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ASYMPTOTICS OF TOEPLITZ, HANKEL, AND TOEPLITZ+HANKEL DETERMINANTS WITH FISHER-HARTWIG SINGULARITIES

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Abstract. We study the asymptotics in n for n-dimensional Toeplitz determinants whose symbols possess Fisher-Hartwig singularities on a smooth background. We prove the general non-degenerate asymptotic behavior as conjectured by Basor and Tracy. We also obtain asymptotics of Hankel determinants on a finite interval as well as determinants of Toeplitz+Hankel type. Our analysis is based on a study of the related system of orthogonal polynomials on the unit circle using the Riemann-Hilbert approach.

1. Introduction

Let $f(z)$ be a complex-valued function integrable over the unit circle with Fourier coefficients

$$f_j = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ij\theta}d\theta, \qquad j = 0, \pm 1, \pm 2, \dots$$

We are interested in the n-dimensional Toeplitz determinant with symbol $f(z)$,

$$D_n(f(z)) = \det(f_{j-k})_{j,k=0}^{n-1}. \tag{1.1}$$

In this paper we consider the asymptotics of $D_n(f(z))$ as $n \to \infty$ and of the related orthogonal polynomials as well as the asymptotics of Hankel, and Toeplitz+Hankel determinants in the case when the symbol $f(e^{i\theta})$ has a fixed number of Fisher-Hartwig singularities [23, 34], i.e., when $f(e^{i\theta})$ has the following form on the unit circle C:

$$f(z) = e^{V(z)} z^{\sum_{j=0}^m \beta_j} \prod_{j=0}^m |z - z_j|^{2\alpha_j} g_{z_j,\beta_j}(z) z_j^{-\beta_j}, \qquad z = e^{i\theta}, \qquad \theta \in [0, 2\pi), \tag{1.2}$$

for some $m = 0, 1, \dots$, where

$$z_j = e^{i\theta_j}, \quad j = 0, \dots, m, \qquad 0 = \theta_0 < \theta_1 < \dots < \theta_m < 2\pi; \tag{1.3}$$

$$g_{z_j,\beta_j}(z) \equiv g_{\beta_j}(z) = \begin{cases} e^{i\pi\beta_j} & 0 \le \arg z < \theta_j \\ e^{-i\pi\beta_j} & \theta_j \le \arg z < 2\pi \end{cases}, \tag{1.4}$$

$$\Re\alpha_j > -1/2, \quad \beta_j \in \mathbb{C}, \quad j = 0, \dots, m, \tag{1.5}$$

and $V(e^{i\theta})$ is a sufficiently smooth function on the unit circle (see below). Here the condition on α_j insures integrability. Note that a single Fisher-Hartwig singularity at z_j consists of a root-type singularity

$$|z - z_j|^{2\alpha_j} = \left|2\sin\frac{\theta - \theta_j}{2}\right|^{2\alpha_j} \tag{1.6}$$

and a jump $g_{\beta_j}(z)$. A point z_j, $j = 1, \dots, m$ is included in (1.3) if and only if either $\alpha_j \neq 0$ or $\beta_j \neq 0$ (or both); in contrast, we always fix $z_0 = 1$ even if $\alpha_0 = \beta_0 = 0$ (note that $g_{\beta_0}(z) = e^{-i\pi\beta_0}$). Observe that for each $j = 1, \dots, m$, $z^{\beta_j} g_{\beta_j}(z)$ is continuous at $z = 1$, and so for each j each "beta" singularity produces a jump only at the point z_j. The factors $z_j^{-\beta_j}$ are singled out to simplify

comparisons with existing literature. Indeed, (1.2) with the notation $b(\theta) = e^{V(e^{i\theta})}$ is exactly the symbol considered in [23, 4, 5, 6, 7, 8, 9, 12, 13, 14, 20, 21, 42]. We write the symbol, however, in a form with $z^{\sum_{j=0}^m \beta_j}$ factored out. The present way of writing $f(z)$ is more natural for our analysis.

A simple example of a symbol of type (1.2) is given by (1.31,1.32) below. Note that finite order zeros also give rise to Fisher-Hartwig singularities: for example, if a sufficiently smooth $f(z)$ has two simple zeros at $0 < \theta_1 < \theta_2 < 2\pi$, then $f(z) = e^{V(z)}|z - z_1||z - z_2|g_{z_1,1/2}(z)g_{z_2,-1/2}(z)\left(\frac{z_1}{z_2}\right)^{-1/2}$ for a suitable $V(z)$.

On the unit circle, $V(z)$ is represented by its Fourier expansion:

$$V(z) = \sum_{k=-\infty}^{\infty} V_k z^k, \qquad V_k = \frac{1}{2\pi}\int_0^{2\pi} V(e^{i\theta})e^{-ki\theta}d\theta. \tag{1.7}$$

The canonical Wiener-Hopf factorization of $e^{V(z)}$ is

$$e^{V(z)} = b_+(z)e^{V_0}b_-(z), \qquad b_+(z) = e^{\sum_{k=1}^{\infty} V_k z^k}, \qquad b_-(z) = e^{\sum_{k=-\infty}^{-1} V_k z^k}. \tag{1.8}$$

In the case that $\alpha_j = \beta_j = 0$, $f(z) = e^{V(z)}$, the classical strong limit theorem of Szegő (in its most general form, see, e.g., [38]) asserts that as $n \to \infty$,

$$D_n(f) = \exp\left\{nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k}\right\}(1 + o(1)), \tag{1.9}$$

provided $V(z) \in H^{1/2} = \{V = \sum_{k=-\infty}^{\infty} V_k z^k : \sum_{k=-\infty}^{\infty} |k||V_k|^2 < \infty\}$.

Fisher and Hartwig [23] were led to single out symbols of type (1.2) based on the solution of a variety of specific problems from statistical mechanics, in particular, the solution of the spontaneous magnetisation problem for the Ising model. Indeed the square of the magnetisation can be expressed as the limit as $n \to \infty$ of a Toeplitz determinant $D_n(f)$ (which represents a 2-spin correlation function at distance n between spins) where the symbol f is a particular example of (1.2) and has the following properties depending on whether temperature T is lower, equal or higher than the critical temperature T_c:

- for $T < T_c$, f has no Fisher-Hartwig singularities;
- for $T = T_c$, f has one singularity at $z_0 = 1$ with $\alpha_0 = 0$, $\beta_0 = -1/2$;
- for $T > T_c$, f has one singularity at $z_0 = 1$ with $\alpha_0 = 0$, $\beta_0 = -1$.

For f of type (1.2), Fisher and Hartwig made a general conjecture in [23] about the asymptotic form of $D_n(f)$,

$$D_n(f) \sim En^{\sigma}e^{nV_0}, \qquad n \to \infty, \tag{1.10}$$

where $\sigma = \sum_{j=0}^m(\alpha_j^2 - \beta_j^2)$, and E is a constant depending on f. Considerable effort has been expended in the mathematics and physics communities in verifying (1.10).

Introduce the seminorm:

$$|||\beta||| = \max_{j,k} |\Re\beta_j - \Re\beta_k|, \tag{1.11}$$

where $1 \le j, k \le m$ if $\alpha_0 = \beta_0 = 0$, and $0 \le j, k \le m$ otherwise. If $m = 0$, set $|||\beta||| = 0$. Note that in the case of a single singularity, we always have $|||\beta||| = 0$.

The case when $|||\beta||| < 1$, i.e., when all $\Re\beta_j$ lie in a single open interval of length 1, namely $\Re\beta_j \in (q - 1/2, q + 1/2)$, for an appropriate $q \in \mathbb{R}$, has now been essentially settled (see, however, Remark 1.7 below): In [42], Widom proved the conjecture when $\Re\alpha_j > -1/2$, and all $\beta_j = 0$. In [4], Basor then verified the conjecture when $\Re\alpha_j > -1/2$, and $\Re\beta_j = 0$. In [12], Böttcher and Silbermann established the result in the case that $|\Re\alpha_j| < 1/2$, $|\Re\beta_j| < 1/2$. Finally, in [21],

Ehrhardt verified the conjecture for $\Re\alpha_j > -1/2$, $|||\beta||| < 1$. In these papers, the explicit form of E was also established (see [21] for a review of these and other related results).

Theorem 1.1. *(Ehrhardt [21]). Let $f(e^{i\theta})$ be defined in (1.2), $V(z)$ be C^∞ on the unit circle, $|||\beta||| < 1$, $\Re\alpha_j > -1/2$, and $\alpha_j \pm \beta_j \neq -1, -2, \dots$ for $j, k = 0, 1, \dots, m$. Then as $n \to \infty$,*

$$D_n(f) = \exp\left[nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k}\right] \prod_{j=0}^{m} b_+(z_j)^{-\alpha_j+\beta_j} b_-(z_j)^{-\alpha_j-\beta_j} \times n^{\sum_{j=0}^{m}(\alpha_j^2-\beta_j^2)} \prod_{0\le j<k\le m} |z_j - z_k|^{2(\beta_j\beta_k-\alpha_j\alpha_k)} \left(\frac{z_k}{z_j e^{i\pi}}\right)^{\alpha_j\beta_k-\alpha_k\beta_j} \times \prod_{j=0}^{m} \frac{G(1+\alpha_j+\beta_j)G(1+\alpha_j-\beta_j)}{G(1+2\alpha_j)} (1+o(1)), \tag{1.12}$$

where $G(x)$ is Barnes' G-function. The double product over $j < k$ is set to 1 if $m = 0$.

Remark 1.2. The branches in (1.12) are determined in the natural way as follows: $b_\pm(z_j)^{-\alpha_j\pm\beta_j} = \exp\{(-\alpha_j \pm \beta_j)\sum_{k=1}^{\infty} V_{\pm k} z^{\pm k}\}$, $(z_k z_j^{-1} e^{-i\pi})^{\alpha_j\beta_k-\alpha_k\beta_j} = \exp\{i(\theta_k - \theta_j - \pi)(\alpha_j\beta_k - \alpha_k\beta_j)\}$, and the remaining branches are principal.

Remark 1.3. In the case of a single singularity, i.e. when $m = 0$ or $m = 1$, $\alpha_0 = \beta_0 = 0$, the theorem implies that the asymptotics (1.12) hold for

$$\Re\alpha_m > -\frac{1}{2}, \qquad \beta_m \in \mathbb{C}, \qquad \alpha_m \pm \beta_m \neq -1, -2, \dots \tag{1.13}$$

In fact, if there is only one singularity and $V \equiv 0$, an explicit formula is known [12] for $D_n(f)$ in terms of the G-functions.

Remark 1.4. If all $\Re\beta_j \in (-1/2, 1/2]$ or all $\Re\beta_j \in [-1/2, 1/2)$, the conditions $\alpha_j \pm \beta_j \neq -1, -2, \dots$ are satisfied automatically as $\Re\alpha_j > -1/2$.

Remark 1.5. Since $G(-k) = 0$, $k = 0, 1, \dots$, the formula (1.12) no longer represents the leading asymptotics if $\alpha_j + \beta_j$ or $\alpha_j - \beta_j$ is a negative integer for some j. A similar situation arises in Theorem 1.13 below if some representations in $\mathcal{M}$ are degenerate. These cases can be approached using Lemma 2.4 below, but we do not address them in the paper.

Remark 1.6. Assume that the function $V(z)$ is analytic. Then the following can be said about the remainder term (for more results and details on the error terms see [16]). If all $\beta_j = 0$, the error term in (1.12) is of order $O(n^{-1})$. If there is only one singularity the error term is also $O(n^{-1})$. In the general case, the error term depends on the differences $\beta_j - \beta_k$. Our methods allow us to calculate several asymptotic terms rather than just the main one presented in (1.12) (and also in (1.27) below). In [16], we show that the expansion (1.12) with analytic $V(z)$ is uniform in all α_j, β_j for β_j in compact subsets of the strip $|\Re\beta_j - \Re\beta_k| < 1$, for α_j in compact subsets of the half-plane $\Re\alpha_j > -1/2$, and outside a neighborhood of the sets $\alpha_j \pm \beta_j = -1, -2, \dots$. It will be clear below that given this uniformity, Theorems 1.20, 1.25 also hold uniformly in the same sense, while for Theorem 1.13 one should replace β_j with $\widetilde{\beta}_j$ (see below) in the condition of uniformity.

Remark 1.7. Theorem 1.1 as proved by Ehrhardt (and as a consequence, Theorems 1.13, 1.20, 1.25 that we proved below) hold for C^∞ functions $V(z)$ on the unit circle. In [16], we extend Theorem 1.1 to less smooth $V(z)$. Namely, it is sufficient that the condition

$$\sum_{k=-\infty}^{\infty} |k|^s |V_k| < \infty \tag{1.14}$$

holds for some s (and hence for all values in $(0,s)$) such that

$$s > \frac{1+\sum_{j=0}^{m}\left[(\Im\alpha_j)^2+(\Re\beta_j)^2\right]}{1-|||\beta|||}. \tag{1.15}$$

In the present work, we show that given Theorem 1.1 with the condition (1.15) on $V(z)$, Theorems 1.20, 1.25 hold for $V(z)$ under a similar condition with m replaced by $r+1$ and contributions from α_0, α_{r+1} appropriately changed, while Theorem 1.13 holds under the condition (1.28) of Remark 1.15 below. The uniformity in α-, β-parameters will also hold provided s is taken large enough.

In [16], we give an independent proof of Theorem 1.1, in the spirit of [19, 28, 32], using a connection of $D_n(f)$ with the system of polynomials orthogonal with weight $f(z)$ (1.2) on the unit circle. These polynomials also play a central role in the proofs presented here.

It follows, in particular, from Theorem 1.1 that all $D_k(f)\neq 0$, $k=k_0,k_0+1\dots$, for some sufficiently large k_0 if $\alpha_j\pm\beta_j\neq -1,-2,\dots$. Then the polynomials $\phi_k(z)=\chi_k z^k+\cdots$, $\widehat{\phi}_k(z)=\chi_k z^k+\cdots$ of degree k, $k=k_0,k_0+1,\dots$, satisfying

$$\frac{1}{2\pi}\int_0^{2\pi}\phi_k(z)z^{-j}f(z)d\theta=\chi_k^{-1}\delta_{jk},\qquad \frac{1}{2\pi}\int_0^{2\pi}\widehat{\phi}_k(z^{-1})z^{j}f(z)d\theta=\chi_k^{-1}\delta_{jk},\qquad z=e^{i\theta},\qquad j=0,1,\dots,k, \tag{1.16}$$

exist. It is easy to see that they are given by the following expressions:

$$\phi_k(z)=\frac{1}{\sqrt{D_kD_{k+1}}}\begin{vmatrix} f_{00} & f_{01} & \cdots & f_{0k}\\ f_{10} & f_{11} & \cdots & f_{1k}\\ \vdots & \vdots & & \vdots\\ f_{k-1\,0} & f_{k-1\,1} & \cdots & f_{k-1\,k}\\ 1 & z & \cdots & z^k\end{vmatrix}, \tag{1.17}$$

$$\widehat{\phi}_k(z^{-1})=\frac{1}{\sqrt{D_kD_{k+1}}}\begin{vmatrix} f_{00} & f_{01} & \cdots & f_{0\,k-1} & 1\\ f_{10} & f_{11} & \cdots & f_{1\,k-1} & z^{-1}\\ \vdots & \vdots & & \vdots & \vdots\\ f_{k0} & f_{k1} & \cdots & f_{k\,k-1} & z^{-k}\end{vmatrix}, \tag{1.18}$$

where

$$f_{st}=\frac{1}{2\pi}\int_0^{2\pi}f(z)z^{-(s-t)}d\theta,\quad s,t=0,1,\dots,k.$$

We obviously have

$$\chi_k=\sqrt{\frac{D_k}{D_{k+1}}}. \tag{1.19}$$

These polynomials satisfy a Riemann-Hilbert problem. In Section 4, we solve the problem asymptotically for large n in case of the weight given by (1.2) with analytic $V(z)$, thus obtaining the large n asymptotics of the orthogonal polynomials. The main new feature of the solution is a construction of the local parametrix at the points z_j of Fisher-Hartwig singularities. This parametrix is given in terms of the confluent hypergeometric function (see Proposition 4.1). A study of the asymptotic behavior of the polynomials orthogonal on the unit circle was initiated by Szegő [39]. Riemann-Hilbert methods developed within the last 20 years allow us to find asymptotics of orthogonal polynomials in all regions of the complex plane (see [18] and many subsequent works by many authors). Such an analysis of the polynomials with an analytic weight on the unit circle was carried out in [35], and for the case of a weight with α_j-singularities but without jumps, in [36]. We

provide, therefore, a generalization of these results. Here we present only the following statement we will need below for the analysis of determinants.

Theorem 1.8. *Let $f(e^{i\theta})$ be defined in (1.2), $V(z)$ be analytic in a neighborhood of the unit circle, and $\phi_k(z) = \chi_k z^k + \cdots$, $\widehat{\phi}_k(z) = \chi_k z^k + \cdots$ be the corresponding polynomials satisfying (1.16). Assume that $|||\beta||| < 1$, $\alpha_j \pm \beta_j \neq -1, -2, \ldots$, $j, k = 0, 1, \ldots, m$. Let*

$$\delta = \max_{j,k} n^{2\Re(\beta_j - \beta_k - 1)} = n^{2|||\beta|||-2}, \tag{1.20}$$

where the indices $j, k = 0$ are omitted if $\alpha_0 = \beta_0 = 0$.

Then as $n \to \infty$,

$$\begin{aligned} \chi_{n-1}^2 = \exp\left[-\int_0^{2\pi} V(e^{i\theta})\frac{d\theta}{2\pi}\right]\Bigg(1 - \frac{1}{n}\sum_{k=0}^{m}(\alpha_k^2 - \beta_k^2) \\ + \sum_{j=0}^{m}\sum_{k\neq j}\frac{z_k}{z_j - z_k}\left(\frac{z_j}{z_k}\right)^n n^{2(\beta_k-\beta_j-1)}\frac{\nu_j}{\nu_k}\frac{\Gamma(1+\alpha_j+\beta_j)\Gamma(1+\alpha_k-\beta_k)}{\Gamma(\alpha_j-\beta_j)\Gamma(\alpha_k+\beta_k)}\frac{b_+(z_j)b_-(z_k)}{b_-(z_j)b_+(z_k)} \\ + O(\delta^2) + O(\delta/n)\Bigg), \end{aligned} \tag{1.21}$$

where

$$\nu_j = \exp\left\{-i\pi\left(\sum_{p=0}^{j-1}\alpha_p - \sum_{p=j+1}^{m}\alpha_p\right)\right\}\prod_{p\neq j}\left(\frac{z_j}{z_p}\right)^{\alpha_p}|z_j - z_p|^{2\beta_p}. \tag{1.22}$$

Under the same conditions,

$$\phi_n(0) = \chi_n\left(\sum_{j=0}^{m} n^{-2\beta_j-1} z_j^n \nu_j \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)}\frac{b_+(z_j)}{b_-(z_j)} + O\left(\left[\delta + \frac{1}{n}\right]\max_k \frac{n^{-2\Re\beta_k}}{n}\right)\right), \tag{1.23}$$

$$\widehat{\phi}_n(0) = \chi_n\left(\sum_{j=0}^{m} n^{2\beta_j-1} z_j^{-n} \nu_j^{-1} \frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)}\frac{b_-(z_j)}{b_+(z_j)} + O\left(\left[\delta + \frac{1}{n}\right]\max_k \frac{n^{2\Re\beta_k}}{n}\right)\right). \tag{1.24}$$

Remark 1.9. The error terms here are uniform and differentiable in all α_j, β_j for β_j in compact subsets of the strip $|\Re\beta_j - \Re\beta_k| < 1$, for α_j in compact subsets of the half-plane $\Re\alpha_j > -1/2$, and outside a neighborhood of the sets $\alpha_j \pm \beta_j = -1, -2, \ldots$. If $\alpha_j + \beta_j = 0$ or $\alpha_j - \beta_j = 0$ for some j, the corresponding terms in the above formulae vanish.

Remark 1.10. Note that the terms with $n^{2(\beta_k-\beta_j-1)}$ in (1.21) become larger in absolute value than the $1/n$ term for $|||\beta||| > 1/2$.

Remark 1.11. With changes to the error estimates, this theorem can be generalized to sufficiently smooth $V(z)$ using (1.19), a well-known representation for orthogonal polynomials as multiple integrals, and similar arguments to those we give in Section 6.2 below.

Our first task in this paper is to extend the asymptotic formula for $D_n(f)$ to arbitrary $\beta_j \in \mathbb{C}$, i.e. for the case when not all $\Re\beta_j$'s lie in a single interval of length less than 1, in other words for

$$|||\beta||| \geq 1.$$

We know from examples (see, e.g., [12, 10, 21]) that in general, the formula (1.12) breaks down. Obviously, the general case can be reduced to $\Re\beta_j \in (q - 1/2, q + 1/2)$ by adding integers to β_j. Then, apart from a constant factor, the only change in $f(z)$ is multiplication with z^ℓ, $\ell \in \mathbb{Z}$. However, as we show in Lemma 2.4, the determinants $D_n(f(z))$ and $D_n(z^\ell f(z))$ are simply related.

They differ just by a factor which involves χ_k, $\phi_k(0)$, $\widehat{\phi}_k(0)$ for large k (these quantities are given by Theorem 1.8), as well as the derivatives of the orthogonal polynomials at 0. The derivatives can be calculated similarly to $\phi_k(0)$, $\widehat{\phi}_k(0)$. Thus it is easy to obtain the general asymptotic formula for $D_n(f(z))$. However, this formula is implicit in the sense that one still needs to separate the main asymptotic term from the others: e.g., if the dimension ℓ of F_n in (2.9) is larger than the number of the leading-order terms in (1.23), the obvious candidate for the leading order in F_n vanishes (this is not the case in the simplest situation given by Theorem 1.18). We resolve this problem below.

Following [10, 21], we define a *FH-representation* of a symbol. Namely, for $f(z)$ given by (1.2) replace β_j by $\beta_j + n_j$, $n_j \in \mathbb{Z}$ if z_j is a singularity (i.e., if either $\beta_j \neq 0$ or $\alpha_j \neq 0$ or both). The integers n_j are arbitrary subject to the condition $\sum_{j=0}^m n_j = 0$. In a slightly different notation from [10, 21], we call the resulting function $f(z; n_0, \dots, n_m)$ a FH-representation of $f(z)$. (The original $f(z)$ is also a FH-representation corresponding to $n_0 = \cdots = n_m = 0$.) Obviously, all FH-representations of $f(z)$ differ only by multiplicative constants. We have

$$f(z) = \prod_{j=0}^m z_j^{n_j} \times f(z; n_0, \dots, n_m). \tag{1.25}$$

We are interested in the FH-representations (characterized by $(n_j)_{j=0}^m$) of f such that $\sum_{j=0}^m (\Re\beta_j + n_j)^2$ is minimal. There is a finite number of such FH-representations and we provide an algorithm for finding them explicitly (see the proof of Lemma 1.12 below). We denote the set of such FH-representations by $\mathcal{M}$. Furthermore, we call a FH-representation degenerate if $\alpha_j + (\beta_j + n_j)$ or $\alpha_j - (\beta_j + n_j)$ is a negative integer for some j. We call $\mathcal{M}$ non-degenerate if it contains no degenerate FH-representations. The set $\mathcal{M}$ can be characterized as follows. For a given $\beta = (\beta_0, \dots, \beta_m)$ let us call

$$O_\beta = \{\widehat{\beta} : \widehat{\beta}_j = \beta_j + n_j, \sum_{j=0}^m n_j = 0\}$$

the *orbit* of β (if $\alpha_0 = \beta_0 = 0$ then we always fix $n_0 = 0$). In other words, it is the set of $\widehat{\beta}$ corresponding to all the FH-representations of f. For any $\widehat{\beta}$ in O_β, we define $|||\widehat{\beta}||| = \max_{j,k} |\Re\widehat{\beta}_j - \Re\widehat{\beta}_k|$ with the range of indicies j, k fixed to be the same as for $|||\beta|||$ (1.11). We have

Lemma 1.12. *There exist only the following 2 mutually exclusive possibilities.*

- *$\exists\widehat{\beta} \in O_\beta$ such that $|||\widehat{\beta}||| < 1$. Then such $\widehat{\beta}$ is unique and it is the unique element of $\mathcal{M} = \{\widehat{\beta}\}$.*
- *$\exists\widehat{\beta} \in O_\beta$ such that $|||\widehat{\beta}||| = 1$. Then there are at least 2 such $\widehat{\beta}$'s and all of them are obtained from each other by a repeated application of the following rule: add 1 to a $\widehat{\beta}_j$ with the smallest real part and subtract 1 from a $\widehat{\beta}_j$ with the largest. Moreover, $\mathcal{M} = \{\widehat{\beta} \in O_\beta : |||\widehat{\beta}||| = 1\}$.*

Proof. Suppose that the seminorm $|||\beta||| > 1$. Then, writing $\beta_s^{(1)} = \beta_s + 1$, $\beta_t^{(1)} = \beta_t - 1$, and $\beta_j^{(1)} = \beta_j$ if $j \neq s, t$, where β_s is one of the beta-parameters with $\Re\beta_s = \min_j \Re\beta_j$, β_t is one of the beta-parameters with $\Re\beta_t = \max_j \Re\beta_j$, we see that $|||\beta^{(1)}||| \leq |||\beta|||$, and f corresponding to $\beta^{(1)}$ is a FH-representation. After a finite number, say r, of such transformations we reduce an arbitrary set of β_j to the situation for which either $|||\beta^{(r)}||| < 1$ or $|||\beta^{(r)}||| = 1$. Note that further transformations do not change the seminorm in the second case, while in the first case the seminorm oscillates periodically taking two values, $|||\beta^{(r)}|||$ and $2 - |||\beta^{(r)}|||$. Thus all the symbols of type (1.2) belong to two distinct classes: the first, for which $|||\beta^{(r)}||| < 1$, and the second, for which $|||\beta^{(r)}||| = 1$. For symbols of the first class, $\mathcal{M}$ has only one member with beta-parameters $\beta^{(r)}$.

Indeed, writing $b_j = \Re\beta_j$, if $-1/2 < b_j^{(r)} - q < 1/2$ for some $q \in \mathbb{R}$ and all j, then for any $(k_j)_{j=0}^m$ such that $\sum_{j=0}^m k_j = 0$ and not all k_j are zero, we have

$$\sum_{j=0}^m (b_j^{(r)}+k_j)^2 = \sum_{j=0}^m (b_j^{(r)})^2 + 2\sum_{j=0}^m (b_j^{(r)}-q)k_j + \sum_{j=0}^m k_j^2 > \sum_{j=0}^m (b_j^{(r)})^2 + \sum_{j=0}^m k_j^2 - |k_j| \geq \sum_{j=0}^m (b_j^{(r)})^2, \tag{1.26}$$

where the first inequality is strict as at least two $k_j \neq 0$. For symbols of the second class, we can find $q \in \mathbb{R}$ such that $-1/2 \leq b_j^{(r)} - q \leq 1/2$ for all j. Equation (1.26) in this case holds with "$>$" sign replaced by "$\geq$". Clearly, there are several FH-representations in $\mathcal{M}$ in this case (they correspond to the equalities in (1.26)) and adding 1 to one of $\beta_s^{(r)}$ with $b_s^{(r)} = \min_j b_j^{(r)} = q - 1/2$ while subtracting 1 from one of $\beta_t^{(r)}$ with $b_t^{(r)} = \max_j b_j^{(r)} = q + 1/2$ provides the way to find all of them.

A simple explicit sufficient, but obviously not necessary, condition for $\mathcal{M}$ to have only one member is that all $\Re\beta_j \mod 1$ be different. □

In Section 6, we prove

Theorem 1.13. *Let $f(z)$ be given in (1.2), $\Re\alpha_j > -1/2$, $\beta_j \in \mathbb{C}$, $j = 0, 1, \ldots, m$. Let $\mathcal{M}$ be non-degenerate. Then, as $n \to \infty$,*

$$D_n(f) = \sum \left(\prod_{j=0}^m z_j^{n_j} \right)^n \mathcal{R}(f(z; n_0, \ldots, n_m))(1 + o(1)), \tag{1.27}$$

where the sum is over all FH-representations in $\mathcal{M}$. Each $\mathcal{R}(f(z; n_0, \ldots, n_m))$ stands for the right-hand side of the formula (1.12), without the error term, corresponding to $f(z; n_0, \ldots, n_m)$.

Remark 1.14. This theorem was conjectured by Basor and Tracy [10] based on an explicit example: see Example 1.31 below. The case when the FH-representation minimizing $\sum_{j=0}^m (\Re\beta_j + n_j)^2$ is unique, i.e. there is only one term in the sum (1.27), was proved by Ehrhardt [21]. Note that this case is exactly the first possibility of Lemma 1.12. Thus, Theorem 1.13 in this case follows from Theorem 1.1 applied to this FH-representation.

Remark 1.15. This theorem relies on Theorem 1.1 and therefore requires $V(z)$ to be C^∞ on the unit circle. As remarked above, we prove in [16] that Theorem 1.1 holds in fact under the condition (1.15). It then follows (see Section 6) that, if $\mathcal{M}$ has several members, Theorem 1.13 holds for any

$$s > \frac{1 + \sum_{j=0}^m \left[(\Im\alpha_j)^2 + \max\left\{ (\Re\widetilde{\beta}_j)^2, (\Re\beta_j^{(r)})^2 \right\} \right]}{1 - |||\widetilde{\beta}|||}, \tag{1.28}$$

where $\widetilde{\beta}_j$ are obtained from $\beta_j^{(r)}$ (see the proof of Lemma 1.12) by subtracting 1 from all $\beta_j^{(r)}$ with the maximal real part and leaving the rest unchanged. The number $|||\widetilde{\beta}||| < 1$ is defined the same way as $|||\widehat{\beta}|||$, $|||\beta|||$ above.

Remark 1.16. The situation when all $\alpha_j \pm \beta_j$ are nonnegative integers, which was considered by Böttcher and Silbermann in [13], is a particular case of the above theorem.

Remark 1.17. The case when *all* the FH-representations of f are degenerate (not only those in $\mathcal{M}$) was considered by Ehrhardt [21] who found that in this case $D_n(f) = O(e^{nV_0} n^r)$, where r is any real number. We can reproduce this result by our methods but do not present it here.

We will now discuss a simple particular case of Theorem 1.13 and present a direct independent proof in this case.

Theorem 1.18 (A particular case of Theorem 1.13)**.** *Let the symbol $f^{\pm}(z)$ be obtained from $f(z)$ (1.2) by replacing one β_{j_0} with $\beta_{j_0}\pm 1$ for some fixed $0\le j_0\le m$. Let $\Re\alpha_j>-\frac{1}{2}$, $\Re\beta_j\in(-1/2,1/2]$, $j=0,1,\dots,m$. Then, for sufficiently large n,*

$$D_n(f^+(z))=z_{j_0}^{-n}\frac{\phi_n(0)}{\chi_n}D_n(f(z)),\qquad D_n(f^-(z))=z_{j_0}^{n}\frac{\widehat{\phi}_n(0)}{\chi_n}D_n(f(z)). \tag{1.29}$$

These formulae together with (1.23,1.24,1.21,1.12) yield the following asymptotic description of $D_n(f^{\pm})$. Let there be more than one singular point z_j and all $\alpha_j\pm\beta_j\neq 0$. For $f^+(z)$, let β_{j_p}, $p=1,\dots,s$ be such that they have the same real part which is strictly less than the real parts of all the other β_j (if any), i.e. $\Re\beta_{j_1}=\dots=\Re\beta_{j_s}<\min_{j\neq j_1,\dots,j_s}\Re\beta_j$. For $f^-(z)$ let one β_{j_p}, $p=1,\dots,s$ be such that $\Re\beta_{j_1}=\dots=\Re\beta_{j_s}>\max_{j\neq j_1,\dots,j_s}\Re\beta_j$. Then the asymptotics of $D_n(f^{\pm})$ are given by the following:

$$D_n(f^+)=z_{j_0}^{-n}\sum_{p=1}^{s}z_{j_p}^{n}\mathcal{R}_{j_p,+}(1+o(1)),\qquad D_n(f^-)=z_{j_0}^{n}\sum_{p=1}^{s}z_{j_p}^{-n}\mathcal{R}_{j_p,-}(1+o(1)), \tag{1.30}$$

where $\mathcal{R}_{j,\pm}$ is the right-hand side of (1.12) (without the error term) in which β_j is replaced by $\beta_j\pm 1$, respectively.

Proof. For simplicity, we present the proof only for $V(z)$ analytic in a neighborhood of the unit circle. Consider the case of $f^-(z)$. It corresponds to one of the β_j shifted inside the interval $(-3/2,-1/2]$. Since

$$z^{\sum_{j=0}^{m}\beta_j-1}=z^{-1}z^{\sum_{j=0}^{m}\beta_j},\qquad g_{\beta_{j_0}-1}(z)=-g_{\beta_{j_0}}(z),\qquad z_{j_0}^{-\beta_{j_0}+1}=z_{j_0}z_{j_0}^{-\beta_{j_0}},$$

we see that

$$f^-(z)=-z_{j_0}z^{-1}f(z).$$

Therefore, using the identity (2.12) below, we obtain

$$D_n(f^-(z))=(-z_{j_0})^nD_n(z^{-1}f(z))=z_{j_0}^{n}\frac{\widehat{\phi}_n(0)}{\chi_n}D_n(f(z)).$$

If, for some j_1, j_2, …, j_s, we have that $\Re\beta_{j_1}=\dots=\Re\beta_{j_s}>\max_{j\neq j_1,\dots,j_s}\Re\beta_j$, then we see from (1.24) that only the addends with $n^{2\beta_{j_1}-1}$, …, $n^{2\beta_{j_s}-1}$ give contributions to the main asymptotic term of $D_n(f^-(z))$. Using Theorem 1.1 for $D_n(f(z))$ and the relation $G(1+x)=\Gamma(x)G(x)$, we obtain the formula (1.30) for $D_n(f^-(z))$. The case of $f^+(z)$ is similar. □

Example 1.19. In [10] Basor and Tracy considered a simple example of a symbol of type (1.2) for which the asymptotics of the determinant can be computed directly, but are very different from (1.12). Up to a constant, the symbol is

$$f^{(BT)}(e^{i\theta})=\begin{cases}-i, & 0<\theta<\pi\\ i, & \pi<\theta<2\pi\end{cases}. \tag{1.31}$$

We can represent $f^{(BT)}$ as a symbol with β-singularities $\beta_0=1/2$, $\beta_1=-1/2$ at the points $z_0=1$ and $z_1=-1$, respectively:

$$f^{(BT)}(z)=g_{1,1/2}(z)g_{-1,-1/2}(z)e^{i\pi/2} \tag{1.32}$$

We see that $f^{(BT)}(z)=f^-(z)$ and $j_0=1$. Therefore by the first part of Theorem 1.18, we have

$$D_n(f^{(BT)}(z))=(-1)^n\frac{\widehat{\phi}_n(0)}{\chi_n}D_n(f(z)),$$

where $\phi_n(z)$, χ_n, $D_n(f(z))$ correspond to $f(z)$ given by (1.2) with $m = 1$, $z_0 = 1$, $z_1 = e^{i\pi}$, $\beta_0 = \beta_1 = 1/2$, $\alpha_0 = \alpha_1 = 0$.

Observing that $s = 2$, $j_1 = j_0 = 1$ and $j_2 = 0$ and using (1.30) we obtain

$$D_n(f^{(BT)}(z)) = (-1)^n((-1)^n\mathcal{R}_{1,-} + \mathcal{R}_{0,-}).$$

Since $\mathcal{R}_{1,-} = \mathcal{R}_{0,-} = (2n)^{-1/2}G(1/2)^2G(3/2)^2(1 + o(1))$, we obtain

$$D_n(f^{(BT)}(z)) = \frac{1 + (-1)^n}{2}\sqrt{\frac{2}{n}}G(1/2)^2G(3/2)^2(1 + o(1)), \tag{1.33}$$

which is the answer found in [10].

As noted by Basor and Tracy, $f^{(BT)}(z)$ has a different FH-representation of type (1.2), namely, with $\beta_0 = -1/2$, $\beta_1 = 1/2$, and we can write

$$f^{(BT)}(z) = -g_{1,-1/2}(z)g_{-1,1/2}(z)e^{-i\pi/2}. \tag{1.34}$$

This fact was the origin of their conjecture. In the notation of Theorem 1.13, the symbol (1.32) has the two FH-representations minimizing $\sum_{j=0}^{1}(\Re\beta_j + n_j)^2$, one with $n_0 = n_1 = 0$ and the other with $n_0 = -1$, $n_1 = 1$.

Note that in the case $\sum_{j=0}^{m}\beta_j = 0$ we can always assume that $\Re\beta_j \in [-1/2, 1/2]$. The beta-singularities then are just piece-wise constant (step-like) functions. This case is relevant for our next result, which is on Hankel determinants.

Let $w(x)$ be an integrable complex-valued function on the interval $[-1, 1]$. Then the Hankel determinant with symbol $w(x)$ is given by

$$D_n(w(x)) = \det\left(\int_{-1}^{1} x^{j+k}w(x)dx\right)_{j,k=0}^{n-1}. \tag{1.35}$$

Define $w(x)$ for a fixed $r = 0, 1, \dots$ as follows:

$$w(x) = e^{U(x)}\prod_{j=0}^{r+1}|x - \lambda_j|^{2\alpha_j}\omega_j(x) \tag{1.36}$$

$$1 = \lambda_0 > \lambda_1 > \cdots > \lambda_{r+1} = -1, \qquad \omega_j(x) = \begin{cases} e^{i\pi\beta_j} & \Re x \le \lambda_j \\ e^{-i\pi\beta_j} & \Re x > \lambda_j \end{cases}, \qquad \Re\beta_j \in (-1/2, 1/2],$$

$$\beta_0 = \beta_{r+1} = 0, \qquad \Re\alpha_j > -\frac{1}{2}, \qquad j = 0, 1, \dots, r+1.$$

where $U(x)$ is a sufficiently smooth function on the interval $[-1, 1]$. Note that we set $\beta_0 = \beta_{r+1} = 0$, $\Re\beta_j \in (-1/2, 1/2]$ without loss of generality as the functions $\omega_0(x)$, $\omega_{r+1}(x)$ are just constants on $(-1, 1)$, and $\omega_j(x; \beta_j + k_j) = (-1)^{k_j}\omega(x; \beta_j)$, $k_j \in \mathbb{Z}$.

In Section 7, we prove

Theorem 1.20. *Let $w(x)$ be defined as in (1.36) with $\Re\beta_j \in \left(-\frac{1}{2},\frac{1}{2}\right)$, $j=1,2,\dots,r$. Then as $n\to\infty$,*

$$
\begin{aligned}
(1.37)\quad D_n(w) &= D_n(1)e^{\left[(n+\alpha_0+\alpha_{r+1})V_0-\alpha_0V(1)-\alpha_{r+1}V(-1)+\frac{1}{2}\sum_{k=1}^{\infty}kV_k^2\right]}\\
&\times\prod_{j=1}^{r}b_+(z_j)^{-\alpha_j-\beta_j}b_-(z_j)^{-\alpha_j+\beta_j}\times e^{\left[2i(n+A)\sum_{j=1}^{r}\beta_j\arcsin\lambda_j+i\pi\sum_{0\le j<k\le r+1}(\alpha_j\beta_k-\alpha_k\beta_j)\right]}\\
&\times 4^{-\left(An+\alpha_0^2+\alpha_{r+1}^2+\sum_{0\le j<k\le r+1}\alpha_j\alpha_k+\sum_{j=1}^{r}\beta_j^2\right)}(2\pi)^{\alpha_0+\alpha_{r+1}}n^{2(\alpha_0^2+\alpha_{r+1}^2)+\sum_{j=1}^{r}(\alpha_j^2-\beta_j^2)}\\
&\times\prod_{0\le j<k\le r+1}|\lambda_j-\lambda_k|^{-2(\alpha_j\alpha_k+\beta_j\beta_k)}\left|\lambda_j\lambda_k-1+\sqrt{(1-\lambda_j^2)(1-\lambda_k^2)}\right|^{2\beta_j\beta_k}\\
&\times\frac{1}{G(1+2\alpha_0)G(1+2\alpha_{r+1})}\prod_{j=1}^{r}(1-\lambda_j^2)^{-(\alpha_j^2+\beta_j^2)/2}\frac{G(1+\alpha_j+\beta_j)G(1+\alpha_j-\beta_j)}{G(1+2\alpha_j)}\left(1+o(1)\right),
\end{aligned}
$$

where $A=\sum_{k=0}^{r+1}\alpha_k$, $V(e^{i\theta})=U(\cos\theta)$, $z_j=e^{i\theta_j}$, $\lambda_j=\cos\theta_j$, $j=0,\dots,r+1$, and the functions $b_\pm(z)$ are defined in (1.8).

Remark 1.21. $D_n(1)$ is an explicitly computable determinant, cf. [43], related to the Legendre polynomials (it can also be written as a Selberg integral)

$$
(1.38)\qquad D_n(1)=2^{n^2}\prod_{k=0}^{n-1}\frac{k!^3}{(n+k)!}=\frac{\pi^{n+1/2}G(1/2)^2}{2^{n(n-1)}n^{1/4}}\left(1+o(1)\right).
$$

To prove Theorem 1.20 we use the fact that $w(x)$ can be generated by a particular class of functions $f(z)$ given by (1.2). Namely, we can find an *even* function f of θ ($f(e^{i\theta})=f(e^{-i\theta})$, $\theta\in[0,2\pi)$) such that

$$
(1.39)\qquad w(x)=\frac{f(e^{i\theta})}{|\sin\theta|},\qquad x=\cos\theta,\quad x\in[-1,1].
$$

We must have (see Section 7 below) that $m=2r+1$, $\theta_0=0$, $\theta_{r+1}=\pi$, $\theta_{m+1-j}=2\pi-\theta_j$, $j=1,\dots r$. If we denote the beta-parameters of $f(z)$ by $\widetilde{\beta}_j$, we obtain $\widetilde{\beta}_0=\widetilde{\beta}_{r+1}=0$, $\widetilde{\beta}_j=-\widetilde{\beta}_{m+1-j}=-\beta_j$, $j=1,\dots,r$. In particular, $\sum_{j=0}^{m}\widetilde{\beta}_j=0$ as remarked above.

In Section 7 we obtain Theorem 1.20 from Theorem 1.1 and the asymptotics for the orthogonal polynomials on the unit circle with weight $f(z)$ using the following connection between Hankel and Toeplitz determinants established by Theorem 2.6 below:

$$
(1.40)\qquad D_n(w(x))^2=\frac{\pi^{2n}}{4^{(n-1)^2}}\frac{(\chi_{2n}+\phi_{2n}(0))^2}{\phi_{2n}(1)\phi_{2n}(-1)}D_{2n}(f(z)),
$$

where $w(x)$ and $f(z)$ are related by (1.39).

Remark 1.22. Asymptotics of a Hankel determinant when some (or all) of β_j have the real part $1/2$ can be easily obtained. For the corresponding $f(z)$ this implies that certain $\Re\widetilde{\beta}_j=-1/2$ and $\Re\widetilde{\beta}_{m+1-j}=1/2$ and the rest $\Re\widetilde{\beta}_k\in(-1/2,1/2)$. Thus, Theorem 1.13 can be used to estimate $D_{2n}(f(z))$. For the asymptotics of $\phi_{2n}(z)$ in this case we need an additional "correction" R_1 term (given by (4.69) below) which is now $O(n^{-2\widetilde{\beta}_j-1})=O(1)$.

Remark 1.23. One can obtain the asymptotics of the polynomials orthogonal on the interval $[-1,1]$ with weight (1.36) by using our results for the polynomials $\phi_k(z)$ orthogonal with the corresponding even weight on the unit circle and a Szegő relation (Lemma 2.5 below) which maps the latter polynomials to the former ones.

Remark 1.24. Asymptotics for a subset of symbols (1.36) which satisfy a symmetry condition and have a certain behaviour at the end-points ± 1 were found by Basor and Ehrhardt in [6]. They use relations between Hankel and Toeplitz determinants which are less general than (1.40) but do not involve polynomials. For some other related results, see [26, 33].

Our final task is to present asymptotics for the so-called Toeplitz+Hankel determinants. We consider the four most important ones appearing in the theory of classical groups and its applications to random matrices and statistical mechanics (see, e.g., [2, 25, 31]) defined in terms of the Fourier coefficients of an even f (evenness implies the matrices are symmetric) as follows:

$$\det(f_{j-k}+f_{j+k})_{j,k=0}^{n-1}, \quad \det(f_{j-k}-f_{j+k+2})_{j,k=0}^{n-1}, \quad \det(f_{j-k}\pm f_{j+k+1})_{j,k=0}^{n-1}. \tag{1.41}$$

There are simple relations [40, 29, 2] between the determinants (1.41) and Hankel determinants on $[-1,1]$ with added singularities at the end-points. These are summarized in Lemma 2.7 below. It is easily seen that if $f(z)$ is an (even) function of type (1.2) then the corresponding symbols of Hankel determinants belong to the class (1.36). Thus a straightforward combination of Lemma 2.7 and Theorem 1.20 (aided by formulae of Section 7) gives the following

Theorem 1.25. *Let $f(z)$ be defined in (1.2) with the condition $f(e^{i\theta})=f(e^{-i\theta})$. Let $\theta_{r+1}=\pi$ and $\Re\beta_j\in\left(-\frac{1}{2},\frac{1}{2}\right)$, $j=1,2,\dots,r$, $\beta_0=\beta_{r+1}=0$. Then as $n\to\infty$,*

$$\begin{aligned}
D_n^{\mathrm{T+H}} &= e^{nV_0+\frac{1}{2}\left[(\alpha_0+\alpha_{r+1}+s+t)V_0-(\alpha_0+s)V(1)-(\alpha_{r+1}+t)V(-1)+\sum_{k=1}^{\infty}kV_k^2\right]} \\
&\times\prod_{j=1}^{r} b_+(z_j)^{-\alpha_j+\beta_j}b_-(z_j)^{-\alpha_j-\beta_j}\times e^{-i\pi\left[\left\{\alpha_0+s+\sum_{j=1}^{r}\alpha_j\right\}\sum_{j=1}^{r}\beta_j+\sum_{1\le j<k\le r}(\alpha_j\beta_k-\alpha_k\beta_j)\right]} \\
&\times 2^{(1-s-t)n+p+\sum_{j=1}^{r}(\alpha_j^2-\beta_j^2)-\frac{1}{2}(\alpha_0+\alpha_{r+1}+s+t)^2+\frac{1}{2}(\alpha_0+\alpha_{r+1}+s+t)}n^{\frac{1}{2}(\alpha_0^2+\alpha_{r+1}^2)+\alpha_0 s+\alpha_{r+1}t+\sum_{j=1}^{r}(\alpha_j^2-\beta_j^2)} \\
&\times\prod_{1\le j<k\le r}|z_j-z_k|^{-2(\alpha_j\alpha_k-\beta_j\beta_k)}|z_j-z_k^{-1}|^{-2(\alpha_j\alpha_k+\beta_j\beta_k)} \\
&\times\prod_{j=1}^{r} z_j^{2\widetilde{A}\beta_j}|1-z_j^2|^{-(\alpha_j^2+\beta_j^2)}|1-z_j|^{-2\alpha_j(\alpha_0+s)}|1+z_j|^{-2\alpha_j(\alpha_{r+1}+t)} \\
&\times\frac{\pi^{\frac{1}{2}(\alpha_0+\alpha_{r+1}+s+t+1)}G(1/2)^2}{G(1+\alpha_0+s)G(1+\alpha_{r+1}+t)}\prod_{j=1}^{r}\frac{G(1+\alpha_j+\beta_j)G(1+\alpha_j-\beta_j)}{G(1+2\alpha_j)}\left(1+o(1)\right),
\end{aligned} \tag{1.42}$$

where $\widetilde{A}=\frac{1}{2}(\alpha_0+\alpha_{r+1}+s+t)+\sum_{j=1}^{r}\alpha_j$ and

$$D_n^{\mathrm{T+H}}=\det(f_{j-k}+f_{j+k})_{j,k=0}^{n-1}, \qquad \textit{with} \quad p=-2n+2, \quad s=t=-\frac{1}{2} \tag{1.43}$$

$$D_n^{\mathrm{T+H}}=\det(f_{j-k}-f_{j+k+2})_{j,k=0}^{n-1}, \qquad \textit{with} \quad p=0, \quad s=t=\frac{1}{2} \tag{1.44}$$

$$D_n^{\mathrm{T+H}}=\det(f_{j-k}\pm f_{j+k+1})_{j,k=0}^{n-1}, \qquad \textit{with} \quad p=-n, \quad s=\mp\frac{1}{2}, \quad t=\pm\frac{1}{2}. \tag{1.45}$$

Remark 1.26. For the case $\Re\beta_j=1/2$ see Remark 1.22 above.

Remark 1.27. For the determinant $\det(f_{j-k}+f_{j+k+1})_{j,k=0}^{n-1}$ in the case when the symbol has no α singularities at $z=\pm 1$ and $|\Re\beta_j|<1/2$, the asymptotics were obtained in [7] (see also [8] if f is non-even, $\alpha_j=0$). Note that for symbols without singularities, i.e. for $f(z)=e^{V(z)}$, the asymptotics of all the above Toeplitz+Hankel determinants (and related more general ones) were found recently in [9].

The results in this paper make it possible to justify various asymptotic formulae that were obtained previously in the literature on the basis of the Basor-Tracy conjecture and conjectures on Hankel and Toeplitz+Hankel determinants. One such application arises in the framework of the random matrix approach in the theory of the Riemann zeta-function and other L-functions (see [30] for a recent survey). Define

$$\phi(z) = \left|2\sin\frac{\theta}{2}\right|^{2k} e^{V(z)}, \quad k \in \mathbb{N}, \tag{1.46}$$

where

$$V(e^{i\theta}) = 2k\left\{\int_1^e u(y)\left(\sum_{j=-\infty}^{\infty} \mathrm{Ci}(|\theta+2\pi j|\ln y \ln X)dy\right) - \ln\left|2\sin\frac{\theta}{2}\right|\right\},$$

$$\mathrm{Ci}(z) = -\int_z^\infty \frac{\cos t}{t}dt,$$

and $u(y)$ is a smooth nonnegative function supported on $[e^{1-1/X}, e]$ and of total mass one. Consider the following average over the orthogonal group $SO(2n)$,

$$E_{SO(2n)}\left(\prod_{j=1}^n \phi(e^{i\theta_j})\right). \tag{1.47}$$

This object was introduced by Bui and Keating in [15] (following [27]) as the random matrix counterpart of a key term contributing to the mean values of certain Dirichlet L-functions in the Katz-Sarnak orthogonal family. The issue is the large n and large X behavior of this average. The question can be resolved with the help of Theorem 1.25. Indeed, one can observe that

$$E_{SO(2n)}\left(\prod_{j=1}^n \phi(e^{i\theta_j})\right) = \frac{1}{2}\det(\phi_{j-k}+\phi_{j+k})_{j,k=0}^{n-1},$$

and that symbol (1.46) is of Fisher-Hartwig type with a single α-singularity at $z_0 = 1$, and $\alpha_0 = k$. A direct application of Theorem 1.25 leads then, for X large, to the following asymptotic behavior as $n \to \infty$ of the average (1.47),

$$E_{SO(2n)}\left(\prod_{j=1}^n \phi(e^{i\theta_j})\right) \sim G(1+k)\left(\frac{\Gamma(1+2k)}{G(1+2k)\Gamma(1+k)}\right)^{1/2}\left(\frac{2n}{e^\gamma \ln X}\right)^{k(k-1)/2}, \tag{1.48}$$

where γ is Euler's constant. Formula (1.48) is precisely the asymptotic form conjectured by Bui and Keating in [15]. In a similar way, Theorem 1.25 provides a justification of an analogous conjecture of Bui and Keating in [15] concerning the average of the same product $\prod_{j=1}^n \phi(e^{i\theta_j})$ over the symplectic group. In the context of the random matrix approach in number theory [30], this means that Theorem 1.25 yields asymptotic formulae which can be used to analyze number-theoretical conjectures in the orthogonal and symplectic cases [15], in the same way that the Toeplitz formulae of Theorem 1.1 were used in the unitary case in [27]. (Note that in [27], the authors only need the version of Theorem 1.1 originally proven by Widom [42]).

Another example is the probability $P_E(n)$ of a ferromagnetic string of length n in the antiferromagnetic ground state in the XY spin chain. For a certain range of parameters, $P_E(n)$ is given by a Toeplitz determinant with 2 β-singularities such that $|||\beta||| = 1$. Thus, Theorem 1.18 verifies the result on $P_E(n)$, $n \to \infty$, presented in [22], which the authors based on the Basor-Tracy conjecture.

In a similar vein, our results can be used to justify the asymptotic results for correlators arising in the theory of the impenetrable Bose gas, that were obtained in [37] on the basis of the Basor-Tracy conjecture.

2. Orthogonal polynomials on the unit circle. Toeplitz and Hankel determinants.

Here we present aspects of the theory of orthogonal polynomials on the unit circle we use in this work. Some of the properties we describe here are well-known (see, e.g. [39], [38]), the others not so. We also adapt the theory to complex weights we need in this work, while in the literature usually only positive weights are considered.

Let $f(z)$ be a complex-valued function integrable over the unit circle, and let $\phi_k(z) = \chi_k z^k + \cdots$, $\widehat{\phi}_k(z) = \chi_k z^k + \cdots$, $k = 0, 1, \ldots$ be a system of polynomials in z of degree k with the same for $\phi_k(z)$ and $\widehat{\phi}_k(z)$ leading coefficients χ_k. These polynomials are called orthonormal on the unit circle with weight $f(z)$ if they satisfy (1.16). If $f(z)$ is positive on the unit circle, it is a classical fact that $D_n(f) \neq 0$ for all $n \geq 1$, and such a system of polynomials exists. In general, suppose that all the Toeplitz determinants D_n, $n = 1, 2, \ldots$ (1.1) are nonzero, $D_0 \equiv 1$. Then the polynomials $\phi_k(z)$ and $\widehat{\phi}_k(z)$ for $k = 0, 1, \ldots$ are given by the explicit formulae (1.17), (1.18) for *all* $k = 1, 2, \ldots$. For $k = 0$ set

$$\phi_0(z) = \widehat{\phi}_0(z) = \chi_0 = 1/\sqrt{D_1}. \tag{2.1}$$

Relations (1.16) are then equivalent to

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_k(z)\widehat{\phi}_m(z^{-1})f(z)d\theta = \delta_{km}, \qquad z = e^{i\theta}, \qquad k, m = 0, 1, \ldots. \tag{2.2}$$

Thus we constructed the system of orthogonal polynomials under condition that all the Toeplitz determinants are nonzero. If we only know that $D_n(f) \neq 0$ for all $n \geq N_0$ with some $N_0 > 0$, then we have the existence of $\phi_k(z) = \chi_k z^k + \cdots$, $\widehat{\phi}_k(z) = \chi_k z^k + \cdots$, $\chi_k \neq 0$, satisfying (1.16) for $k = N_0, N_0 + 1, \ldots$.

Remark 2.1. From (1.17,1.18,2.1) we easily conclude:

a) If $f(z)$ is real on the unit circle, we have $\widehat{\phi}_n(z^{-1}) = \overline{\phi_n(z)}$, $n = 0, 1, \ldots$, on the unit circle.

b) If $f(e^{i\theta}) = f(e^{-i\theta})$, then $\widehat{\phi}_n(z^{-1}) = \phi_n(z^{-1})$.

Lemma 2.2 (Recurrence relations)**.** *Let $D_n(f) \neq 0$, $n \geq 0$. The orthogonal polynomials satisfy the following relations for $n = 0, 1, \ldots$:*

$$\chi_n z\phi_n(z) = \chi_{n+1}\phi_{n+1}(z) - \phi_{n+1}(0)z^{n+1}\widehat{\phi}_{n+1}(z^{-1}); \tag{2.3}$$

$$\chi_n z^{-1}\widehat{\phi}_n(z^{-1}) = \chi_{n+1}\widehat{\phi}_{n+1}(z^{-1}) - \widehat{\phi}_{n+1}(0)z^{-n-1}\phi_{n+1}(z); \tag{2.4}$$

$$\chi_{n+1} z^{-1}\widehat{\phi}_n(z^{-1}) = \chi_n\widehat{\phi}_{n+1}(z^{-1}) - \widehat{\phi}_{n+1}(0)z^{-n}\phi_n(z). \tag{2.5}$$

Moreover,

$$\chi_{n+1}^2 - \chi_n^2 = \phi_{n+1}(0)\widehat{\phi}_{n+1}(0). \tag{2.6}$$

Proof. To prove (2.3) consider the function

$$g(z) = \chi_n\phi_n(z) - \chi_{n+1}z^{-1}\phi_{n+1}(z) + \phi_{n+1}(0)z^n\widehat{\phi}_{n+1}(z^{-1}).$$

We see that it has zero coefficient at z^{-1} and so $g(z)$ is a polynomial in z of degree n. Therefore we can write

$$g(z) = \sum_{k=0}^{n} c_k\phi_k(z),$$

where $c_k = \frac{1}{2\pi}\int_0^{2\pi} g(z)\widehat{\phi}_k(z^{-1})f(z)d\theta$. This integral is easy to calculate using the orthogonality in the form of (1.16) (for example, $\frac{1}{2\pi}\int_0^{2\pi} \phi_{n+1}(z)z^{-1}\widehat{\phi}_k(z^{-1})f(z)d\theta = (\chi_n/\chi_{n+1})\delta_{nk}$), and we obtain that all $c_k = 0$. Thus $g(z) \equiv 0$ and (2.3) is proved.

Similarly, considering $g_1(z) = \chi_n\widehat{\phi}_n(z^{-1}) - \chi_{n+1}z\widehat{\phi}_{n+1}(z^{-1}) + \widehat{\phi}_{n+1}(0)z^{-n}\phi_{n+1}(z)$ we show that $g_1(z) \equiv 0$, which proves equation (2.4).

Collecting the coefficients at z^{n+1} in (2.3) we obtain (2.6).

Finally, multiplying (2.3) by $z^{-n-1}\widehat{\phi}_{n+1}(0)$, and (2.4) by χ_{n+1}, adding the resulting equations together and using (2.6), we obtain (2.5). □

Lemma 2.3 (Christoffel-Darboux identity)**.** *Let $D_n(f) \neq 0$, $n \geq 0$. For any z, $a \neq 0$, $n = 1, 2, \dots,$*

$$(1 - a^{-1}z)\sum_{k=0}^{n-1}\widehat{\phi}_k(a^{-1})\phi_k(z) = a^{-n}\phi_n(a)z^n\widehat{\phi}_n(z^{-1}) - \widehat{\phi}_n(a^{-1})\phi_n(z). \tag{2.7}$$

For any $z \neq 0$, $n = 1, 2, \dots,$

$$\sum_{k=0}^{n-1}\widehat{\phi}_k(z^{-1})\phi_k(z) = -n\phi_n(z)\widehat{\phi}_n(z^{-1}) + z\left(\widehat{\phi}_n(z^{-1})\frac{d}{dz}\phi_n(z) - \phi_n(z)\frac{d}{dz}\widehat{\phi}_n(z^{-1})\right). \tag{2.8}$$

Proof. Consider $(1-a^{-1}z)\widehat{\phi}_k(a^{-1})\phi_k(z)$, for a fixed $k \geq 0$. Using the recurrence relation (2.3) with $n = k$ to express $z\phi_k(z)$ in terms of $\phi_{k+1}(z)$ and $\widehat{\phi}_{k+1}(z^{-1})$, and using (2.5) with $n = k$ to express $a^{-1}\widehat{\phi}_k(a^{-1})$, we obtain:

$$\begin{aligned}(1 - a^{-1}z)\widehat{\phi}_k(a^{-1})\phi_k(z) &= \widehat{\phi}_k(a^{-1})\phi_k(z) - \widehat{\phi}_{k+1}(a^{-1})\phi_{k+1}(z)\\ &+ \frac{\phi_{k+1}(0)}{\chi_{k+1}}z^{k+1}\widehat{\phi}_{k+1}(z^{-1})a^{-k-1}a^{k+1}\widehat{\phi}_{k+1}(a^{-1}) + \frac{\widehat{\phi}_{k+1}(0)}{\chi_k}a^{-k}\phi_k(a)z^{k+1}z^{-k-1}\widehat{\phi}_{k+1}(z)\\ &- \frac{\phi_{k+1}(0)\widehat{\phi}_{k+1}(0)}{\chi_k\chi_{k+1}}a^{-k}\phi_k(a)z^{k+1}\widehat{\phi}_{k+1}(z^{-1}).\end{aligned}$$

Now expressing in the third summand $a^{k+1}\widehat{\phi}_{k+1}(a^{-1})$ from (2.3) with $n = k$ and $z = a$, and in the fourth summand $z^{-k-1}\widehat{\phi}_{k+1}(z)$ from (2.4), and by using (2.6), we obtain

$$\begin{aligned}(1 - a^{-1}z)\widehat{\phi}_k(a^{-1})\phi_k(z) &= \widehat{\phi}_k(a^{-1})\phi_k(z) - \widehat{\phi}_{k+1}(a^{-1})\phi_{k+1}(z)\\ &+ a^{-k-1}\phi_{k+1}(a)z^{k+1}\widehat{\phi}_{k+1}(z^{-1}) - a^{-k}\phi_k(a)z^k\widehat{\phi}_k(z^{-1}).\end{aligned}$$

Summing this over k from $k = 0$ to $n - 1$ yields (2.7). Taking the limit $a \to z$ in (2.7) gives (2.8). □

The next lemma allows us to represent the Toeplitz determinant with symbol $z^\ell f(z)$, where ℓ is any integer, in terms of the one with symbol $f(z)$.

Lemma 2.4. *Let the Toeplitz determinants $D_n(f)$ with symbol $f(z)$ be nonzero for all $n \geq N_0$ with a fixed $N_0 \geq 0$. Let $\Phi_k(z) = \phi_k(z)/\chi_k$, $\widehat{\Phi}_k(z) = \widehat{\phi}_k(z)/\chi_k$, $k = N_0, N_0 + 1, \dots$ be the system of monic polynomials orthogonal on the unit circle with the weight $f(z)$. Fix an integer $\ell > 0$. Then if*

$$F_k = \begin{vmatrix} \Phi_k(0) & \Phi_{k+1}(0) & \cdots & \Phi_{k+\ell-1}(0)\\ \frac{d}{dz}\Phi_k(0) & \frac{d}{dz}\Phi_{k+1}(0) & \cdots & \frac{d}{dz}\Phi_{k+\ell-1}(0)\\ \vdots & \vdots & & \vdots\\ \frac{d^{\ell-1}}{dz^{\ell-1}}\Phi_k(0) & \frac{d^{\ell-1}}{dz^{\ell-1}}\Phi_{k+1}(0) & \cdots & \frac{d^{\ell-1}}{dz^{\ell-1}}\Phi_{k+\ell-1}(0)\end{vmatrix} \neq 0, \qquad k = N_0, N_0 + 1, \dots, n - 1,$$

we have

$$D_n(z^\ell f(z)) = \frac{(-1)^{\ell n} F_n}{\prod_{j=1}^{\ell-1} j!} D_n(f(z)), \qquad n \geq N_0. \tag{2.9}$$

In particular, for $\ell = 1$*, if* $\phi_k(0) \neq 0$*,* $k = N_0, N_0+1, \ldots, n-1$*, we have*

$$D_n(zf(z)) = (-1)^n \frac{\phi_n(0)}{\chi_n} D_n(f(z)), \qquad n \geq N_0. \tag{2.10}$$

Furthermore, if

$$\widehat{F}_k = \begin{vmatrix} \widehat{\Phi}_k(0) & \widehat{\Phi}_{k+1}(0) & \cdots & \widehat{\Phi}_{k+\ell-1}(0) \\ \frac{d}{dz}\widehat{\Phi}_k(0) & \frac{d}{dz}\widehat{\Phi}_{k+1}(0) & \cdots & \frac{d}{dz}\widehat{\Phi}_{k+\ell-1}(0) \\ \vdots & \vdots & & \vdots \\ \frac{d^{\ell-1}}{dz^{\ell-1}}\widehat{\Phi}_k(0) & \frac{d^{\ell-1}}{dz^{\ell-1}}\widehat{\Phi}_{k+1}(0) & \cdots & \frac{d^{\ell-1}}{dz^{\ell-1}}\widehat{\Phi}_{k+\ell-1}(0) \end{vmatrix} \neq 0, \qquad k = N_0, N_0+1, \ldots, n-1,$$

we have

$$D_n(z^{-\ell} f(z)) = \frac{(-1)^{\ell n} \widehat{F}_n}{\prod_{j=1}^{\ell-1} j!} D_n(f(z)), \qquad n \geq N_0. \tag{2.11}$$

In particular, for $\ell = 1$*, if* $\widehat{\phi}_k(0) \neq 0$*,* $k = N_0, N_0+1, \ldots, n-1$*, we have*

$$D_n(z^{-1} f(z)) = (-1)^n \frac{\widehat{\phi}_n(0)}{\chi_n} D_n(f(z)), \qquad n \geq N_0. \tag{2.12}$$

Proof. We give the proof for $\ell = 1$; the generalization is a simple exercise. Recall that since $D_n \neq 0$, $n = N_0, N_0+1, \ldots$, the polynomials $\phi_n(z) = \chi_n z^n + \ldots$, $\chi_n \neq 0$, exist for $n = N_0, N_0+1, \ldots$.

Assume first that $N_0 = 0$. Given the polynomials $\phi_k(z)$ related to the weight $f(z)$, we will need the ones corresponding to the weight $zf(z)$. An analogous construction for polynomials orthogonal on the real line is known as Christroffel's formula (see [39], p. 333). Namely, define $q_k(z)$ by the expression:

$$zq_k(z) = \begin{vmatrix} \phi_k(z) & \phi_{k+1}(z) \\ \phi_k(0) & \phi_{k+1}(0) \end{vmatrix}. \tag{2.13}$$

We see immediately that $q_k(z)$ is a polynomial, and if $\phi_k(0) \neq 0$, it has degree k with leading coefficient $-\chi_{k+1}\phi_k(0)$. Moreover, by orthogonality,

$$\int_0^{2\pi} q_k(z) z^{-j} z f(z) d\theta = 0, \qquad j = 0, 1, \ldots, k-1.$$

For $j = k$,

$$\frac{1}{2\pi}\int_0^{2\pi} q_k(z) z^{-k} z f(z) d\theta = \begin{vmatrix} 1/\chi_k & 0 \\ \phi_k(0) & \phi_{k+1}(0) \end{vmatrix} = \frac{\phi_{k+1}(0)}{\chi_k}.$$

Therefore, for *monic* polynomials $Q_k(z) = q_k(z)/(-\chi_{k+1}\phi_k(0))$,

$$\frac{1}{2\pi}\int_0^{2\pi} Q_k(z) z^{-k} z f(z) d\theta = -\frac{\phi_{k+1}(0)}{\phi_k(0)} \frac{1}{\chi_k \chi_{k+1}} \equiv h_k.$$

Thus, cf. (1.19), the Toeplitz determinant with symbol $zf(z)$, is given by the expression

$$D_n(zf(z)) = \prod_{k=0}^{n-1} h_k = \frac{\phi_n(0)}{\phi_0(0)} \frac{(-1)^n}{\chi_0 \chi_1^2 \cdots \chi_{n-1}^2 \chi_n} = (-1)^n \frac{\phi_n(0)}{\chi_n} D_n(f(z)), \tag{2.14}$$

which is equation (2.10). The case of $z^{-1}f(z)$, i.e. equation (2.12), is obtained similarly by considering $\widehat{\phi}_k(z^{-1})$ instead of $\phi_k(z)$. Namely, we start with the definition

$$z^{-1}\widetilde{q}_k(z^{-1}) = \begin{vmatrix} \widehat{\phi}_k(z^{-1}) & \widehat{\phi}_{k+1}(z^{-1}) \\ \widehat{\phi}_k(0) & \widehat{\phi}_{k+1}(0) \end{vmatrix}$$

and proceed as before.

Suppose now that $D_n(f) \neq 0$ for $n \geq 0$, but F_k, $\widehat{F}_k$ are known to be nonzero only for $k = N_0, N_0+1, \dots, n-1$, with some $N_0 > 0$. Consider the polynomials $F_k(z)$, $\widehat{F}_k(z)$ defined as F_k, $\widehat{F}_k$ with the argument 0 of the orthogonal polynomials replaced with z. Obviously, the set Ω of possible zeros of $F_k(z)$, $\widehat{F}_k(z)$ for $k = 0, 1, \dots, N_0 - 1$ is finite. We now replace (2.13) with

$$(z-t)q_k(z) = \begin{vmatrix} \phi_k(z) & \phi_{k+1}(z) \\ \phi_k(t) & \phi_{k+1}(t) \end{vmatrix},$$

and choose t so that $\phi_k(t) \neq 0$ for $k = 0, 1, \dots, N_0 - 1$. Instead of (2.14) consider the product

$$\prod_{k=0}^{N_0-1} \left[-\frac{\phi_{k+1}(t)}{\phi_k(t)} \frac{1}{\chi_k \chi_{k+1}} \right] \prod_{k=N_0}^{n-1} \left[-\frac{\phi_{k+1}(0)}{\phi_k(0)} \frac{1}{\chi_k \chi_{k+1}} \right]$$

and take the limit $t \to 0$ so that t avoids the set Ω. This proves equation (2.10) under the condition $D_n(f) \neq 0$, $n \geq 0$. We extend the result to a weaker condition $D_n(f) \neq 0$, $n = N_0, N_0+1, \dots$, and thus complete the proof of (2.10), by using the fact that $D_n(f) \neq 0$, $n = 1, 2, \dots$, for positive f on the unit circle, and by a simple continuity argument in α_j and β_j (cf. [28]). The case of $z^{-1}f(z)$ is dealt with similarly. □

We will now establish a connection between a Hankel determinant with symbol on a finite interval and a Toeplitz determinant. First we need a theorem due to Szegő on a relation between polynomials orthogonal on an interval of the real axis and those orthogonal on the unit circle. Szegő considered positive weights on the unit circle, but his theorem is transferred to the general case without much change:

Lemma 2.5. *Let $f(z)$ have the property $f(e^{i\theta}) = f(e^{-i\theta})$, $0 \leq \theta \leq 2\pi$ and let*

$$w(x) = \frac{f(e^{i\theta})}{|\sin\theta|}, \qquad x = \cos\theta.$$

Assume that $D_n(f) \neq 0$, $n \geq N_0$, $N_0 \geq 0$. Then the polynomials $p_n(x) = \varkappa_n x^n + \cdots$, $n = N_0, N_0+1, \dots$ exist which are orthonormal w.r.t. weight $w(x)$ on $[-1,1]$, i.e.,

$$\int_{-1}^{1} p_n(x) x^m w(x) dx = \varkappa_k^{-1} \delta_{nm}, \qquad m = 0, 1, \dots, n, \quad n \geq N_0,$$

and, for $n = N_0, N_0+1, \dots$, there hold the following expressions in terms of the polynomials $\phi_n(z)$ orthogonal w.r.t. $f(z)$ on the unit circle:

$$\varkappa_n = 2^n \chi_{2n} \sqrt{\frac{1 - a_{2n-1}}{2\pi}}, \tag{2.15}$$

$$P_n(x) = \frac{1}{(2z)^n (1 - a_{2n-1})} (\Phi_{2n}(z) + \Phi_{2n}^*(z)), \qquad n \geq N_0, \tag{2.16}$$

where $P_n(z) = p_n(z)/\varkappa_n$, $\Phi_n(z) = \phi_n(z)/\chi_n$, $\Phi_n^*(z) = z^n \Phi_n(z^{-1})$, $a_{n-1} = -\Phi_n(0)$, $n \geq N_0$.

Proof. The condition on Toeplitz determinants immediately implies the existence of the polynomials $\phi_n(z) = \chi_n z^n + \cdots$, $\chi_n \neq 0$, $n = N_0, N_0+1, \dots$ orthogonal w.r.t. $f(z)$ on the unit circle. By Remark (b) above, in the present case of $f(e^{i\theta})$ being an even function of θ, we have $\widehat{\phi}_n(z^{-1}) = \phi_n(z^{-1})$ for all $n \geq N_0$. Now the proof is the same as the argument in the proof of Theorem 11.5 in [39], and we obtain

$$p_n(x) = \frac{1}{\sqrt{2\pi(1-a_{2n-1})}}(z^{-n}\phi_{2n}(z) + z^n\phi_{2n}(z^{-1})). \tag{2.17}$$

Note that $1-a_{2n-1} \neq 0$, $n = N_0, N_0+1, \dots$ as follows from (2.6) which in our case can be rewritten in the form

$$\chi_n^2 = \left(1 - \phi_{n+1}^2(0)/\chi_{n+1}^2\right)\chi_{n+1}^2.$$

We now easily obtain the statement of the lemma from (2.17). □

Note that the recurrence relation (2.5) can be easily rewritten in terms of the monic polynomials in the form (for $f(e^{i\theta})$ an even function of θ and with z is replaced by z^{-1}):

$$\Phi_{n+1}(z) = z\Phi_n(z) - a_n\Phi_n^*(z). \tag{2.18}$$

Replacing here again z by z^{-1} and multiplying both sides by z^{n+1} we obtain

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - a_n z\Phi_n(z). \tag{2.19}$$

Now we are ready to formulate and prove

Theorem 2.6. *[Connection between Toeplitz and Hankel determinants] Let $N_0 \geq 0$ and $D_n(f) \neq 0$ for all $n \geq N_0$. Let the weights $f(z)$ and $w(x)$ be related as in Lemma 2.5. Let, moreover,*

$$D_n(w(x)) = \det\left(\int_{-1}^{1} x^{j+k}w(x)dx\right)_{j,k=0}^{n-1}, \qquad n = N_0, N_0+1, \dots$$

be the Hankel determinant with symbol $w(x)$ on $[-1,1]$. Then, with $\Phi_n(z) = \phi_n(z)/\chi_n$, we have

$$D_n(w(x))^2 = \frac{\pi^{2n}}{4^{(n-1)^2}}(1+\Phi_{2n}(0))^2\frac{D_{2n}(f(z))}{\Phi_{2n}(1)\Phi_{2n}(-1)}, \qquad n = N_0, N_0+1, \dots \tag{2.20}$$

Proof. Assume first $N_0 = 0$. Take equation (2.16) with $n = k+1$ and apply the recurrence relations (2.18,2.19) with $n = 2k+1$ to $\Phi_{2k+2}(z)$ and $\Phi_{2k+2}^*(z)$, respectively. We then obtain

$$P_{k+1}(x) = (2z)^{-k-1}(z\Phi_{2k+1}(z) + \Phi_{2k+1}^*(z)).$$

Now apply again the relations (2.18,2.19) with $n = 2k$ to $\Phi_{2k+1}(z)$ and $\Phi_{2k+1}^*(z)$ here, respectively. The result can be written in the form

$$\Phi_{2k}^*(z) = \frac{(2z)^{k+1}}{1-za_{2k}}P_{k+1}(x) - z\frac{z-a_{2k}}{1-za_{2k}}\Phi_{2k}(z),$$

where we assume that $z \neq 0$ and $1 - za_{2k} \neq 0$. On the other hand, from (2.16) with $n = k$

$$\Phi_{2k}^*(z) = (2z)^k(1-a_{2k-1})P_k(x) - \Phi_{2k}(z)$$

Equating the r.h.s. of the last two equations, we obtain

$$(z^2-1)\Phi_{2k}(z) = (2z)^{k+1}P_{k+1}(x) - (2z)^k(1-a_{2k-1})(1-za_{2k})P_k(x). \tag{2.21}$$

Setting here $z = 1$ (recall from the proof of Lemma 2.5 that $1 \pm a_n \neq 0$, $n = 0, 1, \dots$), we obtain

$$(1-a_{2k-1})(1-a_{2k}) = 2\frac{P_{k+1}(1)}{P_k(1)}.$$

Note that it is a general property of orthogonal polynomials that $P_k(x)$ and $P_{k+1}(x)$ cannot have a zero in common. Similarly, setting $z=-1$, we have

$$(1-a_{2k-1})(1+a_{2k}) = -2\frac{P_{k+1}(-1)}{P_k(-1)}.$$

The product of these two equations yields

$$(1-a_{2k-1})^2(1-a_{2k}^2) = -4\frac{P_{k+1}(1)}{P_k(1)}\frac{P_{k+1}(-1)}{P_k(-1)}.$$

By the relation (2.6), we can substitute here

$$1-a_{2k}^2 = \left(\frac{\chi_{2k}}{\chi_{2k+1}}\right)^2,$$

which gives

$$(1-a_{2k-1})^2 = -4\left(\frac{\chi_{2k+1}}{\chi_{2k}}\right)^2\frac{P_{k+1}(1)P_{k+1}(-1)}{P_k(1)P_k(-1)}. \tag{2.22}$$

This equation together with (2.15) and the well-known expression for $D_n(w(x))$ in terms of the leading coefficients $\varkappa_k^{-2}$ implies

$$D_n(w(x))^2 = \prod_{k=0}^{n-1}\varkappa_k^{-4} = \frac{\pi^{2n}}{4^{n(n-1)}}\frac{(-1)^n}{P_n(1)P_n(-1)}\prod_{k=0}^{2n-1}\chi_k^{-2} = \frac{\pi^{2n}}{4^{n(n-1)}}\frac{(-1)^n}{P_n(1)P_n(-1)}D_{2n}(f(z)). \tag{2.23}$$

Now using (2.16), we obtain

$$P_n(1)P_n(-1) = \frac{(-1)^n}{4^{n-1}(1-a_{2n-1})^2}\Phi_{2n}(1)\Phi_{2n}(-1),$$

and thus finish the proof for $N_0=0$. (Note that (2.22) implies that $\Phi_{2n}(\pm 1)\neq 0$.) The extension to an arbitrary $N_0>0$ is carried out as in the proof of Lemma 2.4. □

We will also need a connection between Hankel and Toeplitz+Hankel determinants. We borrow the idea of the next statement from [40, 29, 2].

Lemma 2.7. *[Connection between Hankel and Toeplitz+Hankel determinants] Let f_j be the Fourier coefficient $f_j = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ij\theta}d\theta$. Let $f(e^{i\theta}) = f(e^{-i\theta})$. Then, for $n=1,2,\dots,$*

$$\det(f_{j-k}+f_{j+k})_{j,k=0}^{n-1} = \frac{2^{n^2-2n+2}}{\pi^n}D_n(v(x)), \tag{2.24}$$

where $D_n(v(x))$ is the Hankel determinant with symbol $v(x) = f(e^{i\theta(x)})/\sqrt{1-x^2}$, $x=\cos\theta$ on $[-1,1]$. Furthermore, again in terms of Hankel determinants with symbols on $x\in[-1,1]$,

$$\det(f_{j-k}-f_{j+k+2})_{j,k=0}^{n-1} = \frac{2^{n^2}}{\pi^n}D_n(f(e^{i\theta(x)})\sqrt{1-x^2}), \tag{2.25}$$

$$\det(f_{j-k}+f_{j+k+1})_{j,k=0}^{n-1} = \frac{2^{n^2-n}}{\pi^n}D_n(f(e^{i\theta(x)})\sqrt{\frac{1+x}{1-x}}), \tag{2.26}$$

$$\det(f_{j-k}-f_{j+k+1})_{j,k=0}^{n-1} = \frac{2^{n^2-n}}{\pi^n}D_n(f(e^{i\theta(x)})\sqrt{\frac{1-x}{1+x}}). \tag{2.27}$$

Proof. Since $f(e^{i\theta}) = f(e^{-i\theta})$, note that for $j, k = 0, 1, \dots,$

$$\frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\cos j\theta \cos k\theta d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})(e^{-i(j+k)\theta} + e^{-i(j-k)\theta})d\theta = f_{j-k} + f_{j+k}. \tag{2.28}$$

Therefore, using the standard expansion (where only the first coefficient is needed to be known explicitly) in non-negative powers of the cosine,

$$\cos k\theta = 2^{k-1}\cos^k\theta + c_{k-2}\cos^{k-2}\theta + c_{k-4}\cos^{k-4}\theta + \cdots, \tag{2.29}$$

we obtain

$$\begin{aligned}\det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1} &= \det\left(\frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\cos j\theta\cos k\theta d\theta\right)_{j,k=0}^{n-1} \\ &= 2^{1+2+\cdots+n-2}\det\left(\frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\cos j\theta\cos^k\theta d\theta\right)_{j,k=0}^{n-1} \\ &= 2^{(n-1)(n-2)}\det\left(\frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\cos^j\theta\cos^k\theta d\theta\right)_{j,k=0}^{n-1} \\ &= 2^{(n-1)(n-2)}\left(\frac{2}{\pi}\right)^n\det\left(\int_0^{\pi} f(e^{i\theta})\cos^j\theta\cos^k\theta d\theta\right)_{j,k=0}^{n-1}.\end{aligned} \tag{2.30}$$

Changing the variable $x = \cos\theta$, $d\theta = -dx/\sqrt{1-x^2}$, we immediately obtain

$$\det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1} = \frac{2^{n^2-2n+2}}{\pi^n}\det\left(\int_{-1}^{1} v(x)x^{j+k}dx\right), \qquad v(x) = \frac{f(e^{i\theta(x)})}{\sqrt{1-x^2}}, \tag{2.31}$$

which is (2.24).

Similarly, using the observations

$$\frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\sin(j+1)\theta\sin(k+1)\theta d\theta = f_{j-k} - f_{j+k+2}, \tag{2.32}$$

$$\frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\cos(j+1/2)\theta\cos(k+1/2)\theta d\theta = f_{j-k} + f_{j+k+1}, \tag{2.33}$$

$$\frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\sin(j+1/2)\theta\sin(k+1/2)\theta d\theta = f_{j-k} - f_{j+k+1}, \tag{2.34}$$

and the expansions in non-negative powers of the cosine of the quantities

$$\frac{\sin(k+1)\theta}{\sin\theta}, \qquad \frac{\cos(k+1/2)\theta}{\cos\frac{\theta}{2}}, \qquad \frac{\sin(k+1/2)\theta}{\sin\frac{\theta}{2}},$$

we obtain (2.25), (2.26), and (2.27). □

Finally, we list some properties of Barnes' G-function (see [3, 41]) we need below. The G-function is an entire function defined, e.g., by the product:

$$G(z+1) = (2\pi)^{z/2}e^{-(z+1)z/2-\gamma_E z^2/2}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^k e^{-z+z^2/(2k)}, \qquad z \in \mathbb{C} \tag{2.35}$$

where γ_E is Euler's constant. $G(z)$ satisfies the recurrence relation:

$$G(z+1) = \Gamma(z)G(z), \qquad G(1) = 1, \tag{2.36}$$

where $\Gamma(z)$ is Euler's G-function. The following representation is useful

$$\int_0^z \ln\Gamma(x+1)dx = \frac{z}{2}\ln 2\pi - \frac{z(z+1)}{2} + z\ln\Gamma(z+1) - \ln G(z+1). \tag{2.37}$$

There holds the identity:

$$2\ln G(1/2) = (1/12)\ln 2 - \ln\sqrt{\pi} + 3\zeta'(-1), \tag{2.38}$$

where $\zeta'(x)$ is the derivative of Riemann's ζ-function. We will also need a doubling formula given by

$$G(2z)\pi^z G(1/2)^2 = G(z)^2 G(z+1/2)^2\Gamma(z)2^{(2z-1)(z-1)}. \tag{2.39}$$

3. Riemann-Hilbert problem

In this section we formulate a Riemann-Hilbert problem (RHP) for the polynomials $\phi_k(z)$, $\widehat{\phi}_k(z)$. We use this RHP in section 5 to find asymptotics of the polynomials.

Let the weight $f(z)$ be given on the unit circle (which, oriented in the positive direction, we denote C) by (1.2). Suppose that the system of orthonormal polynomials satisfying (1.16) exists. Consider the following 2×2 matrix valued function $Y^{(k)}(z) \equiv Y(z)$:

$$Y^{(k)}(z) = \begin{pmatrix} \chi_k^{-1}\phi_k(z) & \chi_k^{-1}\int_C \frac{\phi_k(\xi)}{\xi - z}\frac{f(\xi)d\xi}{2\pi i\xi^k} \\ -\chi_{k-1}z^{k-1}\widehat{\phi}_{k-1}(z^{-1}) & -\chi_{k-1}\int_C \frac{\widehat{\phi}_{k-1}(\xi^{-1})}{\xi - z}\frac{f(\xi)d\xi}{2\pi i\xi} \end{pmatrix}. \tag{3.1}$$

It is easy to verify that $Y(z)$ solves the following Riemann-Hilbert problem:

(a) $Y(z)$ is analytic for $z \in \mathbb{C}\setminus C$.

(b) Let $z \in C\setminus \cup_{j=0}^m z_j$. Y has continuous boundary values $Y_+(z)$ as z approaches the unit circle from the inside, and $Y_-(z)$, from the outside, related by the jump condition

$$Y_+(z) = Y_-(z)\begin{pmatrix} 1 & z^{-k}f(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in C\setminus \cup_{j=0}^m z_j. \tag{3.2}$$

(c) $Y(z)$ has the following asymptotic behavior at infinity:

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^k & 0 \\ 0 & z^{-k} \end{pmatrix}, \qquad \text{as } z\to\infty. \tag{3.3}$$

(d) As $z \to z_j$, $j = 0, 1, \ldots, m$, $z \in \mathbb{C}\setminus C$,

$$Y(z) = \begin{pmatrix} O(1) & O(1) + O(|z - z_j|^{2\alpha_j}) \\ O(1) & O(1) + O(|z - z_j|^{2\alpha_j}) \end{pmatrix}, \qquad \text{if } \alpha_j \neq 0, \tag{3.4}$$

and

$$Y(z) = \begin{pmatrix} O(1) & O(\ln|z - z_j|) \\ O(1) & O(\ln|z - z_j|) \end{pmatrix}, \qquad \text{if } \alpha_j = 0,\ \beta_j \neq 0. \tag{3.5}$$

(Here and below $O(a)$ stands for $O(|a|)$. If $\alpha_0 = \beta_0 = 0$ then $Y(z)$ is bounded at $z = 1$.)

A general fact that orthogonal polynomials can be so represented as a solution of a Riemann-Hilbert problem was noticed in [24] (for polynomials on the line) and extended for polynomials on the circle in [1]. This fact is important because it turns out that the RHP can be efficiently analyzed for large k by a steepest-descent-type method found in [17] and developed further in many subsequent works. Thus, we first find the solution to the problem (a)–(d) for large k (applying this method) and then interpret it as the asymptotics of the orthogonal polynomials by (3.1).

The solution to the RHP (a)–(d) is unique. Note first that $\det Y(z) = 1$. Indeed, from the conditions on $Y(z)$, $\det Y(z)$ is analytic across the unit circle, has all singularities removable, and

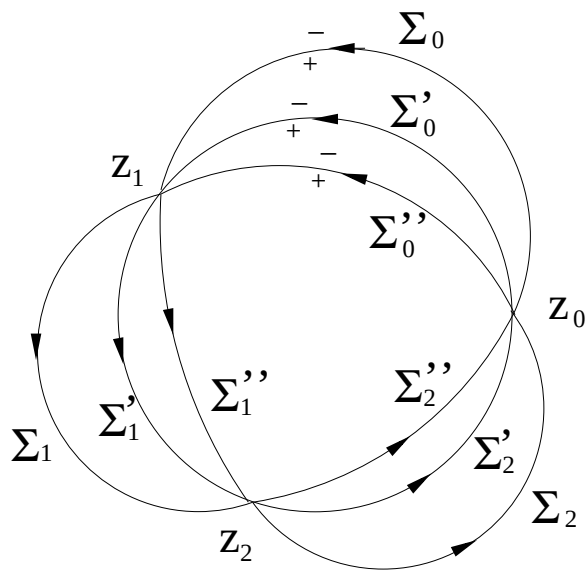

FIGURE 1. Contour for the S-Riemann-Hilbert problem ($m = 2$).

tends to 1 as $z \to \infty$. It is then identically 1 by Liouville's theorem. Now if there is another solution $\widetilde{Y}(z)$, we easily obtain by Liouville's theorem that $Y(z)\widetilde{Y}(z)^{-1} \equiv 1$.

4. Asymptotic analysis of the Riemann-Hilbert problem

In this section we construct an asymptotic solution to the Riemann-Hilbert problem (a) – (d) of Section 3 for large $k = n$ by the steepest descent method. All the steps of the analysis are standard apart from construction of the local parametrix near the points z_j. We always assume that $f(z)$ is given by (1.2). In all Section 4 we also assume for simplicity that $z_0 = 1$ is a singularity. However, the results trivially extend to the case $\alpha_0 = \beta_0 = 0$. In Section 4 we further assume that $V(z)$ is analytic in a neighborhood of the unit circle.

The first step is the following transformation, which normalizes the problem at infinity:

$$T(z) = Y(z) \begin{cases} z^{-n\sigma_3}, & |z| > 1 \\ I, & |z| < 1, \end{cases} \tag{4.1}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. From the RHP for $Y(z)$, we obtain the following problem for $T(z)$:

(a) $T(z)$ is analytic for $z \in \mathbb{C} \setminus C$.
(b) The boundary values of $T(z)$ are related by the jump condition

$$T_+(z) = T_-(z) \begin{pmatrix} z^n & f(z) \\ 0 & z^{-n} \end{pmatrix}, \qquad z \in C \setminus \cup_{j=0}^m z_j, \tag{4.2}$$

(c) $T(z) = I + O(1/z)$ as $z \to \infty$,

and the condition (d) remains unchanged.

Now split the contour as shown in Figure 1. Define a new transformation as follows:

$$S(z) = \begin{cases} T(z), & \text{for } z \text{ outside the lenses,} \\ T(z)\begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^{-n} & 1 \end{pmatrix}, & \text{for } |z|>1 \text{ and inside the lenses,} \\ T(z)\begin{pmatrix} 1 & 0 \\ -f(z)^{-1}z^{n} & 1 \end{pmatrix}, & \text{for } |z|<1 \text{ and inside the lenses.} \end{cases} \tag{4.3}$$

Here $f(z)$ is the analytic continuation of $f(z)$ off C into the inside of the lenses as discussed in Section 4.2 below (see (4.13), (4.14)).

Then the Riemann-Hilbert problem for $S(z)$ is the following:

(a) $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$, where $\Sigma = \cup_{j=0}^{m}(\Sigma_j \cup \Sigma_j' \cup \Sigma_j'')$.

(b) The boundary values of $S(z)$ are related by the jump condition

$$S_+(z) = S_-(z)\begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^{\mp n} & 1 \end{pmatrix}, \qquad z \in \cup_{j=0}^{m}(\Sigma_j \cup \Sigma_j''),$$

where the minus sign in the exponent is on Σ_j, and plus, on Σ_j'',

$$S_+(z) = S_-(z)\begin{pmatrix} 0 & f(z) \\ -f(z)^{-1} & 0 \end{pmatrix}, \qquad z \in \cup_{j=0}^{m}\Sigma_j'.$$

(c) $S(z) = I + O(1/z)$ as $z \to \infty$,

(d) As $z \to z_j$, $j = 0, \dots, m$, $z \in \mathbb{C} \setminus C$ outside the lenses,

$$S(z) = \begin{pmatrix} O(1) & O(1) + O(|z-z_j|^{2\alpha_j}) \\ O(1) & O(1) + O(|z-z_j|^{2\alpha_j}) \end{pmatrix} \tag{4.4}$$

if $\alpha_j \neq 0$, and

$$S(z) = \begin{pmatrix} O(1) & O(\ln|z-z_j|) \\ O(1) & O(\ln|z-z_j|) \end{pmatrix} \tag{4.5}$$

if $\alpha_j = 0$, $\beta_j \neq 0$. The behavior of $S(z)$ for $z \to z_j$ in other sectors is obtained from these expressions by application of the appropriate jump conditions.

Let us encircle each of the points z_j by a sufficiently small disc,

$$U_{z_j} = \{z : |z - z_j| < \varepsilon\}, \tag{4.6}$$

We see that, outside the neighborhoods U_{z_j}, the jump matrix on Σ_j, Σ_j'' $j = 0, \dots, m$ is uniformly exponentially close to the identity. We will now construct the parametrices in $\mathbb{C} \setminus (\cup_{j=0}^{m} U_{z_j})$ and U_{z_j}. We match them on the boundaries ∂U_{z_j}, which yields the desired asymptotics.

4.1. **Parametrix outside the points z_j.** We expect the following problem for the parametrix N in $\mathbb{C} \setminus \cup_{j=0}^{m} U_{z_j}$:

(a) $N(z)$ is analytic for $z \in \mathbb{C} \setminus C$,

(b) with the jump on C

$$N_+(z) = N_-(z)\begin{pmatrix} 0 & f(z) \\ -f(z)^{-1} & 0 \end{pmatrix}, \qquad z \in C \setminus \cup_{j=0}^{m} z_j,$$

(c) and the following behavior at infinity

$$N(z) = I + O\left(\frac{1}{z}\right), \qquad \text{as } z \to \infty.$$

One can easily check directly that the solution to this RHP is given by the formula

$$N(z) = \begin{cases} \mathcal{D}(z)^{\sigma_3}, & |z| > 1 \\ \mathcal{D}(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & |z| < 1 \end{cases}, \tag{4.7}$$

where the Szegő function

$$\mathcal{D}(z) = \exp \frac{1}{2\pi i} \int_C \frac{\ln f(s)}{s-z} ds, \tag{4.8}$$

is analytic away from the unit circle with boundary values satisfying $\mathcal{D}_+(z) = \mathcal{D}_-(z) f(z)$, $z \in C \setminus \cup_{j=0}^m z_j$.

In what follows, we will need a more explicit formula for $\mathcal{D}(z)$. Calculation of the integral (with the help of (4.13) below) gives:

$$\mathcal{D}(z) = \exp\left[\frac{1}{2\pi i}\int_C \frac{V(s)}{s-z} ds\right] \prod_{k=0}^m \left(\frac{z-z_k}{z_k e^{i\pi}}\right)^{\alpha_k+\beta_k} = e^{V_0} b_+(z) \prod_{k=0}^m \left(\frac{z-z_k}{z_k e^{i\pi}}\right)^{\alpha_k+\beta_k}, \qquad |z| < 1. \tag{4.9}$$

and

$$\mathcal{D}(z) = \exp\left[\frac{1}{2\pi i}\int_C \frac{V(s)}{s-z} ds\right] \prod_{k=0}^m \left(\frac{z-z_k}{z}\right)^{-\alpha_k+\beta_k} = b_-(z)^{-1} \prod_{k=0}^m \left(\frac{z-z_k}{z}\right)^{-\alpha_k+\beta_k}, \qquad |z| > 1, \tag{4.10}$$

where V_0, $b_\pm(z)$ are defined in (1.8). Note that the branch of $(z-z_k)^{\pm\alpha_k+\beta_k}$ in (4.9), (4.10) is taken as discussed following equation (4.13) below. In (4.10) for any k, the cut of the root $z^{-\alpha_k+\beta_k}$ is the line $\theta = \theta_k$ from $z = 0$ to infinity, and $\theta_k < \arg z < 2\pi + \theta_k$.

4.2. **Parametrix at z_j.** Let us now construct the parametrix $P_{z_j}(z)$ in U_{z_j}. The construction is the same for all $j = 0, 1, \ldots$. We look for an analytic matrix-valued function in a neighborhood of U_{z_j} which satisfies the same jump conditions as $S(z)$ on $\Sigma \cap U_{z_j}$, the same conditions (4.4,4.5) as $z \to z_j$, and, instead of a condition at infinity, satisfies the matching condition

$$P_{z_j}(z) N^{-1}(z) = I + o(1) \tag{4.11}$$

uniformly on the boundary ∂U_{z_j} as $n \to \infty$.

First, set

$$\zeta = n \ln \frac{z}{z_j}, \tag{4.12}$$

where $\ln x > 0$ for $x > 1$, and has a cut on the negative half of the real axis. Under this transformation the neighborhood U_{z_j} is mapped into a neighborhood of zero in the ζ-plane. Note that $\zeta(z)$ is analytic, one-to-one, and it takes an arc of the unit circle to an interval of the imaginary axis. Let us now choose the exact form of the cuts Σ in U_{z_j} so that their images under the mapping $\zeta(z)$ are straight lines (Figure 2). We add one more jump contour to Σ in U_{z_j} which is the pre-image of the real line Γ_3 and Γ_7 in the ζ-plane. This will be needed below because of the non-analyticity of the function $|z - z_j|^{\alpha_j}$. Note that we can construct two different analytic continuations of this function off the unit circle to the pre-images of the upper and lower half ζ-plane, respectively. Namely, write for z on the unit circle,

$$h_{\alpha_j}(z) = |z - z_j|^{\alpha_j} = (z - z_j)^{\alpha_j/2} (z^{-1} - z_j^{-1})^{\alpha_j/2} = \frac{(z - z_j)^{\alpha_j}}{(z z_j e^{i\ell_j})^{\alpha_j/2}}, \qquad z = e^{i\theta}, \tag{4.13}$$

where ℓ_j is found from the condition that the argument of the above function is zero on the unit circle. Let us fix the cut of $(z-z_j)^{\alpha_j}$ going along the line $\theta=\theta_j$ from z_j to infinity. Fix the branch by the condition that on the line going from z_j to the right parallel to the real axis, $\arg(z-z_j)=2\pi$. For $z^{\alpha_j/2}$ in the denominator, $0<\arg z<2\pi$ (the same convention for roots of z is adopted in (4.15,4.17) below). Then, a simple consideration of triangles shows that

$$\ell_j=\begin{cases}3\pi, & 0<\theta<\theta_j\\ \pi, & \theta_j<\theta<2\pi\end{cases}. \tag{4.14}$$

Thus (4.13) is continued analytically to neighborhoods of the arcs $0<\theta<\theta_j$, and $\theta_j<\theta<2\pi$. In U_{z_j}, we extend these neighborhoods to the pre-images of the lower and upper half ζ-plane (intersected with $\zeta(U_{z_j})$), respectively. The cut of h_{α_j} is along the contours Γ_3 and Γ_7 in the ζ-plane.

For $z\to z_j$, $\zeta=n(z-z_j)/z_j+O((z-z_j)^2)$. We have $0<\arg\zeta<2\pi$, which follows from the choice of $\arg(z-z_j)$ in (4.13).

Denote by Roman numerals the sectors between the cuts in Figure 2. We now introduce the following auxiliary function. First, for $j\neq 0$,

$$\begin{aligned}F_j(z)=e^{\frac{V(z)}{2}}\prod_{k=0}^{m}\left(\frac{z}{z_k}\right)^{\beta_k/2}\prod_{k\neq j}h_{\alpha_k}(z)g_{\beta_k}(z)^{1/2}\\ \times h_{\alpha_j}(z)\begin{cases}e^{-i\pi\alpha_j}, & \zeta\in I,II,V,VI\\ e^{i\pi\alpha_j}, & \zeta\in III,IV,VII,VIII\end{cases},\quad z\in U_{z_j},\quad j\neq 0.\end{aligned} \tag{4.15}$$

The functions $g_{\beta_k}(z)$ are defined in (1.4). The case of U_{z_0} is slightly different because of the branch cut of z^{β_k} and z^{α_k} going along the positive real half-line. Define a step function

$$\widehat{g}_{\beta_0}(z)=\begin{cases}e^{-i\pi\beta_0}, & \arg z>0\\ e^{i\pi\beta_0}, & \arg z<2\pi\end{cases},\qquad z\in U_{z_0}, \tag{4.16}$$

and define

$$\begin{aligned}F_0(z)=e^{\frac{V(z)}{2}}\prod_{k=0}^{m}\left(\frac{z}{z_k}\right)^{\beta_k/2}\prod_{k\neq 0}h_{\alpha_k}(z)g_{\beta_k}(z)^{1/2}\\ \times h_{\alpha_0}(z)\begin{cases}e^{-i\pi\alpha_0}, & \zeta\in I,II\\ e^{i\pi(\alpha_0-\beta_0)}, & \zeta\in III,IV\\ e^{-i\pi(\alpha_0+\beta_0)}, & \zeta\in V,VI\\ e^{i\pi\alpha_0}, & \zeta\in VII,VIII\end{cases},\quad z\in U_{z_0}.\end{aligned} \tag{4.17}$$

It is easy to verify that $F_j(z)$, $j=0,1,\dots$ is analytic in the intersection of each quarter ζ-plane with $\zeta(U_{z_j})$ and has the following jumps:

$$F_{j,+}(z) = F_{j,-}(z)e^{-2\pi i\alpha_j}\qquad \zeta\in\Gamma_1; \tag{4.18}$$

$$F_{j,+}(z) = F_{j,-}(z)e^{2\pi i\alpha_j}\qquad \zeta\in\Gamma_5; \tag{4.19}$$

$$F_{j,+}(z) = F_{j,-}(z)e^{\pi i\alpha_j}\qquad \zeta\in\Gamma_3\cup\Gamma_7. \tag{4.20}$$

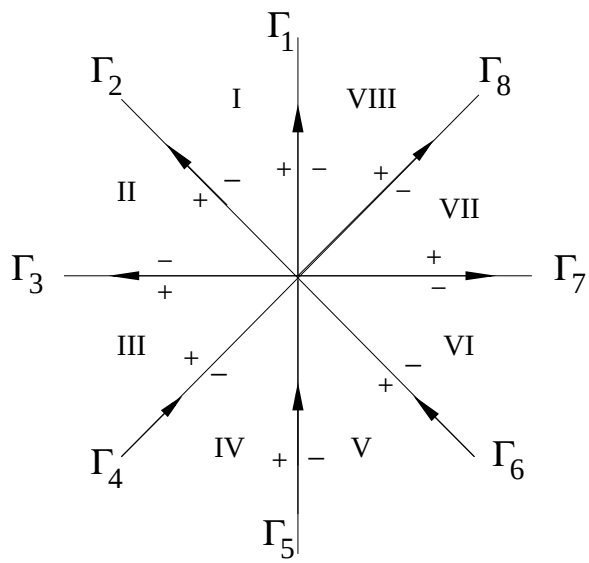

FIGURE 2. The auxiliary contour for the parametrix at z_j.

Comparing (1.2) and (4.15), and using the analytic continuation (see (4.13)) for $f(z)$ off the arcs between the singularities, we obtain the following relations between $f(z)$ and $F_j(z)$:

$$F_j(z)^2 = f(z)e^{-2\pi i\alpha_j} g_{\beta_j}^{-1}(z) \qquad \zeta \in I, II, V, VI; \tag{4.21}$$

$$F_j(z)^2 = f(z)e^{2\pi i\alpha_j} g_{\beta_j}^{-1}(z) \qquad \zeta \in III, IV, VII, VIII. \tag{4.22}$$

for $j \neq 0$. If $j = 0$ the same relations hold with the functions $g_{\beta_0}^{-1}(z)$ replaced by $\widehat{g}_{\beta_0}^{-1}(z)$.

We look for $P_{z_j}(z)$ in the form

$$P_{z_j}(z) = E(z)P^{(1)}(z)F_j(z)^{-\sigma_3} z^{\pm n\sigma_3/2}, \tag{4.23}$$

where plus sign is taken for $|z| < 1$ (this corresponds to $\zeta \in I, II, III, IV$), and minus, for $|z| > 1$ ($\zeta \in V, VI, VII, VIII$). The matrix $E(z)$ is analytic and invertible in the neighborhood of U_{z_j}, and therefore does not affect the jump and analyticity conditions. It is chosen so that the matching condition is satisfied.

It is easy to verify (recall that $P_{z_j}(z)$ has the same jumps as $S(z)$) that $P^{(1)}(z)$ satisfies jump conditions with *constant* jump matrices. Set

$$P^{(1)}(z) = \Psi_j(\zeta). \tag{4.24}$$

Then $\Psi_j(\zeta)$ satisfies a RHP on the contour given in Figure 2:

(a) Ψ_j is analytic for $\zeta \in \mathbb{C} \setminus \cup_{j=1}^{8} \Gamma_j$.

(b) Ψ_j satisfies the following jump conditions:

$$\Psi_{j,+}(\zeta) = \Psi_{j,-}(\zeta) \begin{pmatrix} 0 & e^{-i\pi\beta_j} \\ -e^{i\pi\beta_j} & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{4.25}$$

$$\Psi_{j,+}(\zeta) = \Psi_{j,-}(\zeta) \begin{pmatrix} 0 & e^{i\pi\beta_j} \\ -e^{-i\pi\beta_j} & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_5, \tag{4.26}$$

$$\Psi_{j,+}(\zeta) = \Psi_{j,-}(\zeta) e^{i\pi\alpha_j\sigma_3}, \qquad \text{for } \zeta \in \Gamma_3 \cup \Gamma_7, \tag{4.27}$$

$$\Psi_{j,+}(\zeta) = \Psi_{j,-}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\pm i\pi(\beta_j - 2\alpha_j)} & 1 \end{pmatrix}, \tag{4.28}$$

for $\zeta \in \Gamma_2$ with plus sign in the exponent, for $\zeta \in \Gamma_4$, with minus sign,

$$\Psi_{j,+}(\zeta) = \Psi_{j,-}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\pm i\pi(\beta_j+2\alpha_j)} & 1 \end{pmatrix}, \tag{4.29}$$

for $\zeta \in \Gamma_8$ with plus sign in the exponent, for $\zeta \in \Gamma_6$, with minus sign.

(c) As $\zeta \to 0$, $\zeta \in \mathbb{C} \setminus \cup_{j=1}^{8} \Gamma_j$ outside the lenses,

$$\Psi_j(z) = \begin{pmatrix} O(\zeta^{\alpha_j}) & O(\zeta^{\alpha_j}) + O(\zeta^{-\alpha_j}) \\ O(\zeta^{\alpha_j}) & O(\zeta^{\alpha_j}) + O(\zeta^{-\alpha_j}) \end{pmatrix} \tag{4.30}$$

if $\alpha_j \neq 0$, and

$$\Psi_j(z) = \begin{pmatrix} O(1) & O(\ln|\zeta|) \\ O(1) & O(\ln|\zeta|) \end{pmatrix} \tag{4.31}$$

if $\alpha_j = 0$, $\beta_j \neq 0$. The behavior of $\Psi_j(z)$ for $\zeta \to 0$ in other sectors is obtained from these expressions by application of the appropriate jump conditions.

We will solve this problem explicitly in terms of the confluent hypergeometric function, $\psi(a, c; z)$ with the parameters a, c determined by α_j, β_j. A standard theory of the confluent hypergeometric function is presented, e.g., in the appendix of [28].

The following statement holds.

Proposition 4.1. *Let $\alpha_j \pm \beta_j \neq -1, -2, \dots$ for all j. Then a solution to the above RHP (a)–(c) for $\Psi_j(\zeta)$, $0 < \arg\zeta < 2\pi$, is given by the following function in the sector I:*

$$\Psi_j(\zeta) = \Psi_j^{(I)}(\zeta) = \begin{pmatrix} \zeta^{\alpha_j}\psi(\alpha_j+\beta_j, 1+2\alpha_j, \zeta)e^{i\pi(2\beta_j+\alpha_j)}e^{-\zeta/2} & -\zeta^{\alpha_j}\psi(1+\alpha_j-\beta_j, 1+2\alpha_j, e^{-i\pi}\zeta)e^{i\pi(\beta_j+\alpha_j)}e^{\zeta/2}\frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)} \\ -\zeta^{-\alpha_j}\psi(1-\alpha_j+\beta_j, 1-2\alpha_j, \zeta)e^{i\pi(\beta_j-3\alpha_j)}e^{-\zeta/2}\frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)} & \zeta^{-\alpha_j}\psi(-\alpha_j-\beta_j, 1-2\alpha_j, e^{-i\pi}\zeta)e^{-i\pi\alpha_j}e^{\zeta/2} \end{pmatrix}, \tag{4.32}$$

where $\psi(a, b, x)$ is the confluent hypergeometric function of the second kind, and $\Gamma(x)$ is Euler's Γ-function. The solution in the other sectors is given by successive application of the jump conditions (4.25–4.29) to (4.32).

Remark 4.2. The functions $\zeta^{\pm\alpha_j}$, $\psi(a, b, \zeta)$, and $\psi(a, b, e^{-i\pi}\zeta)$ are defined on the universal covering of the punctured plane $\zeta \in \mathbb{C} \setminus \{0\}$. Recall that the branches are fixed by the condition $0 < \arg\zeta < 2\pi$.

Proof. The condition (c) is verified in the sector I by applying to (4.32) the standard expansion of the confluent hypergeometric function at zero (see, e.g., [11]), namely,

$$\psi(a, c, x) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)}(1+O(x)) + \frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}(1+O(x)), \qquad x \to 0, \qquad c \notin \mathbb{Z}, \tag{4.33}$$

or, to cover also the integer values of c:

$$\psi(a, c, x) = \begin{cases} \frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}(1+O(x\ln x)) + O(1), & \Re c > 1 \\ \frac{\Gamma(1-c)}{\Gamma(1+a-c)}(1+O(x)) + \frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}(1+O(x)), & \Re c = 1, c \neq 1 \\ -\frac{1}{\Gamma(a)}\left(\ln x + \frac{\Gamma'(a)}{\Gamma(a)} - 2\gamma_E\right) + O(x\ln x), & c = 1 \\ \frac{\Gamma(1-c)}{\Gamma(1+a-c)}\left(1 + O(x\ln x) + O(x^{1-c})\right), & \Re c < 1 \end{cases}, \qquad x \to 0, \tag{4.34}$$

where $\gamma_E = 0.5772\ldots$ is Euler's constant.

We verify the condition (c) similarly in the other sectors.

To verify (b), reduce the contour of Figure 2 to the real line, oriented from right to left, by extending the sectors I and IV and collapsing the jump conditions. We then obtain the following reduced RHP:

$$\Psi^{(IV)}_{j,+}(\zeta) = \Psi^{(I)}_{j,-}(\zeta) J_2 J_3 J_4^{-1} = \Psi^{(I)}_{j,-}(\zeta) \begin{pmatrix} e^{i\pi\alpha_j} & 0 \\ 2i\sin(\pi(\beta_j-\alpha_j)) & e^{-i\pi\alpha_j} \end{pmatrix}, \qquad \zeta < 0; \tag{4.35}$$

$$\Psi^{(IV)}_{j,+}(\zeta) = \Psi^{(I)}_{j,-}(\zeta) J_1^{-1} J_8^{-1} J_7^{-1} J_6 J_5 = \Psi^{(I)}_{j,-}(\zeta) \begin{pmatrix} e^{-i\pi(2\beta_j-\alpha_j)} & 2i\sin(\pi(\alpha_j+\beta_j)) \\ 0 & e^{i\pi(2\beta_j-\alpha_j)} \end{pmatrix}, \qquad \zeta > 0,$$

where the jump matrices J_k correspond to jumps on the contours Γ_k, $k = 1, \ldots, 8$ as defined in (4.25–4.29).

The confluent hypergeometric function possesses the following transformation property on the universal covering of the punctured plane:

$$\psi(a, c, e^{-2\pi i}\zeta) = e^{2\pi i a}\psi(a, c, \zeta) - \frac{2\pi i}{\Gamma(a)\Gamma(a-c+1)} e^{i\pi a} e^{\zeta} \psi(c-a, c, e^{-i\pi}\zeta), \tag{4.36}$$

This property is proved in the appendix of [28] (equation (7.30)).

Taking $\Psi^{(I)}_j(\zeta)$ given by (4.32) and applying to it the jump condition for $\zeta < 0$, we obtain using (4.36) and the standard properties of Γ-function the following expressions for the first column of $\Psi^{(IV)}$:

$$\Psi^{(IV)}_{j,11}(\zeta) = \zeta^{\alpha_j}\psi(\alpha_j+\beta_j, 1+2\alpha_j, e^{-2\pi i}\zeta)e^{-\zeta/2} \tag{4.37}$$

$$\Psi^{(IV)}_{j,21}(\zeta) = -\zeta^{-\alpha_j}\psi(1-\alpha_j+\beta_j, 1-2\alpha_j, e^{-2\pi i}\zeta)e^{-i\pi\beta_j}e^{-\zeta/2}\frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)} \tag{4.38}$$

The second column is

$$\Psi^{(IV)}_{j,12}(\zeta) = \Psi^{(I)}_{j,12}(\zeta)e^{-i\pi\alpha_j}, \qquad \Psi^{(IV)}_{j,22}(\zeta) = \Psi^{(I)}_{j,22}(\zeta)e^{-i\pi\alpha_j} \tag{4.39}$$

Now applying to this function the jump condition for $\zeta > 0$ and using again (4.36), we obtain (note that as a result of these manipulations we moved $\zeta \to e^{2\pi i}\zeta$)

$$\Psi^{(I)}_j(\zeta) = \Psi_j(\zeta) \tag{4.40}$$

with $0 < \arg\zeta < \pi$, i.e. the $\Psi^{(I)}_j(\zeta)$ we started with. Thus, (4.32, 4.37) is a solution to the reduced RHP given by the jump condition (4.35). Therefore, (4.32, 4.37–4.39) give a solution to the original RHP for Ψ in the sectors I and IV, respectively; and the solution in the other sectors is reconstructed using (4.25–4.29). Proposition 4.1 is proved. □

We will now match this solution with $N(z)$ on the boundary ∂U_{z_j} for large n. The limit $n \to \infty$, $z \in \partial U_{z_j}$, corresponds to $\zeta \to \infty$, therefore we need the asymptotic expansion of $\Psi_j(\zeta)$. We use the classical result (e.g., [11] or Eq.(7.2) of [28]) for the confluent hypergeometric function:

$$\psi(a, c, x) = x^{-a}[1 - a(1+a-c)x^{-1} + O(x^{-2})], \qquad |x| \to \infty, \qquad -3\pi/2 < \arg x < 3\pi/2. \tag{4.41}$$

Note that these asymptotics can be taken both for $\psi(a, c, \zeta)$ and $\psi(a, c, e^{-i\pi}\zeta)$ for $\zeta \in I$. We apply this result to (4.32) and thus obtain the asymptotics of the solution in the sector I. The "proper" triangular structure of the jump matrices implies that these asymptotics remain the same in the

sector II as well, namely:

$$\Psi_j^{(I)}(\zeta)=\Psi_j^{(II)}(\zeta)=\left[I+\frac{1}{\zeta}\begin{pmatrix}\alpha_j^2-\beta_j^2 & \frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)}e^{i\pi(\beta_j+4\alpha_j)}\\ -\frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)}e^{-i\pi(\beta_j+4\alpha_j)} & -(\alpha_j^2-\beta_j^2)\end{pmatrix}+O(\zeta^{-2})\right]$$
$$\times\zeta^{-\beta_j\sigma_3}e^{-\zeta\sigma_3/2}\begin{pmatrix}e^{i\pi(2\beta_j+\alpha_j)} & 0\\ 0 & e^{-i\pi(\beta_j+2\alpha_j)}\end{pmatrix},\qquad \zeta\to\infty,\quad \zeta\in I,II,\qquad \alpha_j\pm\beta_j\neq -1,-2,\dots \tag{4.42}$$

Furthermore, applying the jump matrices, we obtain the following asymptotics for $\Psi_j(\zeta)$ in the other sectors (here $\Psi_j^{(I)}(\zeta)$ stands for the analytic continuation of the r.h.s. of (4.42) to $0<\arg\zeta<2\pi$) as $\zeta\to\infty$:

$$\Psi_j^{(III)}(\zeta)=\Psi_j^{(IV)}(\zeta) = \Psi_j^{(I)}(\zeta)e^{i\pi\alpha_j\sigma_3}, \tag{4.43}$$

$$\Psi_j^{(V)}(\zeta)=\Psi_j^{(VI)}(\zeta) = \Psi_j^{(I)}(\zeta)\begin{pmatrix}0 & -e^{i\pi\beta_j}\\ e^{-i\pi\beta_j} & 0\end{pmatrix}e^{-i\pi\alpha_j\sigma_3}, \tag{4.44}$$

$$\Psi_j^{(VII)}(\zeta)=\Psi_j^{(VIII)}(\zeta) = \Psi_j^{(I)}(\zeta)\begin{pmatrix}0 & -e^{-i\pi\beta_j}\\ e^{i\pi\beta_j} & 0\end{pmatrix}. \tag{4.45}$$

Now substituting these asymptotics into the condition on E:

$$P_{z_j}(z)N^{-1}(z)=E(z)\Psi_j(\zeta)F_j(z)^{-\sigma_3}z^{\pm n\sigma_3/2}N^{-1}(z)=I+o(1), \tag{4.46}$$

we obtain

$$E(z)=N(z)\zeta^{\beta_j\sigma_3}F_j^{\sigma_3}(z)z_j^{-n\sigma_3/2}\begin{pmatrix}e^{-i\pi(2\beta_j+\alpha_j)} & 0\\ 0 & e^{i\pi(\beta_j+2\alpha_j)}\end{pmatrix},\quad \text{for } \zeta\in I,II, \tag{4.47}$$

$$E(z)=N(z)\zeta^{\beta_j\sigma_3}F_j^{\sigma_3}(z)z_j^{-n\sigma_3/2}\begin{pmatrix}e^{-2\pi i(\beta_j+\alpha_j)} & 0\\ 0 & e^{i\pi(\beta_j+3\alpha_j)}\end{pmatrix},\quad \text{for } \zeta\in III,IV, \tag{4.48}$$

$$E(z)=N(z)\zeta^{-\beta_j\sigma_3}F_j^{\sigma_3}(z)z_j^{n\sigma_3/2}\begin{pmatrix}0 & e^{i\pi(3\alpha_j+2\beta_j)}\\ -e^{-i\pi(3\beta_j+2\alpha_j)} & 0\end{pmatrix},\quad \text{for } \zeta\in V,VI, \tag{4.49}$$

$$E(z)=N(z)\zeta^{-\beta_j\sigma_3}F_j^{\sigma_3}(z)z_j^{n\sigma_3/2}\begin{pmatrix}0 & e^{2\pi i\alpha_j}\\ -e^{-i\pi(\beta_j+\alpha_j)} & 0\end{pmatrix},\quad \text{for } \zeta\in VII,VIII. \tag{4.50}$$

The dependence on z enters into these expressions only via the combination $\mathcal{D}(z)/(\zeta^{\beta_j}F_j(z))$ for $|z|<1$ (i.e., $\zeta\in I,II,III,IV$) and the combination $\mathcal{D}(z)F_j(z)/\zeta^{\beta_j}$ for $|z|>1$ (i.e., $\zeta\in V,VI,VII,VIII$). Expanding the logarithm in (4.12) in powers of $u=z-z_j$, we see immediately from (4.9,4.10,4.15,4.17) that the mentioned combinations, and therefore $E(z)$ have no singularity at z_j. Thus $E(z)$ is an analytic function in U_{z_j}. In what follows, we will need more detailed information about the behaviour of some of these combinations as $u\to 0$. Namely, it is easy to obtain from (4.12,4.9,4.15,4.17) and (4.13) that

$$F_j(z)=\eta_j e^{-3i\pi\alpha_j/2}z_j^{-\alpha_j}u^{\alpha_j}(1+O(u)),\qquad u=z-z_j,\qquad \zeta\in I, \tag{4.51}$$

where

$$\eta_j=e^{V(z_j)/2}\exp\left\{-\frac{i\pi}{2}\left(\sum_{k=0}^{j-1}\beta_k-\sum_{k=j+1}^{m}\beta_k\right)\right\}\prod_{k\neq j}\left(\frac{z_j}{z_k}\right)^{\beta_k/2}|z_j-z_k|^{\alpha_k}, \tag{4.52}$$

and

$$\left(\frac{\mathcal{D}(z)}{\zeta^{\beta_j}F_j(z)}\right)^2 = \mu_j^2 e^{i\pi(\alpha_j-2\beta_j)}n^{-2\beta_j}(1+O(u)), \qquad u=z-z_j, \qquad \zeta\in I, \tag{4.53}$$

$$\mu_j = \left(e^{V_0}\frac{b_+(z_j)}{b_-(z_j)}\right)^{1/2}\exp\left\{-\frac{i\pi}{2}\left(\sum_{k=0}^{j-1}\alpha_k-\sum_{k=j+1}^{m}\alpha_k\right)\right\}\prod_{k\neq j}\left(\frac{z_j}{z_k}\right)^{\alpha_k/2}|z_j-z_k|^{\beta_k}. \tag{4.54}$$

To derive (4.54), we used, in particular, the factorization (1.8). The sums from 0 to -1 and from $m+1$ to m are set to zero.

It is seen directly from (4.47–4.50) that $\det E(z)=e^{i\pi(\alpha_j-\beta_j)}$. Note that as follows by Liouville's theorem from the RHP, $\det\Psi_j(\zeta)=e^{-i\pi(\alpha_j-\beta_j)}$: this function has no jumps, the singularity at zero is removable as $\Re\alpha_j>-1/2$, and the constant value follows from the asymptotics (4.42). Combining these results, we see from (4.23) that $\det P_{z_j}(z)=1$. Comparing the conditions (4.30,4.31) and (4.4,4.5), we see that the singularity of $S(z)P_{z_j}(z)^{-1}$ at $z=z_j$ is at most $O(|z-z_j|^{2\alpha_j})$ or $O(\ln^2|z-z_j|)$. However, by construction of P_{z_j}, the function $S(z)P_{z_j}(z)^{-1}$ has no jumps in a neighborhood of U_{z_j} and hence this singularity is removable. Thus, $S(z)P_{z_j}(z)^{-1}$ is analytic in a neighborhood of U_{z_j}.

Note that the error term in (4.46) $o(1)=n^{-\Re\beta_j\sigma_3}O(n^{-1})n^{\Re\beta_j\sigma_3}$. It is $o(1)$ for $-1/2<\Re\beta_j<1/2$.

This completes the construction of the parametrix at z_j: it is given by the formulae (4.23,4.24,4.47–4.50) and Proposition 4.1.

Considering further terms in (4.42), we can extend (4.46) into the full asymptotic series in inverse powers of n. For our calculations we need to know explicitly the first correction term:

$$P_{z_j}(z)N^{-1}(z) = I+\Delta_1(z)+n^{-\Re\beta_j\sigma_3}O(1/n^2)n^{\Re\beta_j\sigma_3}, \tag{4.55}$$

$$\Delta_1(z)=\frac{1}{\zeta}\begin{pmatrix} -(\alpha_j^2-\beta_j^2) & \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)}\left(\frac{\mathcal{D}(z)}{\zeta^{\beta_j}F_j(z)}\right)^2 z_j^n e^{i\pi(2\beta_j-\alpha_j)} \\ -\frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)}\left(\frac{\mathcal{D}(z)}{\zeta^{\beta_j}F_j(z)}\right)^{-2} z_j^{-n}e^{-i\pi(2\beta_j-\alpha_j)} & \alpha_j^2-\beta_j^2 \end{pmatrix},$$

$$z\in\partial z(I), \qquad \alpha_j\pm\beta_j\neq -1,-2,\dots,$$

where $\partial z(I)$ is the part of ∂U_{z_j} whose ζ-image is in I. As a consideration of the other sectors shows, this expression for $\Delta_1(z)$ extends by analytic continuation to the whole boundary ∂U_{z_j}. As follows from (4.53), it gives a meromorphic function in a neighborhood of U_{z_j} with a simple pole at $z=z_j$.

The error term $O(1/n^2)$ in (4.55) is uniform in z on ∂U_{z_j}.

4.3. **R-RHP.** Throughout this section we assume that $\alpha_j\pm\beta_j\neq -1,-2,\dots$ for all $j=0,1,\dots,m$.

Let

$$R(z)=\begin{cases} S(z)N^{-1}(z), & z\in U_\infty\setminus\Gamma, \qquad U_\infty=\mathbb{C}\setminus\cup_{j=0}^m U_{z_j}, \\ S(z)P_{z_j}^{-1}(z), & z\in U_{z_j}\setminus\Gamma, \qquad j=0,\dots,m. \end{cases} \tag{4.56}$$

It is easy to verify that this function has jumps only on ∂U_{z_j}, and parts of Σ_j, Σ_j'' lying outside the neighborhoods U_{z_j} (we denote these parts without the end-points Σ^{out}, Σ''^{out}). The full contour Γ is shown in Figure 3 where $U_j\equiv U_{z_j}$. Away from Γ, as a standard argument shows, $R(z)$ is analytic. Moreover, we have: $R(z)=I+O(1/z)$ as $z\to\infty$.

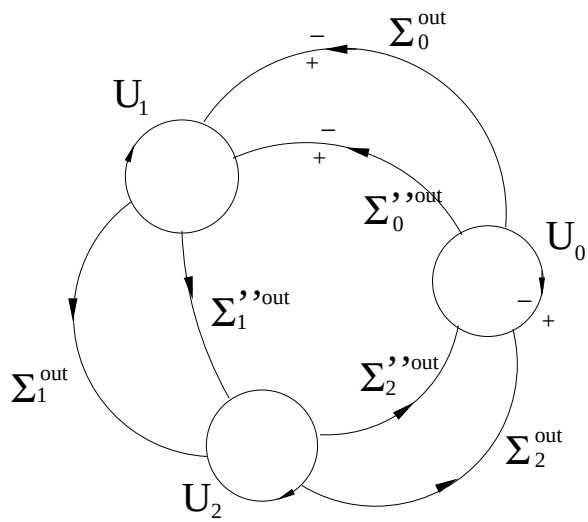

FIGURE 3. Contour Γ for the R and $\widetilde{R}$ Riemann-Hilbert problems ($m = 2$).

The jumps of $R(z)$ are as follows:

$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^{-n} & 1 \end{pmatrix} N(z)^{-1}, \qquad z \in \Sigma_j^{\text{out}}, \tag{4.57}$$

$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^{n} & 1 \end{pmatrix} N(z)^{-1}, \qquad z \in \Sigma_j''^{\text{out}}, \tag{4.58}$$

$$R_+(z) = R_-(z)P_{z_j}(z)N(z)^{-1}, \qquad z \in \partial U_{z_j} \setminus \{\text{intersection points}\}, \tag{4.59}$$

$j = 0, \dots, m.$

The jump matrix on Σ^{out}, Σ''^{out} can be estimated uniformly in α_j, β_j as $I + O(\exp(-\varepsilon n))$, where ε is a positive constant. The jump matrices on ∂U_{z_j} admit a uniform expansion in the inverse powers of n conjugated by $n^{\beta_j\sigma_3}z_j^{-n\sigma_3/2}$ (the first term is given explicitly by (4.55)):

$$I + \Delta_1(z) + \Delta_2(z) + \cdots + \Delta_k(z) + \Delta_{k+1}^{(r)}, \qquad z \in \partial U_{z_j}. \tag{4.60}$$

Every $\Delta_p(z)$, $\Delta_p^{(r)}(z)$, $p = 1, 2, \dots$, $z \in \cup_{j=0}^m \partial U_{z_j}$ is of the form

$$\sum_{j=0}^{m} a_j^{-\sigma_3} O(n^{-p}) a_j^{\sigma_3}, \qquad a_j \equiv n^{\beta_j} z_j^{-n/2}, \tag{4.61}$$

it is of order $n^{2\max_j|\Re\beta_j|-p}$.

To obtain a standard solution of the R-RHP in terms of a Neumann series (see, e.g., [18]) we must have $n^{2\max_j|\Re\beta_j|-1} = o(1)$, that is $\Re\beta_j \in (-1/2, 1/2)$ for all $j = 0, 1, \dots, m$.

However, it is possible to obtain the solution in any open or half-closed interval of length 1, i.e. for $|||\beta||| < 1$. Namely, let $|||\beta||| < 1$ and consider the transformation

$$\widetilde{R}(z) = n^{\omega\sigma_3} R(z) n^{-\omega\sigma_3} \qquad z \in \mathbb{C} \setminus \Gamma, \tag{4.62}$$

where

$$\omega = \frac{1}{2}(\min_j \Re\beta_j + \max_j \Re\beta_j) \tag{4.63}$$

will "shift" all $\Re\beta_j$ (in the conjugation n^{β_j} terms) inside the interval $(-1/2, 1/2)$. Note that $\omega = \Re\beta_{j_0}$ if only one $\Re\beta_{j_0} \neq 0$, and $\omega = 0$ if all $\Re\beta_j = 0$.

In the RHP for $\widetilde{R}(z)$, the condition at infinity and the uniform exponential estimate $I + O(\exp(-\varepsilon n))$ (with different ε) of the jump matrices on Σ^{out}, Σ''^{out} are preserved, while the jump matrices on ∂U_{z_j} have the form:

$$I + n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3} + \cdots + n^{\omega\sigma_3}\Delta_k(z)n^{-\omega\sigma_3} + n^{\omega\sigma_3}\Delta_{k+1}^{(r)}(z)n^{-\omega\sigma_3}, \qquad z \in \partial U_{z_j}, \tag{4.64}$$

where the order of each $n^{\omega\sigma_3}\Delta_p(z)n^{-\omega\sigma_3}$, $n^{\omega\sigma_3}\Delta_p^{(r)}(z)n^{-\omega\sigma_3}$, $p = 1, 2, \ldots$, $z \in \cup_{j=0}^m \partial U_{z_j}$ is

$$O(n^{2\max_j|\Re\beta_j - \omega| - p}).$$

This implies that the standard analysis can be applied to the $\widetilde{R}$-RHP problem in the range $\Re\beta_j \in (q - 1/2, q + 1/2)$, $j = 0, 1, \ldots, m$, for any $q \in \mathbb{R}$, and we obtain the asymptotic expansion

$$\widetilde{R}(z) = I + \sum_{p=1}^{k} \widetilde{R}_p(z) + \widetilde{R}_{k+1}^{(r)}(z), \quad k = 1, 2, \ldots \tag{4.65}$$

In our case the error term

$$\widetilde{R}_{k+1}^{(r)}(z) = O(|\widetilde{R}_{k+1}(z)|) + O(|\widetilde{R}_{k+2}(z)|). \tag{4.66}$$

The functions $\widetilde{R}_j(z)$ are computed recursively. In this paper, we will need explicit expressions only for the first two. Accordingly, set $k = 2$. The function $\widetilde{R}_1(z)$ is found from the conditions that it is analytic outside $\partial U = \cup_{j=0}^m \partial U_{z_j}$, $\widetilde{R}_1(z) \to 0$ as $z \to \infty$, and

$$\widetilde{R}_{1,+}(z) = \widetilde{R}_{1,-}(z) + n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3}, \qquad z \in \partial U. \tag{4.67}$$

The solution is easily written. First denote

$$R_p(z) \equiv n^{-\omega\sigma_3}\widetilde{R}_p(z)n^{\omega\sigma_3}, \qquad R_p^{(r)}(z) \equiv n^{-\omega\sigma_3}\widetilde{R}_p^{(r)}(z)n^{\omega\sigma_3}, \tag{4.68}$$

and write for R:

$$\begin{aligned} R_1(z) &= \frac{1}{2\pi i}\int_{\partial U} \frac{\Delta_1(x)dx}{x - z} \\ &= \begin{cases} \sum_{k=0}^m \frac{A_k}{z - z_k}, & z \in \mathbb{C} \setminus \cup_{j=0}^m U_{z_j} \\ \sum_{k=0}^m \frac{A_k}{z - z_k} - \Delta_1(z), & z \in U_{z_j}, \quad j = 0, 1, \ldots, m. \end{cases}, \qquad \partial U = \cup_{j=0}^m \partial U_{z_j}. \end{aligned} \tag{4.69}$$

where the contours in the integral are traversed in the negative direction, and A_k are the coefficients in the Laurent expansion of $\Delta_1(z)$:

$$\Delta_1(z) = \frac{A_k}{z - z_k} + B_k + O(z - z_k), \qquad z \to z_k, \qquad k = 0, 1, \ldots, m. \tag{4.70}$$

The coefficients are easy to write using (4.55) and (4.53):

$$A_k \equiv A_k^{(n)} = \frac{z_k}{n}\begin{pmatrix} -(\alpha_k^2 - \beta_k^2) & \frac{\Gamma(1+\alpha_k+\beta_k)}{\Gamma(\alpha_k-\beta_k)} z_k^n \mu_k^2 n^{-2\beta_k} \\ -\frac{\Gamma(1+\alpha_k-\beta_k)}{\Gamma(\alpha_k+\beta_k)} z_k^{-n}\mu_k^{-2} n^{2\beta_k} & \alpha_k^2 - \beta_k^2 \end{pmatrix}. \tag{4.71}$$

An expression for B_k is also easy to find, but it is not needed below.

The function $\widetilde{R}_2$ is now found from the conditions that $\widetilde{R}_2(z) \to 0$ as $z \to \infty$, is analytic outside ∂U, and

$$\widetilde{R}_{2,+}(z) = \widetilde{R}_{2,-}(z) + \widetilde{R}_{1,-}(z)n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3} + n^{\omega\sigma_3}\Delta_2(z)n^{-\omega\sigma_3}, \qquad z \in \partial U. \tag{4.72}$$

The solution to this RHP is

$$\widetilde{R}_2(z) = \frac{1}{2\pi i}\int_{\partial U}\left(\widetilde{R}_{1,-}(x)n^{\omega\sigma_3}\Delta_1(x)n^{-\omega\sigma_3} + n^{\omega\sigma_3}\Delta_2(x)n^{-\omega\sigma_3}\right)\frac{dx}{x-z}. \tag{4.73}$$

Further standard analysis (cf. (4.66)) shows that the error term

$$R_3^{(r)}(z) = \begin{pmatrix} O(\delta/n)+O(\delta^2) & O\left(\delta \max_k \frac{n^{-2\Re\beta_k}}{n}\right) \\ O\left(\delta \max_k \frac{n^{2\Re\beta_k}}{n}\right) & O(\delta/n)+O(\delta^2)\end{pmatrix}, \tag{4.74}$$

where δ is given by (1.20).

In particular, as is clear from the above, if there is only one nonzero β_{j_0}, we obtain the expansion of $\widetilde{R}(z)$ purely in inverse integer powers of n valid in fact for all $\beta_{j_0}\in\mathbb{C}$, $\alpha_{j_0}\pm\beta_{j_0}\neq -1,-2,\dots$

It is clear from the construction and the properties of the asymptotic series of the confluent hypergeometric function that the error terms $\widetilde{R}_k^{(r)}(z)$ (4.66), and in particular (4.74), are uniform for all z and for β_j in bounded sets of the strip $q-1/2<\Re\beta_j<q+1/2$, $j=0,1,\dots m$, for α_j in bounded sets of the half-plane $\Re\alpha_j>-1/2$, and for $\alpha_j\pm\beta_j$ away from neighborhoods of the negative integers. Moreover, the series (4.65) is differentiable in α_j, β_j.

For future use note that if $V(z)=V_r(z)+(V(z)-V_r(z))h$, $h\in[0,1]$, and $V_r(z)$ is analytic in a neighborhood of the unit circle, then the error terms are uniform in the parameter $h\in[0,1]$.

5. Orthogonal polynomials. Proof of Theorem 1.8

Using results of the previous section, we can provide a complete asymptotic analysis of the polynomials orthogonal with weight (1.2) on the unit circle with analytic $V(z)$. In this section we will find the asymptotic expressions for χ_n, $\phi_n(0)$, and $\widehat{\phi}_n(0)$.

First, it follows immediately from (3.1) that

$$\chi_{n-1}^2 = -Y_{21}^{(n)}(0). \tag{5.1}$$

Tracing back the transformations $R\to S\to T\to Y$, we obtain for z inside the unit circle and outside the lenses:

$$\begin{aligned} Y(z)=T(z)=S(z)=R(z)N(z) &= n^{-\omega\sigma_3}\widetilde{R}(z)n^{\omega\sigma_3}N(z) \\ &= n^{-\omega\sigma_3}[I+\widetilde{R}_1(z)+\widetilde{R}_2(z)+\widetilde{R}_3^{(r)}(z)]n^{\omega\sigma_3}N(z) \\ &= \left[I+R_1(z)+R_2(z)+R_3^{(r)}(z)\right]\mathcal{D}(z)^{\sigma_3}\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \end{aligned} \tag{5.2}$$

Taking the 21 matrix element and setting $z=0$ we obtain

$$\chi_{n-1}^2 = -Y_{21}^{(n)}(0) = \mathcal{D}(0)^{-1}\left[1+R_{1,22}(0)+R_{2,22}(0)+O(\delta/n+\delta^2)\right], \tag{5.3}$$

where we used the estimate (4.74) for $R_3^{(r)}(z)$.

By (4.9)

$$\mathcal{D}(0)^{-1} = \exp\left[-\int_0^{2\pi}V(\theta)\frac{d\theta}{2\pi}\right] = e^{-V_0}. \tag{5.4}$$

Using (4.69) and (4.71) we obtain

$$R_{1,22}(0) = -\sum_{k=0}^{m}\frac{A_{k,22}}{z_k} = -\frac{1}{n}\sum_{k=0}^{m}\left(\alpha_k^2-\beta_k^2\right). \tag{5.5}$$

Conjugating (4.73) with $n^{\omega\sigma_3}$, setting there $z=0$, and applying (4.69), we obtain:

$$R_2(0) = -\sum_{j=0}^{m} z_j^{-1} \sum_{k\neq j} \frac{A_k A_j}{z_j - z_k} + \frac{1}{n^2} O\begin{pmatrix} 1 & \sum_j n^{-2\beta_j} \\ \sum_j n^{2\beta_j} & 1 \end{pmatrix}. \tag{5.6}$$

From (4.71),

$$(A_k A_j)_{22} = z_k z_j \Big[(\alpha_j^2 - \beta_j^2)(\alpha_k^2 - \beta_k^2) n^{-2} - n^{2(\beta_k - \beta_j - 1)} \left(\frac{z_j}{z_k}\right)^n \frac{\Gamma(1+\alpha_j+\beta_j)\Gamma(1+\alpha_k-\beta_k)}{\Gamma(\alpha_j-\beta_j)\Gamma(\alpha_k+\beta_k)} \frac{\mu_j^2}{\mu_k^2} \Big], \tag{5.7}$$

where μ_j^2 are defined in (4.54).

Substituting the last 3 equations into (5.3), we finally obtain (1.21).

We now turn our attention to $\phi_n(0)$. Using (3.1), we have

$$\phi_n(0) = \chi_n Y_{11}^{(n)}(0) = \chi_n \left(R(0)\mathcal{D}(0)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)_{11} = -\chi_n \mathcal{D}(0)^{-1} \left(R_{1,12}(0) + R_{2,12}^{(r)}(0) \right). \tag{5.8}$$

By (4.69,4.71),

$$R_{1,12}(0) = -\sum_{k=0}^{m} \frac{A_{k,12}}{z_k} = -\frac{1}{n} \sum_{j=0}^{m} n^{-2\beta_j} z_j^n \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)} \mu_j^2, \tag{5.9}$$

and, recalling (4.66), we obtain (1.23).

Finally, starting again with (3.1), we have

$$\begin{aligned} \widehat{\phi}_{n-1}(0) &= -\frac{1}{\chi_{n-1}} \lim_{z\to\infty} \frac{Y_{21}^{(n)}(z)}{z^{n-1}} = -\frac{1}{\chi_{n-1}} \lim_{z\to\infty} \frac{1}{z^{n-1}} (R(z)\mathcal{D}(z)^{\sigma_3} z^{n\sigma_3})_{21} \\ &= -\frac{1}{\chi_{n-1}} \left(\lim_{z\to\infty} z R_{1,21}(z) + O\left(\left[\delta + \frac{1}{n}\right] \max_k \frac{n^{2\Re\beta_k}}{n} \right) \right). \end{aligned} \tag{5.10}$$

We have

$$\lim_{z\to\infty} z R_{21}(z) = \sum_{k=0}^{m} A_{k,21} = -\frac{1}{n} \sum_{j=0}^{m} n^{2\beta_j} z_j^{-n+1} \frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)} \mu_j^{-2}, \tag{5.11}$$

and therefore, recalling (1.21), obtain (1.24).

Note that uniformity and differentiability properties of the asymptotic series of Theorem 1.8 follow from those of the $\widetilde{R}$-expansion of the previous section.

6. Toeplitz determinants. Proof of Theorem 1.13

6.1. **The case of analytic $V(z)$.** First, let $V(z)$ be analytic in a neighborhood of the unit circle. Consider the set $\beta_j^{(r)}$ constructed in the proof of Lemma 1.12. We have to consider only the second class, i.e. $|||\beta^{(r)}||| = 1$. We then have, relabeling $\beta_j^{(r)}$ according to increasing real part,

$$\Re\beta_1^{(r)} = \cdots = \Re\beta_p^{(r)} < \Re\beta_{p+1}^{(r)} \leq \cdots \leq \Re\beta_{m'-\ell}^{(r)} < \Re\beta_{m'-\ell+1}^{(r)} = \cdots = \Re\beta_{m'}^{(r)}, \tag{6.1}$$

for some $p, \ell > 0$. Here m' is the number of singularities: $m' = m+1$ if $z=1$ is a singularity ($\alpha_0 \neq 0$ or $\beta_0 \neq 0$), otherwise $m' = m$. Now consider the symbol (not a FH-representation of f) $\widetilde{f}$ of type (1.2) with beta-parameters denoted by $\widetilde{\beta}$ and given by $\widetilde{\beta}_j = \beta_j^{(r)}$ for $j = 1, \ldots, m'-\ell$, and $\widetilde{\beta}_j = \beta_j^{(r)} - 1$ for $j = m'-\ell+1, \ldots, m'$. It is easy to see that the original symbol f has $\binom{\ell+p}{\ell}$

FH-representations in $\mathcal{M}$ obtained by shifting any ℓ out of $\ell+p$ parameters $\widetilde{\beta}_j$, say $\widetilde{\beta}_{i_1},\dots,\widetilde{\beta}_{i_\ell}$, with the smallest real part to the right by 1. Thus,

$$f(z)=(-1)^\ell\prod_{j=0}^m z_j^{L_j}\times z_{i_1}^{-1}\cdots z_{i_\ell}^{-1}z^\ell\widetilde{f}(z), \tag{6.2}$$

for appropriate L_j.

Let us now and until the end of this section relabel $\widetilde{\beta}_j$, α_j, L_j, and z_j according to increasing real part of $\widetilde{\beta}_j$. Thus, in particular,

$$\Re\widetilde{\beta}_1=\cdots=\Re\widetilde{\beta}_{\ell+p}<\Re\widetilde{\beta}_{\ell+p+1}. \tag{6.3}$$

Assume that the set of all the minimizing FH-representations $\mathcal{M}$ is non-degenerate (see Introduction). This implies that $\alpha_j\pm\widetilde{\beta}_j\neq-1,-2,\dots$.

We now apply Lemma 2.4 (equation (2.9)) to finish the proof of Theorem 1.13. We need to evaluate the determinant F_n, $n\geq N_0$ for a sufficiently large $N_0>0$. First, from (3.1), tracing back the transformations of the RH problem and using (4.69,4.71) we obtain (cf. (5.8,5.9)) for the polynomials orthonormal with weight $\widetilde{f}(z)$:

$$\phi_n(z)/\chi_n=\mathcal{D}(z)^{-1}\rho_n(z),\qquad \rho_n(z)=-\sum_{k=0}^m\frac{A_{k,12}^{(n)}}{z-z_k}+O\left(\left[\delta+\frac{1}{n}\right]n^{-2\Re\widetilde{\beta}_1-1}\right). \tag{6.4}$$

This expansion is uniform and differentiable in a neighborhood of zero. A simple algebra shows that in the determinant

$$F_n=\begin{vmatrix}\phi_n(0)/\chi_n & \phi_{n+1}(0)/\chi_{n+1} & \cdots & \phi_{n+\ell-1}(0)/\chi_{n+\ell-1}\\ \frac{d}{dz}\phi_n(0)/\chi_n & \frac{d}{dz}\phi_{n+1}(0)/\chi_{n+1} & \cdots & \frac{d}{dz}\phi_{n+\ell-1}(0)/\chi_{n+\ell-1}\\ \vdots & \vdots & & \vdots\\ \frac{d^{\ell-1}}{dz^{\ell-1}}\phi_n(0)/\chi_n & \frac{d^{\ell-1}}{dz^{\ell-1}}\phi_{n+1}(0)/\chi_{n+1} & \cdots & \frac{d^{\ell-1}}{dz^{\ell-1}}\phi_{n+\ell-1}(0)/\chi_{n+\ell-1}\end{vmatrix} \tag{6.5}$$

all the terms with the derivatives of $\mathcal{D}(z)$ drop out, and we have

$$F_n=\mathcal{D}(0)^{-\ell}\begin{vmatrix}\rho_n(0) & \rho_{n+1}(0) & \cdots & \rho_{n+\ell-1}(0)\\ \frac{d}{dz}\rho_n(0) & \frac{d}{dz}\rho_{n+1}(0) & \cdots & \frac{d}{dz}\rho_{n+\ell-1}(0)\\ \vdots & \vdots & & \vdots\\ \frac{d^{\ell-1}}{dz^{\ell-1}}\rho_n(0) & \frac{d^{\ell-1}}{dz^{\ell-1}}\rho_{n+1}(0) & \cdots & \frac{d^{\ell-1}}{dz^{\ell-1}}\rho_{n+\ell-1}(0)\end{vmatrix}. \tag{6.6}$$

It is a crucial fact that the size ℓ of this determinant is less than the number of terms, $\ell+p$, in the expansion of $\phi_n(0)/\chi_n$ of the same largest order $O(n^{-2\Re\widetilde{\beta}_1-1})$ (see (1.23) with β_j replaced by $\widetilde{\beta}_j$).

As $|\Re\widetilde{\beta}_j-\Re\widetilde{\beta}_k|<1$, and $\alpha_j\pm\widetilde{\beta}_j\neq-1,-2,\dots$, $j,k=1,\dots,m'$, we obtain for the p'th derivative of $\rho(z)$ from (6.4), (4.71), and (4.54) with β replaced by $\widetilde{\beta}$

$$\begin{aligned}\frac{d^s}{dz^s}\rho_{n+i}(0)&=s!\sum_{k=0}^m\frac{A_{k,12}^{(n+i)}}{z_k^{s+1}}+O\left(\left[\delta+\frac{1}{n}\right]n^{-2\Re\widetilde{\beta}_1-1}\right)\\ &=s!\sum_{j=1}^{\ell+p}d_jz_j^{n+i-s}+O\left(n^{-2\Re\widetilde{\beta}_{\ell+p+1}-1}\right)+O\left(\left[\delta+\frac{1}{n}\right]n^{-2\Re\widetilde{\beta}_1-1}\right),\end{aligned} \tag{6.7}$$

where
(6.8)
$$d_j = n^{-2\widetilde{\beta}_j-1}\frac{\Gamma(1+\alpha_j+\widetilde{\beta}_j)}{\Gamma(\alpha_j-\widetilde{\beta}_j)}e^{V_0}\frac{b_+(z_j)}{b_-(z_j)}\exp\left\{-i\pi\left(\sum_{k=0}^{j-1}\alpha_k-\sum_{k=j+1}^{m}\alpha_k\right)\right\}\prod_{k\neq j}\left(\frac{z_j}{z_k}\right)^{\alpha_k}|z_j-z_k|^{2\widetilde{\beta}_k}.$$

Substituting these expressions into the determinant F_n, we obtain

$$\begin{aligned}(6.9)\quad F_n &= \mathcal{D}(0)^{-\ell}\prod_{s=0}^{\ell-1}s!\sum_{1\le i_1\neq i_2\neq\cdots\neq i_\ell\le\ell+p}d_{i_1}d_{i_2}\cdots d_{i_\ell}z_{i_1}^n\cdots z_{i_\ell}^{n-\ell+1}\prod_{1\le j<k\le\ell}(z_{i_k}-z_{i_j})(1+o(1))\\ &= \mathcal{D}(0)^{-\ell}\prod_{s=0}^{\ell-1}s!\sum_{1\le i_1<i_2<\cdots<i_\ell\le\ell+p}d_{i_1}d_{i_2}\cdots d_{i_\ell}(z_{i_1}\cdots z_{i_\ell})^n\prod_{1\le j<k\le\ell}|z_{i_j}-z_{i_k}|^2(1+o(1)),\end{aligned}$$

as $z_j^{-1}=\overline{z_j}$.

Therefore, by (2.9),

$$\begin{aligned}(6.10)\quad D_n(z^\ell\widetilde{f}(z)) &= \frac{(-1)^{n\ell}F_n}{\prod_{s=0}^{\ell-1}s!}D_n(\widetilde{f}(z))\\ &= (-1)^{n\ell}\mathcal{D}(0)^{-\ell}\sum_{1\le i_1<i_2<\cdots<i_\ell\le\ell+p}d_{i_1}d_{i_2}\cdots d_{i_\ell}(z_{i_1}\cdots z_{i_\ell})^n\prod_{1\le j<k\le\ell}|z_{i_j}-z_{i_k}|^2D_n(\widetilde{f}(z))(1+o(1)).\end{aligned}$$

We now use Theorem 1.1 for $D_n(\widetilde{f}(z))$. Noting, in particular, that $G(1+z)=\Gamma(z)G(z)$ and $\mathcal{D}(0)=e^{V_0}$, we obtain after a straightforward calculation that

$$D_n(z^\ell\widetilde{f}(z)) = (-1)^{n\ell}\sum_{1\le i_1<i_2<\cdots<i_\ell\le\ell+p}(z_{i_1}\cdots z_{i_\ell})^n\mathcal{R}(\dots,\widetilde{\beta}_{i_1}+1,\widetilde{\beta}_{i_2}+1,\dots,\widetilde{\beta}_{i_\ell}+1,\dots),$$

where $\mathcal{R}$ is the r.h.s. of (1.12) where all β_j are replaced with $\widetilde{\beta}_j$ with the exception of β_j, $j=i_1,\dots,i_\ell$ which are replaced as indicated in the argument of $\mathcal{R}$. Note once again that each sum is over indices in the range $1,\dots,\ell+p$ and we use a special numbering of indices (cf. (6.3)). Finally, recalling (6.2), we obtain

$$(6.11)\qquad D_n(f(z)) = \left(\prod_{j=0}^{m}z_j^{L_j}\right)^n\sum_{1\le i_1<i_2<\cdots<i_\ell\le\ell+p}\mathcal{R}(\dots,\widetilde{\beta}_{i_1}+1,\widetilde{\beta}_{i_2}+1,\dots,\widetilde{\beta}_{i_\ell}+1,\dots),$$

which is the statement of Theorem 1.13 for $V(z)$ analytic in a neighborhood of the unit circle.

6.2. **Extension to smooth $V(z)$.** If $V(z)$ is just sufficiently smooth, in particular C^∞, on the unit circle C so that (1.14) holds for s from zero up to and including some $s\ge 0$, we can approximate $V(z)$ by trigonometric polynomials $V^{(n)}(z)=\sum_{k=-p(n)}^{p(n)}V_kz^k$, $z\in C$. First, consider the case when $|||\beta|||=\max_{j,k}|\Re\beta_j-\Re\beta_k|=2\max_j|\Re\beta_j-\omega|<1$, where ω is defined by (4.63). (The indices $j,k=0$ are omitted if $\alpha_0=\beta_0=0$.) We set

$$(6.12)\qquad p=[n^{1-\nu}],\qquad \nu=2\max_j|\Re\beta_j-\omega|+\varepsilon_1=|||\beta|||+\varepsilon_1,$$

where $\varepsilon_1>0$ is chosen sufficiently small so that $\nu<1$ (square brackets denote the integer part).

First, we need to extend the RH analysis of the previous sections to symbols which depend on n, namely to the case when V in f is replaced by $V^{(n)}$. (We will denote such f by $f(z,V^{(n)})$, and the original one, by $f(z,V)$.) We need to have a suitable estimate for the behaviour of the error term in asymptotics with n. For a fixed f, our analysis depended, in particular, on the fact that

$f(z)^{-1}z^{-n}$ is of order $e^{-\varepsilon n}$, $\varepsilon > 0$, for $z \in \Sigma^{\rm out}$ (see Section 4.3), and similarly, $f(z)^{-1}z^n = O(e^{-\varepsilon n})$ for $z \in \Sigma''^{\rm out}$. Here the contours $\Sigma^{\rm out}$, $\Sigma''^{\rm out}$ are outside a *fixed* neighborhood of the unit circle (outside and inside C, respectively). If V is replaced by $V^{(n)}$, let us define the curve Σ outside $\cup_{j=0}^m U_j$, $U_j \equiv U_{z_j}$, by

$$z = \left(1 + \gamma \frac{\ln p}{p}\right) e^{i\theta}, \qquad \gamma > 0, \tag{6.13}$$

and Σ'' outside $\cup_{j=0}^m U_j$ by

$$z = \left(1 - \gamma \frac{\ln p}{p}\right) e^{i\theta}. \tag{6.14}$$

Inside all U_j, the curves still go to z_j as discussed in Section 4.2. Let the radius of all U_j be $2\gamma \ln p/p$. We now fix the value of γ as follows. Using the condition (1.14) we can write (here and below c stands for various positive constants independent of n)

(6.15)

$$\begin{aligned} |V^{(n)}(z)| - |V_0| &\le \sum_{k=-p,\, k\neq 0}^{p} |k^s V_k| \frac{|z|^k}{|k|^s} < c \left(\sum_{k=-p,\, k\neq 0}^{p} |k^s V_k|^2 \right)^{1/2} \left(\sum_{k=1}^{p} \frac{(1 \pm 3\gamma \ln p/p)^{\pm 2k}}{k^{2s}} \right)^{1/2} \\ &< c \left(\sum_{k=1}^{p} \frac{(1 \pm 3\gamma \ln k/k)^{\pm 2k}}{k^{2s}} \right)^{1/2} < c \left(\sum_{k=1}^{p} \frac{1}{k^{2(s-3\gamma)}} \left[1 + O\left(\frac{\ln^2 k}{k} \right) \right] \right)^{1/2}, \end{aligned}$$

where $z \in \Sigma^{\rm out}$, $z \in \partial U_j \cap \{|z| > 1\}$ (with "+" sign in "±"), and $z \in \Sigma''^{\rm out}$, $z \in \partial U_j \cap \{|z| < 1\}$ (with "−" sign). We now set

$$3\gamma = s - (1 + \varepsilon_2)/2, \qquad \varepsilon_2 > 0, \tag{6.16}$$

and then

$$|V^{(n)}(z)| < c, \qquad |b_+(z, V^{(n)})| < c, \qquad |b_-(z, V^{(n)})| < c, \qquad \text{for all } n \tag{6.17}$$

uniformly on $\Sigma^{\rm out}$, $\Sigma''^{\rm out}$, ∂U_j's, and in fact in the whole annulus $1 - 3\gamma \frac{\ln p}{p} < |z| < 1 + 3\gamma \frac{\ln p}{p}$.

It is easy to adapt the considerations of the previous sections to the present case, and we again obtain the expansion (4.60) for the jump matrix of R on ∂U_j. Note that now $|z - z_j| = 2\gamma(1 - \nu) \ln n/n^{1-\nu}$ and $|\zeta(z)| = 2\gamma(1-\nu) n^\nu \ln n (1 + o(1))$ as $n \to \infty$ for $z \in \partial U_j$, and therefore using (4.55), (4.15), (4.17), (4.13), (4.9) and the definition of ν in (6.12), we obtain, in particular,

$$n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3} = O\left(\frac{1}{n^{\varepsilon_1} \ln n} \right), \qquad z \in \cup_{j=0}^m \partial U_j. \tag{6.18}$$

Furthermore, as follows from (6.13), (6.14), (6.17), and (4.57,4.58), the jump matrix on $\Sigma^{\rm out}$ and $\Sigma''^{\rm out}$ is now the identity plus a function uniformly bounded in absolute value by

$$c \left(\frac{n^{1-\nu}}{\ln n} \right)^{2 \max_j |\Re \beta_j|} \left(1 \pm \gamma(1-\nu) \frac{\ln n}{n^{1-\nu}} \right)^{\mp n} < c \exp\left\{ -\frac{\gamma}{2}(1-\nu) n^\nu \ln n \right\} n^{2(1-\nu) \max_j |\Re \beta_j|}, \tag{6.19}$$

where the upper sign corresponds to $\Sigma^{\rm out}$, and the lower, to $\Sigma''^{\rm out}$.

The RH problem for $R(z)$ (see Section 4.3) is therefore solvable, and we obtain $R(z)$ as a series where the first term R_1 is the same as before. For the error term there holds the same estimate for z outside a fixed neighborhood of the unit circle, e.g., at $z = 0$.

This, in particular, implies that the formulae (6.7,6.8) hold for $\widetilde{f}(z, V^{(n)})$ (in $\widetilde{f}$, we substitute $\widetilde{\beta}_j$ for β_j: note that the condition $0 < \nu < 1$ is satisfied).

We will now show that replacing $V^{(n)}$ with V in the symbol of the determinant $D_n(\widetilde{f}(z,V^{(n)}))$ results (under a condition on s) in a small error only, so that (6.10) still holds with V used in $D_n(\widetilde{f}(z,V))$, and $V^{(n)}$ in d_j's and in $D_n(z^\ell\widetilde{f}(z,V^{(n)}))$. Then, proceeding as before, we obtain the statement of the theorem for $D_n(z^\ell\widetilde{f}(z,V^{(n)}))$ as, by (1.14), uniformly on C,

$$b_\pm(z,V^{(n)}) = b_\pm(z,V)\left[1+O\left(\frac{1}{n^{(1-\nu)s}}\right)\right], \qquad |z|=1, \tag{6.20}$$

in (6.8). Recall a standard representation for a Toeplitz determinant with (any) symbol $f(z)$:

$$D_n(f) = \frac{1}{(2\pi)^n n!}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{1\le j<k\le n}|e^{i\phi_j}-e^{i\phi_k}|^2\prod_{j=1}^n f(e^{i\phi_j})d\phi_j. \tag{6.21}$$

We have from this formula, (6.20), and Theorem 1.1 for $D_n(|\widetilde{f}(z,V)|)$ and $D_n(\widetilde{f}(z,V))$, if $s(1-\nu)>1$,

$$\begin{aligned}\Big|D_n(\widetilde{f}(z,V))-D_n(\widetilde{f}(z,V^{(n)})\Big| &< \\ \frac{1}{(2\pi)^n n!}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{1\le j<k\le n}|e^{i\phi_j}-e^{i\phi_k}|^2\prod_{j=1}^n|\widetilde{f}(e^{i\phi_j},V)|d\phi_j &\times\left(\left|1+c/n^{(1-\nu)s}\right|^n-1\right)\\ &< ce^{\Re V_0 n}n^{\sum_{j=1}^m((\Re\alpha_j)^2+(\Im\widetilde{\beta}_j)^2)}(e^{c/n^{(1-\nu)s-1}}-1)\\ &< c\left|e^{V_0 n}n^{\sum_{j=1}^m(\alpha_j^2-\widetilde{\beta}_j^2)}\right|n^{\sum_{j=0}^m((\Im\alpha_j)^2+(\Re\widetilde{\beta}_j)^2)}\frac{1}{n^{(1-\nu)s-1}}\\ &< c\left|D_n(\widetilde{f}(z,V))\right|n^{-((1-\nu)s-1-\sum_{j=0}^m((\Im\alpha_j)^2+(\Re\widetilde{\beta}_j)^2))}.\end{aligned} \tag{6.22}$$

Therefore,

$$D_n(\widetilde{f}(z,V^{(n)})) = D_n(\widetilde{f}(z,V))\left(1+\frac{D_n(\widetilde{f}(z,V^{(n)}))-D_n(\widetilde{f}(z,V))}{D_n(\widetilde{f}(z,V))}\right) = D_n(\widetilde{f}(z,V))(1+o(1)), \tag{6.23}$$

if

$$s > \frac{1+\sum_{j=0}^m((\Im\alpha_j)^2+(\Re\widetilde{\beta}_j)^2)}{1-\nu}. \tag{6.24}$$

Under the condition (6.24) and the ones under which Theorem 1.1 holds, $\alpha_j\pm\widetilde{\beta}_j\neq -1,-2,\dots$, and e.g. C^∞ for V (see Remark 1.7), we then obtain the statement of the theorem for $D_n(z^\ell\widetilde{f}(z,V^{(n)}))$ as mentioned above. The theorem (with Remark 1.15) for $D_n(z^\ell\widetilde{f}(z,V))$, and hence for $D_n(f(z,V))$, immediately follows from an analysis similar to (6.22,6.23) applied to

$$D_n(z^\ell\widetilde{f}(z,V)) = D_n(z^\ell\widetilde{f}(z,V^{(n)}))\left(1-\frac{D_n(z^\ell\widetilde{f}(z,V^{(n)}))-D_n(z^\ell\widetilde{f}(z,V))}{D_n(z^\ell\widetilde{f}(z,V^{(n)}))}\right).$$

The ratio in the brackets is $o(1)$ under the condition (6.24) in which $\widetilde{\beta}_j$ are replaced by $\beta_j^{(r)}$ (and the condition under which Theorem 1.1 holds). As ε_1 can be arbitrary close to zero, this condition together with (6.24) (note that these conditions are consistent with (6.16) and the requirement that $\gamma>0$) and (1.15) for Theorem 1.1 yield the estimate (1.28).

7. Hankel determinants. Proof of Theorem 1.20

Consider the Hankel determinant with symbol $w(x)$ on $[-1,1]$ given by (1.36). In this section we will find its asymptotics using the relation to a Toeplitz determinant established in Theorem 2.6. Let $x=\cos\theta$, $z=e^{i\theta}$, $0\le\theta\le\pi$. In particular,

$$\lambda_j=\cos\theta_j,\quad z_j=e^{i\theta_j},\quad j=0,1,\dots,r+1,\quad 0=\theta_0<\theta_1<\dots<\theta_{r+1}=\pi.$$

First, we find an even function f of the angle θ related to $w(x)$ by (1.39). The Toeplitz determinant $D_{2n}(f(z))$ with this symbol enters the connection formula (2.20). Denote

$$z_j'=e^{(2\pi-\theta_j)i},\qquad j=0,\dots,r+1. \tag{7.1}$$

Then, recalling (1.6), note that

$$|x-\lambda_j|^{2\alpha_j}=|\cos\theta-\cos\theta_j|^{2\alpha_j}=\left|2\sin\frac{\theta-\theta_j}{2}\sin\frac{\theta+\theta_j}{2}\right|^{2\alpha_j}=2^{-2\alpha_j}|z-z_j|^{2\alpha_j}|z-z_j'|^{2\alpha_j}, \tag{7.2}$$

and

$$|\sin\theta|=2^{-1}|z-z_0||z-z_{r+1}|. \tag{7.3}$$

We see that $f(z)$ will have $m+1=2r+2$ singularities at the points $z_0=1$, $z_{r+1}=-1$, z_j, z_j', $j=1,\dots,r$.

Observe that

$$\prod_{j=1}^r\omega_j(x)=e^{-i\pi\sum_{j=1}^r\beta_j}\prod_{j=1}^r z_j^{-\beta_j}z_j'^{\beta_j}\prod_{j=1}^r g_{z_j,-\beta_j}(z)z_j^{\beta_j}g_{z_j',\beta_j}(z)z_j'^{-\beta_j}. \tag{7.4}$$

Note that $\beta_0=\beta_{r+1}=0$ and we have the jumps with $-\beta_j$ at z_j and $+\beta_j$ at z_j'. In particular, the sum over all β's is zero as noted in the introduction. Note that as $\theta_j=\pi/2-\arcsin\lambda_j$, we have in (7.4)

$$e^{-i\pi\sum_{j=1}^r\beta_j}\prod_{j=1}^r z_j^{-\beta_j}z_j'^{\beta_j}=\exp\left(2i\sum_{j=1}^r\beta_j\arcsin\lambda_j\right). \tag{7.5}$$

Collecting the above observations and denoting

$$A=\sum_{j=0}^{r+1}\alpha_j, \tag{7.6}$$

we have by (1.39) (where we single out a multiplicative constant for convenience)

$$f(z)=w(x)|\sin\theta|=C\widetilde{f}(z),\qquad C=2^{-2A-1}\exp\left(2i\sum_{j=1}^r\beta_j\arcsin\lambda_j\right), \tag{7.7}$$

where

$$\widetilde{f}(e^{i\theta})=e^{V(e^{i\theta})}|z-1|^{4\alpha_0+1}|z+1|^{4\alpha_{r+1}+1}\prod_{j=1}^r|z-z_j|^{2\alpha_j}|z-z'_j|^{2\alpha_j}g_{z_j,-\beta_j}(z)z_j^{\beta_j}g_{z_j',\beta_j}(z)z_j'^{-\beta_j}. \tag{7.8}$$

Here $V(e^{i\theta}) = U(\cos\theta)$. Thus $\widetilde{f}(z)$ is the symbol of type (1.2) with $\Re\beta_j \in (-1/2, 1/2]$. Therefore, if $\Re\beta_j \in (-1/2, 1/2)$, $j = 1, \dots, r$, we can apply Theorem 1.1 to $D_{2n}(\widetilde{f}(z))$, and obtain

$$
\begin{aligned}
(7.9)\quad D_{2n}(f(z)) &= C^{2n} D_{2n}(\widetilde{f}(z)) = C^{2n} \exp\left(2nV_0 + \sum_{k=1}^{\infty} kV_k^2\right) b_+(1)^{-4\alpha_0-1} b_+(-1)^{-4\alpha_{r+1}-1} \\
&\times \prod_{j=1}^{r} b_+(z_j)^{-2(\alpha_j+\beta_j)} b_-(z_j)^{-2(\alpha_j-\beta_j)} \times (2n)^{2\sum_{j=1}^{r}(\alpha_j^2-\beta_j^2)+(2\alpha_0+1/2)^2+(2\alpha_{r+1}+1/2)^2} \mathcal{P}(z) \\
&\times \prod_{j=1}^{r} \frac{G(1+\alpha_j+\beta_j)^2 G(1+\alpha_j-\beta_j)^2}{G(1+2\alpha_j)^2} \times \frac{G(1+2\alpha_0+1/2)^2}{G(1+4\alpha_0+1)} \frac{G(1+2\alpha_{r+1}+1/2)^2}{G(1+4\alpha_{r+1}+1)} (1+o(1)),
\end{aligned}
$$

$$
\mathcal{P}(z) = \prod_{0\le j<k\le m} |z_j - z_k|^{2(\widetilde{\beta}_j\widetilde{\beta}_k - \widetilde{\alpha}_j\widetilde{\alpha}_k)} \left(\frac{z_k}{z_j e^{i\pi}}\right)^{\widetilde{\alpha}_j\widetilde{\beta}_k - \widetilde{\alpha}_k\widetilde{\beta}_j},
$$

$$
\Re\alpha_j > -\frac{1}{2}, \qquad \Re\beta_j \in \left(-\frac{1}{2}, \frac{1}{2}\right), \qquad j = 0, 1, \dots, r+1,
$$

where we used the fact that by the symmetry of $V(z)$, $V_k = V_{-k}$, and hence $b_+(z_j) = b_-(z_j')$, $b_-(z_j) = b_+(z_j')$, $b_+(\pm 1) = b_-(\pm 1)$. In the above expression for $\mathcal{P}$, $m = 2r+1$, and the points are numbered as in Theorem 1.1, namely, $\widetilde{z}_0 = z_0 = 1$, $\widetilde{\alpha}_0 = 2\alpha_0 + 1/2$, $\widetilde{\beta}_0 = \beta_0 = 0$; $\widetilde{z}_j = z_j$, $\widetilde{\alpha}_j = \alpha_j$, $\widetilde{\beta}_j = -\beta_j$, $j = 1, \dots, r$; $\widetilde{z}_{r+1} = z_{r+1} = -1$, $\widetilde{\alpha}_{r+1} = 2\alpha_{r+1} + 1/2$, $\widetilde{\beta}_{r+1} = \beta_{r+1} = 0$; $\widetilde{z}_j = e^{2\pi i} z_{m+1-j}^{-1}$, $\widetilde{\alpha}_j = \alpha_{m+1-j}$, $\widetilde{\beta}_j = \beta_{m+1-j}$, $j = r+2, \dots, m$. Expression for $\mathcal{P}$ can be written in terms of λ_j. Namely, it is not difficult to obtain by induction that

$$
\begin{aligned}
(7.10)\quad \mathcal{P} &= 2^{-2(\alpha_0+\alpha_{r+1}+1/4)} \prod_{0\le j<k\le r+1} \left|2\sin\frac{\theta_j-\theta_k}{2}\right|^{-4(\alpha_j\alpha_k-\beta_j\beta_k)} \left|2\sin\frac{\theta_j+\theta_k}{2}\right|^{-4(\alpha_j\alpha_k+\beta_j\beta_k)} \\
&\times \prod_{j=1}^{r} |2\sin\theta_j|^{-2(\alpha_j^2+\beta_j^2+\alpha_j)} \times e^{2i(2A+1)\sum_{j=1}^{r}\beta_j \arcsin\lambda_j} e^{2\pi i \sum_{0\le j<k\le r+1}(\alpha_j\beta_k-\alpha_k\beta_j)}.
\end{aligned}
$$

Assume first that $V(z)$ is analytic. To use (2.20), we need to calculate the asymptotics of the product $\Phi_{2n}(1)\Phi_{2n}(-1)$. In order to do this, consider $Y^{(n)}(z)$ as $z \to z_j$ in such a way that $z \in z(I)$, where $z(I)$ is the pre-image in the z-plane of the sector I of the ζ-plane (see Figure 2 and Section 4.2). Tracing back the transformations of the RHP, we obtain

(7.11)

$$
Y^{(n)}(z) = T(z) = S(z)\begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^n & 1 \end{pmatrix} = (I + R_1^{(r)}(z))P_{z_j}(z)\begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^n & 1 \end{pmatrix}, \qquad z \in z(I),
$$

where the parametrix $P_{z_j}(z)$ at z_j is (see Section 4.2):

$$
(7.12)\qquad P_{z_j}(z) = E(z)\Psi_j(\zeta)F_j(z)^{-\sigma_3} z^{n\sigma_3/2},
$$

with $E(z)$ given by (4.47). Substituting all the expressions into (7.11), we obtain

$$
\begin{aligned}
(7.13)\quad Y^{(n)}(z) &= (I + R_1^{(r)}(z))\mathcal{D}(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\zeta^{\beta_j} F_j(z) z_j^{-n/2}\right)^{\sigma_3} \\
&\times \begin{pmatrix} e^{-i\pi(2\beta_j+\alpha_j)} & 0 \\ 0 & e^{i\pi(\beta_j+2\alpha_j)} \end{pmatrix} \Psi_j(\zeta) \begin{pmatrix} F_j(z)^{-1} & 0 \\ F_j(z)f(z)^{-1} & F_j(z) \end{pmatrix} z^{n\sigma_3/2}.
\end{aligned}
$$

Note that the expansion of $F_j^{-1}(z)$ as $z \to z_j$ is given by (4.51). Using it we further obtain

$$F_j(z)f(z)^{-1} = \eta_j^{-1} e^{i\pi(\beta_j-\alpha_j/2)} z_j^{\alpha_j} u^{-\alpha_j}(1+O(u)), \qquad u = z - z_j, \qquad \zeta \in I. \tag{7.14}$$

Thus,

$$\begin{pmatrix} F_j(z)^{-1} & 0 \\ F_j(z)f(z)^{-1} & F_j(z) \end{pmatrix} = \begin{pmatrix} e^{i\pi\alpha_j} & 0 \\ e^{i\pi(\beta_j-\alpha_j)} & e^{-i\pi\alpha_j} \end{pmatrix} (e^{i\pi\alpha_j/2} z_j^{\alpha_j} u^{-\alpha_j})^{\sigma_3} \eta_j^{-\sigma_3}(1+O(u)). \tag{7.15}$$

To estimate $\Psi(\zeta)$ for $\zeta \to 0$ (i.e., $z \to z_j$), assume first that all $\alpha_j \neq 0$. Substituting (4.34) into (4.32), dropping the terms of order $u^{2\alpha_j}$ in the second column (we will denote thus modified $Y(z)$ by $\widetilde{Y}(z)$) we obtain the following limit for the combination needed in (7.13) (here tilde over the limit sign means that we have to drop $u^{2\alpha_j}$ terms before taking the limit):

$$\begin{aligned} \widetilde{\lim}_{u\to 0} \begin{pmatrix} e^{-i\pi(2\beta_j+\alpha_j)} & 0 \\ 0 & e^{i\pi(\beta_j+2\alpha_j)} \end{pmatrix} \Psi_j(\zeta) \begin{pmatrix} e^{i\pi\alpha_j} & 0 \\ e^{i\pi(\beta_j-\alpha_j)} & e^{-i\pi\alpha_j} \end{pmatrix} (e^{i\pi\alpha_j/2} z_j^{\alpha_j} u^{-\alpha_j})^{\sigma_3} \\ = e^{i\pi(\alpha_j/2-\beta_j)\sigma_3} M n^{\alpha_j\sigma_3}, \end{aligned} \tag{7.16}$$

where

$$M = \begin{pmatrix} \left[e^{i\pi\alpha_j}\frac{1}{\Gamma(\beta_j-\alpha_j)} - e^{-i\pi\alpha_j}\frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)\Gamma(1-\alpha_j-\beta_j)}\right]\Gamma(-2\alpha_j)e^{i\pi\beta_j} & -\frac{\Gamma(2\alpha_j)}{\Gamma(\alpha_j+\beta_j)} \\ \left[e^{-i\pi\alpha_j}\frac{1}{\Gamma(-\beta_j-\alpha_j)} - e^{i\pi\alpha_j}\frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)\Gamma(1-\alpha_j+\beta_j)}\right]\Gamma(-2\alpha_j)e^{i\pi\beta_j} & \frac{\Gamma(2\alpha_j)}{\Gamma(\alpha_j-\beta_j)} \end{pmatrix}. \tag{7.17}$$

This expression can be simplified. Namely, the 11 matrix element

$$\begin{aligned} M_{11} = e^{i\pi(\beta_j-\alpha_j)}\frac{\Gamma(-2\alpha_j)}{\Gamma(\beta_j-\alpha_j)}\left(e^{2\pi i\alpha_j} - \frac{\sin\pi(\beta_j+\alpha_j)}{\sin\pi(\beta_j-\alpha_j)}\right) \\ = \frac{\Gamma(-2\alpha_j)}{\Gamma(\beta_j-\alpha_j)}\frac{\sin(-2\pi\alpha_j)}{\sin\pi(\beta_j-\alpha_j)} = \frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(1+2\alpha_j)}. \end{aligned} \tag{7.18}$$

Similarly,

$$M_{21} = \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(1+2\alpha_j)}. \tag{7.19}$$

Thus

$$M = \begin{pmatrix} \frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(1+2\alpha_j)} & -\frac{\Gamma(2\alpha_j)}{\Gamma(\alpha_j+\beta_j)} \\ \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(1+2\alpha_j)} & \frac{\Gamma(2\alpha_j)}{\Gamma(\alpha_j-\beta_j)} \end{pmatrix}. \tag{7.20}$$

Substituting the just found limit and (4.53) for $(\mathcal{D}(z)/(\zeta^{\beta_j}F_j(z))^2$ into (7.13), we obtain

$$\widetilde{Y}^{(n)}(z_j) = (I + r_j^{(n)})L_j^{(n)}, \qquad L_j^{(n)} = \begin{pmatrix} M_{21}\mu_j\eta_j^{-1}n^{\alpha_j-\beta_j}z_j^n & M_{22}\mu_j\eta_j n^{-\alpha_j-\beta_j} \\ -M_{11}\mu_j^{-1}\eta_j^{-1}n^{\alpha_j+\beta_j} & -M_{12}\mu_j^{-1}\eta_j n^{-\alpha_j+\beta_j}z_j^{-n} \end{pmatrix}, \tag{7.21}$$

where $r_j = R_1^{(r)}(z_j)$, and η_j, μ_j are given by (4.52,4.54).

Note that the matrix $L_j^{(n)}$ has the structure

$$L_j^{(n)} = n^{-\beta_j\sigma_3}\widehat{L}_j^{(n)}n^{\alpha_j\sigma_3}, \tag{7.22}$$

where $\widehat{L}$ depends on n only via the oscillatory terms z_j^n.

From (3.1) and (7.21) at $z_j = 1$,

$$\Phi_{2n}(1) = Y_{11}^{(2n)}(1) = L_{0,11}^{(2n)}(1 + O(n^{-2\max_k\beta_k-1})). \tag{7.23}$$

From (7.21,7.20,4.54,4.52), using the doubling formula for the Γ-function

$$\frac{\Gamma(1+x)}{\Gamma(1+2x)} = \frac{\sqrt{\pi}}{2^{2x}\Gamma(x+1/2)}, \tag{7.24}$$

we obtain the following main term of $\Phi_{2n}(1)$:

$$L_{0,11}^{(2n)} = M_{0,21}\mu_0\eta_0^{-1}(2n)^{2\alpha_0+1/2} = \frac{\sqrt{\pi}e^{(V_0-V(1))/2+i\sum_{j=1}^r(\pi-\theta_j)\beta_j}}{2^{4\alpha_0+1}\Gamma(1+2\alpha_0)}\prod_{j=1}^r\left|2\sin\frac{\theta_j}{2}\right|^{-2\alpha_j}2^{2(\alpha_0-\alpha_{r+1})}n^{2\alpha_0+1/2}. \tag{7.25}$$

Similarly, we obtain

$$\Phi_{2n}(-1) = L_{r+1,11}^{(2n)}(1+O(n^{-2\max_k\beta_k-1})), \tag{7.26}$$
$$L_{r+1,11}^{(2n)} = \frac{\sqrt{\pi}e^{(V_0-V(-1))/2-i\sum_{j=1}^r\theta_j\beta_j}}{2^{4\alpha_{r+1}+1}\Gamma(1+2\alpha_{r+1})}\prod_{j=1}^r\left|2\cos\frac{\theta_j}{2}\right|^{-2\alpha_j}2^{-2(\alpha_0-\alpha_{r+1})}n^{2\alpha_{r+1}+1/2}.$$

Therefore

$$\Phi_{2n}(1)\Phi_{2n}(-1) = \frac{\pi e^{V_0-(V(1)+V(-1))/2+2i\sum_{j=1}^r\beta_j\arcsin\lambda_j}}{2^{4(\alpha_0+\alpha_{r+1})+2}\Gamma(1+2\alpha_0)\Gamma(1+2\alpha_{r+1})} \times\prod_{j=1}^r|2\sin\theta_j|^{-2\alpha_j}n^{2(\alpha_0+\alpha_{r+1})+1}(1+O(n^{-2\max_k\beta_k-1})). \tag{7.27}$$

Substituting (7.9) and (7.27) into (2.20) we obtain (1.37) squared. We use the following observations in the process:

- Since $V_k = V_{-k}$,
$$b_+(\pm1) = e^{(V(\pm1)-V_0)/2}.$$
- The following elementary identity holds

$$\prod_{0\le j<k\le r+1}\left|2\sin\frac{\theta_j-\theta_k}{2}\right|^{-(\alpha_j\alpha_k-\beta_j\beta_k)}\left|2\sin\frac{\theta_j+\theta_k}{2}\right|^{-(\alpha_j\alpha_k+\beta_j\beta_k)} = 2^{-\sum_{0\le j<k\le r+1}\alpha_j\alpha_k}\prod_{0\le j<k\le r+1}|\lambda_j-\lambda_k|^{-(\alpha_j\alpha_k+\beta_j\beta_k)}\left|\lambda_j\lambda_k-1+\sqrt{(1-\lambda_j^2)(1-\lambda_k^2)}\right|^{\beta_j\beta_k} \tag{7.28}$$

- Applying the doubling formula (2.39) we easily obtain that

$$\frac{G(1+2\alpha+1/2)^2}{G(1+4\alpha+1)}\Gamma(1+2\alpha) = 2^{-8\alpha^2-2\alpha}\pi^{2\alpha+1}\frac{G(1/2)^2}{G(1+2\alpha)^2}. \tag{7.29}$$

If $V(z)\equiv V_r(z)$ is real-valued for $z\in C$, and $\alpha_j\in\mathbb{R}$, $i\beta_j\in\mathbb{R}$, $j=0,\dots,m$, then the weight $f(z)$, $z\in C$, is positive, and therefore $D_n(w)$ is positive. Then (1.37) represents the correct branch of the square root. Since $D_n(w)$ is continuous in α_j, β_j, and the parameter h in $V(z)=V_r(z)+(V(z)-V_r(z))h$, $h\in[0,1]$, and the error term is uniform in these parameters (see Section 4.3), the formula (1.37) has the correct sign in general. This finishes the proof for analytic $V(z)$. The extension to smooth $V(z)$ is carried out similarly to the argument in the previous section by using the standard multiple-integral representation of a Hankel determinant.

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