Causal State-Feedback Parameterizations in Robust Model Predictive Control ⋆

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Abstract

In this paper, we investigate the problem of nonlinearity (and non-convexity) typically associated with linear state-feedback parameterizations in the Robust Model Predictive Control (RMPC) for uncertain systems. In particular, we propose two tractable approaches to compute an RMPC controller - consisting of both a causal, state-feedback gain and a control-perturbation component - for linear, discrete-time systems involving bounded disturbances and norm-bounded structured model-uncertainties along with hard constraints on the input and state. Both the state-feedback gain and the control-perturbation are explicitly considered as decision variables in the online optimization while avoiding nonlinearity and non-convexity in the formulation. The proposed RMPC controller - computed through LMI optimizations - is responsible for steering the uncertain system state to a terminal invariant set. Numerical examples from the literature demonstrate the advantages of the proposed scheme.

Key words: Robust Model Predictive Control; optimization under uncertainties; relaxation; S-Procedure; LMI.

1 Introduction

Robust Model Predictive Control (RMPC) strategies have received considerable amount of attention over the past decade. These refer to a class of algorithms which involve optimization to compute control action whilst taking account of system uncertainty/disturbances [9].

Most of the RMPC schemes proposed in the literature can be classified into two categories (or their combinations/variations): open-loop MPC and feedback MPC. Open-loop schemes consider the future input profile as a function of the current state only which, though computationally efficient, is generally too conservative and may cause infeasibility [9]. On the other hand, feedback RMPC schemes consider future inputs as (linear/nonlinear) functions of future predicted states and, therefore, have the advantage of mitigating the effect of uncertainties while potentially avoiding the aforementioned infeasibility problems. Within this category, nonlinear feedback schemes (see e.g. [12]) enjoy reduced conservatism; however, their main drawback is the excessive online computational burden due to the combinatorial nature of the optimization. Therefore, much of the research has been focused towards linear state-feedback RMPC schemes which we discuss next.

To reduce conservatism, a desirable approach in linear state-feedback RMPC is to directly consider the feedback gains as decision variables in the online optimization. However, as noted in [4], the problem with this approach is that formulating such an RMPC problem in the standard way leads to sequences of predicted states and inputs which are nonlinear functions of the state-feedback gains. Therefore, the resulting problem becomes non-convex. A solution to this problem has been proposed in [14] where the state-feedback gains are computed through sequential online optimization based, in part, on the principles of Dynamic Programming. The same approach has been extended, in [15], to systems with scalar uncertainties in their dynamics. The alternative approach involves the use of Q-parameterization-like methods - sometimes also called Youla parameterization - to obtain convexity [17]. The application of such methods in the context of min-max RMPC was introduced in [7]. These results were extended in [4], where the authors showed that for systems involving bounded disturbances, under suitable assumptions, affine state-feedback becomes equivalent to a disturbance-feedback parametrization. Recently, a similar method - based on Q-parameterization - has been proposed in the context of RMPC for systems with stochastic disturbances [13].

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In much of the work described above, the focus has been on systems that involve only disturbances/noise or simple scalar uncertainties. However, as we show in Section 2, in the presence of *both* general model-uncertainties and disturbances (along with state/input constraints), even the state-feedback RMPC formulation employing the aforementioned Q-parameterization-like methods results in nonlinearities and non-convexity. This problem is the main focus of this paper. In particular, we propose two approaches - for systems affected by norm-bounded structured uncertainties as well as disturbances - which both circumvent the aforementioned nonlinearity to yield a state-feedback RMPC scheme based on convex LMI optimizations. The first approach consists of recasting the disturbance as an uncertainty, followed by relaxing the problem using the S-procedure and subsequently using a slack-variable method. In the second approach - which can be considered to be a 'dual' of the first - we re-parameterize the model-uncertainty as a polytopic disturbance to obtain convexity.

This paper is organized as follows. Section 2 provides a description of the system, formulates the general causal RMPC problem subject to uncertainties/disturbances and highlights the associated nonlinearities and computational intractability. In Section 3, we provide an in-depth analysis of the nature of the nonlinearity and subsequently propose an LMI optimization solution based on the use of a slack-variable procedure with system disturbance recast as an uncertainty (approach 1). In Section 4, we provide an alternative solution to the RMPC problem based on the re-parameterization of the uncertainty as a disturbance (approach 2). In Section 5, we illustrate the effectiveness of our algorithms through examples from the literature. We conclude in Section 6.

Notation and background material: The notation we use is fairly standard. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the space of n-dimensional (column) vectors whose entries are in $\mathbb{R}$, $\mathbb{R}^{n\times m}$ denotes the space of all $n\times m$ matrices whose entries are in $\mathbb{R}$ and $\mathbb{D}^n$ denotes the space of diagonal matrices in $\mathbb{R}^{n\times n}$. For $A\in\mathbb{R}^{n\times m}$, A^T denotes the transpose of A. If $A\in\mathbb{R}^{n\times n}$ is symmetric, $\underline{\lambda}(A)$ denotes the smallest eigenvalue of A and we write $A\succeq 0$ if $\underline{\lambda}(A)\geq 0$ and $A\succ 0$ if $\underline{\lambda}(A)>0$. Analogous definitions apply to $\overline{\lambda}(A)$, $A\preceq 0$ and $A\prec 0$. We define the norm of $A\in\mathbb{R}^{n\times m}$ as $\|A\|=\sqrt{\overline{\lambda}(AA^T)}$. For $x,y\in\mathbb{R}^n$, $x<y$ (and similarly $\leq$, $>$ and $\geq$) is interpreted element-wise. The identity matrix is denoted by I with the dimension inferred from the context. Let $z\in\mathbb{R}^n$ and denote the i-th element of z by z_i. Then, $\mathrm{diag}(z)$ is the diagonal matrix whose (i,i) entry is z_i. For square matrices $A_1,\dots,A_m$, $\mathrm{diag}(A_1,\dots,A_m)$ denotes a block diagonal matrix whose i-th diagonal block is A_i. The symbol e_i denotes the ith column of an appropriate identity matrix. If $\boldsymbol{\mathcal{U}}\subseteq\mathbb{R}^{p\times q}$ is a set, then operator $\boldsymbol{\mathcal{B}}$ is such that $\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{U}}$ denotes the unit ball of $\boldsymbol{\mathcal{U}}$. For matrices A and B, $A\otimes B$ denotes the Kronecker product.

In the formulation, we make use of the Schur complement argument. This refers to the result that if $A=A^T$ and $C=C^T\succ 0$ then $\begin{bmatrix} A & B\\ B^T & C\end{bmatrix}\succeq 0$ if and only if $A-BC^{-1}B^T\succeq 0$. To deal with norm-bounded structured uncertainties (usually having repeated and/or full blocks on the diagonal entries), we use the following lemma based on the results in [2].

Lemma 1 *Let $R=R^T,F,E,H$ be real matrices of appropriate dimensions. Let $\widehat{\boldsymbol{\Delta}}$ be a linear subspace and define the associated linear subspace*

$$\widehat{\boldsymbol{\Psi}}=\{(S,T,G):S=S^T\succ 0, T=T^T\succ 0,\ S\Delta=\Delta T,\\ \Delta G+G^T\Delta^T=0,\ \forall\Delta\in\widehat{\boldsymbol{\Delta}}\}.$$

Then, $R+F\Delta(I-H\Delta)^{-1}E+E^T(I-\Delta^TH^T)^{-1}\Delta^TF^T\succ 0$ and $det(I-H\Delta)\neq 0$ for every $\Delta\in\boldsymbol{\mathcal{B}}\widehat{\boldsymbol{\Delta}}$ if there exists a triple $(S,T,G)\in\widehat{\boldsymbol{\Psi}}$ such that

$$\begin{bmatrix} R & E^T+FG^T & FS\\ \star & T+HG^T+GH^T & HS\\ \star & \star & S\end{bmatrix}\succ 0 \tag{1}$$

where $\star$ denotes a term easily inferred from symmetry.

Finally, we refer to the S-procedure which is used to derive simple sufficient LMI conditions (occasionally necessary and sufficient) for the non-negativity or non-positivity of a quadratic function on a set described by quadratic inequality constraints [3,8].

2 Robust MPC problem

In this section, we give a description of the system and the constraints followed by the cost function. We also derive an algebraic formulation of the causal RMPC problem and highlight the associated nonlinearities.

2.1 System Description and Constraints

We consider the following linear discrete-time uncertain system, see e.g. [5],

$$\begin{aligned}\begin{bmatrix} x_{k+1}\\ q_k\\ f_k\\ z_k\end{bmatrix} &= \begin{bmatrix} A & B_u & B_w & B_p\\ C_q & D_{qu} & D_{qw} & 0\\ C_f & D_{fu} & D_{fw} & D_{fp}\\ C_z & D_{zu} & D_{zw} & D_{zp}\end{bmatrix}\begin{bmatrix} x_k\\ u_k\\ w_k\\ p_k\end{bmatrix},\ p_k=\Delta q_k,\\ \begin{bmatrix} q_N\\ f_N\\ z_N\end{bmatrix} &= \begin{bmatrix} \hat{C}_q & 0\\ \hat{C}_f & \hat{D}_{fp}\\ \hat{C}_z & \hat{D}_{zp}\end{bmatrix}\begin{bmatrix} x_N\\ p_N\end{bmatrix},\qquad p_N=\Delta q_N\end{aligned} \tag{2}$$

where $\Delta \in \mathcal{B}\boldsymbol{\Delta}$. Furthermore, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^{n_u}$, $w_k \in \mathbb{R}^{n_w}$, $f_k \in \mathbb{R}^{n_f}$, $z_k \in \mathbb{R}^{n_z}$, $p_k \in \mathbb{R}^{n_p}$ and $q_k \in \mathbb{R}^{n_q}$ are the state, input, disturbance, constrained signal, cost signal and input and output uncertainty vectors, respectively, at prediction step k; all other symbols denote the appropriate distribution matrices. We assume that the pair (A,B_u) is stabilizable. The state x_k is assumed measured and prediction step k belongs to the time set $T_N = \{0, 1, \cdots, N-1\}$ where $N > 0$ is the prediction horizon. We consider a disturbance of the form

$$w_k \in \mathcal{W}_k := \Big\{ w \in \mathbb{R}^{n_w} : -d_k \leq w \leq d_k \Big\} \tag{3}$$

where $d_k > 0,\ k \in T_N$ are given. Furthermore, we consider a norm-bounded structured uncertainty $\Delta \in \boldsymbol{\mathcal{B}\Delta}$ where $\boldsymbol{\Delta} \subseteq \mathbb{R}^{n_p \times n_q}$ is a structured subspace. Note that we allow uncertainties in all the problem data in (2).

It is required, for all $k \in T_N$, to find u_k such that the future constrained outputs satisfy $f_k \leq \bar{f}_k,\ f_N \leq \bar{f}_N$ for all $w_k \in \mathcal{W}_k$ and $\Delta \in \boldsymbol{\mathcal{B}\Delta}$, and the cost function

$$J = \max_{w_k \in \mathcal{W}_k,\ \Delta \in \boldsymbol{\mathcal{B}\Delta}} \quad \sum_{k=0}^{N} (z_k - \bar{z}_k)^T (z_k - \bar{z}_k)$$

is minimized, where $\bar{z}_k,\ k \in T_N$, representing a reference trajectory, is given. Note that f_k may be chosen to represent polytopic constraints on state, output and input.

2.2 Algebraic formulation

As part of our first approach, in order to help linearize the RMPC problem and simplify the presentation, we propose to re-parameterize the disturbance as uncertainty by writing $\mathcal{W}_k$ in (3) as $\mathcal{W}_k = \{\Delta_k^w d_k : \Delta_k^w \in \boldsymbol{\Delta}^w\}$ where $\boldsymbol{\Delta}^w$ is a structured subspace.

Combining the disturbance and uncertainty yields the state dynamics in (2) - but without the entries corresponding to w_k - with the re-definitions, for all $k \in T_N$

$$B_p := [B_p \ \ B_w], \quad \Delta_k := \mathrm{diag}(\Delta, \Delta_k^w),$$
$$q_k := \begin{bmatrix} q_k \\ d_k \end{bmatrix} = \underbrace{\begin{bmatrix} C_q \\ 0 \end{bmatrix}}_{C_q} x_k + \underbrace{\begin{bmatrix} D_{qu} \\ 0 \end{bmatrix}}_{D_{qu}} u_k + \underbrace{\begin{bmatrix} 0 \\ d_k \end{bmatrix}}_{\bar{d}_k}. \tag{4}$$

Let ξ stand for $f,\ \bar{f},\ p,\ q,\ z$ or $\bar{z}$, and define vectors $x = [x_1^T \cdots x_N^T]^T \in \mathbb{R}^{N_n}$, $u = [u_0^T \cdots u_{N-1}^T]^T \in \mathbb{R}^{N_u}$ and $\xi = [\xi_1^T \cdots \xi_N^T]^T \in R^{N_\xi}$, where $N_n = n \times N$, $N_u = n_u \times N$ and $N_\xi = n_\xi \times (N+1)$. Then, by iterating the dynamics in (2), with re-definitions in (4), it can be verified that

$$\begin{bmatrix} x \\ q \\ f \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{A} & 0 & \mathcal{B}_p & \mathcal{B}_u \\ \mathcal{C}_q & I & \mathcal{D}_{qp} & \mathcal{D}_{qu} \\ \mathcal{C}_f & 0 & \mathcal{D}_{fp} & \mathcal{D}_{fu} \\ \mathcal{C}_z & 0 & \mathcal{D}_{zp} & \mathcal{D}_{zu} \end{bmatrix} \begin{bmatrix} x_0 \\ d \\ p \\ u \end{bmatrix}, \ d = \begin{bmatrix} \bar{d}_0 \\ \vdots \\ \bar{d}_{N-1} \end{bmatrix} \tag{5}$$

where $p = \Delta q$, with $\Delta \in \boldsymbol{\mathcal{B}}\widehat{\boldsymbol{\Delta}}$,

$$\widehat{\boldsymbol{\Delta}} := \{\mathrm{diag}(\Delta, \Delta_0^w, \ldots, \Delta, \Delta_{N-1}^w, \Delta) : \Delta \in \boldsymbol{\Delta}, \Delta_k^w \in \boldsymbol{\Delta}^w\}$$

and where all the matrices in (5) can easily be computed by iterating the dynamics in (2).

As mentioned above, we consider a causal state-feedback structure on the RMPC controller (that is, u_i depends only on $x_j, j = 0, \ldots, i$, see e.g. [13]). Therefore, we set

$$u = K_0 x_0 + Kx + v \tag{6}$$

where $K_0 \in \mathbb{R}^{N_u \times n},\ K \in \mathbb{R}^{N_u \times N_n}$ are the current and future state-feedback gain matrices and $v \in \mathbb{R}^{N_u}$ is the control-perturbation sequence, and where causality is captured by the constraint that the matrix $[\,K_0 \ \ K\,]$ is block lower triangular (with $n_u \times n$ blocks). We denote structures of $K_0,\ K,\ v$ as $\mathcal{K}_0,\ \mathcal{K}$ and υ, respectively.

Remark 2 *Note that the state dynamics in (2) can be written as* $x_{k+1} = (A + B_p \Delta C_q) x_k + (B_u + B_p \Delta D_{qu}) u_k + (B_w + B_p \Delta D_{qw}) w_k$. *Hence, we allow uncertainty in all parts of the dynamics as well as in the cost and constraints. Furthermore, the proposed control structure (6) provides flexibility in that the designer may choose to use any combination of the three control terms (for either feedback or open-loop RMPC control). Therefore, the RMPC algorithms proposed in this paper can be readily applied to a broad class of systems.*

Substituting the expression of x in (5) into (6) yields the following expression for u

$$u = \hat{K}_0 x_0 + \hat{K} \mathcal{B}_p p + \hat{v}, \tag{7}$$

where $[\hat{K}_0 \ \ \hat{K} \ \ \hat{v}] := (I - K\mathcal{B}_u)^{-1} [K_0 \ \ K \ \ v + K\mathcal{A}x_0]$.

Note that u is affine in the new variables $(\hat{K}_0, \hat{K}, \hat{v})$ which have the same structure as (K_0, K, v), and which in turn can be recovered as

$$[\,K_0 \quad K \quad v\,] := (I + \hat{K}\mathcal{B}_u)^{-1} [\,\hat{K}_0 \quad \hat{K} \quad \hat{v} - \hat{K}\mathcal{A}x_0\,]$$

Eliminating u from (5) using (7) gives

$$\begin{bmatrix} q \\ f \\ z-\overline{z} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{qp}^{\hat{K}} & \mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} \\ \mathcal{D}_{fp}^{\hat{K}} & \mathcal{D}_{f0}^{\hat{K}_0,\hat{v}} \\ \mathcal{D}_{zp}^{\hat{K}} & \mathcal{D}_{z0}^{\hat{K}_0,\hat{v}} \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \tag{8}$$

$$\left[\begin{array}{c|c} \mathcal{D}_{qp}^{\hat{K}} & \mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} \\ \hline \mathcal{D}_{fp}^{\hat{K}} & \mathcal{D}_{f0}^{\hat{K}_0,\hat{v}} \\ \hline \mathcal{D}_{zp}^{\hat{K}} & \mathcal{D}_{z0}^{\hat{K}_0,\hat{v}} \end{array}\right] := \left[\begin{array}{c|c} \mathcal{D}_{qp}+\mathcal{D}_{qu}\hat{K}\mathcal{B}_p & \mathcal{D}_{qu}\hat{v}+(\mathcal{C}_q+\mathcal{D}_{qu}\hat{K}_0)x_0+d \\ \hline \mathcal{D}_{fp}+\mathcal{D}_{fu}\hat{K}\mathcal{B}_p & \mathcal{D}_{fu}\hat{v}+(\mathcal{C}_f+\mathcal{D}_{fu}\hat{K}_0)x_0 \\ \hline \mathcal{D}_{zp}+\mathcal{D}_{zu}\hat{K}\mathcal{B}_p & \mathcal{D}_{zu}\hat{v}+(\mathcal{C}_z+\mathcal{D}_{zu}\hat{K}_0)x_0-\overline{z} \end{array}\right]$$

where $\overline{z}$ denotes the stacked reference trajectory. Finally, eliminating p using $p = \Delta q$, gives the constraint and cost function signals as $[\, f^T \quad (z-\overline{z})^T\,]^T =$

$$\begin{bmatrix} \mathcal{D}_{fp}^{\hat{K}}\Delta(I-\mathcal{D}_{qp}^{\hat{K}}\Delta)^{-1}\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}}+\mathcal{D}_{f0}^{\hat{K}_0,\hat{v}} \\ \mathcal{D}_{zp}^{\hat{K}}\Delta(I-\mathcal{D}_{qp}^{\hat{K}}\Delta)^{-1}\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}}+\mathcal{D}_{z0}^{\hat{K}_0,\hat{v}} \end{bmatrix} =: \begin{bmatrix} \mathcal{D}_{f0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \\ \mathcal{D}_{z0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \end{bmatrix}.$$

2.3 Minmax formulation

We now formulate the RMPC problem and investigate the associated nonlinearities. Note that the constraint and cost function can respectively be written as

$$f(\hat{K}_0,\hat{K},\hat{v},\Delta) = \mathcal{D}_{f0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \tag{9}$$

$$f_c(\hat{K}_0,\hat{K},\hat{v},\Delta) = (\mathcal{D}_{z0}^{\hat{K}_0,\hat{K},\hat{v},\Delta})^T(\mathcal{D}_{z0}^{\hat{K}_0,\hat{K},\hat{v},\Delta}). \tag{10}$$

Using (9), define the set of all feasible control variables

$$\boldsymbol{\mathcal{U}} = \{(\hat{K}_0,\hat{K},\hat{v}) : e_i^T\mathcal{D}_{f0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \le e_i^T\overline{f}, \forall i\in\mathcal{N}_f, \forall\Delta\}. \tag{11}$$

with $\mathcal{N}_f = \{1,\ldots,N_f\}$. The RMPC problem is to find

$$\phi = \min_{(\hat{K}_0,\hat{K},\hat{v})\,\in\boldsymbol{\mathcal{U}}} \ \max_{\Delta\,\in\boldsymbol{\mathcal{B}\Delta}} \ f_c(\hat{K}_0,\hat{K},\hat{v},\Delta) \tag{12}$$

and a feasible triple $(\hat{K}_0,\hat{K},\hat{v})$ that achieves the minimum. Since the problem is nonconvex, we use a relaxation procedure based on Lemma 1 to minimize an upper bound on the cost function as shown next.

2.4 A semidefinite relaxation of the RMPC problem

The next result uses Lemma 1 to derive sufficient conditions for $(\hat{K}_0,\hat{K},\hat{v})\in\boldsymbol{\mathcal{U}}$ and an upper bound, call it $\overline{f}_c$, on the cost function in (12).

Theorem 3 *Let all variables be as defined above. Then, $f_c(\hat{K}_0,\hat{K},\hat{v},\Delta) \le \overline{f}_c$ and $(\hat{K}_0,\hat{K},\hat{v}) \in \boldsymbol{\mathcal{U}}$ for all $\Delta \in \boldsymbol{\mathcal{B}}\widehat{\boldsymbol{\Delta}}$, if there exist solutions $(S,T,G),(S_i,T_i,G_i) \in \widehat{\boldsymbol{\Psi}}$, $\forall i\in\mathcal{N}_f$, to the following matrix inequalities*

$$\begin{bmatrix} I & \mathcal{D}_{z0}^{\hat{K}_0,\hat{v}} & \mathcal{D}_{zp}^{\hat{K}}G^T & \mathcal{D}_{zp}^{\hat{K}}S \\ \star & \overline{f}_c & (\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}})^T & 0 \\ \star & \star & T+\mathcal{D}_{qp}^{\hat{K}}G^T+G(\mathcal{D}_{qp}^{\hat{K}})^T & \mathcal{D}_{qp}^{\hat{K}}S \\ \star & \star & \star & S \end{bmatrix} \succ 0 \tag{13}$$

$$\begin{bmatrix} e_i^T(\overline{f}-\mathcal{D}_{f0}^{\hat{K}_0,\hat{v}}) & -(\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}})^T+\frac{1}{2}e_i^T\mathcal{D}_{fp}^{\hat{K}}G_i^T & \frac{1}{2}e_i^T\mathcal{D}_{fp}^{\hat{K}}S_i \\ \star & T_i+\mathcal{D}_{qp}^{\hat{K}}G_i^T+G_i(\mathcal{D}_{qp}^{\hat{K}})^T & \mathcal{D}_{qp}^{\hat{K}}S_i \\ \star & \star & S_i \end{bmatrix} \succ 0 \tag{14}$$

PROOF. Using the definitions above, the constraints in (11) can be rearranged in the form, $\forall i\in\mathcal{N}_f$,

$$R_i+F_i\Delta(I-H\Delta)^{-1}E+E^T(I-\Delta^TH^T)^{-1}\Delta^TF_i^T \succ 0, \tag{15}$$

$$\left[\begin{array}{c|c} R_i & F_i \\ \hline E & H \end{array}\right] := \left[\begin{array}{c|c} e_i^T(\overline{f}-\mathcal{D}_{f0}^{\hat{K}_0,\hat{v}}) & \frac{1}{2}e_i^T\mathcal{D}_{fp}^{\hat{K}} \\ \hline -\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} & \mathcal{D}_{qp}^{\hat{K}} \end{array}\right].$$

Using Lemma 1 on (15) yields the inequality (14).

Next, we consider the cost function. Let $\overline{f}_c$ be an upper bound on the cost such that

$$f_c(\hat{K}_0,\hat{K},\hat{v},\Delta) \le \overline{f}_c. \tag{16}$$

Using (10), as well as a Schur complement argument on (16) yields the equivalent requirement

$$\begin{bmatrix} I & \mathcal{D}_{z0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \\ \star & \overline{f}_c \end{bmatrix} \succ 0. \tag{17}$$

Then, using the definition for $\mathcal{D}_{z0}^{\hat{K}_0,\hat{K},\hat{v},\Delta}$, the matrix inequality (17) can be written in the form

$$R_0+F_0\Delta(I-H\Delta)^{-1}E_0+E_0^T(I-\Delta^TH^T)^{-1}\Delta^TF_0^T \succ 0, \tag{18}$$

$$\left[\begin{array}{c|c} R_0 & F_0 \\ \hline E_0 & H \end{array}\right] := \left[\begin{array}{cc|c} I & \mathcal{D}_{z0}^{\hat{K}_0,\hat{v}} & \mathcal{D}_{zp}^{\hat{K}} \\ (\mathcal{D}_{z0}^{\hat{K}_0,\hat{v}})^T & \overline{f}_c & 0 \\ \hline 0 & \mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} & \mathcal{D}_{qp}^{\hat{K}} \end{array}\right].$$

An application of Lemma 1 on (18) yields (13). ■

Theorem 3 then defines an upper-bound on ϕ in (12) as

$$\bar{\phi} = \min\{\overline{f}_c : (\hat{K}_0,\hat{K},\hat{v})\in(\mathcal{K}_0,\mathcal{K},\upsilon),(S,T,G),(S_i,T_i,G_i) \in\widehat{\boldsymbol{\Psi}}, i\in\mathcal{N}_f \text{ s.t. } (13),(14) \text{ are satisfied}\}. \tag{19}$$

We will call a triple $(\hat{K}_0, \hat{K}, \hat{v})$ achieving the bound in (19) an optimal control law for the relaxed Robust MPC (RRMPC) problem. Note that the RRMPC problem (19) is nonlinear in $\hat{K}$ - while being linear in $\hat{K}_0$ and $\hat{v}$. Furthermore, the terms involving $\hat{K}$ are diffused throughout the matrix inequalities of (19) (i.e. (13), (14)). Therefore, the optimal solution requires the use of nonlinear optimization techniques. It can be verified that, in the existing form, the nonlinearity can only be avoided if K is fixed or in the limiting case when the system has no model uncertainty and is subject only to disturbances (see Remark 4). We next propose our first approach to remedy this nonlinearity and hence transform the RRMPC problem into an LMI optimization.

Remark 4 *When the system is subject only to additive disturbance (and no model-uncertainty), the matrix inequalities (13), (14) become linear. To see this, note that for this case, C_q, D_{qu} become zero and therefore, $\mathcal{D}_{qp}^{\hat{K}} = \mathcal{D}_{qp}$ and $\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} = \mathcal{C}_q x_0 + d$ in (8). The variables G, G_i also become zero since Δ is now diagonal. Furthermore, $S = T$ and $S_i = T_i$. Then, effecting the congruence transformation $\mathrm{diag}(I, I, S^{-1}, S^{-1})$ on (13), and considering S^{-1} as a variable, renders (13) linear in $\hat{K}$. A similar procedure can be adopted to linearize (14).*

Remark 5 *It is worth mentioning here that a simple procedure for linearizing the inequalities (13) and (14) is to set $S = S_i = \lambda I_{N_p}$, $T = T_i = \lambda I_{N_q}$ and $G = G_i = 0$, $\forall i$, for a variable $\lambda \in \mathbb{R}$, and subsequently take $\lambda\hat{K}$ as the variable. Though this may be attractive from a computational point of view, the problem is the excessive conservativeness potentially associated with such a restriction which, in turn, may render the problem infeasible (see also the numerical example in Section 5.1).*

3 A linearization procedure for the RRMPC problem - Approach 1

As can be seen from (13) and (14) (which follow from (1)), the terms that include $\hat{K}$ have the form $\hat{K}\mathcal{B}_p X$ where X stands for S, S_i, G, G_i, $i \in \mathcal{N}_f$. To deal with this issue, in Section 3.1 we propose to extend Lemma 1 by introducing slack variables that will allow us to keep only one term in the form $\hat{K}\mathcal{B}_p S_0$, for a free S_0 and for all the matrix inequalities, without excessive loss of the degrees of freedom. Then, in Section 3.2, we propose to treat $\overline{K}(:= \hat{K}\mathcal{B}_p S_0)$ as one decision variable of the optimization, thus linearizing the problem, and allowing us to extract the desired variable $\hat{K}$ from $\overline{K}$.

3.1 An extended S-procedure

In this section, We propose an extended version of Lemma 1 using an approach similar to that used in e.g. [1]. This will enable us to give equivalent necessary and sufficient conditions for (1) in a form that allows us to separate the terms multiplying $\hat{K}$ from other variables.

Theorem 6 *Let all variables be as defined in Lemma 1. Then the following two statements are equivalent:*

(i) There exist $(S, T, G) \in \widehat{\boldsymbol{\Psi}}$ such that (1) is satisfied.
(ii) There exist $(S, T, G) \in \widehat{\boldsymbol{\Psi}}$, $Y = Y^T$, S_0 and G_0 such that $(S_0, T, G_0) \in \widehat{\boldsymbol{\Psi}}_0 \supseteq \widehat{\boldsymbol{\Psi}}$ and $\mathcal{L}_1 :=$

$$\begin{bmatrix} R & E^T & FS_0 & -FG_0^T \\ \star & T+Y & HS_0-R_0 & -HG_0^T+Y_0 \\ \star & \star & S_0+S_0^T-S & -G_0^T-R_0^T+G^T \\ \star & \star & \star & Y_0+Y_0^T-Y \end{bmatrix} \succ 0,$$

$$P := \begin{bmatrix} S & -G^T \\ -G & Y \end{bmatrix} \succ 0.$$

If $S_0,\ G_0,\ R_0$ or Y_0 are constrained, then (ii)→(i).

PROOF. Note first that, for any $Y = Y^T$, we have

$$(1) \Leftrightarrow \begin{bmatrix} R & E^T \\ E & T+Y \end{bmatrix} - \begin{bmatrix} F & 0 \\ H & I \end{bmatrix} \begin{bmatrix} S & -G^T \\ -G & Y \end{bmatrix} \begin{bmatrix} F^T & H^T \\ 0 & I \end{bmatrix} \succ 0 \quad (20)$$

- $(ii) \to (i)$: Taking a Schur complement on (20) yields

$$(1) \;\Leftrightarrow\; \mathcal{L}_2 := \left[\begin{array}{c|c} \begin{bmatrix} R & E^T \\ \star & T+Y \end{bmatrix} & \begin{bmatrix} F & 0 \\ H & I \end{bmatrix} \\ \hline \star & P^{-1} \end{array}\right] \succ 0. \quad (21)$$

Define

$$P_0 = \begin{bmatrix} S_0 & -G_0^T \\ -R_0 & Y_0 \end{bmatrix}.$$

Then, the following identity can be verified

$$P_0^T P^{-1} P_0 = P_0^T + P_0 - P + (P_0^T - P) P^{-1} (P_0 - P). \quad (22)$$

Effecting the congruence transformation $\mathrm{diag}(I, P_0)$ on $\mathcal{L}_2$, followed by the use of identity (22) shows that

$$\mathcal{L}_1 + \begin{bmatrix} 0 \\ I \end{bmatrix} (P_0^T - P) P^{-1} (P_0 - P) \begin{bmatrix} 0 & I \end{bmatrix} \succ 0 \Rightarrow (1)$$

since the last term in (22) is nonnegative.

- $(i) \to (ii)$: Since $S \succ 0$, there exists Y such that $P \succ 0$, e.g. we can take any $Y \succ GS^{-1}G^T$. Therefore, (21) is satisfied. Now let $S_0 = S$ and $G_0 = G$ so that $(S_0, T, G_0) \in \widehat{\boldsymbol{\Psi}}_0$. Then $\mathcal{L}_1 \succ 0$ from (21). ∎

Remark 7 *Theorem 6 introduces slack variables which provide extra degrees of freedom to allow a less conservative change of variables to overcome nonlinearity in (19).*

3.2 Final linearized RRMPC problem

By using Theorem 6 on inequalities (15) and (18) (instead of Lemma 1 as above), it can be seen that the nonlinearities now only have the form $\hat{K}\mathcal{B}_p S_0$ and $\hat{K}\mathcal{B}_p G_0$ for unconstrained S_0 and G_0. Furthermore, no other terms include $\hat{K}$. It follows that by e.g. restricting $S_0 \in \mathbb{S}_0 = \{\lambda_0 I : \lambda_0 \in \mathbb{R}\}$ and $G_0 \in \mathbb{G}_0 = \{0\}$, we immediately ensure linearity by defining $\lambda_0 \hat{K}$ as a new decision variable. However, the 'least-conservative' choice for sets $\mathbb{S}_0$ and $\mathbb{G}_0$ is problem dependent and follows from the fine structure of $\hat{K}$ and $\mathcal{B}_p$. For the sake of clarity of exposition, we we do not go into the details here.

We now propose the following theorem to compute a solution for RRMPC problem through LMI optimization.

Theorem 8 *Let everything be as defined above. Then, $(\hat{K}_0, \hat{K}, \hat{v}) \in \boldsymbol{\mathcal{U}}$ and $f_c(\hat{K}_0, \hat{K}, \hat{v}, \Delta) \leq \overline{f}_c$, for all $\Delta \in \boldsymbol{\mathcal{B}}\widehat{\boldsymbol{\Delta}}$, if there exist solutions (S, T, G), $(S_i, T_i, G_i) \in \widehat{\boldsymbol{\Psi}}$, $Y = Y^T$, $Y_i = Y_i^T$, $Y_0 \in \mathbb{R}^{N_q \times N_q}$, $S_0 \in \mathbb{S}_0$, $G_0 \in \mathbb{G}_0$, $R_0 \in \mathbb{R}^{N_q \times N_p}$, $\forall i \in \mathcal{N}_f$ to following LMIs:*

$$\begin{bmatrix} S & \star \\ -G & Y \end{bmatrix} \succ 0, \quad \begin{bmatrix} S_i & \star \\ -G_i & Y_i \end{bmatrix} \succ 0 \tag{23}$$

$$\begin{bmatrix} I & \star & \star & \star & \star \\ (\mathcal{D}_{z0}^{\hat{K}_0,\hat{v}})^T & \overline{f}_c & \star & \star & \star \\ 0 & \mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} & T+Y & \star & \star \\ (\mathcal{D}_{zp}S_0 + \mathcal{D}_{zu}\overline{K})^T & 0 & Z_0^T & \tilde{S} & \star \\ -G_0\mathcal{D}_{zp}^T & 0 & Y_0^T - G_0\mathcal{D}_{qp}^T & \tilde{G} & \tilde{Y} \end{bmatrix} \succ 0 \tag{24}$$

$$\begin{bmatrix} e_i^T(\overline{f} - \mathcal{D}_{f0}^{\hat{K}_0,\hat{v}}) & \star & \star & \star \\ -\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} & T_i + Y_i & \star & \star \\ \frac{1}{2}(\mathcal{D}_{fp}S_0 + \mathcal{D}_{fu}\overline{K})^T e_i & Z_0^T & \tilde{S}_i & \star \\ -\frac{1}{2}G_0\mathcal{D}_{fp}^T e_i & Y_0^T - G_0\mathcal{D}_{qp}^T & \tilde{G}_i & \tilde{Y}_i \end{bmatrix} \succ 0 \tag{25}$$

where $\tilde{S}_i := S_0^T + S_0 - S_i$, $\tilde{G}_i := -G_0 - R_0 + G_i$, $\tilde{Y}_i := Y_0 + Y_0^T - Y_i$, $\tilde{S} := S_0^T + S_0 - S$, $\tilde{G} := -G_0 - R_0 + G$, $\tilde{Y} := Y_0 + Y_0^T - Y$, $Z_0 := \mathcal{D}_{qp}S_0 + \mathcal{D}_{qu}\overline{K} - R_0$ with $\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}}, \mathcal{D}_{f0}^{\hat{K}_0,\hat{v}}$ and $\mathcal{D}_{z0}^{\hat{K}_0,\hat{v}}$ defined in (8) and where $\overline{K} := \hat{K}\mathcal{B}_p S_0$ (so that $\hat{K}$, and therefore K, can be recovered from $\overline{K}$).

PROOF. The LMIs (25) and (24), along with (23), result from the application of Theorem 6 on (15) and (18), respectively, and the use of definitions given above. ∎

It follows that the RRMPC control law $(\hat{K}_0, \hat{K}, \hat{v})$ can now be computed online - by solving the (convex) problem of minimizing $\overline{f}_c$ subject to the LMI constraints of Theorem 8 - and subsequently applied in a receding horizon manner. Note that the conservatism introduced due to the use of S_0, G_0, R_0 and Y_0 for all the matrix inequalities is potentially much less than that introduced for the case when the same (complete) set of variables is used for all the inequalities (as given in Remark 5). This is also illustrated through a numerical example (Section 5.1).

Remark 9 *In case the considered problem has no feasible solution, then by noting that (25) is linear in $\overline{f}$, an LMI procedure can be adopted for minimally relaxing the constraints so that a control law may be computed.*

4 Causal RMPC - Approach 2

In this section, we formulate our second, computationally less demanding, solution to overcome the nonlinearity in the considered feedback RMPC problem. This scheme can be considered as a 'dual' of Approach 1 (Section 3) in that it involves the re-parameterization of the uncertainty set as a polytopic set similar to the disturbance in (3). It is also inspired from some of the stochastic MPC schemes which, in the interest of tractability, compute bounds on stochastic disturbances and therefore approximate chance constraints with hard constraints (see e.g. [10]). In particular, we propose to compute hard bounds on uncertainty which helps to convexify the RMPC problem and enables computation of optimal $\hat{K}_0$, $\hat{K}$ and $\hat{v}$ through an LMI optimization.

Throughout this section, in the interest of clarity of exposition, we will make the following notational simplifications. Instead of f, we will consider the constraints on state and the input separately. Moreover, a conventional combination of state/input penalty will be considered in the cost function and, without loss of generality, only the regulation problem will be formulated [5]. Finally, we consider disturbance to be uncertainty-free ($D_{qw} = 0$).

Due to the presence of persistent uncertainty and disturbances, the system in (2) cannot be controlled to the origin. The uncertain system state can, at best, be confined to an RPI set Z. Many RMPC schemes incorporate the idea of such RPI terminal sets (see e.g. [12],[6]) since it helps to establish recursive feasibility and stability (see Remarks 14,15). To promote state convergence to Z, we include in our formulation, the terminal state constraint $x_N \in Z$ together with other hard constraints on the input and state. All these are summarized below. The problem-defined state constraints are given by

$$x_k \in \mathcal{X}_k := \left\{ x \in \mathbb{R}^n : \underline{x}_k \le Cx \le \overline{x}_k \right\}, \quad \forall k \in T_I \quad (26)$$

where $T_I := \{1, 2, \cdots, N-1\}$ and $C \in \mathbb{R}^{m \times n}$ can be chosen to represent constraints on linear combinations of the state. Furthermore, we define the terminal state (see Remark 10 below) and input constraints as

$$x_N \in Z := \left\{ x \in \mathbb{R}^n : \underline{x}_N \le Cx \le \overline{x}_N \right\} \quad (27)$$

$$u_k \in \mathcal{U}_k := \left\{ u \in \mathbb{R}^{n_u} : \underline{u}_k \le u \le \overline{u}_k \right\}, \forall k \in T_N. \quad (28)$$

Remark 10 *There exist many algorithms in the literature for the computation of a suitable polyhedral RPI set Z. See e.g. [11], [16] and the references therein.*

4.1 Uncertainty re-parameterization

We first propose to re-parameterize the uncertainty as a disturbance in the following theorem. Subsequently, in Section 4.2, the (re-parameterized) uncertainty is combined with the disturbance and the RMPC scheme is formulated.

Theorem 11 *Let $\mathbb{D}^m$ denote the set of all real $m \times m$ diagonal matrices and let $\mathbb{D}^m_+ := \{D \in \mathbb{D}^m : D \succeq 0\}$. Consider $\Delta \in \boldsymbol{\mathcal{B}\Delta} := \{\mathrm{diag}(\delta_1, \cdots, \delta_n) : \delta_i \in \mathbb{R}, |\delta_i| \le 1\}$. Then, uncertainty vector p_k, in (2), is such that $p^i_k \le \overline{p}^i_k$ for all $i \in \mathcal{N}_p := \{1, \cdots, n_p\}$ and $k = 0, \ldots, N$, if there exist $X^i_k \in \mathbb{D}^m_+$, $U^i_k \in \mathbb{D}^{n_u}_+$ and $0 \le D^i_k \in \mathbb{R}$ such that*

$$\mathcal{L}^i_k(X^i_k, U^i_k, D^i_k, \overline{p}^i_k) := \begin{bmatrix} C^T X^i_k C & \star & \star & \star \\ 0 & U^i_k & \star & \star \\ -\frac{1}{2} e_i^T C_q & -\frac{1}{2} e_i^T D_{qu} & D^i_k & \star \\ -\overline{\underline{x}}^T_k X^i_k C & -\overline{\underline{u}}^T_k U^i_k & 0 & \hat{p}^i_k \end{bmatrix} \succeq 0 \quad (29)$$

$$\mathcal{L}^i_N(X^i_N, D^i_N, \overline{p}^i_N) := \begin{bmatrix} C^T X^i_N C & \star & \star \\ -\frac{1}{2} e_i^T \hat{C}_q & D^i_N & \star \\ -\overline{\underline{x}}^T_N X^i_k C & 0 & \hat{p}^i_N \end{bmatrix} \succeq 0 \quad (30)$$

$$\mathcal{L}^i_0(U^i_0, D^i_0, \overline{p}^i_0) := \begin{bmatrix} U^i_0 & \star & \star \\ -\frac{1}{2} e_i^T D_{qu} & D^i_0 & \star \\ -\overline{\underline{u}}^T_0 U^i_0 & -\frac{1}{2} x_0^T C_q^T e_i & \hat{p}^i_0 \end{bmatrix} \succeq 0 \quad (31)$$

$\forall k \in T_I$, where we have $\hat{p}^i_k := \overline{p}^i_k + \overline{x}^T_k X^i_k \underline{x}_k + \overline{u}^T_k U^i_k \underline{u}_k - D^i_k$, $\hat{p}^i_N := \overline{p}^i_N + \overline{x}^T_N X^i_N \underline{x}_N - D^i_N$, $\hat{p}^i_0 := \overline{p}^i_0 + \overline{u}^T_0 U^i_0 \underline{u}_0 - D^i_0$, $\overline{\underline{x}}_k := \frac{1}{2}(\overline{x}_k + \underline{x}_k)$, $\overline{\underline{u}}_k := \frac{1}{2}(\overline{u}_k + \underline{u}_k)$ and $\overline{\underline{x}}_N := \frac{1}{2}(\overline{x}_N + \underline{x}_N)$.

Similarly, $p^i_k \ge \underline{p}^i_k$, $\forall i \in \mathcal{N}_p$ and $k = 0, \ldots, N$, if there exist $X^i_k \in \mathbb{D}^m_+$, $U^i_k \in \mathbb{D}^{n_u}_+$, $0 \le D^i_k \in \mathbb{R}$ such that (29)-(31) are satisfied with $\overline{p}^i_k$ replaced by $-\underline{p}^i_k$.

PROOF. Using the definition of p_k in (2) (with $D_{qw} = 0$) and an S-procedure, it can be shown that for all $k \in T_I$

$$p^i_k = \overline{p}^i_k - (\overline{x}_k - Cx_k)^T X^i_k (Cx_k - \underline{x}_k) - (\overline{u}_k - u_k)^T U^i_k (u_k - \underline{u}_k) - (1 - \delta_i)^T D^i_k (\delta_i + 1) - y^T_k \mathcal{L}^i_k(X^i_k, U^i_k, D^i_k, \overline{p}^i_k) y_k$$

where $y^T_k := [\, x^T_k \; u^T_k \; \delta^T_i \; 1\,]$, $X^i_k \in \mathbb{D}^n_+$, $U^i_k \in \mathbb{D}^{n_u}_+$, $0 \le D^i_k \in \mathbb{R}$ and $\mathcal{L}^i_k(X^i_k, U^i_k, D^i_k, \overline{p}^i_k)$ given in (29). Thus

$$X^i_k \succeq 0,\; U^i_k \succeq 0,\; D^i_k \succeq 0,\; \mathcal{L}^i_k(X^i_k, U^i_k, D^i_k, \overline{p}^i_k) \succeq 0 \;\Rightarrow\; p^i_k \le \overline{p}^i_k, \;\; \forall i \in \mathcal{N}_p, \; \forall k \in T_I.$$

The LMIs (30) and (31) can analogously be derived for $k = N$ and $k = 0$, respectively. Finally, for the lower bounds the result follows by noting that $p^i_k \ge \underline{p}^i_k$ is equivalent to $-p^i_k \le -\underline{p}^i_k$. ■

Define the vectors $\overline{p}^T := [\, (\overline{p}^\star_0)^T \; (\overline{p}^\star_1)^T \; \cdots \; (\overline{p}^\star_N)^T \,]$ and $\underline{p}^T := [\, (\underline{p}^\star_0)^T \; (\underline{p}^\star_1)^T \; \cdots \; (\underline{p}^\star_N)^T \,]$. Using Theorem 11, the model uncertainty can then be re-parameterized as:

$$p \in \mathcal{P} := \left\{ p \in \mathbb{R}^{N_p} : \underline{p} \le p \le \overline{p} \right\} \quad (32)$$

where, for each $k \in T_N$, we can compute the bounds for each $i \in \mathcal{N}_p$ through the following optimizations:

$$\overline{p}^{i\star}_k = \min\{\, \overline{p}^i_k : (29)/(30)/(31) \text{ is satisfied for corresponding } k, \text{ for } X^i_k \in \mathbb{D}^m_+, U^i_k \in \mathbb{D}^{n_u}_+, D^i_k \ge 0 \}$$

$$-\underline{p}^{i\star}_k = \min\{ -\underline{p}^i_k : (29)/(30)/(31) \text{ is satisfied with } \overline{p}^i_k \text{ replaced by } -\underline{p}^i_k \text{ for } X^i_k \in \mathbb{D}^m_+, U^i_k \in \mathbb{D}^{n_u}_+, D^i_k \ge 0 \}. \quad (33)$$

Remark 12 *Note that for the case of full block uncertainty elements, we have: $p^{iT}_k p^i_k \le q^{iT}_k q^i_k$ where $(.)^i$ denotes the ith block. Hence, the polytopic bounds for full block uncertainty elements can be computed, in a manner similar to Theorem 11, by relaxing the following optimization problems, $\forall i \in \mathcal{N}_p$, $\forall k \in \{0, 1, \cdots, N\}$:*

$$\underline{p}^i_k \le \min_{x_k \in \mathcal{X}_k,\, u_k \in \mathcal{U}_k} q^{iT}_k q^i_k \le \max_{x_k \in \mathcal{X}_k,\, u_k \in \mathcal{U}_k} q^{iT}_k q^i_k \le \overline{p}^i_k.$$

4.2 Computation of $\hat{K}_0$, $\hat{K}$ and $\hat{v}$

In this subsection, we first combine the re-parameterized uncertainty with the disturbance and then derive sufficient conditions (on $\hat{K}_0$, $\hat{K}$, $\hat{v}$) for the satisfaction of constraints and minimization of the cost function.

Using (32), let us introduce the re-definitions:

$$\mathcal{B}_w := [\mathcal{B}_w \;\; \mathcal{B}_p], \qquad \overbrace{[-d_k^T \;\; \underline{p}_k^T]}^{\underline{w}_k^T} \le \overbrace{[w_k^T \;\; p_k^T]}^{w_k^T} \le \overbrace{[d_k^T \;\; \overline{p}_k^T]}^{\overline{w}_k^T}.$$

Therefore, it can be verified that the stacked state-dynamics in (5) can now be written as:

$$x = \mathcal{A}x_0 + \mathcal{B}_u u + \mathcal{B}_w w \tag{34}$$

where $w \in \boldsymbol{\mathcal{W}} := \{w \in \mathbb{R}^{N_w} : \underline{w} \le w \le \overline{w}\}$ and all matrices/vectors are appropriately re-defined.

Theorem 13 *Define the cost function*

$$J(x_0, u, w) := x_N^T P_N x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \tag{35}$$

and let $\mathcal{A}^{\hat{K}_0} := \mathcal{A} + \mathcal{B}_u \hat{K}_0, \underline{\overline{w}} := \frac{1}{2}(\overline{w} + \underline{w}), \widetilde{C} := I_N \otimes C, \widetilde{R} := I_N \otimes R,\ \hat{K}_{\mathcal{B}} := (I + \mathcal{B}_u \hat{K}),\ \widetilde{Q} := \mathrm{diag}(I_{N-1} \otimes Q, P_N),\ \overline{x} = [\overline{x}_1^T \ \cdots \ \overline{x}_N^T]^T,\ \overline{u} = [\overline{u}_0^T \ \cdots \ \overline{u}_{N-1}^T]^T$ *(and analogously for* $\underline{x}$, $\underline{u}$*). Then, there exist feasible* $\hat{K}_0$, $\hat{K}$ *and* $\hat{v}$ *satisfying constraints (26)-(28) and such that* $J(x_0, u, w) \le \overline{f}_c$ *for all* $w \in \boldsymbol{\mathcal{W}}$ *if there exist diagonal solutions* $D_w, \overline{D}_{wx}^i, \underline{D}_{wx}^i,\ i \in \mathcal{N}_x := \{1, \cdots, mN\}$, $\overline{D}_{wu}^j, \underline{D}_{wu}^j,\ j \in \mathcal{N}_u := \{1, \cdots, N_u\}$ *to the following LMIs*

$$\begin{bmatrix} D_w & \star & \star & \star \\ \underline{\overline{w}}^T D_w & \overline{f}_c + \overline{w}^T D_w \underline{w} - x_0^T Q x_0 & \star & \star \\ \hat{K}_{\mathcal{B}} \mathcal{B}_w & \mathcal{B}_u \hat{v} + \mathcal{A}^{\hat{K}_0} x_0 & \widetilde{Q}^{-1} & \star \\ \hat{K} \mathcal{B}_w & \hat{K}_0 x_0 + \hat{v} & 0 & \widetilde{R}^{-1} \end{bmatrix} \succeq 0 \tag{36}$$

$$\mathcal{L}_x^i(\overline{D}_{wx}^i, \hat{K}, \hat{K}_0, \hat{v}, \overline{x}, e_i) := \begin{bmatrix} \overline{D}_{wx}^i & -\overline{D}_{wx}^i \underline{\overline{w}} - \frac{1}{2}\mathcal{B}_w^T \hat{K}_{\mathcal{B}}^T \widetilde{C}^T e_i \\ \star & e_i^T(\overline{x} - \widetilde{C}\mathcal{A}^{\hat{K}_0} x_0 - \widetilde{C}\mathcal{B}_u \hat{v}) + \overline{w}^T \overline{D}_{wx}^i \underline{w} \end{bmatrix} \succeq 0 \tag{37}$$

$$\mathcal{L}_u^j(\overline{D}_{wu}^j, \hat{K}, \hat{K}_0, \hat{v}, \overline{u}, e_j) := \begin{bmatrix} \overline{D}_{wu}^j & -\overline{D}_{wu}^j \underline{\overline{w}} - \frac{1}{2}\mathcal{B}_w^T \hat{K}^T e_j \\ \star & e_j^T(\overline{u} - \hat{K}_0 x_0 - \hat{v}) + \overline{w}^T \overline{D}_{wu}^j \underline{w} \end{bmatrix} \succeq 0, \tag{38}$$

$$\mathcal{L}_x^i(\underline{D}_{wx}^i, \hat{K}, \hat{K}_0, \hat{v}, \underline{x}, -e_i) \succeq 0, \tag{39}$$

$$\mathcal{L}_u^i(\underline{D}_{wu}^j, \hat{K}, \hat{K}_0, \hat{v}, \underline{u}, -e_j) \succeq 0. \tag{40}$$

PROOF. Using (34) and (7), the upper state constraints (26)-(27) can be written as, $\forall w \in \boldsymbol{\mathcal{W}}$:

$$e_i^T \widetilde{C} \hat{K}_{\mathcal{B}} \mathcal{B}_w w \le e_i^T(\overline{x} - \widetilde{C}(\mathcal{A} + \mathcal{B}_u \hat{K}_0)x_0 - \widetilde{C}\mathcal{B}_u \hat{v}).$$

Using the S-procedure, it can be shown that

$$\begin{aligned} &e_i^T \widetilde{C} \hat{K}_{\mathcal{B}} \mathcal{B}_w w - e_i^T(\overline{x} - \widetilde{C}(\mathcal{A} + \mathcal{B}_u \hat{K}_0)x_0 - \widetilde{C}\mathcal{B}_u \hat{v}) = \\ &-(\overline{w} - w)^T \overline{D}_{wx}^i (w - \underline{w}) - y^T \mathcal{L}_x^i(\overline{D}_{wx}^i, \hat{K}, \hat{K}_0, \hat{v}, \overline{x}, e_i) y \end{aligned}$$

where $y^T := [\,w^T \ 1]$, $\overline{D}_{wx}^i$, $\underline{D}_{wx}^i$, $\overline{D}_{wu}^j$, $\underline{D}_{wu}^j$ are diagonal, positive semidefinite matrices and the matrix $\mathcal{L}_x^i(\overline{D}_{wx}^i, \hat{K}, \hat{K}_0, \hat{v}, \overline{x}, e_i)$ is given in (37). It follows that (37) is a sufficient condition for upper state constraints.

Similarly, through application of the S-procedure, it can be shown that (39), (38) and (40) are sufficient for lower state and upper/lower input constraints, respectively. Now, the cost function (35) can be written as:

$$J(x_0, u, w) = y^T X_c^T \widetilde{Q} X_c y + y^T U_c^T \widetilde{R} U_c y + x_0^T Q x_0 \tag{41}$$

where matrix $X_c := [\hat{K}_{\mathcal{B}} \mathcal{B}_w \ \ (\mathcal{B}_u \hat{v} + (\mathcal{A} + \mathcal{B}_u \hat{K}_0)x_0)\,]$, $U_c := [\hat{K}\mathcal{B}_w \ \ \hat{K}_0 x_0 + \hat{v}\,]$ and $y^T := [w^T \ 1]$. In a manner similar to above, using the S-procedure on (41) followed by a Schur complement argument yields LMI (36). ■

It follows from Theorem 13 that the procedure for computing an RMPC controller (i.e. $\hat{K}_0$, $\hat{K}$, $\hat{v}$) which satisfies state and input constraints and minimizes the cost function can be summarized as follows

$$\begin{aligned} \overline{\phi} = \min\{\,\overline{f}_c : (36) - (40) \text{ are satisfied for diagonal} \\ D_w, \overline{D}_{wx}^i, \underline{D}_{wx}^i, \overline{D}_{wu}^j, \underline{D}_{wu}^j, j \in \mathcal{N}_u, i \in \mathcal{N}_x\}. \end{aligned} \tag{42}$$

Note that problem (42) is linear in the variables $(\hat{K}_0, \hat{K}, \hat{v})$ due to the uncertainty re-parameterization proposed in this algorithm. We can now summarize the Approach 2 RMPC algorithm as follows:

Algorithm 1 *Causal RMPC controller - Approach 2*

(1) Read the current state x_0.
(2) Compute polytopic bounds on the uncertainty through LMI problems (33).
(3) Compute K_0, K, v *by solving the LMI problem (42).*
(4) Apply the first control.
(5) If the computed state $x_t \in Z$*, apply the terminal control law* $\kappa_Z(x)$ *for all time, else loop back to (1).*

Remark 14 *Stability analysis of MPC schemes has been the subject of extensive research (see [9] for an excellent survey). The common components to establish RMPC* *stability include a terminal set which is invariant and a terminal cost which serves as a control Lyapunov function. Using an S-procedure, conditions on matrix P_N can readily be derived to ensure that the proposed terminal cost $x_N^T P_N x_N$ is a Lyapunov function over the designed RPI set Z. However, due to space limitations, we do not pursue this here. Instead, the reader is referred to [15, Sec 4.4] for a similar RMPC stability treatment on systems subject to scalar (polytopic) uncertainties.*

Remark 15 *Recursive feasibility of the proposed schemes can be ensured due to the incorporation of the invariant terminal set Z. In particular note that, under the conditions given in [15, Sec 4.4], the optimal control sequence computed at time t can be shifted and subsequently appended with the terminal control law $\kappa_Z(x)$ to yield: $\{u(t+1|t), \cdots, u(t+N|t), \kappa_Z(x)\}$ which remains feasible at next time step $t+1$. See [9] for further details.*

5 Numerical examples

We now consider two examples from the literature to illustrate the effectiveness of the proposed algorithms.

5.1 Example 1

We consider an uncertain version of the unstable process from [14,15]. In particular, we have the system in (2) with:

$$A = \begin{bmatrix} 1 & 0.8 \\ 0.5 & 1 \end{bmatrix}, B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; B_w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, B_p = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$D_{qu} = B_u, \quad C_q = \hat{C}_q = A$$

Furthermore, we consider the uncertainty of the form: $\boldsymbol{\mathcal{B}\Delta} := \{\mathrm{diag}(\delta_1, \delta_2) : \delta_i \in \mathbb{R}, |\delta_i| \leq 1\}$, and the disturbance set is taken to be $\mathcal{W} := \{w \in \mathbb{R}^{n_w} : -1 \leq w \leq 1\}$. The prediction horizon $N = 4$ and the parameters in the cost function (35) are $Q = I$, $R = I$, and $P_N = I$. The constraints on the input and state are given by: $\overline{u}_k = -\underline{u}_k = 3.8 \; \forall k$, and $\overline{x} = -\underline{x} = [3 \;\, 3]^T$, respectively. We linearly tighten constraints down the horizon. Moreover, we set the initial state $x_0 = \overline{x}$. Computing the RPI set and the terminal control law (using the algorithm in [16]) with input constraints $-0.95 \leq u_k \leq 0.95$ and terminal state constraints $|x_N| \leq [1.6 \;\, 1]^T$, yields the polytopes $\overline{x}_N = -\underline{x}_N = [1.55 \;\, 0.89]^T$ and $\kappa_Z(x) = -[0.34 \;\, 0.46]x$.

First of all, applying the proposed algorithm in the open-loop mode (by setting the feedback gain K to zero in (6)) gives infeasibility. Moreover, the feedback algorithm given by Problem (19) (linearized using Remark 5) on the above example also gives infeasibility due to the conservative nature of linearization. Now applying both the proposed schemes - as given by Theorem 8 (with $S_0 = \lambda_0 I$, $S = S_i = \lambda I$, $G_0 = G = G_i = 0$ for all i) and problem (42), respectively - give the simulation results shown in Fig. 1. We note that even with the initial state on the constraint boundary and persistent worst-case uncertainty and disturbances, the proposed algorithms are able to steer the system state to RPI set such that $x_2 \in Z$. The computed control input sequences for Approach 1 and 2 are given by $u_t = [-3.37, \; -2.62]$ and $u_t = [-3.79, \; -2.15]$, respectively.

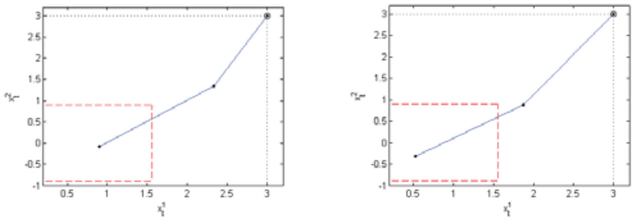

Fig. 1. Results for Approach 1 (left) and Approach 2 with $w_t = \overline{w}cos(t)$ and $\Delta = \mathrm{diag}(1, 1), \; \forall t$

5.2 Example 2

We consider the coupled spring-mass system example from [5]. The mechanical system, shown in Fig. 2, is unstable and has uncertainty in the spring constant value k such that $k_{min} \leq k \leq k_{max}$. The system has four states: x_1 and x_2 are the positions of mass 1 and 2 respectively, and x_3 and x_4 are their respective velocities. The discrete-time dynamics, sampled at 0.1s, are [5]:

$$A = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1k_n & 0.1k_n & 1 & 0 \\ 0.1k_n & -0.1k_n & 0 & 1 \end{bmatrix}, \; B_u = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ 0 \\ -0.1 \\ 0.1 \end{bmatrix}$$
$$C_q = \begin{bmatrix} k_{dev} & -k_{dev} & 0 & 0 \end{bmatrix}, \; D_{qu} = 0$$

where $\delta = \frac{k-k_n}{k_{dev}}$, $k_n = \frac{1}{2}(k_{max} + k_{min})$, and $k_{dev} = \frac{1}{2}(k_{max} - k_{min})$. The spring constant is known to vary between $k_{min} = 0.5$ and $k_{max} = 10$. For the cost, we have $Q = 5$, $R = 1$ and prediction horizon $N = 6$.

The control objective is to make the output (state x_2) track a unit step while providing robustness against persistent variation in spring constant k and respecting the input constraint: $-1 \leq u_k \leq 1$. Fig. 3 shows the simulation results when the system is subjected to a sinusoidal

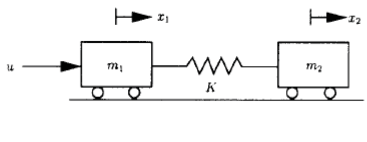

Fig. 2. Coupled spring-mass system

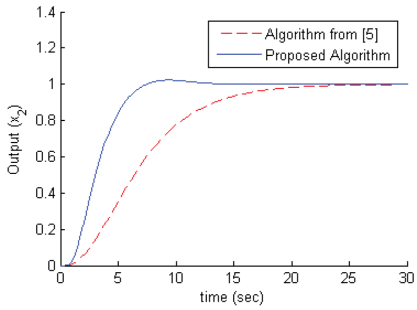

Fig. 3. Results for output step-tracking

uncertainty in the spring constant. We see that the proposed RMPC controller is able to first steer and then maintain the system-output at the desired set-point despite the presence of a persistent uncertainty. The 5% settling time for the output, with the proposed algorithm, is approximately 6.3 sec. For comparison, Fig. 3 also shows the response for infinite horizon RMPC controller proposed in [5] for the same example (red line). Although this algorithm also yields output tracking, however, the response is slower with a 5% settling time of approximately 16.1 sec. Fig. 4 also shows a comparitively faster response in control input for the proposed scheme.

6 Conclusion

We have proposed two algorithms for the (feedback) Robust Model Predictive Control of linear discrete-time systems subject to norm-bounded model-uncertainties and disturbances. The algorithms compute online, through LMI optimization, a constraint-admissible control law (K_0, K, v) that minimizes a cost function.

As shown in Section 2, even with the use of Q-parameterization-like method, the RMPC problem is nonlinear in feedback gain K due to the presence of model-uncertainty. To obtain computational tractability, we have proposed two methods. In the first method, the disturbance is recast as an uncertainty and a slack-variable approach is employed which helps to remove the nonlinearity through a 'less-conservative' change of variables. The second method involves the (online) re-parameterization of the uncertainty as a polytopic disturbance which subsequently leads to convexity. The effectiveness of the proposed schemes has been demonstrated through numerical examples from the literature.

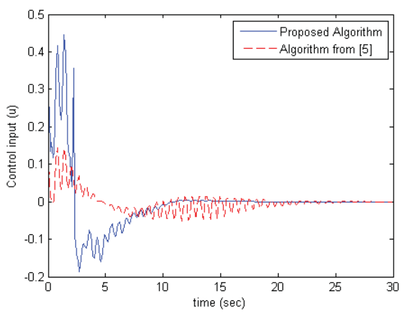

Fig. 4. Control input for Example 2

References

[1] De Oliveira MC, Geromel JC, Bernussou J. Extended H_2 and H_∞ norm characterizations and controller parameterizations for discrete-time systems. *International Journal of Control* 2002; **75**:666–679.

[2] El Ghaoui L, Oustry F,Lebret H. Robust solutions to uncertain semidefinite programs. *SIAM Journal on Optimziation*, 1998, **9:1**:33–52.

[3] Fujie T, Kojima M. Semidefinite Programming Relaxation for Nonconvex Quadratic Programs. *Global Optimization* 1997; **10**:367.

[4] Goulart PJ, Kerrigan EC, Maciejowski JM. Optimization Over State Feedback Policies for Robust Control with Constraints. *Automatica* 2006; **42**:523–533.

[5] Kothare MV, Balarkrishnan V, Morari M. Robust Constrained Model Predictive Control Using Linear Matrix Inequalities. *Automatica* 1996; **32**:1361–1379.

[6] Lee YI, Kouvaritakis B. Constrained Receding Horizon Predictive Control for Systems with Disturbances. *International Journal of Control* 1999; **72**:1027–1032.

[7] Löfberg J. Minimax Approaches to Robust Model Predictive Control. *PhD thesis*, Linköping University, 2003.

[8] Malik U, Jaimoukha IM, Halikias GD, Gungah SK. On the gap between the quadratic integer programming problem and its semidefinite relaxation. *Mathematical Programming* 2006; **107**:505–515.

[9] Mayne DQ, Rawlings JB, Rao CV, Scokaert POM. Constrained Model Predictive Control: Stability and Optimality. *Automatica* 2000; **36**(6):789–814.

[10] Oldewurtel F, Jones CN, Morari M. A Tractable Approximation of Chance Constrained Stochastic MPC based on Affine Disturbance Feedback. *Proceedings of the 47st IEEE Conference on Decision and Control*, 2008.

[11] Rakovic SV, Kerrigan EC, Kouramas KI, Mayne DQ. Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on Automatic Control* 2005; **50**:406–410.

[12] Scokaert POM, Mayne DQ. Min-max Feedback Model Predictive Control for Constrained Linear Systems. *IEEE Transactions on Automatic Control* 1998; **43**:1136–1142.

[13] Skaf J, Boyd SP. Design of Affine Controllers via Convex Optimization. *IEEE Transactions on Automatic Control* 2010; **55**:2476–2487.

[14] Tahir F, Jaimoukha IM. Robust Model Predictive Control through Dynamic State-feedback: An LMI Approach. *Proceedings of the 18th IFAC World Congress*, Milan, Italy, 2011.

[15] Tahir F, Jaimoukha IM. Robust Feedback Model Predictive Control of Constrained Uncertain Systems. *Journal of Process Control* 2013; **23**:189–200.

[16] Tahir F, Jaimoukha IM. Robust Positively Invariant Sets for Linear Systems Subject to model-uncertainty and disturbances. *4th IFAC Conference on Nonlinear Model predictive Control*, The Netherlands, 2012.

[17] Youla D, Jabr H, Bongiorno J. Modern Wiener-Hopf design of optimal controllers-Part II: The Multivariable Case. *IEEE Transactions on Automatic Control* 1976; **3**:319–338.