xrightarrow{\text{id}_{i_* I_{\mathcal{Z}}}} R\pi_{H*} R\mathcal{H}om(i_* I_{\mathcal{Z}}, i_* I_{\mathcal{Z}})[1].$$

The truncation procedure gives unique lifts to the $\tau^{[0,1]}$ truncation of the central and right hand terms. But our $H^1(X, \mathcal{O}_X) = 0$ condition means that $\tau^{[0,1]}$ applied to the central term is zero. \square

Corollary 2.3 completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

By Corollary 2.3 we define the invariants associated to $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$. These invariants give a virtual count of the pairs (1). Then we will relate them to the MNOP invariants $I_{\beta,n}$ counting subschemes $Z \subset X$.

Definition 3.1. Let X be a Calabi-Yau 3-fold with $H^1(X, \mathcal{O}_X) = 0$ and suppose that H is sufficiently positive with respect to β, n in the sense of Definition 2.2.

The perfect obstruction theories of Corollary 2.3 and [Tho00] respectively endow $\mathrm{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ and $I_n(X, \beta)$ with virtual cycles of dimension zero. We define

$$N_{\beta,n}^H := \int_{[\mathrm{Hilb}_{\beta,n}(\mathcal{H}/|H|)]^{\mathrm{vir}}} 1, \quad I_{\beta,n} := \int_{[I_n(X, \beta)]^{\mathrm{vir}}} 1.$$

Since these obstruction theories are symmetric, by [Beh09] the invariants can also be written as weighted Euler characteristics

$$(18) \quad \begin{aligned} N_{\beta,n}^H &= \chi(\mathrm{Hilb}_{\beta,n}(\mathcal{H}/|H|), \nu_{\mathrm{Hilb}_{\beta,n}(\mathcal{H}/|H|)}), \\ I_{\beta,n} &= \chi(I_n(X, \beta), \nu_{I_n(X, \beta)}), \end{aligned}$$

where ν_M denotes the Behrend function [Beh09] of a scheme M . \circlearrowright

By the usual arguments [HT10, JS11, Tho00] these invariants are unchanged under smooth deformation of X .

There is an obvious forgetful map

$$(19) \quad \mathrm{Hilb}_{\beta,n}(\mathcal{H}/|H|) \longrightarrow I_n(X, \beta), \quad (Z \subset H) \mapsto Z.$$

The fiber over $Z \subset X$ is $\mathbb{P}(H^0(\mathcal{I}_Z(H)))$. More globally we have the following result. We use the notation

$$\mathcal{Z} \subset X \times I_n(X, \beta) \xrightarrow{\pi_I} I_n(X, \beta)$$

for the universal subscheme and the projection to the second factor.

Lemma 3.2. *Supposing again that β, n, H satisfy (13), then the map (19) is the projective bundle*

$$\mathrm{Hilb}_{\beta,n}(\mathcal{H}/|H|) \cong \mathbb{P}(\pi_{I*}\mathcal{I}_{\mathcal{Z}}(H)).$$

Proof. The cohomology vanishing condition (13) ensures that $\pi_{I*}\mathcal{I}_{\mathcal{Z}}(H)$ is a vector bundle. Its pullback to $\mathbb{P}(\pi_{I*}\mathcal{I}_{\mathcal{Z}}(H))$ carries a tautological subbundle

$$\mathcal{O}(-1) \hookrightarrow \pi_{I*}\mathcal{I}_{\mathcal{Z}}(H),$$

where again we have suppressed the pullback map. By adjunction we get

$$\pi_I^*\mathcal{O}(-1) \hookrightarrow \mathcal{I}_{\mathcal{Z}}(H)$$

and so a nowhere vanishing section of $\mathcal{I}_{\mathcal{Z}}(H) \boxtimes \mathcal{O}(1) = \mathcal{I}_{\mathcal{Z}}(\mathcal{H})$. This is a family of stable pairs over the base $\mathbb{P}(\pi_{I*}\mathcal{I}_{\mathcal{Z}}(H))$, classified by a map

$$(20) \quad \mathbb{P}(\pi_{I*}\mathcal{I}_{\mathcal{Z}}(H)) \longrightarrow J_n(X, \beta) \cong \mathrm{Hilb}_{\beta,n}(\mathcal{H}/|H|).$$

Conversely, $X \times \mathrm{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ carries the universal family of stable pairs (8). Twisting by $\mathcal{O}(-1)$ (pulled back from $|H|$) gives

$$\mathcal{O}(-1) \xrightarrow{s} \mathcal{I}_{\mathcal{Z}}(H).$$

Pushing down to $\mathrm{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ gives

$$(21) \quad \mathcal{O}(-1) \hookrightarrow \pi_{H*}\mathcal{I}_{\mathcal{Z}}(H).$$

Now $\pi_{H^*}\mathcal{I}_Z(H)$ on $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ is the pullback of the bundle $\pi_{I^*}\mathcal{I}_Z(H)$ on $I_n(X, \beta)$ via the map (19). Therefore the line subbundle (21) is classified by a map $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \rightarrow \mathbb{P}(\pi_{I^*}\mathcal{I}_Z(H))$, the inverse of (20). \square

Corollary 3.3. *For $H^1(X, \mathcal{O}_X) = 0$ and β, n, H satisfying (13), the invariants of Definition 3.1 satisfy*

$$N_{\beta,n}^H = (-1)^{c-1} c \cdot I_{\beta,n},$$

where $c = c(\beta, n, H, X)$ is the topological number

$$c = \chi(\mathcal{I}_Z(H)) = \int_X \left(\frac{1}{6}H^3 + H^2 \text{td}_2(X) - H \cdot \beta \right) - n.$$

Proof. By Lemma 3.2, the map (19)

$$\text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \longrightarrow I_n(X, \beta)$$

is smooth of relative dimension $\chi(\mathcal{I}_Z(H)) - 1$. Therefore by [Beh09, Proposition 1.5(i)], the Behrend function of $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ is the pullback of that of $I_n(X, \beta)$, multiplied by $(-1)^{\chi(\mathcal{I}_Z(H))-1}$. Since the fibers of the map (19) have Euler characteristic $\chi(\mathcal{I}_Z(H))$, the formulas (18) give the result. \square

Corollary 3.3 finishes the proof of Theorem 2.

4. S-DUALITY AND MODULARITY

Our motivation for defining the invariants $N_{\beta,n}^H$ was to try to understand how to define the “supersymmetric BPS invariants associated to D4-D2-D0 systems” studied by string theorists [GSY07, GY07, OSV01]. Their S-duality conjecture is that the generating series of these putative BPS invariants should be *modular*.

Most of the examples of D4-D2-D0 systems studied in [GY07, Sections 2.1–2.6] are of the form (4) above, which is what led to our definition. However it seems that in general one should count *all* (semi)stable torsion sheaves of the right topological type supported on hyperplanes, not just those of the form (4).

Sometimes all of the sheaves (4) are stable, for instance when all members of the linear system $|H|$ are reduced and irreducible. (The hyperplane sections of the quintic threefold have this property, by the Lefschetz hyperplane theorem.) In that case one can think of $N_{\beta,n}^H$ as the contribution of the component $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ of the moduli space of torsion sheaves to the physicists’ numbers.

More generally one would expect to be able to relate our invariants $N_{\beta,n}^H$ to invariants counting more general torsion sheaves via a sequence of wall

crossings.⁴ Ideally these would be in the space of Bridgeland stability conditions on $D(X)$, starting from a stability condition that approximates Joyce–Song stability for the complexes (5), and ending with one approximating Gieseker stability for the quasi-isomorphic sheaves (4).

In combination with Corollary 3.3 this would express the MNOP invariants $I_{\beta,n}$ in terms of modular forms. We plan to return to this in future work.

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⁴In [Tod11] Toda studies some different but remarkable wall crossings. These relate the counting of torsion sheaves to the counting of both ideal sheaves [MNOP06] and stable pairs [PT09]. There may be some connection: a hint is provided by the fact that Fujita’s conjecture (which would determine which H are sufficiently positive in the sense of Definition 2.2) enters into his analysis.