Asymptotics for a determinant with a confluent hypergeometric kernel

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Abstract. We obtain "large gap" asymptotics for a Fredholm determinant with a confluent hypergeometric kernel. We also obtain asymptotics for determinants with two types of Bessel kernels which appeared in random matrix theory.

1 Introduction

Let $K^{(\alpha,\beta)}$ be the operator acting on $L^2(-s,s)$, $s>0$, with kernel

$$K^{(\alpha,\beta)}(u,v)=\frac{1}{2\pi i}\frac{\Gamma(1+\alpha+\beta)\Gamma(1+\alpha-\beta)}{\Gamma(1+2\alpha)^2}\frac{A(u)B(v)-A(v)B(u)}{u-v}, \tag{1}$$

where

$$A(x)=g_\beta^{1/2}(x)|2x|^\alpha e^{-ix}\phi(1+\alpha+\beta,1+2\alpha,2ix),$$
$$B(x)=g_\beta^{1/2}(x)|2x|^\alpha e^{ix}\phi(1+\alpha-\beta,1+2\alpha,-2ix),$$
$$g_\beta(x)=\begin{cases}e^{-\pi i\beta}, & x\geq 0,\\ e^{\pi i\beta}, & x<0.\end{cases}, \qquad \alpha,\beta\in\mathbb{C}, \quad \Re\alpha>-1/2, \quad \alpha\pm\beta\neq -1,-2,\dots$$

Here $\Gamma(x)$ is Euler's Γ-function, and $\phi(a,c,z)$ is the confluent hypergeometric function (see, e.g., [1])

$$\phi(a,c,z)=1+\sum_{n=1}^{\infty}\frac{a(a+1)\cdots(a+n-1)}{c(c+1)\cdots(c+n-1)}\frac{z^n}{n!}. \tag{2}$$

Using the standard recurrence formulae for $\phi(a,c,z)$ (see (36) below), we can rewrite (1) in another form:[3]

$$\begin{aligned}K^{(\alpha,\beta)}(u,v)=\frac{1}{\pi}\frac{\Gamma(1+\alpha+\beta)\Gamma(1+\alpha-\beta)}{(1+2\alpha)\Gamma(1+2\alpha)^2}g_\beta^{1/2}(u)g_\beta^{1/2}(v)e^{-i(u+v)}\frac{4^\alpha|uv|^\alpha}{u-v}\\ \times[u\phi(1+\alpha+\beta,2+2\alpha,2iu)\phi(\alpha+\beta,2\alpha,2iv)-v\phi(1+\alpha+\beta,2+2\alpha,2iv)\phi(\alpha+\beta,2\alpha,2iu)].\end{aligned} \tag{3}$$

The kernel (1) or (3) is called the confluent hypergeometric kernel. For $\alpha\in\mathbb{R}$, $\beta\in i\mathbb{R}$ (in this case the kernel is real, which is easy to see from (1)), it was considered by Borodin and Olshanski

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[3]The case $\alpha=0$ is understood here as a limit $\alpha\to 0$.

in [5], and by Borodin and Deift [4] (Proposition 8.13). This kernel arises in several different, but related, contexts:

First, following [5], consider the space H of infinite Hermitian matrices $(H_{jk})_{j,k=1}^{\infty}$. The $U(\infty)$, the inductive limit of the unitary groups $U(N)$, $N \to \infty$, acts on H by conjugations. A probability Borel measure on H which is invariant under the action of $U(\infty)$ is called ergodic, if any invariant mod 0 set has measure 0 or 1. Consider the space Ω whose elements consist of 2 infinite sequences,

$$\alpha_1^+ \geq \alpha_2^+ \geq \cdots \geq 0, \qquad \alpha_1^- \geq \alpha_2^- \geq \cdots \geq 0, \qquad \text{where} \qquad \sum_{j=1}^{\infty}(\alpha_j^+)^2 + \sum_{j=1}^{\infty}(\alpha_j^-)^2 < \infty,$$

together with 2 extra real parameters γ_1, γ_2, where $\gamma_2 \geq 0$. It turns out that the elements of Ω parametrize the ergodic measures on H. Furthermore, it can be proved that any $U(\infty)$-invariant probability measure on H decomposes on ergodic components, i.e., it can be written as a continuous convex combination of ergodic measures. This *spectral decomposition* is unique and is determined by a probability measure on Ω which is called the *spectral measure* of the original invariant measure.

The space Ω maps to the space $\mathrm{Conf}(\mathbb{R}^*)$ of point configurations on the punctured real line $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ in the following way:

$$(\{\alpha_j^+\}_{j=1,2,\ldots}, \{\alpha_j^-\}_{j=1,2,\ldots}, \gamma_1, \gamma_2) \to (-\alpha_1^-, -\alpha_2^-, \ldots, \alpha_2^+, \alpha_1^+),$$

where possible zeros among $\alpha_j^{\pm}$ are omitted. Under this map spectral measures corresponding to invariant measures on H, push-forwards to measures on $\mathrm{Conf}(\mathbb{R}^*)$, give rise in this way to random particle systems on $\mathbb{R}^*$.

In [5], the authors considered a particular class of $U(\infty)$-invariant measures on H, the Hua-Pickrell measures, and showed that the push-forwards of these measures to $\mathrm{Conf}(\mathbb{R}^*)$ give rise to random particle systems on $\mathbb{R}^*$ which are determinantal with correlation kernels given by $K^{(\alpha,\beta)}(1/u, 1/v)/(uv)$ (see (1)) with parameters $\alpha, i\beta \in \mathbb{R}$.

The kernel (1) is also (see [4]) a particular scaling limit of a kernel $K^{\alpha,\beta,\gamma}(u,v)$ which has a similar structure but with the confluent hypergeometric functions replaced by the hypergeometric functions ${}_2F_1(\alpha,\beta,\gamma;z)$. The kernel $K^{\alpha,\beta,\gamma}(u,v)$ is the correlation kernel for a particle system that arises in the theory of representations of $U(\infty)$: role of the ergodic measures is now played by the indecomposable characters of $U(\infty)$ which are again parametrized by certain sequences together with some extra parameters (see (1.4) in [4]).

The kernel (1) also arises as the correlation kernel for a particle system in a similar way to $K^{\alpha,\beta,\gamma}(u,v)$ above, but in place of irreducible representations of $U(\infty)$, we now consider irreducible (spherical) representations of $U(\infty) \ltimes H(\infty) = \lim_{\to} G(N)$, the inductive limit of the semidirect product $G(N) = U(N) \ltimes H(N)$, where $U(N)$ is the group of $N \times N$ unitary matrices and $H(N)$ denotes $N \times N$ Hermitian matrices: see [5] and references therein.

As we will see in Section 2, the kernel (1) can be obtained as a scaling limit in unitary random matrix ensembles generated by the weight function $f(z,0)$ on the unit circle (given by (20) below) at a point of so-called Fisher-Hartwig singularity. This singularity combines a root-type and a jump-type singularity characterized by the parameters α and β, respectively.

In particular cases, the kernel (1) reduces to a Bessel- and the sine-kernel which attracted much attention mostly because of their interest for random matrices. If $\beta = 0$ the confluent hypergeometric function reduces to Bessel functions (see, e.g., [1]):

$$\phi(\mu, 2\mu, 2ix) = \Gamma\left(\mu + \frac{1}{2}\right) e^{ix} \left(\frac{x}{2}\right)^{-\mu+\frac{1}{2}} J_{\mu-\frac{1}{2}}(x). \tag{4}$$

Therefore, we obtain from (3)

$$K^{(\alpha,0)}(u,v) \equiv K^{(\alpha)}_{Bessel1}(u,v) = \frac{|u|^\alpha |v|^\alpha}{u^\alpha v^\alpha} \frac{\sqrt{uv}}{2} \frac{J_{\alpha+\frac{1}{2}}(u)J_{\alpha-\frac{1}{2}}(v) - J_{\alpha+\frac{1}{2}}(v)J_{\alpha-\frac{1}{2}}(u)}{u-v}. \tag{5}$$

This kernel appeared in [2],[22],[25],[29].

If $\alpha = 0$, $\beta = 0$, then (5) reduces to the sine kernel

$$K^{(0,0)}(x,y) \equiv K_{\sin}(x,y) = \frac{\sin(x-y)}{\pi(x-y)}, \tag{6}$$

the most ubiquitous object of random matrix theory.

Note that the operator $K^{(\alpha,\beta)}$ is trace class (see Appendix), and consider the Fredholm determinant

$$\det(I - K^{(\alpha,\beta)})_{L^2(-s,s)}. \tag{7}$$

Because of the mentioned interpretation of $K^{(\alpha,\beta)}$ with $\alpha, i\beta \in \mathbb{R}$ as the correlation kernel for a particle system produced by a Hua-Pickrell measure, it is easy to see that the Fredholm determinant (7) is the probability that all the $\alpha_j^{\pm}$ are less than $1/s$.

By a random matrix interpretation of the kernel (1), the determinant (7) with $\alpha, i\beta \in \mathbb{R}$ gives the probability, in the bulk scaling limit, that the interval $(-s,s)$ with a Fisher-Hartwig singularity at the center contains no eigenvalues of corresponding unitary random matrix ensembles.

As noticed in [29, 4], the determinant $\det(I - K^{(\alpha,\beta)})_{L^2(0,s)}$ is related to a solution to the Painlevé V equation.[4]

In this paper, we obtain the asymptotics of the Fredholm determinant (7) for large s, i.e., the large gap asymptotics. Our main result is the following.

Theorem 1 *Let $K^{(\alpha,\beta)}$ be the operator with kernel (1) acting on $L^2(-s,s)$, Then, as $s \to +\infty$,*

$$\det(I - K^{(\alpha,\beta)})_{L^2(-s,s)} = \frac{\sqrt{\pi} G^2(1/2) G(1+2\alpha)}{2^{2\alpha^2} G(1+\alpha+\beta) G(1+\alpha-\beta)} s^{-\frac{1}{4}-\alpha^2+\beta^2} e^{-\frac{s^2}{2}+2\alpha s} \left[1 + O\left(\frac{1}{s}\right)\right], \tag{8}$$

where $G(x)$ is Barnes' G-function. This expansion is uniform in compact subsets of the α-half-plane $\Re\alpha > -1/2$ and of the β-plane outside neighborhoods of the points $\alpha \pm \beta = -1, -2, \ldots$.

Remark 2 Setting $\beta = 0$ in (8) and using a doubling formula for the G-function, we obtain the large s-asymptotics for the determinant with kernel (5) where $\Re\alpha > -1/2$:

$$\det(I - K^{(\alpha)}_{Bessel1}) = \frac{1}{(2\pi)^\alpha} G(\alpha + 1/2) G(\alpha + 3/2) s^{-\frac{1}{4}-\alpha^2} e^{-\frac{s^2}{2}+2\alpha s} \left[1 + O\left(\frac{1}{s}\right)\right]. \tag{9}$$

[4]More generally, an ${}_2F_1$-kernel determinant is expressed [4] in terms of a solution to the Painlevé VI equation. For some asymptotic results which use this connection and conjectures see [23].

Setting $\alpha = \beta = 0$ in (8) and using the property $2 \ln G(1/2) = (1/12) \ln 2 - \ln \sqrt{\pi} + 3\zeta'(-1)$, where $\zeta(x)$ is Riemann's zeta-function, we reproduce the result for the sine-kernel determinant:

$$\ln \det(I - K_{\sin}) = -\frac{s^2}{2} - \frac{1}{4} \ln s + \frac{1}{12} \ln 2 + 3\zeta'(-1) + O\left(\frac{1}{s}\right), \quad s \to \infty. \tag{10}$$

The first two terms in (10) were first found by des Cloizeaux and Mehta [6], and the full expansion by Dyson [14]. The calculations in [6, 14] were not fully rigorous. A proof for the first leading term was carried out by Widom [27]. The full asymptotics of the logarithmic derivative $(d/ds) \ln \det(I - K_{\sin})$ were proved by Deift, Its and Zhou in [9]. Finally, the constant term in (10) was proved in [15], [20], [11].

Remark 3 In the present paper, we address only the symmetric case of $L^2(a, b)$ such that $b = -a = s > 0$. However, one can apply our methods to consider non-symmetric cases as well.

In unitary random matrix ensembles at a hard edge of the spectrum (e.g., Jacobi at the edges or Laguerre at zero) local correlations between eigenvalues are expressed in terms of the following Bessel kernel first considered by Forrester [19] (in an equivalent form):

$$K^{(a)}_{Bessel2}(x, y) = \frac{\sqrt{y} J'_a(\sqrt{y}) J_a(\sqrt{x}) - \sqrt{x} J'_a(\sqrt{x}) J_a(\sqrt{y})}{2(x - y)}. \tag{11}$$

In particular, the distribution of the extreme eigenvalue is given in the scaling limit by the Fredholm determinant $\det(I - K^{(a)}_{Bessel2})$, where $K^{(a)}_{Bessel2}$ is the trace-class operator on $L^2(0, s)$, $s > 0$, with kernel (11). In Section 7, we prove the following asymptotic behavior of this determinant.

Theorem 4. *As $s \to +\infty$, we have uniformly in compact subsets of the half-plane $\Re a > -1$:*

$$\det(I - K^{(a)}_{Bessel2})_{L^2(0,s)} = \tau_a s^{-a^2/4} e^{-s/4 + a\sqrt{s}} \left(1 + O(s^{-1/2})\right), \qquad \Re a > -1, \tag{12}$$

where

$$\tau_a = \frac{G(1 + a)}{(2\pi)^{a/2}}. \tag{13}$$

In [26], Tracy and Widom showed that the logarithmic derivative $(d/ds) \ln \det(I - K^{(a)}_{Bessel2})$ is expressed in terms of a solution to Painlevé V equation and used this fact to give a heuristic derivation of (12) with some constant τ_a. Tracy and Widom also conjectured the value of τ_a given in (13) using numerical calculations and comparison with the Dyson asymptotics for the sine-kernel determinants. (In fact, for $a = \mp 1/2$, the Bessel kernel (11) reduces to sine-kernels appearing in orthogonal and symplectic ensembles of random matrices. The sine-kernel (6) appears in unitary ensembles.) Very recently, a proof of the asymptotics (12,13) for the range of the parameter $|\Re a| < 1$ was given by Ehrhardt [16] using operator theory methods.

To prove Theorem 1, we use the approach of [20], [11], [12], where the asymptotics were computed, including the constant terms, of the sine-kernel and the Airy-kernel determinants.

First, in Section 2, using results from [10] we express (see Lemma 6) the Fredholm determinant (7) as a scaling limit of Toeplitz determinants $D_n(\varphi)$ with certain symbols $f(e^{i\theta})$ supported on an arc of the unit circle $\varphi \le \theta < 2\pi - \varphi$ with $\varphi = 2s/n$, $n > s$. The continuation of these symbols into the complex plane has a Fisher-Hartwig singularity at $z = 1$. Theorem 1 then reduces to an asymptotic evaluation of such Toeplitz determinants for large s.

In Section 3 we derive a differential identity (49) for the logarithmic derivative $(d^2/d\varphi^2)\ln D_n(\varphi)$ at $0 \le \varphi < \pi$ in terms of the solution to an associated Riemann-Hilbert problem (in fact, in terms of the associated orthogonal polynomials which are given by this solution). In section 4, we obtain the series expansion of $D_n(\varphi)$ for φ close to π. In Section 5, we solve the Riemann-Hilbert problem asymptotically and thus obtain the asymptotic expression for the r.h.s. of the differential identity (49), namely, we obtain the identity (139) uniformly for $2s/n < \varphi < \pi$, $n > s$, $s > s_0$, with some (large) $s_0 > 0$. Integration of the latter identity w.r.t. φ, using the boundary condition of Section 4, gives the asymptotics of the determinants $D_n(\varphi)$ for any arc with $2s/n < \varphi < \pi$, $n > s$, $s > s_0$, with some $s_0 > 0$, which is sufficient to prove Theorem 1.

In Section 7, we represent the Fredholm determinant with the Bessel kernel (11) as a scaling limit of Hankel determinants related, via a general connection formula of Theorem 2.6. of [10], to the particular case of $D_n(\varphi)$ with $\beta = 0$. The connection formula also involves the polynomials orthogonal w.r.t. $f(z)$ on the circular arc which are represented by matrix elements of the solution to the Riemann-Hilbert problem mentioned above. We prove Theorem 4 by using asymptotic results on these polynomials and on $D_n(\varphi)$ from the previous section as well as an expansion for singular Hankel determinants from [10].

2 Connection with Toeplitz determinants

The aim of this section is to derive an expression for (7) in terms of Toeplitz determinants (Lemma 6 below) and to fix notation for the rest of the paper.

Let E_φ be an arc of the unit circle C oriented counterclockwise:

$$E_\varphi = \{e^{i\theta}, \varphi \le \theta \le 2\pi - \varphi\}, \qquad 0 \le \varphi < \pi. \tag{14}$$

Consider the following function $f(z,\varphi)$ on E_φ:

$$f(z,\varphi) = |z-1|^{2\alpha} z^{\beta} e^{-i\pi\beta}, \qquad z = e^{i\theta} \in E_\varphi \quad \alpha, \beta \in \mathbb{C}, \quad \Re\alpha > -\frac{1}{2}. \tag{15}$$

Note that for $z \in C$

$$|z-1|^{2\alpha} = \frac{(z-1)^{2\alpha}}{z^{\alpha} e^{i\pi\alpha}}, \tag{16}$$

where the cut of $(z-1)^{2\alpha}$ is along $[1,\infty)$, and $0 < \arg(z-1) < 2\pi$. The branches of z^α, z^β are chosen so that $0 < \arg z < 2\pi$. Therefore, we can extend the function $f(z)$ to the complex plane with the cut $[0,\infty)$ by the expression:

$$f(z) = z^{-\alpha+\beta}(z-1)^{2\alpha} e^{-i\pi(\alpha+\beta)} \qquad z \in \mathbb{C} \setminus [0,\infty). \tag{17}$$

Related to the function (15) is a system of orthogonal polynomials $p_k(z;\varphi) = \chi_k(\varphi) z^k + \ldots$, $\widehat{p}_k(z;\varphi) = \chi_k(\varphi) z^k + \ldots$ of degree $k = 0, 1, \ldots$, satisfying

$$\frac{1}{2\pi}\int_{\varphi}^{2\pi-\varphi} p_k(z) z^{-m} f(z) d\theta = \chi_m^{-1}\delta_{km}, \qquad \frac{1}{2\pi}\int_{\varphi}^{2\pi-\varphi} \widehat{p}_k(z^{-1}) z^{m} f(z) d\theta = \chi_m^{-1}\delta_{km}, \tag{18}$$

$$z = e^{i\theta}, \quad m = 0, 1, \dots, k.$$

Note that if the weight function $f(z)$ is not positive on E_φ, the existence of such a system of polynomials is not a priori clear and will be addressed in the situations needed below.

In order to obtain the kernel (1) in a scaling limit, we will need to know the asymptotics of the polynomials

$$q_n(z) \equiv p_n(z; 0), \qquad \widehat{q}_n(z) \equiv \widehat{p}_n(z; 0), \tag{19}$$

corresponding to the weight

$$f(z, 0) = |z-1|^{2\alpha} z^\beta e^{-i\pi\beta}, \qquad z = e^{i\theta}, \qquad 0 \le \theta < 2\pi. \tag{20}$$

In this case the function $f(z)$ possesses a Fisher-Hartwig singularity at the point $z = 1$. The asymptotics of these and more general polynomials were recently analyzed in [10] (in particular, the polynomials exist for sufficiently large degrees) and from those results we obtain the behavior of the polynomials in a neighborhood of the singular point.

Lemma 5 *Let $0 < \varepsilon < 1$, $U_0 = \{z, |z-1| < \varepsilon\}$. Fix the branch of $\ln z = \ln|z| + i\arg z$ by the condition $-\pi < \arg z < \pi$, and the branches of the power functions $w^a = |w|^a \exp\{ia\arg w\}$ by the condition $0 < \arg w < 2\pi$. Then, as $n \to \infty$, $z \in U_0$,*

$$\begin{aligned} q_n(z) = &\left\{ \begin{array}{cc} 1, & z \in \mathbb{C}_+ \cap U_0 \\ e^{-2\pi i(\alpha-\beta)}, & z \in \mathbb{C}_- \cap U_0 \end{array} \right\} (n\ln z)^{\alpha-\beta}(z-1)^{-\alpha+\beta} z^{\alpha-\beta} \\ &\times \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+2\alpha)} \phi(1+\alpha+\beta, 1+2\alpha, n\ln z)\left[1 + O\left(\frac{1}{n}\right)\right], \end{aligned} \tag{21}$$

$$\widehat{q}_n(z^{-1}) = (n\ln z)^{\alpha+\beta}(z-1)^{-(\alpha+\beta)} \frac{\Gamma(1+\alpha-\beta)}{\Gamma(1+2\alpha)} \phi(1+\alpha-\beta, 1+2\alpha, -n\ln z)\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{22}$$

These asymptotics are uniform and differentiable for $z \in U_0$. They are also uniform in compact subsets of the α-half-plane $\Re\alpha > -1/2$ and of the β-plane outside neighborhoods of the points $\alpha \pm \beta = -1, -2, \dots$.

Remark It is easy to check that the singularities in (21), (22) cancel.

Proof. Our polynomials correspond to the special case of [10] with only one singularity located at $z = 1$. Let U_0 be the neighborhood of 1 where the parametrix was constructed in [10] in terms of the confluent hypergeometric function. Let

$$\zeta = n\ln z.$$

Take $\zeta \in I$, where I is the first sector of the image of the neighborhood under the conformal transformation $\zeta = n\ln z$ (see Figure 2 of [10]). Tracing back the Riemann-Hilbert transformations of [10], it is straightforward to obtain

$$\begin{pmatrix} \chi_n^{-1} q_n(z) \\ -\chi_{n-1} z^{n-1} \widehat{q}_{n-1}(z^{-1}) \end{pmatrix} = \left(I + n^{-\Re\beta\sigma_3} O\left(\frac{1}{n}\right) n^{\Re\beta\sigma_3}\right) \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \qquad n \to \infty \tag{23}$$

where

$$\begin{aligned} &Q_1(z) = \zeta^{-\alpha-\beta}(z-1)^{-\alpha+\beta} z^{\alpha-\beta} e^{i\pi(\alpha+\beta)} \Psi_1, \\ &\Psi_1 = -\psi(1-\alpha+\beta, 1-2\alpha, \zeta) \frac{\Gamma(1+\alpha+\beta)}{\Gamma(\alpha-\beta)} + \psi(-\alpha-\beta, 1-2\alpha, e^{-i\pi}\zeta) z^n, \end{aligned} \tag{24}$$

$$
\begin{aligned}
&Q_2 = -\zeta^{\alpha+\beta}(z-1)^{-\alpha-\beta}e^{i\pi(\alpha+\beta)}\Psi_2,\\
&\Psi_2 = \psi(\alpha+\beta, 1+2\alpha, \zeta) - \psi(1+\alpha-\beta, 1+2\alpha, e^{-i\pi}\zeta)\frac{\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+\beta)}e^{-2i\pi\alpha}z^n.
\end{aligned} \tag{25}
$$

Here $\psi(a,c,z)$ is the confluent hypergeometric function of the second kind (see, e.g., [1]), and $O(1/n)$ stands for a 2×2 matrix with the matrix elements of that order.

Applying the following property of the confluent hypergeometric functions:

$$
\psi(a,c,z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\phi(a,c,z) + \frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\phi(a-c+1, 2-c, z). \tag{26}
$$

and Kummer's transformation

$$
\phi(a,c,z) = e^z\phi(c-a,c,-z) \tag{27}
$$

to Ψ_2 in (25) gives

$$
\Psi_2 = \frac{\Gamma(1+\alpha-\beta)}{\Gamma(1+2\alpha)}e^{-i\pi(\alpha+\beta)}z^n\phi(1+\alpha-\beta, 1+2\alpha, -\zeta), \tag{28}
$$

which simplifies the expression for Q_2. To simplify the formula for Q_1, we use (26) and (27) again. We obtain for the combination Ψ_1 in Q_1:

$$
\Psi_1 = \zeta^{2\alpha}\frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+2\alpha)}e^{-i\pi(\alpha+\beta)}\phi(1+\alpha+\beta, 1+2\alpha, \zeta).
$$

Now the expressions (21), (22) of the lemma for $\zeta \in I$ follow easily (noting also that $\chi_n^2 = 1+O(1/n)$ by Theorem 1.8 in [10]). As $q_n(z)$, $\widehat{q}_n(z)$ are polynomials and the asymptotics of [10] hold uniformly for $z \in U_0$, the expressions for $q_n(z)$, $\widehat{q}_n(z)$ extend by continuity to the whole neighborhood U_0 and hold there uniformly. The uniformity properties in α and β follow from the uniformity of the asymptotics in [10]. The multiplier $e^{-2\pi i(\alpha-\beta)}$ for $q_n(z)$ in $\mathbb{C}_-\cap U_0$ appears because of the cut of z^a going through the neighborhood. $\square$

Let $D_n(\varphi)$ denote the Toeplitz determinant with symbol $f(z,\varphi)$:

$$
D_n(\varphi) = \det(f_{j-k})_{j,k=0}^{n-1} = \frac{1}{(2\pi)^n n!}\int_{E_\varphi}\cdots\int_{E_\varphi}\prod_{1\le j<k\le n}|z_j - z_k|^2\prod_{j=1}^n f(z_j,\varphi)\frac{dz_j}{iz_j}, \tag{29}
$$

where f_k are the Fourier coefficients of $f(z,\varphi)$:

$$
f_k = \frac{1}{2\pi}\int_\varphi^{2\pi-\varphi} f(e^{i\theta},\varphi)e^{-ik\theta}d\theta, \quad k = 0, \pm1, \pm2, \ldots
$$

Then the following lemma holds. (For $\alpha=\beta=0$ it reduces to the scaling limit used by Dyson in his analysis of the sine-kernel determinant [14].)

Lemma 6. *Let $s>0$. Then*

$$
\det(I - K^{(\alpha,\beta)}) = \lim_{n\to\infty}\frac{D_n\left(\frac{2s}{n}\right)}{D_n(0)}, \tag{30}
$$

where $K^{(\alpha,\beta)}$ is the operator on $L^2(-s,s)$ with kernel (1).

Proof. Assume first that $\alpha, i\beta \in \mathbb{R}$. Then, as follows, e.g., from (29), $D_n(\varphi) > 0$ for all n, and therefore the polynomials $q_k(z)$, $\widehat{q}_k(z)$ exist for all k (as follows from their determinantal representation: see, e.g., [7]). By a standard argument [24, 7], we first write the term $\prod_{j<k} |z_j - z_k|^2$ in (29) as a product of two Vandermonde determinants whose elements, by a suitable combination of the rows, become the polynomials (19) $q_{k-1}(z_j)/\chi_{k-1}(0)$ and $\widehat{q}_{k-1}(z_j^{-1})/\chi_{k-1}(0)$, $j,k = 1,\dots,n$, respectively. We obtain

$$\prod_{j<k} |z_j - z_k|^2 = \prod_{j=0}^{n-1} \chi_j(0)^{-2} \times \det \left(\sum_{\ell=0}^{n-1} \widehat{q}_\ell(z_j^{-1}) q_\ell(z_k) \right)_{1\le j,k\le n}.$$

Using the well-known expression

$$D_n(0) = \prod_{j=0}^{n-1} \chi_j(0)^{-2},$$

we obtain

$$\frac{D_n(\varphi)}{D_n(0)} = \frac{1}{(2\pi)^n n!} \int_{E_\varphi} \dots \int_{E_\varphi} \det\left(K_n(z_i, z_j)\right)_{1\le i,j\le n} \frac{dz_1}{iz_1} \cdots \frac{dz_n}{iz_n}, \tag{31}$$

where the kernel K_n is given by the expression:

$$\begin{aligned} K_n(z_1,z_2) &= \sqrt{f(z_1,0)f(z_2,0)} \sum_{k=0}^{n-1} \widehat{q}_k(z_1^{-1}) q_k(z_2) \\ &= \sqrt{f(z_1,0)f(z_2,0)} \frac{(z_2/z_1)^n q_n(z_1)\widehat{q}_n(z_2^{-1}) - \widehat{q}_n(z_1^{-1}) q_n(z_2)}{1 - z_2/z_1}. \end{aligned} \tag{32}$$

To obtain the second equality here, we used the Christoffel-Darboux formula: see, e.g., Lemma 2.3. in [10]. Since for sufficiently large n, both the polynomials $q_n(z)$, $\widehat{q}_n(z)$ exist (see Lemma 5) and $D_n(0) \neq 0$ (see (38) below) for complex α and β, equation (31) is extended to the general case of α, β from $\alpha, i\beta \in \mathbb{R}$ by continuity and holds for all sufficiently large n for α and β in a compact set.

As, e.g., in [7] Section 5.4., one shows that the r.h.s. of (31) can be written as the Fredholm determinant $\det(I - K_n)$, where K_n is the operator with kernel (32) acting on $L^2(C \setminus E_\varphi, \frac{dz}{2\pi i z})$; that is

$$D_n(\varphi) = D_n(0) \det(I - K_n)_{L^2(C\setminus E_\varphi)}, \tag{33}$$

where the arc $C \setminus E_\varphi$ is oriented counterclockwise.

We now show that the kernel (1) can be obtained as a scaling limit of (32). Setting $z = e^{\frac{2iu}{n}}$, $u > 0$, in (21), (22), we obtain

$$q_n(e^{\frac{2iu}{n}}) = n^{\alpha-\beta} \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+2\alpha)} \phi(1+\alpha+\beta, 1+2\alpha, 2iu) \left[1 + O\left(\frac{1}{n}\right)\right], \tag{34}$$

$$\widehat{q}_n(e^{-\frac{2iu}{n}}) = n^{\alpha+\beta} \frac{\Gamma(1+\alpha-\beta)}{\Gamma(1+2\alpha)} \phi(1+\alpha-\beta, 1+2\alpha, -2iu) \left[1 + O\left(\frac{1}{n}\right)\right]. \tag{35}$$

Setting now $z = e^{2\pi i + \frac{2iu}{n}}$, $u < 0$, in (21), (22), we obtain the same expressions as the r.h.s. of (34), (35) for the values of the polynomials q_n, $\widehat{q}_n$ at these points.

Let $z(w) = e^{\frac{2iw}{n}}$ if $w > 0$, and $z(w) = e^{2\pi i + \frac{2iw}{n}}$ if $w < 0$. Substituting the just found values of the polynomials into (32), we obtain

$$\lim_{n\to\infty} K_n(z(u), z(v)) \frac{dz(v)}{2\pi i z(v)} = e^{i(v-u)} K^{(\alpha,\beta)}(u,v) dv, \qquad u, v \in \mathbb{R}, \qquad \frac{dz(v)}{2\pi i z(v)} = \frac{dv}{\pi n},$$

where $K^{(\alpha,\beta)}(u,v)$ is given by (1).

Using the following standard recurrence relations for the confluent hypergeometric function

$$\begin{aligned} &a\phi(a+1,c,x) - (a-c+1)\phi(a,c,x) - (c-1)\phi(a,c-1,x) = 0, \\ &c\phi(a+1,c,x) - c\phi(a,c,x) - x\phi(a+1,c+1,x) = 0, \end{aligned} \tag{36}$$

we can rewrite (1) in the form (3).

Now using the estimates (34)–(35) for the polynomials $q_n(z)$, we see that for any $s > 0$ there exists $c(s) > 0$ such that

$$\left| \partial_u^j \partial_v^k \left(\frac{1}{\pi n} K_n(z_1, z_2) - e^{i(v-u)} K^{(\alpha,\beta)}(u,v) \right) \right| \leq \frac{c}{n}, \tag{37}$$

where $u, v \in (-s, s)$, $j, k = 0, 1$. By similar arguments to those of the proof of Corollary 1.3 in [8], the estimate (37) leads to (30). $\Box$

A well-known result on asymptotics of Toeplitz determinants with a single Fisher-Hartwig singularity such that $\alpha \pm \beta \neq -1, -2, \ldots$ (see, e.g., [17, 10]) reads in our case:

$$D_n(0) = n^{\alpha^2 - \beta^2} \frac{G(1+\alpha+\beta) G(1+\alpha-\beta)}{G(1+2\alpha)} (1 + o(1)). \tag{38}$$

This expansion is uniform [10] in compact subsets of the α-half-plane $\Re\alpha > -1/2$ and of the β-plane outside neighborhoods of the points $\alpha \pm \beta = -1, -2, \ldots$. In the next sections we will obtain an expression for $D_n(2s/n)$ for large s, n, $s < n$.

3 Riemann-Hilbert problem and a differential identity

Suppose that the system of orthonormal polynomials satisfying (18) exists and consider the following matrix-valued function:

$$Y^{(n)}(z) = \begin{pmatrix} \chi_n^{-1} p_n(z) & \chi_n^{-1} \int_{C_\varphi} \frac{p_n(\xi)}{\xi - z} \frac{f(\xi) d\xi}{2\pi i \xi^n} \\ \chi_{n-1} z^{n-1} \widehat{p}_{n-1}(z^{-1}) & \chi_{n-1} \int_{C_\varphi} \frac{\widehat{p}_{n-1}(\xi^{-1})}{\xi - z} \frac{f(\xi) d\xi}{2\pi i \xi} \end{pmatrix}, \qquad z \notin C_\varphi, \tag{39}$$

where C_φ is the arc E_φ (14) but oriented clockwise. Denote $z_+ = e^{i\varphi}$, $z_- = e^{i(2\pi - \varphi)}$, the endpoints of the arc.

It is easy to verify directly that $Y(z) = Y^{(n)}(z)$ solves the following Riemann-Hilbert problem:

(a) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus C_\varphi$.

(b) $Y(z)$ has continuous boundary values $Y_+(z)$ as z approaches the inner points of the arc C_φ from the outside of the unit circle, and $Y_-(z)$, from the inside. They are related by the jump condition

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in C_\varphi \setminus \{z_+, z_-\}. \tag{40}$$

(c) $Y(z)$ has the following asymptotic behavior as $z \to \infty$:

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{n\sigma_3}, \qquad \text{where } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{41}$$

(d) Near the endpoints of the arc,

$$Y(z) = O\begin{pmatrix} 1 & \ln|z - z_\pm| \\ 1 & \ln|z - z_\pm| \end{pmatrix}, \tag{42}$$

as $z \to z_\pm$, $\quad z \in \mathbb{C} \setminus C_\varphi$.

The solution (39) to the RHP (a)–(d) is unique. Note first that $\det Y(z) = 1$. Indeed, from the conditions on $Y(z)$, $\det Y(z)$ is analytic across C_φ, has all singularities removable, and tends to 1 as $z \to \infty$. It is then identically 1 by Liouville's theorem. Now if there is another solution $\widetilde{Y}(z)$, we easily obtain by Liouville's theorem that $Y(z)\widetilde{Y}(z)^{-1} \equiv 1$.

A general fact that orthogonal polynomials can be so represented as a solution of a Riemann-Hilbert problem was noticed for polynomials on the real line by Fokas, Its, Kitaev in [18], and extended to polynomials on the circle in [3]. The point of this representation is that the Riemann-Hilbert problem can be efficiently analyzed for large n by a steepest-descent method discovered by Deift and Zhou [13] (and developed further in many subsequent works). This gives the large-n asymptotics of $Y(z)$, and therefore, by (39), the asymptotics of the orthogonal polynomials. We defer the asymptotic analysis in the present case to Section 5. The rest of this section will be devoted to a derivation of a differential identity for $D_n(\varphi)$ in terms of the matrix elements of $Y(z)$.

We start with the following auxiliary lemma (which, in fact, is true for any weight $f(z)$ and a jump contour C_φ).

Lemma 7. *Let the system of polynomials $p_k(z)$, $\widehat{p}_k(z)$, $k = 0, \dots$ satisfying (18) exist. Fix $n \geq 1$. Then we have the following Christoffel-Darboux identity in terms of the function (39):*

$$\sum_{k=0}^{n-1} \widehat{p}_k(z^{-1}) p_k(z) = -z^{-n+1} \lim_{\zeta \to z} \operatorname{tr} \left(\frac{dY(\zeta)}{d\zeta} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y(\zeta)^{-1} \right), \qquad \zeta \notin C_\varphi. \tag{43}$$

Remark. The right hand side of (43) contains only the elements of the first column of Y which are analytic.

Proof. Multiplying the recurrence relation (2.4) of Lemma 2.1. in [10] by z, and replacing n with $n-1$ gives (in the present notation)

$$z\widehat{p}_n(z^{-1}) = \frac{\chi_{n-1}}{\chi_n} \widehat{p}_{n-1}(z^{-1}) + \frac{\widehat{p}_n(0)}{\chi_n} z^{-n+1} p_n(z).$$

Substituting this expression into the r.h.s. of the Christoffel-Darboux identity (2.8) of [10]:

$$\sum_{k=0}^{n-1}\widehat{p}_k(z^{-1})p_k(z) = -np_n(z)\widehat{p}_n(z^{-1}) + z\left(\widehat{p}_n(z^{-1})\frac{d}{dz}p_n(z) - p_n(z)\frac{d}{dz}\widehat{p}_n(z^{-1})\right),$$

we obtain

$$\sum_{k=0}^{n-1}\widehat{p}_k(z^{-1})p_k(z) = -(n-1)\frac{\chi_{n-1}}{\chi_n}z^{-1}p_n(z)\widehat{p}_{n-1}(z^{-1}) \\ -\frac{\chi_{n-1}}{\chi_n}\left[p_n(z)\frac{d}{dz}\widehat{p}_{n-1}(z^{-1}) - \widehat{p}_{n-1}(z^{-1})\frac{d}{dz}p_n(z)\right]. \tag{44}$$

Using the expressions $Y_{11}(z) = \chi_n^{-1}p_n(z)$, $Y_{21} = \chi_{n-1}z^{n-1}\widehat{p}_{n-1}(z^{-1})$, and $\chi_{n-1}\frac{d}{dz}\widehat{p}_{n-1}(z^{-1}) = -(n-1)z^{-n}Y_{21}(z) + z^{-n+1}\frac{d}{dz}Y_{21}(z)$, we obtain from (44)

$$\sum_{k=0}^{n-1}\widehat{p}_k(z^{-1})p_k(z) = z^{-n+1}(Y_{21}\frac{d}{dz}Y_{11} - Y_{11}\frac{d}{dz}Y_{21}), \tag{45}$$

which proves the Lemma. □.

We are now ready to formulate the main result of this section.

Lemma 8. *Let the polynomials $p_k(z)$, $\widehat{p}_k(z)$, $k = N_0, N_0+1, \ldots$, satisfying (18) exist for some $N_0 \geq 0$. Fix $n > N_0$. Let z_+, z_- be the endpoints of the arc C_φ: $z_+ = e^{i\varphi}$, $z_- = e^{i(2\pi-\varphi)}$. Then*

$$\frac{1}{i}\frac{d}{d\varphi}\ln D_n(\varphi) = \operatorname{tr} C_0(C_+ - C_-) + \frac{z_+}{z_+-1}\operatorname{tr} C_1C_+ - \frac{z_-}{z_--1}\operatorname{tr} C_1C_- + \frac{z_++z_-}{z_+-z_-}\operatorname{tr} C_+C_-, \tag{46}$$

where, in terms of the matrix (39),

$$C_0 = -\frac{n+\alpha-\beta}{2}Y(0)\sigma_3Y^{-1}(0), \qquad C_1 = \alpha Y(1)\sigma_3Y^{-1}(1), \tag{47}$$

and the following limits taken for $z \notin C_\varphi$ exist and define $C_\pm$:

$$C_+ = \lim_{z\to z_+}\frac{z^{-n}f(z)}{2\pi i}Y(z)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}Y^{-1}(z), \qquad C_- = -\lim_{z\to z_-}\frac{z^{-n}f(z)}{2\pi i}Y(z)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}Y^{-1}(z), \tag{48}$$

with the extension of f given by (17). Moreover, the second derivative

$$\frac{d^2}{d\varphi^2}\ln D_n(\varphi) = \frac{z_+}{(z_+-1)^2}\operatorname{tr} C_1(C_++C_-) + \frac{4}{(z_+-z_-)^2}\operatorname{tr} C_+C_-. \tag{49}$$

Proof. Assume first that $\alpha, i\beta \in \mathbb{R}$. Then we can set $N_0 = 0$ (cf. proof of Lemma 6). Starting with the representation of a Toeplitz determinant in terms of the leading coefficients of the polynomials p_k:

$$D_n(\varphi) = \prod_{k=0}^{n-1}\chi_k^{-2}(\varphi), \tag{50}$$

we obtain, using first (18) and then integration by parts, that

$$\frac{d}{d\varphi}\ln D_n(\varphi) = -2\sum_{k=0}^{n-1}\frac{\chi_k'(\varphi)}{\chi_k(\varphi)} = -\frac{1}{2\pi}\int_{\varphi}^{2\pi-\varphi}\sum_{k=0}^{n-1}\frac{\partial}{\partial\varphi}\left(p_k(z)\widehat{p}_k(z^{-1})\right)f(z,\varphi)d\theta = \\ -\frac{1}{2\pi}\sum_{k=0}^{n-1}\left[p_k(z_-)\widehat{p}_k(z_-^{-1})f(z_-,\varphi)+p_k(z_+)\widehat{p}_k(z_+^{-1})f(z_+,\varphi)\right]. \tag{51}$$

The previous lemma immediately gives

$$2\pi\frac{d}{d\varphi}\ln D_n(\varphi) = z_+^{-n+1}f(z_+,\varphi)\lim_{z\to z_+}\operatorname{tr}\left(\frac{dY}{dz}(z)\begin{pmatrix}0&1\\0&0\end{pmatrix}Y(z)^{-1}\right) \\ +z_-^{-n+1}f(z_-,\varphi)\lim_{z\to z_-}\operatorname{tr}\left(\frac{dY}{dz}(z)\begin{pmatrix}0&1\\0&0\end{pmatrix}Y(z)^{-1}\right). \tag{52}$$

In the next section we will show that the solution Y of the Riemann-Hilbert problem exists for all $n > N_0$ with N_0 sufficiently large uniformly for α and β in a compact set. Therefore, the identity (52) extends to the general complex α and β from $\alpha, i\beta \in \mathbb{R}$ by continuity.

One could already use the identity (52) for the purposes of the present paper. However, following the philosophy of the Riemann-Hilbert-problem approach [9], we can simplify it further to the form (46) which does not contain derivatives of Y. In order to do this, consider the function

$$\widetilde{Y}(z) = Y(z)\omega(z)^{\sigma_3/2}, \qquad \omega(z) = z^{-n}f(z), \tag{53}$$

where f outside the arc is given by (17). The function $\widetilde{Y}(z)$ is easily seen to be the solution of the problem:

(a) $\widetilde{Y}(z)$ is analytic for $z \in \mathbb{C}\setminus(C_\varphi\cup\mathbb{R}_+)$;

(b) On the contours C_φ and $\mathbb{R}_+$:

$$\widetilde{Y}_+(z) = \widetilde{Y}_-(z)\begin{pmatrix}1&1\\0&1\end{pmatrix}, \quad z\in C_\varphi\setminus\{z_+,z_-\}; \tag{54}$$

$$\widetilde{Y}_+(z) = \widetilde{Y}_-(z)\left(\frac{\omega_+}{\omega_-}\right)^{\sigma_3/2}, \quad z\in(0,+\infty); \tag{55}$$

(c) $\widetilde{Y}(z) = \left(I+O(\frac{1}{z})\right)z^{\frac{n\sigma_3}{2}}f(z)^{\frac{\sigma_3}{2}}, \quad z\to\infty.$

Since $\frac{\omega_+}{\omega_-}$ is constant on $(0,+\infty)$, we see that

$$\widetilde{F}(z) \equiv \frac{d\widetilde{Y}}{dz}\widetilde{Y}^{-1}$$

has no jumps. Since

$$\frac{d\omega}{dz}(z) = \left(-\frac{n}{z}-\frac{\alpha-\beta}{z}+\frac{2\alpha}{z-1}\right)\omega(z), \qquad \frac{d}{dz}\left(\omega(z)^{\sigma_3}\right) = \sigma_3\frac{\omega'(z)}{\omega(z)}\omega(z)^{\sigma_3}, \tag{56}$$

we obtain from the condition (c) for $\widetilde{Y}$ that $\widetilde{F}(z) = O(1/z)$ as $z \to \infty$. The Riemann-Hilbert problem shows that this function can have isolated singularities at z_+, z_-, 0, and 1. First, we obtain using (56):

$$\widetilde{F}(z) = \frac{C_0}{z} + T_0(z), \quad z \to 0; \qquad \widetilde{F}(z) = \frac{C_1}{z-1} + T_1(z), \quad z \to 1, \tag{57}$$

where C_0, C_1 are given by (47), and $T_j(z)$ are Taylor series.

Now let U be a disk of a sufficiently small radius centered at z_+. If $\widehat{Y}$ is defined by the expression

$$\widetilde{Y}(z) = \widehat{Y}(z)\begin{pmatrix} 1 & \frac{1}{2\pi i}\ln(z-z_+) \\ 0 & 1 \end{pmatrix}, \qquad z \in U, \tag{58}$$

then it follows from (54) that $\widehat{Y}(z)$ in U has no jump, and from (42), that its singularity at z_+ is removable. Thus $\widehat{Y}(z)$ is analytic in U. Using (58) we then obtain

$$\widetilde{F}(z) = \frac{C_+}{z-z_+} + T_3(z), \quad z \to z_+; \tag{59}$$

where $T_3(z)$ is a Taylor series and

$$C_+ = \frac{1}{2\pi i}\widehat{Y}(z_+)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\widehat{Y}^{-1}(z_+). \tag{60}$$

Using the definitions (58) and (53), we obtain the expression (48) for C_+. Note that the limit in (48) exists as the logarithmic singularity of $Y(z)$ at z_+ cancels from that expression.

A similar analysis at z_- gives that

$$\widetilde{F}(z) = \frac{C_-}{z-z_-} + T_4(z), \quad z \to z_-; \tag{61}$$

where $T_4(z)$ is again a Taylor series and C_- is defined in (48).

Thus we conclude that $\widetilde{F}(z)$ is a meromorphic function with first-order poles at 0, 1, z_+, z_-, and since $\widetilde{F}(z) = o(1)$ at infinity, we have identically in the complex plane

$$\widetilde{F}(z) = \frac{C_0}{z} + \frac{C_1}{z-1} + \frac{C_+}{z-z_+} + \frac{C_-}{z-z_-},$$

or recalling the definitions of $\widetilde{F}(z)$ and $\widetilde{Y}(z)$,

$$\frac{d\widetilde{Y}}{dz}(z,\varphi) = \left(\frac{C_0}{z} + \frac{C_1}{z-1} + \frac{C_+}{z-z_+} + \frac{C_-}{z-z_-}\right)\widetilde{Y}(z,\varphi), \tag{62}$$

$$\frac{dY}{dz}(z,\varphi) = \left(\frac{C_0}{z} + \frac{C_1}{z-1} + \frac{C_+}{z-z_+} + \frac{C_-}{z-z_-}\right)Y(z,\varphi) - \frac{\omega'(z)}{2\omega(z)}Y(z,\varphi)\sigma_3. \tag{63}$$

Substituting (63) into (52) and noticing that $C_\pm^2 = 0$ gives the identity (46).

To obtain the identity for the second derivative, note first that as follows from the general theory the function $\widetilde{Y}(z)$ is differentiable w.r.t. φ. Similarly to our derivation of (62), we obtain

$$\frac{d\widetilde{Y}}{d\varphi}(z,\varphi)=\left(\frac{-iz_+C_+}{z-z_+}+\frac{iz_-C_-}{z-z_-}\right)\widetilde{Y}(z,\varphi). \tag{64}$$

Equating the derivatives $\frac{d}{dz}\frac{d}{d\varphi}\widetilde{Y}(z,\varphi)=\frac{d}{d\varphi}\frac{d}{dz}\widetilde{Y}(z,\varphi)$, gives by (62) and (64) a compatibility condition on C_0, C_1, C_+, C_-, and their derivatives w.r.t. φ. Equating the coefficients at $1/z$ in this condition gives

$$\frac{d}{d\varphi}C_0+i[C_0,C_+-C_-]=0, \tag{65}$$

where $[A,B]=AB-BA$. Similarly the coefficients at $1/(z-1)$, $1/(z-z_+)$, $1/(z-z_-)$ yield the identities:

$$\frac{d}{d\varphi}C_1+i\left[C_1,\frac{z_+}{z_+-1}C_+-\frac{z_-}{z_--1}C_-\right]=0, \tag{66}$$

$$\frac{d}{d\varphi}C_+-i[C_0,C_+]-\frac{iz_+}{z_+-1}[C_1,C_+]+i\frac{z_++z_-}{z_+-z_-}[C_+,C_-]=0, \tag{67}$$

$$\frac{d}{d\varphi}C_-+i[C_0,C_-]+\frac{iz_-}{z_--1}[C_1,C_-]-i\frac{z_++z_-}{z_+-z_-}[C_+,C_-]=0. \tag{68}$$

Differentiating (46) w.r.t. φ and substituting the above identities for the derivatives of C_0, C_1, C_+, C_- in the resulting expression gives the formula (49). In this calculation, it is convenient to use the elementary algebraic identity:

$$\operatorname{tr}[A,B]C=\operatorname{tr}A[B,C], \tag{69}$$

which equals 0 if any two of A, B, C coincide. □

4 Expansion of $D_n(\varphi)$ as $\varphi\to\pi$.

For a fixed $n\geq 1$ we will now obtain the expansion of $D_n(\varphi)$ as $\varphi\to\pi$. We use the representation (29) of $D_n(\varphi)$ as a multiple integral:

$$D_n(\varphi)=\frac{1}{(2\pi)^n n!}\int_\varphi^{2\pi-\varphi}\cdots\int_\varphi^{2\pi-\varphi}\prod_{1\leq j<k\leq n}|e^{i\theta_j}-e^{i\theta_k}|^2\prod_{j=1}^n|e^{i\theta_j}-1|^{2\alpha}e^{i\theta_j\beta}e^{-i\pi\beta}d\theta_j.$$

For the analysis of $D_n(\varphi)$ as $\varphi\to\pi$, set $\varphi=\pi-\varepsilon$, $\varepsilon>0$. Substituting $\theta_j=\pi+\varepsilon x_j$ in the integrals, we obtain:

$$\begin{aligned}D_n(\varphi)&=\frac{\varepsilon^n}{(2\pi)^n n!}\int_{-1}^1\cdots\int_{-1}^1\prod_{1\leq j<k\leq n}|e^{i\varepsilon x_j}-e^{i\varepsilon x_k}|^2\prod_{j=1}^n|e^{i\varepsilon x_j}+1|^{2\alpha}e^{i\varepsilon x_j\beta}dx_j\\&=\frac{\varepsilon^{n^2}2^{2\alpha n}}{(2\pi)^n}\left(A_n+O(\varepsilon^2)\right),\qquad\text{as }\varepsilon\to 0,\ n\text{ fixed,}\end{aligned} \tag{70}$$

where

$$A_n = \frac{1}{n!}\int_{-1}^{1}\cdots\int_{-1}^{1}\prod_{1\le j<k\le n}(x_j-x_k)^2\prod_{j=1}^{n}dx_j = 2^{n^2}\prod_{k=0}^{n-1}\frac{k!^3}{(n+k)!} \tag{71}$$

is a Selberg integral (or a product $\prod_{k=0}^{n-1}\varkappa_k^{-2}$, where $\varkappa_k$ are the leading coefficients of the orthonormal Legendre polynomials). The error term in (70) is of order ε^2. Indeed, it is easy to see that the expansion of the factors with the absolute value in the integrand in (70) gives an error of order ε^2. The factors $e^{i\varepsilon x_j\beta}$ produce the following term of order ε:

$$\frac{i\varepsilon\beta}{n!}\int_{-1}^{1}\cdots\int_{-1}^{1}\sum_{j=1}^{n}x_j\prod_{1\le j<k\le n}(x_j-x_k)^2\prod_{j=1}^{n}dx_j,$$

which is equal to 0, as the change of variables $x_j\to -x_j$ shows. Therefore, the error term in (70) is indeed $O(\varepsilon^2)$.

The asymptotics of A_n as $n\to\infty$ are [28]:

$$\ln A_n = -n^2\ln 2 + n\ln(2\pi) - \frac{1}{4}\ln n + \frac{1}{12}\ln 2 + 3\zeta'(-1) + o(1), \qquad n\to\infty,$$

where $\zeta'(z)$ is the derivative of Riemann's zeta-function. Therefore,

$$\ln D_n(\varphi) = n^2\ln(\pi-\varphi) + (2\alpha n - n^2)\ln 2 - \frac{1}{4}\ln n + \frac{1}{12}\ln 2 + 3\zeta'(-1) + \delta_n + O_n(\varepsilon^2), \quad \varepsilon = \pi-\varphi. \tag{72}$$

Here δ_n depends only on n and $\delta_n\to 0$ as $n\to\infty$. The term $O_n(\varepsilon^2)\to 0$, as $\varepsilon\to 0$, n fixed.

5 Asymptotic analysis of $Y(z)$

We now analyze the Riemann-Hilbert (RH) problem of Section 3 for $Y(z)$ in the limit of large n. The analysis is similar to that of [20] and [11]. Consider the function

$$\Psi(z) = \frac{1}{2\gamma}\left(z+1+\sqrt{(z-z_+)(z-z_-)}\right), \qquad \gamma = \cos(\varphi/2), \tag{73}$$

which conformally maps the outside of the arc C_φ onto the outside of the unit circle. Note that $\Psi_+\Psi_- = z$ for $z\in C_\varphi\setminus\{z_+,z_-\}$. Furthermore, we see as in [20] that

$$\left|\frac{z}{\Psi(z)^2}\right| < 1, \qquad \text{for } |z|\ne 1. \tag{74}$$

We apply several transformations to the Riemann-Hilbert problem. First, set

$$T(z) = \gamma^{-n\sigma_3}Y(z)\Psi(z)^{-n\sigma_3}. \tag{75}$$

Then we obtain a RH problem which is normalized to I at infinity:

(a) $T(z)$ is analytic for $z\in\mathbb{C}\setminus C_\varphi$.

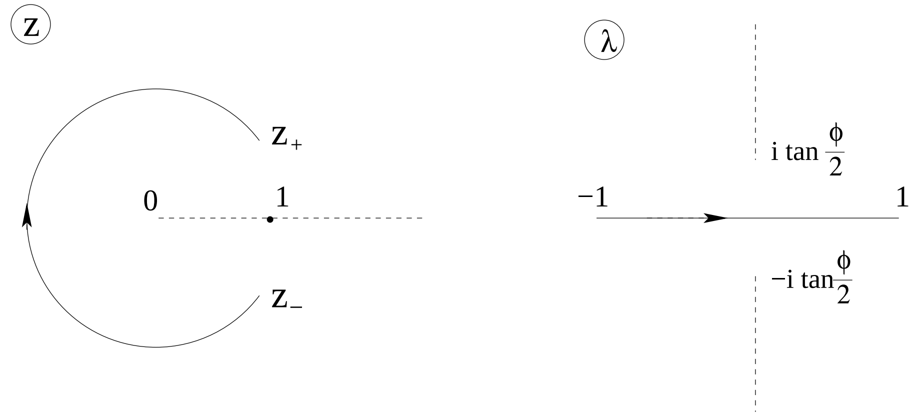

Figure 1: Conformal mapping.

(b) $T(z)$ has L^2 boundary values on C_φ related by the condition

$$T_+(z) = T_-(z) \begin{pmatrix} z^n \Psi_+(z)^{-2n} & f(z) \\ 0 & z^n \Psi_-(z)^{-2n} \end{pmatrix}, \qquad \text{for } z \in C_\varphi \setminus \{z_+, z_-\}. \tag{76}$$

(c)

$$T(z) = I + O\left(\frac{1}{z}\right), \qquad \text{as } z \to \infty. \tag{77}$$

As in [11], we now go over to the variable λ given by the following linear-fractional transformation that maps the arc C_φ onto the interval $[-1, 1]$ with the point $z = z_-$ corresponding to $\lambda = -1$, and $z = z_+$, to $\lambda = 1$:

$$\lambda = \frac{z+1}{z-1} i \tan\frac{\varphi}{2}, \qquad z = \frac{\lambda + i\tan\frac{\varphi}{2}}{\lambda - i\tan\frac{\varphi}{2}}. \tag{78}$$

The complementary to C_φ arc of the unit circle is mapped to $\mathbb{R} \setminus [-1, 1]$. The points $z = 0,\ 1,\ \infty$ are mapped to $\lambda = -i\tan\frac{\varphi}{2},\ \infty,\ i\tan\frac{\varphi}{2}$, respectively. The cut of the function $f(z)$, $(0, 1) \cup (1, +\infty)$ becomes $(-i\tan\frac{\varphi}{2}, -i\infty) \cup (i\infty, i\tan\frac{\varphi}{2})$ in the λ-plane (see Figure 1).

For the case of a varying arc when $\varphi \to 0$ and $\varphi \to \pi$, in the RH analysis, one would need to consider contracting neighborhoods of the end points $z_\pm$ in the z-plane (cf. [20]). This could be carried out. However, in the λ-plane, we can keep neighborhoods of the points $\lambda = \pm 1$ fixed, which considerably simplifies the calculations below. Thus, going over to λ is not essential, but useful.

The problem for T corresponds to the following one in the λ-plane:

(a) $\widetilde{T}(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus [-1, 1]$.

(b) The boundary values of $\widetilde{T}(\lambda)$ on $(-1, 1)$ are related by the condition

$$\widetilde{T}_+(\lambda) = \widetilde{T}_-(\lambda) \begin{pmatrix} \Phi_+(\lambda)^{-2n} & f(z(\lambda)) \\ 0 & \Phi_-(\lambda)^{-2n} \end{pmatrix}, \qquad \text{for } \lambda \in (-1, 1), \qquad \Phi(\lambda) = \frac{\Psi(z(\lambda))}{z(\lambda)^{1/2}}. \tag{79}$$

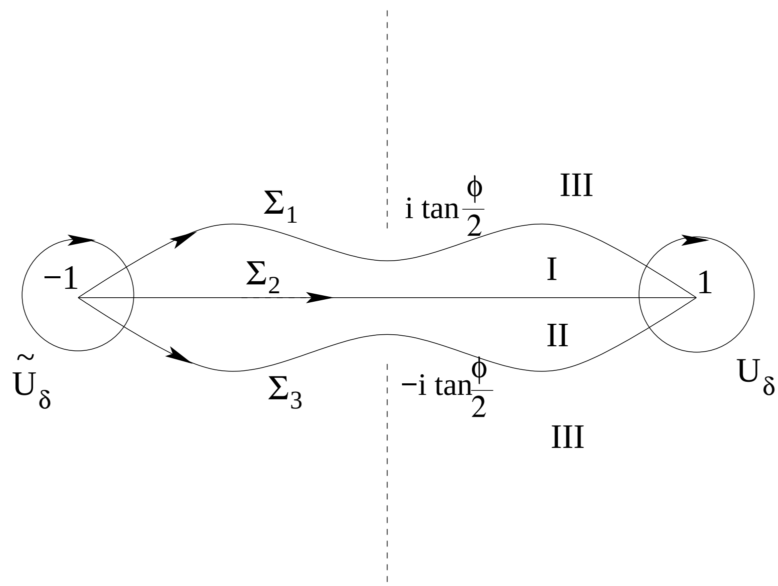

Figure 2: Contour for Riemann-Hilbert problems.

(c)

$$\widetilde{T}(\lambda) = I + O\left(\frac{1}{\lambda}\right), \qquad \text{as } \lambda \to \infty. \tag{80}$$

The solution $\widetilde{T}(\lambda)$ is related to $T(z)$ by the expression:

$$T(z) = T_0 \widetilde{T}(\lambda(z)), \qquad T_0 = \widetilde{T}^{-1}\left(i \tan\frac{\varphi}{2}\right). \tag{81}$$

For the function (17) we have in the λ-variable (we denote it $f(\lambda)$ for simplicity):

$$f(\lambda) \equiv f(z(\lambda)) = e^{-i\pi\beta}\left(2\tan\frac{\varphi}{2}\right)^{2\alpha}\left(\lambda - i\tan\frac{\varphi}{2}\right)^{-\alpha-\beta}\left(\lambda + i\tan\frac{\varphi}{2}\right)^{-\alpha+\beta}. \tag{82}$$

The function $\Phi(\lambda)$:

$$\Phi(\lambda) = \frac{\lambda + i\sin\frac{\varphi}{2}\sqrt{1-\lambda^2}}{\cos\frac{\varphi}{2}(\lambda^2 + \tan^2\frac{\varphi}{2})^{1/2}}. \tag{83}$$

Note that, because of the properties of $\Psi(z)$ discussed above, we have

$$|\Phi(\lambda)| > 1, \quad \lambda \notin \mathbb{R}; \qquad |\Phi_{\pm}(\lambda)| = 1, \quad \lambda \in [-1, 1]. \tag{84}$$

Following the steepest-descent method of Deift and Zhou [13] we now change the RH problem so that the oscillating behavior of the matrix elements in (79) is converted into the exponential decay as $n \to \infty$.

Namely, consider the system of contours shown in Figure 2. Let I be the region bounded by the curves Σ_1 and $\Sigma_2 \equiv (-1, 1)$; region II is the one bounded by the curves Σ_2 and Σ_3; region III is the rest of the complex plane. Define the function $S(\lambda)$ as follows:

in region I,

$$S(\lambda) = \widetilde{T}(\lambda) \begin{pmatrix} 1 & 0 \\ -f(\lambda)^{-1}\Phi(\lambda)^{-2n} & 1 \end{pmatrix}, \tag{85}$$

in region II,

$$S(\lambda) = \widetilde{T}(\lambda) \begin{pmatrix} 1 & 0 \\ f(\lambda)^{-1}\Phi(\lambda)^{-2n} & 1 \end{pmatrix}, \tag{86}$$

in region III,

$$S(\lambda) = \widetilde{T}(\lambda). \tag{87}$$

The Riemann-Hilbert problem for S is then the following:

(a,b) $S(\lambda)$ is analytic in $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3)$ with the following jump conditions on the contours:

$$S_+(\lambda) = S_-(\lambda) \begin{pmatrix} 1 & 0 \\ f(\lambda)^{-1}\Phi(\lambda)^{-2n} & 1 \end{pmatrix}, \qquad \lambda \in \Sigma_1 \cup \Sigma_3, \tag{88}$$

$$S_+(\lambda) = S_-(\lambda) \begin{pmatrix} 0 & f(\lambda) \\ -f(\lambda)^{-1} & 0 \end{pmatrix}, \qquad \lambda \in \Sigma_2 \equiv (-1,1). \tag{89}$$

(c) As $\lambda \to \infty$,

$$S(\lambda) = I + O\left(\frac{1}{\lambda}\right). \tag{90}$$

For S to have these properties, the contours $\Sigma_{1,3}$ should not intersect the real axis and the cuts $(-i\infty, -i\tan\varphi/2)$, $(i\tan\varphi/2, i\infty)$ of $f(\lambda)$.

Below we will investigate the inequality in (84) in more detail and will show that $\Sigma_{1,3}$ can be chosen so that for φ satisfying $2s/n < \varphi < \pi$, $n > s$, $s > s_0$, the jump matrix on $\Sigma_1 \cup \Sigma_3$ is uniformly close to the identity up to an error of order $e^{-\varepsilon s_0}$, $\varepsilon > 0$, outside neighborhoods of the endpoints of the arc. The error is small for s_0 sufficiently large. This suggests that outside some δ-neighborhoods U_δ, $\widetilde{U}_\delta$ of the endpoints, the function S can be approximated by a parametrix which has a jump only on Σ_2. The problem for this parametrix in the outside-the-neighborhoods region is standard [7] and will be presented below. It is solved explicitly. Then we will consider the neighborhoods U_δ, $\widetilde{U}_\delta$, and construct (following [21]) local parametrices there in terms of Bessel functions. We then match the outside and the local parametrices on the boundaries ∂U_δ, $\partial\widetilde{U}_\delta$ for large n, which produces the asymptotic expansion of S in the inverse powers of $n \sin\frac{\varphi}{2}$. The latter expression is large provided again that $2s/n < \varphi < \pi$, $n > s$, $s > s_0$, and s_0 is sufficiently large.

5.1 Outside parametrix

The parametrix outside neighborhoods of $\lambda = \pm 1$ is the solution to the following RH problem:

(a) $N(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus [-1,1]$,

(b) $N(\lambda)$ has L^2 boundary values N_+, N_- on $(-1,1)$ related as follows:

$$N_+(\lambda) = N_-(\lambda) \begin{pmatrix} 0 & f(\lambda) \\ -f(\lambda)^{-1} & 0 \end{pmatrix}, \qquad \lambda \in (-1,1), \tag{91}$$

(c)

$$N(\lambda) = I + O\left(\frac{1}{\lambda}\right), \qquad \text{as } \lambda \to \infty. \tag{92}$$

As is easy to verify, this problem has the following solution:

$$N(\lambda) = \frac{1}{2}(\mathcal{D}_\infty)^{\sigma_3}\begin{pmatrix} a + a^{-1} & -i(a - a^{-1}) \\ i(a - a^{-1}) & a + a^{-1}\end{pmatrix}\mathcal{D}(\lambda)^{-\sigma_3}, \qquad a(\lambda) = \left(\frac{\lambda - 1}{\lambda + 1}\right)^{1/4}, \tag{93}$$

where the branch of the root is chosen so that $a(\lambda) \to 1$ as $\lambda \to \infty$. The Szegő function $\mathcal{D}(\lambda)$ is the solution to the following RH conditions: a) $\mathcal{D}(\lambda)$ is analytic in $\mathbb{C} \setminus [-1, 1]$; b) $\mathcal{D}_+(\lambda)\mathcal{D}_-(\lambda) = f(\lambda)$ for $\lambda \in (-1, 1)$; c) $\mathcal{D}(\lambda) \to \text{const}$ as $\lambda \to \infty$. We have

$$\mathcal{D}(\lambda) = \exp\left(\frac{\sqrt{1-\lambda^2}}{2\pi i}\int_{-1}^{1}\frac{\ln f(\eta)}{\sqrt{1-\eta^2}}\frac{d\eta}{\eta - \lambda}\right), \tag{94}$$

with the integration over the upper ("+") side of the interval $(-1, 1)$: we choose $\sqrt{x} > 0$ for $x > 0$ and $0 < \arg(\lambda \pm 1) < 2\pi$. Finally,

$$\mathcal{D}_\infty = \lim_{\lambda\to\infty}\mathcal{D}(\lambda) = \exp\left(\frac{1}{2\pi}\int_{-1}^{1}\frac{\ln f(\eta) d\eta}{\sqrt{1-\eta^2}}\right). \tag{95}$$

In what follows, we will need an expansion of $D(\lambda)$ at the endpoints ± 1. The integral in (94) can be written as half the integral around a loop encircling $[-1, 1]$. Deforming the loop, we obtain

$$\int_{-1}^{1}\frac{\ln f(\eta)}{\sqrt{1-\eta^2}}\frac{d\eta}{\eta-\lambda} = \frac{\pi i}{(\lambda^2-1)^{1/2}} i \ln f(\lambda) +$$

$$\pi i \lim_{R\to\infty}\left[(\alpha+\beta)\int_{iR}^{i\tan\frac{\varphi}{2}}\frac{id\eta}{(\eta^2-1)^{1/2}}\frac{1}{\eta-\lambda} + (\alpha-\beta)\int_{-iR}^{-i\tan\frac{\varphi}{2}}\frac{id\eta}{(\eta^2-1)^{1/2}}\frac{1}{\eta-\lambda}\right], \tag{96}$$

where $-i(x^2-1)^{1/2} = \sqrt{1-x^2} > 0$ on the upper side of $(-1, 1)$. Expanding $\frac{1}{\eta-\lambda}$ near $\lambda = \pm 1$, we obtain:

$$\int_{iR}^{i\tan\frac{\varphi}{2}}\frac{id\eta}{(\eta^2-1)^{1/2}}\frac{1}{\eta-\lambda} = \pm i - e^{\pm i\varphi/2} + O(\lambda \mp 1), \quad \lambda \to \pm 1; \tag{97}$$

$$\int_{-iR}^{-i\tan\frac{\varphi}{2}}\frac{id\eta}{(\eta^2-1)^{1/2}}\frac{1}{\eta-\lambda} = \pm i + e^{\mp i\varphi/2} + O(\lambda \mp 1), \quad \lambda \to \pm 1. \tag{98}$$

Substituting (97), (98) into (96), we obtain for (94)

$$\begin{aligned}
&\mathcal{D}(\lambda) = f(1)^{1/2}\exp\left(\mathcal{D}_1(1)\left(\frac{\lambda-1}{2}\right)^{1/2} + \mathcal{D}_2(1)\left(\frac{\lambda-1}{2}\right)^{3/2} + O\left(\lambda-1\right)^{5/2}\right), \qquad \lambda \to 1,\\
&\mathcal{D}(\lambda) = f(-1)^{1/2}\exp\left(\mathcal{D}_1(-1)\left(\frac{\lambda+1}{2e^{i\pi}}\right)^{1/2} + \mathcal{D}_2(-1)\left(\frac{\lambda+1}{2e^{i\pi}}\right)^{3/2} + O\left(\lambda+1\right)^{5/2}\right),\\
&\lambda \to -1,
\end{aligned} \tag{99}$$

where

$$\mathcal{D}_1(\pm 1) = 2\left(\alpha\left(1-\sin\frac{\varphi}{2}\right) \pm i\beta\cos\frac{\varphi}{2}\right), \tag{100}$$

and the exact value of $\mathcal{D}_2(\pm 1)$ will not be used as it cancels from the final expressions below. It is clear from the construction that this expansion is uniform in φ as well as in α, β in a compact set.

5.2 Local parametrices

Let U_δ and $\widetilde{U}_\delta$ denote the (nonintersecting) δ-neighborhoods of the points 1 and -1, respectively: see Figure 2. We choose δ to be sufficiently small, see below.

We will now write down essentially known (see [9, 21, 20]) parametrices P, and $\widetilde{P}$ in U_δ and $\widetilde{U}_\delta$, respectively. The parametrices have the same jumps as S inside these neighborhoods and match N on the boundaries ∂U_δ and $\partial\widetilde{U}_\delta$ to the leading order.

Consider the function

$$\omega(\lambda) = \ln^2\Phi(\lambda),$$

which, for a sufficiently small δ, is analytic inside U_δ and maps it conformally onto a neighborhood of zero. It has the following expansion at 1:

$$\omega(\lambda) = 2u\sin^2\frac{\varphi}{2}\left\{1-\frac{2}{3}\left[\cos\varphi+\frac{5}{4}\right]u+O(u^2)\right\}, \quad u=\lambda-1, \quad \lambda\in U_\delta, \tag{101}$$

and

$$\sqrt{\omega} = \sqrt{2}u^{1/2}\sin\frac{\varphi}{2}(1+O(u)), \quad u=\lambda-1.$$

This expansion is uniform in φ.

Consider the following mapping of U_δ:

$$\zeta = n^2\omega(\lambda), \qquad \lambda\in U_\delta. \tag{102}$$

For our analysis below, we need $|\zeta|$ to be uniformly large in φ and λ on the boundary ∂U_δ. We see from (101) that this is indeed so if φ satisfies the condition: $\frac{2s}{n}<\varphi<\pi$, $n>s$, $s>s_0$, with s_0 sufficiently large.

The local parametrix in U_δ is given by the following expression (cf. [9, 21, 20]):

$$P(\lambda) = E(\lambda)Q(n^2\omega(\lambda))e^{-n\sqrt{\omega(\lambda)}\sigma_3}f(\lambda)^{-\sigma_3/2}, \qquad \lambda\in U_\delta, \tag{103}$$

where

$$E(\lambda) = \frac{1}{\sqrt{2}}N(\lambda)f(\lambda)^{\sigma_3/2}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}(\pi n\sqrt{\omega(\lambda)})^{\sigma_3/2}, \tag{104}$$

and the function $Q(\zeta)$ is expressed in terms of modified Bessel and Hankel functions:

1) in the intersection of region I in the λ plane and U_δ

$$Q(\zeta) = \frac{1}{2}\begin{pmatrix} H_0^{(1)}(e^{-i\pi/2}\zeta^{1/2}) & H_0^{(2)}(e^{-i\pi/2}\zeta^{1/2}) \\ \pi\zeta^{1/2}\left(H_0^{(1)}\right)'(e^{-i\pi/2}\zeta^{1/2}) & \pi\zeta^{1/2}\left(H_0^{(2)}\right)'(e^{-i\pi/2}\zeta^{1/2})\end{pmatrix}, \tag{105}$$

2) region II and U_δ

$$Q(\zeta)=\frac{1}{2}\begin{pmatrix} H_0^{(2)}(e^{i\pi/2}\zeta^{1/2}) & -H_0^{(1)}(e^{i\pi/2}\zeta^{1/2}) \\ -\pi\zeta^{1/2}\left(H_0^{(2)}\right)'(e^{i\pi/2}\zeta^{1/2}) & \pi\zeta^{1/2}\left(H_0^{(1)}\right)'(e^{i\pi/2}\zeta^{1/2}) \end{pmatrix}, \tag{106}$$

3) region III and U_δ

$$Q(\zeta)=\begin{pmatrix} I_0(\zeta^{1/2}) & \frac{i}{\pi}K_0(\zeta^{1/2}) \\ \pi i\zeta^{1/2}I_0'(\zeta^{1/2}) & -\zeta^{1/2}K_0'(\zeta^{1/2}) \end{pmatrix}, \tag{107}$$

where $-\pi<\arg(\zeta)<\pi$.

Using the asymptotic expansions of Bessel and Hankel functions for a large argument, one obtains uniformly on ∂U_δ:

$$P(\lambda)N(\lambda)^{-1}=I+N(\lambda)f(\lambda)^{\sigma_3/2}\left\{\frac{1}{8n\sqrt{\omega(\lambda)}}\begin{pmatrix}-1 & -2i\\ -2i & 1\end{pmatrix}-\frac{3}{2^7n^2\omega(\lambda)}\begin{pmatrix}1 & -4i\\ 4i & 1\end{pmatrix}\right.$$
$$\left.+O\left(\left[n\sin\frac{\varphi}{2}\right]^{-3}\right)\right\}f(\lambda)^{-\sigma_3/2}N^{-1}(\lambda)=I+\Delta_1+\Delta_2+O\left(\left[n\sin\frac{\varphi}{2}\right]^{-3}\right),\qquad \lambda\in\partial U_\delta, \tag{108}$$

where Δ_1 and Δ_2 denote the terms with $n\sqrt{\omega(z)}$ and $n^2\omega(z)$, respectively. (Note that $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ are analytic functions in $U_\delta\setminus\{1\}$ with poles of order 1 at $\lambda=1$.) This is an expansion in the inverse powers of $n\sin\frac{\varphi}{2}$, and it holds uniformly for $\frac{2s}{n}<\varphi<\pi$, $n>s$, $s>s_0$, and for λ on the boundary ∂U_δ.

Similarly, we define a conformal mapping for the neighborhood $\widetilde{U}_\delta$:

$$\omega(\lambda)=\ln^2(-\Phi(\lambda)),$$

so that we have

$$\omega(\lambda)=-2u\sin^2\frac{\varphi}{2}\left\{1+\frac{2}{3}\left[\cos\varphi+\frac{5}{4}\right]u+O(u^2)\right\},\quad u=\lambda+1,\quad \lambda\in\widetilde{U}_\delta, \tag{109}$$

and

$$\sqrt{\omega}=-i\sqrt{2}u^{1/2}\sin\frac{\varphi}{2}(1+O(u)),\quad u=\lambda+1.$$

We have for the parametrix in $\widetilde{U}_\delta$:

$$\widetilde{P}(\lambda)=\widetilde{E}(\lambda)\sigma_3Q(n^2\omega(\lambda))\sigma_3e^{-n\sqrt{\omega(\lambda)}\sigma_3}f(\lambda)^{-\sigma_3/2},\quad \lambda\in\widetilde{U}_\delta,$$

with

$$\widetilde{E}(\lambda)=\frac{1}{\sqrt{2}}N(\lambda)f(\lambda)^{\sigma_3/2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}(\pi n\sqrt{\omega(\lambda)})^{\sigma_3/2}.$$

Therefore,

$$\widetilde{P}(\lambda)N(\lambda)^{-1}=I+N(\lambda)f(\lambda)^{\sigma_3/2}\left\{\frac{1}{8n\sqrt{\omega(\lambda)}}\begin{pmatrix}-1 & 2i\\ 2i & 1\end{pmatrix}-\frac{3}{2^7n^2\omega(\lambda)}\begin{pmatrix}1 & 4i\\ -4i & 1\end{pmatrix}\right.$$
$$\left.+O\left(\left[n\sin\frac{\varphi}{2}\right]^{-3}\right)\right\}f(\lambda)^{-\sigma_3/2}N^{-1}(\lambda)=I+\Delta_1+\Delta_2+O\left(\left[n\sin\frac{\varphi}{2}\right]^{-3}\right),\qquad \lambda\in\partial\widetilde{U}_\delta, \tag{110}$$

uniformly in λ and in φ for $\frac{2s}{n}<\varphi<\pi$, $n>s$, $s>s_0$. Similarly to the situation for U_δ, $\Delta_1(\lambda)$, $\Delta_2(\lambda)$ in $\widetilde{U}_\delta\setminus\{-1\}$ are analytic functions with poles of order 1 at $\lambda=-1$.

5.3 Final transformation

Let

$$
\begin{aligned}
R(\lambda) &= S(\lambda)N^{-1}(\lambda), && \lambda \in \mathbb{C} \setminus (\overline{U_\delta \cup \widetilde{U}_\delta} \cup \Sigma_{1,2,3}),\\
R(\lambda) &= S(\lambda)P^{-1}(\lambda), && \lambda \in U_\delta \setminus \Sigma_{1,2,3},\\
R(\lambda) &= S(\lambda)\widetilde{P}^{-1}(\lambda), && \lambda \in \widetilde{U}_\delta \setminus \Sigma_{1,2,3}.
\end{aligned} \tag{111}
$$

Furthermore, set

$$
\widetilde{R} \equiv \mathcal{D}_\infty^{-\sigma_3} R \mathcal{D}_\infty^{\sigma_3}. \tag{112}
$$

It is easy to see that this function has jumps only on ∂U_δ, $\partial\widetilde{U}_\delta$, and parts of Σ_1, and Σ_3 lying outside of the neighborhoods U_δ, $\widetilde{U}_\delta$ (we denote these parts $\Sigma_{1,3}^{\text{out}}$). Namely,

$$
\begin{aligned}
\widetilde{R}_+(\lambda) &= \widetilde{R}_-(\lambda)\mathcal{D}_\infty^{-\sigma_3} N(\lambda) \begin{pmatrix} 1 & 0 \\ f(\lambda)^{-1}\Phi(\lambda)^{-2n} & 1 \end{pmatrix} N(\lambda)^{-1}\mathcal{D}_\infty^{\sigma_3}, && \lambda \in \Sigma_{1,3}^{\text{out}},\\
\widetilde{R}_+(\lambda) &= \widetilde{R}_-(\lambda)\mathcal{D}_\infty^{-\sigma_3} P(\lambda) N(\lambda)^{-1}\mathcal{D}_\infty^{\sigma_3}, && \lambda \in \partial U_\delta,\\
\widetilde{R}_+(\lambda) &= \widetilde{R}_-(\lambda)\mathcal{D}_\infty^{-\sigma_3} \widetilde{P}(\lambda) N(\lambda)^{-1}\mathcal{D}_\infty^{\sigma_3}, && \lambda \in \partial\widetilde{U}_\delta.
\end{aligned} \tag{113}
$$

The jump matrices for $\widetilde{R}$ on ∂U_δ and $\partial\widetilde{U}_\delta$ have the form

$$
I + O(\rho^{-1}), \qquad \rho = n \sin\frac{\varphi}{2},
$$

uniformly in λ (as well as in α, β in compact sets) and in φ provided

$$
\frac{2s}{n} < \varphi < \pi, \qquad n > s, \quad s > s_0. \tag{114}
$$

(A more detailed expansion is given by (108) and (110).)

Let us now estimate the jump matrix on $\Sigma_{1,3}^{\text{out}}$ for φ in the range (114) with s_0 sufficiently large. Below ε' will stand for various positive constants independent of φ, n, and λ. Denote $x = \Re\lambda$, $y = \Im\lambda$, i.e., $\lambda = x + iy$. Choose Σ_1^{out} so that y is small, however, $y > \delta' \sin\frac{\varphi}{2}$ with some fixed $\delta' > 0$. Multiplying (83) with its complex conjugate and expanding in y gives:

$$
|\Phi(\lambda)|^2 = 1 + \frac{2y\sin\frac{\varphi}{2} + O(y^2)}{\sqrt{1-x^2}(\sin^2\frac{\varphi}{2} + x^2\cos^2\frac{\varphi}{2})} \qquad \lambda \in \Sigma_1^{\text{out}}. \tag{115}
$$

Let $|x| > \varepsilon$ for some $\varepsilon > 0$. Then we immediately obtain from (115):

$$
|\Phi(\lambda)|^2 > 1 + \varepsilon' \sin\frac{\varphi}{2},
$$

for $\lambda \in \Sigma_1^{\text{out}}$, and φ in the range (114). Now let $|x| \le \varepsilon$. We parametrise x as follows $x = r\sin\frac{\varphi}{2}$. Therefore $0 < r \le \varepsilon/\sin\frac{\varphi}{2}$. If $r > 1$, we have

$$
|\Phi(\lambda)|^2 > 1 + \frac{r\varepsilon'}{1+r^2} > 1 + \frac{\varepsilon'}{r} > 1 + \varepsilon'\sin\frac{\varphi}{2}.
$$

On the other hand, if $r \le 1$, we have, recalling our condition $y > \delta' \sin\frac{\varphi}{2}$,

$$|\Phi(\lambda)|^2 > 1 + \frac{\delta'\varepsilon'}{1+r^2}.$$

Thus we conclude that the estimate $|\Phi(\lambda)|^2 > 1 + \varepsilon' \sin\frac{\varphi}{2}$ holds uniformly for $\lambda \in \Sigma_1^{\mathrm{out}}$, and φ in the range (114). A similar estimate holds on Σ_3^{out}. As N and f are bounded on $\lambda \in \Sigma_{1,3}^{\mathrm{out}}$ and, in particular, at $\lambda = 0$, these estimates immediately imply that the jump matrix on $\Sigma_{1,3}^{\mathrm{out}}$ can be written as

$$I + O(e^{-\varepsilon'\rho}), \qquad \rho = n\sin\frac{\varphi}{2}, \tag{116}$$

uniformly for $\lambda \in \Sigma_{1,3}^{\mathrm{out}}$, and φ in the range (114).

The above estimates for the jump matrices and the standard analysis of the R-RH problem (see [12]) imply that the R-RH problem is solvable for large s_0 and the solution has the form of the series:

$$\widetilde{R}(\lambda) = I + \sum_{j=1}^{k-1} \widetilde{R}_j(\lambda) + O(\rho^{-k}), \qquad \widetilde{R}_j(\lambda) = O(\rho^{-j}), \tag{117}$$

uniformly for φ in the range (114), for α in a compact subset of the α-half-plane $\Re\alpha > -1/2$, for β in a compact subset of the β-plane, and for all λ.

Explicit expressions for $\widetilde{R}_k$ are obtained by collecting terms of the same order in the jump relations. Thus by (108) we have, in particular, that $\widetilde{R}_1$, $\widetilde{R}_2$ satisfy the following RH problems. The functions $\widetilde{R}_1$, $\widetilde{R}_2$ are analytic in $\mathbb{C} \setminus (\partial U_\delta \cup \partial \widetilde{U}_\delta)$;

$$\begin{aligned} &\widetilde{R}_{1,+}(\lambda) - \widetilde{R}_{1,-}(\lambda) = \mathcal{D}_\infty^{-\sigma_3}\Delta_1(\lambda)\mathcal{D}_\infty^{\sigma_3}, \\ &\widetilde{R}_{2,+}(\lambda) - \widetilde{R}_{2,-}(\lambda) = \widetilde{R}_{1,-}(\lambda)\mathcal{D}_\infty^{-\sigma_3}\Delta_1(\lambda)\mathcal{D}_\infty^{\sigma_3} + \mathcal{D}_\infty^{-\sigma_3}\Delta_2(\lambda)\mathcal{D}_\infty^{\sigma_3}, \quad \lambda \in \partial U_\delta \cup \partial\widetilde{U}_\delta; \end{aligned} \tag{118}$$

and $\widetilde{R}_1 \to 0$, $\widetilde{R}_2 \to 0$ as $\lambda \to \infty$.

As $\Delta_1(\lambda)$ is analytic in neighborhoods of the endpoints except for the simple poles at ± 1, we have an expansion

$$\begin{aligned} \mathcal{D}_\infty^{-\sigma_3}\Delta_1(\lambda)\mathcal{D}_\infty^{\sigma_3} &= \frac{A^{(1)}}{\lambda-1} + A^{(2)} + A^{(3)}(\lambda-1) + O((\lambda-1)^2), \quad \text{as } \lambda \to 1; \\ \mathcal{D}_\infty^{-\sigma_3}\Delta_1(\lambda)\mathcal{D}_\infty^{\sigma_3} &= \frac{B^{(1)}}{\lambda+1} + B^{(2)} + B^{(3)}(\lambda+1) + O((\lambda+1)^2), \quad \text{as } \lambda \to -1, \end{aligned} \tag{119}$$

where A and B are constant matrices. It is now easy to verify that the RH problem for $\widetilde{R}_1$ has the following solution:

$$\widetilde{R}_1(\lambda) = \begin{cases} \frac{A^{(1)}}{\lambda-1} + \frac{B^{(1)}}{\lambda+1}, & \text{for } \lambda \in \mathbb{C} \setminus \overline{U_\delta} \cup \overline{\widetilde{U}_\delta}\ , \\ \frac{A^{(1)}}{\lambda-1} + \frac{B^{(1)}}{\lambda+1} - \mathcal{D}_\infty^{-\sigma_3}\Delta_1(\lambda)\mathcal{D}_\infty^{\sigma_3}, & \text{for } \lambda \in U_\delta \cup \widetilde{U}_\delta\ . \end{cases} \tag{120}$$

For the function Δ_2 we can write similarly:

$$\begin{aligned} \mathcal{D}_\infty^{-\sigma_3}\Delta_2(\lambda)\mathcal{D}_\infty^{\sigma_3} &= \frac{C^{(1)}}{\lambda-1} + C^{(2)} + O(\lambda-1), \quad \text{as } \lambda \to 1; \\ \mathcal{D}_\infty^{-\sigma_3}\Delta_2(\lambda)\mathcal{D}_\infty^{\sigma_3} &= \frac{D^{(1)}}{\lambda+1} + D^{(2)} + O(\lambda+1), \quad \text{as } \lambda \to -1, \end{aligned} \tag{121}$$

where C and D are constant matrices. A similar expression to (120) can now be written for R_2. However, we will need it only in the limit $\lambda \to \pm 1$. We then obtain:

$$\begin{aligned}\widetilde{R}_2(1) &= -\frac{1}{4}[A^{(1)}, B^{(1)}] - \frac{1}{2}(B^{(2)}B^{(1)} + B^{(1)}A^{(2)} - D^{(1)}) + {A^{(2)}}^2 + A^{(3)}A^{(1)} - C^{(2)} \\ \widetilde{R}_2(-1) &= \frac{1}{4}[A^{(1)}, B^{(1)}] + \frac{1}{2}(A^{(2)}A^{(1)} + A^{(1)}B^{(2)} - C^{(1)}) + {B^{(2)}}^2 + B^{(3)}B^{(1)} - D^{(2)},\end{aligned} \tag{122}$$

where $[A, B] = AB - BA$. We also obtain from (120),

$$\widetilde{R}_1(1) = \frac{1}{2}B^{(1)} - A^{(2)}, \qquad \widetilde{R}_1(-1) = -\frac{1}{2}A^{(1)} - B^{(2)}. \tag{123}$$

5.4 Asymptotic form of the differential identity (49)

We will now compute asymptotics of the r.h.s. of (49). First, we determine the components of the matrices $A^{(j)}, \ldots, D^{(j)}$ we need for the calculation below. First, the expansions for $\omega(\lambda)$ in Section 5.2 can be written as

$$\begin{aligned}\frac{1}{\sqrt{\omega(\lambda)}} &= \frac{1}{(2u)^{1/2}\sin\frac{\varphi}{2}}(1 + \omega_{1+}u + \omega_{2+}u^2 + O(u^3)), \qquad u = \lambda - 1, \\ \frac{1}{\sqrt{\omega(\lambda)}} &= \frac{1}{-i(2u)^{1/2}\sin\frac{\varphi}{2}}(1 + \omega_{1-}u + \omega_{2-}u^2 + O(u^3)), \qquad u = \lambda + 1,\end{aligned}$$

with

$$\omega_{1\pm} = \pm\frac{1}{3}\left(\cos\varphi + \frac{5}{4}\right).$$

(The explicit expression for $\omega_{2\pm}$ will not be needed below.) Therefore, expanding $N(\lambda)$ we obtain from (108)

$$\begin{aligned}A^{(1)} &= \frac{1}{16\rho}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \qquad A^{(2)} = \frac{1}{16\rho}\left[\omega_{1+}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} + \Gamma_0\right], \\ \Gamma_0 &= \frac{1}{2}\begin{pmatrix} \gamma_0 & -i(6 + \gamma_0 - 8\mathcal{D}_{1+}) \\ -i(6 + \gamma_0 + 8\mathcal{D}_{1+}) & -\gamma_0 \end{pmatrix}\end{aligned} \tag{124}$$

with

$$\rho = n\sin\frac{\varphi}{2}, \qquad \gamma_0 = -\frac{5}{2} + 4\mathcal{D}_{1+}^2.$$

Here $\mathcal{D}_{1+}$ is given by (100). Furthermore,

$$\begin{aligned}A^{(3)} &= \frac{1}{16\rho}\left[\omega_{2+}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} + \omega_{1+}\Gamma_0 + \Gamma_1\right], \\ \Gamma_1 &= \frac{1}{4}\begin{pmatrix} \gamma_1 & -i(\gamma_2 - 8\mathcal{D}_{2+} - \frac{16}{3}\mathcal{D}_{1+}^3) \\ -i(\gamma_2 + 8\mathcal{D}_{2+} + \frac{16}{3}\mathcal{D}_{1+}^3) & -\gamma_1 \end{pmatrix},\end{aligned} \tag{125}$$

with

$$\gamma_1 = \frac{11}{8} - 2\mathcal{D}_{1+}^2 + 8\mathcal{D}_{1+}\mathcal{D}_{2+} + \frac{4}{3}\mathcal{D}_{1+}^4, \qquad \gamma_2 = \gamma_1 + 8\mathcal{D}_{1+}^2 - 3.$$

We obtain the matrices $B^{(j)}$, $j = 1, 2, 3$, by taking the matrices $A^{(j)}$, $j = 1, 2, 3$, and replacing ρ with $-\rho$; i with $-i$; $\omega_{1,2+}$ with $\omega_{1,2-}$; $\mathcal{D}_{1,2+}$ with $\mathcal{D}_{1,2-}$; and the prefactor $\frac{1}{2}$ of Γ_0 with $-\frac{1}{2}$.

The components of C are also obtained from (108). For C (and similarly for D) we will only need the following combination of matrix elements:

$$C_{21}^{(1)} - C_{12}^{(1)} = \frac{-3i}{2^5\rho^2}, \qquad C_{21}^{(2)} - C_{12}^{(2)} = \frac{-3i}{2^5\rho^2}(2\omega_{1+} + \mathcal{D}_{1+}^2). \tag{126}$$

Similarly,

$$D_{21}^{(1)} - D_{12}^{(1)} = \frac{-3i}{2^5\rho^2}, \qquad D_{21}^{(2)} - D_{12}^{(2)} = \frac{3i}{2^5\rho^2}(-2\omega_{1-} + \mathcal{D}_{1-}^2). \tag{127}$$

We now need to evaluate the asymptotics of $\operatorname{tr} C_+C_-$ and $\operatorname{tr} C_1(C_+ + C_-)$ from (49). Tracing back the transformations of the RH problem, we see that $Y(z)$ for z close to z_+, so that λ is close to 1 in the region III, is given by

$$Y(z) = \gamma^{n\sigma_3} T_0 R P \Psi^{n\sigma_3}, \qquad P(\lambda) = E(\lambda) Q(n^2\omega(\lambda)) e^{-n\sqrt{\omega(\lambda)}\sigma_3} f(\lambda)^{-\sigma_3/2},$$

where E is given by (104), and Q by (107). Substituting this into (48) and expanding Bessel functions at $\zeta = 0$, we obtain:

$$C_+ = \frac{\rho}{2i}\gamma^{n\sigma_3} T_0 \mathcal{D}_\infty^{\sigma_3} \widetilde{R}(1) \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \widetilde{R}(1)^{-1} \mathcal{D}_\infty^{-\sigma_3} T_0^{-1} \gamma^{-n\sigma_3}. \tag{128}$$

Here we can write

$$\widetilde{R}(1) = I + \widetilde{R}_1(1) + \widetilde{R}_2(1) + O(\rho^{-3}), \qquad \widetilde{R}(1)^{-1} = I - \widetilde{R}_1(1) - \widetilde{R}_2(1) + \widetilde{R}_1^2(1) + O(\rho^{-3}).$$

For definitiveness, we assume that the roots in (93) are chosen with the arguments from 0 to 2π, and that the point λ corresponding to the limit in (48) approaches $\lambda = 1$ from above.

For C_- we similarly have

$$C_- = \frac{\rho}{-2i}\gamma^{n\sigma_3} T_0 \mathcal{D}_\infty^{\sigma_3} \widetilde{R}(-1) \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix} \widetilde{R}(-1)^{-1} \mathcal{D}_\infty^{-\sigma_3} T_0^{-1} \gamma^{-n\sigma_3}. \tag{129}$$

Therefore, we now easily obtain

$$\operatorname{tr} C_+C_- = \rho^2 \left[1 + t_1 + t_2 + O\left(\frac{1}{\rho^3}\right)\right], \tag{130}$$

where (recall first (69), and then (123) and the above expressions for A, B)

$$t_1 = i(\widetilde{R}_{1,21}(1) - \widetilde{R}_{1,12}(1) - [\widetilde{R}_{1,21}(-1) - \widetilde{R}_{1,12}(-1)]) = -\frac{2\alpha}{\rho}\left(1 - \sin\frac{\varphi}{2}\right). \tag{131}$$

The expression for t_2 needs more work. First, we obtain it in the form

$$t_2 = i(\widetilde{R}_{2,21}(1) - \widetilde{R}_{2,12}(1) - [\widetilde{R}_{2,21}(-1) - \widetilde{R}_{2,12}(-1)]) + \Omega, \tag{132}$$

where

$$4\Omega = \operatorname{tr}\left\{\widetilde{R}_1(1)\left[\widetilde{R}_1(1), \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\right]\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}\right\}$$
$$+\operatorname{tr}\left\{\widetilde{R}_1(-1)\left[\widetilde{R}_1(-1), \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}\right]\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\right\} + \operatorname{tr}\left\{\left[\widetilde{R}_1(1), \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}\right]\left[\widetilde{R}_1(-1), \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\right]\right\}.$$

Denoting the matrix elements

$$\widetilde{R}_1(1) = \begin{pmatrix} a & ib \\ ic & -a \end{pmatrix}, \qquad \widetilde{R}_1(-1) = \begin{pmatrix} \hat{a} & -i\hat{b} \\ -i\hat{c} & -\hat{a} \end{pmatrix}$$

we obtain after a simple algebraic computation that

$$\Omega = \frac{1}{4}(b - c + \hat{b} - \hat{c})^2 + (a - \hat{a})^2 - (c + \hat{c})(b + \hat{b}) - a(\hat{b} + \hat{c}) - \hat{a}(b + c).$$

Using again (123) and the above expressions for A, B, we deduce from this formula that

$$\Omega = \frac{-1}{2^7\rho^2}(4(\mathcal{D}_{1+}^2 + \mathcal{D}_{1-}^2) + 2(\omega_{1+} - \omega_{1-}) + 5 - 16(\mathcal{D}_{1+} + \mathcal{D}_{1-})^2). \tag{133}$$

On the other hand, we obtain from (122) and the above expressions for A, B, C, D:

$$i(\widetilde{R}_{2,21}(1) - \widetilde{R}_{2,12}(1) - [\widetilde{R}_{2,21}(-1) - \widetilde{R}_{2,12}(-1)]) = \frac{1}{2^7\rho^2}(9 - 22(\omega_{1+} - \omega_{1-}) - 12(\mathcal{D}_{1+}^2 + \mathcal{D}_{1-}^2)). \tag{134}$$

Thus, substituting (133), (134) into (132) and then using the expressions for $\omega_{1\pm}$, $\mathcal{D}_{1+}^2$, we obtain

$$t_2 = \frac{1}{2^5\rho^2}(1 - 6(\omega_{1+} - \omega_{1-}) + 8\mathcal{D}_{1+}\mathcal{D}_{1-}) = \frac{1}{\rho^2}\left(\alpha^2\left(1 - \sin\frac{\varphi}{2}\right)^2 + \beta^2\cos^2\frac{\varphi}{2} - \frac{1}{4}\cos^2\frac{\varphi}{2}\right). \tag{135}$$

Collecting together (131), (135), we finally have

$$\operatorname{tr} C_+C_- = n^2\sin^2\frac{\varphi}{2} - 2\alpha n\sin\frac{\varphi}{2}\left(1 - \sin\frac{\varphi}{2}\right) - \frac{1}{4}\cos^2\frac{\varphi}{2} + \alpha^2\left(1 - \sin\frac{\varphi}{2}\right)^2 + \beta^2\cos^2\frac{\varphi}{2} + O(\rho^{-1}). \tag{136}$$

We now turn our attention to the quantity $\operatorname{tr} C_1(C_+ + C_-)$. Recall that $z = 1$ corresponds to $\lambda = \infty$. We, therefore, immediately obtain

$$C_1 = \alpha Y(1)\sigma_3 Y(1)^{-1} = \alpha\gamma^{n\sigma_3}T_0\sigma_3T_0^{-1}\gamma^{-n\sigma_3}. \tag{137}$$

In $C_+ + C_-$ we now need to take into account only the terms of order no less than $\widetilde{R}_1$. Using (128), (129), and at the last step (131), we obtain

$$\operatorname{tr} C_1(C_+ + C_-) = \alpha n\sin\frac{\varphi}{2}(2 + t_1) + O(\rho^{-1}) = 2\alpha n\sin\frac{\varphi}{2} - 2\alpha^2\left(1 - \sin\frac{\varphi}{2}\right) + O(\rho^{-1}). \tag{138}$$

Substituting this expression and (136) into (49), we finally have

Proposition 9 *As $n \to \infty$,*

$$\frac{d^2}{d\varphi^2}\ln D_n(\varphi) = -\frac{n^2}{4\cos^2\frac{\varphi}{2}} - (\alpha n + \alpha^2)\frac{1 - \sin\frac{\varphi}{2}}{2\cos^2\frac{\varphi}{2}} + \frac{1 + 4(\alpha^2 - \beta^2)}{16\sin^2\frac{\varphi}{2}} + O\left(\frac{1}{n\sin\frac{\varphi}{2}}\right), \tag{139}$$

uniformly for $\frac{2s}{n} < \varphi < \pi$, $n > s$, $s > s_0$, for some $s_0 > 0$, and uniformly in compact subsets of the half-plane $\Re\alpha > -1/2$ and of the plane $\beta \in \mathbb{C}$.

6 Proof of Theorem 1

The asymptotic evaluation of the Toeplitz determinant $D_n(\varphi)$ is based on the integration of the differential identity (139) from φ to $\pi-\varepsilon$ with a small positive ε. We have:

$$(\pi-\varepsilon-\varphi)(\ln D_n)'|_{\pi-\varepsilon} - \ln D_n(\pi-\varepsilon) + \ln D_n(\varphi) = \int_{\varphi}^{\pi-\varepsilon} d\theta \int_{\theta}^{\pi-\varepsilon} I(\phi)d\phi, \tag{140}$$

where $I(\varphi)$ is the r.h.s. of (139). Fix n. Calculating the integral on the r.h.s. of (140), substituting for $\ln D_n(\pi-\varepsilon)$ the expansion (72) (and for $(\ln D_n)'|_{\pi-\varepsilon}$ its derivative), and taking the limit $\varepsilon\to 0$, we obtain

$$\begin{aligned}\ln D_n(\varphi) = n^2\ln\cos\frac{\varphi}{2} + 2(\alpha n+\alpha^2)\ln\left(1+\sin\frac{\varphi}{2}\right) - \frac{1}{4}\ln n + \frac{1}{12}\ln 2 + 3\zeta'(-1)\\ -2\alpha^2\ln 2 - \left(\frac{1}{4}-\beta^2+\alpha^2\right)\ln\sin\frac{\varphi}{2} + O\left(\frac{1}{n\sin\frac{\varphi}{2}}\right) + \delta_n.\end{aligned} \tag{141}$$

uniformly for $\frac{2s}{n}<\varphi<\pi$, $n>s$, $s>s_0$, and where $\delta_n\to 0$ as $n\to\infty$. This expansion is uniform in compact subsets of the α-half-plane $\Re\alpha>-1/2$ and of the β-plane. Now substituting (38) and (141) into (30) and taking the limit $n\to\infty$, we obtain (8). □

7 Bessel kernel. Proof of Theorem 4.

In this section we set $\beta=0$. The weight (15) is then

$$f(z,\varphi)=|z-1|^{2\alpha}, \quad z=e^{i\theta}, \quad \varphi\le\theta\le 2\pi-\varphi, \tag{142}$$

and $f(z,\varphi)=0$ on the rest of the unit circle. Note that f is an even function of the angle θ. Let

$$\omega(x,\varphi)=\frac{f(e^{i\theta},\varphi)}{|\sin\theta|}=2^{\alpha}\frac{(1-x)^{\alpha}}{\sqrt{1-x^2}}, \quad x=\cos\theta. \tag{143}$$

This function is supported on $[-1,\cos\varphi]$. Consider the Hankel determinant with symbol $\omega(x,\varphi)$:

$$\begin{aligned}D_n^H(\varphi) &= \det\left(\int_{-1}^{\cos\varphi} x^{j+k}\omega(x,\varphi)dx\right)_{j,k=0}^{n-1}\\ &= \frac{1}{n!}\int_{-1}^{\cos\varphi}\cdots\int_{-1}^{\cos\varphi}\prod_{1\le j<k\le n}(x_j-x_k)^2\prod_{j=1}^{n}\omega(x_j,\varphi)dx_j.\end{aligned} \tag{144}$$

There holds the following

Lemma 10 *Let $K_{Bessel2}^{(a)}$ be the operator acting on $L^2(0,(2s)^2)$, with kernel (11). Then*

$$\det(I-K_{Bessel2}^{(\alpha-1/2)})_{L^2(0,(2s)^2)} = \lim_{n\to\infty}\frac{D_n^H(\frac{2s}{n})}{D_n^H(0)}. \tag{145}$$

Proof. First, as in the proof of the formula (33) of Lemma 6, one obtains that

$$D_n^H(\varphi) = D_n^H(0)\det(I - \widetilde{K}_n)_{L^2(\cos\varphi,1)} \tag{146}$$

where $\widetilde{K}_n$ is the operator acting on $L^2((\cos\varphi, 1), dx)$ with kernel

$$\widetilde{K}_n(x,y) = \sqrt{\omega(x)\omega(y)}\frac{\varkappa_{n-1}}{\varkappa_n}\frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{x-y}, \tag{147}$$

where $P_k(x) = \varkappa_k x^k + \cdots$, $k = N_0, N_0+1, \ldots$, with some $N_0 \geq 0$, are the polynomials orthonormal on $[-1,1]$ w.r.t. $w(x,0)$:

$$\int_{-1}^{1} P_k(x)x^m\omega(x,0)dx = \varkappa_k^{-1}\delta_{km}, \quad m = 0,1,\ldots,k.$$

The choice of the function (143) implies the following Szegő relations between $P_k(x)$ and the polynomials $q_k(z) = \chi_k z^k + \cdots$, given by (19) (see Lemma 2.5. of [10]):

$$P_k(x) = \frac{1}{\sqrt{2\pi(1 + q_{2k}(0)/\chi_{2k})}}(z^{-k}q_{2k}(z) + z^k q_{2k}(z^{-1})), \tag{148}$$

and for the leading coefficients,

$$\varkappa_k = 2^k\chi_{2k}\sqrt{\frac{1 + q_{2k}(0)/\chi_{2k}}{2\pi}}.$$

Note that $\widehat{q}_n(z) = q_n(z)$ as in our case $f(e^{i\theta}) = f(e^{-i\theta})$.

Similarly to the proof of Lemma 6, we now set $x = \cos(2u/n)$, $y = \cos(2v/n)$, fix $0 < u < s$ and $0 < v < s$ and consider the limit of $\widetilde{K}_n(x,y)$ as $n \to \infty$. We will now show that this double-scaling limit gives the kernel of $K_{Bessel2}^{(\alpha-1/2)}$.

First, it follows from Theorem 1.8. of [10] that

$$\frac{q_{2n}(0)}{\chi_{2n}} = O\left(\frac{1}{n}\right), \qquad \frac{\varkappa_{n-1}}{\varkappa_n} = \frac{1}{2} + O\left(\frac{1}{n}\right).$$

Moreover, we have,

$$x = \cos\left(\frac{2u}{n}\right) = 1 - \frac{2u^2}{n^2} + o(n^{-2}), \qquad z(x) = e^{2iu/n},$$

and similarly for $y = \cos(2v/n)$. Therefore, by (143),

$$\sqrt{\omega(x,0)\omega(y,0)} = \left(\frac{2}{n}\right)^{2\alpha-1}(uv)^{\alpha-1/2}(1 + O(n^{-1})). \tag{149}$$

Moreover, taking $z_1 = e^{2iu/n} \in \mathbb{C}_+$ and using the expression (21) of Lemma 5, we have

$$
\begin{aligned}
&q_{2(n+k)}(z_1) = (2n)^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \phi(1+\alpha, 1+2\alpha, 4iu(1+k/n))(1+O(1/n)) \\
&= (2n)^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\phi(1+\alpha,1+2\alpha,4iu) + \frac{4iuk}{n}\phi'(1+\alpha,1+2\alpha,4iu)\right](1+O(1/n)), \\
&k = 0, -1.
\end{aligned}
$$

Substituting this into (148), we then obtain an expression for $P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)$ in terms of the confluent hypergeometric functions and their derivatives at $\pm 4iu$, $\pm 4iv$. Removing the derivatives with the help of the standard relation

$$
\phi'(a,c,x) = \frac{a}{x}(\phi(a+1,c,x) - \phi(a,c,x))
$$

reducing then, by Kummer's transformation (27), the terms with the arguments $-4iu$, $-4iv$ to functions of the arguments $4iu$, $4iv$, and making use of the following standard recurrence relation

$$
(c-a)\phi(a-1,c,x) + (2a-c+x)\phi(a,c,x) - a\phi(a+1,c,x) = 0,
$$

we obtain

$$
\begin{aligned}
&P_n(x_1)P_{n-1}(x_2) - P_{n-1}(x_1)P_n(x_2) = \frac{i}{\pi}\frac{\Gamma^2(1+\alpha)}{\Gamma^2(1+2\alpha)}(2n)^{2\alpha} e^{-2i(u+v)}\frac{1}{n}\left(1+O\left(\frac{1}{n}\right)\right) \\
&\times \{(u+v)[\phi(1+\alpha,1+2\alpha,4iu)\phi(\alpha,1+2\alpha,4iv) - \phi(1+\alpha,1+2\alpha,4iv)\phi(\alpha,1+2\alpha,4iu)] \\
&+(u-v)[\phi(1+\alpha,1+2\alpha,4iu)\phi(1+\alpha,1+2\alpha,4iv) - \phi(\alpha,1+2\alpha,4iu)\phi(\alpha,1+2\alpha,4iv)]\}\,.
\end{aligned}
\tag{150}
$$

Thus, in the difference $P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)$ the main terms in n dropped out leaving the ones of order $1/n$. Moreover, we did not need to know expressions for the terms $O(1/n)$ in Lemma 5, as their contribution to the terms of order $1/n$ dropped out as well.

We now employ the recurrence relations (36) to express $\phi(\alpha, 1+2\alpha, x)$ and $\phi(1+\alpha, 1+2\alpha, x)$ in terms of $\phi(\alpha, 2\alpha, x)$ and $\phi(1+\alpha, 2+2\alpha, x)$ and then use the connection with Bessel functions (4). We obtain recalling (149):

$$
\widetilde{K}_n(x,y) = \frac{n^2}{2}\frac{[uJ_{\alpha+1/2}(2u)J_{\alpha-1/2}(2v) - vJ_{\alpha+1/2}(2v)J_{\alpha-1/2}(2u)]}{u^2 - v^2}\left(1+O\left(\frac{1}{n}\right)\right),
$$

which leads to

$$
\lim_{n\to\infty}\left(-\frac{1}{2n^2}\right)\widetilde{K}_n(x,y) = -K_{Bessel2}^{(\alpha-1/2)}((2u)^2,(2v)^2), \tag{151}
$$

where $-K_{Besssel2}^{(\alpha-1/2)}$ acts on $((2s)^2, 0)$ (note the reversed direction) and

$$
K_{Bessel2}^{(a)}(x,y) = \frac{\sqrt{x}J_{a+1}(\sqrt{x})J_a(\sqrt{y}) - \sqrt{y}J_{a+1}(\sqrt{y})J_a(\sqrt{x})}{2(x-y)}, \quad a = \alpha - 1/2, \tag{152}
$$

which, by the relation $zJ_{a+1}(z) = aJ_a(z) - zJ'_a(z)$, is equivalent to (11).

The convergence of the determinants follows from the convergence of the kernels as in Lemma 6, and we obtain the statement (145) from (146). □

We now evaluate the r.h.s. of (145). First, note that $D_n^H(0)$ is a Hankel determinant whose symbol $\omega(x,0)$ (143) is supported on $[-1,1]$ and has two Fisher Hartwig singularities at $x=-1$ and $x=1$. Therefore, the asymptotics of $D_n^H(0)$ are given by a particular case of Theorem 1.20 from [10]. Namely,

$$D_n^H(0) = \frac{\pi^{n+\alpha/2}G(1/2)}{G(1/2+\alpha)} 2^{-(n-1)^2-\frac{\alpha^2}{2}+\frac{3\alpha}{2}} n^{\frac{\alpha^2-\alpha}{2}} (1+o(1)), \qquad n\to\infty, \tag{153}$$

uniformly in compact subsets of the half-plane $\Re\alpha > -1/2$.

In order to evaluate $D_n^H(2s/n)$, we use a connection formula between Hankel and Toeplitz determinants given by Theorem 2.6 in [10]. The formula is written in terms of the matrix elements of $Y^{(2n)}(z)$ (39) and for $\varphi = 2s/n$ as follows:

$$\left(D_n^H\left(\frac{2s}{n}\right)\right)^2 = \frac{\pi^{2n}}{2^{2(n-1)^2}} \frac{(1+Y_{11}^{(2n)}(0))^2}{Y_{11}^{(2n)}(1)Y_{11}^{(2n)}(-1)} D_{2n}\left(\frac{2s}{n}\right). \tag{154}$$

The asymptotic expression for the Toeplitz determinant $D_{2n}\left(\frac{2s}{n}\right)$ is given (uniformly in compact subsets of the half-plane $\Re\alpha > -1/2$) by (141) with n replaced by $2n$ and with φ set to be $2s/n$:

$$\begin{aligned}\ln D_{2n}\left(\frac{2s}{n}\right) &= -2s^2 + 4\alpha s - \left(\alpha^2+\frac{1}{4}\right)\ln s + \alpha^2 \ln n - \left(2\alpha^2+\frac{1}{4}\right)\ln 2 \\ &+\frac{1}{2}\ln\pi + 2\ln G(1/2) + O\left(\frac{1}{s}\right) + \hat{\delta}_n(s), \qquad n\to\infty,\end{aligned} \tag{155}$$

where $\hat{\delta}_n(s)\to 0$ as $n\to\infty$.

It now remains to estimate $Y_{11}^{(2n)}(z)$ at $z=-1,0,1$. The λ-images of these points are $\lambda = 0, -i\tan\frac{\varphi}{2}, \infty$, respectively. All of them lie in the regions where $Y^{(2n)}(z)$ is approximated by the outside parametrix $N(\lambda)$. From the expressions (75,81,85–87,111,112,117), we obtain:

$$Y(z)^{(2n)} = \gamma^{2n\sigma_3} N(i\tan\frac{\varphi}{2})^{-1}\left(I + \mathcal{D}_\infty^{\sigma_3} O(\rho^{-1})\mathcal{D}_\infty^{-\sigma_3}\right) N(\lambda(z)) \begin{pmatrix} 1 & 0 \\ b(z) & 1\end{pmatrix} \Psi(z)^{2n\sigma_3}, \tag{156}$$

which is valid for z in neighborhoods of $z=-1,0,1$. In a neighborhood of -1 we assume that $|z|>1$ and then $b(z) = f(z)^{-1}\Psi(z)^{-4n}z^{2n}$. In neighborhoods of 0 and 1, $b(z)=0$.

We will need the values of $\mathcal{D}(\lambda)$ at $\lambda = 0, -i\tan\frac{\varphi}{2}, \infty$. Analyzing the integral in (94) with $\beta=0$, we obtain

$$\mathcal{D}\left(i\tan\frac{\varphi}{2}\right) = D\left(-i\tan\frac{\varphi}{2}\right) = \left(1+\sin\frac{\varphi}{2}\right)^\alpha, \qquad \mathcal{D}(0) = 2^\alpha. \tag{157}$$

From (95) we have

$$\mathcal{D}_\infty = \left(\frac{4\sin\frac{\varphi}{2}}{1+\sin\frac{\varphi}{2}}\right)^\alpha. \tag{158}$$

Noting that in the definition of $N(i\tan\frac{\varphi}{2})$, $a(i\tan\frac{\varphi}{2}) = e^{i(\pi-\varphi)/4}$, we can write the following expression for the 11 element of (156):

$$\begin{aligned}Y_{11}^{(2n)}(z) &= \frac{1}{2}\gamma^{2n}\Psi(z)^{2n}\left(1+\sin\frac{\varphi}{2}\right)^\alpha \Big[\mathcal{D}(\lambda(z))^{-1}\left(e^{i(\varphi-\pi)/4}a(\lambda) + e^{-i(\varphi-\pi)/4}a^{-1}(\lambda)\right) \\ &+ b(z)\mathcal{D}(\lambda(z))i^{-1}\left(e^{i(\varphi-\pi)/4}a(\lambda) - e^{-i(\varphi-\pi)/4}a^{-1}(\lambda)\right)\Big](1+O(\rho^{-1})).\end{aligned} \tag{159}$$

Since by (73),

$$\Psi(-1) = -1, \qquad \Psi(0) = \gamma^{-1}, \qquad \Psi(1) = \gamma^{-1}\left(1 + \sin\frac{\varphi}{2}\right),$$

we obtain from (159)

$$Y_{11}^{(2n)}(-1) = \gamma^{2n}2^{-\alpha}\left(1+\sin\frac{\varphi}{2}\right)^{\alpha}\left[\cos\frac{\varphi}{4}+\sin\frac{\varphi}{4}\right](1+O(\rho^{-1})) = 2^{-\alpha}[1+O(s^{-1})], \quad n > s^2, \tag{160}$$

where the second equation is obtained by substituting $\varphi = 2s/n$, and considering $n > s^2$ and s large. Similarly,

$$Y_{11}^{(2n)}(0) = \sin\frac{\varphi}{2}(1+O(\rho^{-1})) = \frac{s}{n}[1+O(s^{-1})], \quad n > s, \tag{161}$$

and

$$Y_{11}^{(2n)}(1) = \frac{\left(1+\sin\frac{\varphi}{2}\right)^{2(n+\alpha)}}{2^{2\alpha}\sin^{\alpha}\frac{\varphi}{2}}\cos\frac{\varphi-\pi}{4}(1+O(\rho^{-1})) = \frac{e^{2s}n^{\alpha}}{2^{2\alpha+1/2}s^{\alpha}}[1+O(s^{-1})], \quad n > s^2. \tag{162}$$

Note that (160–162) are uniform in compact subsets of the half-plane $\Re\alpha > -1/2$.

Substituting (160–162) and (155) into (154) and taking the square root, we obtain

$$D_n^H\left(\frac{2s}{n}\right) = \pi^{n+\frac{1}{4}}n^{\frac{\alpha^2-\alpha}{2}}2^{-(n-1)^2+\frac{3}{2}\alpha-\alpha^2+\frac{1}{8}}G(1/2)s^{-\frac{1}{2}\left(\alpha-\frac{1}{2}\right)^2}e^{-s^2+(2\alpha-1)s}[1+O(s^{-1})], \quad n > s^2. \tag{163}$$

The branch of the square root is fixed by the fact that $D_{2n}^H > 0$ for $\alpha \in \mathbb{R}$, and by the uniformity of the asymptotic expansion in α. Finally, substituting (163) and (153) into (145), we finish the proof of Theorem 4. □

8 Appendix

Here we show that the operator $K^{(\alpha,\beta)}$ with kernel (1) on $L^2(-s,s)$, where $\Re\alpha > -1/2$ and $\alpha \pm \beta \neq -1, 2, \ldots$, is trace class. Note first that since $g_\beta^{1/2}(x)$ is bounded on $(-s,s)$, it is sufficient to show that the operator $\widehat{K}$ with the following kernel

$$\begin{aligned}\widehat{K}(x,y) = \frac{|x|^\alpha|y|^\alpha}{x-y}[&e^{i(y-x)}\phi(1+\alpha+\beta,1+2\alpha,2ix)\phi(1+\alpha-\beta,1+2\alpha,-2iy)\\ -&e^{i(x-y)}\phi(1+\alpha+\beta,1+2\alpha,2iy)\phi(1+\alpha-\beta,1+2\alpha,-2ix)]\end{aligned} \tag{164}$$

is trace class. Expanding the confluent hypergeometric functions in series,

$$e^{-ix}\phi(1+\alpha+\beta,1+2\alpha,2ix) = \sum_{n=0}^{\infty}\mu_n x^n, \qquad e^{ix}\phi(1+\alpha-\beta,1+2\alpha,-2ix) = \sum_{m=0}^{\infty}\lambda_m x^m,$$

where μ_n, λ_n are determined using (2), we can write the kernel (164) in the form:

$$\widehat{K}(x,y) = |x|^\alpha|y|^\alpha\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mu_n\lambda_m\frac{x^ny^m - y^nx^m}{x-y}. \tag{165}$$

We will now show that the trace norm in $L^2(-s,s)$

$$\||x|^\alpha \frac{x^n y^m - y^n x^m}{x-y}|y|^\alpha\|_1 \leq Cn(m+1)s^{2\Re\alpha+m+n}, \qquad m \geq 0, \quad n \geq 1 \tag{166}$$

for some $C > 0$. Together with the straightforward estimates

$$|\mu_n|, |\lambda_n| \leq \frac{n^b}{n!}, \qquad n \geq 1,$$

for some $b \in \mathbb{R}$, the inequality (166) implies that $\widehat{K}$, and hence $K^{(\alpha,\beta)}$, is trace class.

To prove (166), set first $m = 0$, $n \geq 1$. Then we have for some $C > 0$

$$\begin{aligned} &\||x|^\alpha \frac{x^n - y^n}{x-y}|y|^\alpha\|_1 \leq \sum_{k=0}^{n-1} \||y|^{k+\Re\alpha}\|_{L^2(-s,s)}\||x|^{n-k-1+\Re\alpha}\|_{L^2(-s,s)} = \\ &\sum_{k=0}^{n-1}\left(\frac{2s^{2(k+\Re\alpha)+1}}{2(k+\Re\alpha)+1}\right)^{1/2}\left(\frac{2s^{2(n-k+\Re\alpha)-1}}{2(n-k+\Re\alpha)-1}\right)^{1/2} \leq \\ &C\sum_{k=0}^{n-1} s^{\frac{1}{2}(2(k+\Re\alpha)+1)+\frac{1}{2}(2(n-k+\Re\alpha)-1)} = Cns^{n+2\Re\alpha}, \end{aligned} \tag{167}$$

which gives (166) for $m = 0$, $n \geq 1$. If $m \geq 1$, $n \geq 1$, we can assume that $n > m$ and write

$$|x|^\alpha \frac{x^n y^m - y^n x^m}{x-y}|y|^\alpha = |x|^{\alpha+m}\frac{x^{n-m} - y^{n-m}}{x-y}|y|^{\alpha+m},$$

which is of the same form as (167) with α replaced by $\alpha+m$, and n, by $n-m$. Hence, we complete the proof of (166).

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