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A derivative for complex Lipschitz maps with generalised Cauchy–Riemann equations

Abbas Edalat

Department of Computing, Imperial College London, UK

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ABSTRACT

We introduce the Lipschitz derivative or the L-derivative of a locally Lipschitz complex map: it is a Scott continuous, compact and convex set-valued map that extends the classical derivative to the bigger class of locally Lipschitz maps and allows an extension of the fundamental theorem of calculus and a new generalisation of Cauchy–Riemann equations to these maps, which form a continuous Scott domain. We show that a complex Lipschitz map is analytic in an open set if and only if its L-derivative is a singleton at all points in the open set. The calculus of the L-derivative for sum, product and composition of maps is derived. The notion of contour integration is extended to Scott continuous, non-empty compact, convex valued functions on the complex plane, and by using the L-derivative, the fundamental theorem of contour integration is extended to these functions.

1. Introduction

Complex numbers and complex functions are used in a wide range of areas in engineering, physics as well as mathematics and their role in many fields such as quantum physics and quantum computation is fundamental and indispensable. Many programming languages have a pair of floating point numbers to represent complex numbers. In [31], Knuth proposed the Quarter-imaginary base $2i$, where $i = \sqrt{-1}$, as a system for computing basic arithmetic operations on complex numbers.

In more recent years, there has been a great deal of interesting work by the Blum–Shub–Smale (BSS) model community and also the type two theory (TTE) community in the computability of various subsets of the complex plane and complex mappings, in particular in connection with complex dynamical systems and fractal objects [3,29,30,8]. In [34,35], the computational complexity of Taylor series and the constructive aspects of analytic functions have been examined. Lately, effective analytical continuation and Riemann surfaces have been studied in [43] and a mixed BSS–TTE model has been used to study computability of analytic functions [25].

Our final goal is to develop a domain-theoretic data type for functions of a complex variable, including analytic maps, so that it can simultaneously represent the function and its differential properties. In this paper, we will tackle the task of formulating a derivative for an appropriate family of complex maps.

The class of complex Lipschitz maps, with their rich closure and convergence properties, provides us with a suitable class to develop a complex function data type. In fact, this class contains the fundamental class of piecewise linear maps, is closed under taking absolute value and the basic arithmetic operations on functions. Moreover, as well as being uniformly

E-mail address: a.edalat@ic.ac.uk.
URL: http://www.doc.ic.ac.uk/~ae.

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continuous, Lipschitz maps with uniformly bounded Lipschitz constants are also closed under convergence with respect to the sup norm. Vector real-valued Lipschitz maps are, by Rademacher's theorem, differentiable almost everywhere on finite dimensional Euclidean spaces [10, p. 148], and by Kirszbraun's theorem [24, p. 202] these maps can always be extended from any subset of the Euclidean space to the whole space. Furthermore, complex Lipschitz maps unify the two basic classes of analytic and anti-analytic maps.

The prominent properties of analytic maps are based on complex differentiation. The question therefore arises: how can we capture the differential properties of a complex Lipschitz map? In 1980's, Frank Clarke developed a set-valued derivative for real-valued Lipschitz maps on Euclidean spaces [9]. On finite dimensional Euclidean spaces, the Clarke gradient has non-empty compact and convex subsets of the Euclidean space as its values and it extends the classical derivative of C^1 maps to the bigger class of Lipschitz maps.

Motivated by structures in domain theory, the L-derivative of real Lipschitz maps and a domain for these maps were introduced for functions of a single real variable in [20] by generalising the notion of Lipschitz constant of a function in an open set to a non-empty compact real interval, which gives finitary information about the rate of growth of the function in the open set. The L-derivative of a map at a point is then obtained by collecting together all such finitary differential properties that the map satisfies in its neighbourhoods. This gives a shrinking sequence of non-empty compact sets, in the spirit of interval analysis [33], whose intersection is the L-derivative of the map at the given point. Subsequently, the L-derivative was extended to higher dimensional maps, first as a hyper-rectangular valued map [21], and later as the non-empty, convex and compact set-valued map, which was also defined on general Banach spaces. The L-derivative was then shown to be equivalent to the Clarke gradient for maps on finite dimensional Euclidean spaces [16]. Given that the L-derivative is built by collecting finitary differential properties of a map, it has been used in a number of areas in computer science. In particular, the domain of locally Lipschitz real-valued maps has been used to develop a PCF type programming language for computable and differentiable functions [11]. The question is whether we can do the same for complex Lipschitz maps.

In this paper, we define the Lipschitz derivative or L-derivative of a complex locally Lipschitz map, which is always defined as a non-empty, convex and compact set in the complex plane and is obtained from the set of finitary Lipschitz properties of the map. Compared to the case of real maps, the new complex derivative is formulated differently but is inspired from the real case in that we use a generalisation of the notion of Lipschitz constant this time to a non-empty compact convex subset of the complex plane. At any given point, the L-derivative of a complex Lipschitz map is obtained as the intersection of a shrinking sequence of non-empty convex compact sets, which are the generalised Lipschitz constants of the map in the neighbourhoods of the point and each of which gives a finitary description of the derivative. The L-derivative of a complex map is Scott continuous with respect to the Scott topology of the continuous Scott domain of the non-empty convex and compact subsets of the complex plane partially ordered by reverse inclusion. It extends the classical derivative of complex differentiable maps to complex Lipschitz maps such that for analytic maps the L-derivative at any point would be a singleton, namely the classical derivative at that point. Thus, the L-derivative of an analytic map will be a maximal element of the continuous Scott domain, which also maps every point to a maximal element. This therefore presents another example where a classical notion in mathematical analysis is captured as a subset of the maximal elements of a domain [39,14,13,15,32,18,19].

The L-derivative provides us with a fundamental theorem of calculus for complex Lipschitz maps, a duality between Scott continuous compact, convex valued maps and their Lipschitz primitives. As analytic functions and anti-analytic functions (such as conjugation) are both Lipschitz and anti-analytic maps are non-differentiable, the L-derivative also gives rise to a unifying derivative for both of these fundamental classes.

The calculus of the L-derivative is derived for sum and product; the chain rule for the L-derivative of composition of functions is also developed. We provide some simple examples of the L-derivative of basic maps like conjugation and absolute value that are not differentiable.

We then use the directional derivative of a complex map [27] to define a differential transformation which converts the differential properties of a planar vector valued Lipschitz map into the differential properties of the induced complex map, whose real and imaginary parts are given by the vector valued map. This transformation gives what we call the C-derivative of the complex function by mapping the Clarke gradient of the vector function obtained using the real and imaginary parts of the complex map from a non-empty compact convex subset of the vector space of 2×2 real matrices into the complex plane. We then show that the L-derivative and the C-derivative of a complex Lipschitz map are equal as non-empty compact convex subsets, which gives a set-valued generalisation of the Cauchy–Riemann equations to complex Lipschitz maps. It shows that the L-derivative is the convex hull of a union of disks in the complex plane. For analytic functions, this union is reduced to a single point and we obtain the classical Cauchy–Riemann equations. The extension of Cauchy–Riemann equations to Lipschitz maps provides a common framework to study the differential properties of analytic and anti-analytic maps. In particular we show that the L-derivative of the conjugation map is the unit disk centred at the origin. All these results are extended to functions of several complex variables and thus present a new view on complex differentiation.

Since they were used in the theory of functions by Cauchy and Riemann in the 19th century [40], Cauchy–Riemann equations have been at the basis of Harmonic analysis and Laplace equations with applications in partial differential equations. In the context of the current generalisation of these equations, we are in particular interested in the approximation of analytic maps by piecewise linear functions and construction of a domain and a data type for analytic functions.

As a first application, we define the contour integral of a Scott continuous, compact and convex valued map over a piecewise smooth path (contour) in the complex plane. This integral is again compact and convex valued and we show that, given a fixed contour, the integral operation is Scott continuous on the domain of Scott continuous, compact and convex valued maps. We then extend the fundamental theorem of contour integration to these maps, showing that if a Scott continuous, compact and convex valued map has a primitive with respect to the L-derivative then its contour integral, as a non-empty compact convex set, always contains the difference between the complex number values of the primitive map at the end and the start point of the contour.

We will finally explain in the concluding section how the results in this paper can be used to construct a domain of computation for complex maps in future work.

1.1. The Clarke gradient

We here recall from [9] the definition of the Clarke gradient for scalar and vector real-valued maps on finite dimensional Euclidean spaces. By Rademacher's theorem [10, p. 148], a locally Lipschitz map $h : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is Fréchet differentiable almost everywhere with respect to the Lebesgue measure. Let Ω_h be the nullset where h fails to be differentiable. First assume $m = 1$. Then [9, p. 63]:

$$\partial h(x) = \operatorname{conv}\Big\{ \lim_{j\to\infty} h'(x_j) : x_j \to x,\ x_j \notin \Omega_h \Big\}. \tag{1}$$

The above expression is interpreted as follows. Consider all sequences $(x_j)_{j\geq 0}$, with $x_j \notin \Omega_h$, for $j \geq 0$, which converge to x such that $\lim_{j\to\infty} h'(x_j)$ exists. Then the generalised gradient is the convex hull of all such limits. Note that, in the above definition, since h is locally Lipschitz at x, it is differentiable almost everywhere in a neighbourhood of x and thus there are plenty of sequences $(x_j)_{j\geq 0}$ such that $\lim_{j\to\infty} x_j = x$ and $\lim_{j\to\infty} h'(x_j)$ exists.

Consider now $m > 1$ and the map $h : U \to \mathbb{R}^m$ with components $h_k : U \to \mathbb{R}$ for $1 \leq k \leq m$. Note that h is Lipschitz, say with respect to the max norm, if and only if every component h_k is Lipschitz for $1 \leq k \leq m$. The generalised (Clarke) Jacobian ∂h is defined to be [9, Section 2.6]:

$$\partial h(x) = \operatorname{conv}\Big\{ \lim_{j\to\infty} Jh(x_j) : x_j \to x,\ x_j \notin \Omega_h \Big\}, \tag{2}$$

where $Jh(x)$ denotes the Jacobian of h at $x \in U$. The above formula is to be interpreted like Eq. (1). There are many sequences (x_j) on $U \setminus \Omega_h$, which converge to x such that $Jh(x_j)$ also converges to a limit; the generalised Jacobian $\partial h(x)$ is the convex hull of all such limits. Let the vector space, $\mathbb{R}^{m\times n}$, of $m \times n$ matrices over real numbers be equipped with the Frobenius norm, i.e., $\|M\| = \sqrt{\sum_{j=1}^m \sum_{k=1}^n |M_{jk}|}$. By [9, Proposition 2.6.2], $\partial h_j(x)$ is a non-empty convex compact subset of $\mathbb{R}^{m\times n}$, and the map $\partial h : U \to \mathbf{C}(\mathbb{R}^{m\times n})$ is upper semi-continuous (equivalently Scott continuous), where $\mathbf{C}(\mathbb{R}^{m\times n})$, equipped with its upper topology (equivalently Scott topology), is the space of non-empty compact and convex subsets of the vector space $\mathbb{R}^{m\times n}$. We have: $\partial h(x) \subseteq \partial h_1(x) \times \cdots \times \partial h_m(x)$, where the latter denotes the set of $m \times n$ matrices whose jth row belongs to $\partial h_j(x)$.

1.2. Notations and terminology

We use the standard notions of basic domain theory as in [1,26]. We write $\mathbb{R}$ for the set of real numbers and $\mathbf{I}\mathbb{R} = \{[a,b] \mid a, b \in \mathbb{R}, a \leq b\} \cup \{\mathbb{R}\}$ for the interval domain, i.e. the set of compact, nonempty intervals together with $\mathbb{R}$, ordered by reverse inclusion. The domain of all non-empty compact axis aligned hyper-rectangles in $\mathbb{R}^n$, ordered by reverse inclusion and augmented with the least element $\perp = \mathbb{R}^n$, is denoted by $\mathbf{I}\mathbb{R}^n$, which is the smash product of $\mathbf{I}\mathbb{R}$ with itself n times. Any maximal element of $\mathbf{I}\mathbb{R}^n$ is of the form $\{v\}$ for $v \in \mathbb{R}^n$ and as usual we identify $\{v\}$ and v for convenience.

Given an open subset $a \subseteq X$ of a topological space X and an element $b \in D$ of a continuous Scott domain with bottom $\perp$, the *single step function* $b\chi_a : X \to D$ is defined as $(b\chi_a)(x) = b$ if $x \in a$ and $\perp$ otherwise. Single-step functions are continuous with respect to the Scott topology. Any finite bounded set of single-step functions has a least upper bound, called a *step function*, in the space of all Scott continuous functions of type $X \to D$. The domain of definition of a Scott continuous function $g : X \to D$ is the set of points $x \in D$ with $g(x) \neq \perp$. A continuous Scott domain is a bounded complete countably based continuous dcpo (directed complete partial order). We denote the continuous Scott domain of the nonempty, compact and convex subsets of $\mathbb{R}^n$, taken together with $\mathbb{R}^n$ as a bottom element and ordered by reverse inclusion, by $\mathbf{C}\mathbb{R}^n$. The Scott topology and the upper topology coincide on $\mathbf{C}\mathbb{R}^n$ as well as on $\mathbf{I}\mathbb{R}^n$ [14]. On these domains the way-below relation $\ll$ is given by $A \ll B$ iff $A^\circ \supset B$, where A° is the interior of the compact set $A \subset \mathbb{R}^n$. We denote as $U \to \mathbf{C}\mathbb{R}^n$, where $U \subseteq \mathbb{R}^n$ is open, the domain of all Scott continuous, equivalently upper continuous, functions of type $U \to \mathbf{C}\mathbb{R}^n$, which are ordered pointwise to give a continuous Scott domain [26, Proposition II-4.20(iv)]. We use a canonical basis of $\mathbf{C}\mathbb{R}^n$, consisting of rational compact and convex polyhedra together with the set $\mathbb{R}^n$ as the bottom element. This basis in turn gives a countable and canonical basis of *rational step functions* for $U \to \mathbf{C}\mathbb{R}^n$ generated by single-step functions of the form $b\chi_a$ where a is a rational open hyper-rectangle with faces parallel to the coordinate hyper-planes of $\mathbb{R}^n$ and b is a rational compact convex polyhedra in $\mathbb{R}^n$. The product domain $\mathbf{I}\mathbb{R}^n$ has a canonical basis consisting of all its rational hyper-rectangles and the bottom

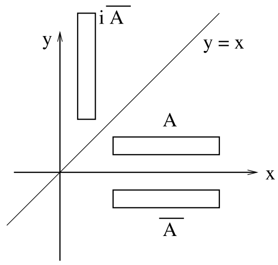

Fig. 1. Representation of an axis aligned rectangle, first reflected through the x-axis and then rotated by $\pi/2$.

element, which in turn provides a countable and canonical basis for $U \to \mathbf{I}\mathbb{R}^n$ similar to the countable canonical basis of $U \to \mathbf{C}\mathbb{R}^n$.

For any $z = x + iy \in \mathbb{C}$, the conjugate complex number is denoted by $\bar{z} = x - iy$; moreover, the same notation is used to denote the pointwise extension of conjugation to sets of complex numbers, i.e., for $C \subseteq \mathbb{C}$, we write $\bar{C} = \{\bar{z} : z \in C\}$. Furthermore, all basic arithmetic operations on complex numbers are extended pointwise to subsets of complex numbers, e.g., for a subset $C \subseteq \mathbb{C}$, we write $iC := \{iz : z \in C\}$. In order to keep the presentation as simple as possible, we use the isomorphism $\mathcal{H} : x + iy \mapsto (x, y)$ to identify the complex plane $\mathbb{C}$, regarded as a group under addition of complex numbers, and the real two dimensional Euclidean plane $\mathbb{R}^2$, regarded as a group under the addition of vectors: we move from one plane to the other plane while suppressing any explicit reference to $\mathcal{H}$ or its pointwise extension to subsets of $\mathbb{C}$ wherever convenient. In particular, an expression containing complex conjugation or multiplication by i can in fact be interpreted, after carrying out the complex number operations, in the real plane, for example, $i\bar{A}$, for a subset $A \subseteq \mathbb{R}^2$, denotes the set $\mathcal{H}[i\overline{[\mathcal{H}^{-1}[A]]}] \subseteq \mathbb{R}^2$, i.e., the subset A is first reflected through the x-axis and then rotated by $\pi/2$ around the origin, i.e., an overall reflection through the line $y = x$; see Fig. 1. For a compact subset $C \subset \mathbb{C}$, we put $\|C\| = \sup\{|z| : z \in C\}$ and we denote by $\operatorname{conv}(A)$ the convex hull of a set $A \subset \mathbb{R}^n$. The transpose of a matrix (or vector) M is denoted by M^T. For a map $F : U \to \mathbb{R}$, where $U \subseteq R^2$ is open, we write the derivative at (x, y), if it exists, as $F'(x, y) = (F'_1(x, y), F'_2(x, y))$, with $F'_1 = \frac{\partial F}{\partial x}$ and $F'_2 = \frac{\partial F}{\partial y}$.

2. L-derivative of complex Lipschitz maps

Throughout this paper, unless otherwise stated, let $U \subseteq \mathbb{C}$ be a non-empty open subset of the complex plane $\mathbb{C}$ often identified with $\mathbb{R}^2$ in this paper, as already mentioned. Consider a complex map $f : U \subseteq \mathbb{C} \to \mathbb{C}$ with $\operatorname{dom}(f) \subseteq U$. The notion of a set-valued Lipschitz constant for f is now defined using the values of the ratio $(f(z_1) - f(z_2))/(z_1 - z_2)$ for complex numbers $z_1 \neq z_2$:

Definition 2.1. We say f has *a Lipschitz constant* $b \in \mathbf{C}(\mathbb{C}) \setminus \{\perp\}$ in a non-empty convex open set $a \subset \operatorname{dom}(f)$ if for all $z_1, z_2 \in a$, with $z_1 \neq z_2$, we have: $\frac{f(z_1)-f(z_2)}{z_1-z_2} \in b$.

Note that the above notion of a Lipschitz constant for a complex map f is quite different from the way the corresponding notion is defined in [20,16,22] for a real map $p : U \subseteq \mathbb{R}^2 \to \mathbb{R}$. It follows immediately that if a map $f : U \to \mathbb{C}$ has a Lipschitz constant $b \in \mathbf{C}(\mathbb{C}) \setminus \{\perp\}$ in a, then it is Lipschitz in a with

$$\left|f(z_1) - f(z_2)\right| \le k|z_1 - z_2|,$$

for all $z_1, z_2 \in a$ where $k = \max\{|z| : z \in b\}$.

The *single tie* of $b\chi_a$, denoted by $\delta(b\chi_a)$, is the collection of all complex-valued functions f defined on U that have a Lipschitz constant b in a.

Proposition 2.2. *Suppose $a \neq \emptyset$ and $b \neq \perp$. We have $\delta(b\chi_a) \supseteq \delta(b'\chi_{a'})$ iff $a' \supseteq a$ and $b \sqsubseteq b'$.*

We define a *tie* to be any family of maps of type $U \to \mathbb{C}$ given by some intersection of single-ties. The next two results make the connection between ties and Scott continuous (equivalently, upper continuous) functions of type $U \to \mathbf{C}(\mathbb{C})$ clear.

Proposition 2.3. *For any indexing set I, the family of single-step functions $(b_i\chi_{a_i})_{i\in I}$ is bounded in $(U \to \mathbf{C}(\mathbb{C}))$ if*

$$\bigcap_{i\in I} \delta(b_i\chi_{a_i}) \neq \emptyset.$$

Proof. Suppose $\bigcap_{i\in I}\delta(b_i\chi_{a_i})\neq\emptyset$ and let $f\in\bigcap_{i\in I}\delta(b_i\chi_{a_i})$. Since $(U\to\mathbf{C}(\mathbb{C}))$ is a continuous Scott domain, we only need to show that for any finite $J\subseteq I$ the set of single-step functions $(b_i\chi_{a_i})_{i\in J}$ is bounded. In fact, if for any $J_0\subseteq J$ we have $\bigcap_{i\in J_0}a_i\neq\emptyset$, then from $f\in\bigcap_{i\in J_0}\delta(b_i\chi_{a_i})$, we obtain $\frac{f(z_1)-f(z_2)}{z_1-z_2}\in\bigcap_{i\in J_0}b_i$ for $z_1,z_2\in\bigcap_{i\in J_0}a_i$ with $z_1\neq z_2$, and it follows that $(b_i\chi_{a_i})_{i\in J}$ is bounded. $\square$

Moreover, we have the following:

Proposition 2.4. *Suppose* $\sup_{i\in I}b_i\chi_{a_i}\sqsubseteq\sup_{j\in J}b_j\chi_{a_j}$. *Then:*

$$\bigcap_{i\in I}\delta(b_i\chi_{a_i})\supseteq\bigcap_{j\in J}\delta(b_j\chi_{a_j}).$$

Proof. We show that $b\chi_a\sqsubseteq\sup_{j\in J}b_j\chi_{a_j}$ implies $\delta(b\chi_a)\supseteq\bigcap_{j\in J}\delta(b_j\chi_{a_j})$ from which the result follows. We have $a\subseteq\bigcup_{j\in J}a_j$. Assume $f\in\bigcap_{j\in J}\delta(b_j\chi_{a_j})$. Fix $w,w'\in a$. Since a is convex, the line segment ww' lies in a and is partitioned by the open subsets a_j for $j\in J$ into a set of disjoint intervals which are crescents generated by these open subsets with respect to the relative Euclidean topology on the line segment ww'. Let z_t for $t=1,\dots,n$ be the boundary points of these intervals ordered from w to w' and put $z_0:=w$ and $z_{n+1}:=w'$. Thus for each $t=0,\dots,n$ the one dimensional interior (z_t,z_{t+1}) of the line segment z_tz_{t+1} lies in one of these crescents. Since for each $z\in a$ we have $b\sqsubseteq\sup_{j:z\in a_j}b_j$, it follows that for each $t=1,\dots,n$ we have $b\sqsubseteq\sup_{j:(z_t,z_{t+1})\subset a_j}b_j$. Thus for any $x,y\in(z_t,z_{t+1})$ and $j\in J$ with $(z_t,z_{t+1})\subseteq b_j$ we have $f(x)-f(y)\in b_j(x-y)$, and hence $f(x)-f(y)\in b(x-y)$. By continuity it follows that this relation also holds for $x,y\in[z_t,z_{t+1}]$. Therefore,

$$f(w)-f(w')=\sum_{t=0}^{n}f(z_t)-f(z_{t+1})\in\sum_{t=0}^{n}b(z_t-z_{t+1})=b\left(\sum_{t=0}^{n}z_t-z_{t+1}\right)=b(w-w'),$$

where the first equality after membership predicate holds because all complex numbers z_t-z_{t+1} have the same direction. Therefore, $f\in\delta(b\chi_a)$. $\square$

Thus, any non-empty tie $\Delta=\bigcap_{i\in I}\delta(b_i\chi_{a_i})$ is uniquely associated with the Scott continuous function

$$g=\sup_{i\in I}b_i\chi_{a_i},$$

and we write $\Delta=\delta(g)$. Therefore, $\delta(g)$ is a family of locally Lipschitz functions, defined on $\bigcup_{i\in I}a_i$, whose local Lipschitz properties are expressible by single-ties provided by the single-step functions below g. In the context of real-valued Lipschitz maps on finite dimensional Euclidean spaces, the notion of the tie of maps associated with g coincides with that of g-Lipschitz maps as studied by Borwein et al. [5–7]. We say a function $g\in(U\to\mathbf{C}(\mathbb{C}))$ is *integrable* if $\delta(g)\neq\emptyset$.

Let $(\mathbf{T}(U),\supseteq)$ be the partial order of ties of Scott continuous functions of type $U\to\mathbf{C}(\mathbb{C})$ ordered by reverse inclusion; as in the real case, here also $(\mathbf{T}(U),\supseteq)$ can be shown to be a dcpo. The set of L-primitives of a Scott continuous function is precisely the tie associated with it. The *L-primitive map* is defined by

$$\int:(U\to\mathbf{C}(\mathbb{C}))\to\mathbf{T}(U)$$
$$g\mapsto\delta(g).$$

The set $\int g$ is the collection of the *L-primitives* of g and the map $\int$ is continuous with respect to the Scott topologies on $(U\to\mathbf{C}(\mathbb{C}))$ and $\mathbf{T}(U)$.

The Lipschitz constants for a map provide us with its local differential properties, which can be collected to define its global derivative.

Definition 2.5. The *Lipschitz derivative* or the *L-derivative* of a Lipschitz function $f:U\to\mathbb{C}$ is the map

$$\mathcal{L}f:U\to\mathbf{C}(\mathbb{C}),$$

defined by

$$\mathcal{L}f=\sup\{b\chi_a: f\in\delta(b\chi_a)\}.$$

From Proposition 2.3, we get:

Corollary 2.6. *The L-derivative of any Lipschitz function is Scott continuous.*

The relationship between the L-derivative and the classical derivative is expressed in the following two results. Recall that a complex map is said to be analytic (or holomorphic) in an open set if it is differentiable at all points in the set; we say that a complex map is analytic at $z \in \mathbb{C}$ if it is analytic in an open neighbourhood of z. Note that a map is analytic in an open set if and only if it has a convergent Taylor series expansion there [2].

Lemma 2.7. *If $f : a \to \mathbb{C}$ is differentiable at $z \in a$ then $f \in \delta(b\chi_a)$ implies $f'(z) \in b$.*

Proof. This follows from $(f(w) - f(z))/(w - z) \in b$ for $w \in a$ with $w \neq z$. $\square$

Proposition 2.8. *If $f : a \to \mathbb{C}$ is analytic in the convex open subset $a \subseteq U$, then the following three conditions are equivalent:*

(i) $f \in \delta(b\chi_a)$,
(ii) $\forall z \in a.\ f'(z) \in b$,
(iii) $b\chi_a \sqsubseteq f'$.

Proof. (i) $\Rightarrow$ (ii). This follows from Lemma 2.7. (ii) $\Rightarrow$ (i). Fix $z_1, z_2 \in a$, with $z_1 \neq z_2$, and take $d \in \mathbf{C}(\mathbb{C})$ with $b \subseteq d^\circ$. Let $z = z_2 - z_1$ and consider the set $T = \{k \in (0, 1] : (f(z_1 + kz) - f(z_1))/kz \in d\}$. Note that T is non-empty since $\lim_{k\to 0^+} = (f(z_1 + kz) - f(z_1))/kz = f'(z_1) \in b \subseteq d^\circ$. We claim that $\sup T = 1$. Suppose $t := \sup T < 1$. Then, by continuity of f and compactness of d, we have: $w_1 := (f(z_1 + tz) - f(z_1))/tz \in d$. Moreover, $f'(z_1 + tz) \in b \subseteq d^\circ$ and thus there exists $s > t$ such that $w_2 := (f(z_1 + sz) - f(z_1 + tz))/((s - t)z) \in d$. Therefore, we obtain

$$\frac{f(z_1 + sz) - f(z_1)}{sz} = \frac{tw_1 + (s - t)w_2}{s}.$$

But since d is convex we have

$$\frac{tw_1 + (s - t)w_2}{s} \in d,$$

which contradicts $t = \sup T$. It follows that

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} \in d.$$

Since this holds for all $d \in \mathbf{C}(\mathbb{C})$ with $b \subseteq d^\circ$, it follows that

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} \in b.$$

(iii) $\iff$ (ii). Obvious. $\square$

We note that the corresponding proof in the real case in [16] for the non-trivial part of Proposition 2.8, i.e., (ii) $\Rightarrow$ (i), relies on the mean value theorem for real maps, which is well known to fail for analytic functions [23]. For example, the analytic map $z \mapsto e^z - 1$ vanishes at $z = 2n\pi i$ for any $n \in \mathbb{Z}$, but its derivative namely $z \mapsto e^z$ never vanishes.

Proposition 2.9. *Consider $f : U \to \mathbb{C}$ and $z \in U$.*

(i) *If f is differentiable at z then $f'(z) \in \mathcal{L}f(z)$.*
(ii) *If f is analytic at z, then $\mathcal{L}f(z) = \{f'(z)\}$.*
(iii) *If the L-derivative $\mathcal{L}f(z)$ is maximal in $\mathbf{C}(\mathbb{C})$ then f is differentiable at z with $\mathcal{L}f(z) = \{f'(z)\}$.*
(iv) *The map f is analytic in U if and only if its L-derivative is a singleton at all points in U.*

Proof. (i) Suppose f is differentiable at z. If $f \in \delta(b\chi_a)$ and $z \in a$, then, by Lemma 2.7, it follows that $f'(z) \in b$. Since this is true for any a and b as such, we obtain $f'(z) \in \mathcal{L}f(z)$.

(ii) Let f be analytic in an open neighbourhood a of z. By Proposition 2.8,

$$f \in \delta(b\chi_a) \iff b\chi_a \sqsubseteq f'.$$

Hence, $f' \sqsupseteq \sup\{b\chi_a : f \in \delta(b\chi_a)\}$. To show equality, let $z \in U$ and put $L := f'(z)$. By the continuity of the derivative $f' : U \to \mathbb{C}$ at z, for each integer $n > 0$, there exists an open ball $a \subseteq U$ with $z \in a$ such that $f'(x) \in R_{1/n}(L)$ for $x \in a$, where $R_r(L)$ is, for $r > 0$, the closed square with sides of length r parallel to the axes and centre $L \in \mathbb{C}$. By Proposition 2.8, we have

$$f' \sqsupseteq \mathcal{L}f \sqsupseteq R_{1/n}(L)\chi_a.$$

Since

$$\bigcap_{n\geq 0} R_{1/n}(L)=\{f'(z)\},$$

we conclude that $\{f'(z)\}=\mathcal{L}f(z)$.

(iii) Let $L:=\mathcal{L}f(z)=\bigcap\{b: f\in\delta(b\chi_a)\ \&\ z\in a\}$ be maximal and assume $\epsilon>0$ is given. Then, $\bigcap\{b: f\in\delta(b\chi_a)\ \&\ z\in a\}\subset B_\epsilon(L)$, where $B_\epsilon(L)$ is the open ball of radius ϵ around $L\in\mathbb{C}$. By the intersection property of compact sets in $\mathbb{R}^2$, there exists $b\in\mathbf{C}(\mathbb{C})$ with $b\subseteq B_\epsilon(L)$ and an open, convex set a with $f\in\delta(b\chi_a)$ and $z\in a$. Thus, for all $w\in a\setminus\{z\}$ we have $\frac{f(z)-f(w)}{z-w}\in b$ and hence

$$\left|\frac{f(z)-f(w)}{z-w}-L\right|<\epsilon,$$

and the result follows.

(iv) This follows immediately from (ii) and (iii). □

Also, as in the real case, we have an extension of the fundamental theorem of calculus for complex maps:

Proposition 2.10. *For a locally Lipschitz map $f:U\to\mathbb{C}$ and Scott continuous $g:U\to\mathbf{C}(\mathbb{C})$ we have*

$$f\in\int g \quad\Longleftrightarrow\quad g\sqsubseteq\mathcal{L}f. \qquad\square$$

Indeed, if g takes maximal values then $g\sqsubseteq\mathcal{L}f$ implies that $\mathcal{L}f$ also takes maximal values and thus, by Proposition 2.9(iii), f is differentiable and thus analytic in U, which gives $g=f'$.

3. Calculus of the L-derivative

For real functions of type $p,q:O\to\mathbb{R}$ where $O\subseteq\mathbb{R}^2$, the Clarke gradient (equivalently the L-derivative) has the following properties for sum and product [9, Propositions 2.3.3 and 2.3.13] at $x\in O$.

$$\mathcal{L}p(x)+\mathcal{L}q(x)\sqsubseteq\mathcal{L}(p+q)(x)$$

$$p(x)\mathcal{L}q(x)+q(x)\mathcal{L}p(x)\sqsubseteq\mathcal{L}(pq)(x).$$

Moreover, each of the above two subset inclusions becomes equality at any point $x\in U$ if at least one of the two Clarke gradients $\mathcal{L}p(x)$ and $\mathcal{L}q(x)$ is a singleton. As for the chain rule, if $q:U\to\mathbb{C}$ and $p:O\to\mathbb{R}$ with $\mathrm{Im}(q)\subseteq O$, then for $x\in U$:

$$(\mathcal{L}p)\circ q(x)\cdot\mathcal{L}q(x)\sqsubseteq\mathcal{L}(p\circ q)(x).$$

In addition, if at least one of $\mathcal{L}q(x)$ and $(\mathcal{L}p)(q(x))$ is a singleton then we obtain equality in the above relation. We will show now that similar relations are satisfied for complex Lipschitz maps.

We say a Lipschitz map $f:U\to\mathbb{C}$ is *maximally differentiable* at a point $z\in U$ if $\mathcal{L}f(z)$ is a singleton. By Proposition 2.9(ii) and (iii), if f is maximally differentiable at z then it is differentiable at z and if f is analytic at z then it is maximally differentiable at z. Examples 4.11 and 4.15, respectively, show there are functions which are maximally differentiable but not analytic at a point, and there are functions which are differentiable but not maximally differentiable at a point. In the sequel, we will use the following notation: given a set $C\in\mathbf{C}(\mathbb{C})$, the ϵ-*neighbourhood* of C is defined as $C_\epsilon=\{w: d(w,C)\leq\epsilon\}$, where $d(w,C)$ is the distance between w and the compact set C.

Proposition 3.1. *Let $f,g:U\to\mathbb{C}$ be Lipschitz maps and $z\in U$. Then we have:*

- ***Sum.*** *$\mathcal{L}f(z)+\mathcal{L}g(z)\sqsubseteq\mathcal{L}(f+g)(z)$, and equality holds at a point $z\in U$ if at least one of f and g is maximally differentiable at z.*
- ***Product.*** *$f(z)\mathcal{L}g(z)+g(z)\mathcal{L}f(z)\sqsubseteq\mathcal{L}(fg)(z)$, and equality holds at a point $z\in U$ if at least one of f and g is maximally differentiable at z.*
- ***The chain rule.*** *Let $g:U_1\to\mathbb{C}$ and $f:U_2\to\mathbb{C}$ with $Im(g)\subseteq U_2$ be Lipschitz maps, where U_1 and U_2 are non-empty open subsets, and let $z\in U_1$. Then:*

  $$\big((\mathcal{L}f)\circ g\big)(z)\mathcal{L}g(z)\sqsubseteq\mathcal{L}(f\circ g)(z),$$

  and we have equality if the following holds: f is maximally differentiable at $g(z)$ or g is maximally differentiable at z.
- ***Scalar multiplication.*** *$\mathcal{L}(kf)(z)=k\mathcal{L}f(z)$.*

- ***Monomials.*** $\mathcal{L}(f(z)^n) = n(f(z))^{n-1}\mathcal{L}f(z)$.
- ***Reciprocal.*** $\mathcal{L}(1/f)(z) = -(\mathcal{L}f(z))/(f(z))^2$, *for* $f(z) \neq 0$.

Proof. Sum. Let $z \in U$ and $b \ll \mathcal{L}f(z) + \mathcal{L}g(z)$. By the Scott continuity of $\mathcal{L}f$ and $\mathcal{L}g$, Proposition 2.10 and the Hausdorff continuity of the Minkowski sum $- \oplus - : \mathbf{C}(\mathbb{C}) \times \mathbf{C}(\mathbb{C}) \to \mathbf{C}(\mathbb{C})$, there is a neighbourhood $O \subseteq U$ of z and $b_1 \ll \mathcal{L}f(z)$ and $b_2 \ll \mathcal{L}g(z)$ such that $f \in \delta(b_1\chi_O)$, $g \in \delta(b_2\chi_O)$ and $b \ll b_1 + b_2 \ll \mathcal{L}f(z) + \mathcal{L}g(z)$. Now for distinct $z_1, z_2 \in O$ we have:

$$\frac{f(z_1) + g(z_1) - f(z_2) - g(z_2)}{z_1 - z_2} = \frac{f(z_1) - f(z_2)}{z_1 - z_2} + \frac{g(z_1) - g(z_2)}{z_1 - z_2} \in b_1 + b_2 \subseteq b.$$

It follows that $b \sqsubseteq \mathcal{L}(f+g)(z)$. Since $b \ll \mathcal{L}f(z) + \mathcal{L}g(z)$ is arbitrary, we obtain the first result. Now assume one of the two maps, say g, is maximally differentiable at z. Let $C \ll \mathcal{L}(f+g)(z)$ and $\epsilon > 0$. Thus, there exists $\delta_0 > 0$ such that

$$\frac{f(w_1) + g(w_1) - f(w_2) - g(w_2)}{w_1 - w_2} \in C, \tag{3}$$

for $|w_j - z| < \delta_0$, $j = 1, 2$, $w_1 \neq w_2$. On the other hand, since $\mathcal{L}g(z) = \{g'(z)\}$, there exists $\delta_1 > 0$ such that

$$\left|g'(z) - \big(g(w_1) - g(w_2)\big)/(w_1 - w_2)\right| < \epsilon,$$

for $|w_j - z| < \delta_1$, $j = 1, 2$, and $w_1 \neq w_2$, and it follows from Relation 3 that

$$\frac{f(w_1) - f(w_2)}{w_1 - w_2} + g'(z) \in C_\epsilon.$$

Thus, $\mathcal{L}(f)(z) \subset C_\epsilon - g'(z)$, i.e., $\mathcal{L}f(z) + g'(z) \subset C_\epsilon$. Since $\epsilon > 0$ is arbitrary, we obtain $\mathcal{L}f(z) + g'(z) \subset C$ and the second result also follows.

Product. This follows similar to the case of sum, by considering the continuity of f and g, the Scott continuity of $\mathcal{L}f$ and $\mathcal{L}g$ and the Hausdorff continuity of the product $- \otimes - : (k, A) \mapsto \{kz : z \in A\} : \mathbb{C} \times \mathbf{C}(\mathbb{C}) \to \mathbf{C}(\mathbb{C})$, together with the identity:

$$\frac{f(w_1)g(w_1) - f(w_2)g(w_2)}{w_1 - w_2} = \frac{f(w_1)[g(w_1) - g(w_2)]}{w_1 - w_2} + \frac{[f(w_1) - f(w_2)]g(w_2)}{w_1 - w_2}$$

for distinct $w_1, w_2 \in U$. The case of equality when g is maximally differentiable at z follows similar to the case of sum.

The chain rule. Fix $z \in U_1$. Then $g(z) \in U_2$. Let $b \ll ((\mathcal{L}f) \circ g)(z) \cdot \mathcal{L}g(z)$. By the Scott continuity of $\mathcal{L}f$ and $\mathcal{L}g$, the continuity of g, and the Hausdorff continuity of the product $- \otimes - : (A, B) \mapsto \{zw : z \in A, w \in B\} : \mathbf{C}(\mathbb{C}) \times \mathbf{C}(\mathbb{C}) \to \mathbf{C}(\mathbb{C})$, there exists a neighbourhood O_1 of z, a neighbourhood O_2 of $g(z)$ and $b_1, b_2 \in \mathbf{C}(\mathbb{C})$ such that $g \in \delta(b_1\chi_{O_1})$ and $f \in \delta(b_2\chi_{O_2})$ with $b \ll b_1 \cdot b_2 \ll ((\mathcal{L}f) \circ g)(z) \cdot \mathcal{L}g(z)$. If g is constant in a neighbourhood of z then the result follows trivially. Otherwise, for $w_1, w_2 \in= O_1 \cap g^{-1}(O_2)$ with $g(w_1) \neq g(w_2)$, we get the identity:

$$\frac{f(g(w_1)) - f(g(w_2))}{w_1 - w_2} = \left(\frac{f(g(w_1)) - f(g(w_2))}{g(w_1) - g(w_2)}\right)\left(\frac{g(w_1) - g(w_2)}{w_1 - w_2}\right).$$

Using this for $w_1, w_2 \in O_1 \cap g^{-1}(O_2)$, it follows that $b \sqsubseteq \mathcal{L}(f \circ g)(z)$ from which the first result follows.

Now suppose g is maximally differentiable at z. If $g'(z) = 0$ then equality follows immediately. Assume $g'(z) \neq 0$. Let $b \ll \mathcal{L}(f \circ g)(z)$. Then there exists a $\delta_0 > 0$ such that

$$\left(\frac{f(g(w_1)) - f(g(w_2))}{g(w_1) - g(w_2)}\right)\left(\frac{g(w_1) - g(w_2)}{w_1 - w_2}\right) = \frac{f(g(w_1)) - f(g(w_2))}{w_1 - w_2} \in b, \tag{4}$$

for $|w_j - z| < \delta_0$, $j = 1, 2$, and $w_1 \neq w_2$. Also, since $\mathcal{L}g(z) = \{g'(z)\}$, for any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that $|g'(z) - (g(w_1) - g(w_2))/(w_1 - w_2)| < \epsilon/(\|\mathcal{L}f \circ g(z)\| + 1)$ for $|w_j - z| < \delta_1$, $j = 1, 2$, and $w_1 \neq w_2$. Then,

$$\left|\frac{f(g(w_1)) - f(g(w_2))}{g(w_1) - g(w_2)}\left(\frac{g(w_1) - g(w_2)}{w_1 - w_2} - g'(z)\right)\right| < \epsilon,$$

for $|w_j - z| < \min(\delta_0, \delta_1)$, $j = 1, 2$, and $g(w_1) \neq g(w_2)$, and it follows that

$$\left(\frac{f(g(w_1)) - f(g(w_2))}{g(w_1) - g(w_2)}\right)g'(z) \in b_\epsilon,$$

and hence

$$\frac{f(g(w_1)) - f(g(w_2))}{g(w_1) - g(w_2)} \in b_\epsilon/g'(z).$$

Thus, $\mathcal{L}f \circ g(z) \subset b_\epsilon/g'(z)$. Since $\epsilon > 0$ is arbitrary, we obtain $\mathcal{L}f \circ g(z) \subset b/g'(z)$, or $\mathcal{L}f \circ g(z)g'(z) \subset b$.

Next suppose f is maximally differentiable at $g(z)$. Then, $(\mathcal{L}f)(g(z)) = \{f'(g(z))\}$. We now assume that $f'(g(z)) \neq 0$, otherwise the result follows immediately. Let $b \ll \mathcal{L}(f \circ g)(z)$ and $\delta_0 > 0$ be such that Eq. (4) is satisfied, and take any $\epsilon > 0$. There exists $\delta_1 > 0$ such that

$$\left| \frac{f(g(w_1)) - f(g(w_2))}{g(w_1) - g(w_2)} - f'\big(g(z)\big) \right| < \frac{\epsilon}{\|\mathcal{L}g(z)\| + 1}$$

for $|w_j - z| < \delta_1$, $j = 1, 2$ and $g(w_1) \neq g(w_2)$, i.e., $f'(g(z))\frac{g(w_1)-g(w_2)}{w_1-w_2} \in b_\epsilon$ and the result follows as in the previous case.

Scalar multiplication. This is trivial to check directly but also follows from the chain rule using the linear map $z \mapsto kz$, which is analytic and hence maximally differentiable with constant derivative with value k.

Monomials. This follows from the chain rule using the analytic map $z \mapsto z^n$.

Reciprocal. This is again a consequence of the chain rule using the analytic map $z \mapsto 1/z$ for $z \neq 0$. □

4. Generalised Cauchy–Riemann equations

Assume that $f : U \to \mathbb{C}$ and write $f = V + iW$ where V and W are the real and imaginary parts of f respectively. For $z \in U$, we write $z = x + iy$ so that $f(z) = V(x, y) + iW(x, y)$. Recall the classical theorem for Cauchy–Riemann equations [2, p. 25]: If $V, W : U \to \mathbb{R}$ are continuously differentiable, then f is analytic in U iff for all points in U,

$$\frac{\partial V}{\partial x} = \frac{\partial W}{\partial y}, \qquad \frac{\partial V}{\partial y} = -\frac{\partial W}{\partial x}, \tag{5}$$

i.e., using our notational convention in Subsection 1.2, $\overline{V'} = i\overline{W'}$. When f is analytic, we have:

$$f' = \left(\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}\right) = \overline{V'}, \qquad f' = \left(\frac{\partial W}{\partial y}, \frac{\partial W}{\partial x}\right) = i\overline{W'}. \tag{6}$$

We will obtain an analogue of these properties for Lipschitz maps. More precisely, we obtain the L-derivative of a complex Lipschitz map in terms of a so-called C-derivative which is obtained, as we will see, by a transformation of the Clarke gradient of its real and imaginary parts. In the case that V and W are C^1 maps, this will generalise the two points given by the partial derivatives of V and W in Eq. (6) to a disk with these two points as a diameter. More generally, this construction leads to the union of disks obtained from the transformed Clarke gradient of the vector valued map $(x, y) \mapsto (V(x, y), W(x, y))^T$. We use the notion of the directional derivative of a complex map presented in [27, Section 6.10], where it is attributed originally (with a different notation) to Bernard Riemann.

Proposition 4.1. *Suppose V and W are differentiable at $(x, y) \in \mathbb{R}^2$ and assume $z = x + iy$. Then,*

$$\lim_{r \downarrow 0} \frac{f(z + re^{i\theta}) - f(z)}{re^{i\theta}} = \frac{1}{2}\big[c_1 + d_2 + i(d_1 - c_2)\big] + \frac{1}{2}e^{-2i\theta}\big[c_1 - d_2 + i(d_1 + c_2)\big],$$

where $(c_1, c_2) = V'(x, y)$ and $(d_1, d_2) = W'(x, y)$.

We denote the circle in $\mathbb{C}$, given in the above proposition, with the pair of points $c = c_1 + ic_2$ and $d = d_1 + id_2$ as the two end points of a diameter by $\mathsf{S}(c, d)$. This circle is named after Kasner, who had called it the *clock* [27, p. 349]. We also call the corresponding disk, denoted by $\mathsf{D}(c, d)$, the Kasner disk, which is the convex hull of $\mathsf{S}(c, d)$. Thus, with the assumptions of Proposition 4.1, we obtain:

Corollary 4.2. *The complex number $\lim_{r\downarrow 0} \frac{f(z+re^{i\theta})-f(z)}{re^{i\theta}}$ lies on the circle $\mathsf{S}(\overline{V'(x, y)}, i\overline{W'(x, y)})$.*

We define the *differential transformation* as the map $T : \mathbb{R}^2 \times \mathbb{R}^2 \times [0, 2\pi) \to \mathbb{C}$, where

$$T(c, d, \theta) = \frac{1}{2}\big[c_1 + d_2 + i(d_1 - c_2)\big] + \frac{1}{2}e^{-2i\theta}\big[c_1 - d_2 + i(d_1 + c_2)\big].$$

Furthermore, we extend T to obtain the binary operation $- \odot - : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbf{C}(\mathbb{C})$ given by

$$c \odot d := \operatorname{conv}\big\{T(c, d, \theta) : \theta \in [0, 2\pi)\big\} = \operatorname{conv}\big(\mathsf{S}(\bar{c}, i\bar{d})\big) = \mathsf{D}(\bar{c}, i\bar{d}), \tag{7}$$

where $\operatorname{conv}(A)$ denotes the convex hull of a set $A \subseteq \mathbb{R}^{2\times 2}$. Since T is a continuous map, it follows that the extended transformation $- \odot -$ is continuous with respect to the Hausdorff metric on $\mathbf{C}(\mathbb{C})$.

Consider now the map $\hat{f} : U \to \mathbb{R}^2$ with $\hat{f}_1 = V$ and $\hat{f}_2 = W$ where $V, W : U \to \mathbb{R}$ (and hence $\hat{f}$ with respect to the max norm) are Lipschitz. We will use the Clarke gradient of the vector map $\hat{f}$ (see Eq. (2)) and the differential transformation to define another notion of derivative for complex Lipschitz maps. We say, in analogy with the case of complex maps, that a Lipschitz $g : U \subset \mathbb{R}^2 \to \mathbb{R}$ is *maximally differentiable* at a point $u \in U$ if $\partial g(u)$ is a single point. At a point where g is maximally differentiable, it will be differentiable but will not be necessarily continuously differentiable [9, Proposition 2.2.4]. We have the following simple property.

Proposition 4.3. *If one of the two maps V and W is maximally differentiable at $u \in U$, then $\partial \hat{f}(u) = \partial V(u) \times \partial W(u)$.*

Proof. Suppose V is maximally differentiable at $u \in U$. Then, any 2×2 matrix $M \in \partial \hat{f}(u)$, given by Eq. (2), will have $\{\partial V(u)\}$ as its first row and some point in $\partial W(u)$ as its second row. Thus, the first row factors out and the result follows. □

Definition 4.4. The *C-derivative* of the complex function $f = V + iW : U \to \mathbb{C}$ at the point $z = x + iy$ is defined as the subset $\partial f(z) \subset \mathbb{C}$ with

$$\partial f(z) = \operatorname{conv} \bigcup \left\{ v \odot w : \begin{bmatrix} v \\ w \end{bmatrix} \in \partial \hat{f}(x, y) \right\},$$

where $\hat{f} : U \to \mathbb{R}^2$ with $\hat{f}_1 = V$ and $\hat{f}_2 = W$.

Note that we are using the same symbol ∂ for the C-derivative ∂f of a complex function f as for the Clarke gradient $\partial \hat{f}$ of a vector real valued function $\hat{f}$. Since $- \odot - : \mathbb{R}^{2\times 2} \to \mathbf{C}(\mathbb{C})$ is continuous with respect to the Hausdorff metric on $\mathbf{C}(\mathbb{C})$, it follows easily that $\partial f(z)$ is compact as well as non-empty and convex. Moreover, $\partial f : U \to \mathbf{C}(\mathbb{C})$ is continuous with respect to the Scott topology (equivalently, upper topology) on $\mathbf{C}(\mathbb{C})$.

We will now show that our two notions of the L-derivative and C-derivative for f coincide. First we need a simple technical lemma. Recall that given a non-empty compact convex set $C \subset \mathbb{R}^n$, its *support function* $S_C : \mathbb{R}^n \to \mathbb{R}$ is defined as $S_C(v) = \sup\{v \cdot x : x \in C\}$ [37,4]. Note that for a unit vector $v \in \mathbb{R}^n$, the value of the support function $S_C(v)$ is simply the supremum of the distance from the origin to the projection of C onto the direction of v. We also have $S_C \le S_B$ if and only if $C \subset B$ [38, p. 37]. Suppose now $P : [a, a + r] \to \mathbb{R}^2$ is a vector-valued function of a real variable, defined almost everywhere with respect to the one dimensional Lebesgue measure on $[a, a + r]$, and suppose its two components $P_1, P_2 : [a, a + r] \to \mathbb{R}$ are both Lebesgue integrable.

Lemma 4.5. *Suppose we have $P(s) \in C$, for almost all $s \in [a, a + r]$, where $C \subseteq \mathbb{R}^2$ is a non-empty compact convex set, then $\frac{1}{r} \int_a^{a+r} P(s)\, ds \in C$.*

Proof. If $P(s) \in C$, it follows that $P(s) \cdot v = S_{P(s)}(v) \le S_C(v)$ for almost all $s \in [a, a + r]$ and all $v \in \mathbb{R}^2$. Thus, by integration we obtain:

$$\left(\frac{1}{r} \int_a^{a+r} P(s)\, ds \right) \cdot v = \frac{1}{r} \int_a^{a+r} P(s) \cdot v\, ds \le S_C(v).$$

Therefore, the support function of $\frac{1}{r} \int_a^{a+r} P(s)\, ds$ is bounded by that of C and the result follows. □

Lemma 4.6. *If $(v_j, w_j)^T \in \mathbb{R}^{2\times 2}$ for $j \in J$, where J is a finite indexing set, and if $v = \sum_{j\in J} \lambda_j v_j$ and $w = \sum_{j\in J} \lambda_j w_j$ where $\lambda_j > 0$ with $\sum_{j\in J} \lambda_j = 1$, then*

$$v \odot w \subset conv\{v_j \odot w_j : j \in J\}.$$

Proof. Denote the centre and radius of the circle $\mathsf{S}(\overline{v}_j, i\overline{w}_j)$ by p_j and r_j respectively. Let $\mathsf{S}_\theta(\overline{v}, i\overline{w})$, for $\theta \in [0, 2\pi)$, be the point on the circle $\mathsf{S}(\overline{v}, i\overline{w})$ with polar coordinate θ, measured anti-clockwise from the real axis with respect to the centre of the circle. The circle $\mathsf{S}(\overline{v}, i\overline{w})$ has in fact its centre at

$$p = \frac{\overline{v} + i\overline{w}}{2} = \sum_{j\in J} \lambda_j \left(\frac{\overline{v}_j + i\overline{w}_j}{2} \right) = \sum_{j\in J} \lambda_j p_j,$$

and has radius

$$r = \frac{|\overline{v} - i\overline{w}|}{2} \le \sum_{j\in J} \lambda_j \left(\frac{|\overline{v}_j - i\overline{w}_j|}{2} \right) = \sum_{j\in J} \lambda_j r_j.$$

Thus, $p \in \operatorname{conv}\{v_j \odot w_j : j \in J\}$ and $r \le \sum_{j\in J} \lambda_j r_j$. We have,

$$\sum_{j\in J}\lambda_j \mathrm{S}_\theta(\overline{v}_j, i\overline{w}_j) = \sum_{j\in J}\lambda_j\big(p + r_j e^{i\theta}\big) = p + \Big(\sum_{j\in J}\lambda_j r_j\Big)e^{i\theta}.$$

Therefore $\mathrm{S}_\theta(\overline{v}, i\overline{w}) = p + re^{i\theta}$ lies on the line segment with end points at p and $\sum_{j\in J}\lambda_j \mathrm{S}_\theta(\overline{v}_j, i\overline{w}_j)$, which are both in $\mathrm{conv}\{v_j \odot w_j : j \in J\}$, and the result follows. □

The following theorem shows that the L-derivative of a complex Lipschitz map coincides with its C-derivative, which is the image of the Clarke gradient of its real and imaginary parts by the differential transformation T. Given the properties of the complex L-derivative, which extend those of the classical derivative, we can regard this theorem as a generalisation of the Cauchy–Riemann equations. Recall that by Carathéodory's theorem, any point of the convex hull of a subset $S \subset \mathbb{R}^m$ lies in the convex hull of at most $m+1$ points in S [12].

Theorem 4.7 *(Generalised Cauchy–Riemann equations). For any Lipschitz map $f : U \to \mathbb{C}$, we have $\mathcal{L}f = \partial f$.*

Proof. First we prove that $\mathcal{L}f \supset \partial f$. Let $f = V + iW$ and $z = x + iy \in U$. Assume that $b \ll \mathcal{L}f(z)$ for some $b \in \mathbf{C}(\mathbb{C})$. By Scott continuity of $\mathcal{L}f$ at z, there exists an open set $a \subset U$ such that $b \ll \mathcal{L}f(w)$ for all $w \in a$. Then $b\chi_a \sqsubseteq \mathcal{L}f$ and thus $f \in \int b\chi_a = \delta(b\chi_a)$ by the Fundamental Theorem of Calculus, Proposition 2.10. Therefore, for all distinct $z_1, z_0 \in a$ we have $\frac{f(z_1)-f(z_0)}{z_1-z_0} \in b$. Assume that $z_j = x_j + iy_j$, for $j = 0, 1$, and that V and W are differentiable at (x_0, y_0). Then, put $z_1 - z_0 = re^{i\theta}$ and fix θ. We have:

$$\begin{aligned}\frac{f(z_1)-f(z_0)}{z_1-z_0} &= e^{-i\theta}\left(\frac{f(z_0+re^{i\theta})-f(z_0)}{r}\right)\\ &= e^{-i\theta}\left(\frac{V(x_0+r\cos\theta, y_0+r\sin\theta) - V(x_0,y_0) + i(W(x_0+r\cos\theta, y_0+r\sin\theta) - W(x_0,y_0))}{r}\right)\end{aligned}$$

In the limit as $r \downarrow 0$, the latter converges to $T(V'(x_0, y_0), W'(x_0, y_0), \theta)$, by Proposition 4.1. Thus, by our assumptions, since θ is arbitrary, we obtain:

$$V'(x_0, y_0) \odot W'(x_0, y_0) \subseteq b. \tag{8}$$

Now take $(v, w)^T \in \partial\hat{f}(x, y)$, with $v, w \in \mathbb{R}^2$, such that there exists (cf. the definition of the Clarke gradient, Eq. (2)), a sequence $(\alpha_n, \beta_n)_{n\geq 0}$ with the following properties:

- V and W are differentiable at (α_n, β_n) for all $n \geq 0$,
- $(\alpha_n, \beta_n) \to (x, y)$ as $n \to \infty$, and
- $\lim_{n\to\infty}(V'(\alpha_n, \beta_n), W'(\alpha_n, \beta_n))^T = (v, w)^T$.

Thus, using Relation (8) for each point (α_n, β_n), we have $V'(\alpha_n, \beta_n) \odot W'(\alpha_n, \beta_n) \subset b$ for all $n \geq 0$. Since the set $\{C \in \mathbf{C}(\mathbb{C}) : C \subset b\}$ is a closed subset of the Vietoris topology, equivalently the Hausdorff metric topology, on $\mathbf{C}(\mathbb{C})$ and since $\odot : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbf{C}(\mathbb{C})$ is continuous with respect to the Hausdorff metric, we conclude that $v \odot w \subset b$ for each pair $(v, w) \in \partial\hat{f}(x, y)$ that satisfies the above three properties. Now take any $(v, w)^T \in \partial\hat{f}(x, y)$. By Carathéodory's theorem, $(v, w)^T$ is in the convex hull of at most five points that satisfy the above three properties. By Lemma 4.6, it follows that $v \odot w \subset b$. Since b is convex, it follows from the definition of the C-derivative 4.4 that $\partial f(z) \subset b$ from which we conclude that $\mathcal{L}f(z) \supset \partial f(z)$.

To prove that $\mathcal{L}f(z) \subset \partial f(z)$ for any $z \in U$, let $b \ll \partial f(z)$. By Scott continuity of ∂f, there exists an open subset $O \subseteq U$, with $z \in O$, such that $\partial f(w) \subseteq b$ for $w \in O$. It follows that if the derivatives of V and W exist at (x, y) with $z = x + iy \in O$ then $V'(x, y) \odot W'(x, y) \subseteq b$. Let $\Omega \subseteq O$ be the null-set (with respect to the Lebesgue measure on $\mathbb{R}^2$) such that V or W fail to be differentiable on Ω. Since, by Rademacher's theorem, $\hat{f}$ is differentiable almost everywhere, Fubini's theorem in polar coordinates (with the origin at the point z) applied to the characteristic function of Ω implies that for almost all $\theta \in [0, 2\pi)$ (with respect to the one dimensional Lebesgue measure), the radial segment given in polar coordinates by $O \cap \{z + re^{i\theta} : r > 0\}$ meets Ω in a null-set. Let θ be such a value, and suppose $z_1 = z + re^{i\theta}$ for $r > 0$. We note that any Lipschitz map coincides with the indefinite Lebesgue integral of its derivative (which exists almost everywhere). Thus, by integrating along the line segment from z to z_1 we obtain:

$$\begin{aligned}\frac{f(z_1)-f(z)}{z_1-z} &= e^{-i\theta}\left(\frac{V(x+r\cos\theta, y+r\sin\theta) - V(x,y) + i[W(x+r\cos\theta, y+r\sin\theta) - W(x,y)]}{r}\right)\\ &= \frac{1}{r}\int_0^r T\big(V'(x+s\cos\theta, y+s\sin\theta), W'(x+s\cos\theta, y+s\sin\theta), \theta\big)\,ds\\ &\in b\end{aligned}$$

where the latter relation holds by Lemma 4.5. By continuity, it follows that, for fixed z, the above relation holds for all $z_1 \in O$ with $z_1 \neq z$. Since $b \ll \partial f(z)$ was arbitrary, the result follows. □

We note from the definitions that

$$\operatorname{conv} \bigcup \left\{ v \odot w : \begin{bmatrix} v \\ w \end{bmatrix} \in \partial \hat{f}(x, y) \right\} = \operatorname{conv} \bigcup \left\{ \mathsf{D}(\overline{v}, i\overline{w}) : \begin{bmatrix} v \\ w \end{bmatrix} \in \partial \hat{f}(x, y) \right\}. \tag{9}$$

Comparing Eq. (9) with the classical Cauchy–Riemann equations (6), we see that for a non-analytic map there exists a 2×2 real matrix $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in \partial \hat{f}(x, y)$ with $\overline{v} \neq i\overline{w}$, which will give rise to a nontrivial disk $\mathsf{D}(\overline{v}, i\overline{w})$ with a non-zero radius. Thus, for Lipschitz maps in general, the L-derivative is given by the convex hull of a union of disks rather than a single point of the complex plane. The following corollaries indicate that we have a generalisation of the classical result.

Corollary 4.8. *Suppose $f = V + iW : U \to \mathbb{C}$ and assume $V, W : U \to \mathbb{R}$ are continuously differentiable. Then $\partial \hat{f} = \partial V \times \partial W$, where*

$$\partial V = V', \qquad \partial W = W',$$

and for any $z \in U$, the value of the L-derivative $\mathcal{L}f(z)$ is the disk $\mathsf{D}(\overline{V'(x, y)}, i\,\overline{W'(x, y)})$.

Thus, the L-derivative of f at any point where the maps V and W are C^1 is a single disk, which is the simplest example of the L-derivative for a non-analytic map. We also obtain another proof for the classical Cauchy–Riemann equations.

Corollary 4.9. *Suppose $f = V + iW : U \to \mathbb{C}$ and assume $V, W : U \to \mathbb{R}$ are continuously differentiable. Then f is analytic in U iff V and W satisfy the classical Cauchy–Riemann equations.*

Proof. If f is analytic in U then $f' = \mathcal{L}f$ in U by Proposition 2.9(ii). Thus, $\mathcal{L}f$ has maximal values at each $z \in U$. Hence, by Corollary 4.8, for all $(x, y) \in U$, the values $\partial V(x, y)$ and $\partial W(x, y)$ are also maximal, i.e., points in $\mathbb{R}^2$, with $\partial V = V'$ and $\partial W = W'$, and we have: $\overline{V'(x, y)} = i\,\overline{W'(x, y)}$ for all $x + iy \in U$, i.e., $V'_1(x, y) - iV'_2(x, y) = W'_2(x, y) + iW'_1(x, y)$, i.e., $\overline{V'(x, y)} = i\overline{W'(x, y)}$, which are precisely the Cauchy–Riemann equations (5).

On the other hand, if the classical Cauchy–Riemann equations hold then, by Corollary 4.8 again, $\mathcal{L}f$ takes maximal values for each $z \in U$. Hence, by Proposition 2.9(iii), $f' = \mathcal{L}f$ exists at each point $z \in U$ and thus f is analytic. □

Since $\partial \hat{f}(x, y) \subset \partial V(x, y) \times \partial W(x, y)$, we always have: $\partial f \subset \operatorname{conv}(\partial V \odot \partial W)$, where as usual $z = x + iy$ and we have extended the map $\odot : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbf{C}(\mathbb{C})$ pointwise to $\odot : \mathbf{C}(\mathbb{C}) \times \mathbf{C}(\mathbb{C}) \to \mathbf{C}(\mathbb{C})$ by

$$C_1 \odot C_2 = \bigcup \{v \odot w : v \in C_1, w \in C_2\}.$$

It follows from the definition that if $\partial \hat{f}(x, y) = \partial V(x, y) \times \partial W(x, y)$, which, by Proposition 4.3, is certainly the case if at least one of V and W is maximally differentiable at (x, y), then we have: $\partial f(z) = \operatorname{conv}(\partial V(x, y) \odot \partial W(x, y))$.

We now present an example for which $\partial f \neq \operatorname{conv}(\partial V \odot \partial W)$.

Example 4.10. Consider $f = V + iW$ where the piecewise linear map $\hat{f} = \left[\begin{smallmatrix} V \\ W \end{smallmatrix}\right] : \mathbb{R}^2 \to \mathbb{R}^2$ is defined, for convenience in matrix notation, as follows:

$$\begin{bmatrix} V(x, y) \\ W(x, y) \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & x \geq 0, \\[1ex] \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & x \leq 0 \;\&\; y \geq |x|, \\[1ex] \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & x \leq 0 \;\&\; 0 \leq y \leq |x|, \\[1ex] \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & x \leq 0 \;\&\; 0 \geq y \geq x, \\[1ex] \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & x \leq 0 \;\&\; y \leq x. \end{cases}$$

This vector function has been used in a different context in [42, Example 6.4]. We calculate:

$$\partial \hat{f}(0, 0) = \operatorname{conv} \left\{ \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \right\}. \tag{10}$$

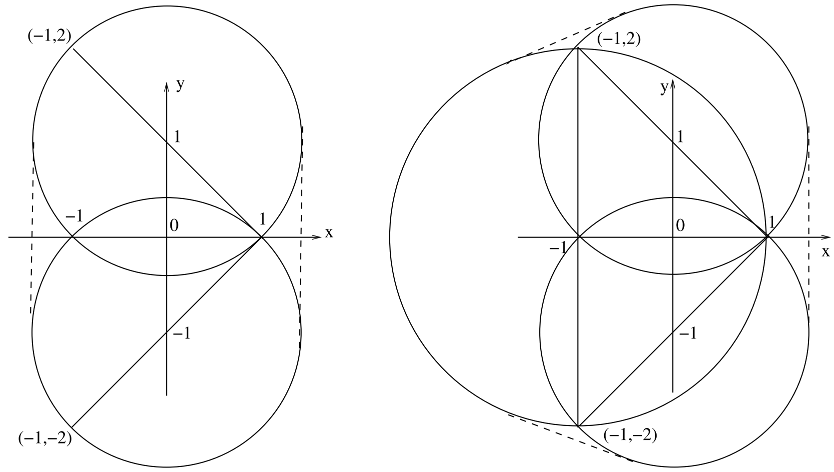

Fig. 2. The L-derivative $\mathcal{L}f(0)$ (left) is a proper subset of $\mathrm{conv}(\partial V(0,0) \odot \partial W(0,0))$ (right).

Note that the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is a convex combination of the two matrices $\left[\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 2 & -1 \end{smallmatrix}\right]$, and thus need not be included in set on the right hand side of Eq. (10). We also have:

$$\partial V(0,0) = \mathrm{conv}\big\{(1,0), (-1,-2), (-1,2)\big\}, \qquad \partial W(0,0) = \mathrm{conv}\big\{(0,1), (-2,-1), (2,-1)\big\}.$$

It is now clear that $\partial \hat{f}(0,0) \neq (\partial V(0,0)) \times (\partial W(0,0))$. Thus, $\partial V(0,0) \times \partial W(0,0)$ is generated as the convex hull of the following nine matrices in $\mathbf{C}(\mathbb{R}^{2\times 2})$:

$$\text{(i)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad \text{(iv)} \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix},$$

$$\text{(v)} \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}, \quad \text{(vi)} \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}, \quad \text{(vii)} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \text{(viii)} \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \quad \text{(ix)} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

Each of the matrices above has two row vectors v and w, say, which give rise to the disk $\mathsf{D}(\overline{v}, i\overline{w}) \subset (\partial V(0,0) \times \partial W(0,0))$. These disks therefore each have a diameter with the following (unordered) pairs of endpoints:

$$\text{(i)} \big\{(1,0), (1,0)\big\}; \quad \text{(ii)} \big\{(1,0), (-1,-2)\big\}; \quad \text{(iii)} \big\{(1,0), (-1,2)\big\}; \quad \text{(iv)} \big\{(-1,2), (1,0)\big\};$$

$$\text{(v)} \big\{(-1,2), (-1,-2)\big\}; \quad \text{(vi)} \big\{(-1,2), (-1,2)\big\}; \quad \text{(vii)} \big\{(-1,-2), (1,0)\big\}; \quad \text{(viii)} \big\{(-1,-2), (-1,-2)\big\};$$

$$\text{(ix)} \big\{(-1,-2), (-1,2)\big\}.$$

Note that (ii) and (vii) are the same pairs, similarly, (iii) and (iv) are the same as well as (v) and (ix). Moreover (i) and (viii) are each a single point. One checks readily that the union of the above disks is the union of the disks represented by the pairs (ii), (iii) and (v). Thus,

$$\partial V(0,0) \odot \partial W(0,0) = \mathrm{conv}\big\{\mathsf{D}\big((1,0), (-1,-2)\big) \cup \mathsf{D}\big((1,0), (-1,2)\big) \cup \mathsf{D}\big((-1,2), (-1,-2)\big)\big\}.$$

Similarly, $\partial \hat{f}(0,0)$ gives rise to four disks each with a diameter with the following pairs of endpoints:

$$\text{(i)} \big\{(1,0), (-1,-2)\big\}; \quad \text{(ii)} \big\{(-1,2), (1,0)\big\}; \quad \text{(iii)} \big\{(-1,-2), (1,0)\big\}; \quad \text{(iv)} \big\{(1,0), (-1,2)\big\}$$

Here, again (i) and (iii), as well as (ii) and (iv) represent the same disks. Thus,

$$\partial f(0) = \mathrm{conv}\big\{\mathsf{D}\big((1,0), (-1,-2)\big) \cup \mathsf{D}\big((1,0), (-1,2)\big)\big\},$$

and we see that $\mathcal{L}f(0) = \partial f(0) \subset \mathrm{conv}(\partial V(0,0) \odot \partial W(0,0))$ is a proper inclusion; see Fig. 2.

Example 4.11. Let $f : z \mapsto |z|^2$. Then, $V(x,y) = x^2 + y^2$ and $W = 0$. Here, V and W are continuously differentiable but the classical Cauchy–Riemann equations only hold at $z = 0$, where f is differentiable with $f' = 0$. We have:

$$\partial V(x,y) = V'(x,y) = 2(x,y), \qquad \partial W(x,y) = W'(x,y) = (0,0).$$

Thus, the extended Cauchy–Riemann equations gives

$$(\mathcal{L}f)(z) = \mathrm{D}(2\bar{z}, 0),$$

i.e., $(\mathcal{L}f)(z)$ is the disk centred at $\bar{z}$ and radius $|z|$. At $z = 0$ we have $(\mathcal{L}f)(0) = \{0\} = \{f'(0)\}$. Hence, f is maximally differentiable at 0 but not analytic there.

Example 4.12. Consider the conjugation map $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = \bar{z}$. Then $V(x, y) = x$ and $W(x, y) = -y$ are both continuously differentiable. At all points $(x, y) \in \mathbb{R}^2$, we have:

$$\partial V(x, y) = V'(x, y) = (1, 0), \qquad \partial W(x, y) = W'(x, y) = (0, -1).$$

Thus, for any $z \in \mathbb{C}$, the L-derivative is the unit disk centred at the origin: $\mathcal{L}f(z) = \mathrm{D}(1, -1)$.

Example 4.13. More generally, consider any anti-analytic map $f : U \to \mathbb{C}$. Recall that we can write $f(z) = h(\bar{z})$ where $h : \overline{U} \to \mathbb{C}$ is analytic, in other words $f = h \circ g$, where g is the conjugation map. Thus, by the chain rule and since analytic maps are maximally differentiable we obtain: $\mathcal{L}f(z) = h'(\bar{z})\mathcal{L}g(z) = h'(\bar{z})\mathrm{D}(1, -1) = \mathrm{D}(h'(\bar{z}), -h'(\bar{z}))$, (using Example 4.12), i.e., the disk centred at the origin and radius $|h'(\bar{z})|$.

Example 4.14. Consider the absolute value map $f : \mathbb{C} \to \mathbb{C}$ with $z \mapsto |z|$. Here, f is nowhere differentiable as the Cauchy–Riemann equations do not hold anywhere. In fact, $V(x, y) = \sqrt{x^2 + y^2}$ and $W(x, y) = 0$, so that, for $z = re^{i\theta} \neq 0$, we have $\frac{\partial V}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$ and $\frac{\partial V}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$, i.e., $V'(x, y) = e^{i\theta}$. At the origin, $\partial V(0)$ is the unit disk centred at 0. Thus, for $z \neq 0$, we have $\mathcal{L}f(z) = \mathrm{D}(\overline{V'(x, y)}, i\overline{W'(x, y)}) = \mathrm{D}(e^{-i\theta}, 0)$, i.e., the disk with the line segment from the origin to $e^{-i\theta}$ as a diameter. For $z = 0$, the L-derivative $\mathcal{L}f(0)$ is the union of disks each of which has a diameter with the origin as one endpoint and a point of the unit disk centred at the origin as the other endpoint. But this union is in fact the unit disk centred at the origin.

Example 4.15. Consider $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = z^2 \sin 1/z$ for $z \neq 0$ and $f(0) = 0$. Then f is differentiable at $z = 0$ with $f'(0) = 0$. In this example, f is differentiable but not maximally differentiable at 0, which can be easily checked. This is similar to the real-valued map $x \mapsto x^2 \sin 1/x : \mathbb{R} \to \mathbb{R}$ for $x \neq 0$, also defined to be 0 at 0.

5. Fundamental theorem of contour integration

As an application for the theory developed so far, we will present a generalisation of the celebrated fundamental theorem of contour integration [41, Theorem 6.7] to complex Lipschitz maps. A path in a connected region $R \subset \mathbb{R}^n$ is a continuous map $p : [a, b] \to R$ with endpoints $p(a)$ and $p(b)$. We say p is piecewise C^1, if p' exists and is continuous except for a finite number of points at which the left and right derivatives of p exist and are limits of p' from right and left respectively. A *contour* is a piecewise C^1 path. A contour is *piecewise linear* if it consists of a finite number of straight line segments. The space $P(U)$ of contours in the region $U \subset \mathbb{R}^n$ is equipped with the C^1 norm:

$$\|p\| = \max\left\{\max_{r\in[a,b]} \|p(r)\|, \max_{p'(r)\text{exists}} \|p'(r)\|\right\}. \tag{11}$$

Recall that the classical theorem states that if $U \subseteq \mathbb{C}$ is an open connected set and $f : U \to \mathbb{C}$ is a continuous function with $F' = f$ for some differentiable $F : U \to \mathbb{C}$, then for any contour, $p : [a, b] \to U$ from z_0 to z_1, we have $\int_p f = F(z_1) - F(z_0)$.

We will extend the fundamental theorem of contour integration to an integrable Scott continuous function $g : U \to \mathbf{C}(\mathbb{C})$. By Proposition 2.10, if g is integrable then there exists $h : U \to \mathbb{C}$, defined on each open connected component of the domain of definition of g in U, such that $g \sqsubseteq \mathcal{L}h$. For the extension theorem, we need to define the integral $\int_p g$ of the convex, compact, non-empty-valued function g along a contour $p : [a, b] \to O$, where O is an open connected component of the domain of definition of g. The integral $\int_p g$ will be a compact and convex subset of $\mathbb{C}$, equivalently $\mathbb{R}^2$, which we will define by specifying its support function.

Recall that a map $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *sublinear* if, (i) it is *positively homogeneous*, i.e., $f(\lambda v) = \lambda f(v)$ for all $\lambda \geq 0$ and all $v \in \mathbb{R}^n$, and, (ii) it is *subadditive*, i.e., $f(u + v) \leq f(u) + f(v)$ for all $u, v \in \mathbb{R}^n$ [38, p. 26]. The following fundamental result characterises support functions in terms of sublinear maps.

Proposition 5.1. *(See [38, p. 38].) A function $h : \mathbb{R}^n \to \mathbb{R}$ is the support function of a (necessarily unique) non-empty compact convex set if and only if h is sublinear.*

Let $S : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$S(v) = \int_a^b \sup\big(\big(g\big(p(t)\big)p'(t)\big)\cdot v\big)\,dt.$$

Note that $g(p(t))p'(t)$ is the pointwise extension of the product of the two complex numbers to the product of the set $g(p(t))$ of complex numbers and the complex number $p'(t)$, whereas the dot product is the pointwise extension of the dot product to that of the planar subset $g(p(t))p'(t)$ and the planar vector v.

Proposition 5.2. *The map S is well-defined and is the support function of a non-empty compact and convex subset of the plane.*

Proof. Since g is Scott continuous and p is piecewise C^1, it follows that, given any $v \in \mathbb{R}^2$, the map $f : t \mapsto \sup(g(p(t))p'(t)) \cdot v : [a, b] \to \mathbb{R}$ is upper-semicontinuous. As any upper semi-continuous functions attains its maximum on any compact set [28, Theorem 1.2], it follows that f is bounded above. On the other hand,

$$-f(t) = -\sup\big(\big(g\big(p(t)\big)p'(t)\big)\cdot v\big) = \inf\big(\big(g\big(p(t)\big)p'(t)\big)\cdot(-v)\big) \le \sup\big(\big(g\big(p(t)\big)p'(t)\big)\cdot(-v)\big),$$

and thus $-f$ is also bounded above. Hence, f is bounded as well as upper semi-continuous and is, therefore, Lebesgue integrable. This shows that S is well-defined. For each $t \in [a, b]$, the set $g(p(t))p'(t)$ is a non-empty, compact and convex set with support function $v \mapsto \sup((g(p(t))p'(t)) \cdot v) : \mathbb{R}^2 \to \mathbb{R}$, which is therefore a sublinear map by Proposition 5.1. Since integration, as a linear operator, preserves the properties of being positively homogeneous and subadditive, it follows that S is also sublinear and hence the support function of a non-empty compact convex set. □

Definition 5.3. The integral of a Scott continuous g with respect to the contour p, which is denoted by $\int_p g$, is the unique non-empty convex, compact set with support function $S : \mathbb{R}^2 \to \mathbb{R}$, as implied by Propositions 5.1 and 5.2. Using our notation for support functions of convex sets we write $S = S_{\int_p g}$.

From the above definition, it follows that for any contour $p : [a, b] \to U$ and any point c with $a \le c \le b$ we have:

$$\int_p g = \int_{p\restriction[a,c]} g + \int_{p\restriction[c,b]} g.$$

Now consider the map

$$\int_p : \big(U \to \mathbf{C}(\mathbb{C})\big) \to \mathbf{C}(\mathbb{C}),$$

where we let $\int_p g = \bot$ if $\mathrm{Im}(p) \subseteq U$ does not hold.

Proposition 5.4. *For any contour p, the map $\int_p : (U \to \mathbf{C}(\mathbb{C})) \to \mathbf{C}(\mathbb{C})$ is Scott continuous.*

Proof. Monotonicity follows immediately from the definition and preservation of lubs is a consequence of the monotone convergence theorem. □

Lemma 5.5. *Let $g \in (U \to \mathbf{C}(\mathbb{C}))$ be a Scott continuous map and suppose $(p_n)_{n\ge0}$ is a sequence of contours with $p_n \to p$ as $n \to \infty$ in the C^1 norm. Then $\int_{p_n} g \to \int_p g$ as $n \to \infty$.*

Proof. The sequence of bounded upper semi-continuous maps $t \mapsto \sup((g(p_n(t))p_n'(t)) \cdot v) : [a, b] \to \mathbb{R}$ converges pointwise to $t \mapsto (\sup(g(p(t))p'(t)) \cdot v) : [a, b] \to \mathbb{R}$. The result follows from the dominated convergence theorem. □

Theorem 5.6 *(Fundamental theorem of contour integration for Lipschitz maps). Suppose $h \in \int g$ is a primitive for the Scott continuous function g. Then for any contour $p : [a, b] \to O$, from $z = p(a)$ to $w = p(b)$, where O is an open connected component of the domain of definition of g, we have*

$$h(w) - h(z) \in \int_p g.$$

Proof. By Lemma 5.5, it suffices to prove the result for any piecewise linear contour $p : [a, b] \to O$. Assume that the end points of the line segments in p are at points $a = t_0 < t_1 <, \ldots, t_i < t_{i+1}, \ldots, t_k = b$. Since h is a primitive of g, it follows, by Proposition 2.10, that $g \sqsubseteq \mathcal{L}h$ with $h(z) = V(x, y) + iW(x, y)$ with $z = x + iy$.

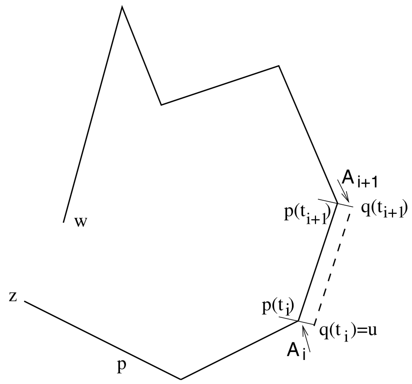

Fig. 3. A piecewise linear contour p from z to w and its parallel transport q in $[t_i, t_{i+1}]$ on the transversal section A_i.

Then V and W are Lipschitz on O and, by Rademacher's theorem, they are differentiable almost everywhere in O with respect to the two dimensional Lebesgue measure. Suppose $\Omega \subseteq O$ is the null-set on which V or W fail to be differentiable. We fix i with $0 \le i \le k$, take a small transversal (Poincaré) section A_i to the line segment from $p(t_i)$ to $p(t_{i+1})$ and consider the parallel transport of p along each point on A_i as follows. See Fig. 3.

Now for each $u \in A_i$ consider the linear contour with one single segment $p_u : [t_i, t_{i+1}] \to O$, where $p_u(t_i) = u$, such that the line segment $p_u[t_i, t_{i+1}]$ is parallel to $p[t_i, t_{i+1}]$, i.e., $p'_u(t) = p'(t)$ for $t_i < t < t_{i+1}$.

By applying Fubini's theorem to the integral of χ_Ω (the indicator function of Ω) with respect to the two dimensional Lebesgue measure over the rectangle bounded between line segments A_i and A_{i+1}, it follows that for almost all $u \in A_i$ with respect to the one dimensional Lebesgue measure on A_i, the intersection of $\Omega \cap p_u([t_i, t_{i+1}])$ has zero one dimensional Lebesgue measure. It follows that for almost all $u \in A_i$, with respect to the one dimensional Lebesgue measure on A_i, the maps V and W are differentiable on the path p_u almost everywhere with respect to the one dimensional Lebesgue measure on this path. Let u be such a point and, to simplify the notation, put $q = p_u$. Then for almost all $t \in [t_i, t_{i+1}]$, both V and W are differentiable on the linear path $q : [t_i, t_{i+1}] \to O$. In the following, let $t \in [t_i, t_{i+1}]$ be such that V and W are both differentiable at $q(t)$, i.e., $V'(q(t))$ and $W'(q(t))$ both exist. We have

$$\frac{d\,h(q(t))}{dt} = q'_1(t)V'_1\big(q(t)\big) + q'_2(t)V'_2\big(q(t)\big) + i\big(q'_1(t)W'_1\big(q(t)\big) + q'_2(t)W'_2\big(q(t)\big)\big). \tag{12}$$

Using $g(q(t)) \sqsubseteq \mathcal{L}h(q(t))$, it follows that

$$\overline{V'\big(q(t)\big)} = V'_1\big(q(t)\big) - iV'_2\big(q(t)\big) \in \mathsf{D}\big(\overline{V'\big(q(t)\big)}, i\overline{W'\big(q(t)\big)}\big) \subseteq \mathcal{L}h\big(q(t)\big) \subseteq g\big(q(t)\big),$$

and similarly,

$$i\overline{W'\big(q(t)\big)} = W'_2\big(q(t)\big) + iW'_1\big(q(t)\big) \in \mathsf{D}\big(\overline{V'\big(q(t)\big)}, i\overline{W'\big(q(t)\big)}\big) \subseteq \mathcal{L}h\big(q(t)\big) \subseteq g\big(q(t)\big).$$

We also compute the following products in the complex plane:

$$a(t) + ib(t) := q'(t)\overline{V'\big(q(t)\big)} = q'_1(t)V'_1\big(q(t)\big) + q'_2(t)V'_2\big(q(t)\big) - i\big(q'_1(t)V'_2\big(q(t)\big) - q'_2(t)V'_1\big(q(t)\big)\big)$$

$$c(t) + id(t) := q'(t)i\overline{W'\big(q(t)\big)} = -q'_2(t)W'_1\big(q\big((t)\big) + q'_1(t)W'_2\big(q(t)\big) + i\big(q'_1(t)W'_1\big(q(t)\big) + q'_2(t)W'_2\big(q(t)\big)\big).$$

From $\mathsf{D}(\overline{V'(q(t))}, i\overline{W'(q(t))}) \subseteq g(q(t))$, by pointwise complex multiplication, we obtain: $q'(t)\mathsf{D}(\overline{V'(q(t))}, i\overline{W'(q(t))}) \subseteq q'(t)g(q(t))$. Also, multiplication by a complex number amounts to a rotation and scaling, thus it sends a disk to another disk, and we obtain:

$$q'(t)\mathsf{D}\big(\overline{V'\big(q(t)\big)}, i\overline{W'\big(q(t)\big)}\big) = \mathsf{D}\big(q'(t)\overline{V'\big(q(t)\big)}, iq'(t)\overline{W'\big(q(t)\big)}\big).$$

Since the two points $a(t) + ib(t), c(t) + id(t) \in \mathsf{D}(q'(t)\overline{V'(q(t))}, iq'(t)\overline{W'(q(t))})$ are the two ends of a diameter of the transformed disk, we deduce that $\frac{d\,h(q(t))}{dt} = a(t) + id(t) \in \mathsf{D}(q'(t)\overline{V'(q(t))}, iq'(t)\overline{W'(q(t))})$; see Fig. 4.

Thus, we obtain:

$$\frac{d\,h(q(t))}{dt} = q'_1(t)V'_1\big(q(t)\big) + q'_2(t)V'_2\big(q(t)\big) + i\big(q'_1(t)W'_1\big(q(t)\big) + q'_2(t)W'_2\big(q(t)\big)\big) \in g\big(q(t)\big)q'(t). \tag{13}$$

Moving to the real plane $\mathbb{R}^2$, we take the scalar product of both sides of Relation (13) with any vector $v \in \mathbb{R}^2$, then take supremum and integrate along the path q from $t = t_i$ to $t = t_{i+1}$, to obtain:

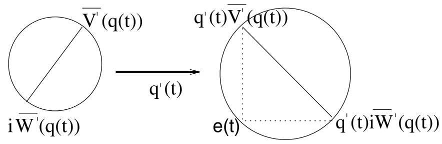

Fig. 4. The disk $\mathsf{D}(\overline{V'(q(t))}, i\overline{W'(q(t))})$ (left) multiplied by $q'(t)$ (right) contains the point $e(t) := \frac{d\,h(q(t))}{dt}$.

$$\big(h\big(q(t_i)\big) - h\big(q(t_{i+1})\big)\big) \cdot v \le \int_{t_i}^{t_{i+1}} \sup\big(\big(g\big(q(t)\big)q'(t)\big) \cdot v\big)\,dt = (S_{\int_{q\restriction[t_i,t_{i+1}]} g}) \cdot v, \tag{14}$$

where we have used Definition 5.3 to write the equality. Since Eq. (14) holds for almost all $u \in A_i$, it follows by continuity and taking the limit as $u \to p(t_i)$ from either side of the path p that

$$\big(h\big(p(t_{i+1})\big) - h\big(p(t_i)\big)\big) \cdot v \le \int_{t_i}^{t_{i+1}} \sup\big(\big(g\big(p(t)\big)p'(t)\big) \cdot v\big)\,dt = (S_{\int_{p\restriction[t_i,t_{i+1}]} g}) \cdot v. \tag{15}$$

Summing the contributions of Eq. (15) for $i = 0, \dots, k$,

$$\big(h(w) - h(z)\big) \cdot v = \sum_{i=0}^{k-1} \big(h(t_{i+1}) - h(t_i)\big) \cdot v \le \sum_{i=0}^{k-1} (S_{\int_{p\restriction[t_i,t_{i+1}]} g}) \cdot v = (S_{\int_{p\restriction[a,b]} g}) \cdot v$$

and it follows that $S_{h(w)-h(z)} \le S_{\int_p g}$, i.e., the support function of $h(w) - h(z)$ is bounded by the support function of $\int_p g$. This implies $h(w) - h(z) \in \int_p g$, as required. □

Corollary 5.7. *Suppose $h \in \int g$ is a primitive for a Scott continuous $g : U \to \mathbf{C}(\mathbb{C})$. Then for any closed contour $p : [a, b] \to O$, with $p(a) = p(b)$, where O is an open connected component of the domain of definition of g, we have*

$$0 \in \int_p g.$$

6. Conclusion and further work

We have defined a simple notion of L-derivative for complex Lipschitz maps, which is given at any point by the intersection of a shrinking sequence of non-empty convex compact subsets of the complex plane. It extends the notion of complex differentiation, fundamental theorem of calculus and fundamental theorem of contour integration to complex Lipschitz maps in a domain-theoretic framework which allows an effective formulation. This is therefore the first step in a domain-theoretic context to bring complex analysis within the sphere of type theory and recursion theory.

A main task for future work would be to develop a domain for complex Lipschitz maps which can represent and approximate any such map and its L-derivative. This would entail the construction of domains for real vector valued functions of several real variables which would extend the results in [22] for multivariable differential calculus. We also need to investigate the question of decidability of the existence of a primitive for rational step functions to obtain an effective version of Theorem 5.6. On this basis, we ask if one can develop denotational semantics for programming languages with a proper complex function data type as done in the case of real functions in [11]. We can also ask, as it was done for the case of real maps in [17]: what is the weakest topology induced on the set of all complex Lipschitz maps by the L-derivative operator? Finally, one can investigate the properties of the complex L-derivative with respect to complex Lipschitz manifolds in differential geometry similar to the application of the Clarke gradient to real Lipschitz manifolds [36].

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