Reflection from a semi-infinite stack of layers using homogenization

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Abstract

A canonical scattering problem is that of a plane wave incident upon a periodic layered medium. Our aim here is to replace the periodic medium by a homogenized counterpart and then to investigate whether this captures the reflection and transmission behaviour accurately at potentially high frequencies.

We develop a model based upon high frequency homogenization and compare the reflection coefficients and full fields with the exact solution. For some material properties it is shown that the asymptotic behaviour of the dispersion curves are locally linear near critical frequencies and that low frequency behaviour is replicated at these critical, high, frequencies. The homogenization approach accurately replaces the periodic medium and the precise manner in which this is achieved then opens the way to future numerical implementation of this technique to scattering problems.

1. Introduction

The exciting and topical field of metamaterials offers to revolutionise optics and acoustics through negative refraction, invisibility, cloaking, shielding and otherwise moulding the flow of light and sound. A key canonical problem is the reflection of waves by a periodic laminate and, for instance, a component of some models of invisibility cloaks [21, 27] is the intimate knowledge of the low frequency homogenized effective properties of a semi-infinite layered medium which is then joined with transformation optics/acoustics [12] to create a cloak. Our aim here is to use the recent high frequency homogenization theory [10] to explore effective medium methodologies for this reflection and transmission problem beyond the low frequency limit.

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Homogenization theory is a route to finding effective media and, for low frequencies or for high contrast, Smith, [25], uses it to recover the effective equation for the layered medium found by [23]. However at higher frequencies, with moderate contrast, there is a lack of an effective dynamic homogenization theory. The classical well-trodden route to replace a microstructured medium with an effective continuum representation is homogenization theory and this is detailed in, for instance [5, 6, 19, 22] and essentially relies upon the wavelength being much larger than the microstructure. Unfortunately, many applications are away from this limit and this motivated the development of high frequency homogenization (HFH) in [10]. HFH breaks free of the low frequency long-wave limitation and, for bulk media, creates effective long-scale equations that encapsulate the microstructural behaviour, which can be upon the same scale as the wavelength, through integrated quantities. The methodology relies upon there being some basic underlying periodic structure so that Bloch waves and standing wave frequencies encapsulate the multiple scattering between elements of the microstructure on the short scale, and this is then modulated by a long-scale function that satisfies an anisotropic frequency dependent partial differential equation. The HFH theory of [10] is not alone: There is considerable interest in creating effective continuum models of microstructured media, in various related fields, that also attempt to break free from the conventional low frequency homogenization limitations: for instance, Bloch homogenization [2, 7, 8, 13]. There is also an associated literature on developing homogenized elastic media, with frequency dependent effective parameters, also applied to periodic media [16, 18]. Indeed these methodologies have been applied the same canonical problem as covered by the present article, in [24, 26], with main focus on the response for the lower dispersion curves.

The HFH approach has been successfully applied to acoustics/electromagnetics [4, 9], elastic plates that support bending waves [3], frames [17] and to discrete media [11]. The advantage of having an effective equation for a microstructured bulk medium or surface is that one need no longer model the detail of each individual scatterer, as they are subsumed into a parameter on the long-scale, and attention can then be given to the overall physics of the structure and one can identify, or design for, novel physics. Having successfully generated effective equations, and verified their accuracy versus infinite periodic media, it is natural to move to modelling finite domains. Although there has been some success in this direction, a key point that is glossed over in these articles, is how to apply boundary conditions on the long-scale equations that correctly match a block of homogenized material to an exterior domain. The model reflection/transmission problem considered here uncovers exactly how this should be done and this is an important detail for numerical implementations of HFH.

Reflection and transmission by a semi-infinite layered medium is a classical problem
Figure 1: The geometry under consideration: an incoming plane wave incident upon a semi-infinite slab of periodic medium that is to be represented by an homogenized continuum.

in wave mechanics and an elegant solution via transfer matrices is possible which was explored in acoustics by [23] and by others, notably in electromagnetics by [28]: an up-to-date exposition of transfer matrices, and their applications, is in [15]. For the simple semi-infinite layered medium it turns out, as we shall see later, that an explicit solution for the reflection and transmission properties can be found: nonetheless the transfer matrix formalism remains useful.

We illustrate the efficiency of the dynamic, high frequency, homogenization theory of [10] for reflection/transmission problems in the simplest setting: a semi-infinite periodic string, where the periodic cell is a bilayer that only varies piecewise in the density and each portion is of equal length, adjoining a semi-infinite perfect string. This is the canonical problem for waves incident (at an angle) on a stack, with varying stiffnesses or widths within the cell, and these extra details can be worked out, but the algebra obscures the main ideas. Therefore we choose to explore the theory via a one-dimensional example.

The plan of this paper is as follows. Initially, in section 2 an exact solution to the one-dimensional reflection/transmission problem of a semi-infinite periodic string is derived by employing a transfer matrix method and verified by a solution constructed from the displacements at the endpoints of each cell. In section 3 dynamic HFH is utilised to produce an effective equation on the macroscale encapsulating the microstructural properties of the medium. The high degree of accuracy of the asymptotics is shown by comparison to the exact solution obtained in the previous section. Dynamic HFH is implemented also to investigate the degenerate case of repeated solutions, which in par-
ticular cases exhibits similar characteristics to the low frequency asymptotics. In section 4 we apply the theory to a finite slab and compare exact solutions with asymptotics. Some concluding remarks are furnished in section 5.

2. Formulation

Our aim is to extract the exact solution for the semi-infinite stack of layers; an essential ingredient is the dispersion relation for an infinite periodic array (see Fig. 1). We briefly recap the properties of an infinite piecewise string (equivalent to a laminate stack with density variation), [20], with

$$u_{xx} + \frac{\omega^2}{\hat{c}^2(x/l)} u = 0 \quad \text{for} \quad x > -1,$$

where $\hat{c}(x/l) = c_0 c(x/l)$ is the wavespeed and $\omega$ is the frequency. We define $\xi = x/l$ with $l = 1$ in this case, and an elementary cell as $-1 \leq \xi < 1$, that periodically repeats. In $-1 \leq \xi < 0$ we have $u_{\xi\xi} + \Omega^2 u = 0$ and in $0 \leq \xi < 1$ we have $u_{\xi\xi} + r^2 \Omega^2 u = 0$, where $\Omega = \omega l/c_0$ is the non-dimensional frequency and $r$ is a positive constant. For an infinite array this elementary cell is all that is required together with quasi-periodic Floquet-Bloch conditions on $\xi = \pm 1$, i.e. $u(-1) = \exp(i 2 \epsilon \kappa) u(1)$ and $u_{\xi}(-1) = \exp(i 2 \epsilon \kappa) u_{\xi}(1)$ with $2 \epsilon$ being the relative width of the elementary cell. The parameter $\kappa$, called the Bloch wavenumber (here rescaled due to the lengthscales chosen), characterizes the phase shift from one cell to the next and is related to the frequency $\Omega$ through an explicit dispersion relation [14]

$$2r[\cos \Omega \cos r \Omega - \cos 2 \epsilon \kappa] - (1 + r^2) \sin \Omega \sin r \Omega = 0. \quad (2)$$

To approach the semi-infinite array we briefly utilise transfer matrices that appear in several different guises; scattering and propagator matrix formulations are equivalent, [1]. We create a vector $v = (u, u_{\xi}/i \Omega)^T$ and connect the field at either end of a cell via

$$v(1) = Q v(-1). \quad (3)$$

The matrix $Q$ is

$$Q = \begin{pmatrix} \cos \Omega r \cos \Omega - \frac{i}{r} \sin \Omega r \sin \Omega & i \cos \Omega r \sin \Omega + \frac{1}{r} \sin \Omega r \cos \Omega \\ i \cos \Omega r \sin \Omega + r \sin \Omega r \cos \Omega & \cos \Omega r \cos \Omega - r \sin \Omega r \sin \Omega \end{pmatrix}. \quad (4)$$

The dispersion relation (2) is deduced from this transfer matrix approach as, for Bloch waves, $v(1) = \exp(i 2 \epsilon \kappa) v(-1)$ so $(Q - \exp(i 2 \epsilon \kappa)I)v(-1) = 0$ and (2) is equivalent to $\det[(Q - \exp(i 2 \epsilon \kappa)I)]=0$, where $I$ is the identity matrix.
The transfer matrix allows one to move effortlessly along the array, for instance given \( \mathbf{v}(-1) \) then \( \mathbf{v}(2N-1) = Q^N \mathbf{v}(-1) \). Taking the \( q_{ij} \) to be the elements of the matrix \( Q \) the Chebyshev identity \([15, 28]\), gives that

\[
Q^N = \begin{pmatrix}
q_{11}U_{N-1} - U_{N-2} & q_{12}U_{N-1} \\
q_{21}U_{N-1} & q_{22}U_{N-1} - U_{N-2}
\end{pmatrix}
\]  

(5)

where

\[
U_N = \frac{\sin((N+1)2\epsilon \kappa)}{\sin(2\epsilon \kappa)}.
\]

The matrix \( Q \) can be diagonalised so that \( Q = V \Lambda V^{-1} \) where \( \Lambda \) is a diagonal matrix of eigenvalues and \( V \) is a matrix created by columns of eigenvalues so

\[
Q^N = V \text{diag}[\exp(i2\epsilon \kappa N), \exp(-i2\epsilon \kappa N)] V^{-1}.
\]

The eigenvalues are identified as a pair of complex conjugates, justifying the opposite signs in the exponents of the exponentials.

The periodic medium is now taken to be semi-infinite lying in \( x > -1 \), and we join this to a semi-infinite string \( u_{xx} + \Omega^2 r_0^2 u = 0 \) for \( x < -1 \) with \( r_0 \) constant. Assuming an incident wave of unit amplitude from minus infinity and a reflected wave with reflection coefficient \( R \) then \( \mathbf{v}(-1) \) is

\[
\mathbf{v}(-1) = \begin{pmatrix} 1 & 1 \\ r_0 & -r_0 \end{pmatrix} \begin{pmatrix} 1 \\ R \end{pmatrix}.
\]  

(6)

Thence, as in \([23]\) one obtains

\[
R = \frac{1 - \chi}{1 + \chi}, \quad \chi = \frac{q_{21}}{r_0(\lambda_1 - q_{22})}
\]  

(7)

where \( \lambda_1 \) is the eigenvalue of \( \Lambda \) for propagating waves, and using (5) the field is found everywhere.

This is the traditional way, \([23]\), of treating the semi-infinite array, but a more direct alternative is to note that everything can be written in terms of the displacements at the endpoints of each cell \([17]\) and this then becomes a fully discrete lattice problem. Propagating solutions in \( x > -1 \), if we set \( u(-1) = \mathcal{T} \), are of the form \( u(2N-1) = \mathcal{T} \exp(\pm i2\epsilon \kappa N) \): the Bloch wavenumber being related to the frequency via (2). An important nuance is that of the sign in the exponential which is positive (negative) for those dispersion branches with positive (negative) group velocity. Using the endpoints
we find that $u_x(2N - 1)$ is given by

$$u_x(2N - 1) = \frac{\Omega}{\sin \Omega} \left( -u_{2N-1} \cos \Omega + \frac{u_{2N-1} \sin \Omega r + ru_{2N+1} \sin \Omega}{\cos \Omega \sin \Omega r + r \cos \Omega r \sin \Omega} \right).$$

(8)

One then uses continuity of $u$ and $u_x$ across $x = -1$ so the reflection and transmission coefficients are related via

$$1 + R = T,$$

(9)

$$ir_0(1 - R) = \frac{T}{\sin \Omega} \left( -\cos \Omega + \frac{\sin \Omega r + r e^{\pm i2\epsilon \kappa} \sin \Omega}{\cos \Omega \sin \Omega r + r \cos \Omega r \sin \Omega} \right),$$

(10)

together these give an explicit solution for $R$ and $T$.

Our aim is, given this exact solution, to determine whether or not one can replace the semi-infinite periodic slab with an effective medium description that is valid for high frequencies and recover accurate asymptotic representations for $R$, $T$ and the field. This is the fundamental canonical question that, when answered, opens up the application of high frequency homogenization to a host of scattering problems.

3. Homogenization

Homogenization, from our perspective and for material constants of order unity, falls into two categories that of the classical long-wave low frequency model and then the high frequency homogenization presented by [10]. We consider both cases here, with more emphasis upon the latter.

The implicit assumption is that there is a small scale, characterized by $l$, and a long scale characterized by $L$ where $\epsilon = l/L \ll 1$. The two new spatial variables $\xi$ and $X$ are treated as independent quantities. The two scale nature of the problem is incorporated using the small and large length scales to define two new independent coordinates namely $X = x/L$, and $\xi = x/l$, where $l$ is taken unity here; this is the method of multiple scales. Equation (1) then becomes,

$$u_{\xi\xi} + \frac{\Omega^2}{c^2(\xi)} u + \epsilon^2 u_{XX} + 2\epsilon u_{\xi X} = 0$$

(11)

with $c(\xi) = 1/r$ for $0 \leq \xi < 1$ and $c(\xi) = 1$ for $-1 \leq \xi < 0$ where $u(x)$ has become $u(\xi, X)$ with $\xi$ and $X$ treated as independent variables. On the short-scale one can identify standing wave solutions at critical frequencies and these solutions are in-phase (periodic) across the elementary cell, so $u(1, X) = u(-1, X)$ and $u_\xi(1, X) = u_\xi(-1, X)$ or alternatively out-of-phase across the periodic cell in a straightforward manner [10]; for these we have $u(1, X) = -u(-1, X)$ and $u_\xi(1, X) = -u_\xi(-1, X)$. Both cases can
be dealt with using the methodology that we present, but we only give results for the periodic cases in detail. This assumption of periodicity on the short-scale then fixes the boundary conditions on the short-scale.

3.1. Low frequency asymptotics

At low frequencies, \( \Omega^2 = \epsilon^2 \Omega_2^2 + \ldots \), we have long waves, and the conventional low frequency viewpoint, and adopt the ansatz for \( u \) that

\[
   u(\xi, X) = u_0(\xi, X) + \epsilon u_1(\xi, X) + \epsilon^2 u_2(\xi, X) + \ldots
\]  

(12)

The term containing \( u_1(\xi, X) \) in equation (12) is omitted from here onwards since the perturbation is of \( O(\epsilon^2) \) and we no longer expect terms of \( O(\epsilon) \) in the \( u(\xi, X) \) expansion. Inserting this ansatz into the governing equation, a hierarchy of equations ensues:

\[
   u_{0\xi\xi} = 0, \quad 2u_{0\xi X} = 0, \quad u_{2\xi\xi} = -\left( u_{0XX} + \frac{\Omega_2^2}{c^2(\xi)} u_0 \right)
\]  

(13)

that are solved order-by-order together with the periodic boundary conditions; \( u_i(1, X) = u_i(-1, X) \) and \( u_{i\xi}(1, X) = u_{i\xi}(-1, X) \) for \( i = 0, 1, 2, \ldots \).

The leading order equation implies that \( u_0(\xi, X) = f(X) \). This has the important corollary that the leading order field does not vary at all on the \( \xi \) scale, i.e. it is constant in each periodic cell in the periodic setting. Herein lies the inherent limitation of the traditional theories which cannot incorporate any local microscale variation. Continuing up the orders, and invoking solvability, one eventually arrives at

\[
   f_{XX} + \left\langle \frac{1}{c^2} \right\rangle \Omega_2^2 f = 0, \quad \text{where} \quad \left\langle \frac{1}{c^2} \right\rangle = \frac{1}{2} \int_{-1}^{1} \frac{1}{c^2(\xi)} d\xi.
\]  

(14)

Unfortunately the knowledge of \( u_0 \) (equivalently \( f \)) is not sufficient to find the asymptotic reflection coefficient and one has to proceed to \( u_2 \).

Since we derived an expression for \( u_0 \), we attempt to find one for \( u_2 \), eventually leading to more accurate asymptotic solutions at higher frequencies near the edges of the Brillouin zone where there is a repeated eigenvalue and the dispersion curves are linear. By integrating (13) twice and applying continuity conditions

\[
   u_2(\xi, X) = \begin{cases} 
   -(f_{XX} + r^2 \Omega_2^2 f) \frac{\xi^2}{2} + \alpha \xi & \text{for } 0 \leq \xi < 1 \\
   -(f_{XX} + \Omega_2^2 f) \frac{\xi^2}{2} + \alpha \xi & \text{for } -1 \leq \xi < 0
   \end{cases}
\]  

(15)
where $\alpha = \Omega^2_2(r^2 - 1)u_0/4$ and $f_{XX} = -\Omega^2_2(r^2 + 1)f/2$. Finally, we arrive at
\begin{align*}
  u(\xi, X) &\sim f(X) + \epsilon^2 u_2(\xi, X) \quad (16) \\
  &\sim f(X)[1 + \epsilon^2 \Omega^2_2(r^2 - 1)/4] \xi(\xi + 1) \quad \text{for} \quad -1 \leq \xi < 0 \\
  &\sim f(X)[1 + \Omega^2_2(r^2 - 1)/4] \xi(\xi + 1) \quad \text{for} \quad -1 \leq \xi < 0. \quad (17)
\end{align*}

Scaling back to the original variable, we have $\xi = x$ and $X = \epsilon x$. In order to match the semi-infinite periodically layered strip with the semi-infinite perfect strip we consider the following in $x < -1$
\begin{equation*}
  u = \exp(i\Omega r_0(x + 1)) + R \exp(-i\Omega r_0(x + 1)) \quad (19)
\end{equation*}
and in $x > -1$
\begin{equation*}
  u = T \left[1 + \Omega^2_2(r^2 - 1)/4 - x(x + 1)\right] f(x) \quad \text{with} \quad f(x) = \exp\left(i\Omega \left(\frac{1 + r^2}{2}\right)^{1/2} (x + 1)\right). \quad (20)
\end{equation*}

Continuity of $u$ and $u_x$ at the junction gives the reflection coefficient, $R$, as $R = (1 - \chi)/(1 + \chi)$ where $\chi = \sqrt{(1 + r^2)/2 + i\Omega(1 - r^2)/4]}/r_0$.

3.2. High frequency homogenization

More interesting is the HFH limit, where the following ansatz is taken [10]:
\begin{equation*}
  \Omega^2 = \Omega^2_0 + \epsilon \Omega^2_0 + \epsilon^2 \Omega^2_2 + \ldots \quad (21)
\end{equation*}
\begin{equation*}
  u(\xi, X) = u_0(\xi, X) + \epsilon u_1(\xi, X) + \epsilon^2 u_2(\xi, X) + \ldots \quad (22)
\end{equation*}
and substituted in (11), where $c = 1/r$ for $0 \leq \xi < 1$ and $c = 1$ for $-1 \leq \xi < 0$. A hierarchy of equations for $u_i(\xi, X)$, with associated boundary conditions from the periodicity in $\xi$, is obtained and solved from the lowest order up. At leading order, the solution is
\begin{equation*}
  u_{0\xi\xi} + \frac{\Omega^2_0}{c^2(\xi)} u_0 = 0 \quad (23)
\end{equation*}
which has a solution of the form $u_0(\xi, X) = f(X)U_0(\xi; \Omega_0)$ and $\Omega_0$ is a standing wave frequency and $U_0(\xi; \Omega_0)$ is the associated Bloch eigenfunction:
\begin{equation*}
  (-r \sin \Omega_0 + p \cos \Omega_0)U_0(\xi; \Omega_0) = \begin{cases} 
    \sin(r\Omega_0 \xi) + p \cos(r\Omega_0 \xi) & \text{for} \quad 0 \leq \xi < 1 \\
    r \sin(\Omega_0 \xi) + p \cos(\Omega_0 \xi) & \text{for} \quad -1 \leq \xi < 0
  \end{cases} \quad (24)
\end{equation*}
with \( p = (r \sin \Omega_0 \pm \sin r \Omega_0) / (\cos \Omega_0 \mp \cos r \Omega_0) \) with the upper/lower signs for periodic/anti-periodic cases respectively. \( U_0 \) is arbitrary up to a multiplicative constant and for definiteness we normalise it to be unity at \( \xi = -1 \).

Continuing to next order, as in [10], we identify \( u_1(\xi, X) \) by introducing a function \( W_1(\xi; \Omega_0) \) as

\[
W_1(\xi; \Omega_0) = \begin{cases} 
\sin(r \Omega_0 \xi) & \text{for } 0 \leq \xi < 1 \\
r \sin(\Omega_0 \xi) & \text{for } -1 \leq \xi < 0,
\end{cases}
\]  

(25)

and then to order \( \epsilon \) the periodic case is

\[
u(\xi, X) = f(X)U_0(\xi; \Omega_0) + \epsilon \frac{df(X)}{dX}U_1(\xi; \Omega_0)
\]

(26)

where \( U_1(\xi; \Omega_0) \) is

\[
U_1(\xi; \Omega_0) = \left( \frac{2W_1(\xi; \Omega_0)}{\sin r \Omega_0 + r \sin \Omega_0} - \xi U_0(\xi; \Omega_0) \right)
\]

(27)

the anti-periodic case is virtually identical. By orthogonality, \( \Omega_1 \) is set to zero.

Assuming that the eigenvalues are isolated and single then an ordinary differential equation (ODE) for \( f(X) \) emerges as

\[
T f_{XX} + \Omega_1^2 f = 0
\]

(28)

where \( T \) is given explicitly as

\[
T = \pm 4 \Omega_0 \frac{\sin \Omega_0 \sin \Omega_0 r}{(r \sin \Omega_0 \mp \sin r \Omega_0)(\cos \Omega_0 \mp \cos r \Omega_0)}
\]

(29)

with upper (lower) signs for the periodic (anti-periodic) standing wave frequencies \( \Omega_0 \) and the sign of \( T \) gives the group velocity direction automatically. The example of a piecewise string is attractive as it is one-dimensional and explicitly solvable. The coefficient \( T \) is found by an integral over the elementary cell \(-1 < \xi < 1 \) in \( \xi \). As shown in [10] this ODE captures the asymptotics of the dispersion relation accurately and models the long-scale behaviour of the effective string. We now need to attach it to the semi-infinite string in \( x < -1 \) and to do this we first transform (28) back into the \( x \) coordinates so

\[
f_{xx} + \frac{(\Omega_1^2 - \Omega_0^2)}{T} f = 0
\]

(30)
Figure 2: The absolute value of the reflection coefficient versus frequency in panel (a) with the associated dispersion curves in panel (b): we chose $r = 1/3$ and $r_0 = \sqrt{5}/3$. The dispersion curves and reflection coefficient repeat periodically in $\Omega$ with period $3\pi$: solid lines represent the exact solutions. In panel (a) the asymptotic results for single eigenvalues and repeated roots are dashed and dotted lines respectively. In panel (b) the asymptotics are represented by small dots and dashed lines for the anti-periodic and periodic cases, respectively. The large dots show the low frequency asymptotics (see section 3.1), and the repeated solutions at $\Omega = 3\pi$ are shown by crosses (see section 3.3).
and so the field in the piecewise string has

$$f(x) = \exp \left( \text{sgn}(T)i \left( \frac{\Omega^2 - \Omega_0^2}{T} \right)^{\frac{1}{2}} (x + 1) \right).$$  \hspace{1cm} (31)

To apply the boundary conditions at $x = -1$ the HFH result for $u$ (26) is transferred back into the $x$ coordinate and is renamed $U_{asy}(x)$ and the field is

$$u(x) = \frac{T_{asy} U_{asy}(x)}{U_{asy}(-1)}$$  \hspace{1cm} (32)

where we normalise for convenience so that $u(-1) = T_{asy}$. Moving back to the single variable $x$ removes $\epsilon$ explicitly from the displacement field, the reason we have gone to first order in the expansion for $u$ is that this is necessary for consistency when generating the derivative $u_x = u_\xi + \epsilon u_X$.

The string in $x < -1$ has solution (19) and applying the continuity conditions at
Figure 4: The real part of $u$ for $\epsilon = 1/2$ in panel (a) for $\Omega_0 = 5.47$ (frequency $\Omega = \sqrt{\Omega_0^2 + \epsilon^2}$) and in panel (b) for $\Omega_0 = 11.128$ (frequency $\Omega = \sqrt{\Omega_0^2 + \epsilon^2}$). To the right of $x = -1$ the solid line represents the exact solution. The locally periodic and anti-periodic spatial behaviour associated with the short-scale is readily identified, in (a) and (b) respectively. The long-scale behaviour given by the asymptotic envelope function $f(x)$ (see section 3.2), is depicted by crosses. For $x < -1$ the rapidly oscillatory field is shown as the solid line. These displacements are shown for $r = 1/3$ and $r_0 = \sqrt{5}/3$. 

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Figure 5: The real part of $u$ is shown for three frequencies: $\Omega = 0.1$ and two frequencies close to the repeated roots at $\Omega_0 = 3\pi$ are shown in panels (a), (b) and (c), respectively. To the right of $x = -1$ the solid line shows the exact solution. The long-scale behaviour given by the asymptotic envelope function $f(x)$ (see section 3.3 and 3.1), is depicted by crosses. For $x < -1$ the rapidly oscillatory field is shown as the solid line. These displacements are shown for $r = 1/3$ and $r_0 = \sqrt{5}/3$. 
Figure 6: The absolute value of the reflection coefficient versus frequency in panel (a) with the associated dispersion curves in panel (b): we chose $r = 2$ and $r_0 = \sqrt{5/2}$. The dispersion curves and reflection coefficient repeat periodically in $\Omega$ with period $2\pi$ and $\pi$ respectively: solid lines represent the exact solutions. In panel (a) the asymptotic results for single eigenvalues and repeated roots are dashed and dotted lines respectively. In panel (b) the asymptotics are represented by small dots and dashed lines for the anti-periodic and periodic cases, respectively. The large dots show the low frequency asymptotics (see section 3.1), and the repeated solutions at $\Omega = \pi$ and $\Omega = 2\pi$ are shown by crosses (see section 3.3).
Figure 7: The real part of $u$ is shown for three frequencies: $\Omega = 0.1$ and values close to the repeated roots $\Omega_0 = \pi$ and $\Omega_0 = 2\pi$ (frequency $\Omega = \Omega_0 + 0.1$) are shown in panels (a), (b) and (c), respectively. The solid line represents the exact solution and locally periodic behaviour is identified in panels (a) and (c) and an anti-periodic in panel (b). The long-scale behaviour, given by the asymptotic envelope function $f(x)$ (see section 3.3 and 3.1), is depicted by crosses. For $x < -1$ the rapidly oscillatory field is shown as the solid lines. These displacements are shown for $r = 2$ and $r_0 = \sqrt{5}/2$. 
\[ x = -1 \] gives the asymptotic relations that \( 1 + R_{\text{asy}} = T_{\text{asy}} \) and

\[ i\Omega r_0(1 - R_{\text{asy}}) = \frac{T_{\text{asy}} U_{\text{asy},x}(-1)}{U_{\text{asy}}(-1)} \] (33)

where

\[ U_{\text{asy},x}(-1) = U_{0\zeta}(-1; \Omega_0) + \epsilon f_X(-1) + \epsilon^2 f_{XX}(-1)U_1(-1, \Omega_0) + \epsilon f_X(-1)U_{1\zeta}(-1; \Omega_0), \] (34)

and then solving these gives the asymptotic reflection and transmission coefficients \( R_{\text{asy}}, T_{\text{asy}} \).

To summarise, the HFH methodology requires the standing wave frequencies and associated Bloch eigensolutions to be identified at the edges of the Brillouin zone. Given this information the homogenized long-scale is governed by the function \( f \) that comes from (28) which contains the microstructural information through the coefficient \( T \). This is then connected to the semi-infinite perfect string through the continuity conditions, that is, \( u \) and \( u_x \) are continuous at the join, which does require the short-scale information which is captured by the Bloch eigenfunction. The algebra is slightly too lengthy to repeat here, but is completely tractable and \( R_{\text{asy}} \) and \( T_{\text{asy}} \) are found explicitly from (33) and are now compared to the exact solutions.

We now show the comparison of this HFH theory to the exact solution to gauge its accuracy. It is worth noting that repeated roots, discussed later, occur when \( r \) or \( 1/r \) are integers and the resulting dispersion curves are locally linear. We choose to illustrate the results for \( r = 1/3 \) and \( r_0 = \sqrt{5}/3 \) (see Fig. 2) as this shows the linear dispersion behaviour at frequencies that are integer multiples of \( 3\pi \) and the reflection coefficient is zero there, \( |R| = 0 \). We also show some results for \( r = 2 \) with \( r_0 = \sqrt{5/2} \) (see Fig. 6) as the linear behaviour occurs for both anti-periodic and periodic standing waves, more typically if \( r \) or \( 1/r \) are non-integer there is no linear behaviour and no repeated roots. In all cases the HFH asymptotics perform very well with a typical set of asymptotes shown in Fig. 3. Low frequency asymptotics are illustrated in Fig. 2, 5, 6 and 7.

We choose to discuss Fig. 2 in detail, which shows the magnitude of the reflection coefficient versus frequency in Fig. 2(a) and the corresponding dispersion curves in Fig. 2(b). The effect of the stop bands in (b) is clearly seen in (a) with perfect reflection occurring for frequencies within them. The case is chosen to have zero reflection with \( r_0 = \sqrt{5}/3 \), i.e. perfect transmission at \( \Omega = 0, 3\pi, ... \) and the problem is \( 3\pi \) periodic in \( \Omega \) as \( r = 1/3 \). The asymptotics of the dispersion curves are given, for the periodic case, by

\[ \Omega \sim \Omega_0 + \frac{T}{2\Omega_0}(\Omega^2 - \Omega_0^2)\kappa^2, \] (35)
and a similar formula for the anti-periodic case, which follow from the HFH equation (30) c.f. [10], which are shown in Fig. 2(b). The corresponding asymptotics for the reflection coefficient using (35) are shown in Fig. 2(a) and in more detail in Fig. 3.

The corresponding displacement field is shown in Fig. 4 for the real part of \( u \) for the exact solution and asymptotic HFH theory, with similar accuracy for the imaginary parts (not shown), for two frequencies. Fig. 4(a) is for a frequency near to a periodic standing wave and Fig. 4(b) near to an anti-periodic standing wave. The choice of a large \( \epsilon, \epsilon = 1/2, \) is meant to indicate how well the asymptotics perform relatively far from the exact standing wave frequency about which they were developed. In both cases the short-wave behaviour is as expected, and that is then modulated by the long-scale \( f(x) \) behaviour shown by crosses; the high frequency nature of the problem is emphasised by the field in the perfect string which is highly oscillatory. It is worth noting that the precise attachment conditions that we identified earlier for the asymptotics are very important as they set the correct phase and amplitude at the join.

In Fig. 5 the displacement given by the real part of the wavefunction at frequencies close to those showing perfect transmission is identical on the long-scale. This insightful observation coming from HFH suggests that the physics related to frequencies near to the origin can be used to interpret the same behaviour occurring at higher frequencies. Indeed one can generalise this, as shown in Fig. 6 and 7, for the choice of material parameters \( r = 2, r_0 = \sqrt{5/2}. \) In this case one obtains repeated roots with locally anti-periodic behaviour at \( \Omega = \pi, \) see Fig. 6(b), but with the same long-scale envelope function as the low frequency and periodic repeated roots case: the resultant displacement fields demonstrating this are shown in Fig. 7.

In more detail, Fig. 7 shows the short-wave behaviour near to the origin, a periodic standing wave and an anti-periodic standing wave, which is given by the exact transfer matrix solution and modulated by the long-scale function \( f(x) \) (in Fig. 7(a) by equation (20) and in Fig. 7(b) and (c) by equation (46)). Comparing Fig. 7(a) to Fig. 7(c), the same revelatory observation as in Fig. 5 is made: the long-scale displacements occurring at a frequency near to the origin are repeated periodically at higher frequencies. Hence, the low frequency asymptotic solution can be translated to a particular higher frequency, i.e. \( \Omega_0 = 4\pi, \) and still be in perfect agreement with the exact solution at that frequency as it exhibits low frequency-like behaviour.

As we have seen repeated eigenvalues occur and have interesting consequencies. For the infinite periodic case this is treated in [10] and we now generate the asymptotic changes required for the semi-infinite array.
3.3. Asymptotics for repeated roots

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\[
u_0(\xi, X) = f^{(1)}(X)u_0^{(1)}(\xi; \Omega_0) + f^{(2)}(X)u_0^{(2)}(\xi; \Omega_0).
\]

By imposing continuity conditions at \( \xi = 0 \), we deduce the following leading order solutions

\[
U_0^{(1)}(\xi; \Omega_0) = \begin{cases} 
\sin r\Omega_0 \xi & \text{for } 0 \leq \xi < 1 \\
r \sin \Omega_0 \xi & \text{for } -1 \leq \xi < 0
\end{cases} \quad (37)
\]

\[
U_0^{(2)}(\xi; \Omega_0) = \begin{cases} 
p \cos r\Omega_0 \xi & \text{for } 0 \leq \xi < 1 \\
p \cos \Omega_0 \xi & \text{for } -1 \leq \xi < 0
\end{cases} \quad (38)
\]

for \( p = (r \sin \Omega_0 \pm \sin r\Omega_0)/(\cos \Omega_0 \mp \cos r\Omega_0) \) with the upper/lower signs for periodic/antiperiodic cases respectively.

Proceeding to the equation of order \( \epsilon \) and applying the compatibility condition to it, two coupled ODEs for \( f^{(1,2)}(X) \) are derived from which the eigenvalue correction \( \Omega_1^2 \) is obtained. The coupled equations [10] are

\[
2f^{(i)}_X \int_{-1}^{1} U_0^{(j)}U_0^{(i)} d\xi \pm \Omega_1^2 \int_{-1}^{1} \left( f^{(j)}U_0^{(j)} + f^{(i)}U_0^{(i)}U_0^{(j)} \right) \frac{d\xi}{c^2(\xi)} = 0 \quad (39)
\]

where \( i, j = 1, 2 \) and \( i \neq j \). One can go one stage further and computing the integrals we obtain

\[
I_1 = \int_{-1}^{1} U_0^{(1)}U_0^{(2)} d\xi = -pr\Omega_0 \left( 1 - \frac{\sin 2\Omega_0}{4\Omega_0} - \frac{\sin 2r\Omega_0}{4r\Omega_0} \right), \quad (40)
\]

\[
I_2 = \int_{-1}^{1} U_0^{(2)}U_0^{(1)} d\xi = pr\Omega_0 \left( 1 + \frac{\sin 2\Omega_0}{4\Omega_0} + \frac{\sin 2r\Omega_0}{4r\Omega_0} \right), \quad (41)
\]

\[
I_3 = \int_{-1}^{1} U_0^{(1)} \frac{d\xi}{c^2(\xi)} = r^2 \left( 1 - \frac{\sin 2\Omega_0}{4\Omega_0} - \frac{\sin 2r\Omega_0}{4r\Omega_0} \right), \quad (42)
\]

\[
I_4 = \int_{-1}^{1} U_0^{(2)} \frac{d\xi}{c^2(\xi)} = p^2 \left( \frac{1 + r^2}{2} + \frac{\sin 2\Omega_0}{4\Omega_0} + \frac{\sin 2r\Omega_0}{4r\Omega_0} \right), \quad (43)
\]

\[
I = \int_{-1}^{1} U_0^{(1)}U_0^{(2)} \frac{d\xi}{c^2(\xi)} = \frac{pr}{4\Omega_0} \left( \cos 2\Omega_0 - \cos 2r\Omega_0 \right). \quad (44)
\]

Evaluation of the above integrals allows the calculation of the correction term \( \Omega_1 \). As an example, we now consider the repeated root \( \Omega_0 = 3\pi \), as shown in Fig. 2, and have \( r = 1/3 \) for which we have \( I_1 = -pr\pi, I_2 = pr\pi, I_3 = 1/9, I_4 = 10p^2/18 \) and \( I = 0 \). In
general the ODEs decouple to give
\begin{equation}
4I_1I_2f^{(1)}_{XX} - \Omega_1^2 I_3I_4f^{(1)} = 0 \quad \text{and} \quad f^{(2)} = f^{(1)}.
\end{equation}

The important point being that we now have effective equations for the repeated root cases. In general, the asymptotic envelope function $f^{(1)}(x)$ for propagating solutions follows as
\begin{equation}
f^{(1)}(x) = \exp \left( \text{sgn}(T)i \left( \frac{\Omega^2 - \Omega_0^2}{\sqrt{T}} \right) (x + 1) \right)
\end{equation}
with $T = -4I_1I_2/(I_3I_4)$. The effectiveness of the asymptotics for repeated roots is illustrated in Fig. 5 and 7.

Returning to the Floquet-Bloch conditions, $f^{(1)}(X) \sim \exp(i\kappa X)$, and $\Omega_1^2 = 18\pi\kappa/\sqrt{5}$ and hence we obtain
\begin{equation}
\Omega \sim 3\pi \pm \frac{3\pi\kappa}{\sqrt{5}}
\end{equation}
which is the linear asymptote presented in Fig. 2(b). The same approach is used successfully for different values of $\Omega$ and $\kappa$ as shown in Fig. 6(b).

### 4. Reflection from a finite number of stacks

Finite regions are a more stringent test upon the methodology, we now extract the exact solution for a finite stack of periodic layers inserted in an infinite homogeneous strip; the key is matching the periodic stack at both ends to the homogeneous material by imposing boundary conditions. This is then compared to the asymptotics.

To approach the finite array we again briefly utilise transfer matrices with vector $\mathbf{v} = (u, u\xi/i\Omega)^T$ and connect the field at the end of a cell where $\xi = -1$ via
\begin{equation}
\mathbf{v}(1) = P_2P_1\mathbf{v}(-1) = Q\mathbf{v}(-1),
\end{equation}
with $Q$ given by (4), and at the other end, $\xi = 2N - 1$, via
\begin{equation}
\mathbf{v}(2N - 1) = Q^N\mathbf{v}(-1),
\end{equation}
with
\begin{equation}
P_1 = \begin{pmatrix}
\cos \Omega & i \sin \Omega \\
i \sin \Omega & \cos \Omega
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
\cos \Omega r & i r \sin \Omega r \\
i r \sin \Omega r & \cos \Omega r
\end{pmatrix},
\end{equation}
The transfer matrix, $Q$, allows one to move effortlessly along the bilayers. We consider each end of the periodic stack independently and then solve simultaneously. At $x = -1$, the periodic stack is attached to a semi-infinite homogeneous strip $u_{xx} + \Omega^2 r_0^2 u = 0$ for
\(x < -1\) with \(r_0\) constant. Assuming an incident wave of unit amplitude from minus infinity and a reflected wave with reflection coefficient \(R\) then \(v(-1)\) is

\[
v(-1) = \begin{pmatrix} 1 & 1 \\ r_0 & -r_0 \end{pmatrix} \begin{pmatrix} 1 \\ R \end{pmatrix} = B_0 \begin{pmatrix} 1 \\ R \end{pmatrix}.
\]

(51)

Thence, as in [23] one obtains

\[
v(1) = QB_0 \begin{pmatrix} 1 \\ R \end{pmatrix}.
\]

(52)

The other end of the periodic stack, \(x = N\), is again joined to a semi-infinite homogeneous strip \(u_{xx} + \Omega^2 r_0^2 u = 0\) for \(x < -1\) with \(r_0\) constant. In this field we need only consider a transmitted wave characterised by a transmission coefficient \(T\). The boundary conditions to be satisfied are given by

\[
v(2N-1) = \begin{pmatrix} 1 & 1 \\ 1 & i r_0 \end{pmatrix} \begin{pmatrix} 0 \\ T \end{pmatrix} = A_0 \begin{pmatrix} 0 \\ T \end{pmatrix}.
\]

(53)

By combining (52) with (53), we deduce the following system of equations with \(R\) and \(T\) both being unknown

\[
\begin{pmatrix} 0 \\ T \end{pmatrix} = A_0^{-1} Q^N B_0 \begin{pmatrix} 1 \\ R \end{pmatrix} = M \begin{pmatrix} 1 \\ R \end{pmatrix}.
\]

(54)

The reflection coefficient, \(R\), is found by \(R = -m_{11}/m_{12}\) with \(m_{11}\) and \(m_{12}\) being entries of matrix \(M\).

Turning to the asymptotic methodology; the long-scale field in the piecewise string has the form

\[
f(x) = A_1 \exp \left( \text{sgn}(T)i \left( \frac{\Omega^2 - \Omega^2_0}{T} \right)^{\frac{1}{2}} (x + 1) \right)
\[+ A_2 \exp \left( -\text{sgn}(T)i \left( \frac{\Omega^2 - \Omega^2_0}{T} \right)^{\frac{1}{2}} (x - (2N - 1)) \right).
\]

(55)

The first term in (55) vanishes at \(x = -1\) and can be viewed as the resulting transmitted wave with transmission coefficient \(A_1\) after the incident wave, propagating from the left hand-side to the right hand-side, hits the first boundary (\(x = -1\)). This wave then reaches the second boundary at the other end of the periodic strip, \(x = 2N - 1\), which causes a reflected wave of amplitude \(A_2\) to travel back along the periodic strip and a new
transmitted wave along the homogeneous strip. The second term in (55) disappears at the second boundary \( x = 2N - 1 \). The boundary conditions are applied in a straightforward generalisation of the semi-infinite case.

Figure 8: The parameters for both (a) and (b) are \( r = 1/3, r_0 = 0.5 \) and the number of bilayers is \( N = 10 \). The dashed line represents the asymptotics and the solid line depicts the transfer matrix solution.

The comparison of the HFH theory to the exact solution is shown in Fig. 8 and 9. In Fig. 8 the magnitude of reflection coefficient versus frequency is illustrated for parameters \( r = 1/3 \) and \( r_0 = 0.5 \), when a normal incident wave is reflected from a finite periodic stack of \( N = 10 \). The asymptotics work very well and in order to see how well they do so we have to look at the blow-up near \( \Omega = 6.6170 \), Fig. 8(b).

Increasing the number of bilayers to \( N = 20 \), Fig. 9, denser oscillations are observed at propagating wave frequencies while the asymptotics seem to be performing even better. The accuracy of the HFH asymptotics in replicating the exact behaviour of the \( |R| \) near standing wave frequencies is clearly demonstrated in the partial enlargement of Fig. 9(a) in Fig. 9(b). The asymptotic solution can be improved even further by solving the equations at higher orders of \( \epsilon \), i.e. \( O(\epsilon^3), O(\epsilon^4) \).
5. Conclusions

The purpose of this article is to demonstrate that one can effectively replace a periodic medium by its high frequency counterpart for reflection and transmission scattering problems. The scattering of the periodic medium is accurately represented by the HFH model: it is evident that the theory works very well. Additionally we are able to show that the standard transfer matrix approach can be replaced by a discrete approach which is more effective in generating the exact solution.

Asymptotically, several interesting features appear. It is notable that one can have repeated roots of the dispersion relation for specific waves of material parameter, interestingly the asymptotics of the dispersion curves are then locally linear and the asymptotic behaviour is closely related to that at low frequencies. This is demonstrated explicitly for several cases with the nuance that the local spatial behaviour of waves being periodic and anti-periodic on the short-scale can be incorporated.

The important detail in capturing the scattering behaviour is to also incorporate the short-scale for the boundary condition, this coupling at the interface is crucial in order to
capture the reflection and transmission behaviour. Now this detail has been understood this opens the way to improved numerical simulations where a block or slab of periodic media is embedded within a homogeneous host medium. Future work will incorporate and develop this further.

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