Stability Robustness in the presence of exponentially unstable isolated equilibria

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Abstract

This note studies nonlinear systems evolving on manifolds with a finite number of asymptotically stable equilibria and a Lyapunov function which strictly decreases outside equilibrium points. If the linearizations at unstable equilibria have at least one positive eigenvalue, then almost global asymptotic stability turns out to be robust with respect to sufficiently small disturbances in the L_∞ norm. Applications of this result are shown in the study of almost global Input-to-State stability.

1 Introduction and Motivations

Stability notions with respect to exogenous signals are a key tool in nonlinear control. On one hand they allow to analyze stability of interconnected systems in terms of Input-Output gains of individual subsystems, see for instance [10]. On the other, they provide quantitative estimates of how the system reacts to exogenous disturbances. Two approaches have been particularly fruitful, both from the purely theoretical point of view as well as successful in several domains of application. These are the so-called H_∞ and Input-to-State Stability framework, [23, 20]. Both approaches extend the classical Lyapunov method, traditionally used to establish internal stability properties, to systems with inputs and outputs. Indeed, in analogy to the classical Lyapunov method, they exploit state-space descriptions of system's dynamics and energy-like functions in order to asses the stability and robustness of a system with respect to internal and external perturbations. The theory is very developed for nonlinear systems

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which are defined on Euclidean spaces and with a globally asymptotically stable equilibrium point (or compact attractor). However, in more general set-ups this is often not the case. For instance smooth systems evolving on manifolds or systems whose attractor is something more complicated than a single equilibrium, typically do not fulfill global asymptotic stability as topological obstructions arise even in the absence of exogenous disturbances. One natural way to relax global requirements is therefore to consider almost global stability notions with respect to a single equilibrium or, more in general, to the non-trivial attractor of interest (for instance multiple equilibria). This entails a deep revision of the analytical techniques involved.

An attempt in this direction was discussed in [1], with a new definition of almost global Input-to-State Stability (aISS) and the proposition of some analytical techniques which may be employed to establish aISS for non-trivial examples of nonlinear systems. The main result in [1] makes use of the so called density functions, which were recently introduced by Rantzer as a natural dual to Lyapunov functions, in the study of almost global stability and attractivity notions, [17, 18]. While software tools to automatically find density functions for certain classes of systems are beginning to become available, [16, 15], recent analysis has also highlighted that explicit closed-form expressions of smooth dual Lyapunov functions in the case of systems with saddle points of negative divergence, [2], might actually not exist in most cases.

The difficulties in finding such functions pushed the authors in the direction of proposing a complementary set of tools for the study of stability robustness in the presence of unstable and antistable invariant sets. The techniques heavily rely on the stable and unstable manifolds theory of dynamical systems, in particular on their time-varying adaptations. This paper was motivated by an open problem publicly posed by one of the authors during the 2009 Oberwolfach meeting in Control Theory, [3], and provides together with a positive answer to the question thereby formulated, a result to address similar questions in general and realistic scenarios.

Just as a motivating example, which will later be discussed in more detail, we recall the question posed in [3]. The system under consideration is a pendulum with friction, of equations

$$\begin{array}{rcl} \dot{\theta} & = & \omega \\ \dot{\omega} & = & -\sin(\theta) - \omega + d, \end{array} \tag{1}$$

whose state variable $x = [\theta, \omega]$ takes values in the manifold $\mathbb{S} \times \mathbb{R}$. For $d = 0$, that is in the absence of exogenous torque disturbances, it is well-known that almost all solutions will converge to the equilibrium $[0, 0]$, corresponding to the pendulum pointing downwards. On the other hand, the upright position of the pendulum $[\pi, 0]$ is an hyperbolic saddle point of negative divergence (divergence is equal to -1 everywhere in state space). Therefore, a zero-measure set of initial conditions (in particular those belonging to the so-called stable manifold which is in this case one-dimensional) give rise to solutions asymptotically approaching the upright equilibrium. One might thus wonder whether, in the presence of non-zero disturbances, almost all initial conditions will give rise to solutions which are ultimately (in positive times) in a ball centered at the downward equilibrium and with radius bounded from above in terms of the L_∞ norm of the applied disturbance, modulated by some $\mathcal{K}$ function[1].

The answer to this question is positive and follows by applying our Main Result. This is a connection between existence of Lyapunov functions with strictly negative derivative and robustness of almost global asymptotic stability to exogenous disturbances of sufficiently small amplitude. While such results are by now well-known and frequently quoted in the case of GAS (see for instance the converse Lyapunov theorems provided in [22]), rather different techniques are needed for almost global stability analysis. Combined with more standard tools for ultimate boundedness or practical Input-to-State Stability analysis, the technique configures a separation principle for claiming almost global Input-to-State Stability. We also show, by means of a one-dimensional example defined on the unit circle (see Section 3.2), that in the presence of unstable equilibria even arbitrarily small disturbances have the potential for changing the qualitative dynamical behaviour of a system (for instance stabilizing an initially unstable equilibrium or creating basins of attraction of positive Lebesgue measure).

2 Problem formulation and Main Result

A0 : Let M be an n-dimensional $\mathcal{C}^2$ connected, orientable[2], Riemannian manifold without boundary M, $f : M \times D \to T_xM$ be a $\mathcal{C}^1$-Lipschitz function and D be a closed subset of $\mathbb{R}^m$.

This note deals with nonlinear systems of the following type:

$$\dot{x}(t) = f(x(t), d(t)) \tag{2}$$

with state x taking value in M. We denote by $X(t, x, t_0; d)$ its solution which is at x at time t_0 and we call unperturbed system the following autonomous ordinary differential equation :

$$\dot{x}(t) = f(x(t), 0) \doteq f_0(x(t)). \tag{3}$$

We assume :

A1 : existence of a nonnegative and proper[3] $\mathcal{C}^1$ function $V : M \to \mathbb{R}$ such that we have[4] :

$$L_{f_0}V\big|_x < 0, \qquad \forall x \in M \,:\, f_0(x) \neq 0. \tag{4}$$

[1] A function $\gamma : [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{K}$ if continuous, increasing and $\gamma(0) = 0$.

[2] Orientability is assumed here to guarantee the existence of a volume form and makes the statement of our results easier. This assumption is not essential however. Without it we can still define a notion of volume by considering a density function, everywhere non-vanishing in M. See for instance [13].

[3] We recall that a function V is proper provided $V^{-1}(K)$ is compact for all compacts K included in the domain of V.

[4] We use the notation $L_fV\big|_{x,d}$ to denote the Lie derivative of V along f at a point x when the perturbation is d.

A2 : any equilibrium x_ℓ which is not asymptotically stable, is isolated and such that at least one eigenvalue of $df_0(x_\ell) : T_{x_\ell}M \to T_{x_\ell}M$ has strictly positive real part, where $df_0(x)$ denotes the differential of f_0 at x.

Notice that (4) implies that stationary points of V are equilibria (the converse need not be true). Also, asymptotically stable equilibria are, by definition, necessarily isolated and with an open basin of attraction. Moreover, if M is compact, it has stationary points (and minima) which, by the previous remark, are necessarily equilibria (respectively asymptotically stable equilibria).

If M is a not compact, let v be a real number arbitrary up to the fact that the compact set

$$\mathfrak{C} \;=\; \{x \,:\, V(x) \leq v\}$$

contains at least one asymptotically stable equilibrium and no equlibrium on its boundary. If M is compact, we let :

$$\mathfrak{C} \;=\; M \;.$$

Since equilibria of the undisturbed system are isolated, $\mathfrak{C}$ contains a finite number $\mathfrak{L}$ of them which we denote by x_ℓ with ℓ ranging in $\{1, 2, \dots, \mathfrak{L}\}$. Also, we denote by E_s the finite set of those which are asymptotically stable.

Proposition 1 *Under assumptions A0 to A2, there exist a real number $\Delta > 0$ and a class $\mathcal{K}$ function γ such that, for each measurable perturbation $d : \mathbb{R} \to D \subset \mathbb{R}^m$ with L^∞ norm smaller than Δ, and for each t_0 in $\mathbb{R}$, there exists a set $\mathfrak{B}_d(t_0) \subset M$ which is countably (n-1)-rectifiable*[5] *and, M being orientable, of zero Riemannian volume such that, any solution $X(t, x, t_0; d)$ of (2) with x in $\mathfrak{C} \setminus \mathfrak{B}_d(t_0)$ is defined at least on $[t_0, +\infty)$ and satisfies :*

$$\limsup_{t\to+\infty} \mathfrak{d}_M(X(t, x, t_0; d), E_s) \;\;\leq\;\; \lim_{t\to+\infty} \operatorname{ess.sup}_{s\geq t} \gamma(|d(s)|) \;, \tag{5}$$

where $\mathfrak{d}_M(x, y)$ denotes the Riemannian distance between x and y in M.

To prove this Proposition, we shall need the following Lemma whose proof is given in appendix. It relies heavily on the results in [4].

Lemma 1 *Let x_ℓ be an isolated equilibrium of the unperturbed system (3) such that at least one eigenvalue of $df_0(x_\ell)$ has strictly positive real part. There exist a neighborhood $\mathcal{P}(x_\ell)$ of x_ℓ, a strictly positive real number $D_p(x_\ell)$, a non-negative integer $p < n$ and a bounded open set $\mathfrak{O}_\ell$ in $\mathbb{R}^p$, such that, for each measurable perturbation $d : \mathbb{R} \to D \subset \mathbb{R}^m$ with L^∞ norm smaller than $D_p(x_\ell)$, a continuous function $\mathfrak{A}_{\ell,d} : \mathbb{R} \times \mathfrak{O}_\ell \to M$ exists such that the map $\xi \mapsto \mathfrak{A}_{\ell,d}(t, \xi)$ is locally Lipschitz (uniformly in t) and any solution $X(t, x, t_0; d)$ defined at least on $[t_0, +\infty)$ and for which there exists s such that*

$$X(t, x, t_0; d) \;\in\; \mathcal{P}(x_\ell) \qquad \forall\, t \geq s$$

necessarily satisfies

$$X(t, x, t_0; d) \;\in\; \mathfrak{A}_{\ell,d}(t, \mathfrak{O}_\ell) \cap \mathcal{P}(x_\ell) \qquad \forall\, t \geq s\,.$$

[5] See [11, 3.2.14] for instance for a definition.

Remark 1

1. In the above statement, when $p = 0$ (as it is always the case for $n = 1$), $\mathbb{R}^0$ denotes the singleton $\{0\}$.
2. For each t in $\mathbb{R}$, the set $\mathfrak{A}_{\ell,d}(t, \mathfrak{O}_\ell)$ is a p rectifiable set. Since p is strictly smaller than n, it has a zero volume.[6]
3. The set $\mathfrak{A}_{\ell,d}(t, \mathfrak{O}_\ell) \cap \mathcal{P}(x_\ell)$ may be empty, in which case no solution exists which is eventually confined within $\mathcal{P}(x_\ell)$. This is the case, for instance, if locally to x_ℓ the system is diffeomorphic to

$$\dot{x}_1 = x_1,\ \dot{x}_2 = x_2^2 + d^2$$

and d has infinite L_2 norm. A suitable Lyapunov function (locally to the equilibrium x_ℓ) is for instance: $V(x_1, x_2) = -x_2 - x_1^2$.

Proof of Proposition 1. Let

- δ be the smallest distance between the equilibria x_ℓ in $\mathfrak{C}$ of the undisturbed system.
- $$F \ = \ 2 \max_{x \in \mathfrak{C}} |f_0(x)|_M, \qquad (6)$$

  where $|v|_M$ denotes the Riemannian norm of a vector field, i.e., for each x in M where v is defined, we have :

$$|v(x)|_M \ = \ \sqrt{v(x)^T g(x) v(x)}\ ,$$

  with g being the Riemannian metric.

If M is a not compact, with our definition of the compact set $\mathfrak{C}$, there exists $\varepsilon_c > 0$ such that :

$$L_f V\big|_{x,0} \ \le \ -\varepsilon_c \qquad \forall x \,:\, V(x) \ = \ v \ .$$

With this, (6) and continuity, we can find a strictly positive real number D_c such that we have, for all d with $|d| \le D_c$,

$$\begin{aligned} \max_{x \,:\, V(x)=v} \ L_f V\big|_{x,d} \ &\le \ 0 \ , \\ \max_{x \in \mathfrak{C}} |f(x,d)|_M \ &\le \ F \ . \end{aligned} \qquad (7)$$

[6]Simply because, for each fixed t, the function $(x, y) \in \mathfrak{O}_\ell \times \mathbb{R}^{n-p} \mapsto F(x, y) \doteq \mathfrak{A}_{\ell,d}(t, x) \in M$ is locally Lipschitz, it maps zero Lebesgue measure subsets of $\mathbb{R}^n$ into zero volume subsets of M. Moreover, $F(\mathfrak{O}_\ell \times \mathbb{R}^{n-p}) = F(\mathfrak{O}_\ell \times \{0\})$ where the subset $\mathfrak{O}_\ell \times \{0\}$ of $\mathbb{R}^p \times \{0\}$ has zero Lebesgue measure in $\mathbb{R}^n$.

It follows that $\mathfrak{C}$ is forward invariant for the system (2) for all perturbation d with L^∞ norm smaller than D_c.

If M is compact, we have $\mathfrak{C} = M$. Therefore $\mathfrak{C}$ is trivially forward invariant for all perturbation d in L^∞_{loc}. In this case, the real number D_c to be used later on can be chosen arbitrarily large (but fixed) and (7) holds again.

To facilitate our forthcoming analysis, we impose also backward completeness. When M is not compact this can be achieved simply by modifying f outside $\mathfrak{C}$ as :

$$f_m(x,d) \;=\; \eta(V(x))\, f(x,d)$$

where η is a C^∞ function satisfying :

$$\begin{array}{rcll} \eta(w) & = & 1 & \text{if} \quad w \leq v \ , \\ & \in & [0,1] & \text{if} \quad v \leq w < v+1 \ , \\ & = & 0 & \text{if} \quad v+1 \leq w \ . \end{array}$$

In order not to overload our notations in this proof, we forget the subscript m for f and we still denote by $X(t,x,t_0,d)$ the solutions of :

$$\dot{x} \;=\; f_m(x(t),d(t)) \ . \tag{8}$$

Actually, this modification is used only at the very end in the construction of the set $\mathfrak{B}_d(t_0)$. Indeed in the remaining part of this proof, we restrict our attention to x in $\mathfrak{C}$, $t \geq t_0$ and d with L^∞ norm smaller than D_c, so there is no difference on $[t_0,+\infty)$ between solutions of (2) and solutions of (8).

Let $B_r(x)$ and $S_r(x)$ denote respectively the Riemannian ball and sphere centered at x and with radius r. Let also :

- $r_e \leq \frac{\delta}{4}$, D_e and ε_m be strictly positive real numbers such that
  1. we have :
  $$\varepsilon_m \;=\; -\frac{1}{2} \max_{x \in \mathfrak{C} \setminus \cup_\ell B_{\frac{r_e}{2}}(x_\ell)} \left. L_f V \right|_{x,0} \ .$$
  2. Then D_e should be such that
  $$\max_{x \in \mathfrak{C} \setminus \cup_\ell B_{\frac{r_e}{2}}(x_\ell)} \left. L_f V \right|_{x,d} \;\leq\; -\varepsilon_m \quad \forall d : \ |d| \leq D_e \ . \tag{9}$$
  3. for each equilibrium x_ℓ which is not asymptotically stable for the undisturbed system, we have
  $$B_{r_e}(x_\ell) \;\subset\; \mathcal{P}(x_\ell) \quad , \qquad D_e \;\leq\; D_p(x_\ell) \tag{10}$$
  where $\mathcal{P}(x_\ell)$ and $D_p(x_\ell)$ are respectively the set and the real number given by Lemma 1.
- $\mathcal{Q}(x_\ell) \subset B_{\frac{r_e}{2}}(x_\ell)$ be a neighborhood of x_ℓ defined as follows :

- If x_ℓ is asymptotically stable, V being strictly decreasing along solutions of the undisturbed system, x_ℓ is a strict local minimum of V. Thus, together with (4), continuity and compactness imply the existence of a compact neighborhood $\mathcal{Q}(x_\ell)$ of x_ℓ which is a connected component of a sublevel set of V and a subset of $B_{\frac{r_e}{2}}(x_\ell)$, and strictly positive real numbers $D_i(x_\ell)$ and $\varepsilon_i(x_\ell)$ so that :

$$\begin{array}{l} L_f V|_{x,d} \; < \; -\varepsilon_i(x_\ell) \\ \forall (x,d) : \; x \in B_{\frac{r_e}{2}}(x_\ell) \setminus \mathcal{Q}(x_\ell) \; , \; |d| \leq D_i(x_\ell). \end{array} \tag{11}$$

- If x_ℓ is not asymptotically stable, we pick $r_i(x_\ell)$ in $(0, \frac{r_e}{2})$ so that, by letting

$$\mathcal{Q}(x_\ell) = B_{r_i(x_\ell)}(x_\ell) \; ,$$

  we have :

$$\max_{x \in \mathcal{Q}(x_\ell)} V(x) - \min_{x \in \mathcal{Q}(x_\ell)} V(x) \; < \; \varepsilon_m \frac{r_e}{F} \tag{12}$$

  The continuity of V guarantees that such an $r_i(x_\ell)$ exists. Then again, continuity and compactness imply the existence of strictly positive real numbers $D_i(x_\ell)$ and $\varepsilon_i(x_\ell)$ so that (11) holds.

With all these definitions, we have :

$$\begin{array}{rl} L_f V|_{x,d} < & -\min\{\varepsilon_m, \varepsilon_i(x_\ell)\} \; < \; 0 \\ \forall (x,d) : & x \in \mathfrak{C} \setminus \bigcup_\ell \mathcal{Q}(x_\ell) \; , \\ & |d| \leq \min\{D_e, D_c, \min_\ell D_i(x_\ell)\}. \end{array} \tag{13}$$

Also any solution which leaves a ball $B_{\frac{r_e}{2}}(x_\ell)$ and reaches a sphere $S_{\frac{r_e}{2}}(x_j)$, with $j \neq \ell$, must "travel" during a time which is at least $\frac{\delta - r_e}{F}$. And during this time the Lyapunov function decreases by an amount which is at least

$$V_{var} \; = \; \varepsilon_m \frac{\delta - r_e}{F} \; \geq \; 4 \varepsilon_m \frac{r_e}{F} \; . \tag{14}$$

From now on, we restrict our attention to perturbations with L^∞ norm smaller than $\min\{D_e, D_c, \min_\ell D_i(x_\ell)\}$.

Pick a solution $X(t, x, t_0; d)$ which at time say t_C is in $\mathfrak{C}$. This compact set being forward invariant, the solution is in it for all times $t \geq t_C$. Since V is lower bounded, we conclude from (13), that this solution must enter or start from one of the sets $\mathcal{Q}(x_\ell)$ and, furthermore, can only spend finite time intervals outside of $\bigcup_\ell \mathcal{Q}(x_\ell)$.

In the following, we shall prove

Claim 1 *There exists a time t_* and an index ℓ_* such that we have :*

$$X(t, x, t_0; d) \in \mathcal{Q}(x_{\ell_*}) \qquad \forall t \geq t_* \; .$$

Assuming for the time being this claim holds true, then 2 cases are possible:
Case 1: x_{ℓ_*} is asymptotically stable. In this case, local asymptotic stability implies local Input-to-State Stability (ISS), for suitable restrictions on inputs and initial conditions. This result is usually stated for systems defined on Euclidean space (see Lemma I.1 in [21]), however, due to its local nature, it can be adapted straightforwardly to systems on manifolds. The estimate in equation (5) is a direct consequence of local ISS.

Case 2: x_{ℓ_*} is not asymptotically stable. In this case, with (10) and Lemma 1, the solution $t \mapsto X(t, x, t_0; d)$ is in $\mathfrak{A}_{\ell_*,d}(t, \mathfrak{O}_{\ell_*}) \cap B_{r_e}(x_{\ell_*})$ for each $t \geq t_*$ and therefore also in $\mathfrak{A}_{\ell_*,d}(i, \mathfrak{O}_{\ell_*}) \cap B_{r_e}(x_{\ell_*})$ for each integer i larger or equal to t_*. But this says that at time t_0 the solution was at x which is in[7] :

$$\mathfrak{C} \bigcap \bigcup_{i \in \mathbb{N} \geq t_0} X(t_0, \mathfrak{A}_{\ell_*,d}(i, \mathfrak{O}_{\ell_*}), i; d)$$

and therefore in the set $\mathfrak{B}_d(t_0)$ defined as :

$$\mathfrak{B}_d(t_0) \ = \ \mathfrak{C} \bigcap \bigcup_{\ell \leq \mathfrak{L} \, , \, i \in \mathbb{N} \geq t_0} X(t_0, \mathfrak{A}_{\ell,d}(i, \mathfrak{O}_\ell), i; d) \ .$$

In other words, if x is not in $\mathfrak{B}_d(t_0)$ then the set $\mathcal{Q}(x_{\ell_*})$ in which the solution $X(t, x, t_0; d)$ ends must be associated to an equilibrium x_{ℓ_*} which is asymptotically stable. Note finally that, since for any given pair (t_0, i), the function

$$x \mapsto X(t_0, x, i; d))$$

is Lipschitz on $\mathfrak{C}$, $\mathfrak{B}_d(t_0)$ is a countable union of images by Lipschitz maps of p-rectifiable sets, with $p \leq n-1$ and therefore is countably $n-1$-rectifiable. So it has zero volume.

To complete our proof, it remains to prove claim 1. Since the solution must enter one of the sets $\mathcal{Q}(x_\ell)$, say at time s, 2 cases are possible:

Case 1: x_ℓ is asymptotically stable. Under such hypotesis $\mathcal{Q}(x_\ell)$ is a connected component of a sublevel set of V which, with (11), is (strictly) forward invariant. So the solution will never leave it in future times.

Case 2: x_ℓ is not asymptotically stable. In this case if the solution $t \mapsto X(t, x, t_0; d)$, which is in the interior of $B_{r_e}(x_\ell)$ at time s, reaches $S_{r_e}(x_\ell)$ at time $\tau > s$, then, as shown below, it will never again enter $\mathcal{Q}(x_\ell)$. Note that since the solution can only spend finite time intervals outside $\bigcup_\ell \mathcal{Q}(x_\ell)$ and the number of x_ℓ is finite, this proves the claim. So for the sake of getting a contradiction, assume the solution does re-enter $\mathcal{Q}(x_\ell)$ at a time $\overline{\tau} > \tau > s$. Two sub-cases are possible.

[7]Recall that in this proof $X(t, x, t_0)$ is a solution of (8), system which by construction is backward complete.

Case 2.1 : It did not enter any other set $\mathcal{Q}(x_k)$ in the interval $(s, \bar{\tau})$. In this case the Lyapunov function has been decreasing whenever the solution was not in $\mathcal{Q}(x_\ell)$. Then because we have $X(s, x, t_0; d) \in \mathcal{Q}(x_\ell)$, $X(\tau, x, t_0; d) \in S_{r_e}(x_\ell)$, and $X(\bar{\tau}, x, t_0; d) \in \partial\mathcal{Q}(x_\ell)$ there exist $\tau_1 < \tau_2 < \tau \leq \tau_3 < \tau_4$ such that we have :

$$\begin{array}{rl} X(\tau_1, x, t_0; d) & \in \partial\mathcal{Q}(x_\ell), \\ \mathfrak{d}_M(x_\ell, X(\tau_2, x, t_0; d)) & = \frac{r_e}{2}, \\ \mathfrak{d}_M(x_\ell, X(\tau, x, t_0; d)) & = r_e, \\ \mathfrak{d}_M(x_\ell, X(\tau_3, x, t_0; d)) & = r_e, \\ \mathfrak{d}_M(x_\ell, X(\tau_4, x, t_0; d)) & = \frac{r_e}{2}. \end{array}$$

Specifically
- at time τ_1, the solution is in the boundary of $\mathcal{Q}(x_\ell)$ and, on the time interval $(\tau_1, \bar{\tau})$, the solution is not in $\mathcal{Q}(x_\ell)$. This implies that on this time interval the Lyapunov function decreases.
- the interval $(\tau_2, \tau]$ is defined so that the solution is in $B_{r_e}(x_\ell) \setminus B_{\frac{r_e}{2}}(x_\ell)$ while it belongs to the sphere $S_{\frac{r_e}{2}}(x_\ell)$ at time τ_2 and to the sphere $S_{r_e}(x_\ell)$ at time τ.
- the interval $[\tau_3, \tau_4)$ (with $\tau_3 \geq \tau$) is defined so that the solution is back to $B_{r_e}(x_\ell) \setminus B_{\frac{r_e}{2}}(x_\ell)$, being in the sphere $S_{r_e}(x_\ell)$ at time τ_3 and in the sphere $S_{\frac{r_e}{2}}(x_\ell)$ at time τ_4.

All this gives
- between τ_1 and τ_2, the solution is not in $\mathcal{Q}(x_\ell)$. So V decreases with some rate which we do not evaluate in this proof.
- between τ_2 and τ, the solution is in $B_{r_e}(x_\ell) \setminus B_{\frac{r_e}{2}}(x_\ell)$ and V decreases by at least $\varepsilon_m(\tau - \tau_2)$ which is lower bounded by $\varepsilon_m \frac{r_e}{2F}$.
- between τ and τ_3, V continues to decrease.
- between τ_3 and τ_4 the solution is in $B_{r_e}(x_\ell) \setminus B_{\frac{r_e}{2}}(x_\ell)$ and V decreases at least by $\varepsilon_m(\tau_4 - \tau_3)$ which is again lower bounded by $\varepsilon_m \frac{r_e}{2F}$.
- finally on the interval $(\tau_4, \bar{\tau})$, the solution is in not in $\mathcal{Q}(x_\ell)$ so V is still decreasing again with some rate which we do not evaluate in this proof.

Using (6) and (7), we have :

$$\begin{array}{l} \mathfrak{d}_M(x_\ell, X(\tau, x, t_0; d)) \\ \leq\ \mathfrak{d}_M(x_\ell, X(\tau_2, x, t_0; d)) + \mathfrak{d}_M(X(\tau, x, t_0; d), X(\tau_2, x, t_0; d)), \\ \mathfrak{d}_M(X(\tau, x, t_0; d), X(\tau_2, x, t_0; d)) \\ \leq\ \int_{\tau_2}^{\tau} |f(X(r, x, t_0; d), d(r))|_M\, dr\ \leq\ F\,(\tau - \tau_2)\ . \end{array}$$

This yields :

$$\begin{array}{rcl} r_e & = & \mathfrak{d}_M(x_\ell, X(\tau, x, t_0; d)) \\ & \leq & \mathfrak{d}_M(x_\ell, X(\tau_2, x, t_0; d))\ +\ F\,[\tau - \tau_2] \\ & = & \dfrac{r_e}{2}\ +\ F\,[\tau - \tau_2]\ . \end{array}$$

Similarly, we have :

$$\frac{r_e}{2}\ \geq\ \mathfrak{d}_M(x_\ell, X(\tau_4, x, t_0; d))$$

$$\begin{aligned} &\geq \mathfrak{d}_M(x_\ell, X(\tau, x, t_0; d)) - F[\tau_4 - \tau] \\ &= r_e - F[\tau_4 - \tau] \end{aligned}$$

and therefore :

$$r_e \leq F[\tau_4 - \tau_2]$$

So, with (9) and (11), we get :

$$\begin{aligned} \min_{x \in \mathcal{Q}(x_\ell)} V(x) &\leq V(X(\overline{\tau}, x, t_0; d)) \\ &\leq V(X(\tau_4, x, t_0; d)) \\ &\leq V(X(\tau_2, x, t_0; d)) - \varepsilon_m [\tau_4 - \tau_2] \\ &\leq V(X(\tau_2, x, t_0; d)) - \varepsilon_m \frac{r_e}{F} \\ &\leq V(X(\tau_1, x, t_0; d)) - \varepsilon_m \frac{r_e}{F} \\ &\leq \max_{x \in \mathcal{Q}(x_\ell)} V(x) - \varepsilon_m \frac{r_e}{F} . \end{aligned}$$

This contradicts (12). So, at least in this case, the solution cannot re-enter $\mathcal{Q}(x_\ell)$.

Case 2.2 : It has entered at least another set $\mathcal{Q}(x_k)$, with $x_\ell \neq x_k$ in the interval $(s, \overline{\tau})$. Let $x_{k_1}, \ldots, x_{k_G}$ be the finite sequence of equilibria corresponding to the neighborhoods $\mathcal{Q}(x_{k_i})$, $i \in \{1, \ldots, G\}$, visited by the solution X before entering $\mathcal{Q}(x_\ell)$ again (where obviously $k_1 = k_G = \ell$). Without loss of generality all of them are distinct from each other (except the first and the last). Clearly none of the x_{k_i}s can be an asymptotically stable equilibrium. Moreover, along any solution leaving a $\mathcal{Q}(x_{k_i})$ to reach $\mathcal{Q}(x_{k_{i+1}})$, the Lyapunov function decreases as long as it belongs to $\left(B_{\frac{r_e}{2}}(x_{k_i}) \setminus \mathcal{Q}(x_{k_i})\right) \bigcup \left(B_{\frac{r_e}{2}}(x_{k_{i+1}}) \setminus \mathcal{Q}(x_{k_{i+1}})\right)$, and as long as it does not belong to $B_{\frac{r_e}{2}}(x_{k_i}) \cup B_{\frac{r_e}{2}}(x_{k_{i+1}})$ it decreases of at least V_{var} (see (14)). Finally, when it exits $\mathcal{Q}(x_{k_{i+1}})$ (i ranging in $1, 2, \ldots, G-1$) the Lyapunov function may have increased, but less than

$$\max_{x \in \mathcal{Q}(x_\ell)} V(x) - \min_{x \in \mathcal{Q}(x_\ell)} V(x) < \varepsilon_m \frac{r_e}{F} < V_{var}.$$

By combining all these considerations, we may estimate that the Lyapunov function between the time the solution leaves $\mathcal{Q}(x_\ell)$ and the time it re-enters it, has decreased of at least $(G-2)(V_{var} - \varepsilon_m r_e/F) + \Delta$. Therefore, we have the following estimations on the various values of the Lyapunov function :

$$\begin{aligned} V(\text{leaving } \mathcal{Q}(x_\ell)) &\leq \max_{x \in \mathcal{Q}(x_\ell)} V(x), \\ V(\text{re-entering } \mathcal{Q}(x_\ell)) &\leq V(\text{leaving } \mathcal{Q}(x_\ell)) \\ &\quad - (G-2)(V_{var} - \varepsilon_m r_e/F) - \Delta, \\ \min_{x \in \mathcal{Q}(x_\ell)} V(x) &\leq V(\text{re-entering } \mathcal{Q}(x_\ell)). \end{aligned}$$

They give :

$$\begin{array}{ll} \max_{x\in\mathcal{Q}(x_\ell)} & V(x) - \min_{x\in\mathcal{Q}(x_\ell)} V(x) \\ & \geq (G-2)\,(V_{var} - \varepsilon_m r_e/F) \;+\; \Delta \;\geq\; V_{var}. \end{array}$$

But, with (14), this contradicts (12).

3 A sufficient condition for almost global Input-to-State Stability

The Main result in the previous Section will be used in order to develop a checkable sufficient condition for the notion of almost global Input-to-State Stability (aISS), recently introduced in [1], which we recall below:

Definition 1 *A system as in (2) is said to be almost globally Input-to-State Stable with respect to a compact subset $A \subset M$ if A is locally asymptotically stable for $d \equiv 0$ and there exists $\tilde{\gamma} \in \mathcal{K}$ such that for each locally essentially bounded and measurable perturbation $d : \mathbb{R} \to \mathcal{D}$, there exists a zero volume set $\tilde{\mathfrak{B}}_d \subset M$ such that, for all $x \in M \backslash \tilde{\mathfrak{B}}_d$, it holds:*

$$\limsup_{t\to+\infty} \mathfrak{d}_M(X(t,x,0;d), A) \leq \tilde{\gamma}(\|d\|_\infty).$$

Notice that in this last inequality we specify $t_0 = 0$, without loss of generality.

Remark 2 This notion is useful in many different contexts, both for systems with $M = \mathbb{R}^n$ (for instance when A is a limit cycle or a set of more than one equilibrium point), as well as for nonlinear systems evolving on manifolds non diffeomorphic to Euclidean space (in which case even A being a single equilibrium requires almost global tools to be handled). Despite its potential interest, few sufficient conditions are available to prove this holds in actual examples. It is also worth pointing out that it is a purely open-loop notion of robustness; as $\tilde{\mathfrak{B}}_d$ is d dependent, letting d be a function of x is generally not possible.

Definition 2 *A system as in (2) fulfills the ultimate boundedness property if there exists a class $\mathcal{K}$ function δ, a constant c and a point $\xi \in M$ such that for each $x \in M$, each $t_0 \in \mathbb{R}$ and each locally essentially bounded and measurable perturbation d, the solution $X(t,x,t_0;d)$ is defined on $[t_0,+\infty)$, and eventually confined to*

$$\{z \in M : \mathfrak{d}_M(z,\xi) \leq \delta(\|d\|_\infty) + c\}.$$

We remark that in the previous definition Ultimate Boundedness could have been equivalently defined by considering the point-set distance to a compact subset of M, rather than a singleton $\{\xi\}$. Our main result for this Section is stated below.

Proposition 2 *Consider a system as in (2) which fulfills assumptions A0 to A2. Assume, in addition, that the set of asymptotically stable equilibria of (3), denoted by E_s, be finite. If ultimate boundedness holds, then, (2) is almost globally ISS with respect to the set E_s.*

Proof. By ultimate boundedness there exist a function $\tilde{\delta}$ of class $\mathcal{K}$ and a constant $\tilde{c}$ such that for each $x \in M$, each $t_0 \in \mathbb{R}$ and each locally essentially bounded and measurable perturbation d, the solution $X(t, x, t_0; d)$ is defined on $[t_0, +\infty)$ and fulfills :

$$\limsup_{t \to +\infty} \mathfrak{d}_M(X(t, x, t_0; d), E_s) \leq \tilde{c} + \tilde{\delta}(\|d\|_\infty). \tag{15}$$

Let the compact set $\mathfrak{C}$ invoked in the main result be selected to contain the set $\{x \in M : \mathfrak{d}_M(x, E_s) \leq \tilde{c} + \tilde{\delta}(1) + 1\}$. Then, let $\Delta > 0$ be given as from our main result. Fix d, as an arbitrary measurable perturbation which is essentially bounded (for unbounded d there is nothing to prove). Since a Riemannian manifold is σ-compact[8], we can pick a monotone increasing sequence of compact subsets of M, $K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots$ with the property that $\bigcup_{n \in \mathbb{N}} K_n = M$. Assume $\|d\|_\infty \leq \min\{\Delta, 1\}$. By virtue of (15), and continuity of solutions with respect to initial conditions, for all $n \in \mathbb{N}$ there exists $T_n > 0$ such that $X(T_n, K_n, 0; d)$ is a subset of $\mathfrak{C}$. Then, applying our main result yields existence of a zero volume set $\mathfrak{B}_{dn} \subset \mathfrak{C}$ such that, for all $x \in K_n$ such that $X(T_n, x, 0; d)$ is in $\mathfrak{C} \backslash \mathfrak{B}_n$, it holds

$$\limsup_{t \to +\infty} \mathfrak{d}_M(X(t, x, 0; d), E_s) \leq \gamma(\|d\|_\infty) \ . \tag{16}$$

Since $x \mapsto X(T_n, x, 0; d)$ is a diffeomorphism which preserves zero volume sets and with inverse $\xi \mapsto X(0, \xi, T_n; d)$, it follows that $\tilde{\mathfrak{B}}_{dn} = X(0, \mathfrak{B}_n, T_n; d)$ has zero volume and, for each $\tilde{x} \in K_n \backslash \tilde{\mathfrak{B}}_{dn}$ (16) holds.

Let $\tilde{\mathfrak{B}} := \bigcup_{n \in \mathbb{N}} \tilde{\mathfrak{B}}_{dn}$. It has zero volume as a countable union of zero volume sets. Moreover, for all $x \in M \backslash \tilde{\mathfrak{B}}$ there exists $n \in \mathbb{N}$ so that $x \in K_n$; thus $x \in K_n \backslash \tilde{\mathfrak{B}} \subset K_n \backslash \tilde{\mathfrak{B}}_{dn}$ and inequality in (16) holds. Finally, almost global ISS follows simply by combining (16) and condition (15) with $\tilde{\gamma}$ given as :

$$\begin{array}{rcll} \tilde{\gamma}(s) & = & \gamma(s) & \text{if} \quad s \leq \min\{\Delta, 1\} \ , \\ & \geq & \tilde{c} \ + \ \tilde{\delta}(s) & \text{if} \quad \min\{\Delta, 1\} < s \ . \end{array}$$

Proposition 3 *Consider a system as in (2), and assume that $W : M \to \mathbb{R}_{\geq 0}$ exists, of class $\mathcal{C}^1$, proper and satisfying:*

$$L_f W|_{x,d} \leq -\alpha(W(x)) + c + \delta(|d|) \tag{17}$$

for all $x \in M$ and all $d \in \mathcal{D}$. Then, system (2) fulfills the ultimate boundedness property.

[8] A Riemannian manifold is locally compact (see [6, Theorem VI.6.6 and page 335]) and paracompact (Stone Theorem) (see [25, Theorem 20.9]). Moreover, a paracompact, locally compact and connected space is σ-compact (see [12, Lemmas 5 and 6]).

Proof. By virtue of (17), it holds:

$$\limsup_{t\to+\infty} \alpha(W(x(t))) \leq 2(c+\delta(|\ d\|_\infty)). \tag{18}$$

As W is proper, taken any $z \in M$, κ of class $\mathcal{K}_\infty$ and a constant c_z exist, so that for all $x \in M$

$$\kappa\left(\mathfrak{d}_M(z,x)\right) \leq W(x) + c_z. \tag{19}$$

Combining (19) and (18) proves ultimate boundedness.

3.1 A planar example: pendulum with friction

Consider the following set of differential equations, describing the motion of a forced pendulum with friction:

$$\begin{array}{rcl} \dot{\theta} & = & \omega \\ \dot{\omega} & = & -a\sin(\theta) - b\omega + d. \end{array} \tag{20}$$

We regard them as a system with state $x = [\theta, \omega]$ taking values on the cylinder $M := \mathbb{S} \times \mathbb{R}$ affected by some exogenous disturbance $d(t)$, whereas a, b are constant positive parameters. The following question was publicly posed in Oberwolfach meeting: is the above system almost globally Input-to-State Stable ? Consider the mechanical energy of the pendulum, that is $W(x) = \omega^2/2 - a\cos(\theta)$. Taking derivatives along (20) yields:

$$\begin{array}{rcl} \dot{W}(x) & = & -b\omega^2 + \omega d \leq -\frac{b}{2}\omega^2 + \frac{1}{2b}d^2 \\ & = & -\frac{b}{2}W(x) - \frac{ab}{2}\cos(\theta) + \frac{1}{2b}d^2 \\ & \leq & -\frac{b}{2}W(x) + c + \frac{1}{2b}d^2 \end{array}$$

with constant $c := ab/2$. By virtue of (17), system (20) fulfills ultimate boundedness. Moreover, it is straightforward to see that, when $d = 0$, (20) has only two equilibria $x_1 = [0, 0]$ and $x_2 = [\pm\pi, 0]$. In particular, x_1 is asymptotically stable, whereas x_2 is an hyperbolic saddle point. Let us denote $E := \{x_1, x_2\}$. In order to build a strict Lyapunov function for (20) we perturb W as follows:

$$V(x) = W(x) + \varepsilon\omega\sin(\theta) \tag{21}$$

for some small parameter ε to be fixed later. Along solutions of the autonomous system V fulfills the following dissipation inequality:

$$\begin{array}{rcl} \dot{V}(x) & = & dV(x)\cdot f(x,0) \\ & = & -b\omega^2 - \varepsilon a\sin^2(\theta) - \varepsilon b\omega\sin(\theta) + \varepsilon\omega^2\cos(\theta) \\ & \leq & -(b-\varepsilon)\omega^2 - \varepsilon a\sin^2(\theta) - \varepsilon b\omega\sin(\theta) < 0 \end{array}$$

for all $x \notin E$, provided $b > \varepsilon$ and $a(b-\varepsilon) > \varepsilon b^2/4$. The previous inequalities can be simultaneously fulfilled by taking ε sufficiently small. Hence, the pendulum equations fulfill all assumptions of our previous result, and we can therefore conclude almost global Input-to-State Stability.

3.2 A scalar counter-example

We show next, by means of a simple scalar example, that the existence of at least one unstable eigenvalue is an assumption which cannot be removed from the Main Result.

Let M be the unit circle $\mathbb{S}$ and θ be the corresponding angular coordinate on $\mathbb{S}$. Consider the system:

$$\dot{\theta} = -\sin^3(\theta) + \sin(\theta)d. \tag{22}$$

For $d = 0$ the system has two equilibria, namely $\theta = 0$ which is asymptotically stable and $\theta = \pi$ which is antistable. Notice that the differential of (22) at $\theta = \pi$ is $df_0(\pi) = 0$, so that the linearized system does not have positive eigenvalues at the unstable equilibrium. We want to show that, even for arbitrarily small input signals it is not true that almost all solutions converge to a neighborhood of 0 whose volume shrinks to 0 as the input perturbation L_∞ norm tends to 0. Indeed, taking constant inputs we obtain $df(\theta, d)|_{\theta=\pi} = -d$ which yields the linearized system:

$$\delta\dot{\theta} = -d\delta\theta.$$

Therefore, for all $d > 0$ we have local asymptotic stability of the equilibrium at $\theta = \pi$. This proves that, no matter how small we pick d there always exists a basin of attraction of positive measure for the equilibrium $\theta = \pi$. This simple example justifies our assumption on df_0.

4 Complementary bibliographical notes

It is worth pointing out that Assumption A1 combined with isolation of equilibria basically amount to an infinitesimal characterization of *gradient-like* systems, introduced by Conley in [9] (original definition entails continuous Lyapunov functions rather than $\mathcal{C}^1$ and is formulated in the context of flows on compact metric spaces rather than differential equations on Riemannian manifolds). They should not be confused with *gradient* systems which are instead often defined on Riemannian manifolds and are for instance studied in [19]. While it is true that a gradient system whose associated Lyapunov function has only isolated stationary points is gradient-like, the converse need not be true.

Gradient-like systems have the peculiarity of having a chain-recurrent set which is of the simplest possible form, namely a totally disconnected set made up by equilibria only. This property was actually used as a definition in [9]. It turns out that when this is the case, then the chain-recurrent set also coincides with the set of points in state space with the property that all continuous functions which are globally non-increasing along the flow are constant along solutions initiated at those points.

Perturbations of gradient-like systems have been studied in the literature, however, this is the first time (to the best of our knowledge) that ISS-like estimates are attempted. In particular, in a recent series of papers [14, 7, 8], small-size time-varying perturbations of gradient-like systems (evolving in Banach spaces) are studied and results derived on the continuity property of the *global* attractor of gradient-like systems. Indeed it is proved that the global attractor of a gradient-like system is "robust" with respect to small time-varying perturbations. In particular it has the same structure of the attractor of the autonomous system, namely the union of globally bounded solutions resulting as perturbation of autonomous systems equilibria and their unstable manifolds. The results presented in this paper differ in several respects from those in the above mentioned papers:

1. Global attractors are the object of interest in [14, 7, 8]; an *attractor* is not necessarily asymptotically stable, and this allows to formulate global convergence notions also in the presence of multiple equilibria (some of which are typically unstable). As we deal with a generalization of almost global asymptotic stability, a global approach is not possible in our case and only almost global ISS estimates and robustness results are guaranteed.

2. In addition, papers [14, 7, 8] assume hyperbolic equilibria (which is not a requirement in this note) and do not immediately apply to dynamical systems evolving on manifolds.

In [5], instead, asymptotically autonomous gradient-like systems are considered and it is shown that their ω-limit sets are necessarily equilibria.

It is worth pointing out that in the recent paper [24], density and Lyapunov functions are jointly used in order to claim almost global Input-to-State Stability properties. The tools developed in this note are meant to complement these techniques which are seemingly hard to use in the context of systems with unstable equilibria of negative divergence, as emphasized in [2].

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Proof of Lemma 1

It suffices to show that the assumptions of Theorem 4.1 in [4] are satisfied.

Let x_ℓ be an equilibrium point of (3) with $df_0(x_\ell)$ having at least one eigenvalue with strictly positive real part. Let ϕ be a coordinate map defined on a neighborhood of x_ℓ in M, with values in $\mathbb{R}^n$ and with a locally Lipschitz inverse. By $\mathcal{x}$ we denote coordinates (in $\mathbb{R}^n$) so that $\phi(x_\ell)$ is the origin. Associated to these coordinates, we have a neighborhood $\mathcal{V} \subset \mathbb{R}^n$ of the origin and a locally C^1-Lipschitz function $\varphi : \mathcal{V} \times \mathbb{R}^m \to \mathbb{R}^n$ satisfying :

$$\varphi(0,0) \;=\; 0$$

and such that the image by ϕ of any solutions of (2) restricted to the preimage by ϕ of $\mathcal{V}$ is a solution of :

$$\dot{\mathcal{x}}(t) \;=\; \varphi(\mathcal{x}(t), d(t)) \; . \tag{23}$$

We define the functions and matrices :

$$\begin{aligned} \sigma(d) &= \varphi(0,d) \; , \\ Q &= \frac{\partial \varphi}{\partial \mathcal{x}}(0,0) \; , \end{aligned}$$

$$
\begin{aligned}
N(d) &= \frac{\partial \varphi}{\partial x}(0,d) \; - \; \frac{\partial \varphi}{\partial x}(0,0) \; , \\
P(x,d) &= \frac{\partial \varphi}{\partial x}(x,d) \; - \; (N(d)+Q) \; .
\end{aligned}
$$

They are defined on $\mathcal{V} \times \mathbb{R}^m$, locally Lipschitz and satisfy

$$
\sigma(0) = 0 \quad , \qquad N(0) = 0 \quad , \qquad P(0,d) = 0 \; .
$$

Then, possibly by restricting our attention to an open subset of $\mathcal{V}$ with a compact closure, there exists $\mathcal{L}$ such that we have for all $x_1, x_2 \in \mathcal{V}$ and all d with $|d| \leq 1$:

$$
\begin{aligned}
|\sigma(d)| \; + \; |N(d)| &\leq \mathcal{L}\,|d|, & (24)\\
|P(x_1,d) - P(x_2,d)| &\leq \mathcal{L}\,|x_1 - x_2|. & (25)
\end{aligned}
$$

Now, from the identity :

$$
\varphi(x,d) \; = \; \varphi(0,d) \; + \; \left(\int_0^1 \frac{\partial \varphi}{\partial x}(\lambda x, d) d\lambda \right) x \; ,
$$

we get :

$$
\varphi(x,d) \; = \; \sigma(d) \; + \; Q\,x \; + \; N(d)\,x \; + \; \psi(x,d) \; ,
$$

where :

$$
\psi(x,d) \; = \; \left(\int_0^1 P(\lambda x, d) d\lambda \right) x \; .
$$

Notice that, with (25), we have :

$$
\begin{aligned}
& |\psi(x_1,d) - \psi(x_2,d)| \\
\leq \; & \int_0^1 \Big| [P(\lambda x_1,d) - P(\lambda x_2,d)] x_1 \Big| \, d\lambda \\
& + \int_0^1 \Big| P(\lambda x_2,d) [x_1 - x_2] \Big| \, d\lambda \\
\leq \; & \int_0^1 \left(\mathcal{L}\lambda |x_1 - x_2| \cdot |x_1| + \mathcal{L}\lambda |x_2| \cdot |x_1 - x_2| \right) d\lambda \\
\leq \; & \mathcal{L} \frac{|x_1| + |x_2|}{2} |x_1 - x_2| \quad \forall d \, : |d| \leq 1 \, \forall x_1, x_2 \in \mathcal{V}.
\end{aligned}
$$

Now, without loss of generality, we can assume that the coordinates x are such that the matrix Q has the following block diagonal structure :

$$
Q \; = \; \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}
$$

where the real parts of the eigenvalues of A are strictly smaller than those of B, the latter being strictly positive. The only case where this structure would not exist would be if Q had all its eigenvalues with the same, and therefore strictly positive, real part. This particular case will be considered at the end of this proof.

The spectral separation for A and B implies the existence of real numbers $K > 0$, $\alpha < \beta$ (with $\beta > 0$) satisfying for all $t \geq 0$:

$$|\exp(At)| \leq K \exp(\alpha t),$$
$$|\exp(-Bt)| \leq K \exp(-\beta t).$$

From these numbers, we define the non negative real number γ as :

$$\gamma = \frac{\beta + \max\{0, \alpha\}}{2} .$$

Corresponding to the decomposition of Q, we have the decompositions :

$$\mathcal{X} = (x, y), \ \sigma = (\sigma_x, \sigma_y), \ N = (N_x, N_y), \ \psi = (f, g),$$

with x of dimension n_x and y of dimension n_y, and (23) reads :

$$\begin{aligned} \dot{x} &= Ax + \sigma_x(d(t)) + N_x(d(t)) \begin{pmatrix} x \\ y \end{pmatrix} + f(x, y, d(t)), \\ \dot{y} &= By + \sigma_y(d(t)) + N_y(d(t)) \begin{pmatrix} x \\ y \end{pmatrix} + g(x, y, d(t)). \end{aligned} \quad (26)$$

This system is defined only on $\mathcal{V} \times \mathbb{R}^m$. We extend it to $\mathbb{R}^n \times \mathbb{R}^m$ as follows. Let L and D_1 be real numbers defined as :

$$\begin{aligned} L &= \tfrac{\beta + 2\gamma - 3\alpha - \sqrt{(\beta - \alpha)^2 + 8(\gamma - \alpha)^2}}{2K}, \\ D_1 &= \min\left\{1, \tfrac{L}{2\mathcal{L}}\right\}. \end{aligned} \quad (27)$$

In particular the choice for L is made to get :

$$\frac{K}{\beta - \gamma - KL} \frac{L(\gamma - \alpha)}{\gamma - \alpha - KL} = \frac{1}{2} .$$

We show next that $L > 0$ follows by taking into account separately the two cases, $\alpha \geq 0$ and $\alpha < 0$ in the definition of γ.

1. $\alpha \geq 0$ yields $\gamma = (\alpha + \beta)/2$ and therefore

$$L = \frac{2(\beta - \alpha) - \sqrt{3(\beta - \alpha)^2}}{2K} = (2 - \sqrt{3}) \frac{\beta - \alpha}{2K} > 0$$

2. $\alpha < 0$ yields $\gamma = \beta/2$ and therefore

$$L = \frac{2\beta - 3\alpha - \sqrt{3\beta^2 + 9\alpha^2 - 10\alpha\beta}}{2K}$$

   Then $L > 0$ follows by noticing that $(2\beta - 3\alpha)^2 - (3\beta^2 + 9\alpha^2 - 10\alpha\beta) = \beta^2 - 2\alpha\beta > 0$.

Similarly, by taking into account separately the cases $\alpha \geq 0$ and $\alpha < 0$, it is easy to show that $\beta - \gamma - KL$ and $\gamma - \alpha - KL$ are strictly positive.

Let also the positive real b satisfy :

$$b \leq \frac{L}{4\mathcal{L}} \tag{28}$$

and be such that the closed ball $\bar{B}_b$ centered at the origin with radius b is contained in $\mathcal{V}$. We let :

$$h(\mathcal{X}) = \begin{cases} 1 & \text{if } |\mathcal{X}| \leq \frac{b}{2}, \\ 2(b - |\mathcal{X}|)/b & \text{if } \frac{b}{2} < |\mathcal{X}| < b, \\ 0 & \text{if } b \leq |\mathcal{X}|. \end{cases}$$

and define :

$$F(t,x,y) = \begin{cases} h((x,y))\Big[\sigma_x(d(t)) & + N_x(d(t))\begin{pmatrix} x \\ y \end{pmatrix} \\ \quad +f(x,y,d(t))\Big] & \text{if } |(x,y)| < b\ , \\ 0 & \text{if } b \leq |(x,y)|, \end{cases}$$

$$G(t,x,y) = \begin{cases} h((x,y))\Big[\sigma_y(d(t)) & + N_y(d(t))\begin{pmatrix} x \\ y \end{pmatrix} \\ \quad +g(x,y,d(t))\Big] & \text{if } |(x,y)| < b\ , \\ 0 & \text{if } b \leq |(x,y)|. \end{cases}$$

With (24), (26), (27) and (28), for any function $t \mapsto d(t)$ in L^∞_{loc} with L^∞-norm $\|d\|_\infty$ smaller than D_1, we have for all (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^n$ and almost all[9] t in $\mathbb{R}$,

$$\begin{aligned} |F(t,x_1,y_1) - F(t,x_2,y_2)| &\leq L\left(|x_1 - x_2| + |y_1 - y_2|\right) , \\ |G(t,x_1,y_1) - G(t,x_2,y_2)| &\leq L\left(|x_1 - x_2| + |y_1 - y_2|\right) , \\ |F(t,0,0)| \ + \ |G(t,0,0)| &\leq \rho(\|d\|_\infty) \end{aligned} \tag{29}$$

where ρ is a class-$\mathcal{K}$ function (which bounds σ). Also any solution of (26) satisfies :

$$\begin{aligned} \dot{x} &= Ax + F(t,x,y) , \\ \dot{y} &= By + G(t,x,y) . \end{aligned} \tag{30}$$

as long as it is in the open ball $B_{\frac{b}{2}}$. As a consequence, any solution of (2) whose image by the coordinate map ϕ stays in $B_{\frac{b}{2}}$ for all times in an interval like $[s, +\infty)$ is a solution of (30) on this interval and can be extended as a solution of (30) on $(-\infty, s]$ This system (30) satisfies all the assumptions of [4, Theorem

[9] This is not for all t as requested in the statement of [4, Theorem 4.1]. But this Theorem holds also in this case.

4.1] except that, here, the right hand side of (29) is not zero. By adapting[10] the proof of this Theorem, we can show the existence of a unique continuous function $s_0 : \mathbb{R} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ such that $x \mapsto s_0(t,x)$ is Lipschitz uniformly in t and any solution $(X(t,(x,y),t_0), Y(t,(x,y),t_0))$ of (30), passing through (x,y) at time t_0, (necessarily defined on $\mathbb{R}$) which satisfies :

$$\sup_{t\geq\tau} |(X(t,(x,y),t_0), Y(t,(x,y),t_0))| \exp(-\gamma t) \;<\; +\infty$$

for some τ satisfies also :

$$Y(t,(x,y),t_0) \;=\; s_0(t, X(t,(x,y),t_0)) \qquad \forall\, t \in \mathbb{R} \;.$$

In view of the relation between solutions of (26) and solutions of (30) and since γ is not negative, this implies that any solution $(X(t,(x,y),t_0), Y(t,(x,y),t_0))$ of (26) which satisfies :

$$(X(t,(x,y),t_0), Y(t,(x,y),t_0)) \;\in\; B_{\frac{b}{2}} \qquad \forall t \geq \tau \tag{31}$$

[10]Since the right hand side of (29) is not zero, we may have $s_0(t,0)$ non zero. Also in the proof of [4, Theorem 4.1], we have to make sure $\|\mu^*\|^+_{\tau,\gamma}$ is bounded. In our case, we get that what is before [4, (63)] becomes :

$$\|\nu\|^+_{\tau,\gamma} \leq K|\xi| \exp(-\gamma\tau) + \frac{K}{\gamma - \alpha - KL}(L\|\mu\|^+_{\tau,\gamma} + \|F(\cdot,0,0)\|^+_{\tau,\gamma}),$$

and [4, (64)] becomes :

$$\begin{aligned}
&|G(t,\nu(t),\mu(t))| \exp(-\gamma t) \\
&\qquad \leq\; L\,\|\nu\|^+_{\tau,\gamma} + L\,\|\mu\|^+_{\tau,\gamma} + |G(t,0,0)| \exp(-\gamma t)\;, \\
&\qquad \leq\; KL|\xi|\exp(-\gamma\tau) + \frac{KL^2}{\gamma-\alpha-KL}\|\mu\|^+_{\tau,\gamma} + L\|\mu\|^+_{\tau,\gamma} \\
&\qquad\quad + \tfrac{KL}{\gamma-\alpha-KL}\|F(\cdot,0,0)\|^+_{\tau,\gamma} + \|G(\cdot,0,0)\|^+_{\tau,\gamma}\;, \\
&\qquad \leq\; KL|\xi|\exp(-\gamma\tau) + \frac{\gamma-\alpha}{\gamma-\alpha-KL}\|\mu\|^+_{\tau,\gamma} \\
&\qquad\quad + \tfrac{KL}{\gamma-\alpha-KL}\|F(\cdot,0,0)\|^+_{\tau,\gamma} + \|G(\cdot,0,0)\|^+_{\tau,\gamma}\;.
\end{aligned}$$

Then, with [4, Lemma 3.6], we have :

$$\begin{aligned}
\|\mu^*\|^+_{\tau,\gamma} &\leq \tfrac{K}{\beta-\gamma-KL}\|G(\cdot,\nu(\cdot),\mu(\cdot))\|^+_{\tau,\gamma}\;, \\
&\leq \tfrac{K}{\beta-\gamma-KL}(KL|\xi|\exp(-\gamma\tau) + \tfrac{\gamma-\alpha}{\gamma-\alpha-KL}\|\mu\|^+_{\tau,\gamma} \\
&\quad + \tfrac{KL}{\gamma-\alpha-KL}\|F(\cdot,0,0)\|^+_{\tau,\gamma} + \|G(\cdot,0,0)\|^+_{\tau,\gamma})
\end{aligned}$$

With (29) and $\|d\|_\infty \leq D_1$, this yields :

$$\begin{aligned}
\|\mu^*\|^+_{\tau,\gamma} &\leq \tfrac{K}{\beta-\gamma-KL}(KL|\xi|\exp(-\gamma\tau) \\
&\quad + \tfrac{L(\gamma-\alpha)}{\gamma-\alpha-KL}\|\mu\|^+_{\tau,\gamma} + \tfrac{\gamma-\alpha}{\gamma-\alpha-KL}\rho(\|d\|_\infty)\exp(-\gamma\tau)\;)
\end{aligned}$$

Hence the fixed point $\mu_{\tau,\xi}$ (and therefore the function $t \mapsto s_0(t,x)$) of the operator $T_{\tau,\xi}$ satisfies the bound :

$$\begin{aligned}
&[1 - \frac{K}{\beta-\gamma-KL}\,\frac{L(\gamma-\alpha)}{\gamma-\alpha-KL}]\|\mu^*_{\tau,\xi}\|^+_{\tau,\gamma} \\
&\qquad \leq\; \frac{K}{\beta-\gamma-KL}(KL|\xi|\exp(-\gamma\tau) + \frac{\gamma-\alpha}{\gamma-\alpha-KL}\rho(\|d\|_\infty)\exp(-\gamma\tau)\;).
\end{aligned}$$

for some τ satisfies also :

$$Y(t,(x,y),t_0) \;=\; s_0(t, X(t,(x,y),t_0)) \qquad \forall t \in \mathbb{R}\ .$$

Notice for instance that, if for all x with norm smaller than $\frac{b}{2}$, and for all τ, we can find $t \geq \tau$ such that $(x, s_0(t,x))$ is not in $B_{\frac{b}{2}}$, then the condition (31) can never be satisfied. Hence the claim of Lemma 1 holds with : $\mathcal{P}(x_\ell) = \phi^{-1}(B_{\frac{b}{2}})$, $p = n_x$, $D_p(x_\ell) = D_1$,

$$\mathfrak{O}_\ell = \{x \in \mathbb{R}^{n_x}\ :\ |x| < b/2\}$$

and

$$\mathfrak{A}_{\ell,d}(t,x) \;=\; \phi^{-1}((x, s_0(t,x)))\ .$$

Now, if Q has all its eigenvalues with the same real part, then there is no exponential dichotomy, viz. no A, x, ..., i.e. $n_y = n$ and the system (26) reduces to :

$$\dot{y} \;=\; B\,y \;+\; \sigma_y(d(t)) \;+\; N_y(d(t))y \;+\; g(y,d(t))\ . \tag{32}$$

Again this system satisfies all[11] the assumptions of Lemma 3.6 in [4]. It follows that there exists a unique continuous function $s_0 : \mathbb{R} \to \mathbb{R}^{n_y}$ such that any solution $Y(t,y,t_0)$ of (32), passing through y at time t_0, (necessarily defined on $\mathbb{R}$) which satisfies :

$$\sup_{t\geq\tau} |Y(t,y,t_0))| \exp(-\frac{\beta}{2}t) \;<\; +\infty$$

for some τ satisfies also :

$$Y(t,y,t_0) \;=\; s_0(t) \qquad \forall\, t \in \mathbb{R}\ .$$

From this, we can conclude the proof as above.

[11] Except that (29) holds only for almost all t which has no consequences.