# COMPUTING GENUS-ZERO TWISTED GROMOV-WITTEN INVARIANTS

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ABSTRACT. Twisted Gromov–Witten invariants are intersection numbers in moduli spaces of stable maps to a manifold or orbifold  $\mathcal X$  which depend in addition on a vector bundle over  $\mathcal X$  and an invertible multiplicative characteristic class. Special cases are closely related to local Gromov–Witten invariants of the bundle, and to genus-zero one-point invariants of complete intersections in  $\mathcal X$ . We develop tools for computing genus-zero twisted Gromov–Witten invariants of orbifolds and apply them to several examples. We prove a "quantum Lefschetz theorem" which expresses genus-zero one-point Gromov–Witten invariants of a complete intersection in terms of those of the ambient orbifold  $\mathcal X$ . We determine the genus-zero Gromov–Witten potential of the type A surface singularity  $\left[\mathbb{C}^2/\mathbb{Z}_n\right]$ . We also compute some genus-zero invariants of  $\left[\mathbb{C}^3/\mathbb{Z}_3\right]$ , verifying predictions of Aganagic–Bouchard–Klemm. In a self-contained Appendix, we determine the relationship between the quantum cohomology of the  $A_n$  surface singularity and that of its crepant resolution, thereby proving the Crepant Resolution Conjectures of Ruan and Bryan–Graber in this case.

#### 1. Introduction

Gromov–Witten invariants of a manifold or orbifold  $\mathcal{X}$  are integrals

$$\int_{[\mathcal{X}_{a,n,d}]^{\mathrm{vir}}} (\cdots)$$

of appropriate cohomology classes against the virtual fundamental class of a moduli space  $\mathcal{X}_{g,n,d}$  of stable maps to  $\mathcal{X}$ . They give the "virtual number" of genus-g degree-d curves in  $\mathcal{X}$  that carry n marked points constrained to lie in certain cycles  $A_1, \ldots, A_n$  in  $\mathcal{X}$ . The cycles  $A_1, \ldots, A_n$  determine the integrand in (1). It is often useful to be able to compute similar integrals

(2) 
$$\int_{[\mathcal{X}_{q,n,d}]^{\mathrm{vir}}} (\cdots) e(F_{g,n,d})$$

which involve in addition the Euler class  $e(F_{g,n,d})$  of an obstruction bundle  $F_{g,n,d}$  over  $\mathcal{X}_{g,n,d}$ .

**Example A.** Let  $E \to \mathcal{X}$  be a vector bundle which is *concave*. This means that

$$H^0(\mathcal{C}, f^*E) = 0$$

for all stable maps  $f: \mathcal{C} \to \mathcal{X}$  of non-zero degree. Let d be non-zero and let  $F_{g,n,d}$  be such that the fiber at the stable map  $f: \mathcal{C} \to \mathcal{X}$  is

$$F_{g,n,d}|_{f:\mathcal{C}\to\mathcal{X}} = H^1(\mathcal{C}, f^*E).$$

Then integrals (2) are Gromov-Witten invariants of the (non-compact) total space of E: they are local  $Gromov-Witten\ invariants\ [18]$ .

**Example B.** Let  $E \to \mathcal{X}$  be a vector bundle which is *convex*. This means that

$$H^1(\mathcal{C}, f^*E) = 0$$

for all genus-zero one-pointed stable maps  $f: \mathcal{C} \to \mathcal{X}$ . Let  $F_{0,1,d}$  be such that

$$F_{0,1,d}|_{f:\mathcal{C}\to\mathcal{X}}=H^0(\mathcal{C},f^{\star}E).$$

Then integrals (2) with g=0 and n=1 give Gromov–Witten invariants of a suborbifold of  $\mathcal{X}$  cut out by a section of E.

In this paper we consider twisted Gromov-Witten invariants. These are integrals

(3) 
$$\int_{[\mathcal{X}_{g,n,d}]^{\mathrm{vir}}} (\cdots) \, \boldsymbol{c}(F_{g,n,d})$$

involving an invertible multiplicative characteristic class<sup>2</sup> c applied to an "obstruction K-class"  $F_{g,n,d} \in K^0(\mathcal{X}_{g,n,d})$ ,

$$F_{g,n,d}|_{f:\mathcal{C}\to\mathcal{X}}=H^0(\mathcal{C},f^\star F)\ominus H^1(\mathcal{C},f^\star F),$$

where F is a vector bundle over  $\mathcal{X}$ . (We give a formal definition in Section 2.3 below.) When c is the trivial characteristic class, these coincide with ordinary Gromov-Witten invariants. The Euler class is not invertible, but nonetheless Examples A and B can be included in this framework as follows. Every vector bundle F carries the action of a torus T which rotates fibers and leaves the base invariant; we can always take  $T = \mathbb{C}^{\times}$ , and if F is the direct sum of line bundles then we can take  $T = (\mathbb{C}^{\times})^{\operatorname{rank} F}$ . The T-equivariant Euler class is invertible over the fraction field of  $H_T^{\bullet}(\{pt\})$ . Example B arises by taking F=E and c to be the T-equivariant Euler class, and then taking the non-equivariant limit. Example A arises by taking F = E and c to be the T-equivariant inverse Euler class, and then taking the non-equivariant limit. Twisted Gromov-Witten invariants also occur in virtual localization formulas [34] for the T-equivariant Gromov-Witten invariants of an orbifold  $\mathcal{Y}$  equipped with the action of a torus T. There  $\mathcal{X}_{q,n,d}$  is part of the T-fixed substack of the moduli stack of stable maps to  $\mathcal{Y}$  and c is the T-equivariant inverse Euler class. If we can compute twisted Gromov-Witten invariants, therefore, then we can compute local Gromov-Witten invariants, genus-zero one-point invariants of complete intersections, and T-equivariant Gromov-Witten invariants. Twisted Gromov–Witten invariants for other choices of c can be interpreted as Gromov-Witten invariants with values in generalized cohomology theories [33].

When  $\mathcal{X}$  is a manifold, one can compute twisted Gromov–Witten invariants using results of Coates–Givental [24]. They prove a "quantum Riemann–Roch theorem" expressing twisted Gromov–Witten invariants of all genera, for any choice of  $\boldsymbol{c}$  and F, in terms of ordinary Gromov–Witten invariants of  $\mathcal{X}$ . From this they deduce a "quantum Lefschetz theorem" which gives simple closed formulas for genus-zero twisted invariants in the case where F is the direct sum of convex line bundles and  $\boldsymbol{c}$  is the T-equivariant Euler class. This implies most of the known mirror theorems for toric complete intersections. The results in [24] are based on a Grothendieck–Riemann–Roch argument, essentially due to Mumford [51] and Faber–Pandharipande [27], and a geometric formalism introduced by Givental [32].

<sup>&</sup>lt;sup>1</sup>One-pointed stable maps are those with n=1.

<sup>&</sup>lt;sup>2</sup>A characteristic class  $\boldsymbol{c}$  is multiplicative if  $\boldsymbol{c}(E_1 \oplus E_2) = \boldsymbol{c}(E_1)\boldsymbol{c}(E_2)$ . It is invertible if  $\boldsymbol{c}(E)$  is invertible in  $H^{\bullet}(\mathcal{Y})$  whenever E is a vector bundle over  $\mathcal{Y}$ . Invertible multiplicative characteristic classes extend to K-theory:  $\boldsymbol{c}(E_1 \oplus E_2) = \boldsymbol{c}(E_1)\boldsymbol{c}(E_2)^{-1}$ .

A quantum Riemann–Roch theorem for orbifolds has been established by Tseng [58] using the Grothendieck–Riemann–Roch theorem of Toen [57]. Tseng also proved a version of quantum Lefschetz in the orbifold setting [58, Theorem 5.15], but this holds only under very restrictive hypotheses on the bundle F.

In this paper we prove a much more general quantum Lefschetz-style result for orbifolds. This is Theorem 4.6 below. It applies whenever F is a direct sum of line bundles, without restriction on the invertible multiplicative class  $\boldsymbol{c}$ , and determines genus-zero twisted Gromov–Witten invariants of an orbifold  $\mathcal X$  in terms of the ordinary Gromov–Witten invariants of  $\mathcal X$ . It removes many of the restrictive hypotheses from Tseng's result and (as did [37]) improves on Coates–Givental when  $\mathcal X$  is a manifold, in that:

- $\bullet$  the characteristic class c does not have to be an Euler class; and
- $\bullet$  the bundle F is not assumed to be convex.

In practice Theorem 4.6 is most useful in the situation of Examples A and B. This gives nothing new in the manifold setting, as these cases were already covered by [24], but the improvement for orbifolds is significant. We illustrate this with several examples. In Section 5 we consider the situation of Example A, computing certain genus-zero Gromov-Witten invariants of the quintic hypersurface in  $\mathbb{P}(1,1,1,1,2)$ . We also prove a quantum Lefschetz theorem for orbifolds, Corollary 5.1 below, which directly generalizes [24, Theorem 2] and [58, Theorem 5.15]. (This suffices, for example, to determine the even-degree part of the small quantum orbifold cohomology algebra of any of the 181 Fano 3-fold weighted projective complete intersections with terminal singularities: see [23, Proposition 1.10].) In Section 6 we consider the situation of Example B, computing in Section 6.2 the genus-zero Gromov-Witten potential of the type A surface singularity  $[\mathbb{C}^2/\mathbb{Z}_n]$ . This has been determined for n=2 by Bryan-Graber [12]; for n=3 by Bryan-Graber-Pandharipande [13]; for n = 4 by Bryan-Jiang [14]. Their methods are quite different from ours. Perroni [53] has studied the small quantum cohomology of orbifolds with transverse ADE singularities, and part of the potential for  $\mathbb{C}^2/\mathbb{Z}_n$  can be extracted from his results. Maulik [48] has computed the genus-zero Gromov-Witten potential and certain higher-genus Gromov-Witten invariants of  $[\mathbb{C}^2/\mathbb{Z}_n]$ . In Section 6.3 we compute certain genus-zero Gromov–Witten invariants of  $[\mathbb{C}^3/\mathbb{Z}_3]$  where  $\mathbb{Z}_3$  acts with weights (1,1,1), verifying predictions of Aganagic– Bouchard-Klemm [1]. In Appendix A, which can be read separately from the main text, we combine results from Section 6.2 with arguments from toric mirror symmetry to prove the Crepant Resolution Conjectures of Ruan and Bryan-Graber for the type A surface singularity  $[\mathbb{C}^2/\mathbb{Z}_n]$ : this is new for  $n \geq 5$ . In Appendix B we prove some foundational results, describing certain aspects of Givental's geometric formalism in terms of non-Noetherian formal schemes.

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#### 2. Preliminaries

In this section we fix notation for orbifold cohomology and orbifold Gromov–Witten theory. These notions were introduced by Chen–Ruan in the symplectic category; an algebraic version of the theory has been developed by Abramovich–Graber–Vistoli. We will assume that the reader is familiar with this material—see [23, Section 2] for a brief overview and the original sources [3, 16, 17] for a comprehensive treatment.

2.1. **Orbifold Cohomology.** We work in the algebraic category, using notation as follows.

 $\mathcal{X}$  a proper smooth Deligne–Mumford stack over  $\mathbb{C}$  with projective coarse moduli space.

 $\mathcal{I}\mathcal{X}$  the inertia stack of  $\mathcal{X}$ . A point of  $\mathcal{I}\mathcal{X}$  is a pair (x,g) with x a point of  $\mathcal{X}$  and  $g \in \operatorname{Aut}_{\mathcal{X}}(x)$ .

 $\mathcal{IX} = \coprod_{i \in \mathcal{I}} \mathcal{X}_i$  the decomposition of  $\mathcal{IX}$  into components; here  $\mathcal{I}$  is an index set

I the involution of  $\mathcal{IX}$  which sends (x,g) to  $(x,g^{-1})$ .

 $H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\mathbb{C})$  the orbifold cohomology groups of  $\mathcal{X}$ . These are the cohomology groups  $H^{\bullet}(\mathcal{I}\mathcal{X};\mathbb{C})$  of the inertia stack.

age a rational number associated to each component  $\mathcal{X}_i$  of the inertia stack. Chen–Ruan call this the degree-shifting number.

 $(\alpha, \beta)_{\text{orb}}$  the orbifold Poincaré pairing  $\int_{\mathcal{I}\mathcal{X}} \alpha \cup I^{\star}\beta$ 

The grading on orbifold cohomology is shifted by the age:  $\alpha \in H^p(\mathcal{X}_i; \mathbb{C})$  has degree  $\deg \alpha = p + 2 \operatorname{age}(\mathcal{X}_i)$ .

2.2. Moduli Spaces of Stable Maps. Let  $\mathcal{X}_{0,n,d}$  denote, as in [23, Section 2.2.1], the moduli stack of n-pointed genus-zero stable maps to  $\mathcal{X}$  of degree  $d \in H_2(\mathcal{X}; \mathbb{Q})$ . This is almost exactly what Abramovich–Graber–Vistoli call the *stack of twisted stable maps*  $\mathcal{K}_{0,n}(\mathcal{X},d)$ . The only difference is that they regard the degree as a curve class on the coarse moduli space of  $\mathcal{X}$ , whereas we regard it as an element of  $H_2(\mathcal{X}; \mathbb{Q})$ . We will not use the term "twisted stable map" as for us "twisted" means something different.

There are evaluation maps  $\operatorname{ev}_i: \mathcal{X}_{0,n,d} \to \overline{\mathcal{IX}}$ , one for each marked point, which take values in the rigidified cyclotomic inertia stack  $\overline{\mathcal{IX}}$ . Since there is a proper étale surjection  $\mathcal{IX} \to \overline{\mathcal{IX}}$ , we can use the evaluation maps to define cohomological pull-backs

$$(\mathrm{ev}_i)^*: H^{\bullet}_{\mathrm{orb}}(\mathcal{X}; \mathbb{C}) \to H^{\bullet}(\mathcal{X}_{0,n,d}; \mathbb{C})$$

even though the maps  $ev_i$  do not take values in the inertia stack IX. We write

$$[\mathcal{X}_{0,n,d}]^{\mathrm{vir}} \in H_{\bullet}(\mathcal{X}_{0,n,d}; \mathbb{C})$$

for the virtual fundamental class of the moduli stack and

$$\psi_i \in H^2(\mathcal{X}_{0,n,d}; \mathbb{C}), \qquad i \in \{1, 2, \dots, n\},\$$

for the first Chern class of the universal cotangent line bundle  $L_i$ . The fiber of  $L_i$  at the stable map  $f: \mathcal{C} \to \mathcal{X}$  is the cotangent line to the coarse moduli space of  $\mathcal{C}$  at the *i*th marked point.

2.3. Twisted and Untwisted Gromov-Witten Invariants. A more detailed account of the material in this section can be found in [58]. Twisted Gromov-Witten invariants are a family of invariants of  $\mathcal{X}$  which depend on an invertible multiplicative characteristic class  $\boldsymbol{c}$  and a vector bundle  $F \to \mathcal{X}$ . Throughout this paper we will take F to be the direct sum of line bundles,

$$F = \bigoplus_{j=1}^{j=r} F^{(j)}.$$

In applications below we will need to take c to be a T-equivariant cohomology class, where the torus  $T = (\mathbb{C}^{\times})^r$  acts on F by scaling the fibers. We write

$$H_T^{\bullet}(\{\mathrm{pt}\}) = \mathbb{C}[\lambda_1, \dots, \lambda_r]$$

where  $\lambda_i$  is Poincaré-dual to a hyperplane in the *i*th factor of  $(\mathbb{CP}^{\infty})^r \cong BT$ . Consider the universal family over  $\mathcal{X}_{0,n,d}$ 

$$\begin{array}{ccc} \mathcal{C}_{0,n,d} & \stackrel{f}{\longrightarrow} & \mathcal{X} \\ \downarrow & & \\ \mathcal{X}_{0,n,d} & & \end{array}$$

and define an element  $F_{0,n,d} \in K^0(\mathcal{X}_{0,n,d})$  by

$$F_{0,n,d} := \pi_! f^* F,$$

where  $\pi_!$  is the K-theoretic push-forward. Genus-zero twisted Gromov-Witten invariants of  $\mathcal{X}$  are intersection numbers of the form

(4) 
$$\langle \alpha_1 \psi^{k_1}, \dots, \alpha_n \psi^{k_n} \rangle_{0,n,d}^{\mathcal{X}, \text{tw}} := \int_{[\mathcal{X}_{0,n,d}]^{\text{vir}}} \boldsymbol{c}(F_{0,n,d}) \cup \prod_{i=1}^n \text{ev}_i^{\star}(\alpha_i) \cdot \psi_i^{k_i}$$

where  $\alpha_1, \ldots, \alpha_n \in H^{\bullet}_{\mathrm{orb}}(\mathcal{X}; \mathbb{C})$ ;  $k_1, \ldots, k_n$  are non-negative integers; and the integral denotes cap product with the virtual fundamental class. If  $\boldsymbol{c}$  is the trivial characteristic class — this is the case of usual, untwisted Gromov–Witten invariants — then we will replace the superscript "tw" by "un".

**Remark 2.1.** The Gromov-Witten invariants defined here coincide with those considered in [58]: we use slightly different stacks of stable maps and also a different definition of the pull-back  $(ev_i)^*$ , but these two differences cancel each other out. The descendant class denoted  $\psi_i$  here is denoted in [58] by  $\bar{\psi}_i$ .

Genus-zero twisted orbifold Gromov–Witten invariants together define a Frobenius manifold, as we now explain. Fix a Kähler class  $\omega$  on  $\mathcal{X}$ . Let  $\mathrm{Eff}(\mathcal{X})$  be the semigroup of degrees of representable maps from possibly-stacky curves to  $\mathcal{X}$  (i.e. of degrees of effective curves in  $\mathcal{X}$ ) and define the Novikov ring  $\Lambda$  to be the completion of the group ring  $\mathbb{C}[\mathrm{Eff}(\mathcal{X})]$  of  $\mathrm{Eff}(\mathcal{X})$  with respect to the additive valuation v,

$$v\left(\sum_{d\in \mathrm{Eff}(\mathcal{X})}a_dQ^d\right)=\min_{a_d\neq 0}\int_d\omega,$$

where  $Q^d$  is the element of  $\mathbb{C}[\mathrm{Eff}(\mathcal{X})]$  corresponding to  $d \in \mathrm{Eff}(\mathcal{X})$ . Note that the completion depends on the choice of Kähler class  $\omega$ . The Frobenius manifold is based on the free  $\Lambda$ -module

$$H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\Lambda) := H^{\bullet}(\mathcal{I}\mathcal{X};\mathbb{C}) \otimes_{\mathbb{C}} \Lambda.$$

To define the pairing, observe that the inertia stack  $\mathcal{I}F$  of the total space of the vector bundle  $F \to \mathcal{X}$  is a vector bundle over  $\mathcal{I}\mathcal{X}$  — the fiber of  $\mathcal{I}F$  over the point  $(x,g) \in \mathcal{I}\mathcal{X}$  consists of the g-fixed subspace of the fiber of F over x — and set

(5) 
$$(\alpha, \beta)_{\text{orb}}^{\text{tw}} = \int_{\mathcal{IX}} \alpha \cup I^{\star} \beta \cup \boldsymbol{c}(\mathcal{I}F).$$

**Example 2.2.** Let  $\mathcal{X} = B\mu_r$  and let  $F \to \mathcal{X}$  be the tautological line bundle. Then  $\mathcal{I}\mathcal{X}$  is the disjoint union of r copies of  $B\mu_r$  where the jth copy,  $0 \le j < r$ , corresponds to the element  $\zeta^j = \exp\left(\frac{2\pi\sqrt{-1}j}{r}\right) \in \mu_r$ . For  $j \ne 0$ ,  $\zeta^j$  acts non-trivially on the fiber of F and so the fiber of  $\mathcal{I}F$  over the jth copy of  $B\mu_r$  is the zero-dimensional vector space. The restriction of  $\mathcal{I}F$  to the zeroth copy of  $B\mu_r$  is F.

Genus-zero twisted Gromov–Witten invariants assemble to give a family of products, defined by

(6) 
$$(\alpha \bullet_{\tau} \beta, \gamma)_{\text{orb}}^{\text{tw}} = \sum_{d \in \text{Eff}(\mathcal{X})} \sum_{n \geq 0} \frac{Q^d}{n!} \langle \alpha, \beta, \gamma, \tau, \tau, \dots, \tau \rangle_{0, n+3, d}^{\mathcal{X}, \text{tw}},$$

parametrized by  $\tau$  in a formal neighbourhood of zero in  $H^{\bullet}_{\rm orb}(\mathcal{X};\Lambda)$ . When c=1, this gives the usual Frobenius manifold structure on orbifold cohomology.

## 3. GIVENTAL'S SYMPLECTIC FORMALISM

In this section we will descibe how to encode genus-zero twisted orbifold Gromov—Witten invariants in a Lagrangian submanifold of a certain symplectic vector space. This idea is due to Givental [33]; it was adapted to the orbifold setting by Tseng [58]. We will describe only the aspects of the theory which we need, referring the reader to [33,58] and the references therein for motivation, context, and further examples of this approach. In particular the genus-zero picture used here is only part of a more powerful formalism involving Gromov—Witten invariants of all genera, and we will not discuss this at all.

**Definition.** For a topological ring R with a non-negative additive valuation  $v: R \setminus \{0\} \to \mathbb{R}_{\geq 0}$ , define the space of *convergent Laurent series* in z to be

$$R\{z,z^{-1}\}:=\left\{\sum_{n\in\mathbb{Z}}r_nz^n\ :\ r_n\in R,\ v(r_n)\to\infty\ \text{as}\ |n|\to\infty\right\}.$$

If R is complete, this becomes a ring<sup>3</sup>. Set

$$R\{z\} := \left\{ \sum_{n \ge 0} r_n z^n : r_n \in R, \ v(r_n) \to \infty \text{ as } n \to \infty \right\}.$$

<sup>&</sup>lt;sup>3</sup>In this case,  $R\{z, z^{-1}\}$  coincides with the completion of  $R[z, z^{-1}]$  under the induced valuation with v(z) = 0.

Consider the space of orbifold-cohomology-valued convergent Laurent series

$$\mathcal{H} := H^{\bullet}_{\mathrm{orb}}(\mathcal{X}; \mathbb{C}) \otimes \Lambda\{z, z^{-1}\}$$

equipped with the  $\Lambda$ -valued symplectic form

$$\Omega^{\text{tw}}(f,g) := \text{Res}_{z=0} \left( f(-z), g(z) \right)_{\text{orb}}^{\text{tw}} dz.$$

We encode genus-zero twisted orbifold Gromov-Witten invariants via the germ of a Lagrangian submanifold  $\mathcal{L}^{tw}$  of  $(\mathcal{H}, \Omega^{tw})$ , defined as follows. Let

$$\{\phi_{\alpha}: 1 \leq \alpha \leq N\}$$
 and  $\{\phi^{\alpha}: 1 \leq \alpha \leq N\}$ 

be  $\Lambda$ -bases for  $H^{\bullet}_{\mathrm{orb}}(\mathcal{X}; \Lambda)$  which are dual with repect to the pairing (5). The submanifold-germ  $\mathcal{L}^{\mathrm{tw}}$  consists of all points of  $\mathcal{H}$  of the form

(7) 
$$-z+t_0+t_1z+t_2z^2+\cdots$$

$$+ \sum_{\substack{d \in \text{Eff}(\mathcal{X}) \\ n > 0}} \sum_{\substack{i_1, \dots, i_n \\ \alpha_1, \dots, \alpha_n}} \sum_{\substack{k \geq 0 \\ 1 \leq \epsilon < N}} \frac{Q^d t_{i_1}^{\alpha_1} \cdots t_{i_n}^{\alpha_n}}{n!} \left\langle \phi_{\alpha_1} \psi^{i_1}, \dots, \phi_{\alpha_n} \psi^{i_n}, \phi_{\epsilon} \psi^k \right\rangle_{0, n+1, d}^{\mathcal{X}, \text{tw}} \frac{\phi^{\epsilon}}{(-z)^{k+1}}$$

where  $t_0 + t_1 z + t_2 z^2 + \dots$  lies in a formal neighbourhood of zero in  $H^{\bullet}_{\text{orb}}(\mathcal{X}; \mathbb{C}) \otimes \Lambda\{z\}$  and  $t_i = \sum_{\alpha} t_i^{\alpha} \phi_{\alpha}$ . If we write  $\mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \dots$  then (7) is

$$-z + \mathbf{t}(z) + \sum_{n \geq 0} \sum_{d \in \text{Eff}(\mathcal{X})} \sum_{1 \leq \epsilon \leq N} \frac{Q^d}{n!} \left\langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi), \frac{\phi_{\epsilon}}{-z - \psi} \right\rangle_{0, n+1, d}^{\mathcal{X}, \text{tw}} \phi^{\epsilon}.$$

The submanifold-germ  $\mathcal{L}^{\text{tw}}$  has extremely special geometric properties, which are listed in [20, Theorem 2.15] and [58, Section 3.1]. These follow from the fact that genus-zero twisted Gromov–Witten invariants satisfy the String Equation, the Dilaton Equation, and the Topological Recursion Relations [33,58].

A Remark on Rigour. The definition of  $\mathcal{L}^{tw}$  just given is not completely rigorous, as we did not spell out exactly what we mean by a formal neighbourhood in an infinite-dimensional vector space. A rigorous definition of  $\mathcal{L}^{tw}$ , as a non-Noetherian formal scheme, is given in Appendix B. There we also establish various geometric properties of  $\mathcal{L}^{tw}$  which will be needed later: see Propositions B.2, B.3, B.4, and Corollary B.7. The rest of this paper can therefore be read in two ways. The reader who is happy to work with an intuitive notion of formal neighbourhood can read the rest of the text as it is, omitting Appendix B. The discussion will then be informal, but no serious confusion should result. The reader who prefers a completely formal approach should at this point skip to Appendix B, and replace the definition (7) above with definition (44) below. The rest of the text can then be read as a series of rigorous arguments within the framework constructed in Appendix B.

## 3.1. The Twisted J-Function. Let

(8) 
$$J^{\text{tw}}(\tau, z) = z + \tau + \sum_{n \ge 0} \sum_{d \in \text{Eff}(\mathcal{X})} \sum_{1 \le \epsilon \le N} \frac{Q^d}{n!} \left\langle \tau, \tau, \dots, \tau, \frac{\phi_{\epsilon}}{z - \psi} \right\rangle_{0, n+1, d}^{\mathcal{X}, \text{tw}} \phi^{\epsilon}.$$

This formal power series in the components  $\tau^1, \ldots, \tau^N$  of  $\tau = \tau^1 \phi_1 + \ldots + \tau^N \phi_N$ , called the *twisted J-function of*  $\mathcal{X}$ , will play an important role below. It takes values in  $\mathcal{H}$  and gives a distinguished family

$$\tau \longmapsto J^{\mathrm{tw}}(\tau,-z), \qquad \tau \text{ in a formal neighbourhood of zero in } H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\Lambda),$$

of elements of  $\mathcal{L}^{tw}$  characterized among such families by the property that

(9) 
$$J^{\text{tw}}(\tau, -z) = -z + \tau + O(z^{-1}).$$

We write  $J^{\mathrm{un}}$ ,  $\mathcal{L}^{\mathrm{un}}$ , and  $\Omega^{\mathrm{un}}$  for the specializations of, respectively,  $J^{\mathrm{tw}}$ ,  $\mathcal{L}^{\mathrm{tw}}$ , and  $\Omega^{\mathrm{tw}}$  to the case  $\boldsymbol{c}=1$ — *i.e.* for the corresponding objects in untwisted Gromov–Witten theory. The untwisted J-function satisfies a system of differential equations

(10) 
$$z \frac{\partial}{\partial \tau^{\alpha}} \frac{\partial}{\partial \tau^{\beta}} J^{\mathrm{un}}(\tau, z) = \sum_{\gamma=1}^{N} c_{\alpha\beta}{}^{\gamma}(\tau) \frac{\partial}{\partial \tau^{\gamma}} J^{\mathrm{un}}(\tau, z)$$

where  $c_{\alpha\beta}^{\ \ \gamma}(\tau)$  are the structure constants of the untwisted multiplication with respect to the basis  $\{\phi_{\epsilon}\}$ :

$$\phi_{\alpha} \bullet_{\tau} \phi_{\beta} \Big|_{\boldsymbol{c}=1} = \sum_{\gamma=1}^{N} c_{\alpha\beta}^{\gamma}(\tau) \phi_{\gamma}.$$

One can see this either as a consequence of the geometric properties of  $\mathcal{L}^{un}$  [33], or directly from the Topological Recursion Relations as in [23, Lemma 2.4] or [52, Proposition 2].

#### 4. General Twists in Genus Zero

In this section we give a formula for a family of elements on the Lagrangian submanifold  $\mathcal{L}^{\text{tw}}$  for the twisted theory. The key ingredient is Tseng's genus-zero orbifold quantum Riemann–Roch theorem [58], so we begin by stating this.

4.1. **Orbifold Quantum Riemann–Roch.** Given a line bundle  $L \to \mathcal{X}$  and a geometric point  $(x,g) \in \mathcal{IX}$ , there is a unique rational number  $f \in [0,1)$  such that g acts on the fiber of L over x by multiplication by  $\exp\left(2\pi\sqrt{-1}f\right)$ . The value of f depends only on the component  $\mathcal{X}_i$  of  $\mathcal{IX}$  containing (x,g). Since F is the direct sum of line bundles,

$$F = \bigoplus_{j=1}^{j=r} F^{(j)},$$

this defines a collection of rational numbers  $f_i^{(j)}$  where  $1 \leq j \leq r$  and  $i \in \mathcal{I}$ . The other ingredients in the statement are the first Chern classes  $\rho^{(j)} \in H^2(\mathcal{X}; \mathbb{C})$  of  $F^{(j)}$ , regarded as elements of orbifold cohomology via the natural inclusion  $\mathcal{X} \to \mathcal{I}\mathcal{X}$ , and the (unique) sequence of parameters  $s_0, s_1, s_2, \ldots$  such that

(11) 
$$c(\cdot) = \exp\left(\sum_{k\geq 0} s_k \operatorname{ch}_k(\cdot)\right).$$

Here  $\operatorname{ch}_k$  is the kth component of the Chern character. We add the variables  $s_k$  to our ground ring, working henceforth over the completion  $\Lambda[s_0, s_1, \ldots]$  of  $\mathbb{C}[\operatorname{Eff}(\mathcal{X})][s_0, s_1, \ldots]$  with respect to the additive valuation v such that

$$v(Q^d) = \int_d \omega, \quad v(s_k) = k + 1.$$

Later we will need the notation

$$s(x) = \sum_{k>0} s_k \frac{x^k}{k!}.$$

Recall that the Bernoulli polynomials  $B_n(x)$  are defined by

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{ze^{zx}}{e^z - 1}.$$

**Theorem 4.1** ([58, Corollary 1]). The transformation  $\Delta^{tw}: \mathcal{H} \to \mathcal{H}$  defined by

$$\Delta^{\text{tw}} = \bigoplus_{i \in \mathcal{I}} \prod_{j=1}^r \exp \left( \sum_{l,m \ge 0} s_{l+m-1} \frac{B_m(f_i^{(j)})}{m!} \frac{(\rho^{(j)})^l}{l!} z^{m-1} \right),$$

where  $\rho^{(j)}$  acts on  $\mathcal{H}$  via the Chen-Ruan orbifold cup product [3,16], gives a linear symplectomorphism between  $(\mathcal{H}, \Omega^{\mathrm{un}})$  and  $(\mathcal{H}, \Omega^{\mathrm{tw}})$ , and

$$\mathcal{L}^{\mathrm{tw}} = \Delta^{\mathrm{tw}} \left( \mathcal{L}^{\mathrm{un}} \right).$$

Here we are implicitly using the facts that

$$H^{\bullet}_{\mathrm{orb}}(\mathcal{X}; \mathbb{C}) = \bigoplus_{i \in \mathcal{I}} H^{\bullet}(\mathcal{X}_i; \mathbb{C})$$

and that the action of  $\rho^{(j)}$  preserves this decomposition. The Chen-Ruan orbifold cup product by  $\rho^{(j)}$  coincides with the ordinary cup product by  $\pi^*\rho^{(j)}$  [58, Lemma 2.3.7], where  $\pi\colon \mathcal{IX}\to\mathcal{X}$  is the natural projection. We define  $s_{-1}$  to be zero.

Remark 4.2. Multiplication by  $\sqrt{c(\mathcal{I}F)}$ , using the usual cup product on  $H^{\bullet}(\mathcal{I}\mathcal{X};\mathbb{C})$ , gives an isomorphism between the symplectic vector spaces  $(\mathcal{H}, \Omega^{\text{tw}})$  and  $(\mathcal{H}, \Omega^{\text{un}})$ . The transformation  $\Delta^{\text{tw}}$  appearing above differs from that in [58] because the transformation there was regarded as an automorphism of  $(\mathcal{H}, \Omega^{\text{un}})$  via this identification  $(\mathcal{H}, \Omega^{\text{tw}}) \cong (\mathcal{H}, \Omega^{\text{un}})$ 

4.2. A Family of Elements of  $\mathcal{L}^{\text{tw}}$ . It will be convenient to break up the untwisted J-function  $J^{\text{un}}(\tau, z)$  into contributions from stable maps of different topological types.

**Definition 4.3.** The topological type of a stable map  $f: \mathcal{C} \to \mathcal{X}$ , where  $\mathcal{C}$  has genus g and marked points  $x_1, \ldots, x_n$  and f has degree  $d \in H_2(\mathcal{X}; \mathbb{Q})$ , is the triple

$$\theta = (q, d, S)$$

where S is the ordered n-tuple consisting the elements of  $\mathcal{I}$  which label the components of the inertia stack picked out by the marked points  $x_1, \ldots, x_n$ .

The topological type is constant on each component of the moduli space  $\mathcal{X}_{0,n,d}$ . We write NETT( $\mathcal{X}$ ) for the set of all topological types of stable maps to  $\mathcal{X}$ , or in other words for the set of effective topological types in  $\mathcal{X}$ , and  $J_{\theta}(\tau, z)$  for the contribution to the untwisted J-function from stable maps of topological type  $\theta$ , so that

$$J^{\mathrm{un}}(\tau, z) = \sum_{\theta \in \mathrm{NETT}(\mathcal{X})} J_{\theta}(\tau, z).$$

**Remark 4.4.** In practice one can determine this decomposition by choosing the basis  $\{\phi_{\alpha}\}$  for  $H^{\bullet}_{\text{orb}}(\mathcal{X};\Lambda)$  so that each  $\phi_{\alpha}$  is supported on exactly one component  $\mathcal{X}_{i(\alpha)}$  of  $\mathcal{I}\mathcal{X}$ . Then the term

$$\frac{Q^d \tau^{\alpha_1} \cdots \tau^{\alpha_n}}{n! (z)^{k+1}} \langle \phi_{\alpha_1}, \dots, \phi_{\alpha_n}, \phi_{\epsilon} \psi^k \rangle_{0, n+1, d}^{\mathcal{X}, \text{un}} \phi^{\epsilon}$$

in the power series expansion (8) of  $J^{\mathrm{un}}(\tau^1\phi_1 + \cdots + \tau^N\phi_N, z)$  contributes to  $J_{\theta}$  only for  $\theta = (0, d, S)$  where  $S = (i(\alpha_1), i(\alpha_2), \dots, i(\alpha_n), i(\epsilon))$ .

**Lemma 4.5.** Let  $\theta \in NETT(\mathcal{X})$  be the topological type (0, d, S) where  $S = (i_1, \dots, i_n)$ . Then

- (1) The orbifold cohomology class  $J_{\theta}(\tau, z)$  is supported on the component  $I(\mathcal{X}_{i_n})$  of  $\mathcal{IX}$ .
- (2) If  $D_i$  is the dilation vector field on  $H^{\bullet}(\mathcal{X}_i; \mathbb{C}) \subset H^{\bullet}_{\mathrm{orb}}(\mathcal{X}; \mathbb{C})$ , so

$$D_i = \sum_{\nu} x^{\nu} \frac{\partial}{\partial x^{\nu}}$$

for any linear co-ordinate system  $(x^{\nu})$  on  $H^{\bullet}(\mathcal{X}_i; \mathbb{C})$ , then

$$D_i J_{\theta}(\tau, z) = n_i J_{\theta}(\tau, z)$$

where  $n_i$  is the number of times that i occurs in  $(i_1, \ldots, i_{n-1})$ .

(3) If  $\rho \in H^2(\mathcal{X}; \mathbb{C})$  is regarded as an orbifold cohomology class via the natural inclusion  $\mathcal{X} \to \mathcal{I}\mathcal{X}$  then

$$z\nabla_{\rho}J_{\theta}(\tau,z) = (\rho + \langle \rho, d \rangle z) J_{\theta}(\tau,z).$$

Here  $\nabla_{\rho}$  is the directional derivative:  $\nabla_{\rho}J_{\theta}(\tau) := \frac{d}{dt}J_{\theta}(\tau + t\rho)|_{t=0}$ .

*Proof.* (1) and (2) follow immediately from Remark 4.4. (3) follows from the Divisor Equation [3, Theorem 8.3.1].

Consider a topological type  $\theta \in \text{NETT}(\mathcal{X})$  with  $\theta = (0, d, S)$  and  $S = (i_1, \dots, i_n)$ . Let  $\bar{\imath}_n \in \mathcal{I}$  be such that the component  $\mathcal{X}_{\bar{\imath}_n}$  is  $I(\mathcal{X}_{i_n})$ , and let

$$N_{\theta}^{(j)} = \left\langle \rho^{(j)}, d \right\rangle - f_{i_1}^{(j)} - f_{i_2}^{(j)} - \dots - f_{i_{n-1}}^{(j)} + f_{\bar{\imath}_n}^{(j)}.$$

Riemann–Roch for orbifold curves implies that  $N_{\theta}^{(j)}$  is an integer: for any stable map  $h: \mathcal{C} \to \mathcal{X}$  of topological type  $\theta$ , the Euler characteristic

$$\chi\left(\mathcal{C}, h^*F^{(j)}\right) = 1 + \langle \rho^{(j)}, d \rangle - f_{i_1}^{(j)} - f_{i_2}^{(j)} - \dots - f_{i_n}^{(j)},$$

and

$$f_{\bar{\imath}_n}^{(j)} = \begin{cases} 0 & \text{if } f_{i_n}^{(j)} = 0\\ 1 - f_{i_n}^{(j)} & \text{if } f_{i_n}^{(j)} \neq 0. \end{cases}$$

Define the modification factor

$$M_{\theta}(z) = \prod_{j=1}^{j=r} \frac{\prod_{-\infty < m \le N_{\theta}^{(j)}} \exp\left[s\left(\rho^{(j)} + \left(m - f_{\bar{\imath}_n}^{(j)}\right)z\right)\right]}{\prod_{-\infty < m \le 0} \exp\left[s\left(\rho^{(j)} + \left(m - f_{\bar{\imath}_n}^{(j)}\right)z\right)\right]}$$

and set

$$I^{\text{tw}}(\tau, z) = \sum_{\theta \in \text{NETT}(\mathcal{X})} M_{\theta}(z) \cdot J_{\theta}(\tau, z).$$

where the multiplication is with respect to the Chen–Ruan orbifold cup product.  $I^{\text{tw}}$  is a formal power series in the components  $\tau^1, \ldots, \tau^N$  of  $\tau$  which takes values in  $\mathcal{H}$ . When  $\mathcal{X}$  is a variety and  $\mathbf{c}$  is the T-equivariant Euler class, it coincides with the *hypergeometric modification*  $I_F$  in [24, Section 7].

Theorem 4.6. The family

$$\tau \mapsto I^{\mathrm{tw}}(\tau, -z)$$

of elements of  $\mathcal{H}$  lies on the Lagrangian submanifold  $\mathcal{L}^{tw}$ .

**Remark.** In the formal framework developed in Appendix B below, Theorem 4.6 is the statement that  $I^{\text{tw}}(\tau, -z)$  is a  $\Lambda[s_0, s_1, \dots][\tau]$ -valued point of  $\mathcal{L}^{\text{tw}}$ .

Proof of Theorem 4.6. We will assume throughout the proof that F is a line bundle and omit the index "(j)", writing  $\rho$  for  $\rho^{(j)}$ ;  $f_i$  for  $f_i^{(j)}$ ;  $N_\theta$  for  $N_\theta^{(j)}$ ; and so on. The proof in the case where F is a direct sum of line bundles requires only notational changes.

Define an element  $G_y(x,z)$  in  $\mathbb{C}[y,x,z,z^{-1}][s_0,s_1,s_2,\ldots]$  by

$$G_y(x,z) := \sum_{l,m>0} s_{l+m-1} \frac{B_m(y)}{m!} \frac{x^l}{l!} z^{m-1}.$$

This satisfies functional equations of gamma-function type:

(12) 
$$G_{y}(x,z) = G_{0}(x+yz,z),$$

(13) 
$$G_0(x+z,z) = G_0(x,z) + s(x).$$

Equality (12) follows from the fact that the coefficient of  $s_k$  in  $G_y(x, z)$  is the degree k part of

$$\left(\sum_{m=0}^{\infty} \frac{B_m(y)}{m!} z^{m-1}\right) \left(\sum_{l=0}^{\infty} \frac{x^l}{l!}\right) = \frac{e^{x+yz}}{e^z - 1},$$

where  $\deg x = \deg z = 1$  and  $\deg y = 0$ . Equality (13) follows from

$$\frac{e^{x+z}}{e^z-1} = \frac{e^x}{e^z-1} + e^x.$$

We need to show that  $I^{\text{tw}}(\tau, -z) \in \mathcal{L}^{\text{tw}}$ . As before, write elements  $\theta \in \text{NETT}(\mathcal{X})$  as  $\theta = (g, d, S)$  with  $S = (i_1, \dots, i_n)$ . Observe that

$$M_{\theta}(-z) = \exp\left(\sum_{m=K}^{m=N_{\theta}} s\left(\rho + (f_{\bar{\imath}_{n}} - m)z\right) - \sum_{m=K}^{m=0} s\left(\rho + (f_{\bar{\imath}_{n}} - m)z\right)\right) \quad \text{for } K \ll 0$$

$$= \exp\left(G_{0}\left(\rho + f_{\bar{\imath}_{n}}z, z\right) - G_{0}\left(\rho + (f_{\bar{\imath}_{n}} - N_{\theta})z, z\right)\right) \quad \text{by (13)}$$

$$= \exp\left(G_{f_{\bar{\imath}_{n}}}\left(\rho, z\right) - G_{0}\left(\rho + (f_{\bar{\imath}_{n}} - N_{\theta})z, z\right)\right) \quad \text{by (12)}.$$

We know that

$$\Delta^{\text{tw}} = \bigoplus_{i \in \mathcal{I}} \exp\Big(G_{f_i}(\rho, z)\Big),$$

that  $\Delta^{\text{tw}}(\mathcal{L}^{\text{un}}) = \mathcal{L}^{\text{tw}}$ , and that  $J_{\theta}(\tau, -z)$  is supported on the component  $\mathcal{X}_{\bar{\imath}_n}$  of  $\mathcal{I}\mathcal{X}$ . It therefore suffices to show that the family

$$\tau \mapsto \sum_{\theta \in \text{NETT}(\mathcal{X})} \exp\left(-G_0\left(\rho + (f_{\bar{\imath}_n} - N_{\theta})z, z\right)\right) J_{\theta}(\tau, -z)$$

of elements of  $\mathcal{H}$  lies in the Lagrangian submanifold  $\mathcal{L}^{\mathrm{un}}$  for the *untwisted* theory.

$$\exp\left(-G_0\left(\rho + (f_{\bar{\imath}_n} - N_\theta)z, z\right)\right) J_\theta(\tau, -z)$$

$$= \exp\left(-G_0\left(\rho - \langle \rho, d \rangle z + f_{i_1}z + f_{i_2}z + \dots + f_{i_{n-1}}z, z\right)\right) J_\theta(\tau, -z),$$

and Lemma 4.5 shows that this is

$$\exp\left(-G_0\left(z\nabla_{\rho}+zD,z\right)\right)J_{\theta}(\tau,-z)$$

where  $D = \sum_{i \in \mathcal{I}} f_i D_i$ . We thus want to show that the family

(14) 
$$\tau \mapsto \exp\left(-G_0\left(z\nabla_{\rho} + zD, z\right)\right) J^{\mathrm{un}}(\tau, -z)$$

lies on  $\mathcal{L}^{\mathrm{un}}$ 

This last statement follows from the geometric properties of  $\mathcal{L}^{un}$  established in Appendix B. Let h be a general point in a formal neighbourhood of -z in  $\mathcal{H}$ :

$$h = -z + \sum_{k=0}^{\infty} t_k z^k + \sum_{k=0}^{\infty} \frac{p_k}{(-z)^{k+1}}, \qquad t_k, p_k \in H_{\mathrm{orb}}^{\bullet}(\mathcal{X}; \Lambda).$$

Then  $\mathcal{L}^{\text{un}}$  is defined by the equations  $E_0 = 0$ ,  $E_1 = 0$ ,  $E_2 = 0$ ,... (c.f. (7) above and (44) below), where

$$E_j(h) := p_j - \sum_{n \ge 0} \sum_{d \in \text{Eff}(\mathcal{X})} \sum_{1 \le \epsilon \le N} \frac{Q^d}{n!} \left\langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi), \psi^j \phi_{\epsilon} \right\rangle_{0, n+1, d}^{\mathcal{X}, \text{un}} \phi^{\epsilon}.$$

Let  $\tau \mapsto J_s(\tau, -z)$  be the family from (14). The application of  $E_j$  to  $J_s(\tau, -z)$  is  $(\tau, s, Q)$ -adically convergent; we want to show that it is zero. It is obvious that  $E_j(J_s(\tau, -z))$  is zero at  $s_0 = s_1 = \cdots = 0$ . Set  $\deg s_k = k+1$  and assume by induction that  $E_j(J_s(\tau, -z))$  vanishes up to degree n in variables  $s_0, s_1, s_2, \ldots$ . Then we have

(15) 
$$\frac{\partial}{\partial s_i} E_j(J_{\mathbf{s}}(\tau, -z)) = d_{J_{\mathbf{s}}(\tau, -z)} E_j\left(z^{-1} P_i(z\nabla, z) J_{\mathbf{s}}(\tau, -z)\right),$$

where

$$P_i(z\nabla, z) = \sum_{m=0}^{i+1} \frac{1}{m!(i+1-m)!} z^m B_m(0) (z\nabla_\rho + zD)^{i+1-m}.$$

The induction hypothesis shows that we can find a family  $\tau \mapsto \widetilde{J}_s(\tau, -z)$  of elements of  $\mathcal{L}^{\mathrm{un}}$  (i.e., in the language of Appendix B, a  $\Lambda[s_0, s_1, \dots][\tau]$ -valued point of  $\mathcal{L}^{\mathrm{un}}$ ) such that  $[\widetilde{J}_s]_+ = [J_s]_+$  and that  $\widetilde{J}_s - J_s$  consists of terms of degree greater than n in  $s_0, s_1, s_2, \dots$  ( $[J]_+$  discards all negative powers of z in J). Then the right hand side of (15) coincides with

$$d_{\widetilde{J}_{\boldsymbol{s}}(\tau,-z)}E_{j}\left(z^{-1}P_{i}(z\nabla,z)\widetilde{J}_{\boldsymbol{s}}(\tau,-z)\right)$$

up to degree n. But this is zero, as repeated applications of Lemma B.1 and Corollary B.7 show that the term in parenthesis is an element of  $T_{\tilde{J}_s(\tau,-z)}\mathcal{L}^{\mathrm{un}}$ .

Hence the left hand side of (15) vanishes up to degree n. This completes the induction step, and the proof.

#### 5. Application 1: Genus-Zero Invariants of Hypersurfaces

It is well-known that, for a complete intersection Y which is cut out of a projective variety X by a section of a direct sum  $E \to X$  of convex line bundles, many genus-zero one-point Gromov–Witten invariants of Y can be obtained as the non-equivariant limit of twisted genus-zero Gromov–Witten invariants of X: one takes F = E and c to be the T-equivariant Euler class. This idea lies at the heart of most proofs of mirror theorems for toric complete intersections [7,30,31,41,43-46]. The same thing holds for complete intersections in orbifolds: given a direct sum  $E \to \mathcal{X}$  of convex line bundles, the non-equivariant limit of a genus-zero twisted one-point Gromov–Witten invariant of  $\mathcal{X}$  (with F = E and c the T-equivariant Euler class) is a genus-zero Gromov–Witten invariant of the complete intersection  $\mathcal{Y} \subset \mathcal{X}$  cut out by a section of E. This is explained in [58, Section 5.2].

5.1. An Example: A Quintic Hypersurface. We illustrate this in the case of the quintic hypersurface in  $\mathbb{P}(1,1,1,1,2)$ , taking  $\mathcal{X} = \mathbb{P}(1,1,1,1,2)$ ; F to be the line bundle  $\mathcal{O}(5) \to \mathcal{X}$ ;  $i: \mathcal{Y} \to \mathcal{X}$  to be the inclusion of the corresponding hypersurface; and  $\mathbf{c}$  to be the T-equivariant Euler class.

The inertia stack  $\mathcal{I}\mathcal{X}$  has two components,

$$\begin{split} \mathcal{X}_0 &\cong \mathbb{P}(1,1,1,1,2) & \text{age } 0, \\ \mathcal{X}_{\frac{1}{2}} &\cong \mathbb{P}(2) & \text{age } 2. \end{split}$$

If  $\mathbf{1}_i$  is the fundamental class of  $\mathcal{X}_i$  and  $p = c_1(\mathcal{O}(1))$ , then

$$\phi_0 = \mathbf{1}_0, \quad \phi_1 = p\mathbf{1}_0, \quad \phi_2 = p^2\mathbf{1}_0, \quad \phi_3 = p^3\mathbf{1}_0, \quad \phi_4 = p^4\mathbf{1}_0, \quad \phi_5 = \mathbf{1}_{\frac{1}{2}}$$

is a basis for  $H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\mathbb{C})$ . Theorem 1.6 in [23] shows that the restriction to the locus  $\tau = tp$  of the untwisted *J*-function  $J^{\mathrm{un}}(\tau, z)$  of  $\mathcal{X}$  is  $^5$ 

$$J^{\mathrm{un}}(tp,z) = ze^{tp/z} \sum_{\substack{d:d \geq 0 \\ 2d \in \mathbb{Z}}} \frac{Q^d e^{dt}}{\prod_{\substack{b:0 < b \leq d \\ \langle b \rangle = (\overline{d})}} Q^d e^{dt}} \mathbf{1}_{\substack{b:0 < b \leq 2d \\ \langle b \rangle = 0}} \mathbf{1}_{\substack{d:d > 0 \\ \langle b \rangle = 0}} \mathbf{1}_{\substack{d:d \geq 0 \\ \langle b \rangle = 0}} \mathbf{$$

This is the sum of the contributions  $J_{\theta}(tp, z)$  where the topological type  $\theta = (0, d, S)$  has either  $S = (0, 0, \dots, 0, 0)$ , in which case

$$M_{\theta}(z) = \prod_{1 \le m \le 5d} (\lambda_1 + 5p + mz),$$

or  $S = (0, 0, ..., 0, \frac{1}{2})$ , in which case

$$M_{\theta}(z) = \prod_{1 \le m \le 5d + \frac{1}{2}} (\lambda_1 + 5p + (m - \frac{1}{2}z)).$$

<sup>&</sup>lt;sup>4</sup>We established notation for *T*-equivariant characteristic classes in Section 2.3.

<sup>&</sup>lt;sup>5</sup>Here and henceforth we write  $\langle r \rangle$  for the fractional part of a rational number r.

Thus Theorem 4.6 implies that the family  $t \mapsto I^{\text{tw}}(tp,-z)$  lies on  $\mathcal{L}^{\text{tw}}$ , where

$$I^{\mathrm{tw}}(tp,z) = z e^{tp/z} \sum_{\substack{d:d \geq 0 \\ 2d \in \mathbb{Z}}} Q^d e^{dt} \frac{\prod_{\substack{b:0 < b \leq 5d \\ \langle b \rangle = \langle d \rangle}} \frac{\langle b \rangle = \langle b \rangle}{\langle b \rangle = \langle d \rangle} \mathbf{1}_{\substack{b:0 < b \leq 2d \\ \langle b \rangle = \langle d \rangle}} \mathbf{1}_{\substack{c:0 < b \leq 2d \\ \langle b \rangle = 0}} \mathbf{1}_{\substack{c:0 < b \leq 2d \\ \langle b \rangle = \langle d \rangle}}.$$

We have

$$I^{\text{tw}}(tp, z) = ze^{tp/z} \left( \mathbf{1}_0 + \frac{60Qe^t}{z} \mathbf{1}_0 + O(z^{-2}) \right)$$
$$= z + tp + 60Qe^t \mathbf{1}_0 + O(z^{-1}),$$

and it follows, as the twisted J-function is characterized by (9), that

$$I^{\text{tw}}(tp, z) = J^{\text{tw}}(tp + 60Qe^t \mathbf{1}_0, z).$$

But the String Equation [3, Theorem 8.3.1] implies that

$$J^{\text{tw}}(\tau + a\mathbf{1}_0, z) = e^{a/z}J^{\text{tw}}(\tau, z),$$

and so

(16) 
$$J^{\text{tw}}(tp,z) = \exp(-60Qe^{t}/z)I^{\text{tw}}(tp,z)$$

$$= ze^{tp/z} \left( \mathbf{1}_{0} + \frac{30Q^{1/2}e^{t/2}}{z^{2}} \mathbf{1}_{\frac{1}{2}} + \frac{(137\lambda_{1} + 265p)Qe^{t}}{z^{2}} \mathbf{1}_{0} + \frac{7650Q^{2}e^{2t}}{z^{2}} \mathbf{1}_{0} + O(z^{-3}) \right).$$

On the other hand, the Divisor Equation gives

$$J^{\text{tw}}(tp,z) = ze^{tp/z} \left( \mathbf{1}_0 + \sum_{d>0} Q^d e^{dt} \left\langle \frac{\phi_{\epsilon}}{z(z-\psi)} \right\rangle_{0,1,d}^{\mathcal{X},\text{tw}} \phi^{\epsilon} \right)$$

where

$$\phi^{i} = \begin{cases} \frac{2p^{4-i}}{\lambda_{1} + 5p} \mathbf{1}_{0} & 0 \le i \le 4\\ 2\mathbf{1}_{\frac{1}{2}} & i = 5 \end{cases}$$

is the basis for  $H^{\bullet}_{\mathrm{orb}}(\mathcal{X}; \mathbb{C}) \otimes \mathbb{C}(\lambda_1)$  which is dual to  $\{\phi_{\epsilon}\}$  under the twisted pairing (5). Expanding (16) in terms of the  $\{\phi^{\epsilon}\}$ , we find that

$$\langle \phi_5 \rangle_{0,1,\frac{1}{2}}^{\mathcal{X},\text{tw}} = 15$$
 
$$\langle \phi_2 \rangle_{0,1,1}^{\mathcal{X},\text{tw}} = \frac{1325}{2}$$
 
$$\langle \phi_3 \rangle_{0,1,1}^{\mathcal{X},\text{tw}} = 475\lambda_1$$
 
$$\langle \phi_4 \rangle_{0,1,1}^{\mathcal{X},\text{tw}} = \frac{137\lambda_1^2}{2}$$
 
$$\langle \phi_3 \rangle_{0,1,2}^{\mathcal{X},\text{tw}} = 19125$$
 
$$\langle \phi_4 \rangle_{0,1,2}^{\mathcal{X},\text{tw}} = 3825\lambda_1.$$

We now take the non-equivariant limit  $\lambda_1 \to 0$ . Since F is convex,  $F_{0,1,d}$  is a vector bundle. In the non-equivariant limit, the twisted Gromov–Witten invariant

$$\langle \phi_{\epsilon} \rangle_{0,1,d}^{\mathcal{X},\text{tw}} = (\text{ev}_{1}^{\star} \phi_{\epsilon} \cup \boldsymbol{c}(F_{0,1,d})) \cap [\mathcal{X}_{0,1,d}]^{\text{vir}}$$

becomes

$$(\operatorname{ev}_1^{\star}\phi_{\epsilon}\cup e(F_{0,1,d}))\cap [\mathcal{X}_{0,1,d}]^{\operatorname{vir}}.$$

Functoriality for the virtual fundamental class [42] implies that

$$e\left(F_{0,1,d}\right)\cap\left[\mathcal{X}_{0,1,d}\right]^{\mathrm{vir}}=j_{\star}\left[\mathcal{Y}_{0,1,d}\right]^{\mathrm{vir}}$$

where  $j: \mathcal{Y}_{0,1,d} \to \mathcal{X}_{0,1,d}$  is the inclusion induced by  $i: \mathcal{Y} \to \mathcal{X}$ , so in the non-equivariant limit

$$\langle \phi_{\epsilon} \rangle_{0,1,d}^{\mathcal{X},\text{tw}} \longrightarrow \langle i^{\star} \phi_{\epsilon} \rangle_{0,1,d}^{\mathcal{Y},\text{un}}.$$

We conclude, for example, that the virtual number of degree- $\frac{1}{2}$  rational curves on  $\mathcal{Y}$  — any such curve passes through the stacky point on  $\mathcal{Y}$  — is 15, and that the virtual number of degree-1 rational curves on  $\mathcal{Y}$  which meet the cycle dual to  $P^2$  is  $\frac{1325}{2}$ .

5.2. A Quantum Lefschetz Theorem for Orbifolds. Let us return now to the general situation of Example A, so that  $F \to \mathcal{X}$  is a direct sum of convex line bundles and c is the T-equivariant Euler class. The key point in our analysis of the quintic hypersurface was that

$$I^{\text{tw}}(t, z) = z + f(t) + O(z^{-1}).$$

This implied the equality  $J^{\text{tw}}(f(t), z) = I^{\text{tw}}(t, z)$ , which expresses genus-zero twisted invariants (on the left-hand side) in terms of untwisted invariants (on the right). In fact, any time we have an equality of the form

(17) 
$$I^{\text{tw}}(t,z) = F(t)z\mathbf{1}_0 + G(t) + O(z^{-1}),$$

where F is an invertible scalar-valued function, G takes values in orbifold cohomology, and  $\mathbf{1}_0 \in H^{\bullet}_{\mathrm{orb}}(\mathcal{X}; \mathbb{C})$  is the identity element, we can deduce that

$$J^{\mathrm{tw}}(\tau(t),z) = \frac{I^{\mathrm{tw}}(t,z)}{F(t)} \qquad \text{where } \tau(t) = \frac{G(t)}{F(t)}.$$

This follows from the fact that  $\mathcal{L}^{\text{tw}}$  is (the germ of) a cone [33,58] — so we can divide (17) by F(t) and still obtain a family of elements of  $\mathcal{L}^{\text{tw}}$  — and the characterization (9) of the twisted J-function. We can assure an equality (17), provided that  $c_1(F)$  is not too positive, by restricting t to lie in an appropriate subspace.

**Corollary 5.1** (Quantum Lefschetz for Orbifolds). Suppose that  $F \to \mathcal{X}$  is a direct sum of line bundles  $F^{(1)}, \ldots, F^{(r)}$  such that each of  $c_1(F^{(1)}), \ldots, c_1(F^{(r)})$ , and  $c_1(\mathcal{X}) - c_1(F)$  are nef. Use notation as in Section 4.1 and write t' for a general point in the subspace

(18) 
$$\left\{ \alpha \in \bigoplus_{i: f_i^{(j)} = 0 \, \forall j} H^{\bullet}(\mathcal{X}_i; \mathbb{C}) : \deg(\alpha) \leq 2 \right\}$$

of  $H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\mathbb{C})$ . Then

(19) 
$$I^{\text{tw}}(t',z) = \sum_{\theta \in \text{NETT}(\mathcal{X})} \prod_{j=1}^{r} \prod_{m=1}^{N_{\theta}^{(j)}} \left(\lambda_j + \rho^{(j)} + \left(m - f_{\bar{\imath}_n}^{(j)}\right)z\right) \cdot J_{\theta}(t',z)$$

(20) 
$$= F(t')z\mathbf{1}_0 + G(t') + O(z^{-1}),$$

for some F and G with F scalar-valued and invertible, and

(21) 
$$J^{\text{tw}}(\tau(t'), z) = \frac{I^{\text{tw}}(t', z)}{F(t')} \qquad where \ \tau(t') = \frac{G(t')}{F(t')}.$$

*Proof.* Since t' is supported on those components  $\mathcal{X}_i$  of the inertia stack such that each  $f_i^{(j)}$  is zero,  $J_{\theta}(t',z)$  vanishes unless each  $N_{\theta}^{(j)} \geq 0$ . This proves (19). The expansion (20) follows by computing the highest powers of z that occur in  $J_{\theta}(t',z)$  and in the modification factor, and using the formula [17, Theorem A] for the virtual dimension of the moduli space of stable maps. The rest was explained above.  $\square$ 

If F is in addition convex then we can pass to the non-equivariant limit, exactly as in Section 5.1, and thereby express genus-zero one-point Gromov–Witten invariants of a complete intersection  $\mathcal Y$  cut out by a section of F in terms of the ordinary Gromov–Witten invariants of  $\mathcal X$ . This approach is used in [23] to compute genus-zero invariants of weighted projective complete intersections. If we assume more — that  $H^1(\mathcal C, f^*F) = 0$  for all topological types  $\theta$  which contribute non-trivially to  $I^{\mathrm{tw}}(t',z)$  — then exactly the same argument allows us to determine those genus-zero (n+1)-point Gromov–Witten invariants of  $\mathcal Y$  which involve n classes coming from (18).

#### 6. Application 2: Genus-Zero Local Invariants

Let G be a finite cyclic group, let  $\mathcal{X} = BG$ , and let  $E \to \mathcal{X}$  be the vector bundle arising from a representation  $\rho: G \to (\mathbb{C}^{\times})^m \subset GL_m(\mathbb{C})$ . Let  $T = (\mathbb{C}^{\times})^m$ . The total space of E is the orbifold  $[\mathbb{C}^m/G]$ , where G acts via  $\rho$ , and so the T-equivariant Gromov–Witten invariants of  $[\mathbb{C}^m/G]$  coincide both with the T-equivariant local Gromov–Witten invariants of E and with twisted Gromov–Witten invariants of  $\mathcal{X}$  where F = E and  $\mathbf{c}$  is the T-equivariant inverse Euler class. In this Section we use Theorem 4.6 to compute T-equivariant Gromov–Witten invariants of  $[\mathbb{C}^2/\mathbb{Z}_n]$ , where  $\mathbb{Z}_n$  acts with weights (n-1,1), and of  $[\mathbb{C}^3/\mathbb{Z}_3]$  where  $\mathbb{Z}_3$  acts with weights (1,1,1). Our starting point is the untwisted J-function of  $B\mathbb{Z}_n$ .

6.1. The Untwisted J-Function of  $B\mathbb{Z}_n$ . Let  $\mathcal{X} = B\mathbb{Z}_n$ . Components of the inertia stack  $\mathcal{I}\mathcal{X}$  are indexed by elements of  $\mathbb{Z}_n$ , and hence by the set of fractions

$$\mathcal{I} = \left\{ \frac{i}{n} : 0 \le i < n \right\}$$

via  $\frac{r}{n} \in \mathcal{I} \mapsto [r] \in \mathbb{Z}_n$ . Each component of  $\mathcal{I}\mathcal{X}$  is a copy of  $B\mathbb{Z}_n$ , and we write  $\mathbf{1}_i$  for the fundamental class of the component  $\mathcal{X}_i$ .

**Proposition 6.1.** Let  $x = x_0 \mathbf{1}_0 + x_1 \mathbf{1}_{\frac{1}{n}} + \dots + x_{n-1} \mathbf{1}_{\frac{n-1}{n}} \in H^{\bullet}_{\mathrm{orb}}(B\mathbb{Z}_n; \mathbb{C})$ . Then

$$J^{\mathrm{un}}(x,z) = z \sum_{k_0,k_1,\dots,k_{n-1}>0} \frac{1}{z^{k_0+k_1+\dots+k_{n-1}}} \frac{x_0^{k_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}}{k_0! \, k_1! \cdots k_{n-1}!} \, \mathbf{1}_{\left\langle \sum_{i=0}^{n-1} \frac{i k_i}{n} \right\rangle}.$$

*Proof.* The Gromov–Witten theory of BG for any finite group G has been completely solved by Jarvis–Kimura [40]. They show in particular that the untwisted quantum orbifold product  $\bullet_{\tau}$  on  $H^{\bullet}_{\text{orb}}(B\mathbb{Z}_n;\mathbb{C})$  is semisimple and independent of  $\tau$ . The untwisted J-function is the unique solution to the differential equations (10) which has the form (9). Thus

$$J^{\mathrm{un}}(\tau, z) = \sum_{\alpha=0}^{n-1} z e^{u^{\alpha}(\tau)/z} \mathfrak{f}_{\alpha}$$

<sup>&</sup>lt;sup>6</sup>One needs to choose a definition of Gromov–Witten invariants of the non-compact orbifold  $[\mathbb{C}^m/G]$ . In the Introduction we defined these to be local Gromov–Witten invariants. One could also define them via virtual localization to T-fixed points using [34]; this gives the same results.

where  $\tau = u^0(\tau)\mathfrak{f}_0 + \cdots + u^{n-1}(\tau)\mathfrak{f}_{n-1}$  is the expansion of  $\tau$  in terms of the basis of idempotents  $\{\mathfrak{f}_{\alpha}\}$  for  $\bullet_{\tau}$ . Applying Jarvis–Kimura's formula [40, Proposition 4.1] for the idempotents completes the proof.

6.2. Genus-Zero Gromov-Witten Invariants of  $[\mathbb{C}^2/\mathbb{Z}_n]$ . Consider now the situation described at the beginning of Section 6 in the case where  $G = \mathbb{Z}_n$ ,  $\mathcal{X} = BG$ , m = 2, and  $\rho : G \to GL_2(\mathbb{C})$  is the representation with weights (n - 1, 1). Twisted Gromov-Witten invariants of  $\mathcal{X}$  here are T-equivariant Gromov-Witten invariants of the type A surface singularity  $[\mathbb{C}^2/\mathbb{Z}_n]$ .

To apply Theorem 4.6, we need to calculate  $I^{\text{tw}}$ . For  $\mathbf{k} = (k_0, k_1, \dots, k_{n-1}) \in \mathbb{Z}^n$ , let

$$a(\mathbf{k}) = \sum_{i=1}^{n-1} \frac{n-i}{n} k_i$$
 and  $b(\mathbf{k}) = \sum_{i=1}^{n-1} \frac{i}{n} k_i$ .

The term

$$\frac{1}{z^{k_0+k_1+\cdots+k_{n-1}}} \frac{x_0^{k_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}}{k_0! k_1! \cdots k_{n-1}!} \mathbf{1}_{\langle b(\mathbf{k}) \rangle}$$

in the untwisted J-function of  $\mathcal{X}$  contributes to  $J_{\theta}(t,z)$  where the topological type  $\theta = (0,0,S)$  has S consisting of some permutation of

$$\underbrace{0,0,\ldots,0}^{k_0},\underbrace{\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}}_{n},\ldots,\underbrace{\frac{n-1}{n},\frac{n-1}{n},\ldots,\frac{n-1}{n}}_{n-1},\ldots,\underbrace{\frac{n-1}{n}}_{n}$$

followed by  $\langle -b(\mathbf{k}) \rangle$ . The corresponding modification factor is

$$M_{k_0,k_1,\ldots,k_{n-1}}(z) := \prod_{l=0}^{\lfloor a(\boldsymbol{k})\rfloor-1} \left(\lambda_1 - \left(\langle a(\boldsymbol{k})\rangle + l\right)z\right) \prod_{m=0}^{\lfloor b(\boldsymbol{k})\rfloor-1} \left(\lambda_2 - \left(\langle b(\boldsymbol{k})\rangle + m\right)z\right).$$

Theorem 4.6 implies that the family  $x \mapsto I^{\text{tw}}(x, -z)$  lies on the Lagrangian submanifold  $\mathcal{L}^{\text{tw}}$ , where  $x = x_0 \mathbf{1}_0 + x_1 \mathbf{1}_{\frac{1}{n}} + \cdots + x_{n-1} \mathbf{1}_{\frac{n-1}{n}}$  and

(22) 
$$I^{\text{tw}}(x,z) = z \sum_{k_0,k_1,\dots,k_{n-1}\geq 0} \frac{M_{k_0,k_1,\dots,k_{n-1}}(z)}{z^{k_0+k_1+\dots+k_{n-1}}} \frac{x_0^{k_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}}{k_0! k_1! \cdots k_{n-1}!} \mathbf{1}_{\langle b(\mathbf{k}) \rangle}.$$

Since

$$\lfloor a(\mathbf{k}) \rfloor + \lfloor b(\mathbf{k}) \rfloor = \begin{cases} \sum_{i=1}^{n-1} k_i & \text{if } n \text{ divides } \sum_{i=1}^{n-1} ik_i \\ \sum_{i=1}^{n-1} k_i - 1 & \text{otherwise} \end{cases}$$

we see that

$$\frac{M_{k_0,k_1,...,k_{n-1}}(z)}{z^{k_1+\cdots+k_{n-1}}} = \frac{\Gamma(1-\langle a(\mathbf{k})\rangle)}{\Gamma(1-a(\mathbf{k}))} \frac{\Gamma(1-\langle b(\mathbf{k})\rangle)}{\Gamma(1-b(\mathbf{k}))} z^{-1} + O(z^{-2})$$

unless n divides  $\sum_{i=1}^{n-1} ik_i$ , in which case

$$\frac{M_{k_0,k_1,\dots,k_{n-1}}(z)}{z^{k_1+\dots+k_{n-1}}} = O(z^{-2}).$$

Thus

$$I^{\text{tw}}(x,z) = z + \tau^0 \mathbf{1}_0 + \tau^1 \mathbf{1}_{\frac{1}{n}} + \dots + \tau^{n-1} \mathbf{1}_{\frac{n-1}{n}} + O(z^{-1}),$$

where

(23) 
$$\tau^{r} = \begin{cases} x_{0} & r = 0\\ \sum_{\substack{k_{1}, \dots, k_{n-1} \geq 0: \\ \langle b(\boldsymbol{k}) \rangle = \frac{r}{\kappa}}} \frac{x_{1}^{k_{1}} x_{2}^{k_{2}} \cdots x_{n-1}^{k_{n-1}}}{k_{1}! k_{2}! \cdots k_{n-1}!} \frac{\Gamma\left(1 - \langle a(\boldsymbol{k}) \rangle\right)}{\Gamma\left(1 - a(\boldsymbol{k})\right)} \frac{\Gamma\left(1 - \langle b(\boldsymbol{k}) \rangle\right)}{\Gamma\left(1 - b(\boldsymbol{k})\right)} & r \neq 0. \end{cases}$$

Since the twisted J-function gives the unique family of elements of  $\mathcal{L}^{tw}$  satisfying (9), it follows that

(24) 
$$I^{\text{tw}}(x,z) = J^{\text{tw}}(\tau^0 \mathbf{1}_0 + \tau^1 \mathbf{1}_{\frac{1}{n}} + \dots + \tau^{n-1} \mathbf{1}_{\frac{n-1}{n}}, z).$$

To calculate genus-zero twisted Gromov–Witten invariants we need to determine the twisted J-function as a function of  $\tau^0, \ldots, \tau^{n-1}$ , and so we need to invert the mirror map

(25) 
$$(x_0, x_1, \dots, x_{n-1}) \mapsto (\tau^0, \tau^1, \dots, \tau^{n-1}).$$

In Appendix A, we prove:

**Proposition 6.2** (cf. Proposition A.6). The inverse to the mirror map (25) is given by

$$x_{i} = \begin{cases} \tau^{0} & i = 0\\ (-1)^{n-i} e_{n-i}(\kappa_{0}, \kappa_{1}, \dots, \kappa_{n-1}) & i \neq 0 \end{cases}$$

where  $e_j$  is the jth elementary symmetric function,  $\zeta = \exp\left(\frac{\pi\sqrt{-1}}{n}\right)$ , and

$$\kappa_k(\tau^1, \dots, \tau^{n-1}) = \zeta^{2k+1} \prod_{r=1}^{n-1} \exp\left(\frac{1}{n} \zeta^{(2k+1)r} \tau^r\right).$$

This Proposition together with (24) determines closed formulas for all genus-zero T-equivariant non-descendant Gromov–Witten invariants of  $[\mathbb{C}^2/\mathbb{Z}_n]$ . These invariants are packaged into a generating function called the Gromov–Witten potential of  $[\mathbb{C}^2/\mathbb{Z}_n]$  — see e.g. [12, Section 1.2] for a definition — which is equal to

$$\mathcal{F}_0^{[\mathbb{C}^2/\mathbb{Z}_n]}(\tau) = \sum_{m>0} \frac{1}{m!} \langle \tau, \tau, \dots, \tau \rangle_{0,m,0}^{\mathcal{X}, \text{tw}}$$

where  $\mathcal{X} = B\mathbb{Z}_n$  and  $\tau = \tau^0 \mathbf{1}_0 + \dots + \tau^{n-1} \mathbf{1}_{\frac{n-1}{n}}$ .

## Proposition 6.3.

$$\mathcal{F}_0^{[\mathbb{C}^2/\mathbb{Z}_n]} = \frac{\left(\tau^0\right)^3}{6n\lambda_1\lambda_2} + \frac{\tau^0}{2n} \sum_{i=1}^{n-1} \tau^i \tau^{n-i} - \lambda_1 G(\tau^1, \dots, \tau^{n-1}) - \lambda_2 G(\tau^{n-1}, \dots, \tau^1)$$

where the derivatives of G are given by

$$\frac{\partial G}{\partial \tau^r}(\tau^1, \dots, \tau^{n-1}) = \sum_{\substack{k_1, \dots, k_{n-1} \ge 0: \\ \langle b(\boldsymbol{k}) \rangle = \frac{n-r}{n}}} \frac{(x_1)^{k_1} \cdots (x_{n-1})^{k_{n-1}}}{n \, k_1! \cdots k_{n-1}!} \times \left(\sum_{m=0}^{\lfloor a(\boldsymbol{k}) \rfloor - 1} \frac{1}{m + \langle a(\boldsymbol{k}) \rangle}\right) \frac{\Gamma(1 - \langle a(\boldsymbol{k}) \rangle)}{\Gamma(1 - a(\boldsymbol{k}))} \frac{\Gamma(1 - \langle b(\boldsymbol{k}) \rangle)}{\Gamma(1 - b(\boldsymbol{k}))}$$

and the relationship between  $x_i$  and  $\tau^r$  is given in Proposition 6.2.

*Proof.* The terms in the potential which involve  $\tau^0$  are determined by

$$\partial_{ au^0}\partial_{ au^i}\partial_{ au^j}\mathcal{F}_0^{[\mathbb{C}^2/\mathbb{Z}_n]}=\left(\mathbf{1}_{rac{i}{n}},\mathbf{1}_{rac{j}{n}}
ight)^{\mathrm{tw}}.$$

The others can be extracted from the  $z^{-1}$  term in (24), using the explicit formula (22) for  $I^{\text{tw}}(t,z)$  and the fact that

$$J^{\text{tw}}(\tau, z) = z + \sum_{i=0}^{n-1} \tau^{i} \mathbf{1}_{\frac{i}{n}} + \frac{1}{z} \left( n \lambda_{1} \lambda_{2} \frac{\partial \mathcal{F}_{0}^{[\mathbb{C}^{2}/\mathbb{Z}_{n}]}}{\partial \tau^{0}} \mathbf{1}_{0} + \sum_{i=1}^{n-1} n \frac{\partial \mathcal{F}_{0}^{[\mathbb{C}^{2}/\mathbb{Z}_{n}]}}{\partial \tau^{i}} \mathbf{1}_{\frac{n-i}{n}} \right) + O(z^{-2}).$$

This equality follows from (8), as the bases

$$\mathbf{1}_0, \mathbf{1}_{\frac{1}{n}}, \mathbf{1}_{\frac{2}{n}}, \mathbf{1}_{\frac{3}{n}}, \dots, \mathbf{1}_{\frac{n-1}{n}}$$
 and  $n\lambda_1\lambda_2\mathbf{1}_0, n\mathbf{1}_{\frac{n-1}{n}}, n\mathbf{1}_{\frac{n-2}{n}}, n\mathbf{1}_{\frac{n-3}{n}}, \dots, n\mathbf{1}_{\frac{1}{n}}$  for  $H^{\bullet}_{\mathrm{orb},T}([\mathbb{C}^2/\mathbb{Z}_n])$  are dual with respect to the twisted pairing (5).

Proposition 6.3 immediately implies an explicit formula for the differential of  $\mathcal{F}_0^{[\mathbb{C}^2/\mathbb{Z}_n]}$ . When n=2, we can integrate this, recovering a result of Bryan–Graber [12].

**Example**  $(n = 2, [\mathbb{C}^2/\mathbb{Z}_2])$ . Propositions 6.2 and 6.3 give

$$x_1 = -\kappa_0 - \kappa_1 = -\sqrt{-1}(e^{\sqrt{-1}\tau^1/2} - e^{-\sqrt{-1}\tau^1/2}) = 2\sin\left(\frac{\tau^1}{2}\right),$$

$$\frac{dG}{d\tau^1} = \sum_{k=1}^{\infty} \frac{(x_1)^{2k+1}}{2^{2k+1}} \frac{((2k-1)!!)^2}{(2k+1)!} \sum_{m=0}^{k-1} \frac{1}{m+\frac{1}{2}}.$$

Using  $(\frac{d}{d\tau^1})^2 = (\frac{d}{dx_1})^2 - \frac{1}{4}(x_1\frac{d}{dx_1})^2$ , we find

$$\left(\frac{d}{d\tau^1}\right)^3 G = \sum_{k=0}^{\infty} \left(\frac{x_1}{4}\right)^{2k+1} \frac{(2k)!}{(k!)^2} = \frac{x_1}{4} \left(1 - \frac{(x_1)^2}{4}\right)^{-1/2} = \frac{1}{2} \tan\left(\frac{\tau^1}{2}\right).$$

6.3. Genus-Zero Gromov-Witten Invariants of  $[\mathbb{C}^3/\mathbb{Z}_3]$ . Consider now the situation described at the beginning of Section 6 in the case where  $G = \mathbb{Z}_3$ ,  $\mathcal{X} = BG$ , m = 3, and  $\rho : G \to GL_3(\mathbb{C})$  is the representation with weights (1, 1, 1). Twisted Gromov-Witten invariants of  $\mathcal{X}$  here are T-equivariant Gromov-Witten invariants of  $[\mathbb{C}^3/\mathbb{Z}_3]$ . We set

$$\alpha(\mathbf{k}) = \frac{k_1}{3} + \frac{2k_2}{3}$$
 where  $\mathbf{k} = (k_0, k_1, k_2) \in \mathbb{Z}^3$ .

Theorem 4.6 implies that the family  $x \mapsto I^{\text{tw}}(x,-z)$  lies on the Lagrangian submanifold  $\mathcal{L}^{\text{tw}}$ , where  $x = x_0 \mathbf{1}_0 + x_1 \mathbf{1}_{\frac{1}{3}} + x_2 \mathbf{1}_{\frac{2}{3}}$  and

$$I^{\text{tw}}(t,z) = z \sum_{k_0,k_1,k_2 \ge 0} \frac{\prod_{\substack{b:0 \le b < \alpha(\mathbf{k}) \\ \langle b \rangle = \langle \alpha(\mathbf{k}) \rangle}} \frac{\prod_{\substack{b:0 \le b < \alpha(\mathbf{k}) \\ \langle b \rangle = \langle \alpha(\mathbf{k}) \rangle}} (\lambda_1 - bz)(\lambda_2 - bz)(\lambda_3 - bz)}{z^{k_0 + k_1 + k_2}} \frac{x_0^{k_0} x_1^{k_1} x_2^{k_2}}{k_0! \, k_1! \, k_2!} \mathbf{1}_{\langle \alpha(\mathbf{k}) \rangle}$$

To obtain an expansion of the form

$$I^{\text{tw}}(x,z) = z + f(x) + O(z^{-1})$$

we restrict to the locus  $x_2 = 0$ , obtaining

$$I^{\text{tw}}(x_0 \mathbf{1}_0 + x_1 \mathbf{1}_{1/3}, z) = z + \tau^0 \mathbf{1}_0 + \tau^1 \mathbf{1}_{\frac{1}{2}} + O(z^{-1})$$

with

(26) 
$$\tau^{0} = x_{0}$$

$$\tau^{1} = \sum_{k>0} \frac{(-1)^{3k} (x_{1})^{3k+1}}{(3k+1)!} \left(\frac{\Gamma(k+\frac{1}{3})}{\Gamma(\frac{1}{3})}\right)^{3}.$$

The twisted J-function is characterized by (9), so

(27) 
$$I^{\text{tw}}(x_0 \mathbf{1}_0 + x_1 \mathbf{1}_{\frac{1}{2}}, z) = J^{\text{tw}}(\tau^0 \mathbf{1}_0 + \tau^1 \mathbf{1}_{\frac{1}{2}}, z).$$

The T-equivariant genus-zero non-descendant potential of  $[\mathbb{C}^3/\mathbb{Z}_3]$  is equal to

$$\mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_3]}(\tau) = \sum_{m \geq 0} \frac{1}{m!} \langle \tau, \tau, \dots, \tau \rangle_{0, m, 0}^{\mathcal{X}, \text{tw}},$$

where  $\mathcal{X} = B\mathbb{Z}_3$  and  $\tau = \tau^0 \mathbf{1}_0 + \tau^1 \mathbf{1}_{\frac{1}{2}} + \tau^2 \mathbf{1}_{\frac{2}{3}}$ .

Proposition 6.4. We have

$$\frac{\partial \mathcal{F}_{0}^{[\mathbb{C}^{3}/\mathbb{Z}_{3}]}}{\partial \tau^{1}} \left( \tau^{0} \mathbf{1}_{0} + \tau^{1} \mathbf{1}_{\frac{1}{3}} \right) = \frac{1}{3} \sum_{j \geq 0} (-1)^{3j} \frac{(x_{1})^{3j+2}}{(3j+2)!} \left( \frac{\Gamma\left(j+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \right)^{3}$$
$$\frac{\partial \mathcal{F}_{0}^{[\mathbb{C}^{3}/\mathbb{Z}_{3}]}}{\partial \tau^{2}} \left( \tau^{0} \mathbf{1}_{0} + \tau^{1} \mathbf{1}_{\frac{1}{3}} \right) = \frac{\tau^{0} \tau^{1}}{3} - \frac{1}{3} \sum_{i \geq 0} \frac{(x_{1})^{3j+1}}{(3j+1)!} \sum_{r=0}^{r=j-1} \frac{\lambda_{1} + \lambda_{2} + \lambda_{3}}{r + \frac{1}{3}}$$

where  $\tau^1$  and  $x_1$  are related by (26).

*Proof.* Since the bases  $\mathbf{1}_0, \mathbf{1}_{\frac{1}{3}}, \mathbf{1}_{\frac{2}{3}}$  and  $3\mathbf{1}_0, 3\mathbf{1}_{\frac{2}{3}}, 3\mathbf{1}_{\frac{1}{3}}$  for  $H^{\bullet}_{\mathrm{orb},T}([\mathbb{C}^3/\mathbb{Z}_3])$  are dual with respect to the twisted pairing (5), we have

$$J^{\mathrm{tw}}(\tau,z) = z + \tau + \frac{3}{z} \frac{\partial \mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial \tau^2} \mathbf{1}_{\frac{1}{3}} + \frac{3}{z} \frac{\partial \mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial \tau^1} \mathbf{1}_{\frac{2}{3}} + O(z^{-2}).$$

The result follows by equating coefficients of  $z^{-1}$  in (27).

We do not know<sup>7</sup> how to invert the mirror map (26). But one can still calculate the first few terms of the series expansion for  $x_1$  in terms of  $\tau^1$ :

$$x_1 = \tau^1 + \frac{(\tau^1)^4}{648} - \frac{29(\tau^1)^7}{3674160} + \frac{6607(\tau^1)^{11}}{71425670400} - \dots$$

and hence extract genus-zero orbifold Gromov–Witten invariants of  $[\mathbb{C}^3/\mathbb{Z}_3]$  one-by-one. For example, if

$$N_{0,k}^{\mathrm{orb}} = \left\langle \mathbf{1}_{\frac{1}{3}}, \mathbf{1}_{\frac{1}{3}}, \dots, \mathbf{1}_{\frac{1}{3}} \right\rangle_{0,3k,0}^{\mathcal{X},\mathrm{tw}}$$

then Proposition 6.4 gives

This agrees with the predictions of Aganagic–Bouchard–Klemm [1, Section 6].

<sup>&</sup>lt;sup>7</sup>A combinatorial formula for the inverse has recently been given by Bayer and Cadman [6].

## Appendix A. The Crepant Resolution Conjecture for Type A Surface Singularities

A long-standing conjecture of Ruan states that if  $\mathcal{X}$  is an orbifold with coarse moduli space X and  $Y \to X$  is a crepant resolution then the small quantum cohomology of Y becomes isomorphic to the small quantum cohomology of  $\mathcal{X}$  after analytic continuation in the quantum parameters followed by specialization of some of the parameters to roots of unity. A refinement of this conjecture, proposed recently by Bryan and Graber [12], suggests that if  $\mathcal{X}$  satisfies a Hard Lefschetz condition on orbifold cohomology then the Frobenius manifold structures defined by the quantum cohomology of  $\mathcal{X}$  and of Y coincide after analytic continuation and specialization of parameters (see also [20] for a Hard Lefschetz condition). This is a stronger assertion: that the biq quantum cohomology of Y coincides with that of  $\mathcal{X}$  after analytic continuation plus specialization, via a linear isomorphism which preserves the (orbifold) Poincaré pairing. In this Appendix we prove these conjectures in the case where  $\mathcal{X}$  is the  $A_{n-1}$  surface singularity  $\left[\mathbb{C}^2/\mathbb{Z}_n\right]$  and Y is its crepant resolution. In fact we prove a more precise statement, Theorem A.1 below, which also identifies an isomorphism and the roots of unity to which the quantum parameters of Y are specialized. We learned this statement from Jim Bryan [10; 12, Conjecture 3.1] and Fabio Perroni [53, Conjecture 1.9; 54].

Our proof of Theorem A.1 is based on mirror symmetry for toric orbifolds. By mirror symmetry we mean the fact, first observed by Candelas et~al.~[15], that one can compute virtual numbers of rational curves in a manifold or orbifold  $\mathcal{X}$ —i.e. certain genus-zero Gromov–Witten invariants of  $\mathcal{X}$ —by solving Picard–Fuchs equations. Following Givental we reinterpret our results from Section 6.2 in these terms, observing that there is a close relationship between a cohomology-valued generating function for genus-zero Gromov–Witten invariants, called the J-function of  $\mathcal{X}$ , and a cohomology-valued solution to the Picard–Fuchs equations called the I-function of  $\mathcal{X}$ . A similar relationship holds for Y: this is Proposition A.3 below. After describing the toric structures of  $\mathcal{X}$  and Y and fixing notation for cohomology and quantum cohomology, we explain below how to extract the quantum products for  $\mathcal{X}$  and Y from the Picard–Fuchs equations. Once we understand this, Theorem A.1 follows easily: the proof is at the end of the Appendix.

A number of cases of Theorem A.1 were already known. Ruan's Crepant Resolution Conjecture was established for surface singularities of type  $A_1$  and  $A_2$  by Perroni [53]. Theorem A.1 was proved in the  $A_1$  case by Bryan-Graber [12], in the  $A_2$  case by Bryan-Graber-Pandharipande [13], and in the  $A_3$  case by Bryan-Jiang [14]. Davesh Maulik has computed the genus-zero Gromov-Witten potential of the type A surface singularity  $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_n]$  for all n (as well as certain higher-genus Gromov-Witten invariants of  $\mathcal{X}$ ) [48] and the reduced genus-zero Gromov-Witten potential of the crepant resolution Y [49]; Theorem 1 should follow from this. The quantum cohomology of the crepant resolutions of type ADE surface singularities has been computed by Bryan-Gholampour [11]. Skarke [55] and Hosono [36] have also studied the  $A_n$  case, from a point of view very similar to ours, as part of their investigations of homological mirror symmetry.

 $\mathcal{X}$  and Y as Toric Orbifolds.  $\mathcal{X}$  is the toric orbifold corresponding to the fan (or stacky fan [8]) in Figure 1(a) and Y is the toric manifold corresponding to the

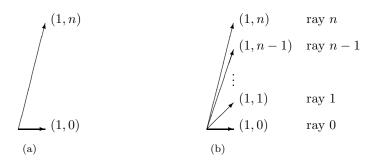


FIGURE 1. (a) The fan for  $\mathcal{X}$ . (b) The fan for Y.

fan in Figure 1(b). Background material on toric manifolds and orbifolds can be found in [4, Chapter VII].

There is an exact sequence

$$0 \longrightarrow \mathbb{Z}^{n-1} \xrightarrow{M^{\mathrm{T}}} \mathbb{Z}^{n+1} \xrightarrow{\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & n \end{pmatrix}} \mathbb{Z}^2 \longrightarrow 0,$$

and hence we can represent the Gale dual of the right-hand map by

$$\mathbb{Z}^{n+1} \xrightarrow{M} \mathbb{Z}^{n-1}$$
,

where

$$M = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

Certain faces of the positive orthant  $(\mathbb{R}_{\geq 0})^{n+1} \subset \mathbb{R}^{n+1}$  project via M to codimension-1 subsets of  $\mathbb{R}^{n-1}$ . The image of the positive orthant is divided by these subsets into chambers, which are the maximal cones of a fan in  $\mathbb{R}^{n-1}$  called the *secondary fan* of Y. Chambers in the secondary fan correspond to toric partial resolutions of  $\mathcal{X}$ . A chamber K corresponds to a fan  $\Sigma$  with rays some subset of the rays of the fan for Y, as follows. Number the rays of the fan for Y as shown in Figure 1(b). For a subset  $\sigma \subset \{0, 1, \ldots, n\}$ , let us write  $\bar{\sigma}$  for the complement  $\{0, 1, \ldots, n\} \setminus \sigma$ ,  $\mathbb{R}^{\sigma}$  for the corresponding co-ordinate subspace of  $\mathbb{R}^{n+1}$ , and say that  $\sigma$  covers K iff  $K \subset M(\mathbb{R}^{\sigma})$ . The fan  $\Sigma$  corresponding to the chamber K is defined by

$$\sigma \in \Sigma \iff \bar{\sigma} \text{ covers } K;$$

the chamber K corresponding to the fan  $\Sigma$  is

$$\bigcap_{\sigma\in\Sigma}M\left(\mathbb{R}^{\bar{\sigma}}\right).$$

We will concentrate on two chambers:  $K_{\mathcal{X}}$ , with rays given by the middle n-1 columns of M, and  $K_Y$  with rays given by the standard basis vectors for  $\mathbb{R}^{n-1}$ .  $K_{\mathcal{X}}$  corresponds to the toric orbifold  $\mathcal{X}$  and  $K_Y$  corresponds to the toric manifold Y.

Let  $\mathcal{M}_{sec}$  be the toric orbifold corresponding to the secondary fan of Y. As  $K_{\mathcal{X}}$  and  $K_Y$  are simplicial, they give co-ordinate patches on  $\mathcal{M}_{sec}$ : the co-ordinates

 $x_1, \ldots, x_{n-1}$  from  $K_{\mathcal{X}}$  and  $y_1, \ldots, y_{n-1}$  from  $K_Y$  are related by

(28a) 
$$y_i = \begin{cases} x_1^{-2} x_2 & i = 1 \\ x_{i-1} x_i^{-2} x_{i+1} & 1 < i < n-1 \\ x_{n-2} x_{n-1}^{-2} & i = n-1. \end{cases}$$

More precisely,  $x_1, \ldots, x_{n-1}$  are multi-valued and the co-ordinate patch  $\mathcal{M}_{\text{sec}}(K_{\mathcal{X}})$  corresponding to the cone  $K_{\mathcal{X}}$  is given by the uniformizing system:

$$\mathcal{M}_{\text{sec}}(K_{\mathcal{X}}) \cong \mathbb{C}^{n-1}/\mu_n, \quad (x_1, x_2, \dots, x_{n-1}) \sim (cx_1, c^2x_2, \dots, c^{n-1}x_{n-1}) \text{ for } c \in \mu_n.$$

The *B-model moduli space*  $\mathcal{M}_B$  is the open subset  $\mathbb{C} \times \mathcal{M}_{sec}(K_{\mathcal{X}})$  of  $\mathbb{C} \times \mathcal{M}_{sec}$ . Denote by  $x_0$  or  $y_0$  the co-ordinate on the first factor  $\mathbb{C}$  of  $\mathbb{C} \times \mathcal{M}_{sec}$ , so that

(28b) 
$$x_0 = y_0.$$

We will refer to the point  $(x_0, x_1, \ldots, x_{n-1}) = (0, 0, \ldots, 0)$  as the large-radius limit point for  $\mathcal{X}$  and the point  $(y_0, y_1, \ldots, y_{n-1}) = (0, 0, \ldots, 0)$  as the large-radius limit point for Y. The co-ordinates  $x_i$  and  $y_j$  are related to each other by (28), so that  $y_0, y_1, \ldots, y_{n-1}$  are co-ordinates on the patch  $\mathbb{C} \times (\mathbb{C}^\times)^{n-1} \subset \mathbb{C} \times \mathcal{M}_{\text{sec}}(K_{\mathcal{X}}) = \mathcal{M}_B$  where each of  $x_1, x_2, \ldots, x_{n-1}$  is non-zero.

**Remark.** In what follows the first factor of  $\mathcal{M}_B$ , which has co-ordinates  $x_0$  or  $y_0$ , will play a rather different role than the second factor. The first factor will correspond under mirror symmetry to  $H^0_{\mathrm{orb}}(\mathcal{X}) \subset H^{\bullet}_{\mathrm{orb}}(\mathcal{X})$  or  $H^0(Y) \subset H^{\bullet}(Y)$ , and the second factor will correspond to  $H^2_{\mathrm{orb}}(\mathcal{X}) \subset H^{\bullet}_{\mathrm{orb}}(\mathcal{X})$  or  $H^2(Y) \subset H^{\bullet}(Y)$ .

**Remark.** It would be more honest to define the B-model moduli space as the product of  $\mathbb{C}$  with the open subset of  $\mathcal{M}_{sec}$  on which the GKZ system associated to Y is non-singular. This set is slightly smaller than  $\mathcal{M}_B$ , as it does not contain the discriminant locus of  $W_{\mathcal{X}}$  or  $W_Y$  which appears below (in the proof of Proposition A.7).

The presentations of  $\mathcal{X}$  as a toric orbifold and Y as a toric variety allow us to write  $\mathcal{X}$  and Y as quotients of open sets  $\mathcal{U}_{\mathcal{X}}$ ,  $\mathcal{U}_{Y} \subset \mathbb{C}^{n+1}$  by  $(\mathbb{C}^{\times})^{n-1}$  — see *e.g.* [4, Chapter VII]. The action of  $T = (\mathbb{C}^{\times})^{2}$  on  $\mathbb{C}^{n+1}$  given by

(29) 
$$(a_0, a_1, \dots, a_n) \stackrel{(s,t)}{\longmapsto} (sa_0, a_1, a_2, \dots, a_{n-1}, ta_n)$$

descends to give T-actions on  $\mathcal{X}$ , X, and Y, and the crepant resolution  $Y \to X$  is T-equivariant. The T-fixed locus on Y is the exceptional divisor. The T-action on  $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_n]$  coincides with that induced by the standard action of T on  $\mathbb{C}^2$ , so the T-fixed locus on  $\mathcal{X}$  is the  $B\mathbb{Z}_n$  at the origin. As in the main text, we write  $H_T^{\bullet}(\{\text{pt}\}) = \mathbb{C}[\lambda_1, \lambda_2]$  where  $\lambda_i$  is Poincaré-dual to a hyperplane in the ith factor of  $(\mathbf{CP}^{\infty})^2 \simeq BT$ .

Orbifold Cohomology of  $\mathcal{X}$  and Cohomology of Y. The T-equivariant orbifold cohomology  $H_{T,\text{orb}}^{\bullet}(\mathcal{X};\mathbb{C})$  is the T-equivariant cohomology of the inertia stack  $\mathcal{I}\mathcal{X}$ .  $\mathcal{I}\mathcal{X}$  has components  $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_{n-1}$ , where

$$\mathcal{X}_k = \left[ \left( \mathbb{C}^2 \right)^g / \mathbb{Z}_n \right]$$
 with  $g = \exp\left( 2k\pi \sqrt{-1}/n \right) \in \mathbb{Z}_n$ .

We have

$$\mathcal{X}_k = \left[\mathbb{C}^2/\mathbb{Z}_n\right]$$
 age = 0 if  $k = 0$ ,  
 $\mathcal{X}_k = B\mathbb{Z}_n$  age = 1 otherwise.

Let  $\delta_i$  be the fundamental class of  $\mathcal{X}_i$ ,  $0 \leq i < n$ ; this gives a  $\mathbb{C}[\lambda_1, \lambda_2]$ -basis for  $H_{T,\text{orb}}^{\bullet}(\mathcal{X};\mathbb{C})$ . The pullback of  $\delta_i$  along the inclusion of the T-fixed locus  $B\mathbb{Z}_n \to \mathcal{X}$  is the class  $\mathbf{1}_{\frac{i}{n}}$  from Section 6.1. The canonical involution I on  $\mathcal{I}\mathcal{X}$  fixes  $\mathcal{X}_0$  and exchanges  $\mathcal{X}_i$  with  $\mathcal{X}_{n-i}$ ,  $1 \leq i < n$ . As I is age-preserving,  $H_{\text{orb}}^{\bullet}(\mathcal{X};\mathbb{C})$  satisfies Hard Lefschetz [12, Definition 1.1; 28].

The cone  $K_Y$  is the Kähler cone for Y and its rays determine a basis  $\gamma_1, \ldots, \gamma_{n-1}$  for  $H^2(Y; \mathbb{Z})$ . The dual basis  $\beta_1, \ldots, \beta_{n-1}$  for  $H_2(Y; \mathbb{Z})$  is positive in the sense of [12, Section 1.2]. If we define  $\gamma_0 = 1$  and choose lifts of  $\gamma_1, \ldots, \gamma_{n-1}$  to T-equivariant cohomology then  $\gamma_i$ ,  $0 \le i < n$ , is an  $\mathbb{C}[\lambda_1, \lambda_2]$ -basis for  $H_T^{\bullet}(Y; \mathbb{C})$ . We choose a standard equivariant lift of each  $\gamma \in H^2(Y; \mathbb{Z})$  in the following way. There is a unique representation  $\rho_{\gamma}$  of  $(\mathbb{C}^{\times})^{n-1}$  such that  $\gamma$  is the first Chern class of the line bundle

$$L_{\gamma} := \mathcal{U}_Y \times_{\rho_{\gamma}} \mathbb{C} \longrightarrow \mathcal{U}_Y / (\mathbb{C}^{\times})^{n-1} = Y.$$

This line bundle  $L_{\gamma}$  admits a T-action such that T acts on  $\mathcal{U}_{Y}$  via (29) and acts trivially on the  $\mathbb{C}$  factor, and the lift  $\gamma \in H^{2}_{T}(Y;\mathbb{Z})$  is the T-equivariant first Chern class of  $L_{\gamma}$ . The columns of M, together with the action (29), define elements  $\omega_{j} \in H^{2}_{T}(Y;\mathbb{C}), 0 \leq j \leq n$ , where

$$\omega_{j} = \begin{cases} \lambda_{1} + \gamma_{1} & j = 0 \\ -2\gamma_{1} + \gamma_{2} & j = 1 \\ \gamma_{j-1} - 2\gamma_{j} + \gamma_{j+1} & 1 < j < n - 1 \\ \gamma_{n-2} - 2\gamma_{n-1} & j = n - 1 \\ \lambda_{2} + \gamma_{n-1} & j = n. \end{cases}$$

The class  $\omega_i$  is the *T*-equivariant Poincaré dual of the toric divisor given in coordinates (29) by  $a_i = 0$ . We have

$$H_T^{\bullet}(Y;\mathbb{C}) = \mathbb{C}[\lambda_1, \lambda_2, \gamma_1, \dots, \gamma_{n-1}] / \langle \omega_i \omega_i : i - j > 1 \rangle.$$

 $\mathcal{X}$  and Y are non-compact but nonetheless one can define (orbifold) Poincaré pairings on the localized T-equivariant (orbifold) cohomology groups

$$H(\mathcal{X}) := H_{T,\mathrm{orb}}^{\bullet}(\mathcal{X}; \mathbb{C}) \otimes \mathbb{C}(\lambda_1, \lambda_2) \quad \text{ and } \quad H(Y) := H_T^{\bullet}(Y; \mathbb{C}) \otimes \mathbb{C}(\lambda_1, \lambda_2)$$

using the Bott residue formula. These pairings take values in  $\mathbb{C}(\lambda_1, \lambda_2)$ , and are non-degenerate. We write  $\{\gamma^i\}$  and  $\{\delta^i\}$  for the bases dual respectively to  $\{\gamma_i\}$  and  $\{\delta_i\}$  under these pairings.

**Gromov–Witten Invariants of**  $\mathcal{X}$  **and** Y. As discussed in [12], even though some moduli spaces of stable maps to  $\mathcal{X}$  or Y are non-compact the T-fixed loci on these moduli spaces are compact and so we can still define  $\mathbb{C}(\lambda_1, \lambda_2)$ -valued Gromov–Witten invariants of  $\mathcal{X}$  and Y using the virtual localization formula of Graber–Pandharipande [34]. For  $\alpha_1, \ldots, \alpha_n \in H(Y), d \in H_2(Y; \mathbb{Z}), \text{ and } i_1, \ldots, i_n \geq 0$ , we set

$$\langle \alpha_1 \psi^{i_1}, \dots, \alpha_n \psi^{i_n} \rangle_d^Y = \int_{[Y_{0,n,d}]^{\text{vir}}} \prod_{j=1}^n \text{ev}_j^{\star} \alpha_j \cdot \psi_j^{i_j}.$$

Here  $\psi_i$  and  $Y_{0,n,d}$  are as in Section 2.3 and the integral is defined by localization to the T-fixed substack, as in [19, Section 3.1; 34, Section 4]. We make a similar definition for  $\mathcal{X}$ ; the discussion around footnote 6 in the main text explains how to

express the resulting correlators  $\langle \alpha'_1 \psi^{i_1}, \dots, \alpha'_n \psi^{i_n} \rangle_0^{\mathcal{X}}$  as twisted Gromov–Witten invariants of  $B\mathbb{Z}_n$ .

Small Quantum Cohomology and Ruan's Conjecture. The small quantum product for  $\mathcal{X}$  is the  $\mathbb{C}(\lambda_1, \lambda_2)$ -algebra defined by

(30) 
$$\delta_i \underset{\text{small}}{\star} \delta_j = \sum_{k=0}^{n-1} \langle \delta_i, \delta_j, \delta_k \rangle_0^{\mathcal{X}} \delta^k.$$

This coincides with the Chen–Ruan orbifold cup product on  $H(\mathcal{X})$  [16]. The small quantum product for Y is the family of  $\mathbb{C}(\lambda_1, \lambda_2)$ -algebras, depending on parameters  $q_1, \ldots, q_{n-1}$ , defined by

(31) 
$$\gamma_i \underset{\text{small}}{\star} \gamma_j = \sum_d \sum_{k=0}^{n-1} \langle \gamma_i, \gamma_j, \gamma_k \rangle_d^Y q_1^{d_1} \cdots q_{n-1}^{d_{n-1}} \gamma^k.$$

where the sum is over classes  $d = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$  with each  $d_i \geq 0$ . It can be obtained from the untwisted product defined in (6) by setting  $\tau = 0$  and  $Q^d = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ . This change in notation reflects a change in perspective: in (6) the variable Q was part of the ground ring  $\Lambda$  and we thought of the product  $\bullet_{\tau}$  as depending formally on Q and  $\tau$ ; here the ground ring is  $\mathbb{C}(\lambda_1, \lambda_2)$  not  $\Lambda$  and we think of  $\star$  as a family of products on H(Y) which varies analytically with  $q_1, \ldots, q_{n-1}$ . It follows from the discussion below that the right-hand side of (31) converges to an analytic function of  $q_1, \ldots, q_{n-1}$  in some neighbourhood of the origin. Ruan's conjecture asserts that there is a linear isomorphism  $H(\mathcal{X}) \to H(Y)$  which identifies the products (30) and (31) after analytic continuation in the  $q_i$  followed by setting the  $q_i$  equal to certain roots of unity.

Big Quantum Cohomology and the Bryan–Graber Conjecture. The big quantum cohomology of  $\mathcal{X}$  is the family of  $\mathbb{C}(\lambda_1, \lambda_2)$ -algebras parametrized by  $\tau \in H(\mathcal{X})$ ,  $\tau = \tau^0 \delta_0 + \tau^1 \delta_1 + \cdots + \tau^{n-1} \delta_{n-1}$ , defined by

(32) 
$$\delta_{i} \underset{\text{big}}{\star} \delta_{j} = \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \frac{1}{m!} \langle \delta_{i}, \delta_{j}, \delta_{k}, \overbrace{\tau, \dots, \tau}^{m} \rangle_{0}^{\mathcal{X}} \delta^{k}.$$

The big quantum cohomology of Y is the family of  $\mathbb{C}(\lambda_1, \lambda_2)$ -algebras parametrized by  $t \in H(Y)$ ,  $t = t^0 \gamma_0 + t^1 \gamma_1 + \cdots + t^{n-1} \gamma_{n-1}$ , defined by

(33) 
$$\gamma_i \underset{\text{big}}{\star} \gamma_j = \sum_{d} \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \frac{1}{m!} \langle \gamma_i, \gamma_j, \gamma_k, \overbrace{t, \dots, t}^m \rangle_d^Y \gamma^k.$$

The first sum here is over classes  $d = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$  with each  $d_i \geq 0$ . It follows from the discussion below that the right-hand sides of (32) and respectively (33) converge to analytic functions of  $\tau^0, \dots, \tau^{n-1}$  and respectively  $t^0, \dots, t^{n-1}$  on appropriate domains. Note that the product (33) differs from the untwisted product  $\bullet_t$  defined in (6) as it does not contain factors of  $Q^d$ . This is better for our purposes, as the Divisor Equation implies that (6) contains redundant information: see [12, Section 2.2] for a discussion of this.

Together with the (orbifold) Poincaré pairings, the big quantum products (32) and (33) define Frobenius manifolds<sup>8</sup> based on  $H(\mathcal{X})$  and H(Y). The Bryan–Graber

<sup>&</sup>lt;sup>8</sup>These Frobenius manifolds are defined over the field  $\mathbb{C}(\lambda_1, \lambda_2)$ .

version of the Crepant Resolution Conjecture asserts that these Frobenius manifolds coincide after analytic continuation in the  $t^i$  and an appropriate change-of-variables. This is our main result.

**Theorem A.1.** The big quantum products (32) for  $\mathcal{X}$  and (33) for Y coincide after analytic continuation in the  $t^i$ , the affine-linear change-of-variables

$$t^{i} = \begin{cases} \tau^{0}, & i = 0\\ -\frac{2\pi\sqrt{-1}}{n} + \sum_{j=1}^{n-1} L^{i}{}_{j}\tau^{j}, & i > 0, \end{cases}$$

where

$$L^{i}_{j} = \frac{\zeta^{2ij} \left( \zeta^{-j} - \zeta^{j} \right)}{n}, \qquad \qquad \zeta = \exp \left( \frac{\pi \sqrt{-1}}{n} \right),$$

and the linear isomorphism

(34) 
$$L: H(\mathcal{X}) \to H(Y)$$

$$\delta_0 \mapsto \gamma_0,$$

$$\delta_j \mapsto \sum_{i=1}^{n-1} L^i{}_j \gamma_i, \qquad 1 \le j < n.$$

Furthermore, the isomorphism (34) matches the Poincaré pairing on H(Y) with the orbifold Poincaré pairing on H(X).

Theorem A.1 establishes Conjecture 3.1 in [12] for the case of polyhedral and binary polyhedral groups of type A, and also Conjecture 1.9 in [53]. The path along which analytic continuation is taken is described after Proposition A.7 below.

The Divisor Equation implies that we can write (33) as

$$\gamma_i \underset{\text{big}}{\star} \gamma_j = \sum_d \sum_{k=0}^{n-1} \langle \gamma_i, \gamma_j, \gamma_k \rangle_d^Y e^{d_1 t_1 + \dots + d_{n-1} t_{n-1}} \gamma^k.$$

To pass from the big quantum cohomology algebras of  $\mathcal{X}$  and Y to the small quantum cohomology algebras, therefore, set  $\tau^i = 0$  and  $e^{t^i} = q_i$ ,  $1 \le i < n$ .

Corollary A.2. The small quantum products (30) for  $\mathcal{X}$  and (31) for Y coincide after analytic continuation in the  $q_i$ , the linear isomorphism (34), and the specialization

$$q_i = \exp\left(-\frac{2\pi\sqrt{-1}}{n}\right),$$
  $1 \le i < n.$ 

Note that from this point of view the specialization  $q_i = c_i$  of quantum parameters to roots of unity arising in Ruan's conjecture just reflects the affine-linear identification of flat co-ordinates

$$t^{i} = \log c_{i} + \sum_{j=1}^{n-1} L^{i}_{j} \tau^{j},$$
  $1 \le i < n.$ 

Mirror Symmetry. As discussed above, by mirror symmetry we mean the fact that one can compute certain genus-zero Gromov–Witten invariants of  $\mathcal{X}$  and Y by solving Picard–Fuchs equations. We now make this precise. Following Givental we compare two cohomology-valued generating functions for genus-zero Gromov–Witten invariants, the J-functions of  $\mathcal{X}$  and Y, with two cohomology-valued solutions to the Picard–Fuchs equations called the I-functions of  $\mathcal{X}$  and Y. Mirror symmetry for us, in this situation, is the statement that the I-function of  $\mathcal{X}$  (or Y) coincides with the I-function of  $\mathcal{X}$  (or I) after a change of variables. After proving this, which is Proposition A.3 below, we then describe how to extract the quantum products (32) and (33) from the Picard–Fuchs equations, and finally explain how this implies Theorem A.1.

The *J*-Functions of  $\mathcal{X}$  and Y. The *J*-function of  $\mathcal{X}$  is

$$J_{\mathcal{X}}(\tau, z) = z + \tau + \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \frac{1}{m!} \left\langle \widehat{\tau, \dots, \tau}, \frac{\delta_k}{z - \psi} \right\rangle_0^{\mathcal{X}} \delta^k$$

where we expand  $1/(z-\psi)$  as  $\sum_m \psi^m/z^{m+1}$ .  $J_{\mathcal{X}}(\tau,z)$  is a function of  $\tau \in H(\mathcal{X})$ ,  $\tau = \tau^0 \delta_0 + \dots + \tau^{n-1} \delta_{n-1}$ , which takes values in  $H(\mathcal{X}) \otimes \mathbb{C}((z^{-1}))$ . It is defined and analytic in an open subset of  $H(\mathcal{X})$  where  $|\tau^1|, \dots, |\tau^{n-1}|$  are sufficiently small; this follows from Proposition A.3 below.

The J-function of Y is

$$J_Y(t,z) = z + t + \sum_{d} \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \frac{1}{m!} \left\langle \overbrace{t,\dots,t}^{m}, \frac{\gamma_k}{z-\psi} \right\rangle_d^Y \gamma^k$$

where the first sum is over  $d = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$  with each  $d_i \geq 0$ .  $J_Y(t,z)$  is a function of  $t \in H(Y)$ ,  $t = t^0\gamma_0 + \cdots + t^{n-1}\gamma_{n-1}$ , which takes values in  $H(Y) \otimes \mathbb{C}((z^{-1}))$ . It is defined and analytic in an open subset of H(Y) where  $\Re(t^i) \ll 0$ ,  $1 \leq i \leq n$ ; this again follows from Proposition A.3.

**Remark.**  $J_{\mathcal{X}}(\tau, z)$  differs from the twisted J-function of  $B\mathbb{Z}_n$  defined in Section 3.1 only in that there we regarded the twisted J-function as a formal series in  $\tau^0, \ldots, \tau^{n-1}$  and here we regard  $J_{\mathcal{X}}(\tau, z)$  as an analytic function of the  $\tau^i$ .  $J_Y(t, z)$  differs from the untwisted J-function of Y defined in Section 3.1 in the same way, and also in that  $J_Y(t, z)$  contains no factors of  $Q^d$ . Using the String Equation and the Divisor Equation, we can write  $J_Y(t, z)$  as

$$e^{t^{0}/z}e^{\left(t^{1}\gamma_{1}+\cdots+t^{n-1}\gamma_{n-1}\right)/z}\left(z\gamma_{0}+\sum_{d}\sum_{k=0}^{n-1}\left\langle \frac{\gamma_{k}}{z-\psi}\right\rangle _{d}^{Y}e^{d_{1}t^{1}+\cdots+d_{n-1}t^{n-1}}\gamma^{k}\right)$$

and so our definition of  $J_Y$  agrees, up to a factor of z, with that in [31, Section 1].

The *I*-Functions of  $\mathcal{X}$  and Y. For  $\mathbf{k} = (k_1, \dots, k_{n-1}) \in \mathbb{Z}^{n-1}$ , let

$$D_{j}(\mathbf{k}) = \begin{cases} -\frac{1}{n} \sum_{i=1}^{n-1} (n-i)k_{i} & j=0\\ k_{j} & 1 \leq j < n\\ -\frac{1}{n} \sum_{i=1}^{n-1} ik_{i} & j=n. \end{cases}$$

Let  $i(\mathbf{k}) = \langle -D_n(\mathbf{k}) \rangle$ , where  $\langle r \rangle$  denotes the fractional part of a rational number r. The *I*-function  $I_{\mathcal{X}}(x,z)$  of  $\mathcal{X}$  is defined to be

$$ze^{x_0/z} \sum_{\substack{k_1,\dots,k_{n-1}>0}} \frac{\prod_{r:D_0(\mathbf{k})< r\leq 0} (\lambda_1+rz) \prod_{s:D_n(\mathbf{k})< s\leq 0} (\lambda_2+sz)}{\frac{\langle r\rangle = \langle D_0(\mathbf{k})\rangle}{z^{k_1+\dots+k_{n-1}}}} \frac{x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}}{k_1! \cdots k_{n-1}!} \delta_{i(\mathbf{k})}.$$

This is a function of  $x=(x_0,\ldots,x_{n-1})\in\mathcal{M}_B, z\in\mathbb{C}^\times$ , and  $\lambda_1,\lambda_2\in\mathbb{C}$  which takes values in  $H^{\bullet}_{T,\mathrm{orb}}(\mathcal{X};\mathbb{C})$ . Each component of  $I_{\mathcal{X}}(x,z)$  with respect to the basis  $\{\delta_i\}$  is an analytic function of  $(x,z,\lambda_1,\lambda_2)$  defined in a domain where  $|x_1|,\ldots,|x_n|$  are sufficiently small and  $x_0,z,\lambda_1,\lambda_2$  are arbitrary. By taking a Laurent expansion at  $z=\infty$  we can regard  $I_{\mathcal{X}}(x,z)$  as an analytic function of  $(x,\lambda_1,\lambda_2)$  which takes values in  $H(\mathcal{X})\otimes\mathbb{C}((z^{-1}))$ .  $I_{\mathcal{X}}(x,z)$  satisfies a system of Picard–Fuchs equations, as follows. Define differential operators

$$\beth_{j} = \begin{cases}
\lambda_{1} - \frac{1}{n} \sum_{i=1}^{n-1} (n-i) z x_{i} \frac{\partial}{\partial x_{i}} & j = 0 \\
z x_{j} \frac{\partial}{\partial x_{j}} & 1 \leq j < n \\
\lambda_{2} - \frac{1}{n} \sum_{i=1}^{n-1} i z x_{i} \frac{\partial}{\partial x_{i}} & j = n.
\end{cases}$$

Then

(35a) 
$$\left(\prod_{j:D_{j}(\mathbf{k})>0} \prod_{m=0}^{D_{j}(\mathbf{k})-1} (\beth_{j} - mz)\right) I_{\mathcal{X}}(x,z) = x_{1}^{k_{1}} \cdots x_{n-1}^{k_{n-1}} \left(\prod_{j:D_{j}(\mathbf{k})<0} \prod_{m=0}^{-D_{j}(\mathbf{k})-1} (\beth_{j} - mz)\right) I_{\mathcal{X}}(x,z).$$

for each  $\mathbf{k} = (k_1, \dots, k_{n-1}) \in \mathbb{Z}^{n-1}$  such that  $i(\mathbf{k}) = 0$ , and

(35b) 
$$z \frac{\partial}{\partial x_0} I_{\mathcal{X}}(x, z) = I_{\mathcal{X}}(x, z).$$

The I-function of Y is

$$I_Y(y,z) = z e^{y_0/z} y_1^{\gamma_1/z} \cdots y_{n-1}^{\gamma_{n-1}/z} \sum_{d} \prod_{j=0}^n \frac{\prod_{m \le 0} (\omega_j + mz)}{\prod_{m \le D'_j(d)} (\omega_j + mz)} y_1^{d_1} \cdots y_{n-1}^{d_{n-1}},$$

where  $y_i^{\gamma_i/z} = \exp(\gamma_i \log y_i/z)$ , the sum is over  $d = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$  with each  $d_i \geq 0$ , and

$$D'_{j}(d) = \begin{cases} d_{1} & j = 0 \\ -2d_{1} + d_{2} & j = 1 \\ d_{j-1} - 2d_{j} + d_{j+1} & 1 < j < n-1 \\ d_{n-2} - 2d_{n-1} & j = n-1 \\ d_{n-1} & j = n. \end{cases}$$

 $I_Y(y,z)$  is a multi-valued function of  $y=(y_0,\ldots,y_{n-1})\in\mathcal{M}_B,\ z\in\mathbb{C}^\times$ , and  $\lambda_1,\lambda_2\in\mathbb{C}$  which takes values in  $H_T^\bullet(Y;\mathbb{C})$ . Each component of  $I_Y(y,z)$  with respect to the basis  $\{\gamma_i\}$  is a multi-valued analytic function of  $(y,z,\lambda_1,\lambda_2)$  defined in a domain where  $|y_1|,\ldots,|y_{n-1}|$  are sufficiently small,  $|z|>\max(|\lambda_1|,|\lambda_2|)$ , and  $y_0$  is arbitrary. By taking a Laurent expansion at  $z=\infty$  we can regard  $I_Y(y,z)$  as a

multi-valued analytic function of  $(y, \lambda_1, \lambda_2)$  which takes values in  $H(Y) \otimes \mathbb{C}((z^{-1}))$ . It also satisfies a system of Picard–Fuchs equations. Define differential operators

Then

(36a) 
$$\left(\prod_{j:D'_{j}(d)>0} \prod_{m=0}^{D'_{j}(d)-1} (\exists_{j}-mz)\right) I_{Y}(y,z)$$

$$= y_{1}^{d_{1}} \cdots y_{n-1}^{d_{n-1}} \left(\prod_{j:D'_{j}(d)<0} \prod_{m=0}^{-D'_{j}(d)-1} (\exists_{j}-mz)\right) I_{Y}(y,z)$$

for every  $d = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$ , and

(36b) 
$$z \frac{\partial}{\partial y_0} I_Y(y, z) = I_Y(y, z).$$

The Picard–Fuchs systems (35) for  $\mathcal{X}$  and (36) for Y coincide under the co-ordinate change (28). Thus there is a global system of Picard–Fuchs equations — a  $\mathcal{D}$ -module over all of  $\mathcal{M}_B$  — which gives (35) near the large-radius limit point for  $\mathcal{X}$  and (36) near the large-radius limit point for Y. This global nature of the Picard–Fuchs system will play a key role in what follows.

**A Mirror Theorem.** By mirror symmetry, we mean the following.

## Proposition A.3.

- (1)  $I_{\mathcal{X}}(x,z)$  and  $J_{\mathcal{X}}(\tau,z)$  coincide after a change of variables expressing  $\tau$  in terms of x.
- (2)  $I_Y(y,z)$  and  $J_Y(t,z)$  coincide after a change of variables expressing t in terms of y.

*Proof.* Part (1) is equation (24):  $I^{\text{tw}}(x,z)$  there coincides with  $I_{\mathcal{X}}(x,z)$  here and  $J^{\text{tw}}(\tau,z)$  there coincides with  $J_{\mathcal{X}}(\tau,z)$  here. The argument that proves Theorem 0.2 in [31] also proves part (2) here. Theorem 0.2 as stated only applies to compact semi-positive toric manifolds, but the proof applies essentially without change to the non-compact toric Calabi–Yau manifold Y.

**Remark.** We learned from Bong Lian that, in unpublished work, he and Chien-Hao Liu have established mirror theorems for non-compact toric Calabi–Yau manifolds using the arguments of [46]. Once again, the proof for compact toric manifolds applies also to the non-compact toric Calabi–Yau case without significant change. This gives an alternative proof of the second part of Proposition A.3.

From equation (23) we know that the change of variables in Proposition A.3 between  $x_0, \ldots, x_{n-1}$  and  $\tau = \tau^0 \delta_0 + \cdots + \tau^{n-1} \delta_{n-1}$  is  $\tau^r = f^r(x)$ ,

$$f^{r}(x) = \begin{cases} x_{0} & r = 0\\ \sum_{\substack{k_{1}, \dots, k_{n-1} \geq 0: \\ \langle b(\boldsymbol{k}) \rangle = \frac{r}{n}}} \frac{x_{1}^{k_{1}} x_{2}^{k_{2}} \cdots x_{n-1}^{k_{n-1}}}{k_{1}! k_{2}! \cdots k_{n-1}!} \frac{\Gamma\left(\langle D_{0}(\boldsymbol{k}) \rangle\right)}{\Gamma\left(1 + D_{0}(\boldsymbol{k})\right)} \frac{\Gamma\left(\langle D_{n}(\boldsymbol{k}) \rangle\right)}{\Gamma\left(1 + D_{n}(\boldsymbol{k})\right)} & r \neq 0. \end{cases}$$

As

$$f^{r}(x) = x_{r} + \text{quadratic}$$
 and higher order terms in  $x_{1}, \ldots, x_{n-1}$ 

the functions  $f^0(x), \ldots, f^{n-1}(x)$  define co-ordinates on a neighbourhood of the large-radius limit point for  $\mathcal{X}$  in  $\mathcal{M}_B$ . We call these *flat co-ordinates for*  $\mathcal{X}$ . Similarly,

$$J_Y(t,z) = z + t^0 \gamma_0 + t^1 \gamma_1 + \dots + t^{n-1} \gamma_{n-1} + O(z^{-1})$$

and

$$I_Y(y,z) = z + g^0(y)\gamma_0 + g^1(y)\gamma_1 + \dots + g^{n-1}(y)\gamma_{n-1} + O(z^{-1})$$

for some functions  $g^0(y), \ldots, g^{n-1}(y)$  with  $g^0(y) = y_0$  and for  $1 \le k < n$ ,

$$g^{k}(y) = \log y_{k} + \text{single-valued analytic function of } y_{1}, \dots, y_{n-1}.$$

The change of variables which equates  $I_Y$  and  $J_Y$  is  $t^i = g^i(y)$ ,  $0 \le i < n$ . The functions  $g^0(y), \ldots, g^{n-1}(y)$  define multi-valued co-ordinates on a neighbourhood of the large-radius limit point for Y; these are the *flat co-ordinates for* Y. Note that the exponentiated flat co-ordinates  $\exp(g^k(y))$  are single-valued.

The J-functions satisfy differential equations which determine the quantum products.

## Proposition A.4.

(1) 
$$z\frac{\partial}{\partial \tau^{i}}z\frac{\partial}{\partial \tau^{j}}J_{\mathcal{X}}(\tau,z) = \sum_{k=0}^{n-1} \left(\delta_{i} \underset{\text{big}}{\star}\right)_{j}^{k} z\frac{\partial}{\partial \tau^{k}}J_{\mathcal{X}}(\tau,z)$$

$$\text{where } \left(\delta_{i} \underset{\text{big}}{\star}\right)_{j}^{k} \text{ are the matrix entries of the product (32).}$$
(2) 
$$z\frac{\partial}{\partial t^{i}}z\frac{\partial}{\partial t^{j}}J_{Y}(t,z) = \sum_{k=0}^{n-1} \left(\gamma_{i} \underset{\text{big}}{\star}\right)_{j}^{k} z\frac{\partial}{\partial t^{k}}J_{Y}(t,z)$$

$$\text{where } \left(\gamma_{i} \underset{\text{big}}{\star}\right)_{j}^{k} \text{ are the matrix entries of the product (33).}$$

*Proof.* Part (2) is well-known: it follows from the Topological Recursion Relations (cf. [25, Chapter 10; 52, Proposition 2]). The proof of (1) is essentially identical, but uses the Topological Recursion Relations for orbifolds [58, Section 2.5.5] instead of the Topological Recursion Relations for varieties. Details here are in [21].  $\Box$ 

From PF to QC. Propositions A.3 and A.4 together show that we can determine the quantum products (32) and (33) by looking at the differential equations satisfied by  $I_{\mathcal{X}}$  and  $I_{Y}$  in flat co-ordinates:

(37) 
$$z \frac{\partial}{\partial \tau^{i}} z \frac{\partial}{\partial \tau^{j}} I_{\mathcal{X}}(x(\tau), z) = \sum_{k=0}^{n-1} \left( \delta_{i \underset{\text{big}}{\star}} \right)_{j}^{k} z \frac{\partial}{\partial \tau^{k}} I_{\mathcal{X}}(x(\tau), z)$$

(38) 
$$z \frac{\partial}{\partial t^{i}} z \frac{\partial}{\partial t^{j}} I_{Y}(y(t), z) = \sum_{k=0}^{n-1} \left( \gamma_{i \underset{\text{big}}{\star}} \right)_{j}^{k} z \frac{\partial}{\partial t^{k}} I_{Y}(y(t), z)$$

A more invariant way to say this is as follows. Let  $\lambda_1$ ,  $\lambda_2$  be fixed complex numbers. If we associate to a vector field  $v = \sum v_k(y) \frac{\partial}{\partial y_k}$  on  $\mathcal{M}_B$  the differential operator  $\sum zv_k(y) \frac{\partial}{\partial y_k}$  then the systems of differential equations (35), (36) define a  $\mathcal{D}$ -module on  $\mathcal{M}_B$ . The characteristic variety  $\mathfrak{V}$  of this  $\mathcal{D}$ -module is a subscheme of  $T^*\mathcal{M}_B$ , and we can read off the quantum products from the algebra of functions  $\mathcal{O}_{\mathfrak{V}}$ . Indeed, choosing flat co-ordinates on a neighbourhood U of the large-radius limit point for  $\mathcal{X}$  in  $\mathcal{M}_B$  identifies  $\mathcal{O}_U$  with analytic functions in  $\tau^0, \ldots, \tau^{n-1}$  and identifies the algebra of fiberwise-polynomial functions on  $T^*U$  with  $\mathcal{O}_U[\xi_0, \ldots, \xi_{n-1}]$ ; here  $\xi_k$  is the fiberwise-linear function on  $T^*U$  given by  $\frac{\partial}{\partial \tau^k}$ . The ideal defining the characteristic variety  $\mathfrak{V}$  is generated by elements

$$P(\tau^0, \dots, \tau^{n-1}, \xi_0, \dots, \xi_{n-1}, 0)$$

where  $P(\tau^0, \dots, \tau^{n-1}, \xi_0, \dots, \xi_{n-1}, z)$  runs over the set of fiberwise-polynomial functions on  $T^*U$  which depend polynomially on z and satisfy

$$P\left(\tau^{0},\ldots,\tau^{n-1},z\frac{\partial}{\partial\tau^{0}},\ldots,z\frac{\partial}{\partial\tau^{n-1}},z\right)I_{\mathcal{X}}(u,z)=0.$$

Equation (37) implies that

$$\mathcal{O}_{\mathfrak{V}}|_{U} = \mathcal{O}_{U}[\xi_{0}, \dots, \xi_{n-1}]/\mathfrak{I}$$

where the ideal  $\Im$  is generated by

$$\xi_i \xi_j - \sum_{k=0}^{n-1} \left( \delta_{i \underset{\text{big}}{\star}} \right)_j^k \xi_k \qquad 0 \le i, j < n.$$

In other words, the quantum cohomology algebra (32) of  $\mathcal{X}$  is the algebra of functions  $\mathcal{O}_{\mathfrak{V}}|_{U}$  on the characteristic variety  $\mathfrak{V}$ , written in flat co-ordinates on U.

Similarly, choosing flat co-ordinates on a neighbourhood V of the large-radius limit point for Y in  $\mathcal{M}_B$  identifies  $\mathcal{O}_V$  with analytic functions in  $t^0, \ldots, t^{n-1}$ , and identifies the algebra of fiberwise-polynomial functions on  $T^*V$  with  $\mathcal{O}_V[\eta_0, \ldots, \eta_{n-1}]$  where  $\eta_k$  is the fiberwise-linear function on  $T^*V$  given by  $\frac{\partial}{\partial t^k}$ . Equation (38) implies that

$$\mathcal{O}_{\mathfrak{V}}|_{V} = \mathcal{O}_{V}[\eta_{0}, \dots, \eta_{n-1}]/\mathfrak{J}$$

where the ideal  $\mathfrak{J}$  is generated by

$$\eta_i \eta_j - \sum_{k=0}^{n-1} \left( \gamma_i \star_{\text{big}} \right)_j^k \eta_k \qquad 0 \le i, j < n,$$

and so the quantum cohomology algebra (33) of Y is the algebra of functions  $\mathcal{O}_{\mathfrak{V}}|_{V}$  on the characteristic variety  $\mathfrak{V}$ , written in flat co-ordinates on V.

The characteristic variety  $\mathfrak V$  is a global analytic object —  $\mathcal O_{\mathfrak V}$  gives an analytic sheaf of  $\mathcal{O}_{\mathcal{M}_B}$ -algebras, defined over all of  $\mathcal{M}_B$  — so to show that the quantum cohomology algebras of  $\mathcal{X}$  and of Y are related by analytic continuation followed by the change-of-variables

$$t^{i} = \begin{cases} \tau^{0}, & i = 0\\ -\frac{2\pi\sqrt{-1}}{n} + \sum_{j=1}^{n-1} L^{i}{}_{j}\tau^{j}, & i > 0 \end{cases}$$

we just need to show that the flat co-ordinates for  $\mathcal{X}$  and for Y are related by analytic continuation followed by the change-of-variables

$$g^{i}(y) = \begin{cases} f^{0}(x), & i = 0\\ -\frac{2\pi\sqrt{-1}}{n} + \sum_{j=1}^{n-1} L_{j}^{i} f^{j}(x), & i > 0. \end{cases}$$

We will do this by showing that  $g^{i}(y)$  and  $f^{j}(x)$  satisfy the same system of differential equations.

The GKZ System Associated to Y. The GKZ system associated to Y is the system of differential equations

(39) 
$$\left(\prod_{j:D'_{j}(d)>0} \prod_{m=0}^{D'_{j}(d)-1} (\mathbb{1}_{j}-m)\right) f$$

$$= y_{1}^{d_{1}} \cdots y_{n-1}^{d_{n-1}} \left(\prod_{j:D'_{j}(d)<0} \prod_{m=0}^{-D'_{j}(d)-1} (\mathbb{1}_{j}-m)\right) f$$

for every  $d = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$ , where

$$\exists_j = \begin{cases} y_1 \frac{\partial}{\partial y_1} & j = 0 \\ -2y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} & j = 1 \\ y_{j-1} \frac{\partial}{\partial y_{j-1}} - 2y_j \frac{\partial}{\partial y_j} + y_{j+1} \frac{\partial}{\partial y_{j+1}} & 1 < j < n-1 \\ y_{n-2} \frac{\partial}{\partial y_{n-2}} - 2y_{n-1} \frac{\partial}{\partial y_{n-1}} & j = n-1 \\ y_{n-1} \frac{\partial}{\partial y_{n-1}} & j = n. \end{cases}$$

## Proposition A.5. Both

- (a)  $f^1(x), \ldots, f^{n-1}(x)$  plus the constant function; and (b)  $g^1(y), \ldots, g^{n-1}(y)$  plus the constant function

form bases of solutions to the GKZ system (39).

Proof. The sets (a) and (b) are evidently each linearly independent. The constant function evidently satisfies (39). The flat co-ordinates  $f^1(x), \ldots, f^{n-1}(x)$ and  $g^1(y), \ldots, g^{n-1}(y)$  are independent of  $\lambda_1, \lambda_2, x_0$ , and  $y_0$ , so they can be extracted from the  $z^0$  terms of  $I_{\mathcal{X}}$  and  $I_Y$  after setting  $\lambda_1 = \lambda_2 = x_0 = y_0 = 0$ . But  $I_{\mathcal{X}}|_{\lambda_1=\lambda_2=x_0=0}$  and  $I_Y|_{\lambda_1=\lambda_2=y_0=0}$  satisfy the systems of differential equations (35a), (36a) with  $\lambda_1$  and  $\lambda_2$  set to zero, and once  $\lambda_1$  and  $\lambda_2$  are set to zero the z-dependence in these differential equations cancels. The resulting system of differential equations in each case is (39).  Any analytic continuation  $\tilde{g}_i(y)$  of  $g^i(y)$  to a neighbourhood of the large-radius limit point for  $\mathcal{X}$  still satisfies (39), so

$$\tilde{g}_i(y) = \sum_{j=1}^{n-1} L^i_{\ j} f^j(x) + m_i$$

for some constants  $L^i_j$  and  $m_i$ . Thus any analytic continuation of  $g^i(y)$  is an affine-linear combination of the flat co-ordinates  $f^1(x), \ldots, f^{n-1}(x)$ . It remains to choose a specific analytic continuation and determine the corresponding constants  $L^i_j$  and  $m_i$ . Before we do so, we prove Proposition 6.2. This follows immediately from:

**Proposition A.6.** Let  $\zeta = \exp\left(\frac{\pi\sqrt{-1}}{n}\right)$ . Let  $\kappa_0(x), \ldots, \kappa_{n-1}(x)$  be the roots of the polynomial

$$W_{\mathcal{X}}(\kappa) = \kappa^n + x_{n-1}\kappa^{n-1} + x_{n-2}\kappa^{n-2} + \dots + x_1\kappa + 1,$$

where the roots are numbered such that as  $x \to 0$ ,  $\kappa_i(x) \to \zeta^{2i+1}$ ,  $0 \le i < n$ . Then another basis of solutions to the GKZ system (39) is given by

(40) 
$$\log \kappa_i(x) - \log \kappa_{i-1}(x), \qquad 1 \le i < n$$

together with the constant function. Furthermore,

(41) 
$$\log \kappa_i(x) = \frac{(2i+1)\pi\sqrt{-1}}{n} + \frac{1}{n} \sum_{k=1}^{n-1} \zeta^{(2i+1)k} f^k(x).$$

*Proof.* Most of this is classical: see for example [47; 56, Theorem 1]. Here we follow [35, Section 6]. Let  $(\kappa, \nu, u, v)$  be co-ordinates on  $\mathbb{C}^{\times} \times \mathbb{C}^{3}$ . By using the Morse–Bott function  $(\kappa, \nu, u, v) \mapsto \Re(\nu(W_{\mathcal{X}}(\kappa) + uv))$  on  $\mathbb{C}^{\times} \times \mathbb{C}^{3}$ , we can construct a Morse cycle  $\Gamma \subset \mathbb{C}^{\times} \times \mathbb{C}^{3}$  as the union of downward gradient flowlines for  $\Re(\nu(W_{\mathcal{X}}(\kappa) + uv))$  from a compact 2-cycle  $\Gamma'$  in the critical set  $Z := \{\nu = 0, W_{\mathcal{X}}(\kappa) + uv = 0\} \subset \mathbb{C}^{\times} \times \mathbb{C}^{3}$ . The integral

$$\gamma(x) = \int_{\Gamma} \exp\left(\nu(W_{\mathcal{X}}(\kappa) + uv)\right) \frac{d\kappa}{\kappa} d\nu \, du \, dv$$

satisfies the differential equations (39). But the integrand here is equal to

$$d\left(\frac{\exp\left(\nu\left(W_{\mathcal{X}}(\kappa) + uv\right)\right)}{W_{\mathcal{X}}(\kappa) + uv} \frac{d\kappa}{\kappa} du \, dv\right)$$

outside Z, and so

$$\gamma(x) = \int_{\Gamma' \subset Z} \frac{\frac{d\kappa}{\kappa} du \, dv}{d \left( W_{\mathcal{X}}(\kappa) + uv \right)}$$
$$= \int_{\Gamma' \subset X} \frac{d\kappa}{\kappa} \frac{du}{u}.$$

Now we take  $\Gamma' \subset X$  to be a vanishing cycle for the function  $W_{\mathcal{X}} + uv$ . Integrating out du/u gives

(42) 
$$\gamma(x) = \int_{C \subset \mathbb{C}^{\times}} \frac{d\kappa}{\kappa} = \log \kappa_m - \log \kappa_l,$$

where  $\kappa_l$ ,  $\kappa_m$  are roots of  $W_{\mathcal{X}}(\kappa) = 0$  and C is a path from  $\kappa_l$  to  $\kappa_m$ . By choosing an appropriate basis of vanishing cycles  $\Gamma'$ , we find the n-1 linearly independent solutions (40); these, together with the constant function, form a basis of solutions to the GKZ system.

It remains to prove (41). We have

$$\kappa_i(x) = \zeta^{2i+1} + O(x_i)$$

and

$$x_k = (-1)^{n-k} e_{n-k}(\kappa_0, \dots, \kappa_{n-1}),$$
  $1 \le k < n.$ 

Since the constant function and (42) are solutions to the GKZ system (39) and  $\kappa_0 \cdots \kappa_{n-1} = (-1)^n$ , each  $\log \kappa_i$  is also a solution to (39). Thus, by Proposition A.6,  $\log \kappa_i$  must be a linear combination of  $f^1(x), \ldots, f^{n-1}(x)$  and a constant:

$$\log \kappa_i = \sum_{r=1}^{n-1} c_{ir} f^r(x) + \text{const.}$$

Since  $\kappa_i = \zeta^{2i+1}$  at x = 0, we have

(43) 
$$\log\left(\frac{\kappa_i}{\zeta^{2i+1}}\right) = \sum_{r=1}^{n-1} c_{ir} f^r(x).$$

On the other hand, differentiating  $W_{\mathcal{X}}(\kappa_i) = 0$  gives

$$(\kappa_i)^k + n(\kappa_i)^{n-1} \frac{\partial \kappa_i}{\partial x_k} + O(x_j) = 0$$

and so

$$\kappa_i(x) = \zeta^{2i+1} - \frac{1}{n} \sum_{k=1}^{n-1} \zeta^{(2i+1)(k-n+1)} x_k + O((x_j)^2).$$

Substituting this into (43) and using the fact that

$$f^{r}(x) = x_{r} + O\left(\left(x_{j}\right)^{2}\right)$$

yields 
$$(41)$$
.

We observed above Proposition A.6 that any analytic continuation of  $g^i(y)$  is an affine-linear combination of the flat co-ordinates  $f^1(x), \ldots, f^{n-1}(x)$ .

**Proposition A.7.** There exists a path from the large-radius limit point for Y to the large-radius limit point for  $\mathcal{X}$  such that the analytic continuation of the flat co-ordinates  $g^i(y)$ ,  $1 \leq i < n$ , along that path satisfy

$$g^{i}(y) = -\frac{2\pi\sqrt{-1}}{n} + \frac{1}{n}\sum_{k=1}^{n-1} \zeta^{2ki} \left(\zeta^{-k} - \zeta^{k}\right) f^{k}(x),$$

where 
$$\zeta = \exp\left(\frac{\pi\sqrt{-1}}{n}\right)$$
.

*Proof.* Consider the polynomial

$$W_Y(\mu) = \mu^n + \mu^{n-1} + y_1 \mu^{n-2} + y_1^2 y_2 \mu^{n-3} + y_1^3 y_2^2 y_3 \mu^{n-4} + \cdots + y_1^{n-1} y_2^{n-2} \cdots y_{n-2}^2 y_{n-1}$$

and number its roots  $\mu_i(y)$ ,  $0 \le i < n$  such that as  $y \to 0$ 

$$\mu_0(y) \to -1$$

$$\mu_1(y) \sim -y_1$$

$$\mu_2(y) \sim -y_1 y_2$$

$$\vdots$$

$$\mu_{n-1}(y) \sim -y_1 y_2 \cdots y_{n-1}$$

We have  $W_{\mathcal{X}}(\kappa) = 0$  if and only if  $W_Y(1/(x_1\kappa)) = 0$ , where  $x_i$  and  $y_j$  are related by (28), so still another basis of solutions to the GKZ system (39) is

$$\log \mu_i(y) - \log \mu_{i-1}(y), \qquad 1 \le i < n,$$

together with the constant function. The solution  $g^i(y)$  is singled out by its behaviour  $g^i(y) = \log y_i + O(y_1, \dots, y_{n-1})$  as  $y \to 0$ , so

$$g^{i}(y) = \log \mu_{i}(y) - \log \mu_{i-1}(y)$$

Along any path from the large-radius limit point for Y to the large-radius limit point for  $\mathcal{X}$ , the root  $\mu_i(y)$  of  $W_Y$  analytically continues to the root  $1/(x_1\kappa_{\sigma(i)}(x))$  of  $W_{\mathcal{X}}$ , for some permutation  $\sigma$  of  $\{0,1,\ldots,n-1\}$ . The group of monodromies around the discriminant locus of  $W_{\mathcal{X}}$  acts n-transitively on the set of roots of  $W_{\mathcal{X}}$ , so we can choose a path such that  $\sigma$  is the identity permutation. Along this path,  $\log \mu_i(y) - \log \mu_{i-1}(y)$  analytically continues to  $\log \kappa_{i-1}(x) - \log \kappa_i(x)$ ,  $1 \leq i < n$ . Applying equation (41) yields

$$g^{i}(y) = -\frac{2\pi\sqrt{-1}}{n} + \frac{1}{n}\sum_{k=1}^{n-1} \zeta^{2ki} \left(\zeta^{-k} - \zeta^{k}\right) f^{k}(x).$$

**Remark.** For an explicit path satisfying the conditions in Proposition A.7, we can concatenate two paths defined as follows. The first runs from  $(y_0, y_1, \ldots, y_{n-1}) = (0, 0, \ldots, 0, 0)$  to  $(y_0, y_1, \ldots, y_{n-1}) = (0, 1, 1, \ldots, 1, 1)$  and is given by  $y_0 = 0$  and

$$W_Y(\mu) = \left(\mu - \left(-1 - \epsilon \rho^2 - \epsilon^2 \rho^3 - \dots - \epsilon^{n-1} \rho^n\right)\right) \prod_{k=1}^{n-1} \left(\mu - \epsilon^k \rho^{k+1}\right), \quad 0 \le \epsilon \le 1,$$

where  $\rho = \exp\left(\frac{2\pi\sqrt{-1}}{n+1}\right)$ . The second runs from  $(x_0, x_1, \dots, x_{n-1}) = (0, 1, 1, \dots, 1, 1)$  to  $(x_0, x_1, \dots, x_{n-1}) = (0, 0, \dots, 0, 0)$ , and is given by  $x_0 = 0$  and

$$W_{\mathcal{X}}(\kappa) = \prod_{k=0}^{n-1} \left( \kappa - \exp\left(\pi\sqrt{-1} \left[ \frac{2k+1}{n} \epsilon' + \frac{2(n-k)}{n+1} (1-\epsilon') \right] \right) \right), \quad 0 \le \epsilon' \le 1.$$

Note that the points  $(y_0, y_1, \dots, y_{n-1}) = (0, 1, 1, \dots, 1, 1)$  and  $(x_0, x_1, \dots, x_{n-1}) = (0, 1, 1, \dots, 1, 1)$  coincide.

The Proof of Theorem A.1. Combining Proposition A.7 with the discussion above equation (39) shows that the quantum cohomology algebras of  $\mathcal{X}$  and Y coincide after analytic continuation along the path specified in Proposition A.7 followed by the affine-linear change-of-variables

$$t^{i} = \begin{cases} \tau^{0}, & i = 0 \\ -\frac{2\pi\sqrt{-1}}{n} + \sum_{j=1}^{n-1} L^{i}{}_{j}\tau^{j}, & i > 0, \end{cases} \qquad L^{i}{}_{j} = \frac{\zeta^{2ij}\left(\zeta^{-j} - \zeta^{j}\right)}{n},$$

and the linear isomorphism

$$L: H(\mathcal{X}) \to H(Y)$$

$$\delta_0 \mapsto \gamma_0,$$

$$\delta_j \mapsto \sum_{i=1}^{n-1} L^i{}_j \gamma_i, \qquad 1 \le j < n.$$

To see that L preserves the Poincaré pairings, first observe that the bases

$$n\lambda_1\lambda_2, \gamma_1, \gamma_2, \dots, \gamma_{n-1}$$
 and  $1, \omega_1, \omega_2, \dots, \omega_{n-1}$ 

for H(Y) are dual with respect to the Poincaré pairing on H(Y). Let  $L^{\dagger}$  denote the adjoint to L with respect to the Poincaré pairing  $(\cdot, \cdot)_Y$  and the orbifold Poincaré pairing  $(\cdot, \cdot)_{\mathcal{X}}$ . It suffices to show that  $(L^{\dagger}\gamma, L^{\dagger}\gamma')_{\mathcal{X}} = (\gamma, \gamma')_Y$  for all  $\gamma, \gamma' \in H(Y)$ . For  $1 \leq i < n$ , we have  $(L^{\dagger}\omega_i, \delta_k)_{\mathcal{X}} = (\omega_i, L\delta_k)_Y = L^i_k$ , and so

$$L^{\dagger}\omega_i = n \sum_{k=1}^{n-1} L^i_{\ k} \delta_{n-k}, \qquad 1 \le i < n.$$

Also  $L^{\dagger}1 = \delta_0$ . Straightforward calculation now gives  $(L^{\dagger}1, L^{\dagger}1)_{\mathcal{X}} = (n\lambda_1\lambda_2)^{-1}$ ,  $(L^{\dagger}1, L^{\dagger}\omega_i)_{\mathcal{X}} = 0$  for  $1 \leq i < n$ , and

$$(L^{\dagger}\omega_i, L^{\dagger}\omega_j)_{\mathcal{X}} = \begin{cases} 0 & \text{if } |i-j| > 1\\ 1 & \text{if } |i-j| = 1\\ -2 & \text{if } i = j \end{cases}$$
 for  $1 \le i, j < n$ .

As the class  $\omega_j$  is the *T*-equivariant Poincaré-dual to the *j*th exceptional divisor we see that  $L^{\dagger}$ , and hence *L*, is pairing-preserving. This completes the proof of Theorem A.1.

Remark. A more conceptual explanation of this result is as follows. One can construct a Frobenius manifold from a variation of semi-infinite Hodge structure [5] (henceforth  $V^{\infty}_{2}HS$ ) together with a choice of opposite subspace<sup>9</sup>. We have argued elsewhere that in certain toric examples one can construct the Frobenius manifold which is the "mirror partner" to the quantum cohomology of Y from a  $V^{\infty}_{2}HS$  parameterized by the B-model moduli space of Y, together with a distinguished opposite subspace associated to the large-radius limit point for Y [20]. (The Frobenius manifold mirror to the quantum cohomology of a toric orbifold  $\mathcal{X}$  birational to Y is given by the same  $V^{\infty}_{2}HS$  but the opposite subspace corresponding to the large-radius limit point for  $\mathcal{X}$ .) One can apply this construction here to get a  $V^{\infty}_{2}HS$  parametrized by  $\mathcal{M}_{B}$ . This  $V^{\infty}_{2}HS$  has the special property that

<sup>&</sup>lt;sup>9</sup>Mirror symmetry often associates to the quantum cohomology of some target space a "mirror family" of manifolds. In this case one can think of the  $V \frac{\infty}{2} HS$  as an analog of the usual variation of Hodge structure on the mirror family, and the opposite subspace as an analog of the weight filtration.

the opposite subspace at the large-radius limit point for Y agrees with the opposite subspace at the large-radius limit point for  $^{10}$   $\mathcal{X}$ . In general the difference between the opposite subspaces at different large radius limit points will be measured by an element of Givental's linear symplectic group, but in this case the corresponding group element maps the opposite subspaces isomorphically to each other. This means that we get a Frobenius manifold over the whole (non-linear) space  $\mathcal{M}_B$ . One can construct flat co-ordinates in a neighbourhood of any point of  $\mathcal{M}_B$ , and the transition functions between such flat co-ordinate patches, such as

$$g^{i}(y) = \sum_{j} L^{i}_{j} f^{j}(x) + \log c_{i},$$

are necessarily affine-linear and (Poincaré) metric-preserving.

**Remark.** It is clear from the proof of Proposition A.7 that changing the path along which analytic continuation is taken will result in a corresponding change in the statements of Theorem A.1 and its Corollary. Hence the co-ordinate change in Theorem A.1 is not unique. This ambiguity can be understood as an automorphism of quantum cohomology. The orbifold fundamental group

$$G := \pi_1^{\operatorname{orb}} \left( \mathcal{M}_B \setminus \{ \operatorname{discriminant locus of} W_{\mathcal{X}} \} \right)$$

acts simply-transitively on the set of homotopy types of paths from the largeradius limit point for Y to that for  $\mathcal{X}$ , and in particular acts transitively (but ineffectively) on the set of all possible co-ordinate changes obtained by analytic continuation. This deserves further study: we just note here the intriguing fact that G is isomorphic to  $\widetilde{A}_{n-1} \rtimes \mathbb{Z}_n$ , which also appears as a subgroup (generated by spherical twists and line bundles) of the group of autoequivalences of  $D_E^b(Y)$ [9,38,39]. Here  $\widetilde{A}_{n-1}$  is the affine braid group and  $D_E^b(Y)$  is the bounded derived category of coherent sheaves on Y supported on the exceptional set E.

## APPENDIX B. GIVENTAL'S LAGRANGIAN CONE AS A FORMAL SCHEME

In the main body of the text, following Givental, we encode genus-zero Gromov–Witten invariants via the formal germ  $\mathcal{L}$  (=  $\mathcal{L}^{\text{tw}}$  or  $\mathcal{L}^{\text{un}}$ ) of an infinite-dimensional submanifold in a symplectic vector space  $\mathcal{H}$ . This submanifold-germ  $\mathcal{L}$  has very special geometric properties, several of which play an essential role in the proof of Theorem 4.6. These geometric properties were discovered in [24,33]. The properties of  $\mathcal{L}$  described in *loc. cit.* should, as stated there, "be interpreted in the sense of formal geometry". It is clear what this means, but an appropriate framework of formal geometry seems to be missing from the literature. In this Appendix we remedy this: we define the formal germ  $\mathcal{L}$  and the formal germ of  $\mathcal{H}$  as non-Noetherian formal schemes. We also establish, within our framework, the geometric properties of  $\mathcal{L}$  which we use above. A more detailed account of the geometry described here will be given in a future paper [22].

This section is unavoidably technical, and we urge the reader unfamiliar with Givental's formalism to begin by reading the much more accessible account [33]. Also, we should emphasize that any originality here is in the framework that we use

 $<sup>^{10}</sup>$ In fact each maximal cone of the secondary fan gives rise to a toric partial resolution Y' of  $\mathcal{X}$ , a large-radius limit point for Y', and an opposite subspace corresponding to this large-radius limit point. All these opposite subspaces agree: we really do get a Frobenius structure defined over all of  $\mathcal{M}_B$ .

rather than in the content of our arguments. The geometric properties of  $\mathcal{L}$  which we discuss have already been established by Givental in [33], and our arguments share a core of ideas with his.

**Topological rings.** In the main body of the text, we considered topological rings equipped with a non-negative additive valuation. Here we will deal with a broader class of topological rings. Let R be a commutative topological ring with a unit. A topology on R is said to be linear if a fundamental neighborhood system of zero in R is given by a descending chain of ideals  $I_0 \supset I_1 \supset I_2 \supset \cdots$  in R. Conversely, any descending chain  $\{I_n\}_{n\geq 0}$  of ideals defines a unique linear topology on R. In this case we say that the topology on R is defined by  $ideals\ \{I_n\}_{n\geq 0}$ . For example, a non-negative additive valuation  $v\colon R\setminus\{0\}\to\mathbb{R}_{\geq 0}$  on an integral domain R gives a linear topology on R defined by the ideals  $I_n=\{r\in R\ ;\ v(r)\geq n\}$ . Throughout Appendix B, a topology on a ring is assumed to be linear, complete and  $Hausdorff^{11}$ .

Define the space of  $convergent\ Laurent\ series$  in z to be

$$R\{z, z^{-1}\} := \left\{ \sum_{n \in \mathbb{Z}} r_n z^n : r_n \in R, r_n \to 0 \text{ as } |n| \to \infty \right\}.$$

When the topology on R is defined by  $\{I_n\}_{n\geq 0}$ , this space is identified with the completion of  $R[z,z^{-1}]$  with respect to the topology defined by  $\{I_n[z,z^{-1}]\}_{n\geq 0}$ . When the topology on R comes from an additive valuation, this definition coincides with that given in Section 3.

Define the ideal of topologically nilpotent elements to be

$$R^{\mathrm{nilp}} := \left\{ r \in R \ : \ \lim_{n \to \infty} r^n = 0 \right\}.$$

When the topology on R is defined by ideals  $\{I_n\}_{n\geq 0}$ , the topology on  $R[\epsilon]/\langle \epsilon^2 \rangle$  is defined by  $\{I_n+I_n\epsilon\}_{n\geq 0}$ , that on R[t] is defined by  $\{I_n[t]+t^nR[t]\}_{n\geq 0}$ , and that on  $R\{z,z^{-1}\}$  is given by  $\{I_n\{z,z^{-1}\}\}_{n\geq 0}$ . One can check that if R is complete and Hausdorff then so are  $R[\epsilon]/\langle \epsilon^2 \rangle$ , R[t], and  $R\{z,z^{-1}\}$ .

**Notation Conventions.** Recall that  $\{\phi_{\alpha}: 1 \leq \alpha \leq N\}$  and  $\{\phi^{\alpha}: 1 \leq \alpha \leq N\}$  are bases for  $H_{\mathrm{orb}}^{\bullet}(\mathcal{X})$  such that  $(\phi_{\alpha}, \phi^{\beta})_{\mathrm{orb}}^{\mathrm{tw}} = \delta_{\alpha}^{\ \beta}$ . We take  $\phi_{1}$  to be the unit class 1, and set  $g_{\alpha\beta} = (\phi_{\alpha}, \phi_{\beta})_{\mathrm{orb}}^{\mathrm{tw}}$  and  $g^{\alpha\beta} = (\phi^{\alpha}, \phi^{\beta})_{\mathrm{orb}}^{\mathrm{tw}}$ . Throughout this Appendix we use Einstein's summation convention for Greek indices, summing repeated Greek (but not Roman) indices over the range  $1, 2, \ldots, N$ .

A formal germ at -z of Givental's space  $\mathcal{H}$ . Recall that  $\Lambda[s_0, s_1, \dots]$  was defined to be the completion of  $\mathbb{C}[\mathrm{Eff}(\mathcal{X})][s_0, s_1, \dots]$  with respect to the non-negative additive valuation v given by

$$v(Q^d) = \int_d \omega, \quad v(s_k) = k+1.$$

Here the  $s_i$  are parameters of the universal invertible multiplicative characteristic class c: see (11). We write  $\Lambda_s := \Lambda[s_0, s_1, \dots]$ . Let  $\mathcal{H}$  be Givental's vector space over  $\Lambda_s$ :

$$\mathcal{H} = H^{\bullet}_{\mathrm{orb}}(\mathcal{X}; \mathbb{C}) \otimes_{\mathbb{C}} \Lambda_{s}\{z, z^{-1}\}.$$

 $<sup>^{11}</sup>$  In the literature on formal schemes, an additional admissibility condition is imposed on R. For example, McQuillan [50] considered the condition that  $R^{\rm nilp}$  is open, i.e. contains some  $I_n$ . Although our examples always satisfy this condition, we do not need this in the explanation below.

We write a general element  $f \in \mathcal{H}$  in the form:

$$f = -z + \mathbf{t}(z) + \mathbf{p}(z), \qquad \mathbf{t}(z) = \sum_{k>0} t_k^{\alpha} \phi_{\alpha} z^k, \qquad \mathbf{p}(z) = \sum_{l>0} p_{l\beta} \frac{\phi^{\beta}}{(-z)^{l+1}}.$$

We regard the coefficients  $t_k^{\alpha}, p_{l\beta} \in \Lambda_s$  as affine co-ordinate functions  $t_k^{\alpha}, p_{l\beta} : \mathcal{H} \to \Lambda_s$ . We define the formal germ  $(\mathcal{H}, -z)$  at -z to be the affine formal scheme over  $\Lambda_s$ :

$$(\mathcal{H}, -z) := \operatorname{Spf} S,$$
  $S := \Lambda_{\mathbf{s}}[t_k^{\alpha}, p_{l\beta} : 1 \le \alpha, \beta \le N; 0 \le k, l < \infty]^{\widehat{}},$ 

where ^means the completion with respect to the valuation

$$v(t_k^{\alpha}) = k + 1, \quad v(p_{l\beta}) = l + 1.$$

We regard the formal scheme  $(\mathcal{H}, -z) = \operatorname{Spf}(S)$  as a functor from the category of topological  $\Lambda_s$ -algebras and continuous  $\Lambda_s$ -algebra homomorphisms to the category of sets:

$$(\mathcal{H}, -z)$$
: (Topological  $\Lambda_s$ -algebras)  $\longrightarrow$  (Sets)  $R \longmapsto \operatorname{Hom}(S, R)$ .

We have

$$(\mathcal{H}, -z)(R) \cong \left\{ -z + \sum_{n \in \mathbb{Z}} r_n^{\alpha} \phi_{\alpha} z^n : r_n^{\alpha} \in R^{\text{nilp}}, r_n^{\alpha} \to 0 \text{ as } |n| \to \infty \right\}.$$

Using this identification, we always write an element of  $(\mathcal{H}, -z)(R)$  in the form  $f = -z + \sum_{n \in \mathbb{Z}} r_n^{\alpha} \phi_{\alpha} z^n = -z + \mathbf{t}(z) + \mathbf{p}(z)$  with  $t_k^{\alpha} = r_k^{\alpha}$ ,  $p_{l\beta} = (-1)^{l+1} r_{-l-1}^{\nu} g_{\nu\beta}$ . The tangent functor  $T(\mathcal{H}, -z)$  is defined to be

$$T(\mathcal{H}, -z)(R) := (\mathcal{H}, -z)(R[\epsilon]/\langle \epsilon^2 \rangle).$$

The tangent space  $T_f(\mathcal{H}, -z)(R)$  at  $f \in (\mathcal{H}, -z)(R)$  is defined to be the preimage of f under the natural map  $(\mathcal{H}, -z)(R[\epsilon]/\langle \epsilon^2 \rangle) \to (\mathcal{H}, -z)(R)$ . We have, for  $f \in (\mathcal{H}, -z)(R)$ ,

$$T_f(\mathcal{H}, -z)(R) \cong \left\{ \sum_{n \in \mathbb{Z}} v_n^{\alpha} \phi_{\alpha} z^n : v_n^{\alpha} \in R, \ v_n^{\alpha} \to 0 \text{ as } |n| \to \infty \right\}$$
$$\cong H_{\mathrm{orb}}^{\bullet}(\mathcal{X}; \mathbb{C}) \otimes_{\mathbb{C}} R\{z, z^{-1}\}.$$

The tangent space  $T_f(\mathcal{H}, -z)$  is equipped with the topology induced from that on  $R\{z, z^{-1}\}$ . This coincides with the "pointwise convergence topology" on  $\operatorname{Hom}(S, R[\epsilon]/\langle \epsilon^2 \rangle) = T(\mathcal{H}, -z)(R)$ , i.e.  $\phi_n \in \operatorname{Hom}(S, R[\epsilon]/\langle \epsilon^2 \rangle)$  converges to  $\phi_{\infty}$  if and only if  $\phi_n(s)$  converges to  $\phi_{\infty}(s)$  in  $R[\epsilon]/\langle \epsilon^2 \rangle$  for every  $s \in S$ .

The Formal Subscheme  $\mathcal{L}^{\text{tw}}$  of  $(\mathcal{H}, -z)$ . We consider genus zero Gromov-Witten theory twisted by the universal multiplicative characteristic class  $\boldsymbol{c}$  in (11). Introduce the double correlator notation:

$$\langle \langle \phi_{\alpha_1} \psi^{k_1}, \dots, \phi_{\alpha_m} \psi^{k_m} \rangle \rangle_{\mathbf{t}}^{\mathrm{tw}} :=$$

$$\sum_{d \in \mathrm{Eff}(\mathcal{X})} \sum_{n \geq 0} \frac{Q^d}{n!} \langle \phi_{\alpha_1} \psi^{k_1}, \dots, \phi_{\alpha_m} \psi^{k_m}, \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle_{0, m+n, d}^{\mathcal{X}, \mathrm{tw}},$$

where the summand is defined to be zero in the unstable range d=0, m+n<3. These correlators are elements of the completion of  $\Lambda_s[t_k^\alpha:1\leq\alpha\leq N;0\leq k<\infty]$ .

Example. The correlator  $\langle\!\langle \ \rangle\!\rangle_{\mathbf{t}}$  with no insertion is the genus-zero descendant potential.

Define elements  $E_{i\alpha} \in S$  and the ideal  $\mathfrak{I} \subset S$  by

$$E_{j\alpha} = p_{j\alpha} - \langle\!\langle \phi_{\alpha} \psi^j \rangle\!\rangle_{\mathbf{t}}^{\mathrm{tw}}, \qquad \qquad \Im = \langle E_{j\alpha} : 1 \le \alpha \le N; 0 \le j < \infty \rangle.$$

The formal subscheme  $\mathcal{L}^{\text{tw}}$  of  $(\mathcal{H}, -z)$  is defined to be

(44) 
$$\mathcal{L}^{\text{tw}} := \operatorname{Spf}\left(S/\overline{\mathfrak{I}}\right),$$

where  $\overline{\mathfrak{I}}$  is the closure of the ideal  $\mathfrak{I}$ . Considering the virtual dimension of the moduli space of stable maps shows that  $E_{j\alpha}$  converges to zero in S as  $j \to \infty$ . Therefore the set  $\mathcal{L}^{\mathrm{tw}}(R)$  of R-valued points of  $\mathcal{L}^{\mathrm{tw}}$  is given by the graph of the functions  $\mathbf{t} \mapsto \langle \! \langle \phi_{\alpha} \psi^{j} \rangle \! \rangle_{\mathbf{t}}^{\mathrm{tw}}$ :

$$\mathcal{L}^{\text{tw}}(R) = \left\{ -z + \mathbf{t}(z) + \boldsymbol{p}(z) \in (\mathcal{H}, -z)(R) : p_{k\alpha} = \langle \! \langle \phi_{\alpha} \psi^{k} \rangle \! \rangle_{\mathbf{t}}^{\text{tw}} \forall k, \alpha \right\}.$$

Example/Definition. Let  $\Lambda_{\mathbf{s}}[\![\tau]\!] := \Lambda_{\mathbf{s}}[\![\tau^1, \dots, \tau^N]\!]$ . The twisted *J*-function (8) is a  $\Lambda_{\mathbf{s}}[\![\tau]\!]$ -valued point on  $\mathcal{L}^{\text{tw}}$ :

$$J^{\text{tw}}(\tau, -z) = -z + \tau + \sum_{k \ge 0} \left\langle \!\! \left\langle \phi_{\alpha} \psi^{k} \right\rangle \!\! \right\rangle_{\mathbf{t} = \tau}^{\text{tw}} \frac{\phi^{\alpha}}{(-z)^{k+1}} \in \mathcal{L}^{\text{tw}}(\Lambda[\tau])$$

characterized by the condition (9).

The tangent functor  $T\mathcal{L}^{\text{tw}}$ , the tangent spaces  $T_f\mathcal{L}^{\text{tw}}(R)$ , and the topologies on them are defined as above. Explicitly, the tangent space  $T_f\mathcal{L}^{\text{tw}}(R)$  at  $f = -z + \mathbf{t}(z) + \mathbf{p}(z) \in \mathcal{L}^{\text{tw}}(R)$  is given by the set of points  $\sum_k \dot{t}_k^{\alpha} \phi_{\alpha} z^k + \sum_k \dot{p}_{k\alpha} \frac{\phi^{\alpha}}{(-z)^{k+1}}$  in  $H_{\text{orb}}^{\bullet}(\mathcal{X}) \otimes R\{z, z^{-1}\}$  satisfying

(45) 
$$\dot{p}_{k\alpha} = \sum_{l,\beta} \dot{t}_l^\beta \left\langle\!\!\left\langle \phi_\beta \psi^l, \phi_\alpha \psi^k \right\rangle\!\!\right\rangle_{\mathbf{t}}^{\mathbf{tw}}.$$

It is easy to check that  $T_f \mathcal{L}^{\text{tw}}(R)$  is a closed subspace of  $T_f(\mathcal{H}, -z) = H^{\bullet}_{\text{orb}}(\mathcal{X}) \otimes R\{z, z^{-1}\}.$ 

The following elementary fact will be useful:

**Lemma B.1.** If  $I(t) \in \mathcal{L}^{tw}(R[t])$  then the derivative  $\frac{dI}{dt}(t)$  lies in  $T_{I(t)}\mathcal{L}^{tw}(R[t])$ .

*Proof.* Observe that there exists an automorphism of  $R[\![t]\!][\epsilon]/\langle \epsilon^2 \rangle$  which sends t to  $t + \epsilon$ . Therefore,  $I(t + \epsilon) = I(t) + \epsilon \frac{dI}{dt}(t)$  belongs to  $\mathcal{L}^{\text{tw}}(R[\![t]\!][\epsilon]/\langle \epsilon^2 \rangle)$ . Thus  $\frac{dI}{dt}(t) \in T_{I(t)}\mathcal{L}^{\text{tw}}(R[\![t]\!])$ .

Special geometric properties of  $\mathcal{L} = \mathcal{L}^{\mathrm{un}}$ . The special case c = 1 (i.e.  $s_0 = s_1 = s_2 = \cdots = 0$ ) gives a formal scheme  $\mathcal{L}^{\mathrm{un}}$ , defined over  $\Lambda$ , which corresponds to untwisted Gromov-Witten theory. We now verify some basic geometric properties of  $\mathcal{L} = \mathcal{L}^{\mathrm{un}}$  which are used in the main body of the text. In the rest of the section we omit the superscript "un", writing  $\mathcal{L}$  for  $\mathcal{L}^{\mathrm{un}}$ ,  $\langle\!\langle \cdots \rangle\!\rangle_{\mathbf{t}}$  for  $\langle\!\langle \cdots \rangle\!\rangle_{\mathbf{t}}^{\mathrm{un}}$ , etc. Also we take R to be a complete, Hausdorff, linearly-topologized  $\Lambda$ -algebra.

The geometric properties of  $\mathcal{L}$  follow from the three universal relations in genuszero orbifold Gromov-Witten theory [58, Section 3.1]:

(DE) 
$$\langle\!\langle \psi \rangle\!\rangle_{\mathbf{t}} = \sum_{k>0} t_k^{\alpha} \langle\!\langle \phi_{\alpha} \psi^k \rangle\!\rangle_{\mathbf{t}} - 2 \langle\!\langle \rangle\!\rangle_{\mathbf{t}},$$

(SE) 
$$\langle \mathbf{1} \rangle_{\mathbf{t}} = \frac{1}{2} (t_0, t_0)_{\text{orb}} + \sum_{k>0} t_{k+1}^{\alpha} \langle \phi_{\alpha} \psi^k \rangle_{\mathbf{t}},$$

(TRR) 
$$\langle \langle \phi_{\alpha} \psi^{k+1}, \phi_{\beta} \psi^{l}, \phi_{\gamma} \psi^{m} \rangle_{t} = \langle \langle \phi_{\alpha} \psi^{k}, \phi_{\nu} \rangle_{t} \langle \langle \phi^{\nu}, \phi_{\beta} \psi^{l}, \phi_{\gamma} \psi^{m} \rangle_{t}.$$

These are called the Dilaton Equation (DE), the String Equation (SE) and the Topological Recursion Relations (TRR), respectively. The Dilaton Equation implies that  $\mathcal{L}$  is a cone:

**Proposition B.2.** For every element  $f \in \mathcal{L}(R)$  and every  $x \in R^{\text{nilp}}$ , we have  $(1+x)f \in \mathcal{L}(R)$ .

*Proof.* Note that  $f \in \mathcal{L}(R)$  can be regarded as an element of  $\mathcal{L}(R[t])$  by the natural inclusion  $R \subset R[t]$ . It suffices to prove that R[t]-valued point  $(1+t)f \in$  $(\mathcal{H}, -z)(R[t])$  belongs to  $\mathcal{L}(R[t])$ . The conclusion then follows by applying the functor  $\mathcal{L}$  to the continuous R-homomorphism:  $R[t] \to R$ ,  $t \mapsto x$ . We write  $f = -z + h = -z + \mathbf{t}(z) + \mathbf{p}(z)$ . Then  $(1+t)f = -z + h_t = -z + \mathbf{t}_t(z) + \mathbf{p}_t(z)$  with  $h_t = -tz + (1+t)h$ ,  $\mathbf{t}_t(z) = -tz + (1+t)\mathbf{t}(z)$  and  $p_t(z) = (1+t)p(z)$ . Because  $f \in \mathcal{L}(R), E_{j\beta}(h_t) = O(t)$ . Assume by induction on n that  $E_{j\beta}(h_t) = O(t^n)$ . We

$$(1+t)\frac{d}{dt}E_{j\beta}(h_t) = (1+t)\left(p_{j\beta} - \sum_{k\geq 0} t_k^{\alpha} \left\langle\!\!\left\langle\phi_{\alpha}\psi^k, \phi_{\beta}\psi^j\right\rangle\!\!\right\rangle_{\mathbf{t}_t} + \left\langle\!\!\left\langle\psi, \phi_{\beta}\psi^j\right\rangle\!\!\right\rangle_{\mathbf{t}_t}\right)$$

$$\equiv \left\langle\!\!\left\langle\phi_{\beta}\psi^j\right\rangle\!\!\right\rangle_{\mathbf{t}_t} - \sum_{k\geq 0} (-t\delta_k^1 \delta_1^{\alpha} + (1+t)t_k^{\alpha}) \left\langle\!\!\left\langle\phi_{\alpha}\psi^k, \phi_{\beta}\psi^j\right\rangle\!\!\right\rangle_{\mathbf{t}_t} + \left\langle\!\!\left\langle\psi, \phi_{\beta}\psi^j\right\rangle\!\!\right\rangle_{\mathbf{t}_t} \mod t^n.$$

In the second line, we used the induction hypothesis  $E_{j\beta}(h_t) = O(t^n)$ . But the second line is zero by the Dilaton Equation. Hence  $E_{j\beta}(h_t) = O(t^{n+1})$ .

The String Equation and the Topological Recursion Relations together imply that the tangent space to  $\mathcal{L}$  at an R-valued point has the structure of an  $R\{z\}$ module:

**Proposition B.3.** The tangent space  $T_f\mathcal{L}(R)$  at  $f \in \mathcal{L}(R)$  is an  $R\{z\}$ -submodule of  $T_f(\mathcal{H}, -z)(R) = H^{\bullet}_{orb}(\mathcal{X}) \otimes R\{z, z^{-1}\}.$ 

*Proof.* Assume that we know  $zT_f\mathcal{L}(R)\subset T_f\mathcal{L}(R)$ . Then for every sequence  $\{a_n\}_{n\geq 0}$ in R with  $\lim_{n\to\infty} a_n = 0$  and  $h \in T_f \mathcal{L}(R)$ , we have

$$a_0h + a_1zh + \dots + a_mz^mh \in T_f\mathcal{L}(R).$$

This element converges to  $(\sum_{n=0}^{\infty} a_n z^n)h$  as  $m \to \infty$ . Since  $T_f \mathcal{L}(R)$  is a closed subspace,  $(\sum_{n=1}^{\infty} a_n z^n)h \in T_f \mathcal{L}(R)$ . Hence  $T_f \mathcal{L}(R)$  is an  $R\{z\}$ -submodule. Now it suffices to show that  $zT_f \mathcal{L}(R) \subset T_f \mathcal{L}(R)$ . Take a tangent vector h = 1

 $\dot{\mathbf{t}}(z) + \dot{\boldsymbol{p}}(z)$  at f. Then we have

$$zh = \sum_{k>1} \dot{t}_{k-1}^{\alpha} \phi_{\alpha} z^{k} - \dot{p}_{0\beta} \phi^{\beta} + \sum_{l>0} (-\dot{p}_{l+1,\beta}) \frac{\phi^{\beta}}{(-z)^{l+1}}.$$

Therefore  $zh \in T_f \mathcal{L}(R)$  is equivalent to the equality:

$$-\dot{p}_{l+1,\beta} = -\dot{p}_{0\nu} \left\langle \!\!\left\langle \phi^{\nu}, \phi_{\beta} \psi^{l} \right\rangle \!\!\right\rangle_{\mathbf{t}} + \sum_{k \geq 1} \dot{t}_{k-1}^{\alpha} \left\langle \!\!\left\langle \phi_{\alpha} \psi^{k}, \phi_{\beta} \psi^{l} \right\rangle \!\!\right\rangle_{\mathbf{t}}.$$

Substituting for  $\dot{p}_{l\beta}$  using (45), one sees that it suffices to show that

$$(46) \qquad \langle\!\langle \phi_{\alpha} \psi^{k+1}, \phi_{\beta} \psi^{l} \rangle\!\rangle_{\mathbf{t}} + \langle\!\langle \phi_{\alpha} \psi^{k}, \phi_{\beta} \psi^{l+1} \rangle\!\rangle_{\mathbf{t}} - \langle\!\langle \phi_{\alpha} \psi^{k}, \phi_{\nu} \rangle\!\rangle_{\mathbf{t}} \langle\!\langle \phi^{\nu}, \phi_{\beta} \psi^{l} \rangle\!\rangle_{\mathbf{t}} = 0.$$

At  $\mathbf{t} = 0$ , we have (writing  $\langle \cdots \rangle_0$  for  $\langle \cdots \rangle_{\mathbf{t}}|_{\mathbf{t}=0}$ )

On the other hand, differentiating in  $t_i^{\gamma}$ , we have

$$\begin{split} \partial_{j,\gamma} \left( \left\langle \! \left\langle \phi_{\alpha} \psi^{k+1}, \phi_{\beta} \psi^{l} \right\rangle \! \right\rangle_{\mathbf{t}} + \left\langle \! \left\langle \phi_{\alpha} \psi^{k}, \phi_{\beta} \psi^{l+1} \right\rangle \! \right\rangle_{\mathbf{t}} \right) \\ &= \left\langle \! \left\langle \phi_{\gamma} \psi^{j}, \phi_{\alpha} \psi^{k+1}, \phi_{\beta} \psi^{l} \right\rangle \! \right\rangle_{\mathbf{t}} + \left\langle \! \left\langle \phi_{\gamma} \psi^{j}, \phi_{\alpha} \psi^{k}, \phi_{\beta} \psi^{l+1} \right\rangle \! \right\rangle_{\mathbf{t}} \\ &= \left\langle \! \left\langle \phi_{\alpha} \psi^{k}, \phi_{\nu} \right\rangle \! \right\rangle_{\mathbf{t}} \left\langle \! \left\langle \phi^{\nu}, \phi_{\gamma} \psi^{j}, \phi_{\beta} \psi^{l} \right\rangle \! \right\rangle_{\mathbf{t}} + \left\langle \! \left\langle \phi_{\beta} \psi^{l}, \phi_{\nu} \right\rangle \! \right\rangle_{\mathbf{t}} \left\langle \! \left\langle \phi^{\nu}, \phi_{\gamma} \psi^{j}, \phi_{\alpha} \psi^{k} \right\rangle \! \right\rangle_{\mathbf{t}} \\ &= \partial_{j,\gamma} \left( \left\langle \! \left\langle \phi_{\alpha} \psi^{k}, \phi_{\nu} \right\rangle \! \right\rangle_{\mathbf{t}} \left\langle \! \left\langle \phi^{\nu}, \phi_{\beta} \psi^{l} \right\rangle \! \right\rangle_{\mathbf{t}} \right), \end{split}$$

where we used (TRR) in the third line. Therefore we have (46).

**Remark.** Let  $R_1, R_2$  be complete  $\Lambda$ -algebras. For a continuous  $\Lambda$ -algebra homomorphism  $\varphi \colon R_1 \to R_2$ , the induced homomorphism  $\varphi_* \colon T_f \mathcal{L}(R_1) \to T_{\varphi(f)} \mathcal{L}(R_2)$  becomes a continuous  $R_1\{z\}$ -module homomorphism.

Define elements  $\tau^{\alpha}(\mathbf{t}) \in S$ ,  $1 \leq \alpha \leq N$ , by

$$\tau^{\alpha}(\mathbf{t}) := \langle \langle \mathbf{1}, \phi^{\alpha} \rangle \rangle_{\mathbf{t}}$$

The String Equation implies that

$$\tau^{\alpha}(\mathbf{t}) = t_0^{\alpha} + \sum_{k>0} t_{k+1}^{\gamma} \left\langle \!\!\left\langle \phi_{\gamma} \psi^k, \phi^{\alpha} \right\rangle \!\!\!\right\rangle_{\mathbf{t}} = t_0^{\alpha} + \text{higher order terms.}$$

Finally we establish the most remarkable property of  $\mathcal{L}$ : that tangent spaces to  $\mathcal{L}$  are parametrized by finitely many parameters  $\tau^1(\mathbf{t}), \ldots, \tau^N(\mathbf{t})$ , and are generated by the derivatives of the J-function as  $R\{z\}$ -modules. This leads us to the  $\mathcal{D}$ -module property of tangent spaces (Corollary B.7).

**Proposition B.4.** The tangent space  $T_f \mathcal{L}(R)$  at  $f = -z + \mathbf{t}(z) + \mathbf{p}(z) \in \mathcal{L}(R)$  is freely generated by the derivatives of the *J*-function

$$\partial_{\alpha} J(\tau, -z)|_{\tau = \tau(\mathbf{t})}, \quad \alpha = 1, \dots, N.$$

as an  $R\{z\}$ -module, where  $J(\tau,z)=J^{\mathrm{un}}(\tau,z)$  is the untwisted J-function.

This is an immediate consequence of the following two lemmas.

**Lemma B.5.** For elements  $r^1, \ldots, r^N \in R^{\text{nilp}}$ , the *J*-function  $J(\tau, -z)$  with  $\tau = \sum_{\alpha} r^{\alpha} \phi_{\alpha}$  gives an *R*-valued point on  $\mathcal{L}$ . The tangent space  $T_{J(\tau, -z)}\mathcal{L}(R)$  is freely generated by the derivatives  $(\partial_{\alpha} J)(\tau, -z)$  as an  $R\{z\}$ -module.

**Lemma B.6.** The tangent space  $T_f \mathcal{L}(R)$  at an R-valued point  $f = -z + \mathbf{t}(z) + \mathbf{p}(z) \in \mathcal{L}(R)$  is the same as the tangent space  $T_{J(\tau(\mathbf{t}), -z)}\mathcal{L}(R)$  at  $J(\tau(\mathbf{t}), -z) \in \mathcal{L}(R)$  as a subspace of  $H_{\text{orb}}(\mathcal{X}) \otimes R\{z, z^{-1}\}$ . Here the ring homomorphism  $\Lambda_s[\![\tau]\!] \to R$  sending  $\tau^{\alpha}$  to  $\tau^{\alpha}(\mathbf{t}) \in R$  gives a point  $J(\tau(\mathbf{t}), -z) \in \mathcal{L}(R)$ .

Proof of Lemma B.5. As we saw earlier, the J-function is a  $\Lambda[\![\tau]\!]$ -valued point on  $\mathcal{L}$ . Thus its derivatives  $\partial_{\alpha}J(\tau,-z)$  belong to the tangent space  $T_{J(\tau,-z)}\mathcal{L}(\Lambda_s[\![\tau]\!])$ , by Lemma B.1. Via the homomorphism  $\Lambda[\![\tau]\!] \to R$  sending  $\tau^{\alpha}$  to  $r^{\alpha} \in R$ , we obtain an R-valued point  $J(\tau,-z) \in \mathcal{L}(R)$  and tangent vectors  $(\partial_{\alpha}J)(\tau,-z) \in T_{J(\tau,-z)}\mathcal{L}(R)$ . Set  $\tau = \sum_{\alpha} r^{\alpha}\phi_{\alpha}$  and write  $[f]_+$  for the non-negative part of the z-series f. From the description (45) of tangent vectors, there exists a one-to-one correspondence between tangent vectors  $\dot{\mathbf{t}}(z) + \dot{\boldsymbol{p}}(z)$  in  $T_{J(\tau,-z)}\mathcal{L}(R)$  and tuples  $\{\dot{t}_k^{\alpha} \in R\}_{k,\alpha}$  satisfying  $\lim_{k\to\infty} \dot{t}_k^{\alpha} = 0$ . It therefore suffices to show that for any given  $\{\dot{t}_k^{\alpha} \in R\}_{k,\alpha}$  satisfying  $\lim_{k\to\infty} \dot{t}_k^{\alpha} = 0$ , there exist unique elements  $c^{\alpha} \in R\{z\}$  such that

(47) 
$$\left[\sum_{\alpha} c^{\alpha} \partial_{\alpha} J\right]_{+} = \sum_{k>0} \dot{t}_{k}^{\alpha} \phi_{\alpha} z^{k}.$$

First we show the existence of  $c^{\alpha}$ . Assume that the topology on R is defined by a descending chain of ideals  $\{I_M\}_{M\geq 0}$ . We will prove the following claim by induction on n:

Claim. There exist  $c(n)^{\alpha} \in R\{z\}$  such that

$$\left[\sum_{\alpha} c(n)^{\alpha} \partial_{\alpha} J\right]_{+} = \sum_{k=0}^{n} \dot{t}_{k}^{\alpha} \phi_{\alpha} z^{k}.$$

Moreover, if  $\dot{t}_k^{\beta} \in I_M$  for all  $0 \le k \le n$  and  $\beta$  and for some M, then  $c(n)^{\alpha} \in I_M\{z\}$ .

The case n=0 is clear from the expansion

$$\partial_{\alpha} J(\tau, -z) = \phi_{\alpha} + \sum_{j>0} \left\langle \!\! \left\langle \phi_{\alpha}, \phi_{\beta} \psi^{j} \right\rangle \!\! \right\rangle_{\tau} \frac{\phi^{\beta}}{(-z)^{j+1}}.$$

Assume that the claim holds for some  $n \geq 0$ . One then has

$$\left[\sum_{\alpha} (\dot{t}_{n+1}^{\alpha} z^{n+1} + c(n)^{\alpha}) \partial_{\alpha} J\right]_{+} = \sum_{k=0}^{n+1} \dot{t}_{k}^{\alpha} \phi_{\alpha} z^{k} - \sum_{j=0}^{n} \dot{t}_{n+1}^{\alpha} \left\langle\!\!\left\langle \phi_{\alpha}, \phi_{\beta} \psi^{j} \right\rangle\!\!\right\rangle_{\tau} (-1)^{j} \phi^{\beta} z^{n-j}.$$

By the induction hypothesis, there exist  $\xi(n)^{\alpha} \in R\{z\}$  such that  $[\sum_{\alpha} \xi(n)^{\alpha} \partial_{\alpha} J]_{+} = \sum_{j=0}^{n} \dot{t}_{n+1}^{\alpha} \langle \tau_{0\alpha} \tau_{j\beta} \rangle_{\tau} (-1)^{j} \phi^{\beta} z^{n-j}$ . Also we have  $\xi(n)^{\alpha} \in I_{M}\{z\}$  if  $\dot{t}_{n+1}^{\beta} \in I_{M}$  for all  $\beta$ . Therefore, we can take  $c(n+1)^{\alpha}$  to be  $c(n)^{\alpha} + \dot{t}_{n+1}^{\alpha} z^{n+1} + \xi(n)^{\alpha}$ . This completes the induction step, and the claim follows.

The above argument shows that  $c(n+1)^{\alpha} - c(n)^{\alpha} \in I_M\{z\}$  if  $\dot{t}_{n+1}^{\beta} \in I_M$  for all  $\beta$ . Therefore  $c(n)^{\alpha}$  converges to some element  $c^{\alpha} \in R\{z\}$  and (47) holds. For the uniqueness of  $c^{\alpha}$ , it suffices to show that if  $[\sum_{\alpha} c^{\alpha} \partial_{\alpha} J]_{+} = 0$  then  $c^{\alpha} = 0$ . Suppose that  $[\sum_{\alpha} c^{\alpha} \partial_{\alpha} J]_{+} = 0$  and that  $c^{\beta} \neq 0$  for some  $\beta$ . Since R is Hausdorff, there exists an M such that  $c^{\beta} \notin I_M\{z\}$ . The equation  $[\sum_{\alpha} c^{\alpha} \partial_{\alpha} J]_{+} = 0$  holds in the ring  $R\{z, z^{-1}\}/I_M\{z, z^{-1}\} = (R/I_M)[z, z^{-1}]$  and  $c^{\beta} \neq 0$  in  $(R/I_M)[z]$ . Comparing the highest order terms in z leads us to a contradiction.

Proof of Lemma B.6. Recall again that a tangent vector at f is given by a set  $\{\dot{t}_k^{\alpha}, \dot{p}_{k\alpha}\}_{k,\alpha}$  in R satisfying  $\lim_{k\to\infty} \dot{t}_k^{\alpha} = \lim_{k\to\infty} \dot{p}_{k\alpha} = 0$  and equation (45). On the other hand, the Topological Recursion Relations imply that:

$$\langle \langle \phi_{\alpha} \psi^{k}, \phi_{\beta} \psi^{l} \rangle \rangle_{\mathbf{t}} = \langle \langle \phi_{\alpha} \psi^{k}, \phi_{\beta} \psi^{l} \rangle \rangle_{\tau(\mathbf{t})}.$$

This is due to Dijkgraaf–Witten [26] (see also [33, Equation 2], [29, Proposition 4.6]). The Lemma follows.  $\Box$ 

**Corollary B.7.** Let I(t) be an R[t]-valued point on  $\mathcal{L}$  and  $\xi(t)$  be a tangent vector at I(t). Then  $z\frac{d\xi}{dt}(t)$  is again a tangent vector at I(t).

*Proof.* Set  $I(t) = -z + \mathbf{t}_t(z) + \mathbf{p}_t(z)$ . By Proposition B.4, we can write  $\xi(t)$  in the form

$$\xi(t) = \sum_{\alpha} c^{\alpha}(t,z) [\partial_{\alpha} J(\tau,-z)]_{\tau=\tau(\mathbf{t}_t)}, \qquad \text{ for some } c^{\alpha}(t,z) \in R[\![t]\!]\{z\}.$$

The Corollary follows from this and the differential equations (10).

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