PRODUCTION PROCESSES
IN
STRONG INTERACTIONS

A thesis presented for the
Degree of Doctor of Philosophy
in the University of London

by

HENRY DAVID DEWAR WATSON, B.A. (Cantab.)

Department of Physics,
Imperial College, London.
November, 1965.

ABSTRACT

The work presented falls into three Parts.

PART I. The production of $Y\bar{Y}$ pairs in $p\bar{p}$ collisions is shown to be consistent with a peripheral model based om K^{*}- exchange. The result is not sensitive to the form of the meson-baryon vertices for comparable values of the coupling constants, and is in accord with the SU(3) symmetry scheme for F-type coupling of the vector mesons and baryons. Certain unsatisfactory features of the model are discussed.

PART II. A critical review of various treatments of absorptive effects in the peripheral model is given. The different approximations made, and the inter-relation of the various results, are examined.

A treatment based on the K-matrix is proposed which has several advantages, amongst which are relativistic validity and the absence of any restriction on the ranges of the forces involved.

PART III. An improved model for $p\bar{p} \rightarrow \Lambda\bar{\Lambda}$ is developed. The $\tilde{U}(12)$ symmetry scheme is used to write down Born amplitudes, on which the requirements of unitarity are then approximately enforced, following the procedure proposed in Part II.

Results are presented for various choices of masses in $\tilde{U}(12)$. In all cases the angular distribution is well

reproduced in form, and in four out of six cases is given in magnitide from the pion-nucleon coupling constant to within 10 - 20 per cent of the experimental value. This represents an essentially no-parameter fit to the data.

CONTENTS

PREFACE

The work presented in this thesis was carried out in the Department of Physics, Imperial College, University of London, between October 1961 and September 1965, under the supervision of Professor P.T. Matthews. The material presented in the text is original, except in so far as explicit reference is made to the work of others, and has not been submitted for a degree in this or any other university.

The work is based on three papers

I) H.D.D. Watson; Nuovo Cimento 29, 1338 (1963).

II) H.D.D. Watson; Phys. Lett. 17, 72 (1965).

III) H.D.D. Watson and J.H.R. Migneron; "$\widetilde{U}(12)$ Absorption Model for $p\bar{p} \rightarrow \Lambda\bar{\Lambda}$" to be published in Phys. Lett.

and has been presented in three corresponding Parts. The author is very indebted to Mr. J.H.R. Migneron for his kind permission to include some of the material of Part III in this work.

In each Part, equations and sub-sections have been numbered starting from one. On the rare occasion when reference is made to an equation or section of another Part, this is shown explicitly, otherwise such references refer to the current Part. Figures are bound in the text as near as possible to the material to which they refer.

The author is sincerely grateful to Professor Matthews for his constant encouragement and guidance, and in particular for several extremely helpful observations. He is also indebted to Professor A. Salam and Drs. J. Charap, T.W. Kibble and R.F. Streater for assistance. Indeed, he has benefitted greatly from discussions with very many members of the Theoretical Physics Group at Imperial College.

This work was made possible by the tenure of a Research Studentship of the Ministry of Education for Northern Ireland, for which the author is very grateful.

PART I

POLE CALCULATION OF $p\bar{p} \rightarrow Y\bar{Y}$.

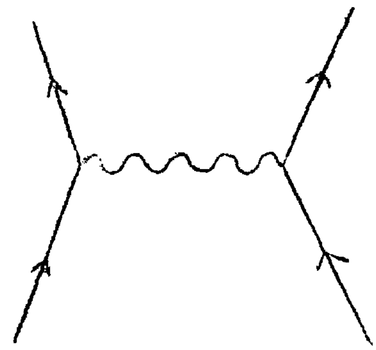
(i) Introduction

In the study of the interactions of elementary particles the primary quantity of physical interest is the differential cross section for scattering. This is given[1] by

$$\frac{d\sigma}{d\Omega} = \rho \frac{|\langle f|M|i\rangle|^2}{F}$$

where ρ is a density of states factor, F the relative flux, and $\langle f|M|i\rangle$ the matrix element for the process under consideration. The problem, from a theoretical viewpoint, is to develop a theory which will permit the calculation of the matrix element.

The formalism relevant to this problem is that of quantum field theory - the so called second quantisation. Historically, the first question to be treated involved the electromagnetic interactions of electrons. It proved possible to develop an iteration procedure to solve the equations of the interacting fields, in which the processes could be adequately described by a second order approximation to the matrix element, the Born approximation. This approximation represents the scattering as arising from the emission by one particle of a virtual "exchange" particle or quantum, which is subsequently absorbed by the second particle. We may represent this process pictorially:-

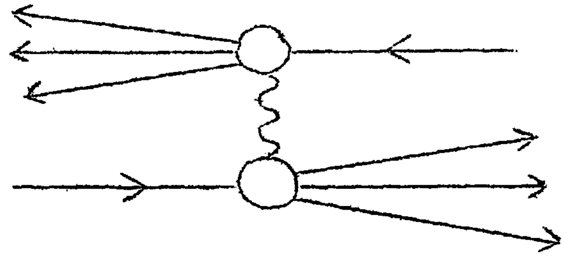


An elegant procedure developed by Feynman permits one to write down the corresponding amplitude in a convenient way. In this work we shall make much use of these methods, and for further details the reader is referred to any standard text-book on field theory.

Early attempts to apply the Born approximation to the calculation of amplitudes involving strongly interacting particles met with little success. In quantum electrodynamics the coupling constant, $G^2/4\pi$, in powers of which one develops the iteration series, is $^1/137$, while for strong interactions $G^2/4\pi \sim 15$. The Born term is simply the leading term in this series, and while the series converges rapidly for electromagnetic interactions, this is not the case for strong interactions. Interest has therefore centred on more sophisticated methods of calculating the amplitudes.

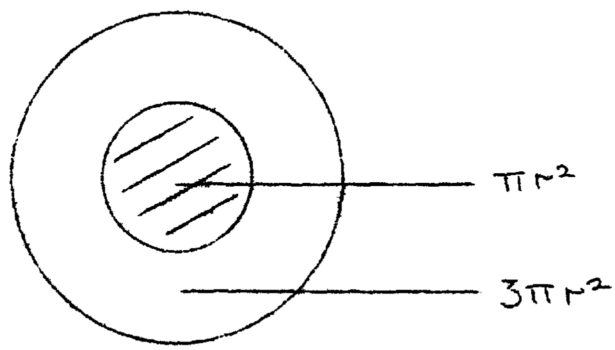
However, in 1958 there was a revival of interest in the Born term approximation. This was associated with the work of many physicists, but especially with Drell[2], Salzman & Salzman[3], and Ferrari & Selleri[4]. The motivation for this new interest was the observation that in a great many interactions at intermediate or high energies the experimentally observed events exhibited a striking concentration in the forward direction. Now any Born term will include, from the propagator, a factor of $1/(q^2 - m^2)$, where q is the four-momentum transfer ($q^2 < 0$ for physical scattering), and m is the mass of the exchanged particle. This term will produce an enhancement of the amplitude in the forward direction, and the suggestion was that this might be the mechanism of the observed forward peaking.

This suggestion is extremely attractive. Events in the forward direction correspond to small momentum transfer, and we would expect these to be events in which one quantum of the lightest available particle is exchanged. The interacting particles just snick one another in passing, as it were, and the reaction products are concentrated in the forward direction:-



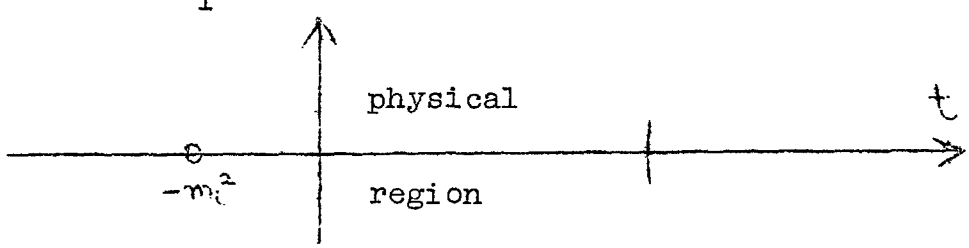
These glancing collisions involve the long range part of the force, and since the range $\sim 1/m$, where m is the mass of the exchanged particle, we see again that the lightest quanta are involved. We would similarly expect the wide angle events to be associated with more violet collisions where the particles collide "head on", which we would expect to correspond to higher order diagrams with multiple particle exchange etc.

Simple geometrical considerations suggest that glancing (long range) interactions are more likely. For example, a target particle will present an area between impact parameters r and $2r$ three times greater than that presented for impact parameter less than r:-



If the Born term is used in this way as an approximation to that part of the amplitude arising from glancing and long range interactions, the resulting model is referred to as the peripheral model. In this work we shall refer to this approximation as the Born term model, or one particle exchange model, or pole approximation, in order to avoid confusion between this approximation and developments of the peripheral model which will also be presented. It must be emphasized again that the Born term is not used as a first term approximation to an iteration solution of the field equations, but simply to represent the long range part of the amplitude.

An equivalent point of view is to consider the analytic properties of the scattering amplitude in the t $(= -q^2)$ - plane. The poles and cuts are concentrated on the negative real axis, simple poles arising at $t_i = - m_i^2$, where m_i is the mass of any particle with the appropriate quantum numbers for exchange. The physical region has $t > 0$, and it is clear that the nearest singularity is a pole corresponding to the lowest value of m_i.



We would expect this pole, or such poles, to dominate the scattering amplitude in the near-by region, which for physical scattering corresponds to low values of t, i.e. forward scattering. The pole term will fall to half its value at $t = 0$, when $t = m^2$, i.e. at an angle θ_o where

$$4p^2 \sin^2 \theta_o/2 = m^2 .$$

If the pole term is to induce a forward peak we require the value of θ_o to be small. Therefore

$$\sin \theta_o/2 \sim \theta/2$$

and $$\theta_o \sim m/p$$

so that for a forwardly peaked amplitude we require

$$p > m . \qquad (1)$$

This condition is just a statement that the colliding particles must be travelling sufficiently fast to be not much deflected by the exchange process, otherwise no forward peak will result and the amplitude will be as much due to short range interactions as peripheral ones. Condition (1) specifies the energy range for the model, which coincides with the so called "optical region".

Yet a third, and still equivalent, statement of the peripheral assumption is to say that we are using the Born term to represent the behaviour of the high partial waves. Under the energy conditions specified, we expect many of these to be involved, and to dominate the interaction. High partial waves correspond to events with large impact parameter, i.e. long range interactions, and other things being equal, are relatively more important in the amplitude due to the weighting factor of $(2\ell + 1)$. (This is an alternative statement of the geometrical preference for glancing collisions given above).

It does not lie within the scope of the present work to give a review of the early work on the peripheral model. The reader is referred to appendix IV of reference (4) for a comprehensive bibliography. More recent developments of, and modifications to, the simple pole model presented here will be dealt with in Part II of this work.

However, we must mention briefly the more direct tests of the one particle exchange hypothesis, which come from an examination of angular correlations between the reaction products[5]. Thus, for example, the exchange of a scalar particle implies that the distribution with respect to an angle ϕ defined by Trieman and Yang[6] should be constant. This distribution is hard to determine experimentally, but in many cases the data is consistent with an isotropic distribution. Again, the Stodolsky-Sakurai[7] model for reactions of the form

$$\pi + N \rightarrow \pi + N^*$$

$$\text{and} \quad K + N \rightarrow K + N^*$$

involving ρ exchange, leads to a decay distribution for the N^* of $1 + 3\cos^2\theta$ with respect to the production normal, and this is found to be quite well observed[7,5].

In this work, we shall be concerned with two particle $\longrightarrow$ two particle reactions. This is not the serious limitation it might appear at first sight. A general feature of multiparticle reactions is the observation of kinematic correlations in the reaction products, which show up clearly in the Dalitz plot. This indicates that the reactions proceed via the formation of resonant states, which subsequently decay. For example, in $K^+ p$ interactions

$$K^+ p \qquad \sim \qquad 7.6 \text{ mb.}$$

$$\left.\begin{array}{l} K^0 \pi^+ p \\ K^+ \pi^0 p \\ K^+ \pi^+ n \end{array}\right\} \qquad \sim \qquad 8.2 \text{ mb.}$$

However, detailed analysis[5] shows that all but a small fraction of the three body events occur through

$$K^+ p \longrightarrow K^{*+} p$$

or

$$K^+ p \longrightarrow K^0 N^{*++} .$$

We include such three body processes as quasi two body reactions.

It is precisely two body processes which we would expect to be amenable to peripheral treatment; more complex many-body-final-state interactions we would expect to be initiated by more violent "head on" collisions.

(ii) The Reaction $p\bar{p} \rightarrow Y\bar{Y}$.

The reaction $p\bar{p} \longrightarrow Y\bar{Y}$ has been studied recently both at CERN[8] and at Brookhaven[9], at energies in the range 3 - 4 Gev/c incident antiproton momentum.

The most striking feature of the observations is the extremely forwardly peaked nature of the differential cross sections for $\Lambda\bar{\Lambda}$, $\Lambda\bar{\Sigma}_0$, $\Sigma_0\bar{\Lambda}$, $\Sigma_+\bar{\Sigma}_+ \rightarrow \Sigma_0\bar{\Sigma}_0$ production. The antihyperons are emitted in a predominantly forward direction with respect to the incident antiproton beam. This is suggestive of a peripheral production mechanism.

An examination of the variation of cross section from channel to channel gives additional evidence in favour of such a mechanism. The experimental values are given in Table I, p. 16. It is seen that the cross-sections for $\Sigma_-\bar{\Sigma}_-$ and $\Xi_-\bar{\Xi}_-$ are appreciably smaller than the others, and these configurations are precisely those that cannot be reached from the initial state by the peripheral exchange of a single quantum. In

$\bar{p}$ momentum in Gev/c	3.0 CERN	Brookhaven	CERN	Brookhaven	CERN
	Cross section in μb				
$\Lambda\bar{\Lambda}$	117	87	77	82	39
$\Lambda\bar{\Sigma}_0 (=\Sigma_0\bar{\Lambda})$	51	28	33.5	35	23
$\Sigma_+\bar{\Sigma}_+$	31	36	23	44	18.5
$\Sigma_0\bar{\Sigma}_0$	< 18		< 22		< 17
$\Sigma_-\bar{\Sigma}_-$	9.5	2	11	8	8
$\Xi_-\bar{\Xi}_-$	2	4	< 1	2	< 1

(Quoted limits on errors are ~ 20 per cent).

$\dfrac{2\,\sigma(\Lambda\bar{\Sigma}_0)}{\sqrt{\sigma(\Lambda\bar{\Lambda})\,\sigma(\Sigma_+\bar{\Sigma}_+)}}$	1.7	1.0	1.6	1.2	1.7

(Errors from quoted limits are typically ± .6)

$4\sqrt{\dfrac{4\,\sigma(\Lambda\bar{\Lambda})}{9\,\sigma(\Sigma_+\bar{\Sigma}_+)}}$	1.14	1.01	1.11	.95	.98

This parameter = 1 for F coupling

= $\frac{1}{3}$ " D " .

TABLE 1.

both cases the exchange of two units of charge would necessarily be involved, and for $\Xi_- \overline{\Xi}_-$ two units of hypercharge also.

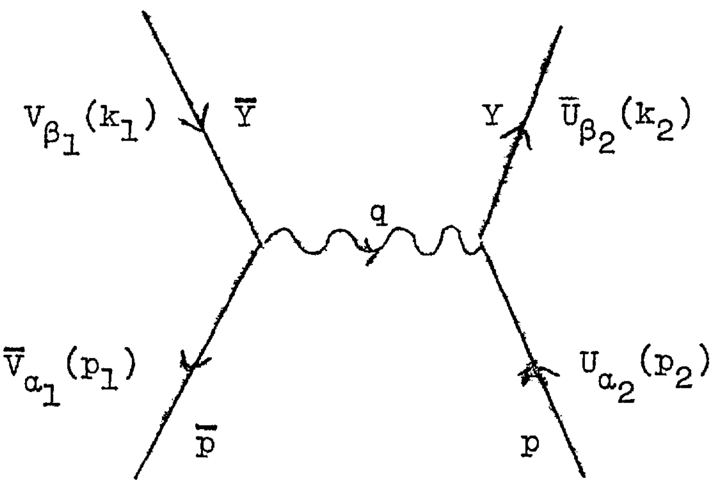
On the other hand, the other five channels are accessible with the exchange of an $I = \frac{1}{2}$, $S = 1$ meson, and there exist two such mesons, K and K^*. The fact that these channels have much larger cross sections is very suggestive that a one particle exchange process is playing a dominant role. The small values of the $\Sigma_- \overline{\Sigma}_-$ and $\Xi_- \overline{\Xi}_-$ cross sections we ascribe to the fact that these reactions are forced to take place through more unlikely complex interactions.

A more detailed analysis of the angular distribution data supports this view. It is precisely in the case of the five reactions permitted via a peripheral process that the strong forward peaking is observed — most of the events are concentrated in $0 < \theta < 60^{\circ}$ — while the distribution is almost isotropic. (Data on $\Xi_- \overline{\Xi}_-$ is inconclusive.)

The energy is sufficiently high for a pole term to produce a forward peak. The centre of mass momentum, p, is $\sim$ 1,100 Mev/c at 3 Gev/c, while $m_K \sim$ 500 Mev, $m_{K^*} \sim$ 900 Mev, so that condition (1) is satisfied. We therefore proceed to investigate in more detail a peripheral model for $Y\overline{Y}$ production involving K and K^* intermediary particles. It need hardly be said that in the present state of our understanding of strong interaction dynamics it would be extremely satisfactory if it were possible to develop a model which would reproduce the observed results to within the 10 - 20 per cent accuracy represented by the neglect of quantities of the order of $\sigma(\Sigma_-\bar{\Sigma}_-)$ and $\sigma(\Xi_-\bar{\Xi}_-)$.

(iii) K-exchange

The lighter of the two possible intermediary particles is the K-meson ($m_K \sim 500$ Mev, $m_{K^*} \sim 900$ Mev), and we therefore proceed to evaluate the $p\bar{p} \rightarrow Y\bar{Y}$ amplitude for K-exchange, i.e. the contribution to the amplitude from the nearest singularity, or the longest range part of the force. The appropriate Feynman diagram is



and the corresponding amplitude

$$\langle \beta_1 \beta_2 | M | \alpha_1 \alpha_2 \rangle = \bar{v}^{p}_{\alpha_1}(p_1)\gamma_5 v^{Y}_{\beta_1}(k_1) \frac{i}{q^2 - m^2} \bar{u}^{Y}_{\beta_2}(k_2)\gamma_5 u^{p}_{\alpha_2}(p_2)$$

where α and β are spin labels, m is the K-meson mass and q $(= k_1 - p_1 = p_2 - k_2)$ is the momentum transfer. The cross section is related to the square of this amplitude, summed over the β's and averaged over the α's:-

$$\frac{d\sigma}{d\Omega} \propto |\langle f | M_{av} | i \rangle|^2 = \frac{1}{4} \sum_{\substack{\alpha_1 \alpha_2 \\ \beta_1 \beta_2}} |\langle \beta_1 \beta_2 | M | \alpha_1 \alpha_2 \rangle|^2$$

$$= \frac{4(p_1 . k_1 - MY)^2}{(q^2 - m^2)^2} \qquad (2)$$

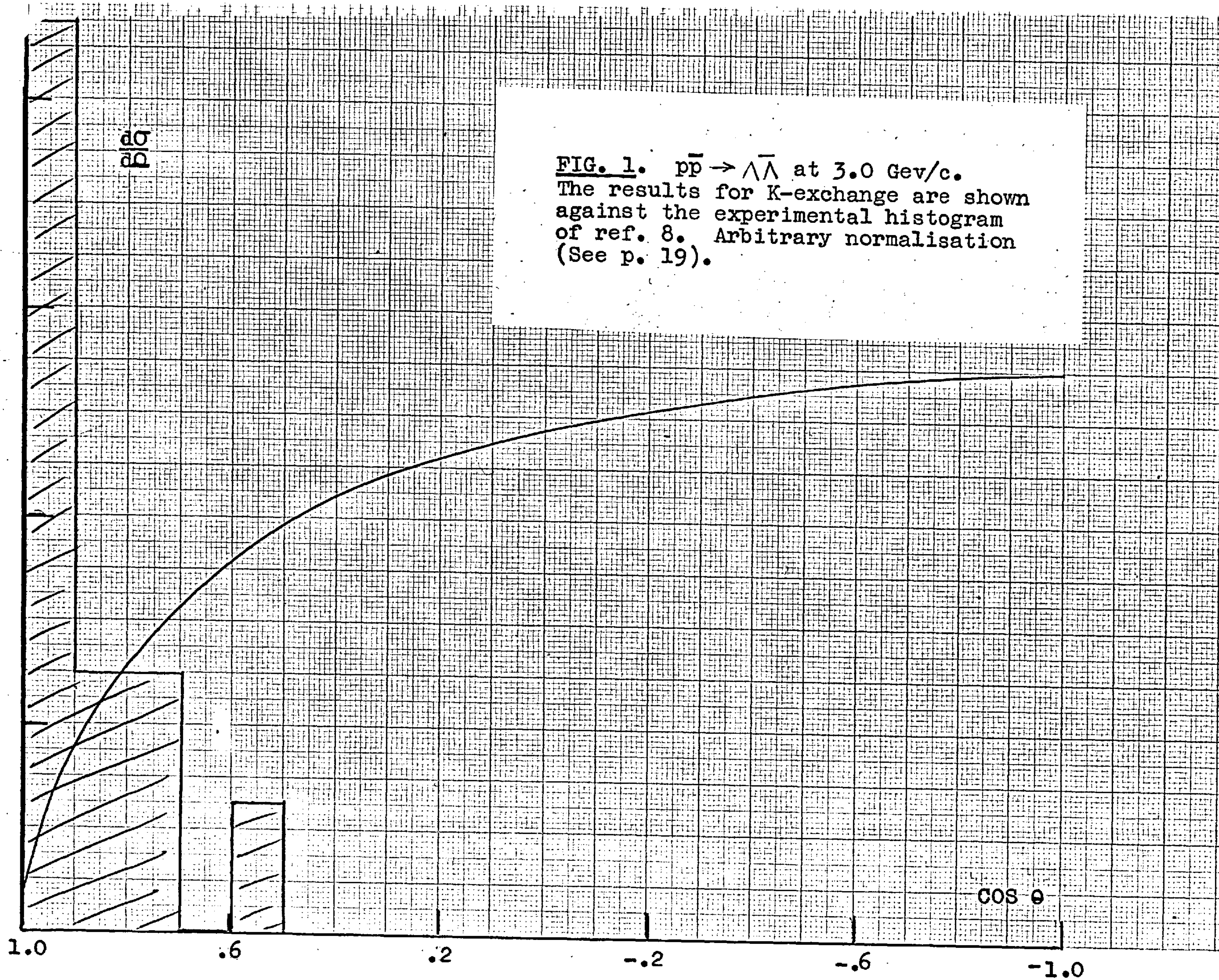
where M is the nucleon and Y the hyperon mass.

In Fig. 1 we have plotted this quamtity in the CM frame for $N\bar{N}$ production, as a function of $\cos\theta$ where θ is the angle between the antiproton and antihyperon ($\cos\theta = 1$ for forward scattering)*. It is seen that the distribution obtained, far from displaying a sharp forward peak, is in fact vanishingly small in the forward direction and rises to a maximum in the backward direction.

The only variation in the form of (2) for different final state combinations of Λ and Σ arises from the mass differences:-

---

* For detailed calculation we confine our attention to the CERN results at 3.0 Gev/c (Ref. (8)).



<u>FIG. 1</u>. $p\bar{p} \rightarrow \Lambda\bar{\Lambda}$ at 3.0 Gev/c.
The results for K-exchange are shown against the experimental histogram of ref. 8. Arbitrary normalisation (See p. 19).

Λ mass (Y) $\sim$ 1115 Mev

Σ " (Y) $\sim$ 1190 Mev

of nucleon mass (M) $\sim$ 940 Mev.

It is quite clear from these values that the angular distributions for the different processes will be entirely similar to Fig. 1, and this is borne out by detailed calculation.

Indeed, we can go further and derive an approximation to the amplitude by setting the hyperon and nucleon masses equal, Y = M. Equation (2) then simplifies

$$|\langle f\,|M_{av}|\,i\rangle|^2 = \frac{4(p^2(1-\cos\theta))^2}{(q^2-m^2)^2}$$

$$= \left[\frac{q^2}{q^2-m^2}\right]^2 .$$

Setting $t = -q^2$,

$$|\langle f\,|M_{av}|\,i\rangle|^2 = \left[\frac{t}{t+m^2}\right]^2 \qquad (3)$$

It is now well known that for pseudoscalar meson exchange between equal mass particles, the amplitude, T , is of the form indicated - $T \sim t/(t+m^2)$. It is also clear that this represents a small backward peaking. This is to be contrasted with the situation for scalar meson exchange, $T \sim (t+4M^2)/(t+m^2)$, which does give a forward peak.

We conclude that a simple model of K-exchange is not in accord with experiment, and we must reject the model. The inclusion of the necessary γ_5 factors introduces additional t-dependent terms in the amplitude which mask the effect of the pole term. Nor is it possible to modify the angular distribution by the assumption of alternative coupling schemes for the K N Y interaction; all such alternative interactions reduce to the original form when the current is written between Dirac spinors -- in other words we have not got enough invariants to construct a genuine alternative interaction. There remains the possibility of assuming the existence of strongly varying form factors at the K N Y vertex, but it seems rather unsatisfactory to invoke on ad hoc grounds the existence of a factor which would have as its sole purpose the conversion of a backward into a forward peak, and we reject this possibility.

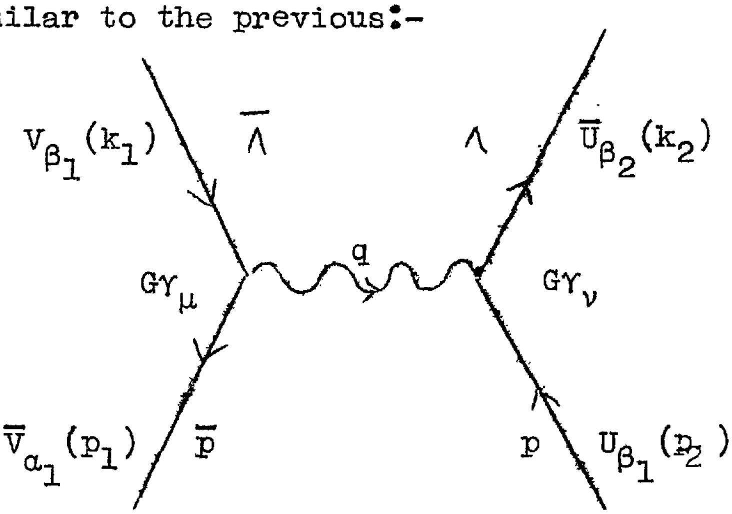
Our conclusion that K-exchange is unimportant in the reaction implies a quantitative limit on the coupling constants; we deduce for example, that $G_{KN}^2 /4\pi \lesssim .5$. This value is smaller than that given by previous estimates(10).

(iv) K^*-exchange

We now investigate a K^*-exchange model. For definiteness we consider first the channel $p\bar{p} \rightarrow \Lambda\bar{\Lambda}$ -- as before, the other cases will follow with only small changes in the masses. We take the effective interaction L as

$$L = G \bar{\psi}_p \gamma_\mu \psi_\Lambda \phi_{+\mu} + h.c. \qquad (4)$$

where $\phi_{+\mu}$ is the positive K^* (1^-) field. The diagram is similar to the previous:-



the corresponding amplitude is

$$\langle \beta_1\beta_2 | M | \alpha_1\alpha_2 \rangle$$

$$= G^2 \bar{V}^p_{\alpha_1}(p_1)\gamma_\mu V^\Lambda_{\beta_1}(k_1) \frac{g_{\mu\nu}-q_\mu q_\nu m^2}{q^2 - m^2} \bar{U}^\Lambda_{\beta_2}(k_2)\gamma_\nu U^p_{\beta_1}(p_2) \qquad (5)$$

where m is now the K^* mass.

Bessis, Itzykson and Jacob[11] have independently carried out the same calculation and have given an amplitude corresponding to (4) which is in disagreement with (5) by a factor of two. It is therefore necessary to expand a little on the derivation of (5).

We have

$$L(x) = G(\bar{\psi}_p \gamma_\mu \psi_\Lambda \phi_{+\mu} + \bar{\psi}_\Lambda \gamma_\mu \psi_p \phi^*_{+\mu}) .$$

The second order amplitude corresponding to the diagram is of the form

$$\frac{1}{2!} \int_{-\infty}^{\infty} d_4 x_1 \, d_4 x_2 \quad T \left\{ L(x_1) \, L(x_2) \right\} .$$

BIJ observed that there are two terms in the time ordered product corresponding to $p\bar{p} \rightarrow \Lambda \bar{\Lambda}$, nam ly

1st term in $L(x_1)$ $\times$ 2nd term in $L(x_2)$

and 2nd term in $L(x_1)$ $\times$ 1st term in $L(x_2)$,

with, in both cases, contractions over the boson fields. Apart from $x_1 \longleftrightarrow x_2$ the terms are the same, and they give rise to identical amplitudes, and BIJ include an extra factor of two in their amplitude on this ground, arguing that one term corresponds to K^* exchange and the second to $\bar{K}^*$ exchange. This is incorrect. The factor of 2 is cancelled by the 1/2! - this is the well known Feynman prescription that one neglects topologically identical diagrams which arise from permuting the x_i .

The differential cross-section corresponding to (5) may be evaluated using standard techniques. In Fig. 2 we have plotted the result for $\Lambda\bar{\Lambda}$ production against the experimental

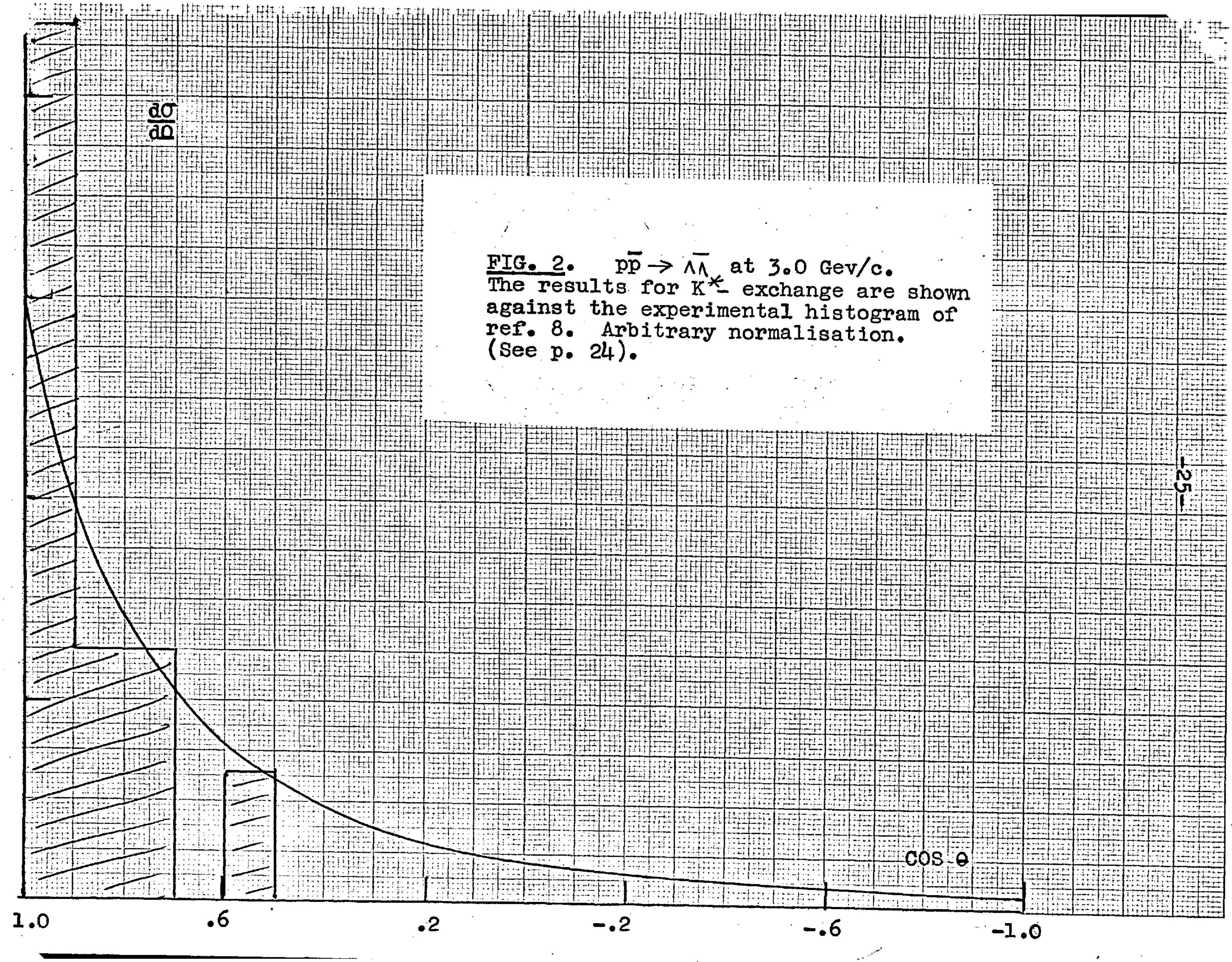


FIG. 2. $p\bar{p} \rightarrow \Lambda\bar{\Lambda}$ at 3.0 Gev/c.
The results for K^* exchange are shown against the experimental histogram of ref. 8. Arbitrary normalisation. (See p. 24).

histogram. It is seen that the main feature of the reaction, the forward peak, is well reproduced. In Figs. 3 and 4 we give the results for $\Lambda \overline{\Sigma_0}$ and $\Sigma_+ \overline{\Sigma_+}$. ($\Sigma_0 \overline{\Lambda}$ is identical to $\Lambda \overline{\Sigma_0}$ by charge conjugation.)

The calculation of the differential cross-section from (5) is a fairly lengthy process, especially when hyperons of different mass are involved. The numerical computation was therefore carried out on a Ferranti Mercury computer. However, if we again set the hyperon and nucleon masses equal, the calculation is much simplified -- in particular the $q_\mu q_\nu / m^2$ terms makes no contribution -- but the results are substantially unaltered. It is doubtful if either the accuracy of the experiments or the state of the theory really warrant carrying the calculations to the accuracy of mass differences. In further work we shall neglect these mass differences.

In the above calculations we have assumed even Λ and Σ parities. When this work was carried out (1962) the Σ parity was not well established. The calculation was repeated for odd $\Sigma \Lambda$ relative parity, by including extra factors of γ_5 at the vertices. In Fig. 5 we have plotted the integrated differential cross-section for $\Sigma_+ \overline{\Sigma_+}$ production for the two cases of odd and even $\Sigma \Lambda$ parity, against the experimental results. The case of even parity is clearly favoured.

This conclusion, however, is dependent on the model, and apart from considerations of the angular distributions

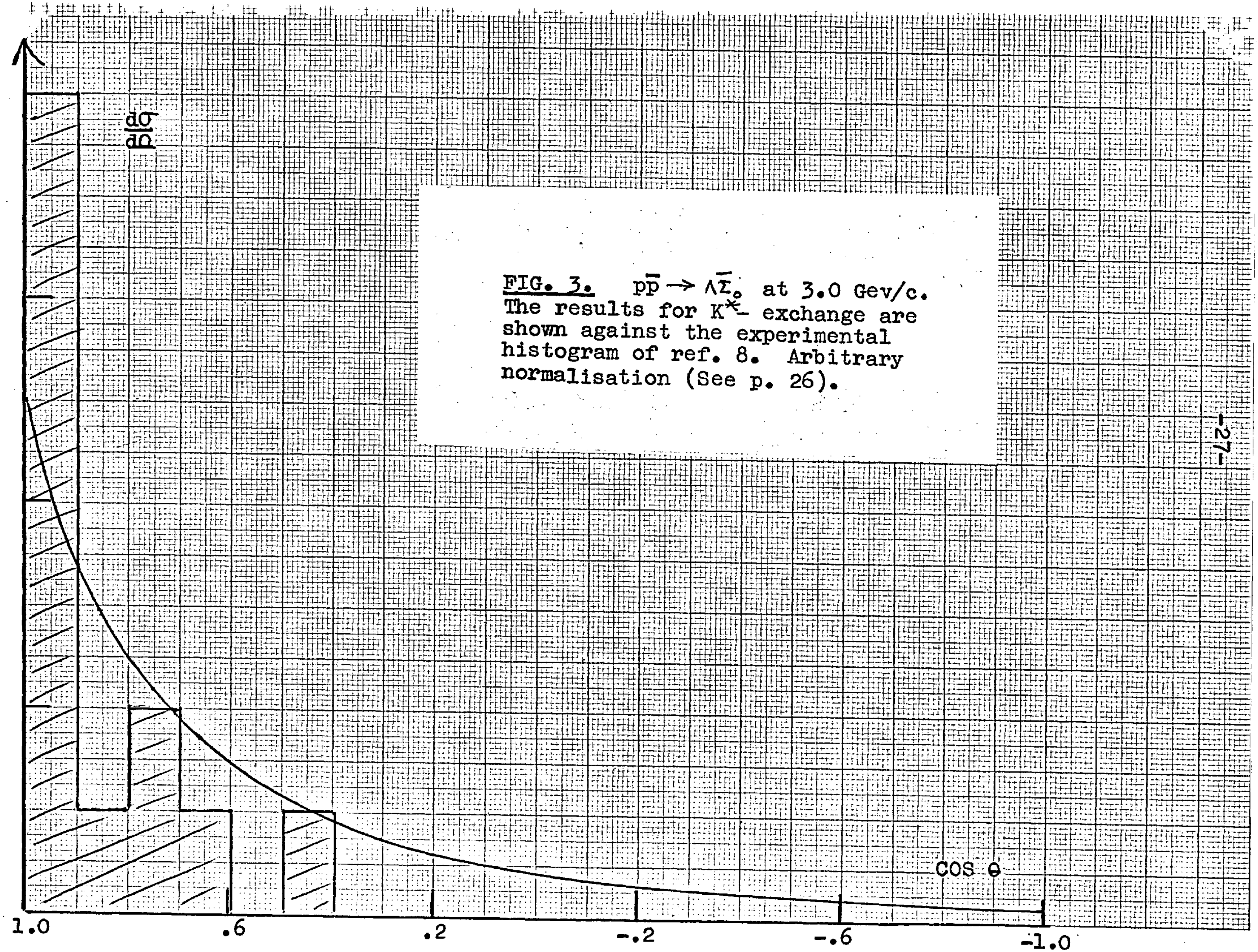


FIG. 3. $p\bar{p} \rightarrow \Lambda\bar{\Sigma}_o$ at 3.0 Gev/c. The results for K^*- exchange are shown against the experimental histogram of ref. 8. Arbitrary normalisation (See p. 26).

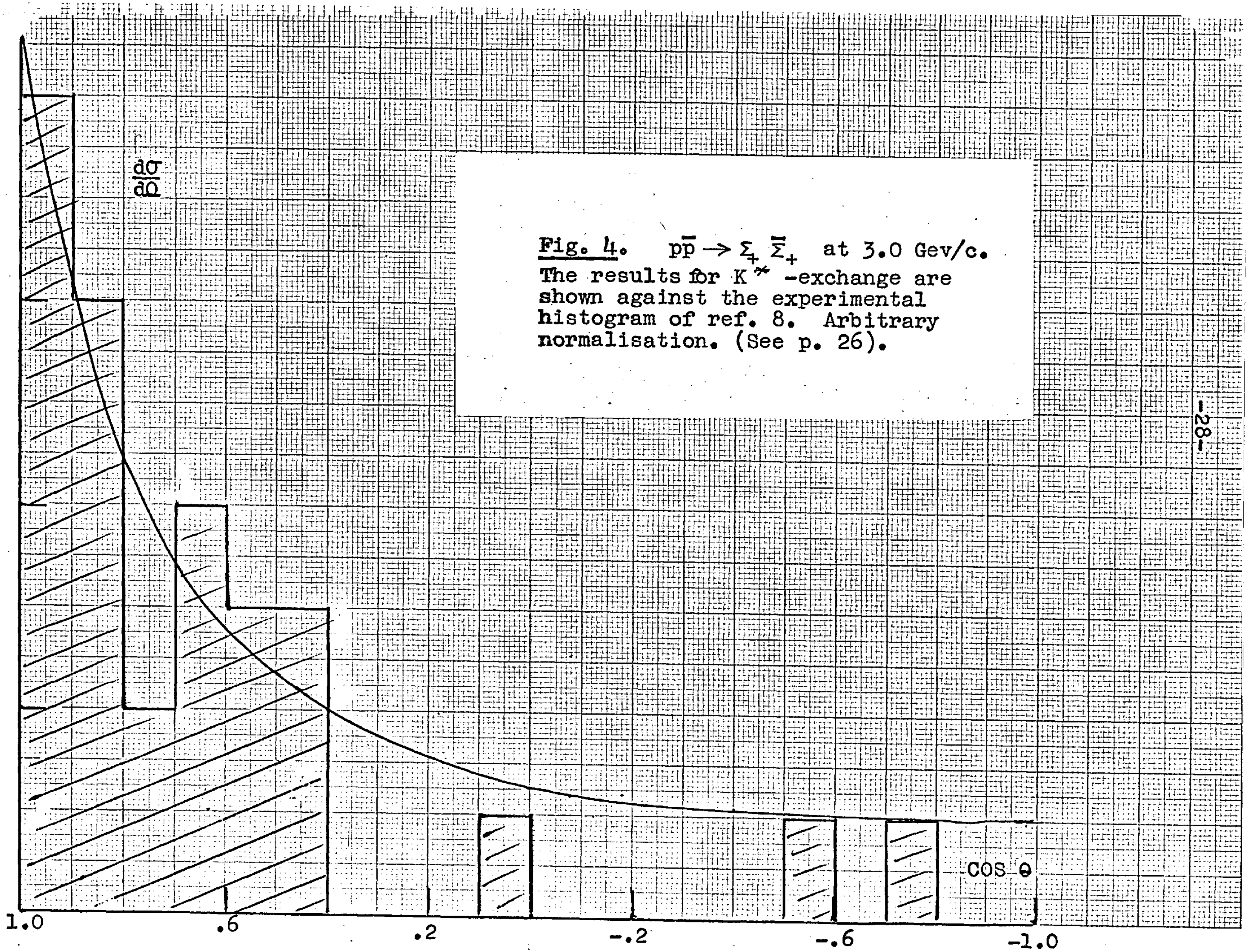


Fig. 4. $p\bar{p} \rightarrow \Sigma_+ \bar{\Sigma}_+$ at 3.0 Gev/c. The results for K^* -exchange are shown against the experimental histogram of ref. 8. Arbitrary normalisation. (See p. 26).

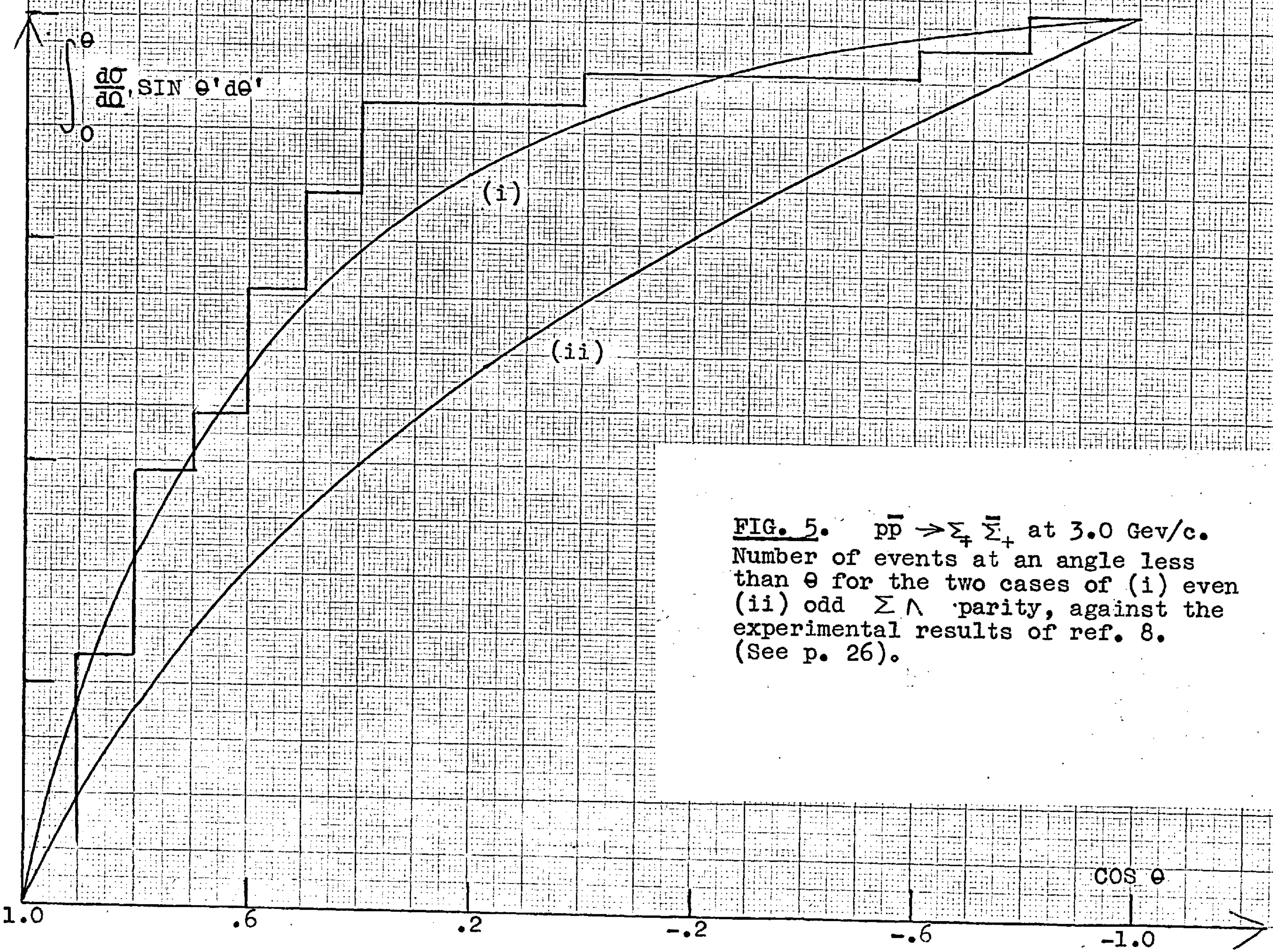


FIG. 5. $p\bar{p} \to \Sigma_{\mp} \bar{\Sigma}_{+}$ at 3.0 Gev/c. Number of events at an angle less than θ for the two cases of (i) even (ii) odd $\Sigma\Lambda$ parity, against the experimental results of ref. 8. (See p. 26).

there remain questions of the magnitudes of the cross-sections, and the sensitivity of the model to alternative coupling schemes, to which we now turn.

(v) Cross-Sections

The $\overset{*}{K}\Lambda N$ and $\overset{*}{K}\Sigma N$ coupling constants, G_Λ and G_Σ, are properly defined in terms of I-space invariant interactions. For the Λ case this is

$$G_\Lambda \bar{\Psi}_N \gamma_\mu \psi_\Lambda \phi_{K^*\mu}$$

of which the term

$$G_\Lambda \bar{\Psi}_p \gamma_\mu \psi_\Lambda \phi_{+\mu} \qquad (6)$$

is relevant here.

For the Σ case the interaction is written

$$G_\Sigma \bar{\Psi}_N \gamma_\mu \underline{\tau} \phi_{K^*\mu} \underline{\Psi}_\Sigma$$

which contains

$$2\, G_\Sigma \bar{\Psi}_p \gamma_\mu \psi_{\Sigma_+} \phi_{o\mu} + G_\Sigma \bar{\Psi}_p \gamma_\mu \psi_{\Sigma_o} \phi_{+\mu} \; . \qquad (7)$$

The coupling constants appropriate to different vertices are therefore as follows

$p\,\Lambda\,K^*$	:	G_Λ
$p\,\Sigma_o\,K^*$	:	G_Σ
$p\,\Sigma_+\,K^*$	:	$\sqrt{2}\,G_\Sigma$.

To the excellent approximation of neglecting the mass difference we can therefore predict the following ratios between the cross-sections:-

$$\begin{aligned}
&\sigma(\Lambda\bar{\Lambda}) && \propto && G_\Lambda^4 \\
&\sigma(\Lambda\bar{\Sigma}_0)\ (=\sigma(\Sigma_0\bar{\Lambda})) && \propto && G_\Lambda^2 G_\Sigma^2 \\
&\sigma(\Sigma_+\bar{\Sigma}_+) && \propto && 4\,G_\Sigma^4 \\
&\sigma(\Sigma_0\bar{\Sigma}_0) && \propto && G_\Sigma^4
\end{aligned} \qquad (8)$$

From this it follows that

$$\frac{2\,\sigma(\Lambda\bar{\Sigma}_0)}{\sqrt{\sigma(\Lambda\bar{\Lambda})\,\sigma(\Sigma_+\bar{\Sigma}_+)}} = 1 \qquad (9)$$

In Table 1 we have given the value of this quantity evaluated for the different sets of experimental data. The error in its value is large, and the limits given are probably little more than a guide. (The experiments are difficult to perform, involving the detection and differentiation of neutral Λ and Σ particles. There appears to be a considerable measure of disagreement between the CERN and Brookhaven results.) It is therefore difficult to draw any firm conclusions from a comparison of (9) with experiment, but we can perhaps say that the results indicate that the model will serve as a basis for development.

The cross section is given in terms of $|\langle f\,|\,M_{av}\,|\,i\rangle|^2$ by

$$\frac{d\sigma}{d\Omega} = G^4 \frac{|\underline{k}_1| \, |\langle f| M_{av} | i\rangle|^2}{|\underline{p}_1| \quad (16 \pi E)^2}$$

in units of $(\text{energy})^{-2}$, where for G^4 the appropriate factor from (8) is inserted. We have three results, the $\Lambda\bar{\Lambda}$, $\Lambda\bar{\Sigma}_0$ and $\Sigma_+\bar{\Sigma}_+$ cross-sections, from which to determine the two coupling constants, $G_\Lambda + G_\Sigma$. At 3.0 Gev/c the best fit has

$$\begin{aligned} G_\Lambda^2 / 4\pi &\sim .20 \\ G_\Sigma^2 / 4\pi &\sim .06 \end{aligned} \qquad (10)$$

With these values, the three observed cross-sections are reproduced with 10 - 20 per cent accuracy. (Very much better fits would be obtained from the Brookhaven data.)

Chan[12] has estimated G_Λ^2 from a study of $\pi^- p \rightarrow \Lambda K^0$. If this calculation is repeated using a more recent value of the K^* width, we find $G_\Lambda^2 / 4\pi \sim 0.22$, in good agreement with our result. (The range of momentum transfer involved at the $K^* \Lambda N$ vertex is similar in this case.)

We can compare the results (10) with the predictions of SU(3) symmetry[13]. As is well known, there are two possible coupling schemes between baryons and mesons, F and D types. (A general linear combination of these couplings is of course possible.) On a simple view, one might expect the vector mesons to be coupled in an F-type manner to the baryons, since in this case ρ , ω , and K^* are coupled to the isospin, hypercharge and strangeness changing currents, which are conserved in the limit of zero mass splitting. Such a (pure F) coupling implies

$$G^2_{\Lambda K^*} : G^2_{\Sigma K^*} = 3 : 1 ,$$

in good agreement with out results (10).

In Table 1 we give values of $\sqrt[4]{\sigma(\Lambda\bar{\Lambda}) / \sigma(\Sigma_+\bar{\Sigma}_+)}$ from the experimental data. This parameter is related to the ratio of the coupling constants as they appear in the interaction Lagrangians, and is normalised to have the value 1 for F-type, and 1/3 for D-type. F-type coupling is clearly preferred in the model; at very least, the proportion of D-coupling must be low.

It is perhaps worthwhile giving explicitly the cross-section ratios predicted on the basis of F-coupling. These are

$$\Lambda\bar{\Lambda} : \Lambda\bar{\Sigma}_0 : \Sigma_+\bar{\Sigma}_+ : \Sigma_0\bar{\Sigma}_0$$
$$= 9 : 3 : 4 : 1 .$$

On the other hand, there is evidence that interactions of scalar mesons involve D-type couplings(14). If we assume a peripheral mechanism of K-exchange with D-coupling, we find, e.g., $\sigma(\Lambda\bar{\Lambda}) : \sigma(\Sigma_+\bar{\Sigma}_+) = 1 : 36$, a result in disagreement with experiment by two orders of magmitude. We see that a K-exchange mechanism would be difficult to reconcile with unitary symmetry ideas, while a K^* mechanism fits well into the SU(3) scheme.

(vi) Alternative Couplings

We now investigate how our conclusions are modified by the inclusion of different possible types of interactions at the K^*-baryon-baryon vertex. Time reversal symmetry does not forbid a term of the q_μ (momentum transfer) form in the case of differing baryon masses, such as N and Y, To make the calculation tractable at this and other points, we work in the unitary symmetric approximation, which imposes a gauge invariance requirement under which a term of this form is inadmissible. In terms of "electric" and "magnetic" form factors[15] the vertex function must then be of the form

$$\frac{1}{4m^2 - q^2} \left(2mG_E(q^2)P_\mu + ikG_M(q^2)r_\mu\right) ,$$

where P_μ is the sum of ingoing and outgoing momenta at the vertex and

$$r_\mu = \frac{i}{2}(\gamma_\mu \not{P} \not{q} - \not{q} \not{P} \gamma_\mu) .$$

$G_E(0)$ and $G_M(0)$ are normalized to one, and k is a dimensionless parameter which gives a measure of the relative strengths of the two types of interaction.

Neglecting the q^2 dependence of G_E and G_M, the interaction is of the form

$$\frac{1}{4m^2 - q^2} (2mP_\mu + ikr_\mu) .$$

In Fig. 6 we have plotted the differential $p\bar{p} \rightarrow Y\bar{Y}$ cross-section for different values of the parameter k. It is seen that the forward peaking is not sensitive to the nature of the interaction for comparable values of the coupling constants, i.e. $k \sim 1$. This conclusion would be strengthened by the inclusion of the q^2 dependence of G_E and G_M, since one would certainly expect this to favour the low q^2 values, as in the case of electromagnetic form factors of the nucleon.

Hand, Miller and Wilson(16) have analysed electron scattering data in terms of electric and magnetic form factors, and fitted these with ρ and ω interactions and a soft core of mass 30 fermi^{-2}, finding $k \;\; = 3.5$, $k_\omega = 1.3$. Unitary symmetry predicts $k_{K^*} = k_\rho = k_\omega$. If k_{K^*} is as large as 3.5 (Fig. 6) the strong forward peaking will no longer be present. (This disappearance of the forward peak with increasing value of k corresponds to the well-known fact that in electromagnetic interactions, the magnetic interaction dominates the wide-angle scattering.) However k_ρ and k_ω are sensitive to the choice of soft core mass, and must be regarded as poorly determined.

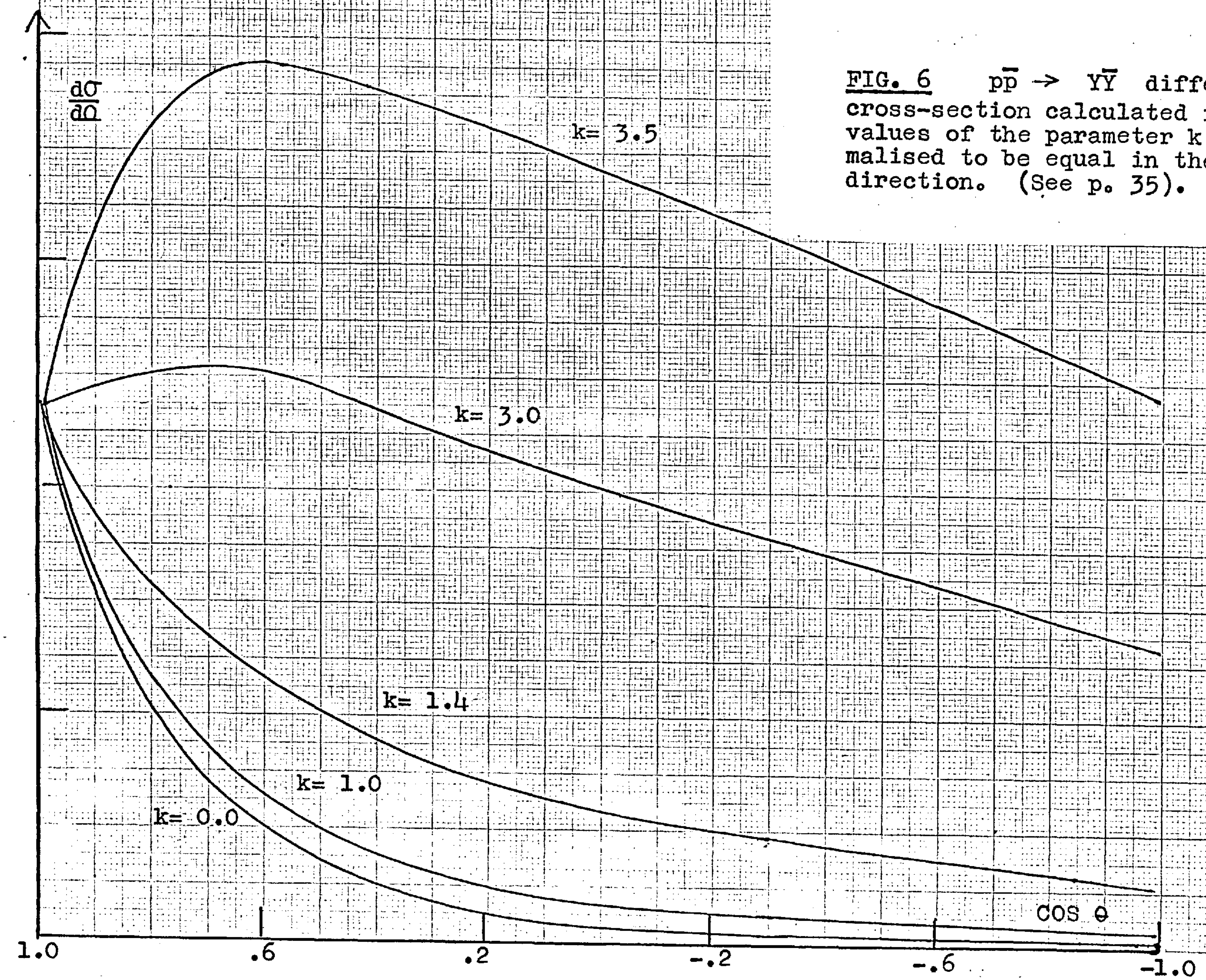


FIG. 6 $p\bar{p} \rightarrow Y\bar{Y}$ differential cross-section calculated for different values of the parameter k, and normalised to be equal in the forward direction. (See p. 35).

(vii) Discussion

The pole term model which we have presented seems able to give a fairly good descriprion of the processes. We have seen that the branching ratios and angular distributions predicted are reasonable representations of the observed quantities, and that the model, which is not unduly sensitive to the form assumed for the baryon-baryon-meson vertices, is in accord with the octet model of unitary symmetry with F-type couplings. However, we must emphasize that the pole model, of which the greatest merit is simplicity, cannot be taken too seriously. We prefer to regard it as a basis on which to build a more satisfactory treatment, for there are serious objections to such a pole term model.

Firstly, Durand and Chiu[17] have pointed out that our argument rejecting K-exchange on the basis of the angular distribution results is unconvincing. The reader will recall that the amplitude for K-exchange was of the approximate form $t/(t + m^2)$, representing a backward concentration of events, while the observed distribution is very much forward peaked. We now observe that this amplitude may be put in the form

$$\frac{t}{t + m^2} = 1 - \frac{m^2}{t + m^2} \qquad (11)$$

It is clear that the second term here has the correct (forwardly concentrated) form. The unit term interferes with this term to produce a total backward concentration. Thus we see that the unit term, representing an S-wave contribution, is dominating the form of the amplitude; if the S wave is omitted, the correct forward form results.

This dominant role of the S-wave is quite in contradiction with the entire spirit and justification of the peripheral model as given in Section (i). We set out with the intention of using the one particle exchange amplitude as an approximation to the long range part of the force, with which we hoped to represent the behaviour of the high partial waves. We now find that (for K) the one particle exchange amplitude is dominated by the S-wave contribution, which in any case we did not expect to be accurately given.

It is a general feature of pole term models that the amplitude obtained is dominated by the low partial waves. It is therefore just not possible to claim that the Born term is an approximation to the long range (i.e. high partial waves) interaction.

This difficulty arises from the fact that the Born term model pays no attention to the requirements of unitarity. For a strong interaction process the Born term model leads to amplitudes which badly violate unitarity for the lower partial waves -- for the S-wave by as much as one or two orders of magnitude in a typical process[18]. Yet not only must the partial wave amplitudes keep below the unitarity limit, but at high energies (corresponding to the peripheral or optical region defined by (1)) when very many competing channels are open, we would expect the lower partial waves to keep well below the unitarity limit - indeed to be very small in just the region where the Born term model gives them as very large.

A related difficulty of the one partille exchange model is evident upon a closer examination of the angular distributions given in Figs. 2, 3 and 4. There is a systematic tendency for the reaction to be more forwardly peaked than predicted - the peripheral amplitude is not sufficiently peripheral. This is a common fault of the Born term model, and to remedy the situation form factors have been invoked at the interaction vertices, with a q^2 dependence chosen to suitably correct the amplitudes. From the present point of view, we remark that the wide angle scattering arises from the low partial waves. We have seen that the low partial waves are incorrectly given by the Born term model and we can only hope that the need for form-factors will disappear when the requirements of unitarity are forced on the model.

At this point we only touch on the question of unitarity. The second Part of this work is devoted to this failing of the Born term model, and we develop a method for attempting to overcome the difficulty by the inclusion of so-called "absorptive effects".

Apart from the fundamental problem of unitarity, there remains in the Born term model considerable arbitriness, of which our present pole calculations provide an example. This arises from several sources. First, there is often more than one quantum available to act as the exchanged particle. Secondly, having made a choice of intermediary particle, on mass or other grounds, we are often faced with a choice of possible forms for the interaction, e.g. "γ_μ" or "$\sigma_{\mu\nu} q_\nu$". There is normally some indication of the form, but usually also considerable freedom of manoeuvre. Finally, in considering branching ratios, or processes involving more than one exchange quantum, we are free to assign ad hoc values to the relevant coupling constants.

When these three sets of choices are put together, there results a very considerable freedom of manoeuvre. One could speculate that it might be possible, with a suitable choice of parameters, to obtain an adequate fit to any set of data. It would be much preferable to start from a higher symmetry scheme in which there was no freedom in these parameters, and to compare the resulting amplitudes against experiment. In Part III of the present work we repeat the calculation of $p\bar{p} \rightarrow \Lambda\bar{\Lambda}$ within the $\widetilde{U}(12)$ theory of Salam, Delbourgo and Strathdee[19].

PART II

ABSORPTIVE EFFECTS

(i) Form Factors

In their early peripheral, or pole term, calculations, first of $NN \rightarrow NN^*$ and later of $\pi N \rightarrow \rho N$, Ferrari & Selleri found that the model did not give predictions which agreed with experiment. The disagreement was of the type we saw in Part I of this work in the case of $\bar{p}p \rightarrow Y\bar{Y}$; the peripheral amplitudes gave too much wide angle scattering. To remedy this discrepancy form factors were introduced into the one particle exchange model. It was argued that the existence of possible structure at the reaction vertices (c.f. EM form factors) and possible unknown renormalisation effects on the propagator permitted the inclusion of theoretically undetermined functions of the squared momentum transfer in the peripheral amplitudes. Such a function we denote by $F(q^2)$ where we have absorbed the three separate effects arising from the two vertices and the propagator into one term, which is normalised by choice of coupling constant so that $F(0) = 1$. $F(q^2)$ is referred to as the form factor, though it clearly represents more than the structure at one vertex, as in the EM case. In general the form factor will be different for different reactions. In practice the form factor was chosen so as to bring the peripheral amplitude into agreement with the data.

We reproduce in Fig. 7 three form factors which have

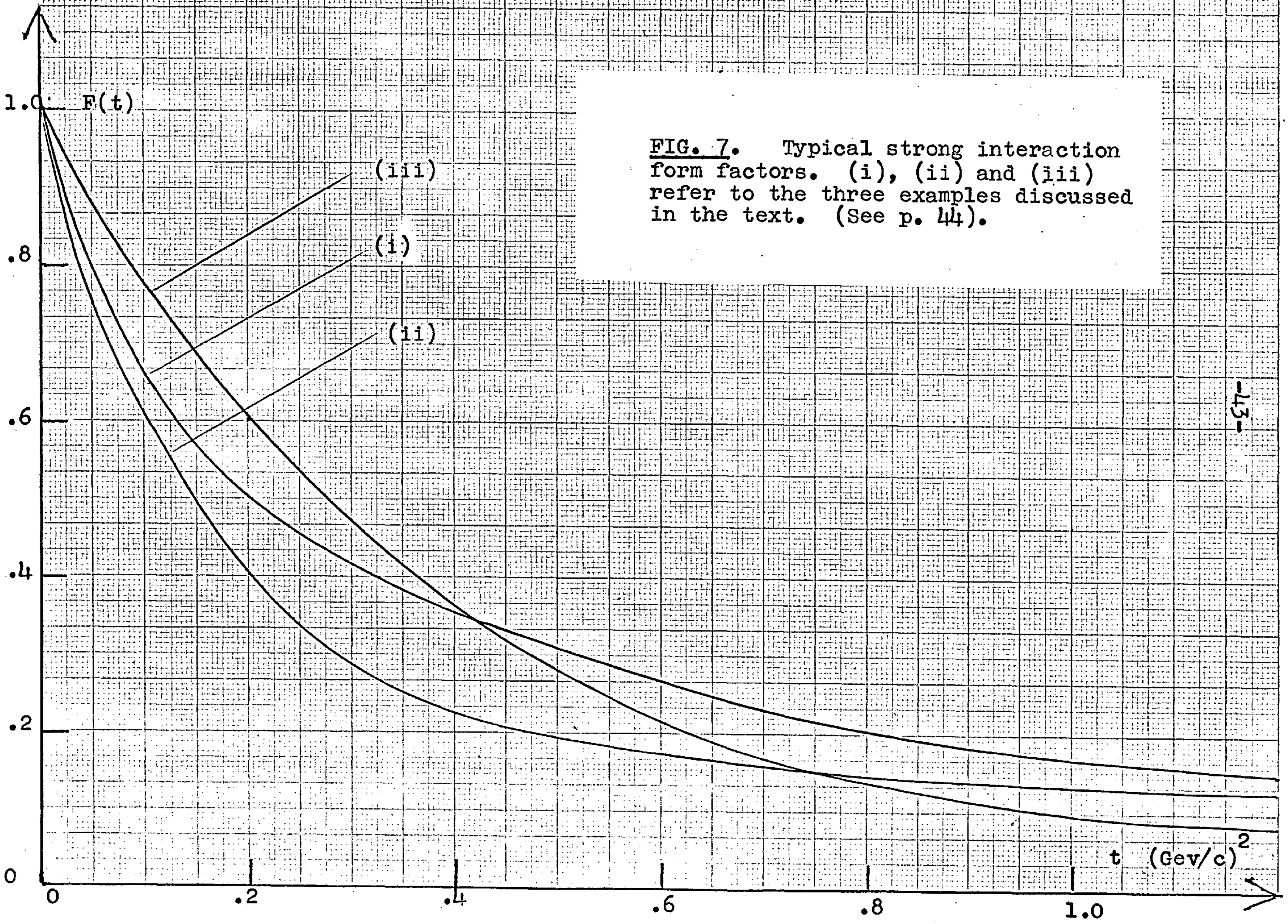


FIG. 7. Typical strong interaction form factors. (i), (ii) and (iii) refer to the three examples discussed in the text. (See p. 44).

been determined in this manner. The functions shown are as follows*

(i) The Ferrari-Selleri type form factor, for one pion exchange in $NN \rightarrow NN^{*}$ and $\pi N \rightarrow \rho N$.

$$F(q^2) = \frac{.72}{1 + (-q^2+\mu^2)4.73\ \mu^2} + \frac{.28}{1 + \left[(-q^2+\mu^2)/32\mu^2\right]^2}$$

where μ is the pion mass. The first of these terms is chosen to give a rapid decrease at small momentum transfer, and the second to reproduce the data at large momentum transfers.

(ii) The pion exchange form factor of Goldhaber et al.(21) for

$$KN \rightarrow K^{*} N^{*}$$

$$F(q^2) = \frac{\alpha^2}{\alpha^2 - q^2}$$

where $\alpha^2 = 0.132\ (\mathrm{Gev}/c)^2$

(iii) The vector meson form factor of Jackson & Pilkuhn(22) for ρ or ω in

$$KN \rightarrow K^{*}N$$

$$KN \rightarrow K\,N^{*}$$

$$F(q^2) = \exp(\lambda q^2)$$

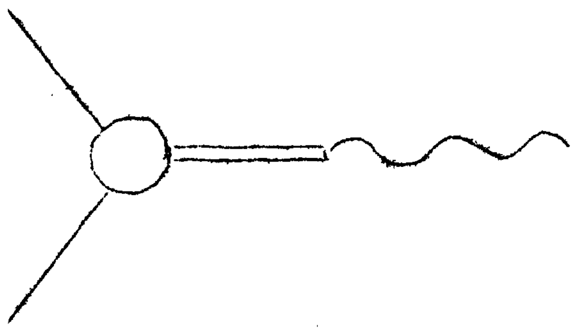
where $\lambda = 2.5\ (\mathrm{Gev}/c)^{-2}$.

---

* Taken from J.D. Jackson & H. Pilkuhn; Nuovo Cimento 33, 906 (1964).

It would at best be unfortunate to have to invoke such <u>ad hoc</u> form factors in order to obtain satisfactory results. However, since the necessary form factors are such rapidly varying functions of q^2, the situation is extremely unsatisfactory. It is not clear that one is doing anything more than curve fitting.

There are also weighty theoretical arguments against the existence of such strong form factors. If the pion form factors are to be ascribed to pole interactions at the vertices of the form



as in EM interactions, then we require the existence of a three pion state at a mass squared $\sim 5m_\pi^2$. However, there is no evidence for such a state up to $\sim 50m_\pi^2$.

Again, if the same assumption is made as to the origin of the vector meson form factors, then we are forced to assume the existence of states with the same quantum numbers as ρ, ω etc., but with <u>lighter</u> masses, even though we start (in the peripheral model) with the supposition that we are exchanging the <u>lightest</u> quanta available.

Altogether, the use of such form factors seems unreasonable. In the pion case, the purpose of the form factor is largely to remove the large S-wave term. As this term is inconsistent with the assumptions underlying the model, and as it badly violates unitarity, this procedure is very questionable. It would be more preferable to find a method of unitarising the model.

The similarity of the form factors which have been introduced by various authors is very striking (c.f. Fig. 7). It is remarkable that such similar forms are found for different exchange particles and different reactions. This circumstance suggests that the mechanism, if such exists, responsible for the sharpening of the forward peak in the peripheral model is <u>independent</u> of the detailed dynamics of the particular reaction under consideration, and is a more general feature of high energy scattering.

This situation is reminiscent of elastic scattering at high energies, of which in all cases the main feature is a sharp forward peak, representing the shadow of the many open inelastic channels, and independent to a large extent of the detailed nature of the particular interaction. This suggests a possible line of development for the peripheral model, namely to find some way of including in the model the effects of the many open channels by taking into account in some manner the strong elastic interactions between the incident and final particles. This is the absorption model.

(ii) Elastic Scattering at High Energies

As the absorption model is based on the characteristics of elastic scattering at high energies, we now give a brief review of these.

We define the high energy region to be that for which

$$p \gg \nu^{-1} \qquad (1)$$

where ν is the range of the elastic interaction. ν is typically $\sim$ 1 f, so (1) implies the Gev range or higher. (This condition should be compared to that of I.1 defining the peripheral region, $p > \mu^{-1}$).

Under these energy conditions the incident particle may be considered localised in a wave packet of smaller dimensions than the target which it "sees". The particle can therefore be thought of as having a definite position; this is just a statement that at high energies quantum mechanics becomes similar in some ways to the classical mechanics of billiard balls - the so-called correspondence principle. Again, under this high energy limit we anticipate that very many inelastic channels are open. If a collision does take place, it will be likely to result in a transition to one of these other channels. From the point of view of the elastic interaction, such a collision results in the loss or absorption of a particle. Collisions which proceed by a direct elastic interaction, e.g. by exchange of quanta leading to the original state, are expected to be relatively few.

The situation is very similar to the passage of a beam of light through an opaque screen, and we therefore speak of this energy region (defined by (1)) as the <u>optical region</u>. The fact that there is no "direct" elastic scattering does not mean that no elastic scattering is observed; on the contrary, the absorption of particles from the beam shows up by diffraction in the formation of an elastic scattering distribution. We therefore also refer to this type of model of elastic scattering at high energies as the optical model - the elastic scattering arises from absorption.

In potential theory the absorption of particles into other channels can be simulated by adding an imaginary part to the potential V. If $\text{Im}\, V < 0$, plane wave solutions display a decreasing amplitude representing absorption. (Equivalent use of a complex impedance to represent absorption is widely known in optics, classical EM theory and wave theory.) In general if $\underline{j}$ is the particle current

$$\nabla . \underline{j} \quad = \quad 2 \, \text{Im} \, V \, |\psi|^2 \quad ,$$

showing that the probability of absorption is α - Im V (Im V < 0). The amplitude is given by

$$f(\theta) \quad = \quad \sum (2\ell + 1) \left(\frac{S_\ell - 1}{2ip} \right) P_\ell(\cos \theta) \qquad (2)$$

where $S_\ell = e^{2i\delta_\ell}$ and δ_ℓ is the phase shift. If V is complex, then so also is δ, and for Im V < 0 we must have Im $\delta > 0$. Setting

$$\delta_\ell \quad = \quad \alpha_\ell + i\beta_\ell \qquad\qquad (\beta_\ell > 0)$$

the optical model assumption is that $\beta_\ell \gg |\alpha_\ell|$, i.e. diffraction effects predominate. Neglecting α_ℓ we have

$$S_\ell \quad = \quad e^{-\beta_\ell} \quad , \qquad (3)$$

i.e. $0 \leqslant S_\ell \leqslant 1$.

The elastic scattering cross section is given by

$$\sigma_{el} \quad = \quad \frac{\pi}{p^2} \sum_\ell (2\ell + 1) \, | S_\ell - 1 |^2$$

so that we see explicitly from (3) that asborption implies elastic scattering. Further, since the inelastic cross-section is

$$\sigma_{in} \quad = \quad \frac{\pi}{p^2} \sum (2\ell + 1)(1 - |S_\ell|^2)$$

we see that in general the larger σ_{in}, the smaller is S_ℓ, and consequently the greater the elastic scattering. (The two are normally of the same order of magnitude).

This process, in which elastic scattering is induced by the absorption of particles into the other channels is just the working of unitarity. Note that since S_ℓ is real, the amplitude f_ℓ is imaginary ($S_\ell = 1 + 2ipf_\ell$). f_ℓ is therefore given by the unitarity relation. In terms of the more familiar covariant amplitude $T(\theta)$, related to $f(\theta)$ by

$$T(\theta) = 8\pi E_c \; f(\theta)$$

this is[23]

$$2 \operatorname{Im} T^{\ell}_{\alpha\alpha} = \sum_{\gamma} \frac{p_\gamma}{4\pi E_c} \left| T^{\ell}_{\alpha\gamma} \right|^2$$

where α, γ, label the channels, p_γ is the momentum in the γ-channel and E_c the total centre of mass energy. It is seen that each open inelastic channel contributes to the imaginary part of $T_{\alpha\alpha}$, and these contributions are additive. The optical model assumes that $T_{\alpha\alpha}$ arises entirely in this way, and therefore $T_{\alpha\alpha}$ is entirely imaginary.

The unitarity requirement gives for forward scattering

$$\operatorname{Im} T_{\alpha\alpha}(0) = 2p\, E_c \; \sigma_{tot}$$

where $\sigma_{tot} = \sigma_{e\ell} + \sigma_{in}$ is the total cross section.

In general

$$\left| T \right| = \sqrt{(\mathrm{Re}T)^2 + (\mathrm{Im}T)^2} \geqslant \mathrm{Im}T$$

$$\therefore \left| T_{\alpha\alpha}(0) \right| \geqslant 2p\, E_c\, \sigma_{tot} \tag{4}$$

the equality holding if, and only if, $T_{\alpha\alpha}(0)$ is imaginary. This provides the most direct check on the optical model assumptions. Though there has recently been some indication of a real part of the forward elastic scattering amplitude(24), the equality in (4) is found to be quite well satisfied for most cases of high energy scattering, and over a wide range of energies (high). This is generally taken as a justification of assuming $T(\theta)$ to be imaginary for all θ, though there is no direct check on this. (It must be stressed, however, that the model which we shall develop does not rely on the assumption of a pure imaginary elastic amplitude – it is only necessary that this amplitude should have a large imaginary part, and this certainly seems well established).

The model we shall develop gives a result (for inelastic processes) which contains elastic scattering matrix elements, e.g. $S^{\ell}_{\alpha\alpha}$, and these must be replaced by appropriate expressions in the light of theoretical and experimental understanding of elastic scattering. For the present work we assume a form for high energy elastic scattering which is widely used and appears to be quite accurate and of fairly general application. This is the Gaussian Model[23],

$$\frac{T(\theta)}{8\pi E_c} = f(\theta) = \frac{i}{2p} \sum_{\ell} (2\ell+1)\, C e^{-\frac{\ell(\ell+1)}{\nu^2 p^2}}\, P_\ell(\cos\theta) \quad (5)$$

where C and ν are parameters, C determining the strength of the interaction and ν giving the range - referred to as the optical model range, sometimes also taken as $\nu/\sqrt{2}$. From (2) and (5) we have

$$\frac{iC}{2p}\, e^{-\frac{\ell(\ell+1)}{\nu^2 p^2}} = \frac{S_\ell - 1}{2\,ip}$$

$$\text{so} \quad S_\ell = 1 - Ce^{-\ell(\ell+1)/\nu^2 p^2} \qquad (6)$$

C must be chosen to make the amplitude go through the optical point, and we find

$$C = \frac{\sigma_{tot}}{2\pi\nu^2} \qquad (7)$$

Experimentally, $C \sim .7 - 1.0$[5].

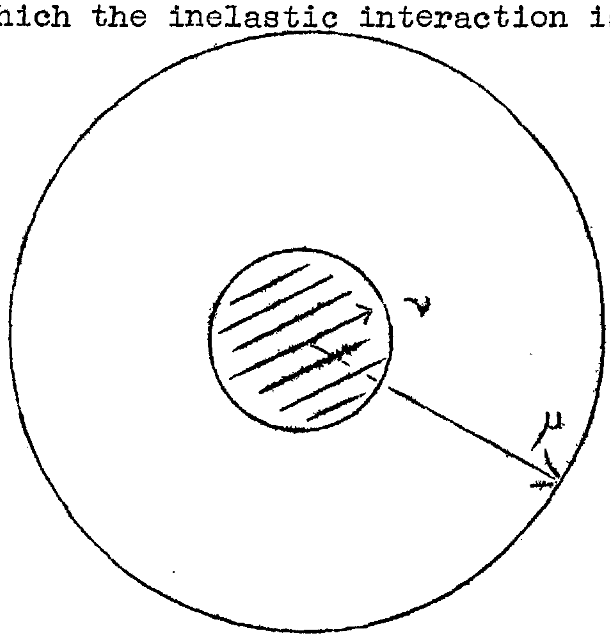
This parameterization leads to a variation of σ with t $(= -q^2)$ for small t as follows:-

$$\frac{d\sigma}{dt} \Big/ \left.\frac{d\sigma}{dt}\right|_{t=0} = e^{-\frac{1}{2}\nu^2 t} .$$

This is of the same form as given by the Regge pole theory of elastic scattering at high energies.

(iii) Absorptive Effects

Having a clear idea of the nature of elastic scattering at high energies, we are now in a position to see how we might expect the main characteristics of elastic scattering to have a pronounced effect on a typical inelastic process, say a transition $\alpha \rightarrow \beta$, where α, β, etc. label the channels. We suppose that the ranges of the elastic interactions in the initial (β) and final (β) states are $\sim \nu$, while the inelastic interaction is of range μ. We will suppose for definiteness that the inelastic process proceeds via the peripheral exchange of quanta of m, so that $\mu \sim m^{-1}$; however, there is in principle no need to make any assumptions concerning the production mechanism. Again, for definiteness, let us suppose $\mu \gg \nu$. The colliding particles therefore appear to one another as having opaque, black centres, only outside of which the inelastic interaction is "seen":-



If the particles interact with impact parameter less than ν, the chances are that they will be absorbed into an alternative channel - from the point of view of the transition $\alpha \rightarrow \beta$, that part of the interaction which would proceed through impact parameters less than ν is effectively lost. On physical grounds we can see that the result would be a marked sharpening of the angular distribution in the forward direction, since we would expect to see the diffraction shadow of the absorptive region even in the inelastic process.

It is therefore expected that any mathematical treatment of this effect will result in decreasing the low partial wave amplitudes. This is precisely the result which we saw in Part I, Section (vii) would be desirable in the peripheral model on very general grounds. It is just these low partial waves which violate unitarity, and lead to the unwanted wide angle scattering. The present idea, that of including the initial and final state elastic amplitudes in the calculation of inelastic processes, would be nothing less than a unitarity correction, since, as we have seen, the elastic amplotudes themselves are due to the large number of open inelastic channels. The reduction of the low partial waves would be just a unitarity damping effect.

If μ is not $\gg$ ν, the situation is not as easily visualised; however, we would expect similar effects.

There have been several independent treatments of the situation discussed here. These treatments start with differing assumptions, proceed by different methods and lead to results (valid under various conditions) which are sometimes inconsistent. It is our intention here to give a critical review of the results to date, which we shall sometimes derive by new methods, and to give a consistent review of the field, inter-relating the various results as much as possible. Finally, in Section (vii) we propose a treatment based on the K-matrix which seems to the author to be the most satisfactory.

However, in order to familiarise the reader with the general nature of the results, and to illustrate their application, we give here a (widely known) formula first written down by Sopkovich[25] and re-derived by Durand and Chiu[26,27], using the distorted wave Born approximation. This relates the modified or corrected partial wave matrix element $T^{\ell}_{\beta\alpha}$ to the Born element $V^{\ell}_{\beta\alpha}$ and the S-matrix elements for elastic scattering in the initial and final states, $S^{\ell}_{\alpha\alpha}$ and $S^{\ell}_{\beta\beta}$. The formula is valid under the conditions

$$p^{-1} \ll \mu \ll \nu \tag{8}$$

(Note that this implies we are in both the "peripheral" and "optical" regions - c.f. I(i) and II(i)).

The relation is

$$T^{\ell}_{\beta\alpha} = \sqrt{S^{\ell}_{\beta\beta}} \; V^{\ell}_{\beta\alpha} \sqrt{S^{\ell}_{\alpha\alpha}} \tag{9}$$

To illustrate the use of (9) let us suppose $S^{\ell}_{\alpha\alpha} = S^{\ell}_{\beta\beta}$, and take for these matrix elements a form appropriate to Gaussian scattering with complete absorption of the lower partial waves (i.e. Eq. (6) with C = 1):

$$S^{\ell}_{\alpha\alpha} = S^{\ell}_{\beta\beta} = 1 - e^{-\ell(\ell+1)/\nu^2 p^2}$$

(9) becomes

$$T^{\ell}_{\beta\alpha} = \left[1 - e^{-\ell(\ell+1)/\nu^2 p^2}\right] V^{\ell}_{\beta\alpha} \quad .$$

We show in Figs. 8, 9, and 10, the effects of this correction on a peripheral amplitude corresponding to the exchange of a vector meson (mass = 890 Mev = m_K) between two particles of nucleonic mass ($N\bar{N}$, $\Lambda\bar{\Lambda}$) at 3.0 Gev/c. (Apart from retaining q dependent terms from the vertices, we have entirely neglected spin.) Fig. 8 shows the correction factor, $1 - e^{-\ell(\ell+1)/\nu^2 p^2}$, against ℓ. Fig. 9 shows the weighted amplitudes (i) $(2\ell+1)\, V^{\ell}_{\beta\alpha}$, and (ii) $(2\ell+1)\, T^{\ell}_{\beta\alpha}$ against ℓ. Fig. 10 shows the corresponding angular distributions (i) Born term (ii) corrected amplitude. The features discussed above are illustrated in the graphs in a quantitative manner, and it will be seen that the corrected results display a striking modification.

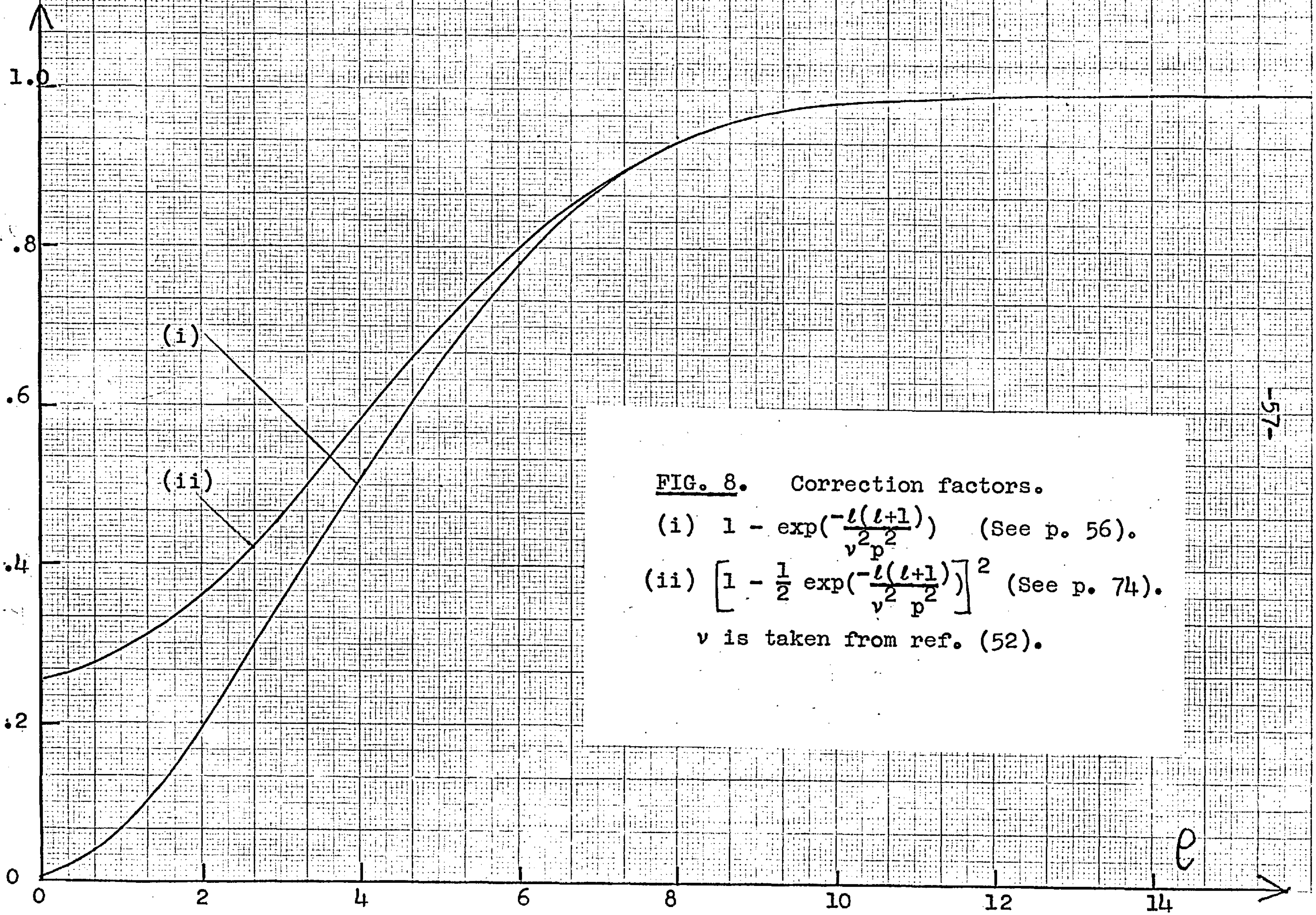


FIG. 8. Correction factors.

(i) $1 - \exp\left(-\frac{\ell(\ell+1)}{\nu^2 p^2}\right)$ (See p. 56).

(ii) $\left[1 - \frac{1}{2}\exp\left(-\frac{\ell(\ell+1)}{\nu^2\ p^2}\right)\right]^2$ (See p. 74).

ν is taken from ref. (52).

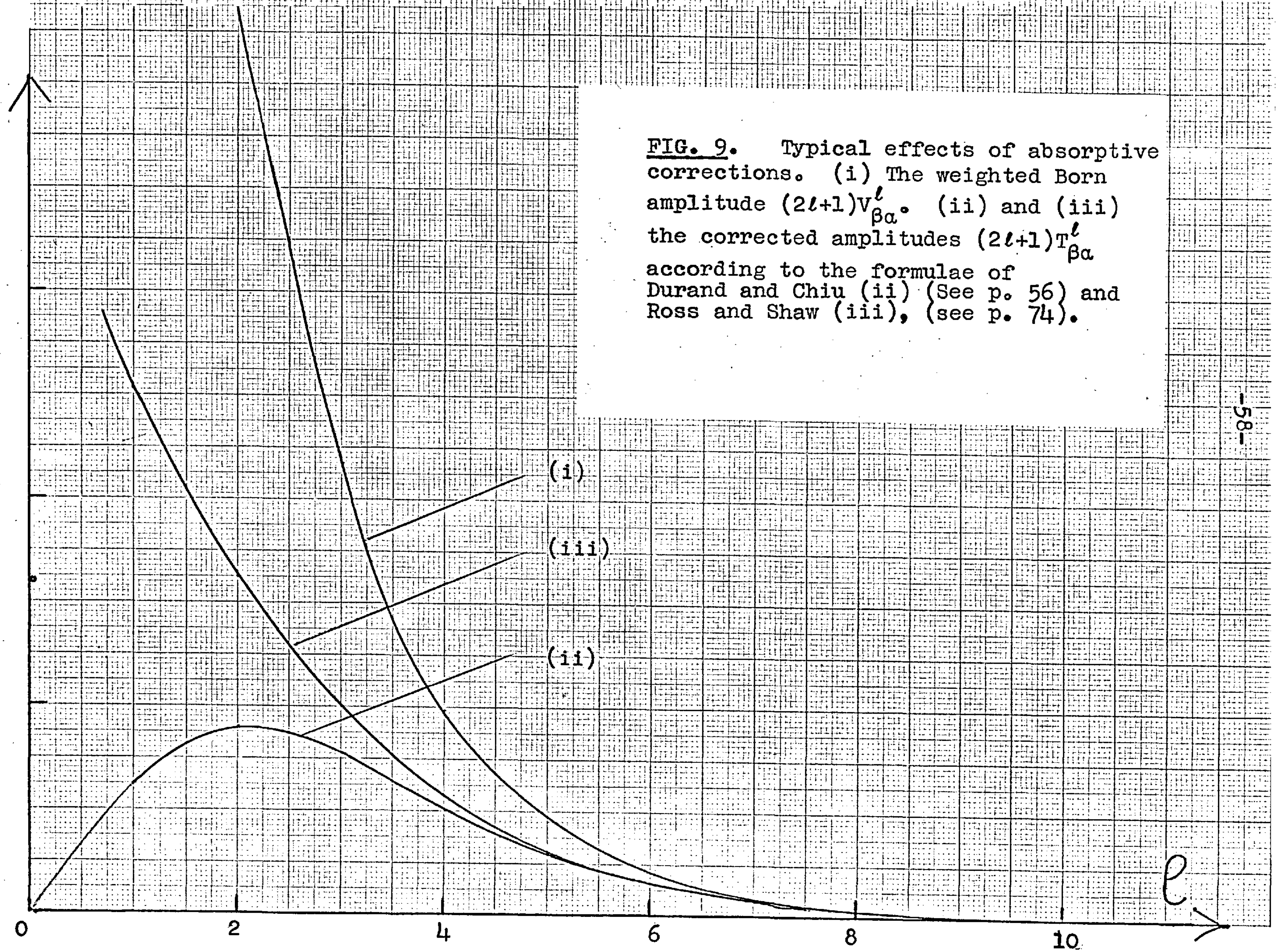


FIG. 9. Typical effects of absorptive corrections. (i) The weighted Born amplitude $(2\ell+1)V^{\ell}_{\beta\alpha}$. (ii) and (iii) the corrected amplitudes $(2\ell+1)T^{\ell}_{\beta\alpha}$ according to the formulae of Durand and Chiu (ii) (See p. 56) and Ross and Shaw (iii), (see p. 74).

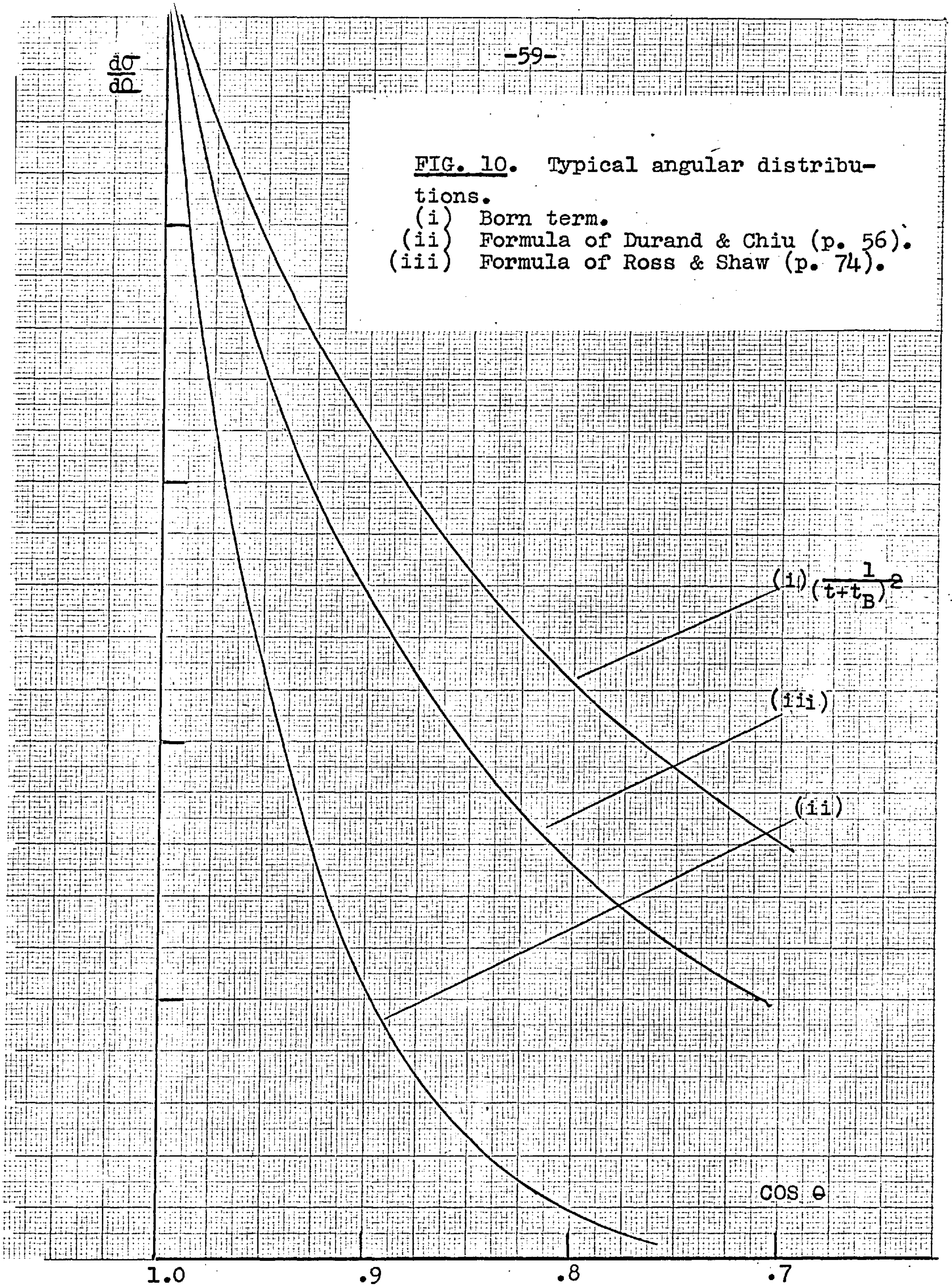


FIG. 10. Typical angular distributions.
(i) Born term.
(ii) Formula of Durand & Chiu (p. 56).
(iii) Formula of Ross & Shaw (p. 74).

(iv) The High Energy WKB Approximation.

We briefly outline here derivations of equation (9) and a similar result holding for $\mu \gg \nu$ which are based on a method due to Gottfried & Jackson[18]. We use the WKB approximation in the Schrodinger equation, so the treatment is non-relativistic. We take the incident beam parallel to OZ (unit vector $\hat{\underline{k}}$) and $\underline{b}$ to be a vector perpendicular to OZ, so that $b (= |\underline{b}|)$ is the impact parameter. The three-momentum transfer we denote by $\underline{\Delta}$. Glauber[29] has shown that the scattering amplitude is given by

$$f(\theta) = \frac{p}{i} \int_0^\infty J_o(\Delta b) \left(e^{i\chi(b)} - 1 \right) b db \qquad (11)$$

where

$$\chi(b) = - \frac{m}{p} \int_{-\infty}^{\infty} V(\underline{b} + \hat{\underline{k}} z) dz \quad . \qquad (12)$$

The Born approximation consists in setting

$$e^{i\chi(b)} - 1 \sim i\chi(b) \equiv 2ip\, B(b) \quad \text{say} \qquad (13)$$

so that (11) becomes

$$f(\theta) = 2p^2 \int_0^\infty J_o(\Delta b) B(b)\, b db \quad . \qquad (14)$$

Under the usual identification

$$pb \longrightarrow \ell + \tfrac{1}{2}$$

$$J_o(\Delta b) \longrightarrow P_\ell(\cos\theta)$$

we recover the usual partial wave expansion

$$f(\theta) = \sum (2\ell+1)\, B_\ell\, P_\ell\, (\cos\theta)$$

Clearly $\chi(b)$ is the phase shift suffered by a particle in traversing the potential V at impact parameter b.

To take into account the effects of absorption we must also include the extra phase shifts suffered by the particle in traversing the absorptive regions, in which we represent the absorption by complex potentials $U_{\alpha\alpha}$ (initial channel) and $U_{\beta\beta}$ (exit channel). We make use of the condition $p \gg \nu$, so that the wave function in the potential U can be taken of the form

$$\psi \sim e^{ikz} \rho(z)$$

where $\rho(z)$ is a slowly varying function of z. Inserting this form into the Schrodinger equation, and neglecting the $\ddot{\rho}$ term (small) we find for ρ

$$\rho(\underline{b} + \hat{\underline{k}}z) = e^{-\frac{im}{p}\int_{-\infty}^{z} (U(\underline{b} + \underline{k}z')dz'} \tag{15}$$

(This is the WKB approximation). The exponent in (15) represents an additional phase shift which we must include in (14). There are two cases:-

<u>Case (i) $p^{-1} \ll \mu \ll \nu$.</u>

In this case the incident particles first traverse the potential $U_{\alpha\alpha}$, then the potential $V_{\beta\alpha}$ (corresponding to $B_{\beta\alpha}$ in (14)), and finally the potential $U_{\beta\beta}$. We use the condition $\mu \ll \nu$ to approximate the first phase shift, say $\delta_{\alpha\alpha}$, by setting $z = 0$ in the limits of integration of (15), i.e. we consider the particle to have traversed the entire $U_{\alpha\alpha}$ potential up to $z = 0$ before it encounters the production potential. (15) gives

$$\delta_{\alpha\alpha}(\underline{b}) = \frac{-m}{p}\int_{-\infty}^{0} U_{\alpha\alpha}(\underline{b} + \hat{\underline{k}}z')dz' \qquad (16)$$

Now we note that the total phase shift at impact parameter b for a particle traversing the potential from $-\infty$ to $+\infty$ is

$$\frac{-m}{p}\int_{-\infty}^{+\infty} U_{\alpha\alpha}(\underline{b} + \hat{\underline{k}}z')dz'$$
$$= 2\delta_{\alpha\alpha}(\underline{b}) \qquad (17)$$

so we see that $\delta_{\alpha\alpha}(\underline{b})$ is just <u>the elastic scattering phase shift in the channel</u> α. The same is true for δ_{β}.

Including these extra phase shifts in (14) we have

$$f_{\beta\alpha}(\theta) = 2p^2\int_0^{\infty} J_0(\Delta b)e^{i\delta_{\beta\beta}} B_{\beta\alpha}(\underline{b})\, e^{i\delta_{\alpha\alpha}}\, bdb \qquad (18)$$

This is the impact parameter representation of the result (9).

Case (ii) $p^{-1} \ll \nu \ll \mu$.

Now the production potential is supposed to be of the greater range. There are two possibilities. The transition $\alpha \rightarrow \beta$ might take place before the particle traverses the absorptive central region, in which case it no longer "sees" the potential $V_{\beta\alpha}$ while leaving the interaction region. The production amplitude is then given by (11), (12) and (13), where the limits of integration in (12) now run from $-\infty \rightarrow 0$, so that $B(\underline{b})$ is replaced by $\frac{1}{2} B(\underline{b})$. The additional phase shift is now that arising from traversing the entire region of the potential $U_{\beta\beta}$ (from (16), with limits $\pm\infty$), i.e. $2\,\delta_{\beta\beta}$. The total contribution from these transitions is

$$f_{\beta\alpha}(\theta) = \tfrac{1}{2}\, 2p^2 \int_0^\infty J_o(\Delta b) e^{2i\delta_{\beta\beta}} B_{\beta\alpha}(\underline{b})\, b db \quad . \qquad (19)$$

There is a similar contribution for transitions made after traversing the central absorptive region

$$f_{\beta\alpha}(\theta) = \tfrac{1}{2}\, 2p^2 \int_0^\infty J_o(\Delta b) B_{\beta\alpha}(\underline{b})\, e^{2i\delta_{\alpha\alpha}}\, b db \quad . \qquad (20)$$

The total result is the sum of (19) and (20). Writing this in the conventional partial wave form

$$f_{\beta\alpha}(\theta) = \sum (2\ell+1)\tfrac{1}{2} \left[S^\ell_{\beta\beta}\, V^\ell_{\beta\alpha} + V^\ell_{\beta\alpha}\, S^\ell_{\alpha\alpha} \right] P_\ell(\cos\theta) \qquad (21)$$

where we have replaced $B_{\beta\alpha}(\underline{b})$ with $V^\ell_{\beta\alpha}$ rather than $B^\ell_{\beta\alpha}$ to conform with an earlier notation. Denoting the corrected partial wave amplitudes by $T^\ell_{\beta\alpha}$ the two results may be stated

$$p^{-1} \ll \nu \ll \mu \qquad T^{\ell}_{\beta\alpha} = \frac{1}{2}\left[S^{\ell}_{\beta\beta}\, V^{\ell}_{\beta\alpha} + V^{\ell}_{\beta\alpha}\, S^{\ell}_{\alpha\alpha}\right] \tag{22}$$

$$p^{-1} \ll \mu \ll \nu \qquad T^{\ell}_{\beta\alpha} = \sqrt{S^{\ell}_{\beta\beta}}\, V^{\ell}_{\beta\alpha}\, \sqrt{S^{\ell}_{\alpha\alpha}} \tag{23}$$

Of these results (23) is of longer standing. It was taken as a plausible ansatz by Sopkovich in 1962[25], and derived under the stated conditions by Durand and Chiu[26, 27] (1964), using involved arguments about the form of the radial wave function in the Schrodinger equation, which also led to (22). Gottfried and Jackson[18] derived (23) by a method similar to ours, which we have extended to deduce (22) also. A derivation of (23) has also been attempted by Ball and Frazer[32] on the basis of S-matrix theory, which work was, however, of an exploratory nature.

Due to the conditions $p \gg \mu^{-1}$, ν^{-1}, (22) and (23) are "high-energy non-relativistic results". In practice (since the conditions imply the Gev range) a relativistic theory is used to calculate V. Clearly, the generalisation of results based on the Schrodinger equation to a highly relativistic situation is an unsatisfactory procedure.

There are more serious limitations on the applicability of these formulae. The elastic scattering radius ν is typically $\sim$ 1f in most cases at high energies. Typical ranges for Born production forces are also of this order of magnitude, e.g. $m_{\pi}^{-1} \sim 2f$, $m_K^{-1} \sim \frac{1}{2}f$.

Therefore, in practice, the physical processes of interest do not fall within the domain of validity of either (22) or (23). This point is often passed over in applications and (23) is taken as a basis for calculation – the form (22) is not at all widely known.

At first sight (22) and (23) are very different results. However, a closer inspection shows they are in many ways similar. There is in both cases the symmetry between α and β; if we set $S^{\ell}_{\alpha\alpha} = S^{\ell}_{\beta\beta} = 1$ in either we recover the Born amplitude; and if we put $V^{\ell}_{\beta\alpha} = 1$ in either with $S_{\alpha\alpha} = S_{\beta\beta}$ we retrieve an S matrix for elastic scattering, as we might intuitively expect.

Further, we note that both (22) and (23) give expressions for $T^{\ell}_{\beta\alpha}$ in terms of $V^{\ell}_{\beta\alpha}$ multiplied by a "correction factor". In the case of (22) this correction factor is

$$\frac{S^{\ell}_{\beta\beta} + S^{\ell}_{\alpha\alpha}}{2} \tag{24}$$

and for (23) it is

$$\sqrt{S^{\ell}_{\beta\beta} \; S^{\ell}_{\alpha\alpha}} \tag{25}$$

We see that (24) is the arithmetic mean of the elastic scattering matrix elements, while (25) is their geometric mean, so all in all (22) and (23) are really rather similar.

We can set

$$S^{\ell} = 1 - 2i\, T'^{\ell} \tag{26}$$

where for the higher partial waves T'^{ℓ} will be small. For these

$$\sqrt{S^{\ell}_{\alpha\alpha}} = (1 - 2i\, T'^{\ell}_{\alpha\alpha})^{1/2} \sim 1 - iT'^{\ell}_{\alpha\alpha}$$

and similarly treat $\sqrt{S^{\ell}_{\beta\beta}}$ so that (23) becomes

$$T'^{\ell}_{\beta\alpha} = V'^{\ell}_{\beta\alpha}\left[1 - i(T'^{\ell}_{\alpha\alpha} + T'^{\ell}_{\beta\beta})\right] \tag{27}$$

In the notation of (26), (22) is identical to (27), so that for the higher partial waves (22) and (23) are equivalent.

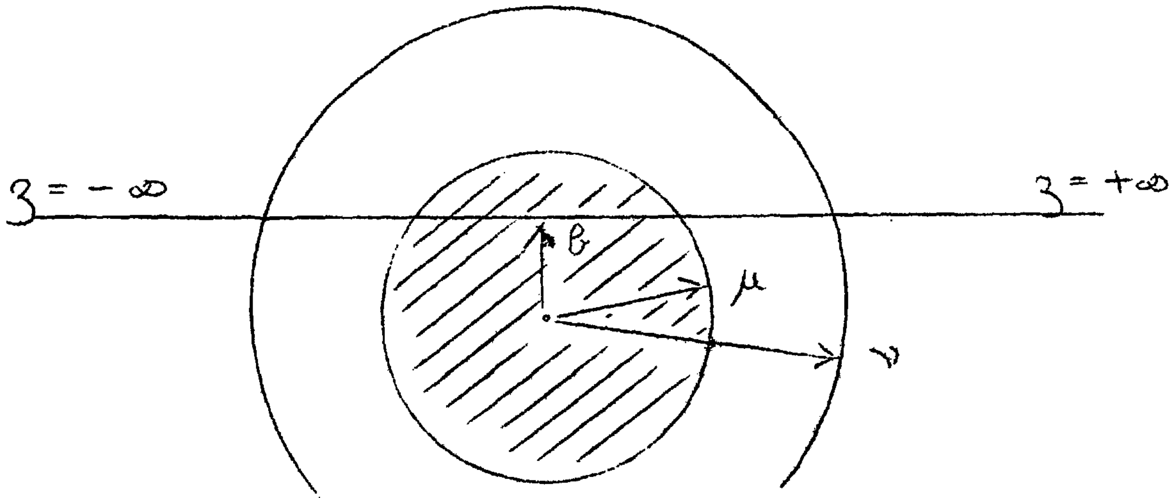
This equivalence of (22) and (23) is true even for the low partial waves if the initial and final state elastic scatterings are similar, $S^{\ell}_{\alpha\alpha} \sim S^{\ell}_{\beta\beta}$, in which case we can set

$$\sqrt{S^{\ell}_{\alpha\alpha}\, S^{\ell}_{\beta\beta}} \sim \frac{S^{\ell}_{\alpha\alpha} + S^{\ell}_{\beta\beta}}{2}$$

which approximation is valid to the second order.

These considerations strongly suggest that the restrictions on the ranges in (22) and (23) are somewhat artificial, and that it might be possible to derive a formula independent of the relative magnitudes of μ and ν, and for practical use this would be an extension of considerable importance.

As we have seen, (22) and (23) differ (at most) only for the low partial waves, and it is precisely for these waves that the derivations of the results is weakest. For $\nu \gg \mu$ the situation is as indicated:-



We have supposed (in our choice of limits in the integrals above) that the particle traverses the absorptive region from $z = -\infty$ to $z = 0$ <u>before</u> it encounters the peripheral force. For low impact parameters, this assumption is not strictly tenable, since the large absorption of partial waves with low b values arises from traversing the extremely absorptive central region, which necessarily coincides with the peripheral region, $b < \mu$. (In other words what we are saying is that the behaviour of the ℓ-th partial wave depends on the details of the details of the potential at distances $\sim \ell/p$; for $\ell/p < \mu$ we can no longer consider the particle as traversing first the absorptive region and secondly the production region).

Similar objections hold for the treatment of low partial waves for $\mu \gg \nu$, though here the approximations are more reasonable. Altogether, it seems quite plausible that a result might be found independent of the ranges.

In addition to the points we have made above ((i) non-relativistic treatment, (ii) the range restrictions, (iii) the approximations for low partial waves), there are additional strong reasons for seeking an alternative treatment of the effect. We have claimed that the inclusion of absorptive effects is a unitarity correction to the Born approximation, yet in the present treatment any relation with unitarity (e.g. through K-matrix formalism) is far from obvious. Again we are essentially including the effects of interactions in the initial and final states on our Born term, yet there is no apparent point of contact with the theory of final state interactions (as developed, e.g. by Watson[30] or Delbourgo[31]). It would be reassuring if we could make contact with K-matrix formalism and/or the theory of interactions in the final state, which steps would bring the work within the framework of conventional scattering theory. We turn, in the following sections to these questions.

A final remark might be added here: the form of (23), involving as it does <u>square roots</u> of matrix elements, seems to the present author quite unusual, and does not lead to simple visualisation. On the other hand we can easily see in an intuitive way how (22) might arise from successive elastic and inelastic interactions.

(v) Initial and Final State Interactions

We therefore now investigate the effects of absorption using the formalism developed by Watson[30] to treat interactions in the final state. (Using the very similar work of Delbourgo, and making parallel assumptions we reach the same results). We confine our attention to the case $\nu \gg \mu$, so that the elastic interactions are in the initial and final states.

Watson's work is based on the Lippmann-Schwinger treatment of scattering[33]. Incoming and outgoing eigenstates of the system are formally related to the corresponding free solutions by equations of the form

$$|E_o\rangle_{\pm} = \Omega_{\pm} |E_o\rangle$$

If we decompose the total Hamiltonian H into

$$H = H_o + \mathcal{V}$$

where H_o is the free Hamiltonian and $\mathcal{V}$ the interaction, then

$$\Omega_{\pm} = 1 + \frac{1}{E_o - H \pm i\varepsilon} \mathcal{V}$$

and $|E_o\rangle_{\pm}$ satisfy

$$|E_o\rangle_{\pm} = |E_o\rangle + \frac{1}{E_o - H_o \pm i\varepsilon} \mathcal{V} |E_o\rangle_{\pm}$$

where formally

$$\frac{1}{E_o - H_o \pm i\varepsilon} = P \frac{1}{E_o - H_o} \mp i\pi\, \delta(E_o - H_o) \qquad (28)$$

so that we have

$$\langle E | E_o\rangle_{\pm} = \langle E | E_o\rangle + P \frac{\langle E | \mathcal{V} | E_o\rangle_{\pm}}{E_o - E} \mp i\pi\, \delta(E_o - E) \langle E | \mathcal{V} | E_o\rangle_{\pm} \qquad (29)$$

The T matrix is given by

$$T = \mathcal{V}\Omega_{+} = \Omega_{-}^{+}\mathcal{V} \qquad (30)$$

and for the Born approximation we set Ω_{+} or Ω_{-} = 1 in (30).

At present we deal with only the initial state elastic interaction. We assume this to arise from a potential v ; in fact v is defined to accurately reproduce the elastic scattering. Then writing

$$\mathcal{V} = V + v$$

we can identify V with the potential $V_{\beta\alpha}$ provided that

$V_{\beta\alpha}$ does not make a large (indirect) contribution to elastic scattering, which we assume to be a valid assumption, since we are as usual at high energies and are supposing that very many channels are contributing to elastic scattering. Watson gives (Eq. 15, ref. 30) for the (total) T matrix:

$$T = V \Omega_{+} \omega_{+} \tag{31}$$

where $\omega_{\pm}$ are the "in" and "out" operators for a Hamiltonian $H = H_o + v$. We can further simplify (31) by setting

$$\Omega_{+} = \Omega_{o+}$$

where $\Omega_{o\pm}$ are "in" and "out" operators for a Hamiltonian $H = H_o + V$,* so that (31) becomes

$$T = V \Omega_{o+} \omega_{+} \tag{32}$$

This equation admits of a ready physical interpretation; due to the condition $\nu \gg \mu$ the term Ω_{+} in (30) has factored into $\Omega_{o+} \omega_{+}$, so that first v acts and then V.

Equation (32) gives

---

* i.e. we drop the second term of Watson's equation (22). This is usual, and is certainly justified in our case since $\nu \gg \mu$, and we can assume that once an interaction through V takes place, no further interaction through v occurs.

$$\langle E_\beta | T | E_\alpha \rangle = \int dE' \langle E_\beta | V \Omega_{0+} | E'_\alpha \rangle \rho(E'_\alpha) \langle E'_\alpha | E_\alpha \rangle_+ \tag{33}$$

where we have merely inserted a complete set of states (not showing explicitly the implied angular integrations), and where

$$|E_\alpha\rangle_+ = \omega_+ |E_\alpha\rangle \quad ,$$

i.e. $|E_\alpha\rangle_+$ is an "in" state of $H = H_0 + V$.

The essential nature of the WKB approximation is that it is an energy shell treatment. The wave suffers shifts in magnitude and in phase, but to the approximation used its energy is unaltered - hence the high energy condition, so that a potential encountered is small in magnitude compared to the particle energy. It is this fact which makes the absorption tractable, since otherwise we would need information (which we do not possess) on the off shell behaviour of the elastic amplitudes. The result obtained contains only physical (energy-shell) values - hence, amongst other consequences, its form as a relation between partial wave amplitudes.

We want to use this energy shell approximation, which we expect to be valid at high energies, to get a useful form for (33). We shall therefore approximate $\langle E'_\alpha | E_\alpha \rangle_+$ by neglecting the principal value integral in (29), so that we set

$$\langle E'_\alpha | E_\alpha \rangle_+ = \langle E'_\alpha | E_\alpha \rangle - i\pi\delta(E_\alpha - E'_\alpha) \langle E'_\alpha | V | E_\alpha \rangle_+ \qquad (34)$$

The neglect of the principal value integral, which is of the form

$$P \int_0^\infty \frac{f(E) \langle E | T | E_\alpha \rangle}{E_\alpha - E} \, dE$$

is argued on the grounds that the range of energies $E_\alpha - E = \Delta E$ in which $\langle E | T | E_\alpha \rangle$ is non-vanishing is much less than the range, $\sim E_\alpha$, permitted by the form of the integral. The range of permitted momentum transfer Δp, is $\Delta p \sim \nu^{-1}$, so that $\Delta p \sim p/m\nu$ while $E_\alpha \sim p^2/m$. If $p \gg \nu^{-1}$, we have $\Delta E \ll E_\alpha$.

Using (30), and replacing $\langle E'_\alpha | E_\alpha \rangle$ by the appropriate δ-function, (34) becomes

$$\langle E'_\alpha | E_\alpha \rangle_+ = \frac{\delta(E_\alpha - E'_\alpha)}{\rho(E_\alpha)} \left(1 - i\pi\rho \langle E_\alpha | T | E_\alpha \rangle \right).$$

Inserting this into (33) the energy integration is now trivial, and we have

$$\langle E_\beta | T | E_\alpha \rangle = \langle E_\beta | V \Omega_{\delta +} | E_\alpha \rangle \left(1 - i\pi\rho \langle E_\alpha | T | E_\alpha \rangle \right)$$

Setting for $V\Omega_{\delta+}$ the Born approximation, V, and including the angular integration which is implied in this equation we find

$$T^{\ell}_{\beta\alpha} = V^{\ell}_{\beta\alpha}\,(1 - i\pi\rho\, T^{\ell}_{\alpha\alpha})$$

or

$$T^{\ell}_{\beta\alpha} = V^{\ell}_{\beta\alpha}\,(1 - i\, T'^{\ell}_{\alpha\alpha})\ .$$

Clearly, including also the final state elastic interaction we have

$$T^{\ell}_{\beta\alpha} = (1 - i\, T'^{\ell}_{\beta\beta})\, V^{\ell}_{\beta\alpha}\,(1 - i\, T'^{\ell}_{\alpha\alpha})$$

which, written in terms of $S_{\alpha\alpha}$ and $S_{\beta\beta}$ to conform with our earlier habit becomes:

$$T^{\ell}_{\beta\alpha} = \left(\frac{1 + S^{\ell}_{\beta\beta}}{2}\right) V^{\ell}_{\beta\alpha}\left(\frac{1 + S^{\ell}_{\alpha\alpha}}{2}\right) \qquad (35)$$

This is our final result. It is the form assumed and used by Ross and Shaw[34]. As is seen, it differs again from our earlier results (22) and (23).

However, for small $T'^{\ell}_{\alpha\alpha}$, $T'^{\ell}_{\beta\beta}$ all three forms become identical: under these conditions we can neglect the term $T'^{\ell}_{\beta\beta}\ V^{\ell}_{\beta\alpha}\ T'^{\ell}_{\alpha\alpha}$ in (35) which then becomes identical to the limit of (23) (i.e. (27)), which is also the form (22).

On the other hand, for large $T'^{\ell}_{\alpha\alpha}$, $T'^{\ell}_{\beta\beta}$, (low ℓ values) this latest form differs appreciably from both (22) and (23). We have plotted out this third result in Figs. 8, 9 and 10 (p. 57-9) for the typical process discussed in section (iii), again assuming

$$S^{\ell}_{\alpha\alpha} = S^{\ell}_{\beta\beta} = 1 - \exp(\ell(\ell+1)/\nu^2 p^2).$$

It will be seen that the form (35) does not lead to such big changes in the peripheral amplitude. Because of the extra term in (35), which is not present in (22), $T^{\ell}_{\beta\alpha}$ does not go to zero as $\ell \to 0$, but to a limit of $(1/4)\ V^{\ell}_{\beta\alpha}$, and the low partial waves give rise to the marked difference between (35) and (27) or (23).

This behaviour of (35) does not seem physically reasonable. E.g. if (to take an extreme case) the elastic interactions were completely absorptive (black) for all ℓ, then according to (35), and contrary to our expectations the peripheral production amplitude is not completely masked out. It is therefore necessary to examine more closely our "energy shell" approximation of this present section, which leads us into K-matrix formalism.

(vi) K-matrix Models

Our approximation of the previous section consisted in neglecting the principal value integral in (33). This implies that for consistency we must also neglect the principal value term in (28); that is to say, we are essentially approximating the free causal propagator ,

$$\Delta = \frac{1}{E_o - H_o + i\varepsilon}$$

$$= P\frac{1}{E_o - H_o} - i\pi\,\delta(E_o - H_o)$$

by $\Delta = -i\pi\,\delta(E_o - H_o)$ (36)

Now the Schrodinger equation gives

$$T = V + V\Delta T \qquad (37)$$

so that with (36) we are taking

$$T = V - i\pi V\delta T \qquad (38)$$

where $\delta = \delta(E_o - H_o)$.

Now the T-matrix is related to the K-matrix by

$$T = K - i\pi K\delta T \quad . \qquad (39)$$

Comparing (38) and (39) we see that the approximation involved is equivalent to taking the K-matrix to be equal to V, the potential. (V is necessarily Hermitian.)

This approximation is well known and widely used in weak coupling theories. It necessarily gives a unitary T-matrix, and the resultant corrections to the Born approximation, $T = V$, give rise to damping effects, e.g. in the theory of the damping of EM transitions. In fact, a systematic development of perturbation theory[35,36] shows that this $(K = V)$ is the correct procedure - it was indeed from these considerations that the K-matrix was discovered.

However the approximation $K = V$ cannot be made for a strong coupling theory. We can illustrate the type of difficulties to which this would lead by considering a simple model with many open channels in which we suppose that the potentials $V_{\alpha\beta}$ are all equal

$$V_{\beta\alpha} = V = K_{\beta\alpha} \tag{40}$$

Now from (39)

$$T = K\,(1 + i\pi\,\delta\,K)^{-1} \tag{41}$$

It is possible to show that if $(1 + i\pi\,\delta\,K)$ is of the form in which all diagonal elements are equal, and all off diagonal elements are equal (which follows from (40)), then $(1 + i\pi\,\delta\,K)^{-1}$ is also of that form, say

$$\begin{aligned}(1 + i\pi\,\delta\,K)^{-1}_{\alpha\beta} &= d \qquad\qquad \alpha \neq \beta \\ &= \Delta \qquad\qquad \alpha = \beta \;.\end{aligned}$$

Then T is of the form

$$T = \begin{pmatrix} V & V & V \\ V & V & \\ V & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \end{pmatrix} \begin{pmatrix} \Delta & d & d & \cdot & \cdot \\ d & \Delta & d & & \\ d & d & \Delta & & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & & \cdot \end{pmatrix}$$

It follows that

$$T_{\alpha\beta} = T_{\alpha\alpha} \qquad \text{all } \alpha, \text{ all } \beta$$

i.e. that the elastic and inelastic amplitudes are the same.* This is a totally incorrect result. We would expect that as the number of channels, N, $\rightarrow \infty$, the elastic amplitude would become large and imaginary, (i.e. display the shadow of the inelastic processes) while the inelastic amplitudes would become small and largely real.

We now show how the result (35) can be derived directly from the K-matrix in the limit of weak coupling. (39) can be written in partial wave form

$$\langle E\ell|T|E\ell\rangle = \langle E\ell|K|E\ell\rangle - i\pi\int dE'\langle E\ell|K|E'\ell\rangle\,\rho(E')\,\delta(E-E')\times\langle E'\ell\,|\,T\,|\,E\ell\rangle$$

$$= \langle E\ell|K|E\ell\rangle - i\pi\,\rho(E)\langle E\ell|K|E\ell\rangle\langle E\ell|T|E\ell\rangle$$

i.e. $T^{\ell} = K^{\ell} - i\pi\rho\, K^{\ell}\, T^{\ell}$ (42)

Following Dalitz (37) we define for a matrix A a corresponding A' such that

---

* (41) can be solved explicitly, in this case, and the result given, and also the unitarity condition verified explicitly. E.J. Squires (private communication) has independently obtained the same result.

$$\langle E | A' | E' \rangle = \sqrt{\pi\rho(E)} \langle E | A | E' \rangle \sqrt{\pi\rho(E')}$$

so that (42) becomes

$$T'^{\ell} = K'^{\ell} - i\, K'^{\ell}\, T'^{\ell} \tag{43}$$

and

$$T'^{\ell} = K'^{\ell} (1 + i\, K'^{\ell})^{-1} \tag{44}$$

In this notation

$$S^{\ell} = 1 - 2i\, T'^{\ell} \tag{45}$$

(c.f. equation (26)). We consider a two channel process, explicitly inverting $(1 + i\, K'^{\ell})$ in (44). We find

$$T'^{\ell}_{\beta\alpha} = K'^{\ell}_{\beta\alpha} / \Delta \tag{46}$$

where $\Delta = \left| (1 + i\, K'^{\ell}) \right|$. (Determinant)

Now from (43)

$$K'^{\ell} = T'^{\ell} (1 - i\, T'^{\ell}) \tag{47}$$

so that from (44) and (47) we have

$$\Delta = 1/ \Delta'$$

where $\Delta' = \left| (1 - iT'^{\ell}) \right|$.

Therefore (46) becomes

$$T'^{\ell}_{\beta\alpha} = K'^{\ell}_{\beta\alpha}\, \Delta'$$

or, in full

$$T'^{\ell}_{\beta\alpha} = K'^{\ell}_{\beta\alpha} \left[(1 - iT'^{\ell}_{\alpha\alpha})(1 - iT'^{\ell}_{\beta\beta}) + T'^{\ell^2}_{\beta\alpha} \right] \qquad (48)$$

For weak coupling, i.e. small $T'^{\ell}_{\beta\alpha}$, we neglect the term $T'^{\ell^2}_{\beta\alpha}$ in (48), and set $K'^{\ell}_{\beta\alpha} = V'^{\ell}_{\beta\alpha}$ so that

$$T^{\ell}_{\beta\alpha} = (1 - iT'^{\ell}_{\beta\beta})\, V^{\ell}_{\beta\alpha}\, (1 - iT'^{\ell}_{\alpha\alpha}) \qquad (49)$$

which is the result (38).

This result, apart from being that used by Ross and Shaw, has also been given by various authors, e.g. Arnold[(38)] and Yonezawa[(39)]. We emphasize that it is only valid for small $T'^{\ell}_{\alpha\alpha}$, $T'^{\ell}_{\beta\beta}$, in which case all results derived by different methods reduce to the same form.

Dietz and Pilkuhn have also proposed setting $K = V$, and, observing that this would not give the elastic scattering (as we saw for a simple model above) have used the many particle reactions to give (via their shadow) the desired form for the elastic scattering. This seems to the present author to be unsatisfactory - in an acceptable theory the presence of very many two-particle channels ought to show up in elastic shadow scattering. Indeed we have seen (I, section ii)) that a very large part of the many body processes can be ascribed to quasi two particle processes.

Yonezawa (<u>loc. cit.</u>) has also suggested solving equation (48) as a quadratic equation for $T'^{\ell}_{\beta\alpha}$. This approach raises new difficulties* and does not appear to be applicable to the many channel situations of physical interest. We want to formulate a K-matrix treatment in which we set $K_{\beta\alpha} = V_{\beta\alpha}$ but assume no knowledge of other K-matrix elements, and to this problem we now turn.

(vii) K-matrix: Alternative Treatment

We wish to propose here an alternative relativistic treatment of the effect which is free from the defects discussed above. Since we are essentially concerned with finding a first order energy shell correction for unitarity to the Born amplitude, one would certainly expect the K-matrix to be the appropriate formalism, and we shall use this. In the spirit of the absorptive peripheral model, we shall retain only those terms which correspond to a reflection of the characteristics of the elastic amplitudes $T_{\alpha\alpha}$, $T_{\beta\beta}$ into the transition amplitude $T_{\beta\alpha}$.

From before we have

$$T' = K' - i K'T'$$

---

* Amongst which are an ambiguity of sign and the fact that we do not recover the Born term as absorption $\rightarrow$ 0.

and

$$T' = K' (1 + i K')^{-1}$$

so that

$$T' = K' - i K'^2/(1 + i K')^{-1} . \qquad (50)$$

Inserting indices to label the states,

$$T'^{\ell}_{\beta\alpha} = K'^{\ell}_{\beta\alpha} - i \sum_{\gamma\gamma'} K'^{\ell}_{\beta\gamma} \left(\frac{1}{1 + i K'}\right)^{\ell}_{\gamma\gamma'} K'^{\ell}_{\gamma'\alpha} .$$

The first term on the right hand side is to be approximated by the Born term, the second represents the unitarity correction. The structure of this correction term is complex, as one would expect. Our approach will be to approximate it by two contributions. We envisage (at high energies) the situation where very many channels are open. We therefore expect a typical inelastic amplitude to be small, whereas the elastic amplitudes, induced by absorption into the many open inelastic channels, will be large. We observe therefore that of all terms of the form

$$\sum_{\gamma} K'^{\ell}_{\beta\gamma} \left(\frac{1}{1 + i K'}\right)^{\ell}_{\gamma\gamma'} = T'^{\ell}_{\beta\gamma'}$$

the largest will be given when $\gamma' = \beta$, and we approximate the summation over γ' by setting $\gamma' = \beta$. This gives a correction

$$i\, T'_{\beta\beta}\, K'_{\beta\alpha} .$$

In a similar manner we can include many terms in the summation over γ by setting $\gamma' = \beta$. This gives a correction

$$i\, K'_{\beta\alpha}\, T'_{\alpha\alpha} \quad .$$

In this way we represent the unitarity correction by what we would expect to be the two largest contributions, so that

$$T'^{\ell}_{\beta\alpha} = K'^{\ell}_{\beta\alpha} - i(T'^{\ell}_{\beta\beta}\, K'^{\ell}_{\beta\alpha} + K'^{\ell}_{\beta\alpha}\, T'^{\ell}_{\alpha\alpha}) \; .$$

It is clear of course that we have neglected many terms. However, it might be argued following various authors(41,42) that it is reasonable to expect the sum

$$\sum_{\gamma' \neq \beta} T'^{\ell}_{\beta\gamma'}\, K'^{\ell}_{\gamma'\alpha}$$

to be small, since it is composed of many small terms, the phases of which might be supposed to vary in a random manner. At any rate, such an approximation is inherent in the idea of absorptive corrections and it should be regarded as an advantage rather than a defect of the present procedure that the nature of the approximation is thus displayed.

Finally, we set $K'_{\beta\alpha} = V'_{\beta\alpha}$, the peripheral assumption, so that

$$T'^{\ell}_{\beta\alpha} = V'^{\ell}_{\beta\alpha} - i(T'^{\ell}_{\beta\beta}\, V'^{\ell}_{\beta\alpha} + V'^{\ell}_{\beta\alpha}\, T'^{\ell}_{\alpha\alpha}) \qquad (51)$$

Since T' is related to S by

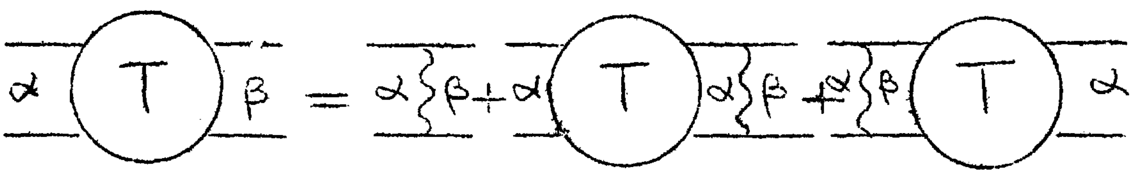
$$S = 1 - 2i\, T'$$

this result may be put in the form

$$T_{\beta\alpha} = \tfrac{1}{2}\left[S^{\ell}_{\beta\beta}\, V^{\ell}_{\beta\alpha} + V^{\ell}_{\beta\alpha}\, S^{\ell}_{\alpha\alpha} \right]$$

and is thus identical to (22).

The expression (51) has considerable intuitive appeal. To the Born term are added two other terms corresponding to the additional effect of the initial and final state interactions, which we can represent diagrammatically:-



The reader will ask why we did not take equation (39)

$$T'^{\ell}_{\beta\alpha} = K'^{\ell}_{\beta\alpha} - i \sum_{\gamma} K'^{\ell}_{\beta\gamma}\, T'^{\ell}_{\gamma\alpha} \tag{52}$$

and directly approximate this by setting $\gamma = \alpha$, obtaining

$$T'^{\ell}_{\beta\alpha} = V'^{\ell}_{\beta\alpha}\,(1 - i\, T'^{\ell}_{\alpha\alpha}) \tag{53}$$

This form is clearly not correct, as it lacks the necessary symmetry between α and β. Again, we could have written (52) (since K and T commute) as

$$T'^{\ell}_{\beta\alpha} = K'^{\ell}_{\beta\alpha} - i \sum_{\gamma} T'^{\ell}_{\beta\gamma}\, K'^{\ell}_{\gamma\alpha} \tag{54}$$

Setting $\Upsilon = \beta$ here gives

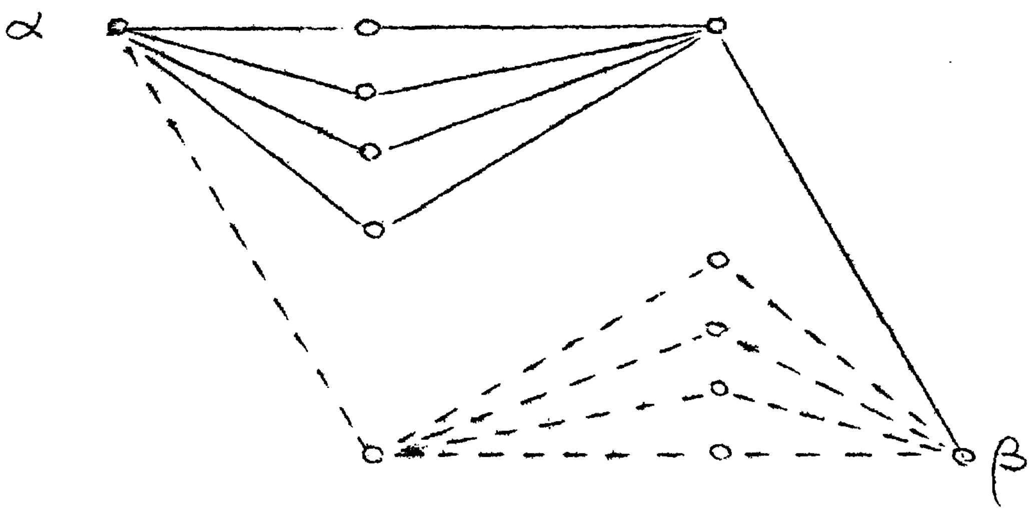
$$T'^{\ell}_{\beta\alpha} = V'^{\ell}_{\beta\alpha} (1 - i \, T'^{\ell}_{\beta\beta}) \quad . \qquad (55)$$

Now we cannot argue that (53) and (54) are identical since in general $T_{\beta\beta} \neq T_{\alpha\alpha}$. One might suppose that the correct form was a symmetrised form of (53) and (55), i.e.

$$T'^{\ell}_{\beta\alpha} = V'^{\ell}_{\beta\alpha} (1 - \tfrac{1}{2} i (T'^{\ell}_{\alpha\alpha} + T'^{\ell}_{\beta\beta})) \qquad (56)$$

but this cannot be correct since it is a purely optional procedure. In addition (56) is wrong, since for complete initial and final state absorption $T'^{\ell}_{\beta\alpha}$ does not go to zero.

We have therefore to be more careful, and accordingly we have worked in a fashion such that the symmetry between $\alpha + \beta$ is displayed at all stages. We see in this way that both the extra terms of (53) and (55) should properly be included. Nor is it the case that we are counting terms twice in (51). We can schematically indicate the origin of our two extra terms, denoting $T'^{\ell}_{\beta\beta} V'^{\ell}_{\beta\alpha}$ by full lines, and $V'^{\ell}_{\beta\alpha} T'^{\ell}_{\alpha\alpha}$ by broken :-



The main features of the present treatment are its complete relativistic validity, and the absence of any restrictions on the ranges of the interactions,involved. In addition, our derivation of (51) does not imply any unreasonable form for the elastic amplitudes, and is clearly valid for the case of physical interest, i.e. many channels open.

We summarise our conclusions on the various approaches in the next section.

(viii) Discussion & Conclusions

We briefly summarise in Table 2 (p. 88) the various results which we have discussed in the previous seven sections.

We noted that all the results give the same result for

$S^{\ell}_{\alpha\alpha}$, $S^{\ell}_{\beta\beta} \sim 1$, i.e. weak absorption or high partial waves. However, the contribution of low partial waves (high absorption) is of crucial importance to the peripheral model; these are so very large in the Born term that it matters greatly whether they are completely damped out or merely reduced by a factor of, say, one quarter. It is therefore important to assess the relative merits of the various results.

We saw that the form (35) involved approximations that were only valid for weak couplings, and that this particular result seemed unreasonable in that a substantial inelastic amplitude still remained even if both the elastic interactions were completely "black" for all ℓ. We also anticipated (in view of the similarity of (22) and (23)) that it would be possible to find a result independent of the ranges μ and ν. The derivation of (22) which we have given in the preceding section is indeed independent of these. This treatment is in addition fully relativistic, and, depending as it does on K-matrix formalism, is within the conventional framework of high energy physics, unlike either the distorted wave Born approximation or the WKB approximation. In addition, the form (22) admits of a ready "intuitive" interpretation.

For all these reasons it seems to the present author that (22) provides the most satisfactory basis for calculation.

Form for $T^{\ell}_{\beta\alpha}$	Numbered in text	Relativistic or not	Conditions	Method	Author
$\sqrt{S^{\ell}_{\beta\beta}}\,V^{\ell}_{\beta\alpha}\sqrt{S^{\ell}_{\alpha\alpha}}$	23	non-rel.	–	–	Sopkovich
		"	$p^{-1} \ll \mu \ll \nu$	Distorted wave Born approx.	Durand & Chiu
		"	"	WKB	Gottfried & Jackson
$\frac{1}{2}\left[S^{\ell}_{\beta\beta}V^{\ell}_{\beta\alpha} + V^{\ell}_{\beta\alpha}S^{\ell}_{\alpha\alpha}\right]$	22	non-rel.	$p^{-1} \ll \nu \ll \mu$	DWBA	Durand & Chiu
		"	"	WKB	Watson (following Gottfried & Jackson)
		relativistic	high E	K-matrix	Watson
$\frac{(1+S^{\ell}_{\beta\beta})V^{\ell}_{\beta\alpha}(1+S^{\ell}_{\alpha\alpha})}{4}$	35	non-rel.	$p^{-1} \ll \mu \ll \nu$	initial & final state interactions	Ross & Shaw
		relativistic	high E	K-matrix	Arnold
		"	–	"	Yonezawa

TABLE 2

An important ommission in all our previous discussion is that of spin considerations. In the K-matrix formalism, however, this may readily be included. It merely involves having extra labels for our matrix elements. Denoting, in a symbolic manner, the various helicities involved in the helicity representation of Jacob and Wick[43] by λ, (22) becomes

$$\langle \lambda_\beta | T^J_{\beta\alpha} | \lambda_\alpha \rangle = \frac{1}{2}\left[\sum_{\lambda'_\beta} \langle \lambda_\beta | S^J_{\beta\beta} | \lambda'_\beta \rangle \langle \lambda'_\beta | V^J_{\beta\alpha} | \lambda_\alpha \rangle + \sum_{\lambda'_\alpha} \langle \lambda_\beta | V^J_{\beta\alpha} | \lambda'_\alpha \rangle \langle \lambda'_\alpha | S^J_{\alpha\alpha} | \lambda_\alpha \rangle \right] \qquad (52)$$

Gottfried & Jackson[18] have stressed the importance of the proper inclusion of spin, even where the angular distribution is concerned. Different helicity amplitudes will be modified in a way which depends on their exact form (i.e. composition in terms of partial wave amplitudes), and it is not satisfactory to take some "mean" amplitude, neglecting spin -- amongst other things, spurious diffraction zeros may be introduced.

PART III

$\widetilde{U}(12)$ ABSORPTION MODEL FOR $p\bar{p} \rightarrow \Lambda\bar{\Lambda}$

(i) Introduction

In this final part of the work we present a calculation of $p\bar{p}\rightarrow\Lambda\bar{\Lambda}$ using a much improved model, in the light of the discussion of Section (vii) of Part I. The treatment is still based on the Born amplitude, i.e. the "peripheral" idea, but we make use of the $\tilde{U}(12)$ symmetry scheme[19] to write down completely unambiguous and realistic vertices, and we fully incorporate the effects of absorption which we have discussed at length in Part II.

In choosing to apply the absorption model to $p\bar{p}\rightarrow\Lambda\bar{\Lambda}$ we have borne in mind the most serious practical difficulty in the use of the model, namely, that one requires information on the nature of the elastic interactions in both the entrance and exit channels. In general (as in the present case) one knows nothing about the elastic scattering characteristics of the final state channel. However, p and Λ both being spin one-half baryons with approximately equal mass, it seems plausible to take $\Lambda\bar{\Lambda}$ elastic scattering as identical to $p\bar{p}$ - one would certainly be surprised if the two interactions were very dissimilar.

Our pole approximations of Part I were carried out in 1963, and it is necessary briefly to summarise developments in the theoretical study of $p\bar{p}\rightarrow\Lambda\bar{\Lambda}$ between that time and the present (1965). Several authors have investigated the process:-

(i) Chan[44] confirmed our result of Part I in a study which established that (at least at present energies) the Reggeisation of K^* would not affect the model.

(ii) Durand and Chiu[17] applied their absorption formula, and confirmed the dominant role of K^* - exchange against K-exchange, finding the latter to be too forwardly peaked. (See Part II of the present work. It was indeed this work of Durand and Chiu which established the importance of absorptive effects (1964)).

(iii) Cohen-Tannoudji and Navelet[45] repeated the calculation of Durand and Chiu taking spin fully into account. K^* was again found to play the dominant role, though the conclusion in this case was on the grounds that K-exchange was not sufficiently forwardely peaked.

(iv) All the works mentioned above assumed a γ_μ type coupling at the $K^* N \Lambda$ vertex (c.f. I, Section (vi)). In fact, since 1963 the evidence for a large magnetic type interaction of vector mesons has somewhat hardened — for example, see the "ρ -photon analogy" of Stodolsky and Sakurai[7]. Högassen and Högassen[46] generalised the K^*-exchange absorption model to include mixed couplings, and concluded that if the "magnetic" term was more than approximately the same size as the "electric" one, the model would no longer fit (this is roughly in accord with our findings in

Section (vi) Part I). However, it seems possible that the ratio "magnetic":"electric" may be as much as 3 - 4 : 1 .

All of these authors admit the possibility of a mixture of K and K^* terms, and it might be possible to fit the data this way even with a large K^* "magnetic" coupling. This is an example of the considerable freedom of manoeuvre one has in the choice of parameters for peripheral calculations in the absence of any restricting principle. It would seem much preferable to start with the $\tilde{U}(12)$ symmetry in which the vertices are fixed and to compare the results against experiment.

In the $\tilde{U}(12)$ scheme the baryons are degenerate and all of mass m, as also are the mesons (mass μ). The ratio "magnetic" : "electric" is (see Section (iii)) $1 + \frac{2m}{\mu} : 1$, i.e. $\sim 3 : 1$, and this is realistic.

From the point of view of the $\tilde{U}(12)$ theory the reaction $p\bar{p} \to \Lambda\bar{\Lambda}$ must be considered as, on the face of it, fairly promising for treatment in a peripheral approximation. The mass difference between p (938 Mev) and Λ (1115 Mev) is small, while the difference between the K mass (494 Mev) and that of K^* (891 Mev) is at least much less than the full variation of the meson masses, e.g. from the pion (135 Mev) to ϕ (1019 Mev). In addition, the experimental results are as peripheral as one could hope. The $\tilde{U}(12)$ S-matrix is known to violate unitarity, but with the absorptive correction we are approximately enforcing unitarity on our peripheral amplitude. Altogether, it would seem that if the $\tilde{U}(12)$ absorption model is inadequate for $p\bar{p}\rightarrow\Lambda\bar{\Lambda}$, it might not be adequate for any reaction.

To help the reader follow the line of the following somewhat algebraically involved work, we give an outline of the various steps of the calculation.

To include spin, we must know the sixteen amplitudes involved, $\langle \pm \pm | \phi | \pm \pm \rangle$. In the next section (Section (ii)) we show that because of C and P invariance only six of these are independent, say, ϕ_i , $i = 1, \ldots\ldots, 6$. In Section (iii) we give the $\tilde{U}(12)$ prescriptions for the currents involved, and the corresponding amplitudes written in terms of Dirac spinors. From these we then find explicitly the various helicity amplitudes, $\phi_i(\theta)$, which involves tedious algebra (Section (iv)). In Section (v) we show how to project out from the $\phi_i(\theta)$ the partial wave helicity amplitudes, ϕ_i^j. Our absorption formalism then gives us (Section (vi)) a set of "unitarised" partial wave amplitudes, $\bar{\phi}_i^{\,j}$ in terms of the ϕ_i^j. In the same section we see how to arrive at the corrected cross-section. Finally, our results are presented and discussed in Section (vii).

(ii) The Requirements of C and P Invariance.

In what follows, we make much use of widely known and elegant helicity formalism developed by Jacob and Wick for the treatment of particles with spin[43]. We label the helicities of the particles involved as follows:-

particle	$\bar{\Lambda}$	Λ	$\bar{p}$	p
helicity	λ_3	λ_4	λ_1	λ_2

The λ's give the spin projections along the direction of motion. The amplitude for scattering through an angle θ may be expanded in terms of the amplitudes for scattering in states of specified total angular momentum j :

$$\langle\lambda_3\lambda_4|\phi(\theta)|\lambda_1\lambda_2\rangle = \sum_j (2j+1)\langle\lambda_3\lambda_4|\phi^j|\lambda_1\lambda_2\rangle \; d^j_{\lambda\mu}(\theta) \qquad (1)$$

where $\lambda = \lambda_1 - \lambda_2$

and $\mu = \lambda_3 - \lambda_4$

and where we have taken the azimuthal angle ϕ to be zero without loss of generality.

The states $|j\;\lambda_1\;\lambda_2\rangle$ transform under the parity operator as

$$P|j\;\lambda_1\;\lambda_2\rangle = |j\;-\lambda_1\;-\lambda_2\rangle$$

The assumption of P invariance (this is a strong interaction process) therefore gives from (1),

$$\langle\lambda_3\lambda_4|\phi(\theta)|\lambda_1\lambda_2\rangle = \sum_j (2j+1)\langle-\lambda_3-\lambda_4|\,\phi^j|-\lambda_1-\lambda_2\rangle\, d^j_{\lambda\mu}(\theta)$$

Using the symmetry property of the d functions

$$d^j_{\lambda\mu}(\theta) = (-1)^{\lambda-\mu}\, d^j_{-\lambda-\mu}(\theta)$$

we have

$$\langle\lambda_3\lambda_4|\phi(\theta)|\lambda_1\lambda_2\rangle = \sum (2j+1)(-1)^{\lambda-\mu}\langle\lambda_3-\lambda_4|\,\phi^j\,|-\lambda_1-\lambda_2\rangle d^j_{-\lambda-\mu}(\theta)$$

$$= (-1)^{\lambda-\mu}\langle-\lambda_3-\lambda_4|\,\phi(\theta)|-\lambda_1-\lambda_2\rangle \qquad (2)$$

C-invariance gives us

$$\langle\lambda_3\lambda_4|\phi^j|\lambda_1\lambda_2\rangle = \langle\lambda_4\lambda_3|\phi^j|\lambda_2\lambda_1\rangle \qquad (3)$$

i.e. a particle of helicity λ is transformed into an antiparticle of helicity λ. Making use again of the property of the d functions quoted above, we find in a similar way that C-invariance gives

$$\langle\lambda_3\lambda_4|\phi(\theta)|\lambda_1\lambda_2\rangle = (-1)^{\lambda-\mu}\langle\lambda_4\lambda_3|\phi(\theta)|\lambda_2\lambda_1\rangle \qquad (4)$$

In the present case we cannot make use of T-invariance, since this would relate the amplitudes for $p\bar{p}\to\Lambda\bar{\Lambda}$ to those for the inverse reaction, $\Lambda\bar{\Lambda}\to p\bar{p}$.

Using (2) and (4) we find that the sixteen amplitudes have the following form, in an obvious notation.

$p\bar{p}$ / $\Lambda\bar{\Lambda}$	++	+−	−+	−−
++	ϕ_1	ϕ_5	$-\phi_5$	ϕ_2
+−	ϕ_6	ϕ_3	ϕ_4	ϕ_6
−+	$-\phi_6$	ϕ_4	ϕ_3	$-\phi_6$
−−	ϕ_2	ϕ_5	$-\phi_5$	ϕ_1

There are thus six independent amplitudes. This situation should be compared with the case of nucleon nucleon scattering, where T-invariance gives one extra restriction and the number of amplitudes is reduced to five. The table above is also that given in reference 11.

(iii) The $\tilde{U}(12)$ Interaction

The $\tilde{U}(12)$ predictions for those parts of the pseudoscalar and vector currents relevant to the interactions of the eightfold baryons with the mesons are

$$J_5 = (1 + \frac{2m}{S}) \frac{P^2}{4m^2} (\bar{N} \gamma_5 N)_D + \frac{2}{3}F$$

$$J_\mu = (1 + \frac{q^2}{2Vm}) \frac{P_\mu}{2m} (\bar{N} N)_F + (1 + \frac{2m}{V})(\bar{N} \frac{r_\mu}{4m^2} N)_D + \frac{2}{3}F$$

q_μ and P_μ are defined in terms of the incoming and outgoing baryon momenta, p_μ and p_μ' :

$$q_\mu = p_\mu - p_\mu' \qquad \text{(momentum transfer)}$$

$$P_\mu = p_\mu + p_\mu'$$

Note that $P^2 + q^2 = 4m^2$. (m is the baryon mass.)

r_μ is defined by

$$r_\mu = \varepsilon_{\mu\nu k\lambda} P_\nu q_k \gamma_\lambda \gamma_5 ,$$

and $P_\mu/2m$ and $r_\mu/4m^2$ are the conventional forms for "electric" and "magnetic" interactions(15). D and F refer to the SU(3) invariant symmetric and antisymmetric combinations.

In the $\tilde{U}(12)$ theory $S = V = \mu$, the "meson mass". However, we wish to admit the possibility of setting S and V to be different, and of thus splitting the scalar and vector octets "by hand".

The $p\Lambda K$ and $p \Lambda K^*$ interaction Lagrangians are of the form

$$L_S \sim \bar{p}\, j_5 \Lambda\, \phi_\mu^{K^+} \qquad (+\text{ h.c.})$$

$$L_V \sim \bar{p}\, j_\mu \Lambda\, \phi_\mu^{K^{*+}} \qquad (+\text{ h.c.})$$

From (6) we have

$$j_5 = (1 + \frac{2m}{S}) \frac{P^2}{4m^2} \gamma_5$$

$$j_\mu = F_E \frac{P_\mu}{2m} + F_M \frac{r_\mu}{4m^2}$$

where

$$F_E = (1 + \frac{q^2}{2mV})$$

$$F_M = (1 + \frac{2m}{V}) \quad .$$

As is well known(47), the SU(3) combinations give rise to the following Lagrangians

$$D \Longrightarrow -\frac{1}{\sqrt{3}} \bar{p} \, j_5 \wedge \phi_5^{K^+}$$

$$F \Longrightarrow -\sqrt{3} \, \bar{p} \, j_5 \wedge \phi_5^{K^+}$$

$$\therefore \quad D + \tfrac{2}{3}F \Longrightarrow (-\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}}) \, \bar{p} \, j_5 \wedge \phi_5^{K^+}$$

$$= -\sqrt{3} \, \bar{p} \, j_5 \wedge \phi_5^{K^+}$$

so that in this case F and $D + \frac{2}{3}F$ give rise to the same factor of $-\sqrt{3}$. (We have taken the scalar case here; for K^*, the argument is identical). We therefore have

$$L_S = -\sqrt{3} \, \bar{p} \, j_5 \wedge \phi_5^{K^+}$$

$$L_V = -\sqrt{3} \, \bar{p} \, j_\mu \wedge \phi_\mu^{K^{*+}} .$$

For subsequent algebraic work it is convenient to re-write j_μ in terms of γ_μ and P_μ rather than r_μ and P_μ. (This conversion is normal before explicit calculation is carried out[48].) The appropriate expression may be derived fairly readily[48], and is

$$j_\mu = (1 - \frac{q^2}{4m^2}) F_E \gamma_\mu - (F_M - F_E) \frac{P_\mu}{2m} .$$

Explicitly,

$$F_M - F_E = (1 + \frac{2m}{V}) - (1 + \frac{q^2}{2mV})$$
$$= \frac{2m}{V} (1 - \frac{q^2}{4m^2})$$

Finally, including a factor of $-\sqrt{3}$ from SU(3) considerations we have

$$j_\mu = \sqrt{3} (1 - \frac{q^2}{4m^2}) \left[-(1 + \frac{2m}{V}) \gamma_\mu + \frac{1}{V} P_\mu \right] \qquad (7)$$

For j_5 we have

$$j_5 = -\sqrt{3} (1 - \frac{q^2}{4m^2})(1 + \frac{2m}{S}) \gamma_5 \qquad (8)$$

where we have replaced P^2 by $(1 - q^2/4m^2)$.

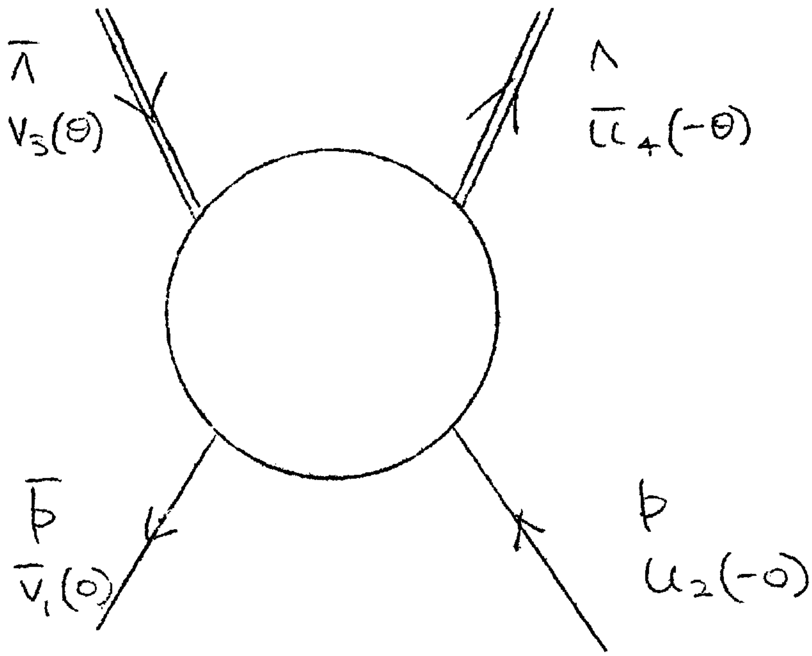
We make some observations on the form of these currents. They both contain an extra "kinematical" form factor, $(1 - q^2/4m^2)$, which is a special feature of $\widetilde{U}(12)$, in addition to the types of term one would expect, on general grounds, to be present. Since for physical processes $q^2 < 0$, this factor will tend to emphasize backward scattering. Setting $q^2 = -2p^2(1 - \cos\theta)$, we may estimate the importance of this term by putting $p = m$, which implies an energy $\sim$ 3 GeV. We find

$$\left(1 - \frac{q^2}{4m^2}\right) \sim \tfrac{1}{2}(3 - \cos\theta)$$

and therefore the term varies between 1 $(\theta = 0)$ and 2 $(\theta = \pi)$. The cross-section will contain $(1 - q^2/4m^2)^4$, and therefore the backward events are 16 times more favoured than the forward. We will see later that the $\tilde{U}(12)$ amplitude with no absorptive corrections rises steadily with θ, in contrast to the experimental results. This feature arises from the term $(1 - q^2/4m^2)$. On the other hand, we can see that this term will be almost completely removed by the absorptive corrections since it will contribute mainly to the very low partial waves which are destroyed in the absorption model. We merely comment on these features here to give the reader a qualitative understanding of the results which we present later.

(iv) The Helicity Amplitudes $\phi_i(\theta)$.

We label our particles as shown below



The indices 1, 2, 3, 4 correspond to λ_1, λ_2, λ_3, λ_4, where the λ's take the values ± 1, corresponding to helicities $\pm \frac{1}{2}$. For brevity, we refer to λ as the helicity. By $U(-\theta)$ etc. we mean a particle travelling in the direction physically opposite to that specified by θ, $\phi = 0$. (This notation will not lead to any confusion). Note that we have chosen the incoming anti-proton to correspond to $\theta = 0$, while θ gives the angle between incoming antiproton and outgoing anti-lambda. This corresponds to the experimental arrangement.

We now construct explicitly the helicity states involved. We use the metric

$$g_{\mu\nu} = (1 , -1 , -1 , - 1)$$

and take for our γ matrices the Dirac representation:-

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1$$

$$\gamma_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \sigma_i$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

A four momentum is specified, in the centre of mass, by

$$p = (E, \underline{p})$$

and we set

$$\varepsilon = \sqrt{E + m} \quad .$$

In this notation solutions of the Dirac equation for particles moving in the direction $\underline{p}$ with spins parallel or antiparallel to the z-axis are

$$U_{\uparrow}(\underline{p}) = \begin{pmatrix} \varepsilon \\ \frac{\underline{p}\cdot\underline{\sigma}}{E} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$U_{\downarrow}(\underline{p}) = \begin{pmatrix} \varepsilon \\ \frac{\underline{p}\cdot\underline{\sigma}}{\varepsilon} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If $\underline{p}$ is parallel to the z-axis ($\theta = 0$), then these states are clearly helicity states, with $\underline{p}.\underline{\sigma} = \pm p$. Denoting

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ by } \chi_{+1} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ by } \chi_{-1}$$

we have

$$U_\lambda(0) = \begin{pmatrix} \varepsilon \\ \frac{\lambda p}{\varepsilon} \end{pmatrix} \chi_\lambda$$

The states with helicity λ moving in the opposite direction are clearly given by

$$U_\lambda(-0) = \begin{pmatrix} \varepsilon \\ \frac{\lambda p}{\varepsilon} \end{pmatrix} \chi_{-\lambda} \qquad (9)$$

since $\underline{\sigma}.\underline{p}$ is unaltered, but the absolute direction of spin is reversed.

A state moving in direction θ may be obtained by applying a rotation through θ around the y-axis

$$U_\lambda(\theta) = \begin{pmatrix} \varepsilon \\ \frac{\lambda p}{\varepsilon} \end{pmatrix} e^{-\frac{1}{2}i\sigma_2\theta} \chi_\lambda$$

Explicitly

$$e^{-\frac{1}{2}i\sigma_2\theta} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}$$

where

$$C = \cos\theta/2$$

$$S = \sin\theta/2 \quad .$$

Similarly

$$U_\lambda(-\theta) = \begin{pmatrix} \varepsilon \\ \frac{\lambda p}{\varepsilon} \end{pmatrix} e^{-\frac{1}{2}i\sigma_2\theta} \chi_{-\lambda}$$

$$\therefore \quad \bar{U}_\lambda(-\theta) \quad = \quad \chi^+_{-\lambda} \quad e^{+\frac{1}{2}i\sigma_2\theta} \; (\varepsilon, \; -\frac{\lambda p}{\varepsilon}) \qquad (10)$$

The antiparticle states are related to the particle states by

$$V_\pm \quad = \quad \mp \, i \, \gamma_5 \, U_\mp \quad .$$

This gives

$$V_\lambda(0) \quad = \begin{pmatrix} \frac{p}{\varepsilon} \\ -\lambda\varepsilon \end{pmatrix} \quad \chi_{-\lambda}$$

while

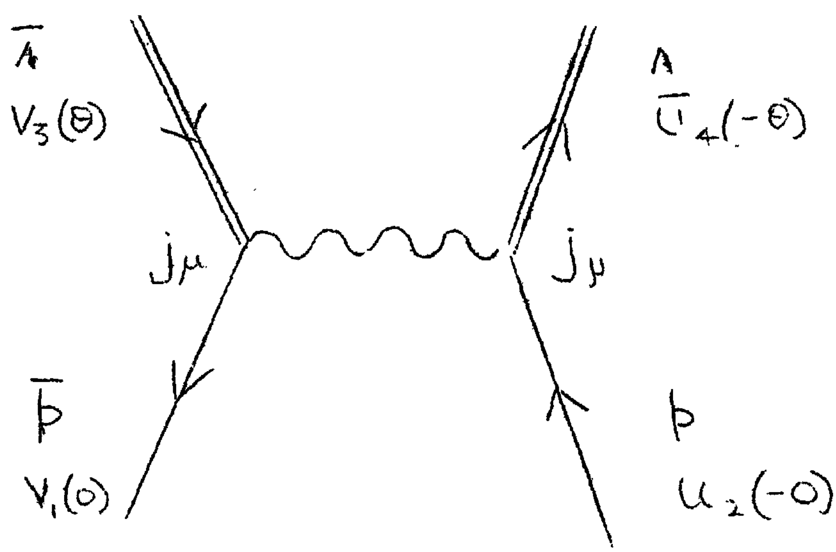
$$\bar{V}_\lambda(0) \quad = \quad \chi^+_{-\lambda} \; (\frac{p}{\varepsilon}, \quad \lambda\varepsilon) \qquad (11)$$

and

$$V_\lambda(\theta) \quad = \quad \begin{pmatrix} \frac{p}{\varepsilon} \\ -\lambda\varepsilon \end{pmatrix} e^{-\frac{1}{2}i\sigma_2\theta} \quad \chi_{-\lambda} \qquad (12)$$

Equations (9), (10), (11) and (12) give the explicit forms of the four states in which we are interested.

We consider first the contribution to the helicity amplitudes from the K^* exchange process. The diagram is



The corresponding amplitude is

$$\langle \lambda_3\lambda_4|\phi(\theta)|\lambda_1\lambda_2\rangle = \bar{V}_1(0)\, j_\mu\, V_3(\theta)\ \frac{1}{q^2-V^2}\,\bar{U}_4(-\theta) j_\mu\, U_2(-0) \tag{13}$$

$$= \frac{1}{q^2-V^2}\ \langle \lambda_3\lambda_4|T(\theta)|\lambda_1\lambda_2\rangle \text{ say} \tag{14}$$

The $q_\mu\, q_\nu\, /V^2$ term which occurs in the 1^- propagator vanishes in the present case, because of the assumed equality of the p and Λ masses. j_μ is given by (7). Substituting into (13) we find that $T(\theta)$ is composed of four terms:

$$\frac{\langle \lambda_3\lambda_4|T(\theta)|\lambda_1\lambda_2\rangle}{3(1 - q^2/4m^2)^2} =$$

$$\begin{aligned}
&(1 + \tfrac{2m}{V})^2 \left[V_1(0)\gamma_\mu\, V_3(\theta)\ \bar{U}_4(-\theta)\gamma_\mu\, U_2(-0)\right] \\
+ &\ \frac{1}{V^2} \left[V_1(0)P'_\mu\, V_3(\theta)\ \bar{U}_4(-\theta)\ P_\mu\, U_2(-0)\right] \\
-&(1 + \tfrac{2m}{V})\frac{1}{V}\begin{bmatrix} V_1(0)\ \not{P}'\, V_3(\theta)\ \bar{U}_4(-\theta)\ U_2(-0) \\ + V_1(0)\ V_3(\theta)\ \bar{U}_4(-\theta)\ \not{P}'\ U_2(-0)\end{bmatrix}
\end{aligned} \tag{15}$$

Note that where the P_μ term in j_μ occurs at the anti-particle vertex we have denoted it by P'_μ. Since an anti-particle of physical four momentum p is associated with a solution of the Dirac equation of four momentum $-p$, we must set

$$P'_\mu = -(p^{\bar{p}}_\mu + p^{\bar{\Lambda}}_\mu)$$

$$\text{cf.} \quad P_\mu = (p^{p}_\mu + p^{\Lambda}_\mu)$$

where on the right hand side we have all physical momenta. Since we are working in the CM frame it follows that

$$\begin{aligned} P'_o &= -P_o \\ P'_i &= P_i \end{aligned} \tag{16}$$

The four terms in (15) must be explicitly evaluated using equations (9) - (12) and setting for the γ-matrices their appropriate forms (i.e. the Dirac representation). We shall illustrate this procedure in one case, and quote the other results. We consider the term

$$\bar{V}_1(0) \not{P} V_3(\theta)\ \bar{U}_4(-\theta)\ U_2(-0)\ .$$

First take $\bar{V}_1(0)\ P\ V_3(\theta) = \bar{V}_1(0)(P_o\gamma_o - P_i\gamma_i)V_3(\theta)$.

Consider first the contribution from γ_o where

$$\gamma_o = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \otimes 1$$

$\therefore$ γ_o contribution

$$= P_o\left(\tfrac{p}{\varepsilon} \quad \lambda_1\varepsilon\right)\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} \frac{p}{\varepsilon} \\ -\lambda_3\varepsilon \end{pmatrix}\chi^+_{-\lambda_1}\ e^{-\frac{1}{2}i\sigma_2\theta}\chi_{-\lambda_3}$$

$$= \left\{(E - m) + \lambda_1\lambda_3(E + m)\right\}\begin{pmatrix} C & -S \\ S & C \end{pmatrix}$$

$$= 2P_o\begin{pmatrix} EC & mS \\ -mS & EC \end{pmatrix}$$

where the columns lable λ_1 and the rows λ_3 .

Similarly the γ_i contribution is

$$-2p\begin{pmatrix} P_1S + P_3C & 0 \\ 0 & P_1S + P_3C \end{pmatrix}$$

so that

$$\overline{V}_1(0)\, \not{P}'\, V_3(\theta) = 2\left(\begin{array}{c|c} P_oEC - P_1pS - P_3pC & P_omS \\ \hline -P_omS & P_oEC - P_1pS - P_3pC \end{array}\right) \qquad (17)$$

In the same way we find

$$\overline{U}_4(-\theta)\, U_2(-0) = 2\begin{pmatrix} mC & -ES \\ ES & mC \end{pmatrix} \qquad (18)$$

Putting (17) and (18) together, we find that the contribution of the term $\overline{V}\not{P}'V\ \overline{U}\,U$ to the helicity amplitudes, as defined in the matrix notation of Section (ii) is:-

$\overline{V}\not{P}'V\,\overline{U}\,U$ =

$$4\left(\begin{array}{c|c} mC & -ES \\ \hline ES & mC \end{array}\right) \otimes \left(\begin{array}{c|c} P_oEC - P_1pS - P_3pC & P_omS \\ \hline -P_omS & P_oEC - P_1pS - P_3pC \end{array}\right) \qquad (19)$$

(We have omitted here various factors, e.g. the propagator term, the kinematical term in the current and the mass terms).

If (19) is evaluated it will <u>not</u> give the correct symmetry properties, as derived in Section (ii). To check these, we must add (19) to the corresponding term

$$\bar{V}\, V\, \bar{U} \not{P}'\, U \quad =$$

$$4 \left(\begin{array}{c|c} \begin{matrix} P_0' EC \\ P_1' \, pS \\ P_3' \, pC \end{matrix} & -P_0' mS \\ \hline P_0' \, mS & \begin{matrix} P_0' EC \\ P_1' \, pS \\ P_3' \, pC \end{matrix} \end{array} \right) \otimes \left(\begin{array}{c|c} -mC & -ES \\ \hline ES & -mC \end{array} \right)$$

(19) and (20) together satisfy all the symmetry requirements, if we bear in mind the result (16). We now list the results for the three terms which remain in (15) if (19) and (20) are taken together. We set

$$\begin{aligned} A &= \left(1 + \frac{2m}{V}\right) \\ B &= \frac{1}{V} \\ X &= \frac{3\left(1 - \frac{q^2}{4m^2}\right)^2}{q^2 - V^2} \\ x &= \cos\theta \\ 2E &= \text{total CM energy} \end{aligned} \qquad (21)$$

and our normalisation is such that

$$\frac{d\sigma}{d\Omega}_{(\text{unpolarised})} = \frac{1}{2}\,\frac{M}{(16\pi E)^2} \qquad (22)$$

with
$$M = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 + 2|\phi_5|^2 + 2|\phi_6|^2 \qquad (23)$$

<u>$\bar{V}\,\gamma_\mu\, V\, \bar{U}\, \gamma_\mu\, U$</u>

$$\phi_1 = A^2X\left[8(E^2 - m^2) + 2m^2(1 + x)\right]$$
$$\phi_2 = A^2X\left[-2m^2\,(1 - x)\right]$$
$$\phi_3 = A^2X\left[2(2E^2 - m^2)(1 + x)\right]$$
$$\phi_4 = -\phi_2$$
$$\phi_5 = A^2X\left[2Em\,\sin\theta\right]$$
$$\phi_6 = -\phi_5\,.$$

<u>$\bar{V}\,\not{P}'\,V\,\bar{U}\,U \;+\; \bar{V}\,V\,\bar{U}\,\not{P}'\,U$</u>

$$\phi_1 = -ABX\left[16\,mC^2(E^2 + p^2)\right]$$
$$\phi_2 = -ABX\left[-8E^2m\,(1 - x)\right]$$
$$\phi_3 = \phi_1$$
$$\phi_4 = -\phi_2$$
$$\phi_5 \quad -ABX\left[8E^3\,\sin\theta\right]$$
$$\phi_6 = -\phi_5$$

$\overline{V}\,P'_\mu\,V\,\overline{U}\,P_\mu\,U$

$$\phi_1 = B^2X\left[2m^2(4E^2 + 2p^2(1 + x))(1 + x)\right]$$

$$\phi_2 = B^2X\left[-2E^2(4E^2 + 2p^2(1 + x))(1 - x)\right]$$

$$\phi_3 = \phi_1$$

$$\phi_4 = -\phi_2$$

$$\phi_5 = B^2X\left[2Em(4E^2 + 2p^2(1 + x))\sin\theta\right]$$

$$\phi_6 = -\phi_5 .$$

For the scalar term we find, with

$$(1 + \frac{2m}{S}) = C \qquad (24)$$

$\overline{V}\,\gamma_5\,V\,\overline{U}\,\gamma_5\,U$

$$\phi_1 = 0$$

$$\phi_2 = C^2X\left[-2p^2(1 - x)\right]$$

$$\phi_3 = 0$$

$$\phi_4 = \phi_2$$

$$\phi_5 = 0$$

$$\phi_6 = 0$$

In the way in which we have set out these results, the terms in the square brackets correspond exactly to the invariant couplings which are underlined at the head of each set of results. (The Dirac spinors are normalised so that $\bar{U} U = - \bar{V} V = 2m$).

The greatest possible care has been taken to ensure that the forms of the helicity amplitudes given here are correct. It is clear that the results are dimensionally correct. In each case, all 16 amplitudes have been derived in full, and the C and P symmetries (see Section (ii)) have been fully verified. In addition we note that all helicity changing amplitudes (ϕ_2, ϕ_4, ϕ_5, ϕ_6) vanish in the forward direction, $x = 1$, as is necessary.

It has been further checked that the scalar interaction and the $\gamma_\mu - \gamma_\mu$ vector interaction helicity amplitudes reproduce the results given in Part I of this work. Certain terms have also been checked against the special case considered in reference (44). The general vector term has also been compared numerically with the results given in Part I, and found to agree. Finally, the expressions given here have been independently derived by J.H.R. Migneron.

The contributions to each $\phi_i(\theta)$ from all terms (including vector and scalar) must be added to give the total $\widetilde{U}(12)$ helicity amplitudes, $\phi_i(\theta)$, i.e. interference takes place. It is quite clear that it is well-nigh impossible to proceed algebraically, as the next step in the work must be to project out the partial wave amplitudes ϕ_i^j by integration of expressions of the form $d_{\mu\nu}^j$ (cos θ) ϕ_i^j (cos θ). We have therefore programmed the results given here for an IBM 7090 computer. In any case, we see no virtue in attempting to deal with extremely lengthy algebraic expressions. Such a procedure would be of no assistance in understanding the underlying physics, while the possibility of error would be greatly increased, and we would certainly also be forced into rather drastic approximations in order to proceed. It should therefore be understood that from this point onwards we regard the functions $\phi_i(\theta)$ as known. They are determined numerically from the equations given here.

(v) The Partial Wave Helicity Amplitudes ϕ_i^j.

We now show how to project the partial wave amplitudes for ϕ_i^j from the amplitudes $\phi_i(\theta)$. We first note that from the explicit expressions for the $\phi_i(\theta)$ given in the last section we see $\phi_5(\theta) = \phi_6(\theta)$. We have therefore only to deal with five independent amplitudes - - $i = 1....5$. For convenience, we give again here the notation we have adopted from Section (ii) onwards in labelling these amplitudes:-

$$\begin{aligned}
\phi_1 &= \langle ++ | \phi | ++ \rangle \\
\phi_2 &= \langle ++ | \phi | -- \rangle \\
\phi_3 &= \langle +- | \phi | +- \rangle \\
\phi_4 &= \langle +- | \phi | -+ \rangle \\
\phi_5 &= \langle ++ | \phi | +- \rangle
\end{aligned} \tag{25}$$

Inserting appropriate values for λ and μ in equation (1) we have

$$\begin{aligned}
\phi_1(x) &= \sum (2j+1)\, \phi_1^j\, d_{00}^j(x) \\
\phi_2(x) &= \sum (2j+1)\, \phi_2^j\, d_{00}^j(x) \\
\phi_3(x) &= \sum (2j+1)\, \phi_3^j\, d_{11}^j(x) \\
\phi_4(x) &= \sum (2j+1)\, \phi_4^j\, d_{-11}^j(x) \\
\phi_5(x) &= \sum (2j+1)\, \phi_5^j\, d_{10}^j(x)
\end{aligned} \tag{26}$$

Using the orthogonality of the $d_{\lambda\mu}$ functions

$$\int_{-1}^{+1} d_{\lambda\mu}^{j}(x)\ d_{\lambda\mu}^{j'}(x)\ dx = \left(\frac{2}{2j+1}\right) \delta_{jj'}$$

we may invert the equations (26), finding

$$\phi_1^j = \tfrac{1}{2}\int_{-1}^{1} \phi_1(x)\, d^j_{00}(x)\, dx$$

$$\phi_2^j = \tfrac{1}{2}\int_{-1}^{1} \phi_2(x)\, d^j_{00}(x)\, dx$$

$$\phi_3^j = \tfrac{1}{2}\int_{-1}^{1} \phi_3(x)\, d^j_{11}(x)\, dx$$

$$\phi_4^j = \tfrac{1}{2}\int_{-1}^{1} \phi_4(x)\, d^j_{-11}(x)\, dx$$

$$\phi_5^j = \tfrac{1}{2}\int_{-1}^{1} \phi_5(x)\, d^j_{10}(x)\, dx$$

These integrals are to be evaluated numerically for different j values. Since we are at high energies, large j values will contribute significantly. It is therefore necessary to find some way of computing readily the functions $d^j_{\lambda\mu}(x)$.

We show how the $d^j_{\lambda\mu}$ functions may be written quite generally in terms of Legendre polynomials. The resulting expressions contain no derivatives of Legendre polynomials, but the procedure does not appear to be well known.

We first write the $d^j_{\lambda\mu}$ functions in terms of the Jacobi polynomial(49)

$$d^j_{\lambda\mu}(x) = \left[\frac{(j+\lambda)!(j-\lambda)!}{(j+\mu)!(j-\mu)!}\right]^{1/2} C^{\lambda+\mu} S^{\lambda-\mu} P^{\lambda-\mu,\lambda+\mu}_{j-\lambda}(x)$$

where, as previously

$$C = \cos\theta/2 \qquad S = \sin\frac{\theta}{2} .$$

For our particular values of λ and μ, this gives

$$\begin{aligned} d^j_{oo} &= P^{oo}_j \\ d^j_{11} &= C^2 P^{o2}_{j-1} \\ d^j_{-11} &= S^2 P^{2o}_{j-1} \\ d^j_{1o} &= \sqrt{\frac{j+1}{j}}\, C\, S\, P^{11}_{j-1} . \end{aligned} \tag{28}$$

We note first that[50]

$$P^{oo}_j = P_j \tag{29}$$

In general we can write any function $P^{m,n}_j$ as a linear sum of Legendre polynomials. This is possible because the $P^{m,n}_j$ obey recurrence relations which permit one to write $P^{m,n}_j$ in terms of $P^{m-1,n}_{j+1}$ and $P^{m-1,n}_j$, or alternatively to write $P^{m,n}_j$ in terms of $P^{m,n-1}_{j+1}$ and $P^{m,n-1}_j$ (Rainville, locus. cit. Equation (13), p. 265, and equation (11), p. 264). This procedure can be repeated until we have $m = 0$, $n = 0$, in which case the Jacobi polynomials are just Legendre polynomials.

For the polynomials occurring in (28) we find in this way

$$P^{o2}_{j-1} = \frac{2\left[jP_{j+1} + (2j+1)P_j + (j+1)P_{j-1}\right]}{(2j+1)(1+x)^2}$$

$$P^{2o}_{j-1} \quad \frac{2\left[jP_{j+1} - (2j+1)P_j + (j+1)P_{j-1}\right]}{(2j+1)(1-x)^2} \tag{30}$$

$$P^{1,1}_{j-1} = \frac{2j\left[P_{j-1} - P_{j+1}\right]}{(2j+1)(1-x^2)} \quad .$$

Inserting (28) and (30) into (27) we have*

$$\phi^j_1 = \tfrac{1}{2}\int_{-1}^{1} \phi_1(x)\, P_j(x)\, dx$$

$$\phi^j_2 = \tfrac{1}{2}\int_{-1}^{1} \phi_2(x)\, P_j(x)\, dx$$

$$\phi^j_3 = \tfrac{1}{2}\int_{-1}^{1} \frac{\phi_3(x)}{1+x} \left[\frac{jP_{j+1} + (2j+1)P_j + (j+1)P_{j-1}}{(2j+1)}\right] dx$$

$$\phi^j_4 = \tfrac{1}{2}\int_{-1}^{1} \frac{\phi_4(x)}{1-x} \left[\frac{jP_{j+1} - (2j+1)P_j + (j+1)P_{j-1}}{(2j+1)}\right] dx$$

$$\phi^j_5 = \tfrac{1}{2}\int_{-1}^{1} \frac{\phi_5(x)}{\sqrt{1-x^2}} \frac{\sqrt{j(j+1)}}{(2j+1)} \left[P_{j-1} - P_{j+1}\right] dx \tag{30}$$

---

* These equations correspond closely to the set given by Goldberger, Grisaru, MacDowell and Wong[51]. The expression corresponding to ϕ^j_5 is given with opposite sign; this is because of a difference in definition of $d^j_{\lambda\mu}(\theta)$, $d^j_{\lambda\mu}(\theta) \rightarrow d^j_{\lambda\mu}(-\theta) = (-1)^{\lambda-\mu}\, d^j_{\lambda\mu}(\theta)$, which affects only ϕ^j_5. However the expressions corresponding to ϕ^j_3 and ϕ^j_4 in this work are in error by a factor of (2j+1) in one of the terms.

These equations permit the determination of the ϕ_i^j. The $\phi_i(x)$ are available from our previous work, and the $P_j(x)$ may be readily generated numerically.

(vi) Absorptive Effects

We now proceed to derive an expression for the corrected, or unitarised amplitudes $\bar{\phi}_i^j$ in terms of the Born amplitudes ϕ_i^j. The $\bar{\phi}_i^j$ differ from the ϕ_i^j in that the former include the absorptive unitarity corrections discussed at length in Part II of this work.

According to equation (52) of Part II the $\bar{\phi}_i^j$ are given by

$$\langle \beta_1\beta_2 | \bar{\phi}^j_{\beta\alpha} | \alpha_1\alpha_2 \rangle$$

$$= \frac{1}{2} \left[\sum_{\beta_1'\beta_2'} \langle \beta_1\beta_2 | S^j_{\beta\beta} | \beta_1'\ \beta_2' \rangle \langle \beta_1'\ \beta_2' | \phi^j_{\beta\alpha} | \alpha_1\ \alpha_2 \rangle + \sum_{\alpha_1'\alpha_2'} \langle \beta_1\ \beta_2 | \phi^j_{\beta\alpha} | \alpha_1'\ \alpha_2' \rangle \langle \alpha_1'\ \alpha_2' | S^j_{\alpha\alpha} | \alpha_1\alpha_2 \rangle \right] \tag{32}$$

where we have dropped the index i in the ϕ^j and $\bar{\phi}^j$ and displayed the spin dependence explicitly, and labelled the channels α ($p\bar{p}$) and β ($\Lambda\bar{\Lambda}$) and the helicities $\alpha_1\ \alpha_2$, $\beta_1\ \beta_2$ etc. to conform with our earlier notation.

We must insert into (32) appropriate values for the elastic scattering S-matrix elements. We therefore make several assumptions about the form of the elastic scattering amplitudes.

(i) We assume $S^j_{\alpha\alpha}$ and $S^j_{\beta\beta}$ only involve non-helicity changing terms. This is the conventional assumption[5,26,27]. It is justified on the grounds that the elastic scattering amplitude is (experimentally) largest in the forward direction, where the helicity changing amplitudes vanish, and therefore the helicity changing amplitudes are presumably fairly unimportant in general.* (We are, of course, confining our attention once again to the high energy region).

(ii) We take $\Lambda\bar{\Lambda}$ elastic scattering to be identical to $p\bar{p}$ elastic scattering, which assumption we foresaw in Section (i). As we noted there, it is a plausible approximation.

(iii) We assume the elastic scattering diagonal S-matrix elements are all equal.

---

* In the view of the present author, it would seem more reasonable to assume that the spin-changing amplitudes vanished for elastic scattering at high energies. This is not the same assumption as (i) above, except in the forward direction. An investigation of this point is being undertaken, but it lies outside the scope of the present work.

(iv) We take as a parameterization of the experimental results for $p\bar{p}$ scattering a Gaussian model with complete absorption of the low partial waves (see II, Section(ii)):-

$$S_\ell = 1 - e^{-\ell(\ell+1)/\nu^2 p^2}$$

with $\nu^{-1} = 190$ Mev/c at 3.7 Gev/c. (52)

(v) We set

$$\langle \alpha_1 \alpha_2 | S^j_{\alpha\alpha} | \alpha_1 \alpha_2 \rangle = \langle \beta_1 \beta_2 | S^j_{\beta\beta} | \beta_1 \beta_2 \rangle$$
$$= 1 - e^{-j(j+1)/\nu^2 p^2}$$

for all $\alpha_1 \alpha_2$, $\beta_1 \beta_2$.

(v) does not exactly follow from (iii) and (iv). We have assumed for S^j the same form as S^ℓ, and since in general $\ell \neq j$ this assumption introduces possible errors. However a closer examination suggests that the identification of ℓ with j is quite well justified. In general the amplitude for total angular momentum j will be a weighted mean of the amplitudes with orbital angular momenta $j+1$, j and $j-1$, and S_j will not be very different from $S_{\ell=j}$, even for the low partial waves. (This is confirmed in reference (45)).

With assumptions (i) - (v), (32) now greatly simplifies, and we have

$$\langle \beta_1\beta_2 | \bar{\phi}^j_{\beta\alpha} | \alpha_1\alpha_2 \rangle = \left\{1 - e^{-j(j+1)/\nu^2 p^2}\right\} \langle \beta_1\beta_2 | \phi^j_{\beta\alpha} | \alpha_1\alpha_2 \rangle$$

Reverting to our notation of labelling the different helicity amplitudes by ϕ^j_i we have

$$\bar{\phi}^j_i = \left(1 - e^{-j(j+1)/\nu^2 p^2}\right) \phi^j_i \qquad (33)$$

With (33), we have now all the equations necessary to evaluate the unitarised (or absorption model) differen--tial cross section. The ϕ^j_i are given by (31). (33) gives the $\bar{\phi}^j_i$. The corresponding unitarised $\bar{\phi}_i(\theta)$ are given by equations identiaal to (26), where the summation over j is done numerically until it converges. The differential cross-section is then given by (22) and (23), where $\phi_i(\theta)$ is replaced by $\bar{\phi}_i(\theta)$. Before presenting the results, we give some details of the numerical methods used.

Note on numerical methods.

The integration in (31) was performed using a 40 point Gaussian approximation, i.e. we set

$$\int_{-1}^{1} f(x)\, dx = \sum_{i=1}^{40} A_i\, f(x_i) \qquad (34)$$

where the 40 points x_i and the corresponding weighting functions A_i are given in standard textbooks on numerical methods. (34) is exact if $f(x)$ is a polynomial of degree less than 80.

We therefore require the values of $\phi_i(x)$ and $P_j(x)$ at the points x_i. The $\phi_i(x)$ are readily obtained from the equations of Section (iv). $P_j(x_i)$ may be readily generated from the recurrence relation

$$jP_j(x_i) = (2j - 1)x_i P_{j-1}(x_i) - (j - 1)P_{j-2}(x_i)$$

with
$$P_0(x_i) = 1$$
$$P_1(x_i) = x_i \; .$$

It was found that the summations in (26) had largely converged with $j_{max} = 15$, and we therefore set $j_{max} = 30$.

The construction of the Legendre polynomials, the integration routine, and the convergence of the summation over j -- and indeed the consistency of our entire set of equations -- may be tested by projecting out the ϕ_i^j from the $\phi_i(x)$ using (31), and resuming in (26) to check that the $\phi_i(x)$ are accurately reproduced. We invariably found this check to work correctly to within a greater accuracy than one part in 10^4. The complete calculation merely involves repeating the same procedure with at one point the inclusion of an extra factor (c.f. 33), and we therefore have the greatest confidence in the numerical work.

(vi) Results and Discussion

The most detailed experimental results to date* appear to be those of Baltay et al.[53], at an incident antiproton laboratory momentum of 3.7 Gev/c. We have therefore carried out our calculations at a corresponding energy.

We are now faced with the problem of the choice of S and V, the scalar and vector meson masses. The mean mass of the 0^- and 1^- mesons (the "$\widetilde{U}(12)$ mass") is 610 MeV ; the mean masses of the 0^- and 1^- mesons are respectively 370 and 850 MeV (the "SU(3) masses"), while the physical masses involved are 494 MeV (K) and 891 Mev (K^*); clearly the variation involved in these values is far from negligible, and one is in some difficulty as to how best to proceed.

We have therefore performed the necessary calculations

---

* September, 1965.

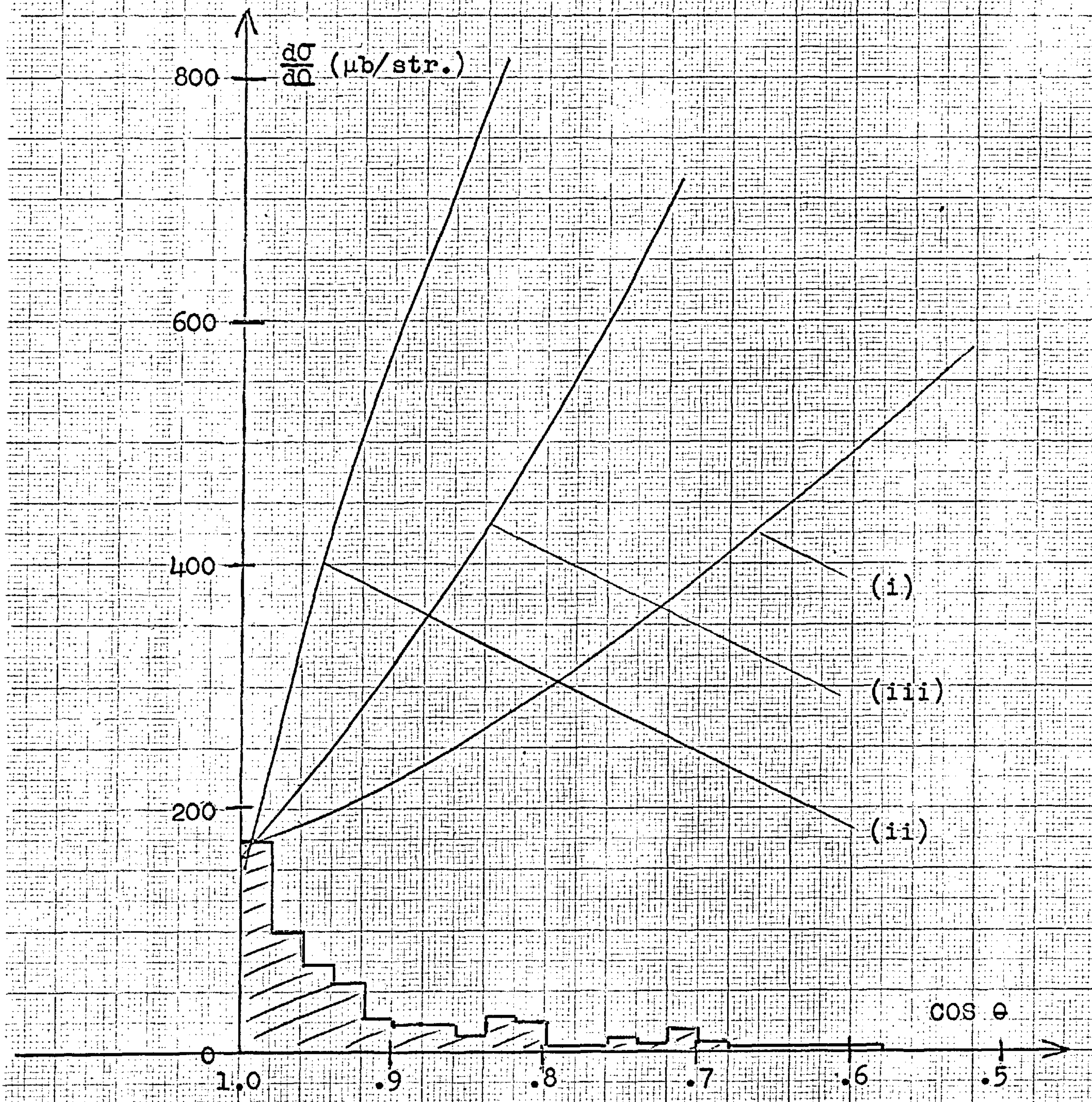
for the three choices of S and V which appear to be possible.

(i)	$\widetilde{U}(12)$ masses	:	S = 610 MeV.
			V = 610 MeV.
(ii)	SU(3) masses	:	S = 370 MeV.
			V = 850 MeV.
(iii)	Physical masses	:	S = 494 MeV.
			V = 891 MeV.

We have plotted the results obtained for the above cases (i), (ii) and (iii) in Figs. 11 and 12, against the experimental histogram. Fig. 11 gives the results without the inclusion of absorptive effects, while Fig. 12 gives the absorption model results. The curves shown have been normalised to give the correct number of events in the first interval of the experimental histogram.

It is seen that in the absence of absorptive corrections, the $\widetilde{U}(12)$ results display a totally unacceptable angular form, as we foresaw in Section (iii). In contrast, the absorption model results give an adequate description of the experimental results in all three cases. The fits to the data are not perfect, indeed there is evidence of still too much wide angle scattering, but the results are nevertheless substantially accurate and should be compared with those given in Part I.

In principle, the absolute value of the differential cross-section is given from $\widetilde{U}(12)$ in terms of the π N N



<u>FIG. 11</u>. The $\tilde{U}(12)$ Born term predictions for $p\bar{p} \to \Lambda\bar{\Lambda}$ at 3.0 Gev/c. against the experimental results of ref. (53). (i) $\tilde{U}(12)$ masses, (ii) SU(3) masses, (iii) physical masses. (See p. 124).

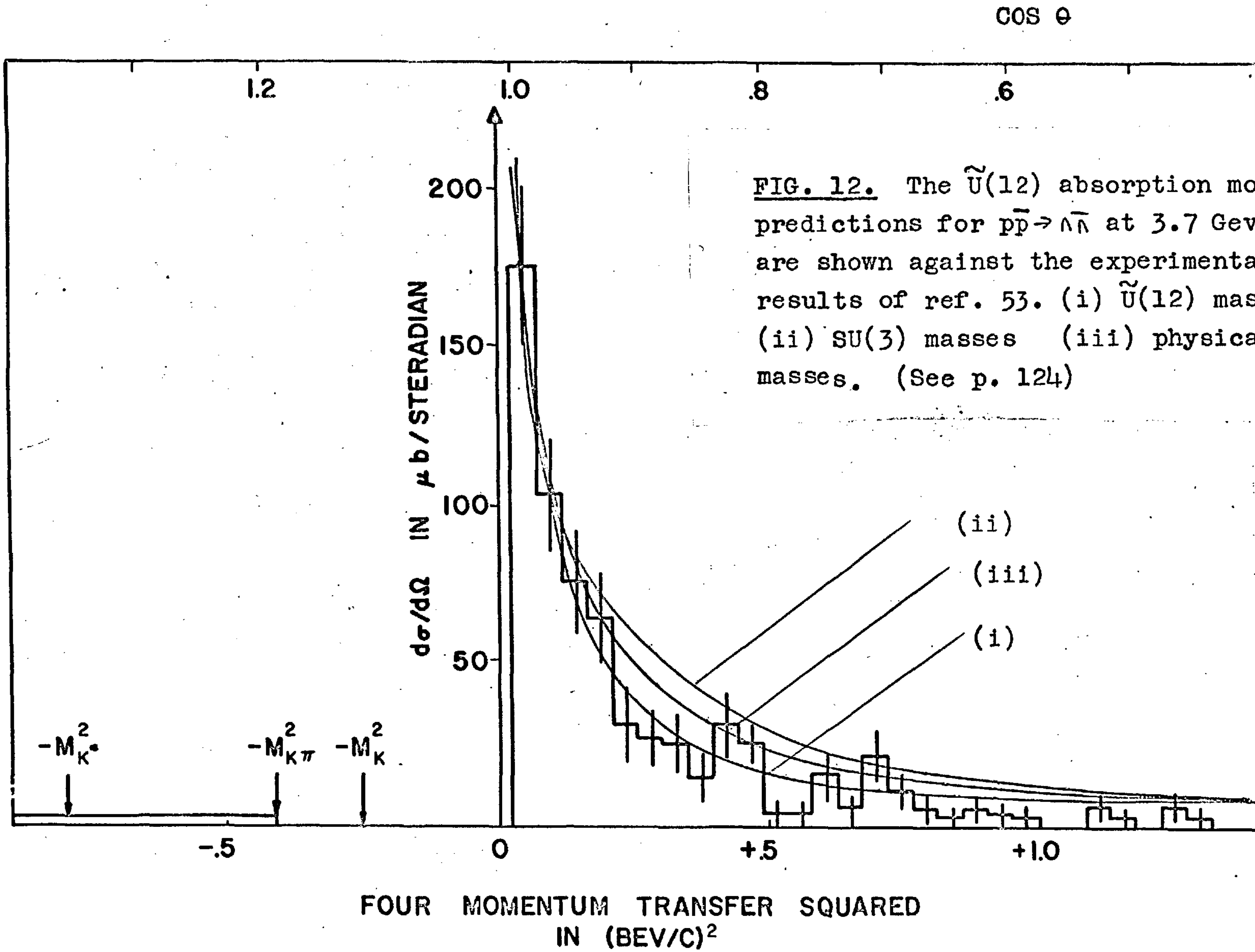


FIG. 12. The $\widetilde{U}(12)$ absorption model predictions for $p\bar{p} \to \Lambda\bar{\Lambda}$ at 3.7 Gev/c are shown against the experimental results of ref. 53. (i) $\widetilde{U}(12)$ masses (ii) SU(3) masses (iii) physical masses. (See p. 124)

coupling constant. For ease of comparison, we calculate the values of $G^2_{\pi N N}/4\pi$ which would be necessary in cases (i), (ii) and (iii) to give the correct normalisation in the first experimental interval, i.e. that displayed in Fig. 12.

We therefore determine first the appropriate values of the $\tilde{U}(12)$ coupling constant G. $G_{\pi N N}$ is then given by

$$G_{\pi N N} = \left(1 + \frac{2m}{\mu}\right) \frac{5}{3} G \quad ,$$

where μ is the $\tilde{U}(12)$ meson mass. In practice, one is, as usual, in some doubt as to the choice of μ. The pion mass itself is anomalously low (135 MeV). We have therefore determined $G_{\pi N N}$ for each of our three results with two alternative choices of μ: (i) for $\mu = 370$ MeV, the mean 0^- mass, (ii) for $\mu = S$, the particular 0^- mass used in each case. The results are given below.

S,V (Mev)	$G^2_{\pi NN}/4\pi$ (Setting μ=370)	$G^2_{\pi NN}/4\pi$ (Setting μ = S)
(i) S = 610 V = 610	14.1	6.3
(ii) S = 370 V = 850	13.2	13.2
(iii) S = 494 V = 891	23.6	14.8

It is known that $G^2_{\pi NN}/4\pi \sim 15$. Of our results, only one value (6.3) is particularly bad, and this arises from setting μ = 610 MeV, a value which has little relation to the mean 0^- mass, and less to that of the pion.

Though it would clearly be possible to select from our various results an optimum fit to the data, we do not think that this would be a justifiable procedure. Any such selection would have to be made on the basis of a "prescription" for dealing with the masses in $\tilde{U}(12)$ which could only be reached after a comparison of results for many different reactions against experiment. Our present results would be only one set to be taken into account.

Nevertheless, it is clear that the $\tilde{U}(12)$ absorption model presented here gives a satisfactory result for the angular distribution, and, at least in four out of the six cases discussed, gives correctly in terms of the π N N coupling constant the absolute value of the differential cross-section to an accuracy of 10 - 20 per cent. If we consider that we are dealing with a reaction involving strange particles, and that there are no free parameters in the model at all, this must be considered a very satisfactory result.

We have come a long way since our simple pole calculations of Part I, and have seen how two of the most serious defects of the peripheral model which were discussed there, namely the arbitrariness in the choice of coupling schemes and constants and the tendency of the model to give too much wide angle scattering, may be overcome -- in the former case by invoking a higher symmetry scheme, and in the latter by taking account of absorptive effects.

Nevertheless, our work raises many questions. There is the problem of mass splitting in $\tilde{U}(12)$. Again, the absorption model appears still to be too sensitive to the behaviour of low partial waves; the high energy behaviour of the model, and the possibility of making alternative assumptions as to the form of the spin dependence of the elastic scattering amplitudes require investigation. There is also the interesting possibility of a unified treatment of elastic and inelastic processes along the lines suggested by Byers and Yang[54]. However, these questions, and indeed the extension of the present calculation to other $Y\bar{Y}$ final states*, lie outside the scope of this work.

---

* This is being undertaken by J.H.R. Migneron.

REFERENCES

(1) F. Slazman & G. Salzman; Phys. Rev. 112, 1344 (1958).

(2) S.D. Drell; Phys. Rev. Letters 5, 342 (1960); Rev. Mod. Phys. 33, 458 (1961).

(3) F. Salzman & G. Salzman; Phys. Rev. 120, 599 (1960).

(4) E. Ferrari & F. Selleri, Suppl. del Nuovo Cimento 24, 453 (1962).

(5) J.D. Jackson; Rev. Mod. Phys. 37, 485 (1965).

(6) S.B. Treiman & C.N. Yang; Phys. Rev. Letters 8, 140 (1962).

(7) L. Stodolsky & J.J. Sakurai; P.R.L. 31, 360 (1964); L. Stodolsky, Phys. Rev. 134, B 1099 (1964).

(8) (i) CERN internal publication DD/EXP/62/22: "A Study of hyperon - anti-hyperon production by 3, 3.6 and 4 Gev/c antiprotons in hydrogen".
(ii) B. Musgrave et al.; Nuovo Cimento 35, 735 (1965).

(9) (i) C. Baltay et al., Proc. International Conf. on Nucleon Structure, Stanford, 1963.
(ii) C. Baltay et al.; preprint, July, 1965 (BNL 9276).

(10) See, e.g.
(i) P.T. Matthews & A. Salam; Phys. Rev. 110, 565 (1958).
(ii) S. Iwao; Nuovo Cimento 23, 516 (1962).

(11) D. Bessis, C. Itzykson & M. Jacob; Nuovo Cimento 27, 376 (1963).

(12) Chia-Hwa Chan; Phys. Rev. Letters 6, 383 (1961).

(13) (i) Y. Ne'eman; Nucl. Phys. 26, 222 (1961).
(ii) M. Gell-Mann; Phys. Rev. 125, 1067 (1962).

(14) J.J. Sakurai; Proc. International School of Physics "Enrico Fermi", Varenna, Italy, 1962.

REFERENCES (Contd.)

(15) (i) K.J. Barnes; Phys. Lett. 1, 166 (1962).

(ii) L.N. Hand, D.G. Miller & R. Wilson; Phys. Rev. Lett. 8, 110 (1962).

(16) L.N. Hand, D.G. Miller & R. Wilson; privately circulated paper.

(17) L. Durand & Y.T. Chiu; Phys. Rev. Lett. 12, 399 (1964); erratum; 13, 45 (1964).

(18) K. Gottfried & J.D. Jackson; Nuovo Cimento 34, 735 (1964).

(19) A. Salam, R. Delbourgo & J. Strathdee; Proc. Roy. Soc., A 284, 146 (1965).

(20) U. Amaldi & F. Selleri; Nuovo Cimento 31, 360 (1964).

(21) G. Goldhaber et al.; Phys. Lett. 6, 62 (1963).

(22) J.D. Jackson & H. Pilkuhn; Nuovo Cim. 33, 906 (1964).

(23) P.T. Matthews; Brandeis Lecture notes, 1963.

(24) See, e.g., S. Brandt et al., Phys. Rev. Lett, 10, 413 (1963).

(25) N.J. Sopkovich; Nuovo Cimento 26, 186 (1962).

(26) L. Durand, III & Yam Tsi Chiu; Phys. Rev. 137, B1530 (1965).

(27) L. Durand, III & Yam Tsi Chiu; Phys. Rev. 139, B646 (1965).

(28) (i) G. Wentzel; Zeits. f. Physik 38, 518 (1926).
(ii) H.A. Kramers; Zeits. f. Physik 39, 828 (1926).
(iii) L. Brillouin; Comptes Rendus 183, 24 (1926).

(29) R.J. Glauber; "Lectures in Theoretical Physics", Boulder, 1958, Vol. 1 (Interscience, London & New York, 1959).

REFERENCES (Contd.)

(30) Kenneth M. Watson; Phys. Rev. 88, 1163 (1952).

(31) R. Delbourgo; Nucl. Phys. 38, 281 (1962).

(32) J.S. Ball & W.R. Frazer; Phys. Rev. Letters 14, 746 (1965).

(33) B.A. Lippmann & Julian Schwinger; Phys. Rev. 79, 469 (1950).

(34) Marc H. Ross & Gordon L. Shaw; Phys. Rev. Lett. 12, 627 (1964).

(35) W. Pauli; "Meson Theory of Nuclear Forces", Interscience, New York, 1946.

(36) W. Heitler; "The Quantum Theory of Radiation", O.U.P., 1953.

(37) R.H. Dalitz; "Strange Particles and Strong Interactions", O.U.P., 1962.

(38) Richard C. Arnold; Phys. Rev. 136, B1388 (1964).

(39) M. Yonezawa; Nuovo Cimento 19, 365 (1965).

(40) K. Dietz & H. Pilkuhn; Nuovo Cimento 37, 1561 (1965).

(41) A. Dar & W. Tobocman; Phys. Rev. Lett. 12, 511 (1964).

(42) E.J. Squires; Nuovo Cimento 34, 1328 (1964).

(43) M. Jacob & G.C. Wick; Annals of Physics 7, 404 (1959).

(44) C.H. Chan; Phys. Rev. 133, B431 (1964).

(45) C. Cohen-Tannoudji & H. Navelet; Nuovo Cimento 37, 1511 (1965).

(46) H. Högassen & J. Högassen, CERN preprint 6S/829/5 Th 561.

(47) J.J. Sakurai; Proc. International School of Physics "Enrico Fermi", Varenna, Italy, 1962.

REFERENCES (Contd.)

(48) K.J. Barnes; Ph.D. Thesis, Imperial College, (University of London).

(49) A.R. Edmonds; "Angular Momentum in Quantum Mechanics", Princeton University Press, 1957.

(50) E.D. Rainville; "Special Functions", 1960.

(51) M.L. Goldberger, M.T. Grisaru, S.W. MacDowall & D.Y. Wong; Phys. Rev. 120, 2250 (1960) (Equations 4.22 - 4.25).

(52) O. Czyzewski et al.; Proc. Sienna Conf., 1963, p. 252.

(53) C. Baltay et al.; preprint, July, 1965 (BNL 9276).

(54) N. Byers & C.N. Yang; preprint, September 1965.