On moduli spaces of periodic monopoles and gravitational instantons

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Abstract

The topic of this thesis is the study of moduli spaces of periodic monopoles (with singularities), *i.e.* (singular) solutions to the Bogomolny equation (the dimensional reduction of the anti-self-duality equation to 3 dimensions) on $\mathbb{R}^2 \times \mathbb{S}^1$. Using arguments from physics, Cherkis and Kapustin gave strong evidence that 4–dimensional moduli spaces of (singular) periodic monopoles yield examples of gravitational instantons (*i.e.* complete hyperkähler 4–manifolds with decaying curvature) of type ALG. Recently, Hein constructed ALG metrics by solving a complex Monge-Ampère equation on the complement of a fibre in a rational elliptic surface.

The thesis is the first step in a programme aimed to verify Cherkis and Kapustin's predictions and understand them in relation to Hein's construction. More precisely:

(i) We construct moduli spaces of periodic monopoles (with singularities) and show that they are smooth hyperkähler manifolds for generic choices of parameters.

(ii) For each admissible choice of charge and number of singularities (and under additional conditions on the parameters in certain cases), we show that moduli spaces of periodic monopoles (with singularities) are non-empty by gluing methods.

After presenting these results, we will conclude the thesis with an outline of the other steps in the programme.

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Contents

Introduction

The topic of this thesis is the study of moduli spaces of *periodic monopoles* (with singularities). Magnetic monopoles on a Riemannian 3–manifold (X, g) are gauge equivalence classes of pairs (A, Φ) satisfying the Bogomolny equation

$$*F_A = d_A \Phi.$$

Here F_A is the curvature of a connection A on a principal G–bundle $P \to X$ (G a compact Lie group) and Φ, the Higgs field, is a section of the adjoint bundle ad(P). The Bogomolny equation arises as the dimensional reduction of the anti-self-duality equation to 3 dimensions.

When X is compact smooth monopoles reduce to pairs of a flat connection on P and a covariantly constant section of ad(P), so in order to find non-flat solutions it is necessary to consider a non-compact base manifold X and/or allow for (point) singularities of the fields. Periodic monopoles are solutions to the Bogomolny equation on $X = \mathbb{R}^2 \times \mathbb{S}^1$ endowed with its standard flat metric.

Working on a non-compact manifold, it is necessary to impose boundary conditions; if this is done appropriately, one can attach to each solution to the Bogomolny equation an integer k, which is a topological invariant of $(A, \Phi)|_{\partial X}$ and is called the *charge* of the monopole. Moreover, one can allow singularities at n distinct points $p_1, \dots, p_n$. If the parameters defining the boundary conditions are chosen in such a way to exclude the existence of reducible solutions, monopoles of charge k with n fixed singularities at $p_1, \dots, p_n$ form a smooth moduli space $\mathcal{M}_{n,k}$.

Moduli spaces of magnetic monopoles have extremely rich geometric properties even in the simplest case $X = \mathbb{R}^3$. In particular, when the base 3–manifold is flat and the boundary conditions are chosen so that infinitesimal deformations are square-integrable, the L^2 scalar product defines a hyperkähler metric on the moduli space of magnetic monopoles.

In the mid-1980s Atiyah and Hitchin [7] studied the moduli space of centred charge 2 $SU(2)$ monopoles on $\mathbb{R}^3$. Its double cover, the Atiyah–Hitchin manifold, is a simply connected 4–dimensional hyperkähler manifold with an isometric action of $SO(3)$ induced by rotations on $\mathbb{R}^3$. Thus the metric has cohomegeneity one and Atiyah and Hitchin were able to write an explicit formula: The Atiyah–Hitchin manifold is a complete 4–dimensional hyperkähler manifold with cubic volume growth and which locally near infinity is asymptotic to a circle fibration over the complement of a ball in $\mathbb{R}^3$.

Complete 4–dimensional hyperkähler manifolds with decaying curvature are called *gravitational instantons*. By specifying the asymptotic geometry at infinity, one can distinguish four families of gravitational instantons, ALE, ALF, ALG and ALH: ALE spaces are asymptotic to $\mathbb{R}^4$ with its flat metric up to a finite cover; ALF, ALG and ALH gravitational instantons are locally asymptotic to a $\mathbb{T}^{4-m}$–fibration over the complement of a ball in $\mathbb{R}^m$ with $m = 3, 2, 1$, respectively. These particular classes of 4–dimensional complete hyperkähler manifolds are believed to be relevant in the study of degenerations of Einstein metrics. As a simple prototypical example, think of the Kummer construction of the Ricci-flat metric on a K3 surface by gluing rescaled Eguchi–Hanson spaces to resolve the singularities of $\mathbb{T}^4/\mathbb{Z}_2$, *cf.* [104], [72] and [39].

In the early 2000's Cherkis and Kapustin suggested that moduli spaces of solutions to dimensional reductions of the Yang–Mills anti-self-duality (ASD) equations are "a natural place to look for gravitational instantons" [30]. In [32], they considered moduli spaces of centred charge 2 monopoles with singularities on $\mathbb{R}^3$ as examples of gravitational instantons of type ALF. In [29, 34] periodic monopoles (with singularities) are introduced: Moduli spaces of centred charge 2 periodic monopoles (with singularities) are conjectured to be 4–dimensional complete hyperkähler manifolds and, relying on physical arguments, an asymptotic formula for the L^2–metric is derived in [33]. This formula shows that the metric is of type ALG, in the sense that it has quadratic volume growth and the asymptotic geometry is that of a 2–torus bundle over the complement of a ball in $\mathbb{R}^2$ (but the modulus of the torus goes to infinity in the upper half-plane). By a topological constraint, the number of singularities cannot exceed twice the charge; thus there are five different moduli spaces of periodic monopoles which are expected to be gravitational instantons of type ALG. Finally, moduli spaces of monopoles (with singularities) on $\mathbb{R} \times \mathbb{T}^2$ and $\mathbb{T}^3$ are expected to yield examples of gravitational instantons of type ALH.

Recently, rigorous constructions of ALF, ALG and ALH gravitational instantons have been carried out by directly solving the complex Monge–Ampère equation. The most extensive list of examples in the ALG case, thought to be exhaustive up to certain deformations, has been obtained by Hein [53] by removing an anticanonical divisor from a rational elliptic surface; the singular type of the divisor removed (a fibre of the anticanonical elliptic fibration) determines the asymptotics of the Kähler Ricci-flat metric on its complement.

Following the general scheme of the so-called Hitchin–Kobayashi correspondence, Cherkis and Kapustin [29, 34] suggest that there exists a map, conjecturally an isomorphism, from the moduli space of centred charge 2 periodic monopoles to the complement of a fibre in a rational elliptic surface; the type of the fibre removed depends on the number of singularities of the monopoles.

The work presented in this thesis is motivated by the desire to verify Cherkis and Kapustin's predictions about moduli spaces of periodic monopoles and compare them with Hein's results. A wide range of questions has been left open by Cherkis and Kapustin's work. The main results of this thesis answer two of these questions: In Chapter 3 we show that moduli spaces of periodic monopoles (with singularities) are smooth hyperkähler manifolds for generic choices of parameters, provided they are non-empty. In Chapter 4, we present a gluing construction which shows that periodic monopoles (with singularities) exist for each compatible choice of charge and number of singularities.

Overview of the chapters

Chapter 1. In the first introductory chapter, intended to provide basic definitions, set the notation and motivate the work presented in this thesis, we discuss two different topics: On one side the theory of magnetic monopoles on 3–manifolds, on the other that of gravitational instantons. Linking the two, we place at the centre of the discussion the Atiyah–Hitchin manifold. The material presented in the chapter is completely standard and well-known and is organised with the aim of describing the main features of the Atiyah–Hitchin metric.

In Section 1 we present the classification of rotationally symmetric complete hyperkähler 4–manifolds. There are only three non-flat examples: the Eguchi–Hanson, Taub–NUT and Atiyah–Hitchin metrics.

In Sections 2 and 3 we provide the necessary background on the theory of magnetic monopoles on 3–manifolds (definitions, deformation theory, the L^2–metric on the moduli space). Most of the content of these sections is valid both for monopoles on $\mathbb{R}^3$ and for periodic monopoles.

The L^2–metric on moduli spaces of monopoles on $\mathbb{R}^3$ (and $\mathbb{R}^2 \times \mathbb{S}^1$) is hyperkähler by virtue of an infinite dimensional hyperkähler quotient construction, as we explain in Section 4. A simple finite dimensional example of this general construction of hyperkähler metrics is provided by the Eguchi–Hanson space. The discussion of the geometry of this example leads us to introduce the class of gravitational instantons of type ALE, completely classified by Kronheimer [64, 65].

In Section 5, in order to understand the asymptotic geometry of the Atiyah–Hitchin metric, we introduce the Gibbons–Hawking ansatz. As a simple illustration of this construction, we discuss the Taub–NUT metric as the prototypical example of a gravitational instanton of type ALF.

Finally, the last two sections shift the attention to gravitational instantons of type ALG. Section 6 reviews some aspects of Hein's construction [53] of a Ricci-flat Kähler metric on the complement of a fibre in a rational elliptic surface. In particular, we explain how the type of the fibre removed determines the asymptotics of the Ricci-flat metric. In Section 7 we discuss Cherkis and Kapustin's predictions about the properties of moduli spaces of periodic monopoles. These two final sections provide the motivation for the work presented in the rest of the thesis.

Chapter 2. In the second chapter we introduce periodic monopoles (with singularities). The main goal of the chapter is to review Cherkis and Kapustin's definitions of boundary conditions for smooth [29] and singular [34] periodic monopoles.

In Section 1 we study periodic Dirac monopoles, *i.e.* solutions to the Bogomolny equation on $\mathbb{R}^2 \times \mathbb{S}^1$ with structure group $U(1)$ and an isolated singularity at a point. The materials presented in this section will be used in two different ways in the rest of the thesis. On one side, periodic Dirac monopoles yield the asymptotic models used to define boundary conditions for non-abelian monopoles. On the other hand, a sum of periodic Dirac monopoles is one of the building blocks in the gluing construction of Chapter 4. Hence we devote some care to derive precise asymptotic expansions for a periodic Dirac monopole both at infinity and at its singularity.

In Section 2 we define boundary conditions for non-abelian periodic monopoles. The fact that the Green's function of $\mathbb{R}^2 \times \mathbb{S}^1$ grows logarithmically at infinity has the consequence that periodic monopoles have infinite energy even when there are no singularities. We should also point out that it is important to choose structure group $SO(3)$ (or $U(2)$) instead of $SU(2)$ when singularities are allowed at a finite collection of points.

Chapter 3. In Chapter 3 we introduce the analytical tools required to work with Cherkis and Kapustin's definitions. As a first application, we study the deformation theory of periodic monopoles and show that the moduli spaces of periodic monopoles (with singularities) are smooth hyperkähler manifolds for a generic choice of parameters specifying the boundary conditions.

In Section 1 we review the work of Kronheimer [67] on monopoles with Dirac type singularities: A monopole on a 3–ball with a singularity at the origin modelled on a Dirac monopole can be lifted via the Hopf map to an $\mathbb{S}^1$–invariant instanton on the 4–ball. Instead of reducing the theory to the 4–dimensional one, however, we decided to work directly in 3–dimensions, relying on Lockhart–McOwen's theory of weighted Sobolev spaces. It turns out that this choice is better suited to the gluing construction of Chapter 4.

In Sections 2 and 3 we introduce function spaces to deal with the two sources of non-compactness in the problem: The singularities and the end of $\mathbb{R}^2 \times \mathbb{S}^1$. We define appropriate weighted spaces, check that the necessary embedding and multiplication properties hold and discuss the basic elliptic theory for the operators governing the deformation theory of monopoles.

Finally, in Section 4 we apply these analytic results to the construction of the moduli spaces of periodic monopoles (with singularities). After a brief discussion of reducibility, we prove the following theorem.

Theorem (*cf.* Theorem 3.4.8)**.** *For generic choices of parameters defining the boundary conditions, the moduli space $\mathcal{M}_{n,k}$ of charge k periodic monopoles with n singularities is a smooth hyperkähler manifold.*

Chapter 4. The last chapter contains the main result of this thesis, an existence theorem for periodic monopoles (with singularities).

Theorem (*cf.* Theorem 4.7.2)**.** *For each charge k and each number of singularities n satisfying the topological constraint $n \leq 2k$, there exist choices of the parameters specifying the boundary behaviour of the fields such that the moduli space $\mathcal{M}_{n,k}$ of periodic monopoles of charge k with n singularities is non-empty.*

The theorem is proved via gluing methods. Monopoles on $\mathbb{R}^3$ (without singularities) were themselves constructed via gluing methods in a seminal work by Taubes [60]. Moreover, this approach is close to the physical intuition that allowed Cherkis and Kapustin to guess the asymptotic behaviour of the metric on the moduli space.

The starting point of Taubes's construction is that charge 1 monopoles on $\mathbb{R}^3$ are completely explicit: Up to translations and scaling there exists a unique solution, localised around the origin in $\mathbb{R}^3$. The gluing construction then shows that in a certain region of the moduli space a charge k monopole can be thought of as a superposition of k charge 1 particle-like components. This has to be thought of as a converse to the following compactness statement for finite energy monopoles: A sequence of monopoles with uniformly bounded energy can fail to converge in the moduli space only because some particle components move off to infinity leaving behind a monopole of lower charge.

A number of new features/difficulties appear when trying to implement the construction for periodic monopoles. First of all, not even charge 1 periodic monopoles are explicitly known. In fact, numerical experiments of Ward [107] show that such monopoles are localised around *two* rather than a single point when the ratio α between the mass of the monopole and the period of the $\mathbb{S}^1$ factor in the base is negative and large in absolute value. The mass of the monopole is the constant term in the expansion of the Higgs field Φ at infinity. On the contrary, if α is positive and large charge 1 periodic monopoles are expected to be well-approximated by a scaled charge 1 monopoles on $\mathbb{R}^3$.

As a consequence, the construction of a charge k periodic monopole as a superposition of k charge 1 monopoles can be carried out only when the charge 1 constituents have large positive mass. There are two ways of arranging for this to happen. On one side one can construct periodic monopoles of charge k and with an arbitrary number of singularities $n \leq 2k$ when the mass is sufficiently large. In fact, it is conceivable that monopoles with high mass exist on any complete Riemannian 3–manifold satisfying appropriate conditions. More interestingly, the fact that the

Green's function of $\mathbb{R}^2 \times \mathbb{S}^1$ grows logarithmically (and therefore that monopoles have infinite energy) implies that periodic monopoles can "bubble off" at infinity: When the number n of singularities satisfies $n < 2(k-1)$, we will construct periodic monopoles with arbitrary mass and such that the k charge 1 components are more and more concentrated around their centres as these move off to infinity.

The chapter is organised as follows. In Section 2 we describe an initial singular configuration given by a sum of periodic Dirac monopoles. The gluing construction is a desingularisation of this initial solution, by gluing rescaled Euclidean charge 1 monopoles: In Section 3 we collect the main properties of these monopoles and in Section 4 construct a family of initial approximate solutions to the Bogomolny equation. We deform these into genuine monopoles by means of the Implicit Function Theorem. The crucial step is to study the linearised equation; we carry out this step in Sections 5 and 6, adapting the analysis of Chapter 3. Finally, in Section 7 we state and prove a precise existence result.

In the final Section 8 we return to the questions left open at the end of Chapter 1. We briefly describe how the results of this thesis represent only the first steps in a programme aimed to answer those questions.

Chapter 1

Moduli spaces of monopoles and gravitational instantons

The aim of this introductory chapter is to fix the notation and provide the motivation for the work presented in this thesis. We approach the material through the discussion of the main properties of the Atiyah–Hitchin manifold; each different point of view provides the chance to introduce well-known results on the theory of magnetic monopoles on 3–manifolds and that of gravitational instantons.

In Section 1 we introduce the Atiyah–Hitchin metric as one of the rotationally symmetric complete hyperkähler 4–manifolds. Sections 2 and 3 contain background material on magnetic monopoles. In Section 4 we recall the hyperkähler quotient construction, first in the finite dimensional setting, discussing the Eguchi–Hanson metric and the family of gravitational instantons of type ALE; then we interpret moduli spaces of monopoles on a flat 3–manifold as hyperkähler quotients of an infinite dimensional quaternionic affine space. The aim of Section 5 is to describe the asymptotic geometry of the Atiyah–Hitchin metric. In order to do this, it is necessary to introduce the Gibbons–Hawking ansatz for hyperkähler 4–manifolds with a tri-holomorphic vector field; we illustrate this construction discussing the Taub–NUT metric and other examples of gravitational instantons of type ALF. Section 6 reviews Hein's construction of ALG metrics on the complement of a fibre in a rational elliptic surface. Finally, in Section 7 we discuss the work of Cherkis and Kapustin on moduli spaces of periodic monopoles (with singularities) and how it points in the direction of a connection with Hein's results.

1.1 Rotationally symmetric hyperkähler 4–manifolds

Identify $SU(2)$ with $\mathbb{S}^3 \subset \mathbb{C}^2$ via $h\colon\ (z_1, z_2) \longmapsto \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}$. It will be convenient to use Euler angles

$$(z_1, z_2) = \left(e^{\frac{i}{2}(\theta+\psi)} \cos\left(\tfrac{\phi}{2}\right), e^{\frac{i}{2}(\theta-\psi)} \sin\left(\tfrac{\phi}{2}\right) \right),$$

with $\theta \in [0, 2\pi)$, $\phi \in [0, \pi)$ and $\psi \in [0, 4\pi)$. A basis $\{\eta_1, \eta_2, \eta_3\}$ of left invariant 1–forms on $SU(2)$ is defined by the Maurer–Cartan form $h^{-1}dh = \eta_1\sigma_1 + \eta_2\sigma_2 + \eta_3\sigma_3$, where:

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\begin{cases} \eta_1 = -\sin\psi\, d\phi + \cos\psi\, \sin\phi\, d\theta \\ \eta_2 = \cos\psi\, d\phi + \sin\psi\, \sin\phi\, d\theta \\ \eta_3 = d\psi + \cos\phi\, d\theta \end{cases} \tag{1.1.1}$$

Since $[\sigma_i, \sigma_j] = -\epsilon_{ijk}\,\sigma_k$, where ϵ_{ijk} is the anti-symmetric symbol, the Maurer–Cartan equation reads $d\eta_i = \frac{1}{2}\sum_{j,k} \epsilon_{ijk}\,\eta_j \wedge \eta_k$. (1.1.1) defines left-invariant 1–forms on $SO(3)$ by restricting the range of ψ to $[0, 2\pi)$.

If (M^4, g) is a Riemannian 4–manifold with an isometric action of $G = SU(2)$ or $SO(3)$ such that the orbits are generically 3–dimensional, parametrise M with Euler angles and a radial coordinate $s \in I \subset \mathbb{R}$ and write the metric in Bianchi IX form [45]

$$g = f^2\, ds^2 + a_1^2\, \eta_1^2 + a_2^2\, \eta_2^2 + a_3^2\, \eta_3^2, \tag{1.1.2}$$

where f and a_i are functions of s only. By redefining s, one can always assume that $f = a_1 a_2 a_3$, but other normalisations will be equally useful.

At the end of this section we will restrict our attention to three rotationally symmetric 4–manifolds for which the generic 3–dimensional orbits are diffeomorphic to $SU(2), SO(3)$ and $SO(3)/\mathbb{Z}_2$, respectively. In the latter case we quotient $SO(3)$ by the involution

$$(\theta, \phi, \psi) \longmapsto (\theta + \pi, \pi - \phi, -\psi). \tag{1.1.3}$$

Working explicitly with a metric in Bianchi IX form, we recall few standard facts about Riemannian 4–manifolds. Some of the references we follow are [6], [12, Chapters 1 and 13], [54, Chapter 1, §8], [61, Chapters 2, 3 and 7] and [71]. Introduce the orthonormal co-frame

$$\theta_0 = f\,ds, \qquad \theta_1 = a_1\,\eta_1, \qquad \theta_2 = a_2\,\eta_2, \qquad \theta_3 = a_3\,\eta_3. \tag{1.1.4}$$

Notice that only η_1 is invariant under the involution (1.1.3). However, at least on the open set where orbits are 3–dimensional one can always consider a double cover on which (1.1.4) are all well-defined.

In Cartan's formalism, we view the Levi–Civita connection of g as the $SO(4)$–connection $\omega = (\omega_{ab})_{a,b=0}^3$ on $T^*M \to M$ defined by $d\theta_a + \sum_{b=0}^3 \omega_{ab} \wedge \theta_b = 0$. Hence

$$\omega_{01} = -\frac{\dot{a}_1}{f a_1}\,\theta_1 \qquad \omega_{23} = \frac{a_2^2 + a_3^2 - a_1^2}{2a_1 a_2 a_3}\,\theta_1 \tag{1.1.5}$$

together with the analogous expressions obtained by cyclic permutations of $1, 2, 3$.

The curvature operator $\mathcal{R}_g \colon \bigwedge^2 \to \bigwedge^2$ is recovered from the curvature F_ω of ω by the formula $\mathcal{R}_g \alpha = \langle F_\omega, \alpha \rangle_{\mathfrak{so}_4}$, where, using the metric, we identify 2–forms with skew-adjoint endomorphisms of $\bigwedge^1$. With this identification, the decomposition of $\bigwedge^2 = \bigwedge^+ \oplus \bigwedge^-$ into self-dual and anti-self-dual forms, the ± 1 eigenspaces of the Hodge–$*$ operator, corresponds to the decomposition of the non-simple Lie algebra $\mathfrak{so}_4 = \mathfrak{so}_3^+ \oplus \mathfrak{so}_3^-$. If we choose the orientation on M defined by $\theta_0 \wedge \theta_1 \wedge \theta_2 \wedge \theta_3$, the forms

$$\phi_i^\pm = \theta_0 \wedge \theta_i \pm \theta_j \wedge \theta_k, \tag{1.1.6}$$

for $i = 1, 2, 3$ and j, k chosen so that $\epsilon_{ijk} = 1$, yield a G–invariant basis of $\bigwedge^\pm$.

Under the decomposition $\bigwedge^2 = \bigwedge^+ \oplus \bigwedge^-$ the curvature operator takes the form

$$\mathcal{R}_g = \begin{pmatrix} W^+ + \frac{\mathrm{Scal}}{12}\,\mathrm{id}_3 & \mathring{\mathrm{Ric}} \\ \mathring{\mathrm{Ric}} & W^- + \frac{\mathrm{Scal}}{12}\,\mathrm{id}_3 \end{pmatrix} \tag{1.1.7}$$

where $W^+ + W^-$ is the Weyl curvature of g, Scal the scalar curvature and $\mathring{\mathrm{Ric}}$ the trace-less Ricci curvature. The metric g is said to be *anti-self-dual* if $W^+ = 0$; *Einstein* if $\mathring{\mathrm{Ric}} = 0$, in which case Scal is forced to be a constant. We are interested in Riemannian manifolds (M, g) which are both anti-self-dual and Ricci flat, *i.e.* such that the curvature of the Levi-Civita connection on $\bigwedge^+$ vanishes.

Now notice that $SO(4) \to SO(3) \times SO(3)$ is a double cover, where the map is defined by the action of $SO(4)$ on $\bigwedge^{\pm}$. Suppose that M is simply connected: If $\bigwedge^{+}$ is flat, the holonomy group of g is contained in $SU(2)$, the subgroup of $SO(4)$ acting trivially on $\bigwedge^{+}$. Thus (M, g) is a simply connected hyperkähler manifold:

Definition 1.1.1. We say that (M^{4n}, g) is a *hyperkähler* manifold if there exists a triple of parallel complex structures J_1, J_2, J_3 such that g is Hermitian with respect to each of them and $J_1 J_2 = J_3$.

If this is the case, for each $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{S}^2$ $J_\alpha = \alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3$ also is a parallel complex structure and g is Hermitian with respect to J_α. To each complex structure $J = J_\alpha$ we associate a Kähler form ω_J by $\omega_J(\,\cdot\,,\,\cdot\,) = g(J\,\cdot\,,\,\cdot\,)$. The metric g is Hermitian with respect to J if and only if J is skew-adjoint with respect to g (and therefore ω_J is a 2–form as claimed). We say that ω_J is a Kähler form if $d\omega_J = 0$. Given an almost complex structure and a Hermitian metric on M^{2n}, the condition $\nabla J = 0$ is in fact equivalent to J being integrable and ω_J a Kähler form. Summarising, a hyperkähler manifold (M, g, J_1, J_2, J_3) admits a 2–sphere of parallel complex structures J_α and corresponding Kähler forms ω_{J_α}. In the lowest possible dimension $4n = 4$, since each Kähler form is self-dual, the triple $(\omega_{J_1}, \omega_{J_2}, \omega_{J_3})$ yields a trivialisation of $\bigwedge^{+}$ by closed forms.

A compact hyperkähler 4–manifold is either a flat torus or a K3 surface endowed with Yau's Ricci flat metric. We are interested in complete non-compact hyperkähler 4–manifolds with decaying curvature at infinity.

Definition 1.1.2. A *gravitational instanton* is a complete non-compact 4–dimensional hyperkähler manifold (M^4, g) with $\int_M |\mathcal{R}_g|^2 < \infty$.

Now specialise to a metric g in Bianchi IX form. Because of the rotational symmetry, on the open set where the orbits are 3–dimensional the equations for self-duality and Ricci-flatness of the metric reduce to second order ODEs. The problem of finding rotationally symmetric hyperkähler manifolds was initially studied by Gibbons and Pope [45] and completely solved by Atiyah and Hitchin [7, Chapter 9]. Gibbons and Pope showed how the second order equations for the coefficients f, a_1, a_2, a_3 can be integrated to a system of first order ODEs. This can be seen as follows.

Suppose that M is simply connected: If g is Ricci-flat and anti-self-dual, the $SO(3)$–bundle $\bigwedge^{+}$ is trivial. By working on a finite cover if necessary, assume that the generic orbit is diffeomorphic to $G = SU(2)$ or $SO(3)$ so that we can use the moving frame (1.1.4). Then, up to gauge transformations, the component ω^+ of the connection form $\omega = \omega_{ab}$ acting on $\bigwedge^+$ vanishes. On the other hand, the group of symmetries G acts on the sphere of Kähler forms ω_J: The action is either trivial or the standard action of $\mathrm{Ad}(G) = SO(3)$ on $\mathbb{S}^2$ by rotations. Fix a hyperkähler structure (J_1, J_2, J_3) and write the corresponding triple of Kähler forms as $(\omega_{J_1}, \omega_{J_2}, \omega_{J_3}) = u\,(\phi_1^+, \phi_2^+, \phi_3^+)$, where $u\colon\ I \times G \to SO(3)$ is a gauge transformation and ϕ_i^+ is defined in (1.1.6). Since the Kähler forms are closed, $\omega^+ = u^{-1}du$. Moreover, since ω has no θ_0–component by (1.1.5), u is independent of the radial coordinate s and by choosing (J_1, J_2, J_3) appropriately we can assume that $u = \mathrm{id}$ if the action of G on $\bigwedge^+$ is trivial and $u = \mathrm{Ad}$ otherwise. Together with (1.1.5) and the definition of the 1–forms (1.1.4) in terms of the Maurer–Cartan form of G, we obtain

$$2\,\frac{\dot{a}_1}{a_1} = a_2^2 + a_3^2 - a_1^2 - 2\varepsilon\, a_2 a_3 \tag{1.1.8}$$

and the other two equations given by cyclic permutation of $1, 2, 3$. Here we used the normalisation $f = a_1 a_2 a_3$ and $\varepsilon = 0, 1$ depending on whether $u = \mathrm{id}$ or $u = \mathrm{Ad}$.

Proposition 1.1.3. *The unique complete rotationally symmetric hyperkähler* 4*–manifolds with* 3*–dimensional generic orbits and which are not flat are:*

1. *The Eguchi–Hanson metric, with $G = SO(3)$, $\varepsilon = 0$ and* 3*–dimensional orbits diffeomorphic to $SO(3)$;*
2. *The Taub–NUT metric, with $G = SU(2)$, $\varepsilon = 1$ and $SU(2)$ as the generic orbit;*
3. *The Atiyah–Hitchin metric, with $G = SO(3)$, $\varepsilon = 1$ and* 3*–dimensional orbits diffeomorphic to $SO(3)/\mathbb{Z}_2$.*

The classification is stated in [46, Proposition 2.7] but it follows from the work of Gibbons–Pope [45] and Atiyah–Hitchin [7, Chapter 9]. Gibbons and Pope found the solutions to (1.1.8) which correspond to the first two metrics. They are well-known examples of gravitational instantons (known previously to [45]) and both have an additional $SO(2)$–isometry: $a_1 = a_2$ and the ODEs can be solved explicitly. We will discuss the geometry of these two examples in some details later in the Chapter, when we will use them to illustrate some general constructions of hyperkähler metrics. A few years later, Atiyah and Hitchin found the third solution to

(1.1.8) yielding a non-flat complete metric. The coefficients f, a_1, a_2, a_3 are explicit expressions of elliptic integrals. With the normalisation $f(s) = -\frac{b(s)}{s}$, the Atiyah–Hitchin manifold (AH) is diffeomorphic to $(\pi, +\infty) \times SO(3)/\mathbb{Z}_2$ compactified at $s = \pi$ by adding a minimal 2–sphere.

If a formula for the AH metric could be found by studying the system (1.1.8), Atiyah and Hitchin knew that such a metric—a complete non-flat hyperkähler 4–manifold acted upon by $SO(3)$ with generic 3–dimensional orbits and distinct from the EH and the Taub–NUT metric—had to exists. The AH manifold is the double cover of the moduli space of centred charge 2 $SU(2)$ magnetic monopoles on $\mathbb{R}^3$, which we now introduce.

1.2 The Bogomolny equation

Let (X, g) be an oriented Riemannian 3–manifold and $P \to X$ a principal G–bundle, where G is a compact Lie group. In the rest of the thesis we will almost exclusively consider bundles with structure group $G = U(1), SU(2), U(2)$ or $SO(3)$. Equivalently, we can work on vector bundles associated to P via some representation of G. For example, we will consider line bundles L if $G = U(1)$, complex rank 2 Hermitian vector bundles E if $G = U(2)$—with trivial determinant if $G = SU(2)$—and real oriented 3–dimensional Riemannian vector bundles V when $G = SO(3)$.

Definition 1.2.1. Magnetic monopoles are gauge equivalence classes of solutions (A, Φ) to the *Bogomolny equation*

$$* F_A = d_A \Phi. \tag{1.2.1}$$

Here $*$ is the Hodge star operator of (X, g); F_A is the curvature of a connection A on the principal G–bundle P; Φ, the *Higgs field*, is a section of the adjoint bundle $\mathrm{ad}(P)$; finally, the equivalence is defined with respect to the action of the gauge group $\mathrm{Aut}(P)$.

Remark. The Bogomolny equation arises as the dimensional reduction of the anti-self-duality equation to 3 dimensions: (A, Φ) is a solution to (1.2.1) on (X, g) if and only if $\hat{A} = A + \Phi \otimes ds$ is an anti-self-dual (ASD) connection on $X \times \mathbb{R}_s$ invariant under translations along the s-axis. Here we endow $X \times \mathbb{R}_s$ with the product metric and the volume form $ds \wedge \mathrm{dvol}_g$ and $\hat{A}$ is ASD if $*F_{\hat{A}} = -F_{\hat{A}}$.

A first immediate consequence of equation (1.2.1) is

$$\begin{cases} d_A^* F_A = [d_A\Phi, \Phi], \\ d_A^* d_A \Phi = 0. \end{cases} \tag{1.2.2}$$

In particular, when X is compact smooth monopoles are necessarily trivial: A is a flat connection and Φ a parallel section (and therefore, if it is non-zero, it defines a reduction of the structure group). In order to find interesting solutions to (1.2.1) one has to consider a non-compact base manifold X, in the sense that either

(i) X is complete or

(ii) we allow for singularities of the fields (A, Φ),

or both, as it will be the case in the next chapters. The classical case of smooth monopoles on $\mathbb{R}^3$, which is the focus of this and the next few sections, and the rich geometric properties of their moduli spaces have been investigated from many different points of view; the standard reference is Atiyah and Hitchin's book [7]. Monopoles with and without singularities have also been studied on 3–manifolds X with different geometries: Hyperbolic monopoles were introduced by Atiyah [5], Braam reduced the study of monopoles on an asymptotically hyperbolic manifold X to that of $\mathbb{S}^1$–invariant ASD connections on a conformal compactification [23]; partial results were established by Floer [40, 41] for asymptotically Euclidean X; more recently, Kottke initiated the study of monopoles on asymptotically conical 3–manifolds [63]. Monopoles with singularities were first considered by Kronheimer [67]; the dimension of the moduli space of singular monopoles over a compact manifold X was computed by Pauly [87]; Charbonneau and Hurtubise considered monopoles with singularities on the product of a compact Riemann surface with a circle [27].

The system (1.2.2) is the Euler–Lagrange equation of the Yang–Mills–Higgs functional (or *energy*)

$$\mathcal{A}(A, \Phi) = \frac{1}{2} \int_X |F_A|^2 + |d_A\Phi|^2. \tag{1.2.3}$$

Using $\frac{1}{2}|*F_A \pm d_A\Phi|^2 = \frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\Phi|^2 \pm \langle d_A\Phi, *F_A\rangle$ and the Bianchi identity, rewrite $\mathcal{A}(A,\Phi)$ as

$$\mathcal{A}(A,\Phi) = \frac{1}{2}\int_X |*F_A \pm d_A\Phi|^2 \mp \int_X d\,\langle \Phi, F_A\rangle. \tag{1.2.4}$$

Hence a monopole (or an anti-monopole) minimises the Yang–Mills–Higgs energy amongst pairs (A,Φ) with the same "boundary" term $\int_X d\,\langle\Phi, F_A\rangle$.

Now specialise to the case $X = \mathbb{R}^3$, $G = SU(2)$. P is a trivial bundle and A, Φ are a 1 and a 0–form with values in $\mathfrak{su}_2$. We will assume that the energy $\mathcal{A}(A,\Phi)$ is finite and impose the boundary condition $\lim_{|x|\to+\infty}|\Phi| = 1$. The quantity

$$-\frac{1}{4\pi}\lim_{R\to\infty}\int_{\partial B_R}\langle\Phi, F_A\rangle$$

is the degree of the map $|\Phi|^{-1}\Phi\colon \partial B_R \to \mathbb{S}^2 \subset \mathfrak{su}_2$ for large enough R. By (1.2.3) and (1.2.4), if (A,Φ) is a solution to (1.2.1) the degree needs to be a non-positive integer $-k$, where $k \in \mathbb{Z}_{\geq 0}$ is called the *charge* of (A,Φ). Moreover, since $\mathcal{A}(A,\Phi) = 4\pi k$, either $k > 0$ or Φ is covariantly constant and A is a flat reducible connection.

By continuity, if the boundary condition $\lim_{|x|\to+\infty}|\Phi| = 1$ is satisfied then Φ doesn't vanish outside of a compact set. Thus the trivial rank 2 complex vector bundle E on $\mathbb{R}^3$ splits over this exterior region as a direct sum $E \simeq H^k \oplus H^{-k}$ of eigenspaces of Φ. Since $|\Phi|^{-1}\Phi$ has degree $-k$, H is the radial extension of the inverse of the Hopf line bundle over $\mathbb{S}^2$. The reason for this confusing choice of notation is that $H \to \mathbb{S}^2 \simeq \mathbb{P}^1$ has positive degree, *i.e.* it has holomorphic sections. Fix $k \in \mathbb{Z}_{>0}$ and let $\mathcal{C}_k$ be the space of smooth pairs (A,Φ) on the trivial $SU(2)$–bundle over $\mathbb{R}^3$ with finite energy, charge k and such that $\lim_{|x|\to+\infty}|\Phi| = 1$.

A gauge transformation $g \in C^\infty(\mathbb{R}^3; SU_2)$ acts on a pair $c = (A,\Phi) \in \mathcal{C}_k$ changing c into $c + (d_1 g)g^{-1}$, where

$$d_1 g = -\left(d_A g, [\Phi, g]\right) \in \Omega(\mathbb{R}^3; \mathfrak{su}_2). \tag{1.2.5}$$

Here we introduced the notation $\Omega = \Omega^1 \oplus \Omega^0$. Denote by $\mathcal{G}$ the space of bounded gauge transformations such that $(d_1 g)g^{-1} \in L^2$ and let $\mathcal{G}_0$ be the subspace of gauge transformations which are asymptotic to the identity. Observe that under the asymptotic isomorphism $E \simeq H^k \oplus H^{-k}$, gauge transformations $g \in \mathcal{G}$ are only allowed to approach a non-trivial value in the $U(1)$–subgroup of $SU(2)$ which fixes Φ, *i.e.* $\mathcal{G}$ is an extension of $\mathcal{G}_0$ by $U(1)$.

Regard the Bogomolny equation (1.2.1) as a map $\Psi\colon\ \mathcal{C}_k \to \Omega^1(\mathbb{R}^3; \mathfrak{su}_2)$. Following the notation of [7], the moduli space M_k (respectively, N_k) of framed (unframed) charge k monopoles is by definition $M_k = \Psi^{-1}(0)/\mathcal{G}_0$ ($N_k = \Psi^{-1}(0)/\mathcal{G}$). Analytic results of Taubes [98] guarantee that M_k is a smooth manifold of dimension $4k$. Moreover, the circle parametrising the asymptotic isomorphism $E \simeq H^k \oplus H^{-k}$ acts on M_k with quotient N_k.

It remains to define the moduli space of *centred* monopoles and show that it carries a hyperkähler metric (M_k carries one as well, in fact). To proceed any further we need to study the deformation theory of monopoles.

1.3 Deformation theory of monopoles

The discussion that follows applies both to $\mathbb{R}^3$ as well as to any oriented Riemannian 3–manifold (X, g). In particular, the set-up described here will be used in the following chapters to study monopoles on $\mathbb{R}^2 \times \mathbb{S}^1$.

To fix the notation in this more general situation, let $P \to X$ be a principal G–bundle and denote by $\mathcal{C}$ the infinite dimensional space of smooth pairs $c = (A, \Phi)$, where A is a connection on $P \to X$ and $\Phi \in \Omega^0(X; \operatorname{ad} P)$ a Higgs field. Since X is not compact, elements $c \in \mathcal{C}$ have to satisfy appropriate boundary conditions, which we suppose to be included in the definition of $\mathcal{C}$. $\mathcal{C}$ is an affine space: The underlying vector space is the space of sections $\Omega_{\text{ad}} = \Omega^1(X; \operatorname{ad} P) \oplus \Omega^0(X; \operatorname{ad} P)$ satisfying appropriate decay conditions. Let $\mathcal{G}$ be the group of bounded smooth sections of $\operatorname{Aut}(P)$ which preserve the chosen boundary conditions, *i.e.* such that $c + (d_1 g)g^{-1} \in \mathcal{C}$ for all $c \in \mathcal{C}$.

Consider the map $\Psi\colon\ \mathcal{C} \to \Omega^1(X; \operatorname{ad} P)$ defined by $(A, \Phi) \mapsto *F_A - d_A\Phi$. By fixing a base point $c = (A, \Phi) \in \mathcal{C}$ we write $\Psi(A + a, \Phi + \psi) = \Psi(c) + d_2(a, \psi) + (a, \psi) \cdot (a, \psi)$ for all $(a, \psi) \in \Omega_{\text{ad}}$. The linearisation d_2 of Ψ at c and the quadratic term are defined by:

$$d_2(a, \psi) = *d_A a - d_A\psi + [\Phi, a] \tag{1.3.1}$$

$$(a, \psi) \cdot (a, \psi) = *[a, a] - [a, \psi] \tag{1.3.2}$$

The gauge group $\mathcal{G}$ acts naturally on $\mathcal{C}$ and the map Ψ is gauge equivariant. The linearisation at c of the action of $\mathcal{G}$ on $\mathcal{C}$ is the operator $d_1\colon\ \Omega^0(X;\operatorname{ad}P)\to\Omega_{\text{ad}}$ defined as in (1.2.5). Couple d_2 with d_1^* to obtain an elliptic operator

$$D = D_c = d_2 \oplus d_1^*\colon\ \Omega_{\text{ad}} \longrightarrow \Omega_{\text{ad}}. \tag{1.3.3}$$

The moduli space $\mathcal{M}$ of monopoles in $\mathcal{C}$ is defined as $\mathcal{M}=\Psi^{-1}(0)/\mathcal{G}$. Suppose that $c=(A,\varphi)$ is a solution to the Bogomolny equation and consider the elliptic complex

$$\Omega^0(X;\operatorname{ad}P) \xrightarrow{d_1} \Omega_{\text{ad}} \xrightarrow{d_2} \Omega^1(X;\operatorname{ad}P) \tag{1.3.4}$$

(this is a complex precisely when $\Psi(A,\Phi)=0$). Standard theory [38, Chapter 4] shows that $\mathcal{M}$ is a smooth manifold if—after choosing Sobolev completions of the spaces of ad(P)–valued forms so that Ψ and the action of gauge transformations $\mathcal{G}\times\mathcal{C}\to\mathcal{C}$ extend to smooth maps of Banach spaces and (1.3.4) is a Fredholm complex—the cohomology groups of (1.3.4) in degree 0 and 2 vanish. Then the tangent space $T_{[c]}\mathcal{M}$ at the point $[c]$ is identified with $\ker D_c$, *i.e.* the cohomology of (1.3.4) in degree 1.

We can interpret $D=D_c$ as a twisted Dirac operator on Ω_{ad}. Given a 1–form α and a k–form β on X, the Clifford multiplication

$$\gamma(\alpha)\beta = \alpha\wedge\beta - \alpha^\sharp\lrcorner\,\beta \tag{1.3.5}$$

together with the metric induced by g and the Levi-Civita connection make the bundle of differential forms on X into a Dirac bundle. Then we define a twisted Dirac operator $\not{D}_A$ on Ω_{ad} by

$$\Omega^1\oplus\Omega^0 \xrightarrow{(\text{id},*)} \Omega^1\oplus\Omega^3 \xrightarrow{\gamma\circ\nabla_A} \Omega^2\oplus\Omega^0 \xrightarrow{(*,\text{id})} \Omega^1\oplus\Omega^0. \tag{1.3.6}$$

The operator D of (1.3.3) is $D=\tau\not{D}_A+[\Phi,\cdot\,]$, where τ is a sign operator with $\tau=1$ on 1–forms and $\tau=-1$ on 0–forms. From this point of view, the product (1.3.2) is the multiplication on Ω_{ad} obtained combining Clifford multiplication on the 1–form part and the Lie bracket on ad(P). The formal L^2–adjoint of D is $D^*=D-2[\Phi,\cdot\,]$ and we have Weitzenböck formulas:

Lemma 1.3.1. *If (X,g) is flat*

$$DD^* = \nabla_A^*\nabla_A - ad(\Phi)^2 + \Psi \qquad D^*D = DD^* + 2d_A\Phi,$$

where $\Psi = *F_A - d_A\Phi$ and $d_A\Phi$, as ad(P)–valued 1–forms, act on Ω_{ad} by the multiplication $\cdot$ obtained from the Clifford multiplication and the Lie bracket.

Proof. An explicit calculation yields

$$DD^*(a,\psi) = \big(\triangle_A a - *[*F_A, a] - \text{ad}(\Phi)^2 a, \triangle_A\psi - \text{ad}(\Phi)^2\psi\big) + \Psi\cdot(a,\psi),$$

where $\triangle_A = d_A^* d_A + d_A d_A^*$. The standard Weitzenböck formula for unitary connections implies the first equation. The second identity follows from $D = D^* + 2[\Phi,\,\cdot\,]$ and the identity $[D, \text{ad}(\Phi)](a,\psi) = d_A\Phi\cdot(a,\psi)$. □

Remark. If the metric g on X is not flat, there is an additional term in both formulas involving the Ricci curvature of g acting on 1–forms.

As a final remark in this general setting, observe that if one fixes boundary conditions so that infinitesimal deformations are L^2–integrable, the L^2–product restricted to $\ker D$ defines a Riemannian metric on the moduli space $\mathcal{M}$. If $X = \mathbb{R}^3$ (or $X = \mathbb{R}^2 \times \mathbb{S}^1$) this L^2*–metric* is hyperkähler.

1.4 Hyperkähler quotient construction

In this section we are going to show that the L^2–metric on the moduli space M_k of charge k $SU(2)$ monopoles on $X = \mathbb{R}^3$ is hyperkähler. At least at the formal level, the discussion carries over to $\mathbb{R}^2 \times \mathbb{S}^1$ and moduli spaces of periodic monopoles satisfying appropriate boundary conditions.

Via the Clifford multiplication (1.3.5) (composed with the Hodge–$*$ operator as in (1.3.6) so to define a map $\Omega_{\text{ad}} \to \Omega_{\text{ad}}$) the three parallel 1–forms dx_1, dx_2, dx_3 yield three endomorphisms of TM_k. Indeed, $D \circ \gamma(dx_h) = \gamma(dx_h) \circ D$. Since dx_1, dx_2, dx_3 is an orthonormal basis of 1–forms, the three endomorphisms $\gamma(dx_h)$ are orthogonal almost complex structures with $\gamma(dx_1)\gamma(dx_2) = \gamma(dx_3)$. Formally speaking, the fact that this triple defines a hyperkähler structure on the moduli space M_k follows from an infinite dimensional hyperkähler quotient construction.

Recall (*cf.* for example [81, Chapter 8]) that given a symplectic manifold (M, ω) with a Hamiltonian action of a Lie group G, which we assume compact and connected, there exists an equivariant function $\mu\colon M \to \mathfrak{g}^*$, the *moment map*, such that $d\mu(X) = \omega(v_X, \cdot)$ for all $X \in \mathfrak{g}$. Here $\mathfrak{g}^*$ is the dual of the Lie algebra of G, acted upon by G via the coadjoint representation, and for all $X \in \mathfrak{g}$, v_X is the associated vector field on M. The relevance of moment maps is that they allow to define the notion of a quotient in the symplectic category: Given an element $\zeta \in \mathfrak{g}^*$ fixed by the coadjojnt action, the quotient $\mu^{-1}(\zeta)/G$ is again a "symplectic space", the *symplectic quotient* (or symplectic reduction) of M at ζ. If ζ is a regular point for μ and G acts freely on $\mu^{-1}(\zeta)$ then $\mu^{-1}(\zeta)/G$ is a symplectic manifold. When the quotient is not a smooth manifold, it is still possible to define the structure of a symplectic stratified space, *cf.* Lerman–Sjamaar [96]. A special case of this construction is when (M, ω, J) is a Kähler manifold and G acts preserving the Kähler structure. Then, away from the singular set, the symplectic quotient $\mu^{-1}(\xi)/G$ carries a Kähler structure induced by that of M.

Now let (M, g, J_1, J_2, J_3) be a hyperkähler manifold with associated triple of Kähler forms $(\omega_1, \omega_2, \omega_3)$. Suppose that a connected compact Lie group G acts on M by tri-holomorphic Hamiltonian isometries. Then the three moment maps μ_1, μ_2, μ_3 fit together to form a hyperkähler moment map $\mu\colon M \to \mathfrak{g}^* \otimes \mathbb{R}^3$. For each complex structure $I = J_\alpha$ for some $\alpha \in \mathbb{S}^2 \subset \mathbb{R}^3$ it is convenient to split μ into a real and complex moment map $\mu_{\mathbb{R}} = \mu_I$ and $\mu_{\mathbb{C}} = \mu_J + i\mu_K$, where J is an anticommuting complex structure and $K = IJ$. Hitchin et al. [58] show that for any element $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in \mathfrak{g}^* \otimes \mathbb{R}^3$ fixed by the coadjoint action of G the smooth part of the quotient $M_\zeta = \mu^{-1}(\zeta)/M$ carries an induced hyperkähler structure.

1.4.1 Finite dimensional quotients: the Eguchi–Hanson metric

To illustrate the hyperkähler quotient construction, we consider the Eguchi–Hanson (EH) metric, one of the three rotationally symmetric hyperkähler 4–manifolds of Proposition 1.1.3 and the simplest example of a gravitational instanton of type ALE.

Let $V = \mathbb{C}^2$ be endowed with the standard flat Euclidean Kähler structure and consider $M = V \oplus V^* = T^*V$. M is a hyperkähler vector space with a Kähler form $\omega_{\mathbb{R}}$ induced by that of V and other two given by the real and imaginary part, respectively, of the canonical holomorphic symplectic form $\omega_{\mathbb{C}} = dv \wedge d\alpha$ on the cotangent bundle, $(v, \alpha) \in M$. Let $\mathbb{S}^1$ act on M by $e^{i\theta}\mathrm{id} \oplus e^{-i\theta}\mathrm{id}$. A hyperkähler moment map is $\mu_{\mathbb{R}}(v, \alpha) = |v|^2 - |\alpha|^2$ and $\mu_{\mathbb{C}}(v, \alpha) = \alpha(v)$.

In order to identify the hyperkähler quotient $M_{(1,0)} = \mu^{-1}(1, 0)/\mathbb{S}^1$, extend the action of $\mathbb{S}^1$ to an action of the complexification $\mathbb{C}^*$. Observe that $M_{(1,0)} = \left(\mu_{\mathbb{C}}^{-1}(0) \cap \{v \neq 0\}\right)/\mathbb{C}^*$ because for all (v, α) with $v \neq 0$ there exists a unique $t > 0$ such that $\mu_{\mathbb{R}}(tv, t^{-1}\alpha) = 1$. Then, dualising the standard identification of the tangent space of $\mathbb{P}^1$ at a line $\ell \subset V$ with $\mathrm{Hom}\,(\ell, V/\ell)$, we have $M_{(1,0)} = T^*\mathbb{P}^1$.

Remark 1.4.1 (The Kempf–Ness theorem)**.** The identification of $M_{(1,0)}$ with the quotient $\left(\mu_{\mathbb{C}}^{-1}(0) \cap \{v \neq 0\}\right)/\mathbb{C}^*$ by the complexified group $\mathbb{C}^*$ is not a coincidence, but an application of the Kempf–Ness Theorem [62]. Let W be a Hermitian vector space with a unitary action of a compact connected Lie group G induced by a linear action of the complexification $G^{\mathbb{C}}$. Let X be a $G^{\mathbb{C}}$–invariant affine variety $X \subset W$. The choice of a character $\chi\colon G^{\mathbb{C}} \to \mathbb{C}^*$ yields on one side an element $i\zeta_{\mathbb{R}} = d\chi|_{\mathfrak{g}} \in i\mathfrak{g}^*$, where $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, on the other a GIT notion of stability. Let $\mu_{\mathbb{R}}$ be a moment map for the action of G on X. Then one can consider the Kähler quotient $\mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}})/G$ and at the same time the GIT quotient $X^{ps}/G^{\mathbb{C}}$, where X^{ps} denotes the set of χ–polistable objects in X. The Kempf–Ness theorem states that the two quotients are homeomorphic. Moreover, if the Kähler quotient is a smooth manifold, then the Kähler structure induced on $\mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}})/G$ is compatible with the holomorphic structure on the GIT quotient: The complex structures coincide and the Kähler form represents the first Chern class of the polarisation induced from the construction of the GIT quotient. We refer to [81, Chapter 8] and [101] for the details. In the concrete simple example we considered, $X = \mu_{\mathbb{C}}^{-1}(0)$ and $\chi = \mathrm{id}$.

Returning to EH, since $\mathbb{S}^1$ is abelian we can take the quotient at a different level set of the moment map. For example, $\mu^{-1}(\zeta_{\mathbb{R}}, 0)/\mathbb{S}^1$ for $\zeta_{\mathbb{R}} \neq 0$ is $T^*\mathbb{P}^1$ with $\zeta_{\mathbb{R}}$ determining the size of the zero-section. By solving the ODEs (1.1.8) we can express the EH metric explicitly in Bianchi IX form (1.1.2)

$$g = \frac{ds^2}{1 - \left(\frac{a}{s}\right)^4} + \frac{s^2}{4}\left(\eta_1^2 + \eta_2^2\right) + \frac{s^2}{4}\left[1 - \left(\frac{a}{s}\right)^4\right]\eta_3^2 \tag{1.4.1}$$

where $a^2 = 4|\zeta_{\mathbb{R}}|$. The generic 3–dimensional orbit has to be $SO(3)$ for the metric to be smooth at $s = a$ and therefore g is asymptotic to $\mathbb{C}^2/\mathbb{Z}_2$ at infinity.

We can also move the value $\zeta_{\mathbb{C}}$ of the complex moment map. If $\zeta_{\mathbb{C}} \neq 0$ the quotient is biholomorphic to a smooth quadric in $\mathbb{C}^3$. When $\zeta_{\mathbb{C}} = 0$ we have a map from $M_{(\zeta_{\mathbb{R}},0)}$ to $\mathbb{C}^2/\mathbb{Z}_2$ which is an isomorphism away from the origin, replaced in $M_{(\zeta_{\mathbb{R}},0)}$ with a sphere of self-intersection -2 (the zero-section of $T^*\mathbb{P}^1$). Thus M_ζ, $\zeta \neq 0$, is a resolution of singularities of $\mathbb{C}^2/\mathbb{Z}_2$. The chosen level set $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ of the moment map determines the values of $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$ on the class of the exceptional divisor.

The EH metric is the easiest non-flat example of an ALE (asymptotically locally Euclidean) space. In Definition 1.1.2 we introduced the notion of a gravitational instanton. One can distinguish different classes of gravitational instantons prescribing the asymptotics of the metric at infinity. Recall that by the Cheeger–Gromoll splitting theorem [28], non-flat gravitational instantons have a unique end.

One of the first quantities to consider when prescribing the asymptotic behaviour of the metric is the volume growth of geodesic balls. By the Bishop–Gromov volume comparison (*cf.* for example [75]), the volume $\mathrm{Vol}(B_r)$ of a geodesic ball of radius r in a complete Riemannian manifold (M, g) with Ric ≥ 0 is bounded by the volume of an Euclidean ball of same radius and dimension. The gravitational instantons of type ALE are those with maximal volume growth.

Definition 1.4.2. Let (M, g, J_1, J_2, J_3) a gravitational instanton. We say that M is of type ALE if there exists a compact set $K \subset M$ and a finite cover $\psi\colon \mathbb{R}^4 \setminus B_R \to M \setminus K$ for some $R > 0$ such that $|\nabla^k(g_{\text{flat}} - \psi^* g)| = O(r^{-4-k})$, $k \geq 0$. Here the norm, ∇ and r are all taken with respect to the flat metric g_{flat} on $\mathbb{R}^4$.

The fact that a gravitational instanton with volume growth r^4 satisfies this asymptotic behaviour follows by a theorem of Bando–Kasue–Nakajima [11]. ALE gravitational instantons have all been constructed as finite-dimensional hyperkähler quotients and completely classified by Kronheimer.

Theorem 1.4.3 (Kronheimer [64, 65])**.** *Let Γ be a finite group of $SU(2)$ and let $X = X_\Gamma$ be a minimal resolution of $\mathbb{C}^2/\Gamma$. Denote by U the subset of $H^2(X;\mathbb{R})\otimes\mathbb{R}^3$ of those triples $(\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha_1(\Sigma)^2 + \alpha_2(\Sigma)^2 + \alpha_3(\Sigma)^2 \neq 0$ for all $\Sigma \in H_2(X;\mathbb{Z})$ with $\Sigma^2 = -2$. Then:*

(i) For all $\alpha \in U$ there exists an ALE hyperkähler structure on X such that the triple of Kähler classes $([\omega_1], [\omega_2], [\omega_3]) \in H^2(X;\mathbb{R}) \otimes \mathbb{R}^3$ coincides with α.

(ii) Every ALE space is diffeomorphic to X_Γ for some $\Gamma < SU(2)$ and the triple of Kähler classes lies in U.

(iii) Given a diffeomorphism f of X and $\alpha \in U$, there exists a tri-holomorphic isometry between the ALE hyperkähler structures induced by α and $f^*\alpha$.

1.4.2 Infinite dimensional quotients: the Atiyah–Hitchin manifold

Return to the moduli spaces M_k, N_k of framed and unframed charge k $SU(2)$ monopoles on $\mathbb{R}^3$. We are going to interpret the Bogomolny equation as a moment map. Consider the following situation: A compact Lie group G acts on the quaternionic vector space $M = \mathfrak{g} \otimes \mathbb{H}$ by the adjoint representation. Here $\mathfrak{g}$ is endowed with the scalar product defining a bi-invariant metric on G. Write elements of M as $A = A_0 + A_1 i + A_2 j + A_3 k$. Then, identifying $\mathfrak{g}$ with its dual using the metric, a moment map is given by $\mu_1(A) = [A_0, A_1] + [A_2, A_3]$ and μ_2, μ_3 defined by cyclic permutation of $1, 2, 3$. Formally, setting $A_0 = -\Phi$ and replacing A_i with the covariant derivative $\nabla^A_{e_i}$ of a connection A in the direction $e_i \in \mathbb{R}^3$, we regard the Bogomolny equation (1.2.1) as the vanishing of the moment map $\Psi\colon \mathcal{C} \to \Omega^1(\mathbb{R}^3; \mathfrak{su}_2)$ for the action of $\mathcal{G}_0$ on $\mathcal{C}$ and M_k as the corresponding hyperkähler quotient.

In order to prove that this formal picture yields an hyperkähler sructure on M_k, one considers the constant-coefficient Kähler forms

$$\omega_h(\xi, \xi') = \int_{\mathbb{R}^3} \langle \gamma(dx_h)\xi, \xi' \rangle$$

on $\Psi^{-1}(0)$ corresponding to the almost complex structures $\gamma(dx_h)$, $h = 1, 2, 3$. It is enough to show that these forms descend to the quotient M_k. With the analytical set-up of [98], this follows from the fact that $\ker d_1^*$ is a slice for the action of $\mathcal{G}_0$ on $\mathcal{C}$ and that, assuming the integration by parts poses no problem,

$$\omega_h(\xi, d_1 u) = \int_{\mathbb{R}^3} \langle \gamma(dx_h)\xi, d_1 u \rangle = \int_{\mathbb{R}^3} \langle d_1^* \circ \gamma(dx_h)\, \xi, u \rangle = 0$$

whenever ξ is a tangent vector to $\Psi^{-1}(0)$, *i.e.* $d_2\xi = 0$. This follows from the identities $d_1 u = D^*(0, u)$, $(0, u) \in \Omega_{\text{ad}}$, and $D \circ \gamma(dx_h) = \gamma(dx_h) \circ D$.

Now the vector field $(d_A\Phi, 0)$—which lies in $\ker D$ by (1.2.2)—generates an $\mathbb{S}^1$–action on M_k and $M_k \to N_k$ is a principal circle bundle. Translations on the base induce an action of $\mathbb{R}^3$ on M_k generated by the vector fields $v \lrcorner (F_A, d_A\Phi) = -\gamma(v^\flat)\, (d_A\Phi, 0)$ for all $v \in \mathbb{R}^3$. Let X be any vector field induced by a translation; for all but two conjugate complex structures J, the vector field JX too is induced by a translation. Then both X and JX preserve the hyperkähler structure of M_k; since J is parallel, ∇X is both Hermitian and skew-Hermitian with respect to J and is forced to vanish. On the other hand, Atiyah and Hitchin [7, Chapter 2] show that $\pi_1(N_k) \simeq \mathbb{Z}_k$. It follows that a k–fold covering $\widetilde{M}_k$ of M_k is an isometric product $\mathbb{S}^1 \times \mathbb{R}^3 \times \widetilde{M}^0_k$, where $\mathbb{R}^3 \times \widetilde{M}^0_k$ is the universal cover of N_k. Since the tangent space to the flat factor is the quaternionic sub-bundle spanned by $(d_A\Phi, 0)$, $\widetilde{M}^0_k$ is a simply connected hyperkähler manifold of dimension $4k-4$, the k–fold cover of the moduli space M^0_k of *centred* monopoles. Finally, the action of rotations on $\mathbb{R}^3$ induces an isometric action of $SO(3)$ on $\widetilde{M}^0_k$ such that the action on the sphere of complex structures is the standard action of $SO(3)$ on $\mathbb{S}^2$ by rotations.

The Atiyah–Hitchin (AH) manifold is $\widetilde{M}^0_2$, the simply connected double cover of the moduli space of centred charge 2 $SU(2)$ monopoles on $\mathbb{R}^3$. It carries a rotationally symmetric complete hyperkähler metric. The completeness of the metric is deduced from the analytic results of Taubes [98] and Uhlenbeck Compactness [105] (*cf.* also [108, Theorem A'] for the statement in the non-compact case).

1.5 Asymptotic geometry of the Atiyah–Hitchin metric

In the previous sections we have reviewed the construction of the AH manifold as the simply connected double cover of the moduli space of centred charge 2 monopoles on $\mathbb{R}^3$. In Section 1.1 we alluded to the geometry of AH in terms of the $SO(3)$–orbits structure: There is a unique 2–dimensional orbit, a minimal sphere, onto which AH retracts; by studying the behaviour of the solution to the system of ODEs (1.1.8), Atiyah and Hitchin show that for the metric to remain smooth as the 3–dimensional orbits collapse to the 2–sphere, the former have to be diffeomorphic to $SO(3)/\mathbb{Z}_2$.

In this section we describe the asymptotic geometry of the AH metric. It is necessary to introduce the Gibbons–Hawking ansatz [43], a set of adapted coordinates to express a hyperkähler metric with a tri-holomorphic Killing vector field.

Gibbons–Hawking ansatz. Let $U \subset \mathbb{R}^3$ be an open set endowed with the Euclidean metric and coordinates (y_1, y_2, y_3). Let $\pi\colon M \to U$ be a principal $U(1)$–bundle, with fibre-wise $\mathbb{S}^1$–action generated by the vector field X. We normalise X so that its period is 2π. Let θ be a connection 1–form on M, *i.e.* an $i\mathbb{R}$–valued

$\mathbb{S}^1$–invariant 1–form on X such that $\theta(X) = i$. Then the curvature $d\theta = \pi^*\alpha$ for a 2–form α on U such that $\frac{1}{2\pi i}\alpha$ represents the first Chern class of the bundle $\pi\colon M \to U$. Suppose that there exists a positive harmonic function h on U such that $*dh = -i\alpha$. Set $\theta_0 = -i\theta$ and define 2–forms on M:

$$\begin{aligned}\omega_1 &= dy_1 \wedge \theta_0 + h\, dy_2 \wedge dy_3\\ \omega_2 &= dy_2 \wedge \theta_0 + h\, dy_3 \wedge dy_1\\ \omega_3 &= dy_3 \wedge \theta_0 + h\, dy_1 \wedge dh_2\end{aligned}$$

It is easy to check that $\omega_1^2 = \omega_2^2 = \omega_3^2 \neq 0$, $\omega_i \wedge \omega_j = 0$ if $i \neq j$ and $d\omega_j = 0$. The triple then defines a hyperkähler structure on M. Indeed, the real and complex 2–forms ω_1 and $\Omega_1 = \omega_2 + i\omega_3 = (dy_2 + idy_3) \wedge (\theta_0 - ih\, dy_1)$ satisfy

(i) ω_1 is non-degenerate;

(ii) Ω_1 is locally decomposable and non-vanishing;

(iii) $\omega_1 \wedge \Omega_1 = 0$;

(iv) $\frac{1}{4}\Omega_1 \wedge \overline{\Omega}_1 = \omega_1^2$;

(v) $d\omega_1 = 0 = d\Omega_1$.

Then, as in [59, §2], Ω_1 defines a complex structure J_1 such that a 1–form a is of type $(1,0)$ if and only if $a \wedge \Omega_1 = 0$ (the complex structure is integrable because Ω_1 is closed) and ω_1 is a Kähler form with respect to J_1. Notice that (y_1, y_2, y_3) is a hyperkähler moment map for the $\mathbb{S}^1$–action. Since $J_1(\partial_{y_2}) = \partial_{y_3}$ and $hJ_1(\partial_{y_1}) = X$, the hyperkähler metric defined by $g(u,v) = \omega_1(u, J_1 v)$ is

$$g = h\left(dy_1^2 + dy_2^2 + dy_3^2\right) + h^{-1}\theta_0^2. \tag{1.5.1}$$

Taub–NUT metric and gravitational instantons of type ALF. As a simple explicit illustration of the Gibbons–Hawking construction we discuss the Taub–NUT metric, which has already appeared together with EH and AH in Proposition 1.1.3.

Consider the Hopf bundle $\pi\colon \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$, where the $\mathbb{S}^1$–action on $\mathbb{R}^4 \simeq \mathbb{C}^2$ is $e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2)$. Thus $U = \mathbb{R}^3 \setminus \{0\}$, the moment map $\pi\colon \mathbb{C}^2 \to \mathbb{R}^3$ is

$$(z_1, z_2) \mapsto \left(|z_1|^2 - |z_2|^2, 2z_1 z_2\right) \in \mathbb{R} \oplus \mathbb{C} \tag{1.5.2}$$

and $\theta_0 = \frac{1}{2}\eta_3$, where η_3 is the left-invariant form on $\mathbb{S}^3 = SU(2)$ defined in (1.1.1). A positive harmonic function on $\mathbb{R}^3 \setminus \{0\}$ with $*dh = d\theta_0$ has to be of the form $h = 2m + \frac{1}{2\rho}$, where ρ is the radius function on $\mathbb{R}^3$ and $m \in \mathbb{R}_{\geq 0}$. By a change of coordinates one checks that the metric g_m obtained as in (1.5.1) extends smoothly to $\mathbb{R}^4$. If $m = 0$ g_m is the Euclidean metric, while $m > 0$ yields the Taub–NUT metric of Proposition 1.1.3.

The parameter m can be changed by rescaling: If $\Phi_\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2)$ for some $\lambda > 0$, then $\Phi_\lambda^* g_m = \lambda^2 g_{\lambda^2 m}$. As $m \to 0$ g_m converges to the flat metric on $\mathbb{C}^2$ (in other words, the tangent cone to Taub–NUT at the origin is the flat Euclidean space). However, at infinity the asymptotic geometry of Taub–NUT differs radically from the Euclidean geometry. Notice that, because of the rotational symmetry, the 3–spheres $\{|z|^2 = \rho = \text{const}\}$, where $|z|$ is the Euclidean distance from the origin in $\mathbb{C}^2$, are geodesic spheres with respect to the Taub–NUT metric. Moreover, the radius of the sphere $\{\rho = \text{const}\}$ with respect to g_m is $2\rho \int_0^1 \sqrt{2m + \frac{1}{2t\rho}}\, t\, dt \sim \sqrt{2m}\rho$ as $\rho \to \infty$. Thus the volume of large geodesic balls B_r in Taub–NUT is proportional to r^3. Asymptotically $(\mathbb{R}^4, g_m)$ is a circle fibration over flat 3–space with fibres of finite length approaching $\frac{2\pi}{\sqrt{2m}}$.

The asymptotic geometry of the Taub–NUT metric leads us to define another class of gravitational instantons, the ALF (asymptotically locally flat) spaces: A gravitational instanton (M^4, g) is said to be of ALF type if there exists a compact set $K \subset M$ such that $M \setminus K$ is the total space of a circle fibration over the complement of a ball in $\mathbb{R}^3$ or $\mathbb{R}^3/\{\pm \text{id}\}$ and the metric can be written as

$$g = \pi^* g_{\mathbb{R}^3} + \theta_0^2 + O(\rho^{-\tau})$$

for some $\tau > 0$. Here θ_0 is a connection 1–form on the $\mathbb{S}^1$–bundle $M \setminus K$. The necessity to allow for a circle bundle over $\mathbb{R}^3/\mathbb{Z}_2$ will be clear in a moment, when we will discuss the asymptotics of the AH metric. In particular, an ALF metric has cubic volume growth.

Beyond the flat $\mathbb{S}^1 \times \mathbb{R}^3$ and the Taub–NUT metric, an infinite family of ALF metrics, known as multi-Taub–NUT metrics, can be obtained from the Gibbons–Hawking ansatz (1.5.1) with the harmonic function $h = 2m + \sum_{j=1}^k \frac{1}{2|x - x_k|}$ for k distinct points $x_1, \ldots, x_k$ in $\mathbb{R}^3$ and $m > 0$. If $m = 0$ we still obtain complete hyperkähler metrics: They are of type ALE; the case $k = 1$ is $\mathbb{C}^2$ with its flat metric,

$k = 2$ is EH and the others are known as multi-Eguchi–Hanson metrics.

We say that an ALF metric is of *cyclic type* if the asymptotic geometry is that of a circle fibration over $\mathbb{R}^3$; of *dihedral type* if the base of the fibration is $\mathbb{R}^3/\mathbb{Z}_2$. The multi-Taub–NUT metrics are of cyclic type. Different constructions of metrics of dihedral type are available in the literature; conjecturally they all agree. Cherkis and Kapustin [32] suggested that the L^2–metric on moduli spaces of centred charge $2\ U(2)$ monopoles with singularities on $\mathbb{R}^3$ is an ALF metric of dihedral type. We will come back to this point at the end of this chapter, but in a moment we will see that AH belongs to this class (it is referred to as an ALF metric of dihedral type D_1). The metrics arising in [32] have been studied via twistor methods by Cherkis and Hitchin [31]. More recently, ALF metrics of dihedral type have been constructed by Auvray [8, 9] via PDE methods. Conjecturally, the list of known examples covers all possibilities for ALF gravitational instantons.

The Taub–NUT metric can also be constructed as the hyperkähler quotient of $\mathbb{C}^2\times\mathbb{C}^*\times\mathbb{C}$ with respect to the $\mathbb{S}^1$–action $e^{i\theta}\cdot(z_1,z_2,w_1,w_2) = (e^{i\theta}z_1, e^{-i\theta}z_2, e^{i\theta}w_1, w_2)$, *cf.* [47]. Here $\mathbb{C}^2\times\mathbb{C}^*\times\mathbb{C}$ is endowed with the complete flat hyperkähler structure:

$$\omega = \omega_1 = \frac{i}{2}\left(dz_1\wedge d\overline{z}_1 + dz_2\wedge d\overline{z}_2 + \frac{dw_1}{w_1}\wedge\frac{d\overline{w}_1}{\overline{w}_1} + dw_2\wedge d\overline{w}_2\right)$$

$$\Omega = \omega_2 + i\omega_3 = dz_1\wedge dz_2 + \frac{dw_1}{w_1}\wedge dw_2$$

From this description one can see that Taub–NUT and Euclidean $\mathbb{R}^4$ are isomorphic as holomorphic symplectic manifolds in every complex structure, a fact that was first observed by LeBrun [69]. By a hyperkähler rotation, which corresponds to an actual rotation of the $\mathbb{R}^3$ base in Gibbons–Hawking coordinates, it is enough to exhibit the isomorphism with respect to the complex structure J_1 induced by ω and Ω above. As usual, we write the moment map $\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})$ where

$$\mu_{\mathbb{R}}(z_1,z_2,w_1,w_2) = |z_1|^2 - |z_2|^2 + \log|w_1| \qquad \mu_{\mathbb{C}}(z_1,z_2,w_1,w_2) = 2z_1z_2 + 2w_2.$$

By the Kempf–Ness theorem of Remark 1.4.1, $\mu^{-1}(0)/\mathbb{S}^1 = \mu_{\mathbb{C}}^{-1}(0)/\mathbb{C}^*$ (this can be checked directly in this concrete case). Consider the map $\psi\colon \mu_{\mathbb{C}}^{-1}(0) \to \mathbb{C}^2$, $\psi(z_1,z_2,w_1,-z_1z_2) = \big(w_1^{-1}z_1, w_1z_2\big)$. Since $\psi^*(d\zeta_1\wedge d\zeta_2) = \Omega|_{\mu_{\mathbb{C}}^{-1}(0)}$, ψ induces the required isomorphism of holomorphic symplectic manifolds between the quotient $\mu_{\mathbb{C}}^{-1}(0)/\mathbb{C}^*$ and $\mathbb{C}^2$.

Aymptotics of the Atiyah–Hitchin metric. The explicit formula for the AH metric found by solving the system of ODEs (1.1.8) involves elliptic integrals. In [7] an asymptotic formula for the coefficients of the metric in Bianchi IX form (1.1.2) is given: Up to exponentially decaying terms, a_1, a_2, a_3 behave for large s as $a_1(s) = a_2(s) = s\sqrt{1-\frac{2}{s}}$ and $a_3(s)^{-1} = -2\sqrt{1-\frac{2}{s}}$. The 3–dimensional $SO(3)$–orbits in AH are diffeomorphic to $SO(3)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$–action is given by (1.1.3). Then on the double cover $\mathbb{R}_{>0} \times SO(3)$, the left-invariant form $i\eta_3$ is a connection form on the circle bundle $SO(3) \to \mathbb{S}^2$. With the substitution $s = 2\rho$, the asymptotic expression for the AH metric can be written in Gibbons–Hawking coordinates

$$\frac{1}{4}g_{\mathrm{AH}} = \left(1-\frac{1}{\rho}\right)\left(d\rho^2 + \rho^2 g_{\mathbb{S}^2}\right) + \left(1-\frac{1}{\rho}\right)^{-1}\theta_0^2 \tag{1.5.3}$$

where $\theta_0 = \eta_3$ and we have to quotient by (1.1.3). Hence AH is an ALF metric of dihedral type as claimed: Up to a double cover, the metric looks like a Taub–NUT metric with negative *mass* -2. Here we call mass the coefficient of $\frac{1}{2\rho}$ in the expansion of the harmonic function h; with our normalisations, it coincides with the opposite of the first Chern class of the circle bundle at infinity. Notice that the tri-holomorphic $\mathbb{S}^1$–action does not descend to the quotient by the involution (1.1.3) of $SO(3)$: AH does not admit any tri-holomorphic vector field.

1.6 Gravitational instantons from rational elliptic surfaces

Starting with rotationally symmetric complete hyperkähler 4–manifolds we introduced some examples of gravitational instantons and distinguished them depending on their asymptotics: Of the three metrics of Proposition 1.1.3 EH is a gravitational instanton of type ALE, Taub–NUT and AH are ALF metrics.

In analogy with the Kummer construction of the Ricci-flat metric on a K3 surface (*cf.* [39]), Hitchin [56] suggested to construct complete hyperkähler metrics of volume growth k by resolving the orbifold $\left(\mathbb{R}^k \times \mathbb{T}^{4-k}\right)/\mathbb{Z}_2$, $k = 1, 2, 3$. For example, the resolution of singularities of $\left(\mathbb{R}^3 \times \mathbb{S}^1\right)/\mathbb{Z}_2$ carries an ALF metric of dihedral type constructed by Hitchin [56] via twistor methods and by Biquard and Minerbe [19] via PDE techniques. In fact, a folklore conjecture states that "most" complete hyperkähler 4–manifolds, probably under the finite energy assumption

$\int |\mathcal{R}_g|^2 < \infty$, are locally asymptotic to $\mathbb{R}^k \times \mathbb{T}^{4-k}$. A gravitational instanton is then said to be of type ALE, ALF, ALG or ALH depending on whether $k = 4, 3, 2$ or 1.

The most extensive list of gravitational instantons with slower than cubic volume growth has been obtained by Hein [53] and the aim of this section is to review some aspects of his construction. We will focus on the gravitational instantons of [53] with quadratic volume growth. As we will see, they fall into two groups:

(i) Seven families of genuine ALG metrics: Asymptotically they converge to a flat 2–torus bundle over a flat 2–dimensional cone. The cone however does *not* have to be a quotient of $\mathbb{R}^2$.

(ii) A list of four more degenerate asymptotic geometries; these are still 2–torus fibrations such that the asymptotic metric restricts to a flat metric on the fibre, but the modulus of the torus goes to infinity in the upper half plane.

In our previous discussion, symmetries have played a major role, to the point that explicit formulas of some ALE and ALF metrics have been written down. Exploiting symmetries in the case of gravitational instantons of slower than cubic volume growth seems more problematic: Let (M^4, g) be a complete hyperkähler manifold with a nowhere vanishing Killing vector field X. Then X acts on the sphere of compatible complex structures and either this action is trivial (and X is a tri-holomorphic vector field) or X preserves exactly two conjugate complex structures and acts as a rotation in the plane orthogonal to these. In the former case, if M is simply connected a theorem of Bielawski [14, Corollary 2] shows that (M, g) is either flat or isometric to one of the ALE multi-Eguchi–Hanson or ALF multi-Taub–NUT metrics. In the latter case, locally the metric can be expressed in terms of a single potential u on $\Omega \times I \subset \mathbb{R}^2 \times \mathbb{R}$ which satisfies $\triangle_{\mathbb{R}^2} u - \partial^2_{tt}(e^u) = 0$ [70, Proposition 1]. Some of the metrics constructed by Hein admit a Killing vector field of the second type, but the only known global solutions to the PDE for the potential give rise to the flat metric, EH, Taub–NUT and AH (*cf.* [51, Remark 7.5(i)]). As a consequence, the construction of [53] has to rely on more sophisticated techniques.

Complete Ricci flat Kähler metrics on the complement of an anti-canonical divisor. Our starting point is yet another point of view on the AH manifold.

Theorem 1.6.1 (Donaldson [36]). *Given an identification* $\mathbb{R}^3 \simeq \mathbb{R} \times \mathbb{C}$ *there exists a diffeomorphism between the moduli space* M_k *of framed charge* k *monopoles on* $\mathbb{R}^3$ *and* R_k*, the space of rational maps* $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ *of degree* k *such that* $f(\infty) = 0$.

The philosophy behind the theorem is a general strategy to study moduli spaces of solutions to the anti-self-duality equations. Results of this type, which can be thought of as a Kempf–Ness theorem in infinite dimensions, are often referred to as the Hitchin–Kobayashi correspondence. As Donaldson writes in the introduction to [36], the strategy to prove a result such as Theorem 1.6.1 is analogous to the proof of the Kempf–Ness theorem:

(i) Decompose the ASD equations (recall that we can think of them as a moment map) into a real and a complex one; the complex equation is invariant under the larger group of complex gauge transformations. For the Bogomolny equation, once a direction in $\mathbb{R}^3$ is chosen so that we identify $\mathbb{R}^3 \simeq \mathbb{C}_z \times \mathbb{R}_t$ one can write

$$\begin{cases} F_{\mathbb{C}} = \nabla_t \Phi \\ \left[\nabla^{0,1}, \nabla_t - i\Phi\right] = 0 \end{cases} \tag{1.6.1}$$

where ∇ is the covariant derivative induced by A, $\nabla^{0,1}$ its $(0,1)$–part with respect to the complex coordinate z and $F_{\mathbb{C}}$ is the curvature of the connection restricted to the plane $\mathbb{C} \times \{t\}$.

(ii) One then shows (by variational methods or a parabolic flow) that for each solution to the complex equation there exists an element in its complex orbit, unique up to real gauge transformations, such that the real equation is also satisfied. At this stage it is necessary to introduce a notion of stability and relate the existence of a solution to the full ASD equations to the stability of the complex orbit.

(iii) Use algebro-geometric methods to study the moduli space of solutions to the complex equation.

By Theorem 1.6.1, to each monopole of charge 2 we associate a rational map $f(z)$ of degree 2. A normal form for f is given by $f(z) = \frac{a_0 + a_1 z}{b_0 + b_1 z + z^2}$ where numerator and denominator are coprime polynomials, *i.e.* the resultant

$$a_0^2 - a_0 a_1 b_1 + a_1^2 b_0 \neq 0.$$

Then M_2 is diffeomorphic to the complement of $x_0\,(x_0x_1^2 - x_1x_2x_3 + x_2^2x_4) = 0$ in $\mathbb{P}^4$. By taking the divisor at infinity $x_0 = 0$ with multiplicity 2, M_2 is identified with the complement of an anti-canonical divisor in $\mathbb{P}^4$. Moreover, under the diffeomorphism of Theorem 1.6.1 the action of $\mathbb{S}^1 \times \mathbb{R}^3 \simeq \mathbb{C}^* \times \mathbb{C}$ on $M_2 \simeq R_2$ is $(\lambda, z_0) \cdot f(z) = \lambda f(z - z_0)$ [7, Chapter 2]. Thus the moduli space of centred monopoles M_2^0 is also the complement of an anti-canonical divisor, namely the complement of $x^2v^2 + y^2uv = 0$ in $\mathbb{P}^1_{[x:y]} \times \mathbb{P}^1_{[u:v]}$. Finally, by scaling in the line $[x : y]$, the AH manifold $\widetilde{M_2^0}$ can be described as the affine variety in $\mathbb{C}^3$ of equation $x^2 + y^2u = 1$.

Let X be a compact Kähler manifold and $D \subset X$ a divisor such that there exists a holomorphic volume form Ω with poles of order $\beta \in \mathbb{N}$ along D. If $D \in |-K_X|$ then by definition such an Ω exists with $\beta = 1$. Assuming the existence of an appropriate initial complete background metric ω_0 on $M = X \setminus D$, Tian and Yau's method [102, 103] can be applied to construct a Ricci-flat Kähler metric on M by solving the complex Monge–Ampère equation $\omega^m = \alpha\,\Omega \wedge \overline{\Omega}$, $\alpha > 0$, on the complement of D. For the method to apply it is necessary to assume that ω_0 is already an approximate solution at infinity, with a certain rate. Notice that the choice of the background ω_0 is not obvious nor unique: We saw that $\mathbb{C}^2 = \mathbb{P}^2 \setminus \mathbb{P}^1$ supports two different complete hyperkähler metrics with the same holomorphic volume form. Only under special assumptions on (X, D) Tian and Yau are able to make an ansatz to construct ω_0. For example, in [103] one assumes that $\beta > 1$ and D is smooth and admits a positive Kähler–Einstein metric; then ω is an asymptotically conical Ricci-flat Kähler metric.

Conversely, Yau [110] asks which complete Ricci-flat Kähler manifolds admit projective compactifications. All currently known examples with finite topology are biholomorphic to quasi-projective varieties. Note, however, that Anderson, Kronheimer and LeBrun [4] constructed complete hyperkähler 4–manifolds with infinite topology: Given a sequence of points $p_j = (x_j, 0, 0) \in \mathbb{R}^3$ such that $\sum_{j=1}^{\infty} \frac{1}{|x_j|} < \infty$, the Gibbons-Hawking ansatz with the harmonic function $h = \sum_{j=1}^{\infty} \frac{1}{2|x-p_j|}$ yields a complete hyperkähler 4–manifold (M, g) with an isometric tri-holomorphic $\mathbb{S}^1$–action. M has infinitely generated second homology: Each segment in $\mathbb{R}^3$ joining p_j to p_{j+1}, which are fixed points of the $\mathbb{S}^1$–action, lifts in M to a sphere (a holomorphic sphere in complex structure J_1).

In [53] Hein constructs gravitational instantons applying Tian–Yau's method to rational elliptic surfaces.

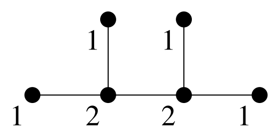
Rational elliptic surfaces. We follow Miranda's lecture notes [80] and Hein's Thesis [51]. A complex surface X is said to be elliptic if it admits a map $\pi\colon\ X \to C$ onto a smooth complex curve such that the generic fibre is a smooth curve of genus 1. If X has a holomorphic section σ the generic fibre becomes a smooth elliptic curve. We say that X is a minimal elliptic surface if there are no (-1)–curves contained in the fibres. A *rational elliptic surface* is a complex surface which is birationally equivalent to $\mathbb{P}^2$ and which admits a minimal elliptic fibration with a section. All rational elliptic surfaces can be constructed in the following way. Let C_1 be a smooth plane cubic and C_2 a second distinct cubic. Then the pencil $\{\lambda_1 C_1 + \lambda_2 C_2 \,|\, [\lambda_1 : \lambda_2] \in \mathbb{P}^1\}$ has $C_1 \cdot C_2 = 9$ base points (counted with multiplicities). After blowing them up we obtain a rational elliptic surface $\pi\colon\ X \to \mathbb{P}^1$. X is a minimal elliptic surface because we blew-up just enough to resolve all the tangencies of the pencil. X has at least a section: The (-1)–curve obtained in the last blow-up. Finally, the class of a fibre is anti-canonical.

If X is a rational elliptic surface not all fibres can be smooth elliptic curves because $\chi\left(\mathbb{P}^2\#9\overline{\mathbb{P}}^2\right) = 12$. The possible singular fibres of rational elliptic surfaces have been classified by Kodaira. They are distinguished by the monodromy: Work locally with a minimal elliptic surface $\pi\colon\ X \to \triangle$ over a disc with a section σ and assume that all fibres except possibly the one over the origin are smooth elliptic curves. Using σ, one can describe the restriction $X|_{\triangle^*}$ of X to the punctured disc as $\pi\colon\ (\triangle^* \times \mathbb{C})/\Lambda \to \triangle^*$, for a family of lattices $\Lambda \subset \mathbb{C}$ defined by (possibly multi-valued) holomorphic functions τ_1, τ_2 on $\triangle^*$. The monodromy is the representation of the fundamental group of $\triangle^*$ on the mapping class group of the smooth fibre. We can think of it as the conjugacy class of the matrix $A \in SL(2,\mathbb{Z})$ generating the action of $\pi_1(\triangle^*)$ on the oriented pair (τ_1, τ_2). We won't reproduce Kodaira's list, for which we refer to [80, Table I.4.1 and I.4.2], but limit ourselves to discuss two series of examples which cover all possible asymptotics of Hein's metrics with quadratic volume growth.

Example (Isotrivial rational elliptic surfaces)**.** Let E be a smooth elliptic curve admitting a $\mathbb{Z}_r$–subgroup of automorphisms for $r = 2, 3, 4$ or 6. Thus E is any elliptic curve if $r = 2$; a Weierstrass equation for E is $y^2 = x^3 + x$ if $r = 4$, with $\mathbb{Z}_4$–action generated by $(x, y) \mapsto (-x, iy)$; if $r = 3$ or 6, $E\colon\ y^2 = x^3 + 1$ and the

$\mathbb{Z}_3$ or $\mathbb{Z}_6$–action is generated by $(x, y) \mapsto (e^{2\pi i/3}x, y)$ or $(x, y) \mapsto (e^{2\pi i/3}x, -y)$, respectively. Now consider the orbifold $(\mathbb{P}^1 \times E)/\mathbb{Z}_r$, where the cyclic group $\mathbb{Z}_r$ acts diagonally on $\mathbb{P}^1$ and E. Resolve the singularities and blow down all (-1)–curves in the fibres to obtain a rational elliptic surface with only two singular fibres over 0 and ∞ and such that all smooth fibres are isomorphic: We say that X is an isotrivial fibration. Corresponding to $r = 2, 3, 4, 6$ this construction yields four pairs of singular fibres—(I_0^*, I_0^*), (II, II^*), (III, III^*) and (IV, IV^*) in Kodaira's notation. Unless $r = 2$, the two fibres in each pair are different because the $\mathbb{Z}_r$–action on $\mathbb{P}^1$ has different weights at 0 and ∞. By removing the fibre of non-$*$-type in each pair, one obtains a crepant resolution of $T^*E/\mathbb{Z}_r$.

Example (Singular fibres of type I_b^*)**.** In terms of pencils of cubics a singular fibre of type I_0^* can be realised by taking a smooth cubic C_1 and $C_2 = 2L + M$, for two distinct lines L, M in $\mathbb{P}^2$ such that C_1 intersects L transversally in three distinct points p_1, p_2, p_3 and M in a further three points q_1, q_2, q_3 (possibly infinitely close). Then the fibre of X corresponding to C_2 is a union of five rational curves intersecting with dual graph $\widetilde{D}_4$: One central sphere meeting the others at four distinct points. By making the intersection points p_i's and q_j's coalesce onto each other and onto the intersection point of L and M we obtain fibres of Kodaira type I_b^*, $0 \leq b \leq 4$: They are unions of rational curves intersecting with dual graph the extended Dynkin diagram $\widetilde{D}_{4+b}$. For example, if p_2 and p_3 coincide—so that now C_2 intersects L transversally at p_1, but tangentially at p_2—we obtain a fibre of type I_1^* corresponding to the Dynkin diagram $\widetilde{D}_5$:



The integer attached to each vertex is the multiplicity of the corresponding rational curve in the fibre (thought of as a divisor).

Semi-flat metrics in a neighbourhood of a singular fibre. Let $\pi\colon X \to \mathbb{P}^1$ be a rational elliptic surface and $D = \pi^{-1}(\infty)$. We want to explain how the Kodaira type of D determines the asymptotics of the Ricci-flat metric constructed by Hein on $X \setminus D$. The models for the metric at infinity are provided by certain semi-flat metrics, *i.e.* such that they restrict to a flat metric on each fibre. We recall the formulas of Hein [53, §3] (*cf.* also Gross–Wilson [50, §2]) for these semi-flat metrics and then discuss what they look like in the two lists of examples we considered.

Let $\pi\colon X \to \mathbb{P}^1$ be a rational elliptic surface and restrict the fibration to a neighbourhood $\triangle$ of ∞. Fix an affine coordinate z on $\triangle$ centred at $\infty \in \mathbb{P}^1$. Then $\pi\colon X \to \triangle$ is a minimal elliptic fibration with a section such that all fibres except possibly $D = \pi^{-1}(0)$ are smooth. Kodaira constructs a normal form for such an elliptic surface for each type of exceptional fibre D. We assume that $\pi\colon X \to \triangle$ is in such a normal form and identify $X|_{\triangle^*}$ with $(\triangle^* \times \mathbb{C}_w)/(\tau_1\mathbb{Z} + \tau_2\mathbb{Z})$ as before. Without loss of generality we assume that $\mathrm{Im}(\overline{\tau_1}\tau_2) > 0$. Recall that we have a holomorphic symplectic form Ω on X with simple poles along D. In the coordinates z, w we write $\Omega = f(z) dz \wedge dw$ for a holomorphic function f on $\triangle^*$. Given $\varepsilon > 0$ construct a semi-flat metric $\omega = \omega_{sf,\varepsilon}$ using the following ingredients:

1. For each $z \in \triangle^*$ define a flat Kähler metric $\omega_{z,\varepsilon}$ on $\pi^{-1}(z)$ by choosing a dual basis $\xi_1(z), \xi_2(z)$ to $\tau_1(z), \tau_2(z)$ and setting $\omega_{z,\varepsilon} = \varepsilon\, \xi_1(z) \wedge \xi_2(z)$. Changing basis to $dw, d\overline{w}$ yields $\omega_{z,\varepsilon} = \frac{i}{2} W dw \wedge d\overline{w}$, with $W = \frac{\varepsilon}{\mathrm{Im}(\overline{\tau_1}\tau_2)}$.

2. Define $g_{\triangle^*,\varepsilon}$ as the unique Riemannian metric on $\triangle^*$ such that the pairing $T^{1,0}\triangle^* \times (\triangle^* \times \mathbb{C}) \to \mathbb{C}$ induced by Ω is isometric with respect to the Hermitian metrics induced by $\omega_{z,\varepsilon}$ and $g_{\triangle^*,\varepsilon}$. Thus $g_{\triangle^*,\varepsilon} = W^{-1}|f(z)|^2\, |dz|^2$.

3. The family of lattices $\tau_1\mathbb{Z} + \tau_2\mathbb{Z}$ defines a flat connection on the trivial bundle $\triangle^* \times \mathbb{C}$ by declaring τ_1 and τ_2 flat sections. The associated connection 1–form is
$$\Gamma\, dz = \frac{1}{\mathrm{Im}(\overline{\tau_1}\tau_2)}\big(\mathrm{Im}(\overline{\tau_1}w) d\tau_2 - \mathrm{Im}(\overline{\tau_2}w) d\tau_1\big).$$

The semi-flat metric is then

$$\omega = \omega_{sf,\varepsilon} = \frac{i}{2} W^{-1}|f(z)|^2\, dz \wedge d\overline{z} + \frac{i}{2} W(dw - \Gamma dz) \wedge (d\overline{w} - \overline{\Gamma} d\overline{z}) \qquad (1.6.2)$$

It is a hyperkähler metric such that $\omega^2 = \frac{1}{4}\Omega \wedge \overline{\Omega}$ and the volume of $\pi^{-1}(z)$ is ε.

Example (Isotrivial fibrations)**.** In the isotrivial case of Example 1.6 if we remove the fibre of non-$*$-type, the semi-flat metric coincides with the flat metric on $T^*E/\mathbb{Z}_r$. In fact, in this case Hein's Ricci flat metric on $X \setminus D$ can also be obtained from the Kummer type construction of Biquard–Minerbe [19], gluing rescaled ALE spaces to resolve the singularities of the flat orbifold. When we remove the fibre of

$*$–type in each pair, the Ricci flat metric is asymptotic to the twisted product of E endowed with the flat metric of volume ε and of a flat 2–dimensional cone which is not a quotient of $\mathbb{C}$, *cf.* [53, Theorem 1.5 (ii)] .

Example (Fibre of type I_b^*)**.** If the fibre removed is of type I_b^*, $0 \leq b \leq 4$, it is convenient to work on a double cover $\pi\colon\ X \to \triangle$ such that $X|_{\triangle^*}$ is isomorphic to $(\triangle^* \times \mathbb{C})/(\tau_1\mathbb{Z}+\tau_2\mathbb{Z})$ with $\tau_1 = 1$, $\tau_2 = \frac{2b}{2\pi i}\log z$. Up to first order, the holomorphic volume form is $\Omega = \frac{dz}{z^2} \wedge dw$. Kodaira's normal form is obtained by taking the quotient by the involution $(z, w) \mapsto (-z, -w)$ and resolving the singularities.

As for the asymptotic formula (1.5.3) for the AH metric, when pulled back to the double cover, the semi-flat metric (1.6.2) admits a tri-holomorphic $\mathbb{S}^1$–action and, following [50], we can rewrite it in Gibbons–Hawking coordinates. First notice that the imaginary part of $W(dw - \Gamma\, dz)$ is closed and therefore there exists a function $t\colon\ \triangle^* \times \mathbb{C} \to \mathbb{R}$, unique up to the addition of a constant, such that $-W^{-1}dt = \mathrm{Im}(dw - \Gamma\, dz)$. Then $\pi\colon\ (\triangle^* \times \mathbb{C})/\tau_1\mathbb{Z} \to \triangle^* \times \mathbb{R}_t$ is a principal $\mathbb{S}^1$–bundle. Explicitly, assuming $|z| < 1$, $t = \frac{2\pi\varepsilon \mathrm{Im}(w)}{2b\log|z|}$. Taking the quotient by $\tau_2\mathbb{Z}$ we obtain a principal $\mathbb{S}^1$–bundle $(\triangle^* \times \mathbb{C})/(\tau_1\mathbb{Z} + \tau_2\mathbb{Z}) \to \triangle^* \times \mathbb{R}/\varepsilon\mathbb{Z}$. Its Euler class evaluated on $|z| = $ const is $\pm 2b$, depending on the orientation. Now set $h = W^{-1}$, $dw - \Gamma\, dz = \theta_0 - ih\, dt$ and use polar coordinates $re^{i\theta} = \frac{1}{z}$. The semi-flat metric (1.6.2) is now written in Gibbons–Hawking coordinates (1.5.1)

$$g_{sf,\varepsilon} = \frac{2b\log r}{2\pi\varepsilon}\left(dr^2 + r^2 d\theta^2 + dt^2\right) + \frac{2\pi\varepsilon}{2b\log r}\theta_0^2 \tag{1.6.3}$$

Remark. For the Ricci flat metric on the complement of a fibre of type I_b^* Hein [53, Theorem 1.5 (iii)] computes that the volume growth is $\mathrm{Vol}(B_r) \sim r^2$, the injectivity radius decays as $(\log r)^{-1/2}$ and the curvature $|\mathcal{R}_g| \sim r^{-2}(\log r)^{-1}$ as $r = \mathrm{dist}(x_0, \cdot) \to \infty$.

1.7 Gravitational instantons from periodic monopoles

In this final section we close the circle and return to moduli spaces of monopoles and their hyperkähler L^2–metric. Between the late 90's and the early 2000's—when our knowledge of gravitational instantons was essentially limited to Kronheimer's ALE spaces, Gibbons–Hawking's multi-Taub–NUT and the AH manifold—Cherkis and Kapustin suggested that moduli spaces of solutions to dimensional reductions of the Yang–Mills ASD equations are “a natural place to look for gravitational instantons” [30]. More precisely, in [32, 33, 35] 4–dimensional moduli spaces of monopoles (with singularities) on $\mathbb{R}^3$, $\mathbb{R}^2 \times \mathbb{S}^1$ and $\mathbb{R} \times \mathbb{T}^2$ are (expected to be) gravitational instantons with cubic, quadratic and sub-quadratic volume growth, respectively. The fact that these moduli spaces are hyperkähler manifolds when non-empty follows as in Section 1.4.

Cherkis and Kapustin’s first observation is that, passing to structure group $U(2)$ or $SO(3)$ instead of $SU(2)$, one can introduce solutions to the Bogomolny equation with singularities at a finite number of distinct points and still expect smooth moduli spaces with a complete L^2–metric. To give an idea of the nature of the singularities (precise definitions will be given in the next chapter), consider the case of singular monopoles on $\mathbb{R}^3$; in [31,32] moduli spaces of centred charge 2 monopoles with singularities on $\mathbb{R}^3$ give rise to ALF metrics of dihedral type. Fix n distinct points $p_1, \dots, p_n \in \mathbb{R}^3$ and let $\pi\colon\ M \to \mathbb{R}^3$ be the multi-Taub–NUT manifold with $p_1, \dots, p_n$ as the fixed points of the $\mathbb{S}^1$–action. Kronheimer [67] defines monopoles (A, Φ) with singularities at $p_1, \dots, p_n$ in such a way that the ASD connection obtained by lifting (A, Φ) through π extends to a finite energy $\mathbb{S}^1$–invariant instanton on M (*cf.* Section 3.1).

In [29,34] Cherkis and Kapustin introduced *periodic monopoles* (with singularities), *i.e.* solutions to the Bogomolny equation (1.2.1) on $\mathbb{R}^2 \times \mathbb{S}^1$ endowed with its standard flat metric. We will review Cherkis and Kapustin’s definitions in Section 2.2 and define the moduli space $\mathcal{M}_{k,n}$ of charge k $SO(3)$ periodic monopoles with n singularities in Section 3.4. Cherkis and Kapustin studied periodic monopoles from the point of view of the Nahm Transform, a sort of Fourier Transform for solutions to the ASD equations on $\mathbb{R}^4$ invariant under a group of translations. We will follow a direct analytic approach and prove in Section 3.4 that $\mathcal{M}_{k,n}$ is a smooth hyperkähler manifold. We expect its dimension to be $4k - 4$ and the L^2–metric to be complete.

Now restrict to the case of charge $k = 2$. For topological reasons (*cf.* Section 2.2) the number of singularities cannot exceed $2k$. Hence we have five candidate gravitational instantons to investigate. Two of Cherkis and Kapustin’s predictions are of particular relevance at this point:

1. In [29, 34] Cherkis and Kapustin suggest what the analogue of the Hicthin–Kobayashi correspondence of Theorem 1.6.1 should be in the case of periodic

monopoles: Picking the complex structure corresponding to the identification $\mathbb{R}^2 \times \mathbb{S}^1 \simeq \mathbb{C} \times \mathbb{S}^1$, the moduli space $\mathcal{M}_{2,n}$ should be identified with a rational elliptic surface with an I^*_{4-n} fibre removed.

2. In [77] Manton gave a physical computation of the asymptotic metric on the moduli space of charge 2 monopoles on $\mathbb{R}^3$ without singularities, which recovers the explicit asymptotic expression (1.5.3) for the AH metric. In [33] Cherkis and Kapustin extend this computation to the periodic case. The formula they obtain has the shape of Hein's semi-flat metric (1.6.3).

Three obvious questions arise from these predictions: Can we prove that $\mathcal{M}_{2,n}$ is diffeomorphic to the complement of a I^*_{4-n} in a rational elliptic surface? Can we derive a rigorous asymptotic formula for the L^2–metric on $\mathcal{M}_{2,n}$ and show that it is semi-flat at infinity? What is the relation with the Ricci flat metric constructed by Hein?

Chapter 2

Periodic monopoles (with singularities)

In this chapter we begin the study of periodic monopoles (with singularities). These are solutions to the Bogomolny equation (1.2.1) on $\mathbb{R}^2 \times \mathbb{S}^1$ endowed with its standard flat metric. We will allow isolated singularities of the fields at a finite number of points. (Singular) periodic solutions to the Bogomolny equation with appropriate boundary conditions have been introduced by Cherkis and Kapustin in [29, 34] and the aim of this chapter is to discuss their definitions.

The simplest situation to consider is that of abelian solutions to (1.2.1), *i.e.* the case when the structure group is $G = U(1)$. We will see that non-trivial abelian solutions, called *periodic Dirac monopoles*, have an isolated singularity at a point in $\mathbb{R}^2 \times \mathbb{S}^1$ and are characterised by a topological invariant, called the charge.

Next we will introduce non-abelian periodic monopoles. Since we work on a non-compact base manifold, we have to fix boundary conditions. Cherkis and Kapustin define boundary conditions for smooth $SU(2)$ periodic monopoles in [29] and for monopoles with singularities and structure group $U(2)$ or $SO(3)$ in [34]. In both cases, the reducible solutions induced by periodic Dirac monopoles of appropriate charges provide the asymptotic models for the non-abelian monopoles both near the singularities and at infinity.

Notation. In the rest of the thesis X will denote $\mathbb{R}^2 \times \mathbb{S}^1$.

2.1 Periodic Dirac monopole

When the structure group $G = U(1)$, the Bogomolny equation (1.2.1) reduces to a linear equation: The Higgs field Φ is a harmonic function such that $\frac{*d\Phi}{2\pi i}$ represents the first Chern class of a line bundle. Global solutions are necessarily trivial: On $\mathbb{R}^3$ they are given by pairs $(A, \Phi) = (0, v)$ while on X by $(A, \Phi) = (ib\, dt, v)$, where $v \in \mathbb{R}$, $b \in \mathbb{R}/\mathbb{Z}$ and we don't distinguish between a connection and the associated connection 1–form. We call such pairs flat (or vacuum) abelian monopoles. Non-trivial abelian solutions are obtained if one allows an isolated singularity.

Definition 2.1.1. Fix a point $q \in \mathbb{R}^3$ and let H_q denote the radial extension of the inverse of the Hopf line bundle to $\mathbb{R}^3 \setminus \{q\}$. Fix $k \in \mathbb{Z}$ and $v \in \mathbb{R}$. The *Euclidean Dirac monopole* of charge k and mass v with singularity at q is the abelian monopole (A_q, Φ_q) on H_q^k, where

$$-i\Phi_q = v - \frac{k}{2|x-q|},$$

$x \in \mathbb{R}^3$, and A_q is the $SO(3)$–invariant connection on H_q^k with curvature $*d\Phi_q$.

It is easy to check that the above definition is consistent, in the sense that

$$\int_{|x-q|=\text{const.}} *d\Phi_q = 2\pi i k = 2\pi i\, c_1(H_q^k) \cdot [\mathbb{S}_q^2],$$

where the class $[\mathbb{S}_q^2]$ of a 2–sphere enclosing the point q generates $H_2(\mathbb{R}^3 \setminus \{q\}, \mathbb{Z})$.

Periodic Dirac monopoles are defined in a similar way. Fix coordinates $(z,t) \in \mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z} = X$ and a point $q = (z_0, t_0) \in X$. Line bundles of a fixed degree on $X \setminus \{q\}$ differ by tensoring by flat line bundles. We can distinguish connections with the same curvature by comparing their holonomy around loops $\gamma_z := \{z\} \times \mathbb{S}_t^1$ for $z \neq z_0$. Set $\theta_q = \arg(z - z_0)$ and fix an origin in the circle parametrised by θ_q. It will follow from the discussion below (*cf.* Remark 2.1.4) that the holonomy around γ_z of a connection on a degree k line bundle over $X \setminus \{q\}$ is of the form $e^{-ik\theta_q} e^{-2\pi i b}$ for some $b \in \mathbb{R}/\mathbb{Z}$. Denote by L_q the degree 1 line bundle on $X \setminus \{q\}$ with connection A_q whose holonomy around γ_z is $e^{-i\theta_q}$. Any line bundle of degree 1 is of the form $L_q \otimes L_b$ for some flat line bundle L_b. Hence $L_q \otimes L_b$ is equipped with the connection $A_q + ib\, dt$ whose holonomy around γ_z is $e^{-i\theta_q} e^{-2\pi i b}$.

Definition 2.1.2. Fix a point $q \in X$. The *periodic Dirac monopole* of charge $k \in \mathbb{Z}$, with singularity at q and twisted by the flat line bundle $L_{v,b}$ for some $v \in \mathbb{R}$ and

$b \in \mathbb{R}/\mathbb{Z}$ is the pair (A, Φ) on $L_q^k \otimes L_{v,b}$, where

$$-i\Phi = v + kG_q$$

and up to gauge transformations (*i.e.* the addition of an exact 1–form) the connection $A = kA_q + ib\,dt$. Here G_q defined in (2.1.1) below is a Green's function of X with singularity at q.

We devote the rest of this section to the derivation of asymptotic expansions for the Green's function G_q and the connection A_q, both at infinity and close to the singularity. It turns out that the periodic abelian Bogomolny equation has already been considered by Gross and Wilson [50]: Using the Green's function of X in the Gibbons–Hawking ansatz produces an incomplete hyperkähler metric, the Ooguri–Vafa metric [83], defined on a neighbourhood of a singular fibre of Kodaira type I_1 in an elliptic fibration. In [50] the Ooguri–Vafa metric is used as a model in a gluing construction aimed at recovering the Ricci-flat metric on an elliptic K3 surface with fibres of small volume. Some of the asymptotic expansions we will need have already been computed in [50, Lemma 3.1].

The Green's function of $\mathbb{R}^2 \times \mathbb{S}^1$. By taking coordinates centred at $q \in X$, we can assume that the singularity is located at $q = 0$. We use polar coordinates $z = re^{i\theta} \in \mathbb{C}$. Consider the series

$$G(z,t) = -\frac{1}{2}\sum_{m\in\mathbb{Z}}\left[\frac{1}{\sqrt{r^2 + (t-2m\pi)^2}} - a_{|m|}\right], \tag{2.1.1}$$

where

$$a_{|m|} = \frac{1}{2|m|\pi} \text{ if } m \neq 0 \qquad a_0 = 2\frac{\log 4\pi - \gamma}{2\pi}$$

(γ is the Euler–Mascheroni constant, $\gamma = \lim_{n\to\infty}\sum_{k=1}^{n} k^{-1} - \log n$).

Lemma 2.1.3. *The series (2.1.1) converges uniformly on compact sets of $X \setminus \{0\}$ to a Green's function of X with singularity at 0.*

(i) Whenever $z \neq 0$, G can be expressed as

$$G(z,t) = \frac{1}{2\pi}\log r - \frac{1}{2\pi}\sum_{m\in\mathbb{Z}^*} K_0(|m|r)e^{imt},$$

where K_0 is the second modified Bessel function.

(ii) There exists a constant $C_1 > 0$ such that

$$\left|\nabla^k\left(G(z,t)-\frac{1}{2\pi}\log r\right)\right| \leq C_1 e^{-r}$$

for all $r \geq 2$ and $k = 0, 1, 2$.

(iii) There exists a constant $C_2 > 0$ such that

$$\left|\nabla^k\left(G(z,t)-\frac{a_0}{2}+\frac{1}{2\rho}\right)\right| \leq C_2 \rho^{2-k}$$

for all (z,t) with $\rho = \sqrt{r^2+t^2} < \frac{\pi}{2}$ and $k = 0, 1, 2$.

Proof. The convergence and the expansion in (i) are proved in [50, Lemma 3.1 (a),(b)]. When looking for a solution to $\triangle G = -2\pi\delta$, decompose in Fourier modes

$$G(z,t) = \sum_{m\in\mathbb{Z}} G_m(z)\, e^{imt} \qquad \delta(z,t) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \delta_{\mathbb{R}^2}(z)\, e^{imt}.$$

For all $m \geq 0$, G_m is then a solution to $\triangle G_m + m^2 G_m = -\delta_{\mathbb{R}^2}$ in $\mathbb{R}^2$. It follows that $G_0(z) = \frac{1}{2\pi}\log r$ and $G_m(z) = -\frac{1}{2\pi}K_0(|m|r)$ for all $m \neq 0$ and therefore (2.1.1) defines a Green's function of X with singularity at 0.

The estimate in (ii) follows from the exponential decay of the second modified Bessel function K_0 and its derivatives. Indeed, by [1, 9.6.27 and 9.6.28] $\frac{d}{dx}K_0(x) = -K_1(x)$ and $\frac{d}{dx}K_1(x) = -K_0(x) - \frac{1}{x}K_1(x)$; and by [1, 9.8.6 and 9.8.8] there exists $C_1 > 0$ such that for all $x \geq 2$

$$\sqrt{x}e^x K_0(x) \leq C_1, \qquad \sqrt{x}e^x K_1(x) \leq C_1.$$

To prove (iii), using the generating function $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$ of the Legendre polynomials P_n [1, §22.1 and 22.2], one can express $[r^2 + (t-2m\pi)^2]^{-\frac{1}{2}}$ as an infinite sum of these (this is the classical multipole expansion). Summing over $m \in \mathbb{Z}^*$ and rearranging yields

$$\frac{1}{2}\sum_{m\in\mathbb{Z}^*}\left[\frac{1}{\sqrt{r^2+(t-2m\pi)^2}} - a_{|m|}\right] = \sum_{l=1}^{+\infty}\frac{\zeta(2l+1)}{(2\pi)^{2l+1}}P_{2l}(\cos\phi)\rho^{2l},$$

where ζ is the Riemann zeta function [1, 23.2.1] and ϕ the polar angle in a set of spherical coordinates centred at the origin, *i.e.* such that $t = \rho\cos\phi$. The series converges for all $\rho < 2\pi$, but ρ is well defined and smooth in a ball of radius smaller than π, the injectivity radius of X. □

The connection. Fix a constant $v \in \mathbb{R}$ and consider the Higgs field $\Phi = iv + iG$. The 2–form $i * dG$ represents the curvature of a line bundle $L = L_q$ over $X \setminus \{q\}$; indeed, it is easy to check that $\frac{1}{2\pi i} * dG$ is an integral form. A connection $A = A_q$ on L is uniquely determined up to the addition of a closed 1–form. The action of gauge transformations is the addition of exact forms, so the gauge equivalence class of A is uniquely determined up to the addition of an imaginary multiple of dt, corresponding to tensoring L by a flat line bundle.

Remark 2.1.4. In order to calculate the holonomy of A around a loop $\gamma_z = \{z\}\times\mathbb{S}^1_t$ one can use Lemma 2.1.3.(i) to show that $d\left(\int_{\gamma_z} A\right) = -\int_{\gamma_z} F_A = i\int_{\gamma_z} r(\partial_r G)dt = i\,d\theta_q$. Here, as before, $\theta_q = \arg(z - z_0)$ if $q = (z_0, t_0) \in X$.

In a neighbourhood of the singularity L is isomorphic to the inverse of the Hopf line bundle extended radially from a small sphere $\mathbb{S}^2$ enclosing the origin. At infinity L is isomorphic to the radial extension of a line bundle of degree 1 over the torus $\mathbb{T}^2_\infty$. Let us discuss representatives for the connection in these asymptotic models.

- Introduce spherical coordinates $(z, t) = (\rho\sin\phi\, e^{i\theta}, \rho\cos\phi)$ on a 3-ball B_σ around the singularity. The cover of $\mathbb{S}^2$ given by $U_\pm = \mathbb{S}^2\setminus(0, 0, \pm 1)$ together with the transition function $e^{i\theta}$ from U_+ to U_- define the inverse H of the Hopf line bundle. The unique connection A^0 on H with harmonic curvature $\frac{i}{2}\,\mathrm{dvol}_{\mathbb{S}^2}$ is defined by $\frac{i}{2}(\pm 1 - \cos\phi)d\theta$ on $U_\pm$, respectively. We extend it radially to a $U(1)$–connection, still denoted by A^0, on the punctured ball $B_\sigma \setminus \{0\}$.
- Consider the connection $A^\infty = -i\frac{t}{2\pi}d\theta$ on the trivial line bundle $\underline{\mathbb{C}}$ over $\mathbb{S}^1_\theta \times \mathbb{R}_t$. If $(e^{i\theta}, t, \xi) \in \underline{\mathbb{C}}$, the map $\tau(e^{i\theta}, t, \xi) = (e^{i\theta}, t + 2\pi, e^{i\theta}\xi)$ satisfies $\tau^* A^\infty = A^\infty$. Define a line bundle with connection over $\mathbb{T}^2_{\theta,t}$ as the quotient $(\underline{\mathbb{C}}, A^\infty)/\tau$ and extend it radially to $(\mathbb{R}^2 \setminus B_R) \times \mathbb{S}^1$, for any $R > 0$.

Any connection A on L with $F_A = *d\Phi$ is asymptotically gauge equivalent to A^0 as $\rho \to 0$. As $r \to \infty$, up to gauge transformations, A is asymptotic to $A^\infty + i\alpha\, d\theta + ib\, dt$ for some $\alpha, b \in \mathbb{R}/\mathbb{Z}$. The monodromy of this limiting connection is $e^{-i\theta-2\pi ib}$ around the circle $\{\theta\}\times\mathbb{S}^1_t$ and $e^{it-2\pi i\alpha}$ around the circle $\mathbb{S}^1_\theta\times\{t\}$. While b can be chosen arbitrarily, α is fixed by the Bogomolny equation (1.2.1). Indeed, (2.1.1) implies that $\partial_t G(z,t)=0$ if $t\in\pi\mathbb{Z}$: By (1.2.1) the connection A restricted to the plane $\{t=\pi\}$ is flat. On the other hand, as we approach infinity the limiting holonomy of A on larger and larger circles $\{r=\text{const}, t=\pi\}$ converges to $e^{i(\pi-2\pi\alpha)}$. Thus $\alpha=\frac{1}{2}$ modulo $\mathbb{Z}$.

Lemma 2.1.5. *Fix parameters $(v,b)\in\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let (A,Φ) be a solution to (1.2.1) such that $\Phi=i\,(v+G)$ and the holonomy of A around circles $\{re^{i\theta}\}\times\mathbb{S}^1_t$, $r\neq 0$, is $e^{-i\theta-2\pi ib}$.*

(i) In the region where $r\geq 2$ the connection A is gauge equivalent to

$$A^\infty+\frac{i}{2}\,d\theta+ib\,dt+a$$

for a 1–form a such that $d^*a=0=\partial_r\lrcorner\, a$ and $|a|+|\nabla a|=O(e^{-r})$.

(ii) In a ball of radius $\frac{\pi}{2}$ centred at the singular point $z=0=t$, A is gauge equivalent to A^0+a' where $|a'|+\rho|\nabla a'|=O(\rho^2)$ and $d^*a'=0=\partial_\rho\lrcorner\, a'$.

Proof. (i) Write $\Phi=i\left(v+\frac{1}{2\pi}\log r\right)+\psi$ and solve (1.2.1) in a radial gauge. Write $A=A^\infty+a$, where $a=a_\theta d\theta+a_t dt$ solves $da=*d\psi$, *i.e.* :

$$\begin{cases}\partial_r a_\theta=r\partial_t\psi\\ \partial_r a_t=-\frac{1}{r}\partial_\theta\psi=0\\ \partial_\theta a_t-\partial_t a_\theta=r\partial_r\psi\end{cases}$$

Since $|\psi|+|\nabla\psi|=O(e^{-r})$, we solve the system integrating along rays: Up to exponentially decaying terms, a has a flat limit $a^\infty=a^\infty_\theta d\theta+a^\infty_t dt$ over the torus at infinity. By holonomy considerations as above, up to gauge transformations $a^\infty_\theta=\frac{i}{2}$ and $a^\infty_t=ib$. Then set $a_\theta-a^\infty_\theta=-\int_r^\infty r\partial_t\psi$ and $a_t=a^\infty_t$. Using these expressions one can check that a is a solution to the system above because ψ is harmonic; moreover, $d^*a=0$ because ψ is independent of θ. Finally, the decay of ψ and its gradient imply the desired estimates.

(ii) In spherical coordinates on a 3–ball of radius $\frac{\pi}{2}$ around the point $z=0=t$ we solve (1.2.1) in a radial gauge. As above, write $\Phi=i\left(v+\frac{a_0}{2}-\frac{1}{2\rho}\right)+\psi'$

with $|\psi'| + \rho|\nabla\psi'| = O(\rho^2)$. Then $A = A^0 + a'$, where a' is a solution to

$$\begin{cases} \partial_\rho a'_\theta = -\sin\phi\,(\partial_\phi\psi') = O(\rho^2) \\ \partial_\rho a'_\phi = 0 \\ d^{\mathbb{S}^2} a' = i\rho^2(\partial_\rho\psi')\ \mathrm{dvol}_{\mathbb{S}^2} = O(\rho^3) \end{cases}$$

(here $d^{\mathbb{S}^2}$ denotes exterior derivative on the 2–sphere $\rho = $ const.). By using the decay properties of ψ' and its derivative and the fact that ψ' is harmonic and independent of θ, integrate along rays to solve the system and check that a' satisfies the desired properties. □

The action of translations, rotations and scaling. Given an arbitrary point $q = (z_0, t_0)$ in X the same formulas describe the asymptotic behaviour of the periodic Dirac monopole (A_q, Φ_q) with singularity at q in coordinates centred at q. It will be useful to express the behaviour of (A_q, Φ_q) at large distances from q in a fixed coordinate system.

Lemma 2.1.6. *For $r \geq 2|z_0|$ we have*

$$\frac{1}{i}\Phi_q(z,t) = v + \frac{1}{2\pi}\log r - \frac{1}{2\pi}\mathit{Re}\left(\frac{z_0}{z}\right) + O(r^{-2})$$
$$A_q(z,t) = A^\infty + ib\,dt + i\,\frac{t_0+\pi}{2\pi}\,d\theta - \frac{i}{2\pi}\mathit{Im}\left(\frac{z_0}{z}\right)dt + O(r^{-2}).$$

Proof. Write $z = re^{i\theta}$ and $z_0 = r_0 e^{i\theta_0}$ and expand the logarithm for $r > r_0$

$$\log|z - z_0| = \log r - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{r_0}{r}\right)^n\cos\left[n(\theta-\theta_0)\right] = \log r - \mathrm{Re}\left(\frac{z_0}{z}\right) + O\left(\frac{r_0^2}{r^2}\right).$$

Together with Lemma 2.1.3.(ii), this proves the asymptotic expansion for the Higgs field. In order to derive an asymptotic expansion for the connection A_q, solve the abelian Bogomolny equation (1.2.1) using the asymptotic expansion for Φ. As in the proof of Lemma 2.1.5.(i), it follows that for large r the connection A_q is gauge equivalent to

$$A^\infty + ib\,dt + i\alpha\,d\theta - \frac{i}{2\pi}\mathrm{Im}\left(\frac{z_0}{z}\right)dt + O(r^{-2})$$

for some $\alpha, b \in \mathbb{R}/\mathbb{Z}$. As before, the holonomy on circles $r = $ const, $t = t_0 + \pi$ has to be trivial and therefore $2\pi\alpha = t_0 + \pi$ modulo $2\pi\mathbb{Z}$. □

The choice of the parameters $(v, b) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ is related to rotations and dilations: By a rotation in the z–plane, we can always assume that $b = 0$. On the other hand, let X_ℓ denote $\mathbb{C} \times \mathbb{R}/2\pi\ell\mathbb{Z}$ and, given any $\lambda > 0$, consider the homothety

$$h_\lambda : X_1 \longrightarrow X_\lambda$$

of ratio λ. We saw that the Bogomolny equation is the dimensional reduction of the ASD equation, which is conformally invariant. Then, forcing the Higgs field to scale as a 1–form, $(h_\lambda^* A, \lambda\, h_\lambda^* \Phi)$ is a monopole on X_1 if and only if (A, Φ) solves the Bogomolny equation on X_λ. Now, given a periodic Dirac monopole (A_q, Φ_q) with mass v, set $\lambda = v + \frac{a_0}{2}$. Then as $v \to \infty$

$$\lambda^{-1} h_{\lambda^{-1}}^* \Phi \longrightarrow i \left(1 - \frac{1}{2\sqrt{r^2 + t^2}} \right)$$

and $X_\lambda \to \mathbb{R}^3$, *i.e.* the limit $v \to \infty$ corresponds to the limit $\mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^3$ and in this limit a periodic Dirac monopole converges to an Euclidean Dirac monopole.

2.2 Boundary conditions

Having described the abelian periodic solutions to the Bogomolny equation, we can proceed to state and discuss the boundary conditions for periodic monopoles (with singularities) introduced by Cherkis and Kapustin in [29] and [34]: Periodic monopoles will be required to approach periodic Dirac monopoles of appropriate charges both at infinity and at the singularities. As a motivation for this choice, we recall the boundary conditions for $SU(2)$ monopoles on $\mathbb{R}^3$ without singularities.

The Euclidean case. In Section 1.2, we pointed out that the Bogomolny equation arises as a first order equation satisfied by absolute minima of the Yang–Mills–Higgs functional (1.2.3). It is therefore natural to consider configurations $c = (A, \Phi)$ with finite energy. In [60, Chapter IV, Part II] Taubes shows that critical points of (1.2.3) with finite energy satisfy the following asymptotic conditions. First of all, $|\Phi| \to v$ as $|x| \to \infty$. By (1.2.2) and the maximum principle, $|\Phi| \leq v$ everywhere on $\mathbb{R}^3$ and either the strict inequality holds or $|\Phi| \equiv v$. If we suppose that (A, Φ) is non-trivial, then $v \neq 0$ and $|\Phi(x)| < v$ for all $x \in \mathbb{R}^3$. By rescaling we can assume that $v = 1$.

Remark. Conversely, in the recent [100, §2.(d)] Taubes shows that if (A, Φ) solves the Bogomolny equation and $\lim_{|x|\to\infty} |\Phi| = 1$ then (A, Φ) has finite energy.

It follows that outside of a compact set, the trivial rank 2 complex vector bundle E on $\mathbb{R}^3$ splits into a direct sum $E \simeq H^k \oplus H^{-k}$ of eigenspaces of Φ and there exists an asymptotic gauge in which the Higgs field is given by

$$\Phi = \left(1 - \frac{k}{\rho}\right)\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + O(\rho^{-2}).$$

Here we chose the normalisation $|A|^2 = -2\,\mathrm{Trace}(A^2)$ for the $SU(2)$–invariant product on $\mathfrak{su}(2)$; $k \in \mathbb{Z}_{\geq 0}$ is the charge of (A, Φ). Moreover, $|d_A\Phi| = O(\rho^{-2})$ as $\rho \to \infty$ and the curvature of A approaches the curvature of the $SO(3)$–invariant connection on $H^k \oplus H^{-k}$. In other words, at infinity a finite energy monopole on $\mathbb{R}^3$ is asymptotic to an Euclidean Dirac monopole up to terms of order $O(\rho^{-2})$.

As observed in [7, Remark 1, page 47], in the same asymptotic gauge the term of order exactly ρ^{-2} of the Higgs field takes the form

$$\Phi = \left(1 - \frac{k}{\rho} - k\,\frac{\langle x, q\rangle}{\rho^3}\right)\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + O(\rho^{-3}).$$

This same $q \in \mathbb{R}^3$ fixes the term $-k\,\frac{\langle x\times q, dx\rangle}{\rho^3}\frac{\Phi}{|\Phi|}$ of order exactly ρ^{-2} of the connection A as well. We call the parameter q the *centre* of the monopole, because $\frac{1}{|x-q|} = \frac{1}{\rho} + \frac{\langle x,q\rangle}{\rho^3} + O(\rho^{-3})$ as $\rho \to \infty$. Thus one can define moduli spaces M^0_k, M_k of centred and uncentred monopoles depending on whether the term of order exactly ρ^{-2} in the expansion of (A, Φ) at infinity is fixed. Both spaces carry a Riemannian metric induced by the L^2–norm of infinitesimal deformations. As we shall see, in the periodic case only the moduli space of centred monopoles consists of L^2 deformations.

Modelling their choices on the Euclidean case, Cherkis and Kapustin define boundary conditions for periodic monopoles replacing Euclidean Dirac monopoles with periodic ones. Before giving the precise definitions, we need to address the issue of which structure group to consider.

The structure group: $U(2)$ vs. $SO(3)$. Limiting ourselves to compact Lie groups of rank 2, the simplest choice would be to take $G = SU(2)$. However, in order to introduce singularities of the fields while hoping to obtain moduli spaces which are complete smooth manifolds, it is necessary to pick $SO(3)$ as structure group. To see this, consider as in Kronheimer [67] $SU(2)$ monopoles with singularities on $\mathbb{R}^3$. The singularity model at a point p is given by an Euclidean Dirac monopole $H^l_p \oplus H^{-l}_p$ for some $l \in \mathbb{Z}_{>0}$. The adjoint bundle then splits as $\underline{\mathbb{R}} \oplus H^{2l}_p$. Kronheimer shows that the moduli space of $SU(2)$ monopoles on $\mathbb{R}^3$ with non-abelian charge 1 and one singularity p as above can be identified with the hyperkähler manifold obtained from the Gibbons-Hawking ansatz (1.5.1) with the harmonic function $h = 2m + \frac{2l}{2|x-p|}$: It has a singularity at p modelled on $\mathbb{C}^2/\mathbb{Z}_{2l}$.

In [34] Cherkis and Kapustin define periodic $U(2)$ and $SO(3)$–monopoles with singularities. We briefly discuss the relation between the two choices of structure group, following Braam–Donaldson [22, §1.1-1.2, Part II] and Donaldson [37, §5.6].

Notation. Given a collection S of n distinct points $p_1, \dots, p_n \in X$ set $X^* = X \setminus S$.

Let $V \to X^*$ be an $SO(3)$–bundle: By a result of Whitney [109, §III.7], isomorphism classes of $SO(3)$–bundles over a CW–complex of dimension at most 3 are completely classified by the second Stiefel–Whitney class w_2. The second homology of X^* is generated by the classes of 2–spheres $\mathbb{S}^2_{p_i}$ each enclosing the point $p_i \in S$. We fix the isomorphism class of V by requiring that $w_2(V) \cdot [\mathbb{S}^2_{p_i}] = 1$ for all $i = 1, \dots, n$. V does not lift to an $SU(2)$–bundle whenever $n > 0$.

However, V does always lift to a $U(2)$–bundle: The unique obstruction is the image of $w_2(V)$ under the Bockstein map $H^2(X^*; \mathbb{Z}_2) \to H^3(X^*; \mathbb{Z}) = 0$. Hence $w_1(V)$ always lifts to an integral class $c_1(E) \equiv w_2(V)$ (mod 2) representing the first Chern class of a rank 2 Hermitian complex vector bundle $E \to X^*$.

The adjoint bundle $\mathfrak{g}_E$ splits into a direct sum $\mathfrak{g}_E = \underline{\mathbb{R}} \oplus \mathfrak{g}^{(0)}_E$ of a trivial real line bundle, the trace part, and the trace-less part $\mathfrak{g}^{(0)}_E \simeq V$, which is a $PU(2) \simeq SO(3)$ bundle. A pair (A, Φ) on E satisfying the Bogomolny equation induces an abelian monopole (A_{tr}, Φ_{tr}) on $\det(E)$ and an $SO(3)$–monopole $(A^{(0)}, \Phi^{(0)})$ on V. Conversely, by fixing the abelian monopole (A_{tr}, Φ_{tr}), we would like to lift an $SO(3)$–monopole on V to a $U(2)$–monopole on E. We need to discuss gauge transformations. Denote by P_V and P_E the principal $SO(3)$ and $U(2)$–bundles associated with V and E, respectively. Gauge transformations acting on V are sections of the bundle $P_V \times_{\mathrm{Ad}} SO(3)$, while, since we fix the central part of pairs (A, Φ) on E, it is natural to consider the group of automorphisms of P_E with determinant 1. Denote these two groups by $\mathcal{G}_V$ and $\mathcal{G}_E$, respectively. Now let $\mathcal{C}_V$ be the space of pairs (A, Φ) on V and $\mathcal{C}_E$ the space of pairs on E with fixed central part (A_{tr}, Φ_{tr}). The map $\mathcal{C}_E/\mathcal{G}_E \longrightarrow \mathcal{C}_V/\mathcal{G}_V$ is a double cover, with $H^1(X^*;\mathbb{Z}_2) \simeq \mathbb{Z}_2$ as the group of deck transformations. Indeed we have an exact sequence

$$1 \to \mathcal{G}_E \longrightarrow \mathcal{G}_V \longrightarrow H^1(X^*;\mathbb{Z}_2) \to 1$$

which we think as follows. Identify $\mathcal{G}_E$ with the group of sections of $P_V \times_{\mathrm{Ad}} SU(2)$. Each element $g \in \mathcal{G}_V$ defines a double cover of X^* by taking the doubly–valued graph of g in $\mathcal{G}_E$ and therefore a class in $H^1(X^*;\mathbb{Z}_2)$, trivial if and only if g can be lifted to a section of $\mathcal{G}_E$. Very concretely, the action of $H^1(X^*;\mathbb{Z}_2)$ is given by tensoring E with the flat line bundle $L_{\frac{1}{2}}$ with connection $\frac{i}{2}\,dt$ whose holonomy around circles $\gamma_z = \{z\} \times \mathbb{S}^1_t$ is $-\mathrm{id}$. Up to gauge transformations, the square of this flat line bundle is trivial as a line bundle with connection and therefore tensoring by $L_{\frac{1}{2}}$ doesn't affect the central nor the trace-free part of a pair (A, Φ) on E.

We conclude that, up to a finite cover, it makes no difference to consider $U(2)$–monopoles with fixed central part and $SO(3)$–monopoles. We will mainly work with structure group $G = SO(3)$.

Boundary conditions for $SO(3)$–monopoles. We begin with some preliminary notational remarks. First, with our normalisation $|A|^2 = -2\,\mathrm{Trace}\,(A^2)$ of the scalar product on $\mathfrak{su}(2)$, the isomorphism $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$ via the adjoint representation is an isometry. Secondly, observe that if $V \to X^*$ is a rank 3 real oriented Riemannian vector bundle and P is the principal $SO(3)$–bundle of orthonormal frames of V, then $V \simeq \mathrm{ad}\,P$. Finally, a reducible $SO(3)$–bundle V is an oriented Riemannian rank 3 vector bundle with a decomposition $V \simeq \underline{\mathbb{R}} \oplus M$ for an $SO(2)$–bundle M. We denote by $\hat{\sigma}$ the trivialising unit-norm section of the first factor. We will use the isomorphism $V \simeq \mathrm{ad}\,P$ to identify $\hat{\sigma}$ with $[\sigma_3, \cdot\,]$, where σ_3 is defined in (1.1.1), in a local trivialisation $\mathrm{ad}\,P \simeq U \times \mathfrak{su}_2$ over an open set U. In this sense we will talk of diagonal and off-diagonal sections of V to denote the sections of the two factors in the decomposition $V \simeq \underline{\mathbb{R}} \oplus M$.

Fix a collection S of n distinct points $p_1, \dots, p_n \in X$ and an $SO(3)$–bundle V on X^* with the topology described above. We also fix an origin and a frame in

$X \simeq \mathbb{C} \times \mathbb{S}^1$ and use coordinates $(z,t) \in \mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z}$ with $z = x + iy = re^{i\theta}$.

Definition 2.2.1. Given a non-negative integer $k_\infty \in \mathbb{Z}_{\geq 0}$, parameters $(v,b) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and a point $q = (\mu, \alpha) \in X$, let $\mathcal{C} = \mathcal{C}(p_1, \ldots, p_n, k_\infty, v, b, q)$ be the space of smooth pairs $c = (A, \Phi)$ of a connection A on V and a section Φ of ad $P \simeq V$ satisfying the following boundary conditions.

1. For each $p_i \in S$ there exists a ball $B_\sigma(p_i)$ and a gauge $V|_{B_\sigma(p_i)\setminus\{p_i\}} \simeq \underline{\mathbb{R}} \oplus H_{p_i}$ such that (A, Φ) can be written

$$\Phi = -\frac{1}{2\rho_i}\,\hat{\sigma} + \psi \qquad A = A^0\,\hat{\sigma} + a$$

with $\xi = (a, \psi) = O(\rho_i^{-1+\tau})$ and $|\nabla_A \xi| + |[\Phi, \xi]| = O(\rho_i^{-2+\tau})$ for some rate $\tau > 0$. Here ρ_i is the distance from p_i and A^0 is the $SO(3)$–invariant connection on H_{p_i}.

2. There exists $R > 0$ and a gauge $V \simeq \underline{\mathbb{R}} \oplus \left(L_q^{k_\infty} \otimes L_{v,b}\right)$ over $(\mathbb{R}^2 \setminus B_R) \times \mathbb{S}^1$ such that (A, Φ) can be written

$$\Phi = \left[v + \frac{k_\infty}{2\pi}\log r - \frac{k_\infty}{2\pi}\mathrm{Re}\left(\frac{\mu}{z}\right)\right]\hat{\sigma} + \psi$$

$$A = \left[b\,dt + k_\infty A^\infty + \frac{k_\infty}{2\pi}(\alpha + \pi)d\theta - \frac{k_\infty}{2\pi}\mathrm{Im}\left(\frac{\mu}{z}\right)dt\right]\hat{\sigma} + a$$

with $\xi = (a, \psi) = O(r^{-1-\tau})$ and $|\nabla_A \xi| + |[\Phi, \xi]| = O(r^{-2-\tau})$ for some $\tau > 0$. Here A^∞ is the connection on L_q of Lemma 2.1.5.

Remark. (i) Cherkis and Kapustin [34] have $\tau = 1$ both at the singularity and at infinity as the rate of convergence of (A, Φ) to the model Dirac monopoles.

(ii) We will return to the definition of monopoles with Dirac type singularities in Section 3.1, where we will review Kronheimer's approach [67] and see that locally singular monopoles can be lifted via the Hopf map to $\mathbb{S}^1$–invariant instantons on a 4–ball.

(iii) At infinity, it is expected that, in analogy with the case of $\mathbb{R}^3$, $|\Phi|$ has an asymptotic expansion as a harmonic function up to exponentially decaying terms and therefore one can take $\tau = 1$ as in [34]. We modelled our definition on [18, Theorem 0.1], where Biquard and Jardim study the asymptotic behaviour of doubly-periodic instantons with quadratic curvature decay.

(iv) In Section 3.4 we will define moduli spaces of periodic monopoles fixing an initial background pair (A, Φ) satisfying these boundary conditions (but not necessarily the Bogomolny equation) and considering pairs of the form $(A, \Phi) + (a, \psi)$ with (a, ψ) in certain weighted Sobolev spaces.

We collect some comments on Definition 2.2.1.

- There is a topological constraint on the choice of the charge at infinity k_∞. Since $[\mathbb{T}_\infty]$ is homologous to the sum $[\mathbb{S}^2_{p_1}] + \ldots + [\mathbb{S}^2_{p_n}]$ and k_∞ (mod 2) is the value of the second Stiefel–Whitney class $w_2(V)$ on $[\mathbb{T}_\infty]$, we must have $k_\infty \equiv n$ modulo 2. We then define a non-negative integer k by $2k = k_\infty + n$, and call it the (non-abelian) *charge* of the $SO(3)$–monopole $c = (A, \Phi)$.

- If $k_\infty = 0$ we require that $v > 0$, so that Φ still defines a reduction $V \simeq \underline{\mathbb{R}} \oplus L$ of the structure group to $SO(2)$ both at infinity and close to the singularities.

- A priori one could allow singularities with different higher charges, as in Pauly [87] for monopoles on compact 3–manifolds. However, we saw that we cannot expect the moduli space to be a complete smooth manifold in that case.

- Non-trivial periodic monopoles have infinite energy (1.2.3). We can still say that periodic monopoles minimise the Yang–Mills–Higgs functional $\mathcal{A}$ in the following sense: (A, Φ) is a monopole if and only if for any compact set $K \subset X^*$ and any pair (A', Φ') such that $\int_{\partial K} \langle \Phi', F_{A'} \rangle = \int_{\partial K} \langle \Phi, F_A \rangle$

$$\mathcal{A}(A, \Phi; K) \leq \mathcal{A}(A', \Phi'; K).$$

  Here the notation $\mathcal{A}(A, \Phi; K)$ means that we evaluate the integrals on the compact set K. Moreover, the equality holds if and only if (A', Φ') is also a monopole.

- We call the parameter q in Definition 2.2.1 the *centre* of the monopole. Differently from the Euclidean case, only moduli spaces of centred periodic monopoles carry a Riemannian metric induced by the L^2–norm of infinitesimal deformations. Notice that the boundary conditions of Definition 2.2.1 depend on the choice of an origin and a frame in X.

- Finally, one should make precise the choice of gauge group. The analysis of the next chapter implies that in order to develop a Fredholm theory for the deformation complex of a periodic monopole (with singularities) it is necessary to allow gauge transformations which approach a constant diagonal matrix in the asymptotic decomposition $V \simeq \underline{\mathbb{R}} \oplus \left(L_q^{k_\infty} \otimes L_{v,b}\right)$. Thus we will study only moduli spaces of unframed periodic monopoles.

Periodic monopoles with structure group $U(2)$. In order to define boundary conditions for $U(2)$ periodic monopoles with singularities it is necessary to make additional choices which determine both a lift of $w_2(V)$ to an integral class and the central part (A_{tr}, Φ_{tr}) of the monopoles.

For each singularity $p_i \in S$ choose $e_i \in \{\pm 1\}$ and therefore a partition $n = n_+ + n_-$. Fix integers $k_1 \geq k_2$ and parameters $v_1 \geq v_2 \in \mathbb{R}$, $b_1, b_2 \in \mathbb{R}/\mathbb{Z}$ and $q_1, q_2 \in X$ such that $v_1 > v_2$ if $k_1 = k_2$. Let E be a Hermitian rank 2 complex vector bundle on X^* and (A, Φ) a pair (connection, Higgs field) on E. Cherkis and Kapustin require that the triple (E, A, Φ) is asymptotically gauge equivalent to reducible bundles

$$\begin{cases} H_{p_i}^{e_i} \oplus \underline{\mathbb{C}} \text{ in a punctured neighbourhood of } p_i, \\ \\ \left(L_{q_1}^{k_1} \otimes L_{v_1,b_1}\right) \oplus \left(L_{q_2}^{k_2} \otimes L_{v_2,b_2}\right) \text{ as } r \to +\infty \end{cases}$$

each endowed with the reducible pairs induced by the corresponding Dirac monopole as in Definition 2.2.1.

By taking traces, the central part (A_{tr}, Φ_{tr}) is an abelian monopole on X^*, and therefore necessarily of the form

$$\det E = L_{v_1+v_2, b_1+b_2} \otimes \bigotimes_{i=1}^{n} L_{p_i}^{h_i},$$

for some integers h_i, $i = 1, \dots, n$. In order to agree with the fixed behaviour of E around p_i one has $h_i = e_i$. Then we can use Lemma 2.1.6 to compare the asymptotic expansion of this sum of periodic Dirac monopoles with the chosen boundary conditions at infinity. Requiring that $L_{v_1+v_2,b_1+b_2} \otimes \bigotimes_{i=1}^{n} L_{p_i}^{e_i} \simeq L_{v_1+v_2,b_1+b_2} \otimes L_{q_1}^{k_1} \otimes L_{q_2}^{k_2}$ outside of a compact set imposes constraints on the choice of parameters:

$$k_1 + k_2 = n_+ - n_- \qquad \sum_{i=1}^{n} e_i \, p_i = q_1 + q_2,$$

where the last sum is taken in $\mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z}$. The first equality follows by comparing the charges; the second one by looking at the terms of order $\frac{1}{r}$ in the expansions for the connection and the Higgs field.

Remark (Periodic monopoles with structure group of higher rank)**.** Cherkis and Kapustin define boundary conditions for $U(m)$–monopoles for $m > 2$ [34] in a similar way. The behaviour at a singularity is still given by an asymptotic isomorphism $E \simeq H_{p_i}^{e_i} \oplus \underline{\mathbb{C}}^{m-1}$ with $e_i \in \{\pm 1\}$. In the higher rank case, each $U(m)$–monopole gives rise to a periodic Dirac monopole and a $PU(m) = SU(m)/\mathbb{Z}_m$–monopole via the splitting of the adjoint bundle into trace and trace-free part. In analogy with the second Stifel-Whitney class, e_i modulo m represents the obstruction to lift the $PU(m)$–bundle to an $SU(m)$–bundle around the ith singularity. Notice that if $m > 2$ singularities with $e_i = 1$ and $e_i = -1$ remain topologically different even after passing to structure group $PU(m)$.

Chapter 3

Moduli spaces

The content of this chapter is divided into three parts. The first two are devoted to introduce the analytical tools needed to work with Cherkis and Kapustin's definitions. We work locally around the singularities and on the big end of X^*, studying the deformation theory of monopoles with Dirac type singularities in the former case, working with a periodic Dirac monopole as a background in the latter.

In Section 1 we review the work of Kronheimer on Euclidean monopoles with Dirac-type singularities: Via the Hopf map, these correspond to $\mathbb{S}^1$–invariant instantons on $\mathbb{R}^4$. Pauly extended this approach to study moduli spaces of (necessarily singular) monopoles on compact 3–manifolds. However, we preferred to study monopoles with Dirac-type singularities directly in 3 dimensions relying on Lockhart–McOwen's theory of weighted Sobolev spaces [76]. Even if Melrose's b–calculus yields the most complete and sharp version of the results of [76] (Theorem 3.2.10 below is Theorems 5.60 and 6.5 in [79]), Lockhart–McOwen's original approach is sufficient for our purposes. In fact we will give direct proofs of all the results we need by elementary methods, even if they follow from the general theory. In Section 2 we introduce the relevant function spaces and show that they are well-behaved to study gauge theory. We then study the Dirichlet problem for the Laplacian DD^*, where D is the Dirac operator (1.3.3) twisted by the Euclidean Dirac monopole.

Section 3 deals with the analysis on the end of $\mathbb{R}^2 \times \mathbb{S}^1$ with a periodic Dirac monopole as a background. A fairly general framework to deal with Fredholm operators on $\mathbb{R}^n \times K$, where K is a compact manifold, (and more in general on manifolds with fibred boundary) is provided by the Φ–calculus of Mazzeo–Melrose

[78]. However, the Dirac operator 1.3.3 twisted by a periodic Dirac monopole is not fully-elliptic in the sense of [78] and we are forced to define weighted Sobolev spaces adapted to the problem. As a way of motivation, we begin by studying the Dirichlet problem for the Laplacian on an exterior domain in $\mathbb{R}^2$; this discussion also illustrates some of the consequences of the parabolicity of $\mathbb{R}^2 \times \mathbb{S}^1$. Recall that a complete Riemannian manifold is said to be *parabolic* if it doesn't admit a positive Green's function. As pointed out by Li [74, §2], "In general the methods in dealing with function theory on these [parabolic and non-parabolic] manifolds are different". The rest of Section 3 has a similar structure to Section 2: We introduce the appropriate function spaces, study the continuity of various products between them and show that we can solve the Dirichlet problem for the operator DD^* twisted by a periodic Dirac monopole in the appropriate spaces and with arbitrary boundary data.

In the final section we apply these analytical preliminaries to give a rigorous definition and construction of the moduli spaces of periodic monopoles (with singularities). We fix a smooth background pair (A, Φ) satisfying the boundary conditions of Definition 2.2.1 and consider pairs of the form $(A, \Phi) + (a, \psi)$, with (a, ψ) lying in the appropriate weighted Sobolev space. After a brief discussion of reducibility, we develop a Fredholm theory for the deformation complex (1.3.4) and show that moduli spaces of periodic monopoles (with singularities), if non-empty, are smooth hyperkähler manifolds for generic choices of the parameters specifying the boundary conditions.

3.1 Hopf lift of a monopole with a Dirac type singularity

In this section we study monopoles with Dirac type singularities on a punctured 3–ball. We review the approach of Kronheimer [67], who showed that the Hopf fibration induces a bijection between monopoles on $\mathbb{R}^3$ with Dirac type singularities and $\mathbb{S}^1$–invariant instantons on $\mathbb{R}^4$ (endowed with either the flat or Taub–NUT metric). The same lift procedure has been extended by Pauly [87] to study (necessarily singular) monopoles on compact 3–manifolds. We will consider $SO(3)$ monopoles, but the discussion can be adapted to other structure groups.

Let $B^3 = B_\sigma(0)$ be a ball in $\mathbb{R}^3$. Fix complex coordinates (z_1, z_2) on $\mathbb{C}^2 \simeq \mathbb{R}^4$ and consider the Hopf projection $\pi\colon\ B^4 \rightarrow B^3$ of (1.5.2). Here $B^4 = B_{\sqrt{\sigma}}(0) \subset$

$\mathbb{R}^4$. We express the Euclidean metric on $B^4 \setminus \{0\}$ in Gibbons-Hawking coordinates (1.5.1) using the harmonic function $h = \frac{1}{2\rho}$, where ρ is the distance from $0 \in \mathbb{R}^3$.

Let $V \to B^3 \setminus \{0\}$ be an $SO(3)$–bundle and (A, Φ) a connection and Higgs field on V. Define a connection $\hat{A}$ on $\hat{V} = \pi^* V \to B^4 \setminus \{0\}$ by

$$\hat{A} = \pi^* A - \pi^* \left(h^{-1}\Phi\right) \otimes \theta_0. \tag{3.1.1}$$

Explicit model example: The Euclidean Dirac monopole. Let us first consider the explicit local model for a Dirac type singularity: $V = \underline{\mathbb{R}} \oplus H^k$ and (A, Φ) is the Euclidean Dirac monopole of charge k, mass λ and singularity at the origin as in Definition 2.1.1.

Use coordinates $(z_1, z_2) = \left(r_4 e^{i\theta_1} \cos\varphi, r_4 e^{i\theta_2} \sin\varphi\right)$ on B^4 and spherical coordinates $(\rho e^{i\theta} \sin\phi, \rho\cos\phi)$ on B^3. The Hopf projection is then given by

$$\pi(z_1, z_2) = \left(r_4^2 \sin(2\varphi) e^{i(\theta_1+\theta_2)}, r_4^2 \cos(2\varphi)\right).$$

Let A^0 denote the $SO(3)$–invariant connection on H. Then $\theta_0 = ds - A^0$, where s is a fibre coordinate on π. Since $r_4^2 \theta_0 = |z_1|^2 d\theta_1 - |z_2|^2 d\theta_2$, in local trivialisations for π:

(i) On $\mathbb{C} \times \mathbb{C}^* \subset \mathbb{R}^4$ we choose a fibre coordinate $s = -\theta_2$ and therefore

$$\theta_0 = -d\theta_2 + \cos^2(\varphi) d(\theta_1 + \theta_2) = ds + \pi^* \left(\frac{1}{2}(1 + \cos\phi)\, d\theta\right).$$

(ii) Similarly on $\mathbb{C}^* \times \mathbb{C}$ choose $s = \theta_1$ and

$$\theta_0 = d\theta_1 - \sin^2(\varphi) d(\theta_1 + \theta_2) = ds - \pi^* \left(\frac{1}{2}(1 - \cos\phi)\, d\theta\right).$$

It follows that $\hat{A} = -\lambda r_4^2 \left(ds - \pi^* A^0\right) \otimes \hat{\sigma} + k\, ds \otimes \hat{\sigma}$. The first term is a multiple of the smooth 1–form

$$\eta = \frac{i}{2}\left(z_1 d\overline{z}_1 - \overline{z}_1 dz_1\right) - \frac{i}{2}\left(z_2 d\overline{z}_2 - \overline{z}_2 dz_2\right), \tag{3.1.2}$$

while the gauge transformation

$$g = \begin{cases} e^{k\theta_1\sigma_3} & \text{if } z_1 \neq 0 \\ e^{-k\theta_2\sigma_3} & \text{if } z_2 \neq 0 \end{cases} \tag{3.1.3}$$

where σ_3 is defined in (1.1.1), yields an isomorphism $\hat{V} \simeq B^4 \times \mathfrak{su}(2)$ such that $g(\hat{A}) = g\hat{A}g^{-1} - (dg)g^{-1} = -\lambda\eta \otimes \hat{\sigma}$. It follows that $\hat{A}$ extends to a smooth $\mathbb{S}^1$–invariant reducible connection on the trivial bundle $B^4 \times \mathfrak{su}(2)$. The $\mathbb{S}^1$–action is defined by

$$e^{is} \cdot (z_1, z_2, X) = \left(e^{is} z_1, e^{-is} z_2, \mathrm{Ad}\left(e^{ks\sigma_3}\right) X\right) \tag{3.1.4}$$

for $(z_1, z_2) \in \mathbb{C}^2$ and $X \in \mathfrak{su}(2)$. Hence the charge of the Dirac monopole is the weight of the $\mathbb{S}^1$–action on the fibre of $\pi^* V$ over the origin. Finally, as expected by dimensional reduction considerations, $d\hat{A} = -\lambda\, d\eta \otimes \hat{\sigma}$ is an ASD form on B^4.

The general case. Now consider an $SO(3)$–bundle $V \to B^3 \setminus \{0\}$ and an arbitrary pair (A, Φ) with a Dirac type singularity at the origin, in the sense made precise below. Let $\hat{A}$ be the connection on $\hat{V} \to B^4 \setminus \{0\}$ obtained from (A, Φ) as in (3.1.1). With $\Psi = *F_A - d_A\Phi$, we compute

$$F_{\hat{A}} = \pi^*\left(h * d_A(h^{-1}\Phi)\right) - \pi^*\left(d_A(h^{-1}\Phi)\right) \wedge \theta_0 + \pi^*\left(*\Psi\right). \tag{3.1.5}$$

Using the expression (1.5.1) for the flat metric on $\mathbb{R}^4$ in Gibbons–Hawking coordinates and choosing the orientation in which $\pi^* \mathrm{dvol}_{\mathbb{R}^3} \wedge \theta_0$ is positive, a computation of the Hodge star operator yields $2F^+_{\hat{A}} = \pi^*(*\Psi) + \pi^*(h^{-1}\Psi) \wedge \theta_0$. Thus $\hat{A}$ is ASD if and only if (A, Φ) is a monopole. Another immediate consequence of (3.1.5) is

$$\int_{B^4} |F_{\hat{A}}|^2 \,\mathrm{dvol}_{\mathbb{R}^4} = 2\pi \int_{B^3} \left(2|d_A(h^{-1}\Phi)|^2 + h^{-2}|\Psi|^2\right) h\, \mathrm{dvol}_{\mathbb{R}^3}.$$

Now assume

(i) $\Psi = 0$.

(ii) $\int_{B^3} |d_A(h^{-1}\Phi)|^2 h\, \mathrm{dvol}_{\mathbb{R}^3} < \infty$.

(iii) $h^{-1}|\Phi| \to k \in \mathbb{Z}_{>0}$.

In view of (ii) Uhlenbeck's Removable Singularities Theorem [106, Theorem 4.1] implies that, after gauge transformation, the connection $\hat{A}$ extends to an ASD connection on B^4. By the uniqueness of this extension, the $\mathbb{S}^1$–action on $\hat{V}$ extends over B^4 and $\hat{A}$ is $\mathbb{S}^1$–invariant. It remains to show that the weight of the representation $\mathbb{S}^1 \to SO(3)$ on the fibre of the extension of $\hat{V}$ over the origin $0 \in B^4$ is k: Up to gauge transformations, the action is given by (3.1.4), with $k' \in \mathbb{Z}_{\geq 0}$; the fact that $k' = k$ follows from the converse construction, from an $\mathbb{S}^1$–invariant $\hat{A}$ on B^4 to a monopole (A, Φ) on $B^3 \setminus \{0\}$ with a Dirac type singularity at the origin.

Hence suppose that $\hat{V} \to B^4$ is an $SO(3)$–bundle with a given $\mathbb{S}^1$–action covering the Hopf one on B^4. Choose a smooth gauge such that the $\mathbb{S}^1$–action is given by (3.1.4) where $k \in \mathbb{Z}_{\geq 0}$ is the weight of the action on the fibre over the origin. Let $\hat{A}$ be a smooth $\mathbb{S}^1$–invariant ASD connection $\hat{A}$ on $\hat{V}$. We use the inverse of (3.1.3) to write

$$g^{-1}(\hat{A}) = \pi^* a - \pi^* \psi \otimes \eta + g^{-1}(dg)$$

where η is defined in (3.1.2) and a, ψ are a 1 and 0–form on $B^3 \setminus \{0\}$ with values in $V = \hat{V}/\mathbb{S}^1$. By the definition of g and η, $V \simeq \underline{\mathbb{R}} \oplus H^k$ and $g^{-1}(\hat{A})$ is the lift (3.1.1) of a pair (A, Φ) on V such that $(A, \Phi) = k(A^0, \Phi^0)\, \hat{\sigma} + (a, \psi)$. Finally, using the expression for the flat metric in Gibbons–Hawking coordinates to compute norms (*cf.* Definition 3.1.3.(ii) below), $h^{-1}(|a|^2 + |\psi|^2) = |\hat{A}|^2 < \infty$. In particular $h^{-1}|\Phi| \to k$ as $\rho \to 0$.

Remark 3.1.1. For the rough estimate $h^{-1}(|a|^2 + |\psi|^2) < \infty$ we did not use the $\mathbb{S}^1$–invariance nor the anti-self-duality of $\hat{A}$. For example, imposing $\mathbb{S}^1$–invariance one can show that the diagonal component of (a, ψ) is bounded at the origin.

Summarising, we proved:

Proposition 3.1.2 (Lemma 3.5 of [67]). *A smooth pair (A, Φ) is a monopole on $B^3 \setminus \{0\}$ such that*

(i) $h^{-1}|\Phi| \to k \in \mathbb{N}$ *as* $\rho \to 0$*, and*

(ii) $\int_{B^3} |d_A(h^{-1}\Phi)|^2 h \,\mathrm{dvol}_{\mathbb{R}^3} < \infty$

if and only if $\hat{A}$ defined by (3.1.1) is gauge equivalent to a smooth $\mathbb{S}^1$–invariant ASD connection on B^4 and the $\mathbb{S}^1$–action on the fibre over the origin of the extension of $\pi^* V$ has weight k.

Remark. Pairs (A, Φ) satisfying the boundary conditions of Definition 2.2.1 have finite weighted energy (ii). Conversely, we showed that if (i)-(ii) are satisfied and

(A, Φ) is a monopole then there exists a gauge such that $(A, \Phi) = k(A^0, \Phi^0)\,\hat{\sigma} + O(\rho^{-1+\tau})$ for any rate $\tau \leq \frac{1}{2}$.

As in Kronheimer [67], we can replace the integral bound (ii) in Proposition 3.1.2 with the condition $d\,(h^{-1}|\Phi|) < \infty$: If (ii) is satisfied then the curvature $|d_A(h^{-1}\Phi)| = 2|F_{\hat{A}}|$ is bounded and Kato's inequality implies $d\,(h^{-1}|\Phi|) < \infty$. Conversely, since h is harmonic away from the origin, we write

$$h|d_A(h^{-1}\Phi)|^2 = -\frac{1}{2}d^*d(h^{-1}|\Phi|^2)$$

so that (ii) becomes equivalent to $\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(0)} *d(h^{-1}|\Phi|^2) < \infty$. Using the bound $d\,(h^{-1}|\Phi|) < \infty$ together with the assumption (i) in Proposition 3.1.2 one then shows that the limit above is equal to πk^2.

Remark. (i) Proposition 3.1.2 continues to hold if B^4 is endowed with the Taub–NUT metric.

(ii) If we work on $\mathbb{R}^3$ and allow n singularities $p_1, \ldots, p_n$, we can extend this local correspondence to a global one. Indeed, we can take $h = v + \sum_{i=1}^n G_{p_i}$, where $G_{p_i} = (2|x - p_i|)^{-1}$ is the Green's function of $\mathbb{R}^3$ with singularity at p_i and $v \geq 0$. Then monopoles with singularities on $\mathbb{R}^3$ correspond to $\mathbb{S}^1$–invariant ASD connections on the corresponding multi-Eguchi–Hanson or multi-Taub–NUT space. In the periodic case (and on a compact 3–manifold, *cf.* [87, §2.3]) such a global correspondence cannot hold, because X is parabolic, *i.e.* its Green's function is *not* everywhere positive.

Deformation theory of $\mathbb{S}^1$–invariant instantons on B^4. Having clarified some aspects of the definition of monopoles with an isolated Dirac type singularity, we move on to study the local deformation theory. Proposition 3.1.2 suggests to lift the whole local deformation problem to $\mathbb{R}^4$ via the Hopf projection (1.5.2) and study deformations of $\mathbb{S}^1$–invariant instantons instead. This is the approach adopted by Pauly in [87] to construct moduli spaces of singular monopoles on compact 3–manifolds.

More precisely, let (A, Φ) be a pair of a connection and a Higgs field on a $SO(3)$–bundle V over $B^3 \setminus \{0\}$ such that the conditions (i)-(ii) of Proposition 3.1.2 are verified. Let $\hat{A}$ be the connection (3.1.1) lifted from (A, Φ) and fix a gauge as in (3.1.3) such that $\hat{A}$ and the circle action extend over the origin in B^4. We can use this gauge and a trivialisation of $\pi^*V \to B^4$ to define Sobolev spaces $W^{k,p}_{\mathbb{S}^1}$ of $\mathbb{S}^1$–invariant forms with values in π^*V and do gauge theory with these function spaces: Gauge transformations are $\mathbb{S}^1$–invariant $W^{k,p}$–maps $B^4 \to SO(3)$ acting on connections of class $W^{k-1,p}_{\mathbb{S}^1}$. For a well-defined theory it is necessary to assume $kp > 2$ so that gauge transformations are continuous.

For future reference, we spell out a "dictionary" between the deformation theory of monopoles on $B^3 \setminus \{0\}$ and that of $\mathbb{S}^1$–invariant instantons upstairs on B^4. Let V be an $SO(3)$–bundle over $B^3 \setminus \{0\}$ and set $\hat{V} = \pi^*V$ as before, where π is the Hopf projection (1.5.2). In Section 1.3 we set up the deformation theory of monopoles: We introduced the deformation complex (1.3.4) and the Dirac operator $D = d_1^* \oplus d_2$ (1.3.3), where d_1 and d_2 are the linearisation of the action of gauge transformations and the Bogomolny equation, respectively. Similarly, the deformation theory of an ASD connection $\hat{A}$ is governed by a Dirac operator

$$\hat{D} := 2d^+_{\hat{A}} \oplus d^*_{\hat{A}} : \Omega^1(B^4; \hat{V}) \to \Omega^+(B^4; \hat{V}) \oplus \Omega^0(B^4; \hat{V}), \tag{3.1.6}$$

where Ω^+ denotes the space of self-dual forms.

In the definition below we use the notation $B^* = B^3 \setminus \{0\}$, norms are taken with respect to the flat metric both on B^3 and B^4 and h is the harmonic function used to express the flat metric on $\mathbb{R}^4$ in Gibbons–Hawking coordinates.

Definition 3.1.3. (i) If $u \in \Omega^0(B^*; V)$ and $\alpha \in \Omega^1(B^*; V)$ set $\hat{u} = \pi^*u$ and $\hat{\alpha} = \pi^*(*h\alpha) + \pi^*\alpha \wedge \theta_0$. Then $|u| = |\hat{u}|$ and $|\hat{\alpha}| = |\alpha|$.

(ii) If $\xi = (a, \psi) \in \Omega(B^*; V)$ define a 1–form $\hat{\xi}$ with values in $\hat{V}$ by:

$$\hat{\xi} = \pi^*a - \pi^*(h^{-1}\psi) \otimes \theta_0$$

Then $|\hat{\xi}|^2 = h^{-1}\left(|a|^2 + |\psi|^2\right)$.

(iii) With the identifications above, we have $\widehat{d_1 u} = d_{\hat{A}}\hat{u}$ and $\widehat{d_2^*\alpha} = d^*_{\hat{A}}\hat{\alpha}$.

(iv) Similarly,

$$d^*_{\hat{A}}\hat{\xi} = \pi^*(h^{-1}d_1^*\xi), \qquad 2d^+_{\hat{A}}\hat{\xi} = \pi^*(*d_2\xi) + \pi^*(h^{-1}d_2\xi) \wedge \theta_0.$$

In other words, under the identifications given by $\hat{\cdot}$, the Dirac operator $\hat{D}$ and its adjoint $\hat{D}^*$ correspond to $h^{-1}D$ and D^*, respectively. As in Lemma 1.3.1 there are Weitzenböck formulas for the Dirac operators $\hat{D}$ twisted by a connection $\hat{A}$.

Lemma 3.1.4 (*cf.* [42, Appendix C]). *Let $\hat{A}$ be a smooth connection on $\hat{V} \to B^4$, $\hat{u} \in \Omega^+(B^4;\hat{V}) \oplus \Omega^0(B^4;\hat{V})$ and $\hat{\xi} \in \Omega^1(B^4;\hat{V})$. Then*

$$\hat{D}\hat{D}^*\hat{u} = \nabla^*_{\hat{A}}\nabla_{\hat{A}}\hat{u} + F^+_{\hat{A}} \cdot \hat{u} \qquad \hat{D}^*\hat{D}\hat{\xi} = \nabla^*_{\hat{A}}\nabla_{\hat{A}}\hat{\xi} + F^-_{\hat{A}} \cdot \hat{\xi}.$$

In view of Definition 3.1.3, pushing them down to 3 dimensions, the standard 4–dimensional Sobolev norms yield weighted Sobolev norms adapted to the metric structure induced by the Hopf fibration on $B^3\backslash\{0\}$. To be specific, $(B^3 \setminus \{0\}, g, d\mu)$ is a metric measure space, *i.e.* $g = h\, g_{\mathbb{R}^3}$ is a Riemannian metric and $d\mu = h^{-\frac{1}{2}}\, \mathrm{dvol}_g = h\, \mathrm{dvol}_{\mathbb{R}^3}$ a measure. Then the weighted estimates deduced from standard elliptic estimates in 4–dimensions can be re-derived directly in 3–dimensions exploiting this structure. For example, from this (perverse) point of view one regards the standard Laplacian $\triangle$ on $\mathbb{R}^3$ as the φ–Laplacian $e^{-2\varphi}\triangle u = \triangle_\varphi u = \triangle_g u - \langle \nabla\varphi, \nabla u\rangle_g$, where φ is defined by $h = e^{2\varphi}$, and uses a Bochner formula in which the Bakry–Emery Ricci tensor [10] of $(B^3 \setminus \{0\}, g, d\mu)$ appears naturally.

There is in fact a theory of weighted Sobolev spaces which allows to work directly in 3–dimensions in a more sensible way, using a Dirac monopole as a background for the analysis. This is the route we will undertake. Some of its advantages are:

(i) In terms of decay at the puncture, we will work with stronger norms than the one obtained from $W^{k,p}_{\mathbb{S}^1}$–norms. Upstairs in 4 dimensions the stronger decay is forced by the $\mathbb{S}^1$–invariance, which is never really used in the estimates of [87]. For example, if g is a continuous gauge transformation then the $\mathbb{S}^1$–invariance forces $g(0)$ to lie in the $SO(2)$–subgroup of elements of $SO(3)$ which commute with the generator of the circle action.

(ii) Working directly in 3 dimensions we can take $k = p = 2$ because of the Sobolev embedding $W^{2,2} \hookrightarrow C^0$.

(iii) In the next chapter we will apply the analysis developed here to a gluing problem. Weighted Sobolev spaces yield uniform estimates for the inverse of the linearised operator.

Of course, our choice has the disadvantage of forcing us to work a bit harder to set up the definitions and develop the analysis, instead of relying on standard Sobolev theory. This is the content of the next section.

3.2 Monopoles with Dirac singularities and weighted Sobolev spaces

This section is entirely devoted to introduce the analytical tools necessary to study monopoles with Dirac type singularities directly in 3 dimensions. The models for our analysis are the work of Biquard [15, 16] on singular connections on punctured Riemann surfaces but also the work of Kronheimer–Mrowka [66] and Råde [89–91] on ASD connections with codimension 2 singularities. We will rely on Lockhart–McOwen's theory of Sobolev weighted spaces and elliptic operators on asymptotically cylindrical manifolds [76]. The Laplacian DD^* is a simple prototypical example of the operators to which this theory applies, but we will be able to derive all the results we need directly by elementary methods.

3.2.1 Function spaces for gauge theory

Definition 3.2.1. Let B^* be the punctured unit ball in X and $E \rightarrow B^*$ a Riemannian vector bundle endowed with a metric connection A. Given $\delta \in \mathbb{R}$ define the space $W^{m,p}_{\rho,\delta}$ as the closure of the space of sections $u \in C^\infty(B^*; E)$ vanishing in a neighbourhood of the origin with respect to the norm:

$$\|u\|^p_{W^{m,p}_{\rho,\delta}} = \sum_{j=0}^{m} \int \left|\rho^{-\delta-\frac{3}{p}+j}\nabla^j_A u\right|^p \mathrm{dvol}_{\mathbb{R}^3}$$

We will use the notation $L^p_{\rho,\delta}$ for $W^{0,p}_{\rho,\delta}$.

Remark 3.2.2. (i) $\rho^\beta \in L^p_{\rho,\delta}$ if and only if $\beta > \delta$.

(ii) Pass to the conformal cylinder $(0, +\infty) \times \mathbb{S}^2$ with metric

$$g_{cyl} = d\tau^2 + g_{\mathbb{S}^2} = \frac{d\rho^2}{\rho^2} + g_{\mathbb{S}^2},$$

where we set $\tau = -\log \rho$. Then $u \in W^{m,p}_{\rho,\delta}$ if and only if $e^{\delta\tau}u \in W^{m,p}_{cyl}$, where the last symbol denotes the standard Sobolev space defined with respect to the cylindrical metric.

The latter observation and the lemmas below are useful tools to work with these weighted spaces.

Lemma 3.2.3 (*cf.* [66, Lemma 3.1])**.** *If* $u \in W^{m,p}_{loc}(B^*)$ *and* $\|u\|_{W^{k,p}_{\rho,\delta}} < \infty$ *then* $u \in W^{k,p}_{\rho,\delta}$.

Proof. By standard arguments we can always assume that u is smooth. It is simple to verify that the sequence $\chi_n u$ converges to u in $W^{m,p}_{\rho,\delta}$, where $\chi_n(\cdot) = \chi(n\cdot)$ and χ is a smooth cut-off function with $\chi \equiv 0$ in a neighbourhood of the origin. $\square$

Lemma 3.2.4 (*cf.* [15, Theorem 1.2])**.** *For all* $\delta \neq 0$ *there exists* $C_\delta > 0$ *such that*

$$\|u\|_{W^{1,p}_{\rho,\delta}} \leq \frac{1}{|\delta|}\|\nabla_A u\|_{L^p_{\rho,\delta-1}}$$

for all $u \in C^\infty_0(B^*)$. *If* $\delta > 0$ *it is not necessary to require* $u \equiv 0$ *on* ∂B.

Proof. Suppose that u is a smooth section over B^* vanishing in a neighbourhood of the origin. We are going to show by an integration by parts that the norm of the radial derivative of u controls the norm of u. Since u vanishes in a neighbourhood of the origin:

$$\begin{aligned}\int_{B^*} \rho^{-\delta p-3}|u|^p \,\mathrm{dvol}_{\mathbb{R}^3} &= \int_0^1 \frac{d}{d\rho}\left(-\frac{\rho^{-\delta p}}{\delta p}\right)\left(\frac{1}{\rho^2}\int_{\partial B_\rho} |u|^p \rho^2 \,\mathrm{dvol}_{\mathbb{S}^2}\right) d\rho \\ &= -\frac{1}{\delta p}\int_{\partial B}|u|^p + \frac{1}{\delta}\int_{B^*} \rho^{-\delta p-2}|u|^{p-1}(\partial_\rho |u|)\,\mathrm{dvol}_{\mathbb{R}^3}\end{aligned}$$

The boundary term is non-positive if either $\delta > 0$ or $u|_{\partial B} \equiv 0$. Hence if one of these conditions is satisfied we have:

$$\int_{B^*} \rho^{-\delta p-3}|u|^p \,\mathrm{dvol}_{\mathbb{R}^3} \leq \frac{1}{|\delta|}\int_{B^*} \left(\rho^{-\delta p-3}|u|^p\right)^{\frac{p-1}{p}} \left(\rho^{-(\delta-1)p-3}|\nabla|u|\,|\right)^{\frac{1}{p}}$$

Conclude using Kato's inequality $|\nabla|u|\,| \leq |\nabla_A u|$ and Hölder's inequality. $\square$

Below we define spaces for gauge theory on the punctured ball modelled on the spaces $W^{m,2}_{\rho,\delta}$. Let V be the reducible $SO(3)$–bundle $V = \underline{\mathbb{R}} \oplus H^k \to B^*$ endowed with a pair $c = k(A^0, \Phi^0)\,\hat{\sigma}$ induced by an Euclidean Dirac monopole of charge k, mass 0 and singularity at the origin. For a V–valued form u we will write $u = u_D \oplus u_T$ in the decomposition of $V = \underline{\mathbb{R}} \oplus H^k$ into diagonal and off-diagonal part. We use covariant weighted $W^{m,2}_{\rho,\delta}$–norms for sections of V. Norms of V–valued differential forms are defined similarly by taking the $W^{m,2}_{\rho,\delta}$ norm of each component of the form.

Definition 3.2.5. Let $c = k(A^0, \Phi^0)\,\hat{\sigma}$ be a Dirac monopole on $V = \underline{\mathbb{R}} \oplus H^k \to B^*$ and fix $\delta > 0$.

(i) Define the gauge group $\mathcal{G}^0_\delta$ as the set of automorphisms g of V such that $(d_1 g)g^{-1} \in L^2_{\rho,\delta-1}$ and $\nabla^2_A g \in L^2_{\rho,\delta-2}$.

(ii) Define $\mathcal{C}^0_\delta$ as the space of configurations $c+(a,\psi)$ on V with $(a,\psi) \in W^{1,2}_{\rho,\delta-1}$.

(iii) Define a space $\widetilde{W}^{2,2}_{\rho,\delta}$ of infinitesimal gauge transformations as

$$\widetilde{W}^{2,2}_{\rho,\delta} = \left\{(u_D, u_T) \in L^2_{\rho,-\delta} \oplus L^2_{\rho,\delta} \,|\, \nabla_A u \in L^2_{\rho,\delta-1}, \nabla^2_A u \in L^2_{\rho,\delta-2}\right\}.$$

The fact that $\mathcal{G}^0_\delta$ is a group, at the moment unjustified, is Corollary 3.2.8.(a) below.

Remark. 1. The superscript ^{0} indicates that we work close to one of the singularities and has to be understood in opposition to the superscript $^\infty$ used to denote objects defined on the big end of X^*.

2. By the definition of d_1, $g \in \mathcal{G}^0_\delta$ satisfies $\nabla_A g \in L^2_{\rho,\delta-1}$, $(g\Phi g^{-1} - \Phi) \in L^2_{\rho,\delta-1}$ and $\nabla^2_A g \in L^2_{\rho,\delta-2}$. By Lemma 3.2.6 below g is continuous and has a well-defined limit over $0 \in B$; the condition $(g\Phi g^{-1} - \Phi) \in L^2_{\rho,\delta-1}$ forces this limiting value to lie in the stabiliser of Φ.

3. Since Φ acts by $-i\frac{k}{2\rho}$ on the off-diagonal component u_T and trivially on the diagonal u_D, $\widetilde{W}^{2,2}_{\rho,\delta}$ can be defined globally using the equivalent norm:

$$\|u\|_{\widetilde{W}^{2,2}_{\rho,\delta}} \sim \|u\|_{L^2_{\rho,-\delta}} + \|\nabla_A u\|_{L^2_{\rho,\delta-1}} + \|[\Phi, u]\|_{L^2_{\rho,\delta-1}} + \|\nabla^2_A u\|_{L^2_{\rho,\delta-2}}$$

4. Similarly, for a V–valued form $u \in \Omega(B^*; V)$ we define its $\widetilde{W}^{2,2}_{\rho,\delta}$–norm by

$$\|u\|^2_{\widetilde{W}^{2,2}_{\rho,\delta}} = \|u\|^2_{L^2_{\rho,-\delta}} + \|(\nabla_A u, [\Phi, u])\|^2_{L^2_{\rho,\delta-1}} + \|(\nabla_A(D^* u), [\Phi, D^* u])\|^2_{L^2_{\rho,\delta-2}}$$

If $u \in \Omega^0(B^*;V)$, $D^*(0,u) = -(d_A u, [\Phi, u])$ and, using the definition of Φ, this norm is equivalent to the one of Definition 3.2.5.(iii).

The following lemma helps to understand the definition of the space $\widetilde{W}^{2,2}_{\rho,\delta}$.

Lemma 3.2.6. *Fix $\delta > 0$. There are continuous embeddings $\widetilde{W}^{2,2}_{\rho,\delta} \hookrightarrow C^0$ and $W^{2,2}_{\rho,\delta} \hookrightarrow \rho^\delta C^0$. Moreover, $\|u - u(0)\|_{W^{2,2}_{\rho,\delta}} \leq C\|u\|_{\widetilde{W}^{2,2}_{\rho,\delta}}$ for all $u \in \widetilde{W}^{2,2}_{\rho,\delta}$.*

Proof. The first claim is proved in three steps:

1. By the Sobolev embedding in 3 dimensions and the assumption $\delta > 0$, if $u \in \widetilde{W}^{2,2}_{\rho,\delta}$ then $\rho^{-\delta+\frac{1}{2}}\nabla_A u \in L^p$ for all $2 \leq p \leq 6$.

2. If $\delta \geq \frac{1}{2}$ conclude immediately that $\nabla_A u \in L^p$ for all $2 \leq p \leq 6$. Otherwise, by Hölder's inequality $\nabla_A u \in L^p$ for all $3 < p < \frac{3}{1-\delta}$.

3. By Kato inequality and Morrey's estimate [48, Theorem 7.19] $u \in C^{0,\alpha}(B)$ for all $\alpha \in (0,\delta)$.

The second statement follows from Remark 3.2.2.(ii) and the Sobolev embedding with respect to the cylindrical metric, while the last claim is Lemma 3.2.4. □

Lemma 3.2.7. *Assume that all weighted spaces below are spaces of sections of V and the product on $V \simeq ad(P_V)$ is induced by the Lie bracket of $\mathfrak{su}_2$. If $\delta > 0$ the following are continuous maps:*

1. $W^{1,2}_{\rho,\delta-1} \hookrightarrow L^6_{\rho,\delta-1}$

2. $\widetilde{W}^{2,2}_{\rho,\delta} \hookrightarrow C^0(B)$

3. $\widetilde{W}^{2,2}_{\rho,\delta} \times L^2_{\rho,\delta-2} \to L^2_{\rho,\delta-2}$

4. $\widetilde{W}^{2,2}_{\rho,\delta} \times \widetilde{W}^{2,2}_{\rho,\delta} \to \widetilde{W}^{2,2}_{\rho,\delta}$

5. $\widetilde{W}^{2,2}_{\rho,\delta} \times W^{1,2}_{\rho,\delta-1} \to W^{1,2}_{\rho,\delta-1}$

6. $W^{1,2}_{\rho,\delta-1} \times W^{1,2}_{\rho,\delta-1} \to L^2_{\rho,\delta-2}$

In the last two cases the maps $\widetilde{W}^{2,2}_{\rho,\delta} \to W^{1,2}_{\rho,\delta-1}$ and $W^{1,2}_{\rho,\delta-1} \to L^2_{\rho,\delta-2}$ obtained by fixing the second factor are compact.

Proof. The embeddings 1 and 2 follow from the Sobolev embedding theorem with respect the cylindrical metric and Lemma 3.2.6, respectively.

The continuity of the products in 3–6 follows easily using the embeddings 1–2, Hölder's inequality and the assumption $\delta > 0$. The statement about the compactness of the maps induced by 5–6 also follows combining compactness properties of standard Sobolev embeddings together with 1–2. Some details follow.

We prove 6 first. Hölder's inequality implies

$$\|\xi \cdot \eta\|_{L^2_{\rho,\delta-2}} = \|\rho^{-\delta+\frac{1}{2}}(\xi \cdot \eta)\|_{L^2} \leq \|\rho^{-\delta+\frac{1}{2}}\xi\|_{L^6}\|\eta\|_{L^3} = \|\xi\|_{L^6_{\rho,\delta-1}}\|\eta\|_{L^3}$$

and similarly

$$\|\eta\|_{L^3} \leq \text{diam}(B)^{\delta}\|\eta\|^{\frac{1}{2}}_{L^6_{\rho,\delta-1}}\|\eta\|^{\frac{1}{2}}_{L^2_{\rho,\delta-1}}.$$

The continuity of the product $W^{1,2}_{\rho,\delta-1} \times W^{1,2}_{\rho,\delta-1} \to L^2_{\rho,\delta-2}$ now follows from the embedding in 1. The compactness of the induced map $W^{1,2}_{\rho,\delta-1} \to L^2_{\rho,\delta-2}$ is deduced by writing

$$\|\xi\cdot(\eta_i-\eta_{i'})\|_{L^2_{\rho,\delta-2}} \leq \|\xi\|_{L^6_{\rho,\delta-1}(B_\sigma)}\|\eta_i-\eta_{i'}\|_{L^3(B_\sigma)}+\|\xi\|_{L^6_{\rho,\delta-1}(B\setminus B_\sigma)}\|\eta_i-\eta_{i'}\|_{L^3(B\setminus B_\sigma)}$$

and using the fact that $\|\xi\|_{L^6_{\rho,\delta-1}(B_\sigma)} \to 0$ as $\sigma \to 0$ together with the compactness of the embedding $W^{1,2} \hookrightarrow L^3$.

In view of the embedding in 2, the continuity of the map in 3 is immediate. For the statement in 4, observe that in the decomposition $u = u_D + u_T$ the product takes the form:

$$(u_D + u_T) \cdot (v_D + v_T) = (u_T \cdot v_T) + (u_D \cdot v_T + u_T \cdot v_D)$$

Therefore $\|(u \cdot v)_D\|_{L^2_{\rho,-\delta}} \leq \|u\|_{L^\infty}\|v\|_{L^2_{\rho,-\delta}}$ and

$$\|(u \cdot v)_T\|_{L^2_{\rho,\delta}} \leq \sqrt{2}\|u_D\|_{L^\infty}\|v_T\|_{L^2_{\rho,\delta}} + \sqrt{2}\|v_D\|_{L^\infty}\|u_T\|_{L^2_{\rho,\delta}}.$$

The rest of the proof of 4 and 5 follows easily making use of 6.

To prove the compactness of the map $\widetilde{W}^{2,2}_{\rho,\delta} \to W^{1,2}_{\rho,\delta-1}$ induced by 5, write:

$$\begin{aligned}\|(u_i - u_{i'}) \cdot \xi\|_{W^{1,2}_{\rho,\delta-1}} \leq{}& \|u_i - u_{i'}\|_{C^0(B\setminus B_\sigma)}\|\xi\|_{W^{1,2}_{\rho,\delta-1}} + \|u_i - u_{i'}\|_{C^0}\|\xi\|_{W^{1,2}_{\rho,\delta-1}(B_\sigma)} \\ &+ \|\nabla_A(u_i - u_{i'}) \cdot \xi\|_{L^2_{\rho,\delta-2}}\end{aligned}$$

Now use the compactness of the embedding $W^{2,2} \hookrightarrow C^0$ on $B \setminus B_\sigma$, the fact that $\|\xi\|_{W^{1,2}_{\rho,\delta-1}(B_\sigma)} \to 0$ as $\sigma \to 0$ and the compactness of the map $W^{1,2}_{\rho,\delta-1} \to L^2_{\rho,\delta-2}$ induced by the multiplication by $\xi \in W^{1,2}_{\rho,\delta-1}$. □

Corollary 3.2.8. *For all* $\delta > 0$

(a) $\mathcal{G}^0_\delta$ *is a Banach Lie group which acts smoothly on* $\mathcal{C}^0_\delta$.

(b) The map

$$\Psi : (A, \Phi) + \xi \longmapsto \Psi(0) + d_2\xi + \xi \cdot \xi$$

is a smooth map $\xi \in W^{1,2}_{\rho,\delta-1} \longrightarrow L^2_{\delta-2}$.

In other words, the spaces $\mathcal{C}^0_\delta$ and $\mathcal{G}^0_\delta$ are well-suited to study gauge theory. Our next task is to develop a Fredholm theory for the deformation complex (1.3.4) using the weighted spaces $W^{m,p}_{\rho,\delta}$ just introduced. In the next section we are going to show that we can find a range of values for $\delta > 0$ such that the Laplacian DD^* (coupled to Dirichlet boundary conditions) is an isomorphism $DD^* \colon \widetilde{W}^{2,2}_{\rho,\delta} \to L^2_{\rho,\delta-2}$. This result will find its applications to the construction of the moduli space of periodic monopoles in Section 3.4.

3.2.2 Elliptic theory

We continue to work with the reducible pair $(A, \Phi) = k(A^0, \Phi^0)\,\hat{\sigma}$ given by an Euclidean Dirac monopole of charge k, zero mass and singularity at the origin. By changing variables to $\tau = -\log\rho$ the punctured ball $B^* = B_\sigma \setminus \{0\}$ becomes the half cylinder $Q = (T, +\infty) \times \mathbb{S}^2$, where $T = -\log\sigma$. The operator $\rho^2 DD^*$ has the form

$$\rho^2 DD^* u = -\ddot{u} + \dot{u} + Lu =: \mathcal{L}u \tag{3.2.1}$$

where the dots denote derivatives with respect to τ. L is the positive self-adjoint operator on $\mathbb{S}^2$ $L = \left(\triangle_{\mathbb{S}^2}, \nabla^*_A\nabla_A + \frac{k^2}{4}\right)$ in the decomposition $V = \underline{\mathbb{R}} \oplus H^k$. Here $\nabla^*_A\nabla_A$ is the Laplacian of the connection $A = kA^0$ on $H^k \to \mathbb{S}^2$.

$\mathcal{L}$ is a translation-invariant operator on the cylinder Q. In view of Remark 3.2.2.(ii), we want to study its mapping properties between weighted Sobolev spaces $\mathcal{L}\colon e^{-\delta\tau}W^{2,2}_{cyl} \to e^{-\delta\tau}L^2_{cyl}$. Lockhart–McOwen's theory [76] deals precisely with this kind of elliptic operators and their perturbations on cylinders and asymptotically cylindrical manifolds. By restricting to the special concrete case of (3.2.1) we will be able to give direct proofs of Lockhart–McOwen's results. In addition to the original [76], our references are Pacard [84], Pacard–Rivière [85] and Pacini [86].

Since we will study a boundary value problem, we introduce the appropriate spaces for the boundary data:

Definition 3.2.9. Let $\partial\widetilde{W}^{2,2}_{\rho,\delta}$ be the closure of $C^\infty(\partial B; V|_{\partial B})$ with respect to the norm

$$\|\varphi\|_{\partial\widetilde{W}^{2,2}_{\rho,\delta}} = \inf \|\tilde{\varphi}\|_{\widetilde{W}^{2,2}_{\rho,\delta}},$$

where the infimum is taken over all $\tilde{\varphi} \in C^\infty(B^*; V)$ such that $\tilde{\varphi}|_{\partial B} \equiv \varphi$.

We associate to the operator $\mathcal{L}$ of (3.2.1) a discrete set of weights, called *exceptional*, as follows. Since L is a self-adjoint positive operator its eigenvalues form a discrete sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ Moreover, we can select an orthonormal basis of $L^2(\mathbb{S}^2; \underline{\mathbb{R}} \oplus H^k)$ given by eigensections ϕ_j of L. Every solution to $\mathcal{L}u = 0$ can be written

$$u = \sum_{j=1}^{\infty} \left(A_j^+ e^{-\gamma_j^+ \tau} + A_j^- e^{-\gamma_j^- \tau} \right) \phi_j$$

where $\gamma_j^\pm$ are the two solutions to $\gamma^2 + \gamma - \lambda_j$, *i.e.* $\gamma_j^\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda_j}$. Define the set of exceptional weights of the operator $\mathcal{L}$ to be the collection $\mathcal{D}(\mathcal{L})$ of all $\gamma_j^\pm$, $j \geq 0$.

Theorem 3.2.10 (Lockhart–McOwen [76, Theorem 6.3])**.** *The operator*

$$e^{-\delta\tau} W^{2,2}_{cyl} \longrightarrow e^{-\delta\tau} L^2_{cyl} \oplus \partial W^{2,2}_{cyl},$$

defined by $u \longmapsto \mathcal{L}u \oplus u|_{\partial Q}$ *is Fredholm for all* $\delta \notin \mathcal{D}(\mathcal{L})$*. If* $\delta_1 < \delta_2$ *are both in the complement of* $\mathcal{D}(\mathcal{L})$ *then the difference of indices is* $i_{\delta_1} - i_{\delta_2} = \#\{\gamma \in (\delta_1, \delta_2) \cap \mathcal{D}(\mathcal{L})\}$*, where we count with multiplicities. Here the multiplicity of an element* $\gamma \in \mathcal{D}(\mathcal{L})$ *is the multiplicity of the corresponding eigenvalue of* L*.*

Moreover, the same result holds if $\mathcal{L}$ *is replaced by an operator whose coefficients decay in* C^0 *to those of* $\mathcal{L}$ *as* $\tau \to \infty$*.*

Here $\partial W^{2,2}_{cyl}(\partial Q)$ is defined in a way similar to Definition 3.2.9.

In the next lemma we calculate the set of exceptional weights $\mathcal{D}(\mathcal{L})$. Since we could always pick $\delta > 0$ sufficiently small, this wouldn't be strictly necessary: On the diagonal part $\mathcal{L}$ coincides with the scalar Laplacian; using $|\Phi| = \frac{k}{2\rho}$, on the off-diagonal part we know a priori that the first eigenvalue of $\nabla_A^*\nabla_A - \mathrm{ad}^2(\Phi)$ restricted to ∂B_ρ is greater than $\frac{k^2}{4}$.

Lemma 3.2.11. *The exceptional weights $\gamma_j^\pm \in \mathcal{D}(\mathcal{L})$ are:*

$$\begin{cases} \gamma_j^+ = j + \dfrac{|m|}{2} \\ \gamma_j^- = -j - 1 - \dfrac{|m|}{2} \end{cases}$$

for $j = 0, 1, 2, 3, \dots$ each with multiplicity $2j + |m| + 1$. Here we take $m = 0$ for the operator restricted to the diagonal component and $m = k$ when we restrict $\mathcal{L}$ to sections of H^k.

Proof. The eigenvalues of the Laplacian $\nabla_A^*\nabla_A$ of the $SO(3)$–invariant connection mA^0 on H^m have been calculated by Kuwabara [68, Theorem 5.1]. The computation is easily carried out in the framework of Section 3.1. Pulling back via the Hopf projection $\pi\colon\ \mathbb{S}^3 \to \mathbb{S}^2$ (1.5.2), regard sections of H^m as functions $f\colon\ \mathbb{S}^3 \to \mathbb{C}$ satisfying

$$f(e^{is}z_1, e^{-is}z_2) = e^{ims}f(z_1, z_2) \tag{3.2.2}$$

for all $(z_1, z_2) \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$ and $e^{is} \in \mathbb{S}^1$. Via this lift, the connection mA^0 is defined as $\nabla_X f = \widetilde{X} \cdot f$, where, for all tangent vector X to $\mathbb{S}^2$, $\widetilde{X}$ is the horizontal lift on $\mathbb{S}^3$ with respect to the standard connection on the Hopf bundle π. Since the round metric on $\mathbb{S}^3$ can be written as $g_{\mathbb{S}^3} = \frac{1}{4}\pi^* g_{\mathbb{S}^2} + \eta^2$, where η is the restriction of the 1–form (3.1.2) to the sphere,

$$\nabla_A^*\nabla_A = \frac{1}{4}\triangle_{\mathbb{S}^3} + \frac{1}{4}\xi^2,$$

where $\triangle_{\mathbb{S}^3}$ is the standard Laplacian on the 3–sphere and ξ is the Hopf vector field (*i.e.* ξ is the generator of the $\mathbb{S}^1$–action on $\mathbb{S}^3$ and is normalised so that $\eta(\xi) = 1$). Selecting the spherical harmonics that satisfy the $\mathbb{S}^1$–equivariance (3.2.2), Kuwabara proves that the eigenvalues of $\nabla_A^*\nabla_A$ are

$$\frac{l(l+2) - m^2}{4}, \qquad l = |m| + 2j, \text{ for } j = 0, 1, 2, \dots$$

each with multiplicity $l + 1$. Hence the eigenvalues of L are $\frac{l(l+2)}{4}$, where we take

$m = 0$ on the diagonal component and $m = k$ on the off-diagonal part. The Lemma follows. □

In particular, 0 is an exceptional weight with multiplicity 1 (the constant functions) for the operator $\mathcal{L}$ restricted to the diagonal part, while none of the weights in the interval $(-1 - \frac{|k|}{2}, \frac{|k|}{2})$ is exceptional for the operator restricted to the off-diagonal part.

Proposition 3.2.12. *Fix $0 < \delta < \min\{1, \frac{|k|}{2}\}$. The Dirichlet problem*

$$\begin{cases} \nabla_A^* \nabla_A u - ad^2(\Phi)u = f \\ u|_{\partial B} = \varphi \end{cases}$$

has a unique solution $u \in \widetilde{W}^{2,2}_{\rho,\delta}$ for all $f \in L^2_{\rho,\delta-2}$ and $\varphi \in \partial\widetilde{W}^{2,2}_{\rho,\delta}$. Moreover there exists a constant C independent of u, f, φ such that:

$$\|u\|_{\widetilde{W}^{2,2}_{\rho,\delta}} \leq C \left(\|f\|_{L^2_{\rho,\delta}} + \|\varphi\|_{\partial\widetilde{W}^{2,2}_{\rho,\delta}} \right)$$

Proof. The Proposition is a corollary of Theorem 3.2.10. We will nonetheless sketch a direct elementary proof.

Step 1. First we show that a weighted elliptic estimate holds, *i.e.* working on the half cylinder Q, we prove:

$$\|u\|_{e^{-\delta\tau}W^{2,2}_{cyl}} \leq C \left(\|\mathcal{L}u\|_{e^{-\delta\tau}L^2_{cyl}} + \|u|_{\partial Q}\|_{\partial W^{2,2}_{cyl}} + \|u\|_{e^{-\delta\tau}L^2_{cyl}} \right)$$

Choose $n_0 \in \mathbb{N}$ such that $n_0 - 2 > T > n_0 - 3$ and apply standard interior elliptic estimates in the region $(n-1, n+1) \times \mathbb{S}^2$ for $n \geq n_0$ and elliptic estimate close to the boundary in $(T, n_0) \times \mathbb{S}^2$:

$$\|u\|_{W^{2,2}_{cyl}(n-1,n+1)} \leq C \left(\|\mathcal{L}u\|_{L^2_{cyl}(n-2,n+2)} + \|u\|_{L^2_{cyl}(n-2,n+2)} \right)$$
$$\|u\|_{W^{2,2}_{cyl}(T,n_0)} \leq C \left(\|\mathcal{L}u\|_{L^2_{cyl}(T,n_0+1)} + \|u|_{\partial Q}\|_{\partial W^{2,2}_{cyl}} + \|u\|_{L^2_{cyl}(T,n_0+1)} \right)$$

Here $W^{m,p}_{cyl}(a,b)$ denotes the $W^{m,p}_{cyl}$–norm on the region $(a,b) \times \mathbb{S}^2$. Multiply the first inequality by $e^{\delta n}$, sum over $n \in \mathbb{N}_{\geq n_0}$ and add the resulting estimate to $e^{\delta n_0}$ times the second inequality.

Step 2. Given an operator $\mathcal{L}$ of the form (3.2.1) one can explicitly solve the boundary value problem in the statement of the Lemma, *cf.* for example Caffarelli–Hardt–Simon [25, Theorem 1.1] and Leon Simon [93, Part I, §5].

The proof is by separation of variables and elementary. Let $\{\gamma_j^{\pm}\}$ be the set of exceptional weights of the operator $\mathcal{L}$. Assuming that L is positive, $\gamma_j^+ \geq 0$ for all j. Suppose now that $\delta \notin \mathcal{D}(\mathcal{L})$ and $\delta > \gamma_1^-$ and let J be the integer such that $\gamma_J^+ < \delta < \gamma_{J+1}^+$. Denote by $\Pi_J\colon\ L^2(\partial Q; V|_{\partial Q}) \rightarrow L^2(\partial Q; V|_{\partial Q})$ the projection onto the sum of eigenspaces of L corresponding to eigenvalues λ_j for $j \geq J+1$.

We claim that given $f \in e^{-\delta\tau}L^2_{cyl}$ and $\varphi \in \partial W^{2,2}_{cyl}$ there exists a unique solution $u \in e^{-\delta\tau}L^2_{cyl}$ to

$$\begin{cases} \mathcal{L}u = f \\ \Pi_J u|_{\{\tau=T\}} = \Pi_J\varphi \end{cases}$$

and that there exists $C>0$ such that:

$$\|u\|_{e^{-\delta\tau}L^2_{cyl}} \leq C\left(\|f\|_{e^{-\delta\tau}L^2_{cyl}} + \|\varphi\|_{\partial W^{2,2}_{cyl}}\right)$$

Indeed, decompose u and f into Fourier modes with respect to the basis of eigenfunctions of L and then solve the resulting ODEs:

$$u_j = \alpha_j e^{-\gamma_j\tau} - e^{-\gamma_j\tau}\int_{\beta_j}^{\tau} e^{(2\gamma_j+1)s}\int_s^{+\infty} e^{-(\gamma_j+1)t} f_j(t)\, dt\, ds$$

where we set $\gamma_j = \gamma_j^+$. The constants of integration α_j, β_j have to be chosen so that the expression is well defined for $f \in e^{-\delta\tau}L^2_{cyl}$ and so that $u \in e^{-\delta\tau}L^2_{cyl}$:

$$\begin{cases} \alpha_j = 0,\ \beta_j = \infty & \text{if } j \leq J \\ \alpha_j = \langle\varphi,\varphi_j\rangle,\ \beta_j = T & \text{if } j \geq J+1 \end{cases}$$

The estimate now follows by a direct calculation, *cf.* [84, Proposition 6.2.1].

Step 3. Step 2 concludes the proof in the case of the off-diagonal part because the first exceptional weight is $\frac{|k|}{2}$. On the diagonal part, going back to the ball B

in $\mathbb{R}^3$, the solution $u \in W^{2,2}_{\rho,\delta}$ obtained in Step 2 satisfies

$$u|_{\partial B_\sigma} = \int_{B_\sigma} \left(\frac{1}{\rho} - \frac{1}{\sigma} \right) f + \varphi - \langle \varphi, 1 \rangle$$

Define $v = u - \int_{B_\sigma} \left(\frac{1}{\rho} - \frac{1}{\sigma} \right) f + \langle \varphi, 1 \rangle$. Then

$$\left| \int_{B_\sigma} \left(\frac{1}{\rho} - \frac{1}{\sigma} \right) f \right| \leq C\sigma^\delta \|f\|_{L^2_{\rho,\delta-2}}$$

and $v \in \widetilde{W}^{2,2}_{\rho,\delta}$. □

Remark. The argument in Step 3 also shows why the operator fails to be Fredholm when $\delta \in \mathcal{D}(\mathcal{L})$: The range is not closed. Indeed, we have just seen that if $u \in W^{2,2}_{\rho,\delta}$, $\delta \geq 0$, then

$$\langle u|_{\partial B_\sigma}, 1 \rangle = \int_{B_\sigma} \left(\frac{1}{\rho} - \frac{1}{\sigma} \right) \triangle u.$$

However, if $\delta = 0$ the RHS is not continuous on $L^2_{\rho,\delta-2}$.

Finally we introduce the possibility of considering a pair (A, Φ) of the form

$$(A, \Phi) = k(A^0, \Phi^0)\, \hat{\sigma} + (a, \psi),$$

where (A^0, Φ^0) is the Euclidean Dirac monopole of charge 1, 0 mass and singularity at the origin and $\xi = (a, \psi)$ is a lower order term in the sense specified below. We denote by $\mathcal{L}'$ the operator obtained as in (3.2.1) by substituting (A, Φ) to $k(A^0, \Phi^0)\, \hat{\sigma}$. If (A, Φ) is a solution to the Bogomolny equation then $\mathcal{L}' = \rho^2 DD^*$. We consider $\mathcal{L}'$ as a perturbation of the translational invariant operator $\mathcal{L}$.

Corollary 3.2.13. *Fix $0 < \delta < \min\{1, \frac{|k|}{2}\}$ and let $\mathcal{L}'$ be the operator* (3.2.1) *associated with the pair $(A, \Phi) = k(A^0, \Phi^0)\, \hat{\sigma} + \xi$ with $\xi \in W^{1,2}_{\rho,\delta-1}$. Then there exists $\sigma' < \sigma$ and C depending on ξ such that:*

$$\|u\|_{\widetilde{W}^{2,2}_{\rho,\delta}} \leq C \left(\|\rho^{-2}\mathcal{L}'u\|_{L^2_{\rho,\delta-2}} + \|u\|_{\widetilde{W}^{2,2}_{\rho,\delta}(B \setminus B_{\sigma'})} \right)$$

Moreover, the Dirichlet problem of Proposition 3.2.12 with $\mathcal{L}$ replaced by $\mathcal{L}'$ is a Fredholm operator $\widetilde{W}^{2,2}_{\rho,\delta} \to L^2_{\rho,\delta-2} \oplus \partial\widetilde{W}^{2,2}_{\rho,\delta}$ of index 0.

Proof. Observe that there exists a constant $C > 0$ such that:

$$\|\rho^{-2}(\mathcal{L}' - \mathcal{L})u\|_{L^2_{\rho,\delta-1}} \leq C\|\xi\|_{W^{1,2}_{\rho,\delta-1}}\|u\|_{\widetilde{W}^{2,2}_{\rho,\delta}}$$

Indeed the difference of the two operators has the form $\xi\cdot\nabla_A u + \nabla_A\xi\cdot u + \xi\cdot\xi\cdot u$ so the estimate follows from the continuity of $W^{1,2}_{\delta-1}\times W^{1,2}_{\delta-1}\to L^2_{\delta-2}$ and $\widetilde{W}^{2,2}_{\delta}\hookrightarrow C^0$ in Lemma 3.2.7.

Now choose σ' so that $\|\xi|_{B_{2\sigma'}}\|_{W^{1,2}_{\rho,\delta-1}}$ is sufficiently small and let χ be a cut-off functions supported in $B_{2\sigma'}$ and with $\chi\equiv 1$ on $B_{\sigma'}$. Apply the estimate of Proposition 3.2.12 to χu:

$$\|\chi u\|_{\widetilde{W}^{2,2}_{\rho,\delta}} \leq C\|\rho^{-2}\mathcal{L}(\chi u)\|_{L^2_{\rho,\delta-2}} \leq C\|\rho^{-2}\mathcal{L}'(\chi u)\|_{L^2_{\rho,\delta-2}} + C\|\xi|_{B_{2\sigma'}}\|_{W^{1,2}_{\rho,\delta-1}}\|\chi u\|_{\widetilde{W}^{2,2}_{\rho,\delta}}$$

and use the fact that

$$\|\rho^{-2}\mathcal{L}'(\chi u)\|_{L^2_{\rho,\delta-2}} \leq C\left(\|\rho^{-2}\mathcal{L}'u\|_{L^2_{\rho,\delta-2}} + \|u\|_{\widetilde{W}^{2,2}(B\setminus B_{\sigma'})}\right).$$

Finally, the last statement in the Corollary follows because $\rho^{-2}\mathcal{L}'$ differs from $\rho^{-2}\mathcal{L}$ by a compact operator by Lemma 3.2.7. □

Remark. The point of the estimate is of course that over $B\setminus B_{\sigma'}$ the weight function ρ is uniformly bounded above and below. If ξ is smooth, the $\widetilde{W}^{2,2}_{\rho,\delta}$–norm in the RHS of the estimate can be replaced by the L^2–norm on a slightly larger compact subset of B^* by standard elliptic regularity for the operator $\mathcal{L}'$.

3.3 Analysis on the big end of X^*

Having developed a local theory for monopoles with Dirac type singularities, we move on to discuss the framework to tackle the analysis on the big end of X^*. The local model is provided in this case by a periodic Dirac monopole, or better its asymptotic form analysed in Lemma 2.1.3 and 2.1.5: We work on the $SO(3)$–bundle $V = \underline{\mathbb{R}}\oplus(L_{v,b}\otimes L_q^{k_\infty})$ endowed with the reducible pair (A_∞,Φ_∞) induced by a periodic Dirac monopole of centre q, charge k_∞ and vacuum asymptotic parameters v, b. We will drop the subscript $_\infty$ for most of the section, hoping not to create too much confusion.

Fix $R > 0$ so that for $r \geq R$ we can write $|\Phi| = v + \frac{k_\infty}{2\pi}\log r + O(r^{-1})$. Hence we can find a constant $c = c(R, v, q) > 0$ such that

$$|\Phi| \geq c \qquad |d_A \Phi| \leq \frac{c}{r} \tag{3.3.1}$$

if $r \geq R$. For this, recall that we assume $v > 0$ if $k_\infty = 0$. Let U_R be the open exterior domain $\mathbb{R}^2 \setminus \overline{B}_R$; we will drop the subscript $_R$ when it is not essential in the discussion. If u is a section of V we write $u = u_D + u_T$ in the decomposition into diagonal and off-diagonal part. Then in the region $U \times \mathbb{S}^1$

$$|[\Phi, u]|^2 \geq c|u_T|^2. \tag{3.3.2}$$

By Fourier analysis with respect to the circle variable t we can further decompose $u_D = \Pi_0 u_D + \Pi_\perp u_D$ into $\mathbb{S}^1$–invariant and oscillatory part. On each circle $\{z\} \times \mathbb{S}^1_t$ the following Poincaré inequality holds:

$$\int_{\mathbb{S}^1} |\nabla(\Pi_\perp u_D)|^2 \geq \int_{\mathbb{S}^1} |\Pi_\perp u_D|^2 \tag{3.3.3}$$

The inequalities (3.3.2) and (3.3.3) suggest that, via the Weitzenböck formula Lemma 1.3.1, we have extremely good control of the off-diagonal and oscillatory piece of u in terms of $DD^* u$. On the other hand, analytic complications arise from the $\mathbb{S}^1$–invariant diagonal piece $\Pi_0 u_D$ because $\mathbb{R}^2$ is parabolic. To understand and deal with this issue we will introduce appropriate weighted spaces. Recall that we are looking for good Banach spaces in which to invert the operator $DD^* = \nabla_A^* \nabla_A - \mathrm{ad}^2(\Phi)$. On the diagonal part DD^* reduces to the scalar Laplacian. We motivate our choice of weighted spaces by studying the Dirichlet problem for the Laplacian on an exterior domain in $\mathbb{R}^2$.

3.3.1 The Laplacian on an exterior domain in $\mathbb{R}^2$

What follows combines the general framework of [76, 84–86] with results of Amrouche, Girault and Giroire [3], who study the Dirichlet and Neumann exterior problems for the Laplacian on $\mathbb{R}^n$ in the framework of weighted Sobolev spaces. From the latter article we retain in particular the use of a logarithmic factor in the weight function to deal with the exceptional weight $\delta = 0$.

Fix $R > 0$ and work on the exterior domain $U = U_R \subset \mathbb{R}^2$. Define weight functions

$$\omega(z) = \sqrt{2+r^2}, \qquad \hat{\omega}(z) = \omega(z)\log(2+r^2). \tag{3.3.4}$$

For future reference we spell out that ω satisfies

$$|\nabla\omega| \leq 1, \qquad -\omega\triangle\omega + |\nabla\omega|^2 = 2 \tag{3.3.5}$$

The reason to introduce the logarithmic factor in $\hat{\omega}$ is the following Poincaré inequality.

Lemma 3.3.1. *There exists a constant $C = C(R)$ such that*

$$\begin{aligned} \|\hat{\omega}^{-1}u\|_{L^2} &\leq C\|\nabla u\|_{L^2} \\ \|\omega^{-(\delta+1)}\, u\|_{L^2} &\leq \frac{C}{|\delta|}\|\omega^{-\delta}\,\nabla u\|_{L^2} \end{aligned}$$

for all $\delta \neq 0$ and all $u \in C_0^\infty\left(\overline{U}\right)$ subject to the additional restriction $u|_{\partial U} = 0$ if $\delta \geq 0$.

Proof. The estimates are analogous to Lemma 3.2.4. We prove only the first inequality; the second one follows by a similar integration by parts. If $\delta < 0$ the boundary term appearing in the integration by parts has the correct sign to give the inequality without assuming $u \equiv 0$ on ∂U.

It is enough to observe that $-2r\hat{\omega}^{-2} = \frac{d}{dr}\left(\frac{1}{\log(2+r^2)}\right)$. Then, integrating by parts and applying Hölder's inequality,

$$\begin{aligned} \int \hat{\omega}^{-2}u^2 &= -\frac{1}{2}\int d\left(\frac{1}{\log(2+r^2)}\right) \wedge \frac{u^2 * dr}{r} = \int \frac{u(\partial_r u)}{r\log(2+r^2)} \\ &\leq C\left(\int \hat{\omega}^{-2}u^2\right)^{1/2}\left(\int |\nabla u|^2\right)^{1/2} \end{aligned}$$

with $C = \frac{\sqrt{2+R^2}}{R}$. □

Definition 3.3.2. Given $\delta \in \mathbb{R}$ and $m \in \mathbb{N}$ we define Sobolev spaces $W^{m,2}_{\omega,\delta}$:

1. If $\delta \neq 0$ denote by $W^{m,2}_{\omega,\delta}(\mathbb{R}^2)$, $W^{m,2}_{\omega,\delta}(U)$, $\mathring{W}^{m,2}_{\omega,\delta}(U)$ the completion of $C_0^\infty(\mathbb{R}^2)$, $C_0^\infty\left(\overline{U}\right)$, $C_0^\infty(U)$ with respect to the norm

$$\|u\|^2_{W^{m,2}_{\omega,\delta}} = \sum_{j=0}^{m} \|\omega^{-\delta-1+j}\nabla^j u\|^2_{L^2}.$$

For simplicity we will write $L^2_{\omega,\delta}$ for $W^{0,2}_{\omega,\delta}$.

2. If $\delta = 0$ we make the same definition using the norm:

$$\|u\|^2_{W^{m,2}_{\omega,0}} = \|\hat{\omega}^{-1}u\|^2_{L^2} + \|\nabla u\|^2_{W^{m-1,2}_{\omega,-1}}$$

3. Finally, denote by $W^{-1,2}_{\omega}(U)$ the dual of $\mathring{W}^{1,2}_{\omega,0}(U)$.

Now, Lemma 3.3.1 can be stated: $\|\omega^{-\delta}\nabla u\|_{L^2}$ defines an equivalent norm on $\mathring{W}^{1,2}_{\omega,\delta}(U)$. The parabolicity of $\mathbb{R}^2$ comes into play when trying to extend this statement to $W^{1,2}_{\omega,\delta}(\mathbb{R}^2)$. Indeed, a complete Riemannian manifold admits a Hardy-type inequality

$$\int u^2\phi \leq \int |\nabla u|^2$$

for a strictly positive ϕ and all $u \in C^\infty_0$ if and only if it is non-parabolic (*cf.* [26] and references therein). In the parabolic case, it is necessary to restrict to functions of mean value zero.

Lemma 3.3.3 (Anrouche–Girault–Giroire [2, Corollary 8.4])**.** *For all $\delta > 0$ there exists $C = C_\delta > 0$ such that*

$$\int_{\mathbb{R}^2} |u|^2\omega^{-2(1+\delta)} \leq C\int_{\mathbb{R}^2} |\nabla u|^2\omega^{-2\delta}$$

for all $u \in C^\infty_0$ with $\int_{\mathbb{R}^2} u\,\omega^{-2(\delta+1)} = 0$.

Proof. We give an outline of the proof of Amrouche–Girault–Giroire. (The weighted space $W^{1,2}_{\alpha,\beta}$ of [2] coincides with our $W^{1,2}_{\omega,\delta}$ if $\alpha = -\delta$ and $\beta = 0$.)

Assume by contradiction that there exists a sequence $u_i \in W^{1,2}_{\omega,\delta}$ such that $\int_{\mathbb{R}^2} u_i\,\omega^{-2(\delta+1)} = 0$, $\|u_i\|_{W^{1,2}_{\omega,\delta}} = 1$ and $\|\omega^{-\delta}\nabla u\|_{L^2} \to 0$. Then $u_i \rightharpoonup 0$ in $W^{1,2}_{\omega,\delta}$ and in order to get a contradiction we need to show that the convergence actually occurs in the strong sense. This follows easily using a partition of unity subordinate to the cover $\mathbb{R}^2 = B_2 \cup (\mathbb{R}^2 \setminus B_1)$, the compactness of the embedding $W^{1,2}(K) \hookrightarrow L^2(K)$ for a compact set K and Lemma 3.3.1. Indeed if $\chi \in C^\infty_0(B_2)$ with $\chi \equiv 1$ on B_1, then on one side χu_i converges strongly to 0 by Rellich's compactness. On the other hand, by Lemma 3.3.1:

$$\|\omega^{-(1+\delta)}(1-\chi)u_i\|_{L^2} \leq C_\delta\|u_i\|_{L^2(B_2\setminus B_1)} + C_\delta\|\omega^{-\delta}\nabla u_i\|_{L^2}$$

where we used the fact that ω and $\nabla\chi$ are uniformly bounded on the compact region B_2. Rellich's compactness implies that the first term converges to 0 and, by the hypothesis $\|\omega^{-\delta}\nabla u_i\|_{L^2} \to 0$, we conclude that $\|\omega^{-(\delta+1)}u_i\|_{L^2} \to 0$ and therefore obtain a contradiction to $\|u_i\|_{W^{1,2}_{\omega,\delta}} = 1$. $\square$

Remark. An alternative proof in the case $\delta > \frac{1}{2}$ can be deduced from Bobkov–Ledoux [21, Theorem 3.1]. More generally, Hein [52, Theorem 1.2(i)] proves similar weighted Poincaré and Sobolev inequalities on complete manifolds with a polynomial volume growth condition.

This lemma will find application in the next chapter. For the moment return to the exterior domain U. Lemma 3.3.1 allows to solve the Dirichlet problem for the Laplacian on U in the Sobolev space $W^{2,2}_{\omega,0}$. The space $\partial W^{2,2}_{\omega,\delta}(\partial U)$ is defined as in Definition 3.2.9.

Lemma 3.3.4. *There exists $C > 0$ such that the following holds. Given $\varphi \in \partial W^{2,2}_{\omega,0}$ and $f \in W^{-1,2}_{\omega} \cap L^2_{\omega,-2}$ there exists a unique $u \in W^{2,2}_{\omega,0}$ satisfying*

$$\begin{cases} \triangle u = f & \text{in } U, \\ u = \varphi & \text{on } \partial U. \end{cases}$$

Moreover,

$$\|u\|_{W^{2,2}_{\omega,0}} \leq C\left(\|f\|_{W^{-1,2}_{\omega}} + \|f\|_{L^2_{\omega,-2}} + \|\varphi\|_{\partial W^{2,2}_{\omega,0}}\right).$$

Proof. The existence of a weak solution $u \in W^{1,2}_{\omega,0}$ with

$$\|u\|_{W^{1,2}_{\omega,0}} \leq C\left(\|f\|_{W^{-1,2}_{\omega}} + \|\varphi\|_{\partial W^{1,2}_{\omega,0}}\right)$$

follows by variational methods. Indeed, we can always reduce to vanishing boundary conditions by extending φ to $\tilde{\varphi} \in W^{1,2}$ with compact support in $\overline{U}$ and such that $\|\tilde{\varphi}\|_{W^{1,2}} \leq \|\varphi\|_{\partial W^{1,2}_{\omega,0}}$ and then replacing u with $u + \tilde{\varphi}$ and f with $f + \triangle\tilde{\varphi}$. Now exploit the first inequality of Lemma 3.3.1 to find a minimiser of the functional $\frac{1}{2}\int|\nabla u|^2 - \langle u, f\rangle$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the dual pairing between $\mathring{W}^{1,2}_{\omega,0}(U)$ and its dual.

In order to estimate the norm of the second derivative, combine the following a priori estimate with standard elliptic estimates in a neighbourhood of ∂U. Suppose that $u \in C^\infty_0(\mathbb{R}^2)$ and integrate ω^2 against the Bochner formula $-dd^*\left(\frac{1}{2}|\nabla u|^2\right) =$

$|\nabla^2 u|^2 + \langle \nabla u, \nabla \triangle u \rangle$. Integrations by parts yield

$$\int \omega^2 |\triangle u|^2 - \int \omega^2 |\nabla^2 u|^2 = 2 \int \omega(\triangle u) \langle \nabla u, \nabla \omega \rangle + \int \left(\omega \triangle \omega + |\nabla \omega|^2 \right) |\nabla u|^2$$

which in turn implies

$$\|\omega \nabla^2 u\|_{L^2} \leq C \left(\|\triangle u\|_{L^2_{\omega,-2}} + \|\nabla u\|_{L^2} \right).$$

by Hölder's inequality and (3.3.5). □

The use of the space of distributions $W^{-1,2}_{\omega}$ is inevitable: $L^2_{\omega,-2} \not\subset W^{-1,2}_{\omega}$ since $\delta = 0$ is an exceptional weight in the sense of Lockhart–McOwen's theory. In order to avoid the use of Sobolev spaces with a negative number of derivatives we can introduce weighted norms with a small $\delta < 0$.

Definition 3.3.5. We say that $u \in \widetilde{W}^{2,2}_{\omega,\delta}$ if $u \in L^2_{\omega,-\delta}$ and $\nabla u \in W^{1,2}_{\omega,\delta-1}$.

As in Lemma 3.2.6, the main point of this definition is to enlarge the space $W^{2,2}_{\omega,\delta}$, $\delta < 0$, with constant functions. The lemma below clarifies the reason for this choice.

Lemma 3.3.6. *For all $\delta \in (-1, 0)$ there exists a constant $C = C_\delta > 0$ such that the following holds. Given $f \in L^2_{\omega,\delta-2}$ and $\varphi \in \partial\widetilde{W}^{2,2}_{\omega,\delta}$ there exists a unique $u \in \widetilde{W}^{2,2}_{\omega,\delta}$ satisfying*

$$\begin{cases} \triangle u = f & \textit{in } U, \\ u = \varphi & \textit{on } \partial U. \end{cases}$$

Moreover

$$\|u\|_{\widetilde{W}^{2,2}_{\omega,\delta}} \leq C \left(\|f\|_{L^2_{\omega,\delta-2}} + \|\varphi\|_{\partial\widetilde{W}^{2,2}_{\omega,\delta}} \right).$$

Proof. The proof is similar to the one of Proposition 3.2.12. First one shows that

$$\|u\|_{W^{2,2}_{\omega,\delta}} \leq C \left(\|\triangle u\|_{L^2_{\omega,\delta-2}} + \|\varphi\|_{\partial\widetilde{W}^{2,2}_{\omega,\delta}} + \|u\|_{L^2_{\omega,\delta}} \right)$$

either by passing to the conformal cylinder $\mathbb{R} \times \mathbb{S}^1$ and arguing as in Step 1 of the proof of Proposition 3.2.12 by patching standard elliptic estimates over annuli $(n-2, n+2) \times \mathbb{S}^1$ or directly by integrations by parts as in Lemma 3.3.4.

In a second step one finds the unique solution $u \in L^2_{\omega,\delta}$ to

$$\begin{cases} \triangle u = f \\ u|_{\partial B_R} = \varphi + c \end{cases}$$

where $c \in \mathbb{R}$ depends on f and φ, together with the estimate

$$\|u\|_{L^2_{\omega,\delta}} \leq C \left(\|f\|_{L^2_{\omega,\delta-2}} + \|\varphi\|_{\partial W^{2,2}_{\omega,\delta}} \right)$$

This can be done by separation of variables as in Step 2 of the proof of Proposition 3.2.12.

Combining these first two steps one obtains a solution $u \in W^{2,2}_{\omega,\delta}$. It satisfies:

$$u|_{\partial B_R} = -\frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus B_R} f \log\left(\frac{r}{R}\right) + \varphi - \frac{1}{2\pi} \int_{\partial B_R} \varphi$$

The proof is completed as in Step 3 of the proof of Proposition 3.2.12 observing that the first term defines a bounded linear functional on $L^2_{\omega,\delta-2}$ whenever $\delta < 0$. □

3.3.2 Function spaces for gauge theory

With these analytical preliminaries in mind, we define weighted Sobolev spaces for gauge theory adapting Definition 3.3.2. We work in the setting specified at the beginning of the section: The reducible $SO(3)$–bundle $V = \underline{\mathbb{R}} \oplus (L_{v,b} \otimes L_q^{k_\infty})$ over the exterior domain $U \times \mathbb{S}^1$ endowed with the reducible pair $(A, \Phi) = (A_\infty, \Phi_\infty)$ induced by a periodic Dirac monopole. In particular we assume that (3.3.1) is satisfied for some $c > 0$.

Definition 3.3.7. Extend ω as an $\mathbb{S}^1$–invariant weight function on $\mathbb{R}^2 \times \mathbb{S}^1$. For a smooth V–valued form $u \in \Omega\left(U \times \mathbb{S}^1; V\right)$ and $\delta \in \mathbb{R}$ we define norms:

(i) $\|u\|_{L^2_{\omega,\delta}} = \|\omega^{-(\delta+1)} u\|_{L^2}$

(ii) $\|u\|^2_{W^{1,2}_{\omega,\delta}} = \int \omega^{-2\delta-2}|u|^2 + \omega^{-2\delta}\left(|\nabla_A u|^2 + |[\Phi, u]|^2\right)$

(iii) $\|u\|^2_{W^{2,2}_{\omega,\delta}} = \|u\|^2_{L^2_{\omega,\delta}} + \|(\nabla_A u, [\Phi, u])\|^2_{L^2_{\omega,\delta-1}} + \|(\nabla_A(D^*u), [\Phi, D^*u])\|^2_{L^2_{\omega,\delta-2}}$

(iv) $\|u\|^2_{\widetilde{W}^{2,2}_{\omega,\delta}} = \|u\|^2_{L^2_{\omega,-\delta}} + \|(\nabla_A u, [\Phi, u])\|^2_{L^2_{\omega,\delta-1}} + \|(\nabla_A(D^*u), [\Phi, D^*u])\|^2_{L^2_{\omega,\delta-2}}$

The corresponding weighted Sobolev spaces are defined as the closure of the space of smooth compactly supported forms with respect to these norms.

Remark. (i) Since (A,Φ) is a solution to the Bogomolny equation, the $W^{1,2}_{\omega,\delta}$–norm of a compactly supported form $u \in C^\infty_0(U\times\mathbb{S}^1)$ is equivalent to $\|u\|_{L^2_{\omega,\delta}} + \|D^*u\|_{L^2_{\omega,\delta-1}}$ by the Weitzenböck formula Lemma 1.3.1 for DD^*.

(ii) In view of (3.3.2) and (3.3.3), if $u \in W^{1,2}_{\omega,\delta}$ then $\Pi_\perp u_D, u_T \in L^2_{\omega,\delta-1}$.

(iii) In particular, the only difference between the spaces $W^{2,2}_{\omega,\delta}$ and $\widetilde{W}^{2,2}_{\omega,\delta}$ consists in the chosen weighted L^2–norm of $\Pi_0 u_D$. It follows from the proof of Lemma 3.3.9 below that when $\delta<0$ we have an extension

$$0 \to W^{2,2}_{\omega,\delta} \to \widetilde{W}^{2,2}_{\omega,\delta} \to \mathbb{R}\,\hat{\sigma} \to 0.$$

Definition 3.3.8. Fix $\delta<0$.

(i) $\mathcal{G}^\infty_\delta$ is the space of sections g of Aut(V) over the exterior domain $U\times\mathbb{S}^1$ such that $(d_1 g)g^{-1} \in W^{1,2}_{\omega,\delta-1}$.

(ii) $\mathcal{C}^\infty_\delta$ is the space of pairs (A,Φ) on V of the form $(A_\infty,\Phi_\infty)+(a,\psi)$, where $\xi=(a,\psi)$ is a section of $(\bigwedge^1\oplus\bigwedge^0)\otimes V$ of class $W^{1,2}_{\omega,\delta-1}$.

(iii) Infinitesimal gauge transformations are elements of $\widetilde{W}^{2,2}_{\omega,\delta-2}(U\times\mathbb{S}^1;V)$.

Remark. Our definitions essentially coincide with those adopted by Biquard–Jardim [18] to study doubly periodic (*i.e.* on $\mathbb{R}^2\times\mathbb{T}^2$) instantons with quadratic curvature decay.

Lemma 3.3.9. *Fix* $\delta\in(-1,0)$.

(i) If $\xi = \Pi_0\xi_D + \Pi_\perp\xi_D + \xi_T \in W^{1,2}_{\omega,\delta-1}$ *is a* V*–valued differential form then*

$$\omega^{-\delta}\Pi_0\xi_D,\, \omega^{-\delta+1}\Pi_\perp\xi_D,\, \omega^{-\delta+1}\xi_T \in L^p$$

for all $2\le p\le 6$ *and the inclusions are continuous.*

(ii) $\widetilde{W}^{2,2}_{\omega,\delta} \hookrightarrow C^0$ *is a continuous embedding.*

The following products are continuous:

(iii) $\widetilde{W}^{2,2}_{\omega,\delta} \times \widetilde{W}^{2,2}_{\omega,\delta} \to \widetilde{W}^{2,2}_{\omega,\delta}$

(iv) $\widetilde{W}^{2,2}_{\omega,\delta} \times W^{m,2}_{\omega,\delta-2+m} \to W^{m,2}_{\omega,\delta-2+m}$ *for* $m = 0, 1$

(v) $W^{1,2}_{\omega,\delta-1} \times W^{1,2}_{\omega,\delta-1} \to L^2_{\omega,\delta-2}$

Moreover, the maps $\widetilde{W}^{2,2}_{\delta} \to W^{m,2}_{\delta-2+m}$ *and* $W^{1,2}_{\omega,\delta-1} \to L^2_{\omega,\delta-2}$ *induced by (iv) and (v) by fixing the second argument are compact.*

Here the products are those induced by the Lie bracket on $\mathfrak{su}(2)$ *under the identification* $V \simeq \mathrm{ad}\, P$.

Proof. (i) It is a consequence of the Sobolev embedding theorem $W^{1,2} \hookrightarrow L^6$ in 3 dimensions and the fact that if $\xi \in W^{1,2}_{\omega,\delta-1}$ then $\omega^{-\delta+1}\Pi_\perp \xi_D, \omega^{-\delta+1}\xi_T \in L^2$.

(ii) For the oscillatory and off-diagonal part this is a consequence of the standard Sobolev embedding $W^{2,2} \hookrightarrow C^0$. In fact we have a bit more: $\omega^{-(\delta-1)}u \in W^{2,2}$ if $\Pi_0 u_D = 0$, so that $u \in \omega^{\delta-1}C^0$.

On the other hand, suppose that $u = \Pi_0 u_D$ so that we can work on $U \subset \mathbb{R}^2$. First of all we can replace ω with r because the two weights are equivalent (with a constant depending on R) on U. If $\nabla u \in W^{1,2}_{\omega,\delta-1}$, $r^{-\delta+1}\nabla u \in W^{1,2}_{cyl}$, where the latter is the standard Sobolev space with respect to the cylindrical metric $r^{-2}g_{\mathbb{R}^2}$. Thus $r^{-\delta+1}\nabla u \in L^p_{cyl}$ for all $p \in [2,\infty)$ by the standard Sobolev embedding. By an inversion $r = \frac{1}{\rho}$ we consider the function $\tilde{u}(\rho e^{i\theta}) = u(\rho^{-1}e^{i\theta})$ defined on a punctured ball $B_{1/R} \subset \mathbb{R}^2$: It is integrable because $u \in L^2_{\omega,-\delta}$ and $\delta > -1$ ($\delta > -2$ would be enough). Moreover, $\tilde{u}$ has gradient in L^p for all $p < \frac{2}{1+\delta}$. Since $\delta < 0$ we can choose $p > 2$ and apply Morrey's estimate [48, Theorem 7.19] to show that $\tilde{u}$, and therefore u, is continuous. In particular there exists a well-defined limit of $u_\infty = \lim_{r\to\infty} u(re^{i\theta})$ and, by Lemma 3.3.1, $u - u_\infty \in W^{2,2}_{\omega,\delta}$.

The rest of the Lemma now follows easily in a way similar to Lemma 3.2.7. It is crucial to observe that terms of the form $u_D \cdot v_D$ do not appear in the products. □

Corollary 3.3.10. *For all* $\delta \in (-1, 0)$, $\mathcal{G}^\infty_\delta$ *is a Banach Lie group acting smoothly on* $\mathcal{C}^\infty_\delta$.

The map $\Psi\colon \mathcal{C}^\infty_\delta \to L^2_{\omega,\delta-2}(U \times \mathbb{S}^1; \bigwedge^1 \otimes V); (A, \Phi) \mapsto *F_A - d_A\Phi$ *is smooth.*

3.3.3 Elliptic theory

As in Section 3.2.2, we want to study the equation $DD^*u = \nabla_A^*\nabla_A u - \mathrm{ad}(\Phi)^2 u = f$ for $f \in L^2_{\omega,\delta-2}$ and $u \in \widetilde{W}^{2,2}_{\omega,\delta}$ with $\delta < 0$ sufficiently close to 0. At first we continue to work with the pair (A_∞, Φ_∞) induced by a periodic Dirac monopole and extend to a perturbation of this asymptotic model in a second step.

Proposition 3.3.11. *There exists $-1 \leq \delta_0 < 0$ and $R_0 > 0$ such that if either*

(i) $\delta \in (\delta_0, 0)$ and $R > 0$ is arbitrary or

(ii) $\delta \in (-1, 0)$ is arbitrary and $R \geq R_0$,

then the following holds. For all $f \in L^2_{\omega,\delta-2}$ and $\varphi \in \partial\widetilde{W}^{2,2}_{\omega,\delta}$ there exists a unique solution $u \in \widetilde{W}^{2,2}_{\omega,\delta}$ to the Dirichlet problem

$$\begin{cases} DD^*u = f & \text{in } U_R \times \mathbb{S}^1 \\ u = \varphi & \text{on } \partial U_R \times \mathbb{S}^1 \end{cases}$$

Moreover there exists a constant $C = C(\delta) > 0$ independent of u and f such that

$$\|u\|_{\widetilde{W}^{2,2}_{\omega,\delta}} \leq C \left(\|f\|_{L^2_{\omega,\delta-2}} + \|\varphi\|_{\partial\widetilde{W}^{2,2}_{\omega,\delta}} \right).$$

Proof. By Lemma 3.3.6 the Proposition holds for all $\delta \in (-1, 0)$ and $R > 0$ if $f = \Pi_0 f_D$. Hence we assume that $\Pi_0 f_D = 0 = \Pi_0 u_D$. Then (3.3.2) and (3.3.3) imply the existence of a constant $c > 0$ such that

$$c \int_{\mathbb{S}^1} |u|^2 \leq \int_{\mathbb{S}^1} |\nabla_A u|^2 + |[\Phi, u]|^2. \tag{3.3.6}$$

Since $L^2_{\omega,\delta-2} \subset L^2$ we obtain a solution $u \in L^2$ to the Dirichlet problem by direct minimisation of the functional $\frac{1}{2}\int |\nabla_A u|^2 + |[\Phi, u]|^2 - \int \langle u, f\rangle$. We want to show that actually $u \in \widetilde{W}^{2,2}_{\omega,\delta}$.

Step 1. We can always reduce to the case $\varphi = 0$ by extending φ to $\widetilde{\varphi} \in \widetilde{W}^{2,2}_{\omega,\delta}$ such that $\|\widetilde{\varphi}\|_{\widetilde{W}^{2,2}_{\omega,\delta}} \leq \|\varphi\|_{\partial\widetilde{W}^{2,2}_{\omega,\delta}}$ and replacing u with $u - \widetilde{\varphi}$ and f with $f - DD^*\widetilde{\varphi}$. The estimates can now be proved by integration by parts.

Step 2. Since u vanishes on the boundary a first integration by parts yields (all integrals are taken over $U_R \times \mathbb{S}^1$):

$$\int \omega^{-2\delta} \langle \nabla_A^* \nabla_A u - \mathrm{ad}^2(\Phi)u, u\rangle = \int \omega^{-2\delta} \left(|\nabla_A u|^2 + |[\Phi, u]|^2 \right) - 2\delta \int \omega^{-2\delta-1} \langle \nabla_A u, u \otimes d\omega \rangle$$

By Hölder's inequality, (3.3.5) and (3.3.6):

$$\left| 2\delta \int \omega^{-2\delta-1} \langle \nabla_A u, u \otimes d\omega\rangle \right| \leq C|\delta| \|\omega^{-1}\|_{L^\infty} \int \omega^{-2\delta} \left(|\nabla_A u|^2 + |[\Phi, u]|^2 \right)$$

Thus if $|\delta|$ is sufficiently small or if R is sufficiently large we deduce

$$\| \left(\nabla_A u, [\Phi, u] \right) \|_{L^2_{\omega,\delta-1}} \leq C \|DD^* u\|_{L^2_{\omega,\delta-2}}.$$

In other words, in view of (3.3.6) and the definition of D^*, we proved

$$\|u\|_{L^2} + \|\omega^{-\delta} D^* u\|_{L^2} \leq C \|DD^* u\|_{L^2_{\omega,\delta-2}}.$$

Step 3. Notice that if χ is a smooth function supported in a compact set $K \subset U_R \times \mathbb{S}^1$, then

$$\|DD^*(\chi u)\|_{L^2_{\omega,\delta-2}} \leq C \left(\|\nabla^2 \chi\|_{L^2} \|u\|_{L^2(K)} + \|\nabla \chi\|_{L^2} \|\nabla_A u\|_{L^2(K)} + \|DD^* u\|_{L^2_{\omega,\delta-2}} \right)$$

and similarly $\|\omega^{-\delta} D^*(\chi u)\|_{L^2} \leq C \left(\|\nabla \chi\|_{L^2} \|u\|_{L^2(K)} + \|\omega^{-\delta} D^* u\|_{L^2} \right)$.

Choose $\chi \in C^\infty$ with $\chi \equiv 1$ on $\{r \leq R+1\}$ and $\chi \equiv 0$ if $r \geq R+2$. Write $u = \chi u + (1-\chi)u$. By Step 2 and standard elliptic regularity close to the boundary (*cf.* for example [48, Theorem 8.12]),

$$\|\chi u\|_{\widetilde{W}^{2,2}_{\omega,\delta}} \leq C \|\chi u\|_{W^{2,2}} \leq C \left(\|DD^*(\chi u)\|_{L^2} + \|\chi u\|_{L^2} \right) \leq C \|f\|_{L^2_{\omega,\delta-2}}.$$

Hence we reduced to prove an estimate

$$\|(\nabla_A \xi, [\Phi, \xi])\|_{L^2_{\omega,\delta-2}} \leq C \left(\|D\xi\|_{L^2_{\omega,\delta-2}} + \|\xi\|_{L^2_{\omega,\delta-1}} \right)$$

for $\xi = D^*\big((1-\chi)u\big)$, *i.e.* with ξ vanishing in a neighbourhood of $\partial U_R \times \mathbb{S}^1$.

Step 4. The Weitzenböck formula for D^*D in Lemma 1.3.1 implies

$$\frac{1}{2}d^*d\left(|\xi|^2\right) = -|\nabla_A\xi|^2 - |\Phi\xi|^2 + \langle D^*D\xi, \xi\rangle - 2\langle d_A\Phi\cdot\xi, \xi\rangle. \tag{3.3.7}$$

Integrate this Bochner-type identity against $\omega^{-2\delta+2}$ and integrate by parts:

$$\begin{aligned}\int \omega^{-2\delta+2}\left(|\nabla_A\xi|^2 + |[\Phi,\xi]|^2\right) \leq & \int \omega^{-2\delta+2}|D\xi|^2 - 2\int \omega^{-2\delta+2}\langle d_A\Phi\cdot\xi,\xi\rangle \\ & + 2(1-\delta)\int \omega^{-2\delta+1}\langle D\xi, d\omega\cdot\xi\rangle \\ & + 2(1-\delta)\int \omega^{-2\delta+1}\langle \nabla_A\xi, d\omega\otimes\xi\rangle \\ & + 2(1-\delta)^2\int \omega^{-2\delta}|\xi|^2\end{aligned} \tag{3.3.8}$$

Consider the term $\int \omega^{-2\delta+2}\langle d_A\Phi\cdot\xi,\xi\rangle$. Since (A,Φ) is reducible we can assume that $\xi = \xi_T$. Moreover, by (3.3.1) $\omega|d_A\Phi| \leq c$. Then Hölder's and Young's inequality with $\varepsilon > 0$ imply

$$\left|\int \omega^{-2\delta+2}\langle d_A\Phi\cdot\xi,\xi\rangle\right| \leq \frac{c}{\varepsilon_1}\int \omega^{-2\delta}|\xi|^2 + c\varepsilon_1\int \omega^{-2\delta+2}|\xi|^2$$

for any $\varepsilon_1 > 0$. Moreover, by (3.3.1)

$$c\varepsilon_1\int \omega^{-2\delta+2}|\xi|^2 \leq \varepsilon_1\int \omega^{-2\delta+2}|[\Phi,\xi]|^2.$$

Secondly, by Hölder's inequality

$$\left|\int \omega^{-2\delta+1}\langle D\xi, d\omega\cdot\xi\rangle\right| \leq \|\omega^{-\delta+1}D\xi\|_{L^2}\|\omega^{-\delta}\xi\|_{L^2} \leq \frac{1}{2}\|\omega^{-\delta+1}D\xi\|_{L^2}^2 + \frac{1}{2}\|\omega^{-\delta}\xi\|_{L^2}^2$$

because $|d\omega| \leq 1$ by (3.3.5).

Similarly, for any $\varepsilon_2 > 0$:

$$\begin{aligned}\left|\int \omega^{-2\delta+1}\langle \nabla_A\xi, d\omega\cdot\xi\rangle\right| & \leq \|\omega^{-\delta+1}\nabla_A\xi\|_{L^2}\|\omega^{-\delta}\xi\|_{L^2} \\ & \leq \varepsilon_2\|\omega^{-\delta+1}\nabla_A\xi\|_{L^2}^2 + \frac{1}{\varepsilon_2}\|\omega^{-\delta}\xi\|_{L^2}^2\end{aligned}$$

Now choose $\varepsilon_1, \varepsilon_2 < 1$ so that the appropriate terms can be absorbed in the LHS of (3.3.8) and obtain the required estimate. □

Remark 3.3.12. For later use, notice that the a priori estimate in Step 2 holds for any $\delta \in \mathbb{R}$.

We extend the result to perturbations of the asymptotic model as in Corollary 3.2.13.

Corollary 3.3.13. *Let* $(A_\infty, \Phi_\infty) + \xi \in \mathcal{C}^\infty_\delta$*, for some* $\delta \in (\delta_0, 0)$*, be a pair on* $V \to U \times \mathbb{S}^1$*. Here* $\delta_0 \in (-1, 0)$ *is such that Proposition 3.3.11 holds. Let* DD^* *be the second order elliptic operator associated with* (A, Φ)*. Then there exists* $R' > R$ *and* C *depending on* ξ *such that:*

$$\|u\|_{\widetilde{W}^{2,2}_{\omega,\delta}} \leq C\left(\|DD^* u\|_{L^2_{\omega,\delta-2}} + \|u\|_{W^{2,2}(R \leq r \leq R')}\right)$$

where $\|u\|_{W^{2,2}(R\leq r\leq R')}$ *denotes the* $\widetilde{W}^{2,2}_{\omega,\delta}$*–norm of* u *over the region* $(B_{R'} \setminus B_R) \times \mathbb{S}^1$ *where* ω *is uniformly bounded above and below.*

Moreover, the Dirichlet problem

$$DD^* \oplus \cdot|_{\partial U \times \mathbb{S}^1} \colon \widetilde{W}^{2,2}_{\omega,\delta} \longrightarrow L^2_{\omega,\delta-2} \oplus \partial\widetilde{W}^{2,2}_{\omega,\delta}$$

is a Fredholm map of index zero for all $\delta \in (\delta_0, 0)$*.*

Proof. The corollary is a consequence of the compactness of the maps induced by the products (iv)–(v) in Lemma 3.3.9. ☐

Remark. If $(A, \Phi) \in \mathcal{C}^\infty_\delta$ is a solution to the Bogomolny equation (or if $\Psi(A, \Phi)$ is sufficiently small in some appropriate sense), the Weitzenböck formula Lemma 1.3.1 implies that the Dirichlet problem

$$DD^* \oplus \cdot|_{\partial U \times \mathbb{S}^1} \colon \widetilde{W}^{2,2}_{\omega,\delta} \longrightarrow L^2_{\omega,\delta-2} \oplus \partial\widetilde{W}^{2,2}_{\omega,\delta}$$

is injective and therefore an isomorphism.

3.4 Construction of the moduli spaces

We are now in the position to apply the analysis developed in the previous sections to the construction of the moduli spaces of $SO(3)$ (and $U(2)$) periodic monopoles (with singularities). We begin by making precise definitions of the spaces of connections, Higgs fields and gauge transformations. We work out explicit definitions for the case of $SO(3)$ monopoles. Monopoles with structure group $U(2)$ are treated in a similar way.

- Fix a collection S of n distinct points $p_1, \ldots, p_n \in X$. Let $V \to X^* = X \setminus S$ be an $SO(3)$–bundle such that $w_2(V) \cdot [\mathbb{S}^2_{p_i}] = 1$. Denote by P the associated principal $SO(3)$–bundle.

  Choose parameters $k_\infty \in \mathbb{Z}_{\geq 0}$ with $k_\infty \equiv n \pmod 2$ and $v, b \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$, $q \in X$, with $v > 0$ if $k_\infty = 0$. Let $\mathcal{C} = \mathcal{C}(p_1, \ldots, p_n, k_\infty, v, b, q)$ be the space of smooth pairs of a connection and a Higgs field on V as in Definition 2.2.1.

- Fix a smooth pair $c = (A, \Phi) \in \mathcal{C}$. We will refer to c as the background pair. We can pick the approximate solution to the Bogomolny equation (1.2.1) that will be constructed in Section 4.4. It has the following properties:

  (i) There exists $\sigma > 0$ and a gauge over $B_\sigma(p_i) \setminus \{p_i\}$ such that (A, Φ) can be written as $(A^0, \Phi^0)\,\hat{\sigma} + (a, \psi)$, with $(a, \psi) = O(1)$, purely diagonal in the decomposition $V \simeq \underline{\mathbb{R}} \oplus H_{p_i}$ and such that (A, Φ) is an exact solution to (1.2.1) on $B_\sigma(p_i) \setminus \{p_i\}$.

  (ii) There exists $R > 0$ and a gauge over $U_R \times \mathbb{S}^1$ such that (A, Φ) can be written as $(A_\infty, \Phi_\infty) + (a, \psi)$, with $(a, \psi) = O(r^{-2})$, purely diagonal in the decomposition $V \simeq \underline{\mathbb{R}} \oplus (L_{v,b} \otimes L_q^{k_\infty})$ and such that (A, Φ) is an exact solution to (1.2.1) on $U_R \times \mathbb{S}^1$.

- Given such a pair $c = (A, \Phi)$ and chosen preferred gauges over $B_\sigma(p_i) \setminus \{p_i\}$ and $U_R \times \mathbb{S}^1$ as above, we use (A, Φ) as a background to define spaces $W^{m,2}_{\rho,\delta_1}$ and $W^{m,2}_{\omega,\delta_2}$ of forms with values in $V|_{B_\sigma(p_i)}$ and $V|_{U_R \times \mathbb{S}^1}$ as in Definitions 3.2.1 and 3.3.7.

  We say that a V–valued form $u \in L^2_{loc}$ on X^* belongs to the global weighted Sobolev space $L^2_{\delta_1,\delta_2}$ if, in the preferred gauges around each singularity and at infinity, $u|_{B_\sigma(p_i)} \in L^2_{\rho,\delta_1}$ and $u|_{U_R \times \mathbb{S}^1} \in L^2_{\omega,\delta_2}$. We define a norm on $L^2_{\delta_1,\delta_2}$ by taking the maximum of the semi-norms $\|u|_{B_\sigma(p_i)}\|_{L^2_{\rho,\delta_1}}$, $\|u|_{U_R \times \mathbb{S}^1}\|_{L^2_{\omega,\delta_2}}$ and $\|u|_{K_{\frac{\sigma}{2},2R}}\|_{L^2}$. Here $K_{\sigma,R} = \big(\overline{B}_R \times \mathbb{S}^1\big) \setminus \bigcup_{i=1}^n B_\sigma(p_i)$. The spaces $\widetilde{W}^{2,2}_{\delta_1,\delta_2}$, $W^{2,2}_{\delta_1,\delta_2}$ and $W^{1,2}_{\delta_1,\delta_2}$ are defined in a similar way.

- Fix $0 < \delta_1 < \frac{1}{2}$ and $\delta_0 < \delta_2 < 0$, where δ_0 is given by Proposition 3.3.11. Define $\mathcal{C}_{\delta_1,\delta_2}$ as the space of pairs of a connection and a Higgs field on V of the form $c+\xi$ with $\xi \in W^{1,2}_{\delta_1-1,\delta_2-1}$.

- The group $\mathcal{G}_{\delta_1,\delta_2}$ of gauge transformations is defined as the space of sections g of $P \times_{\mathrm{Ad}} SO(3)$ such that $c+(d_1 g)g^{-1} \in \mathcal{C}_{\delta_1,\delta_2}$. The Lie algebra of $\mathcal{G}_{\delta_1,\delta_2}$ is the space of sections of V of class $\widetilde{W}^{2,2}_{\delta_1,\delta_2}$.
- $(A,\Phi) \mapsto *F_A - d_A\Phi$ defines a smooth map $\Psi\colon\ \mathcal{C}_{\delta_1,\delta_2} \to L^2_{\delta_1-2,\delta_2-2}(X^*; \bigwedge^1 \otimes V)$.

The fact that $\mathcal{G}_{\delta_1,\delta_2}$ is a group of continuous gauge transformations acting smoothly on $\mathcal{C}_{\delta_1,\delta_2}$ and that Ψ is a smooth map follows from Corollaries 3.2.8 and 3.3.10.

3.4.1 Reducible pairs

We open a brief parenthesis to discuss reducibility of monopoles in the spaces just defined. Below we will show that the moduli space $\mathcal{M}_{\delta_1,\delta_2} = \Psi^{-1}(0)/\mathcal{G}_{\delta_1,\delta_2}$ is a smooth manifold in a neighbourhood of each irreducible pair (A,Φ).

Definition 3.4.1. A pair (A,Φ) is said to be *reducible* if $V \simeq \underline{\mathbb{R}} \oplus M$ for an $SO(2)$–bundle $M \to X^*$ and (A,Φ) is induced by an abelian monopole on M.

Equivalently, a pair (A,Φ) is irreducible if and only if there does not exist any non-trivial solution to $\nabla_A u = 0 = [\Phi, u]$.

It turns out that the existence of reducible monopoles in $\Psi^{-1}(0) \subset \mathcal{C}_{\delta_1,\delta_2}$ depends only on the parameters $p_1,\dots,p_n$, k_∞ and q used to define boundary conditions in Definition 2.2.1. Denote by $c_{v,b}$ the abelian flat monopole $(ib\,dt, v)$ and with c_p the periodic Dirac monopole of charge 1 with singularity at $p \in X$ as in Definition 2.1.2. Recall that we defined $k = \frac{k_\infty+n}{2} \in \mathbb{Z}_{\geq 0}$ as the non-abelian charge of the $SO(3)$–pair $(A,\Phi) \in \mathcal{C}_{\delta_1,\delta_2}$.

Lemma 3.4.2. *If $n<k$ every monopole in $\mathcal{C}_{\delta_1,\delta_2}$ is irreducible.*

If $n \geq k$, reducible monopoles in $\mathcal{C}_{\delta_1,\delta_2}$ are in one to one correspondence with subsets $\{p_{i_1},\dots,p_{i_k}\}$ of $S=\{p_1,\dots,p_n\}$ of cardinality k and such that $p_{i_1}+\ldots+p_{i_k} = \frac{1}{2}\left(\sum_{i=1}^n p_i + k_\infty q\right)$ in X. After reordering the p_i's if necessary, assume that $\{p_1,\dots,p_k\}$ satisfies this condition. Then the unique reducible monopole corresponding to this choice is

$$c_{v,b} + \sum_{i=1}^{k} c_{p_i} - \sum_{i=k+1}^{n} c_{p_i}.$$

Proof. If $c = (A, \Phi) \in \mathcal{C}_{\delta_1,\delta_2}$ is a reducible monopole then $\Phi = \varphi\,\hat{\sigma}$ for a harmonic function φ on X^* with prescribed behaviour at the punctures and at infinity. Here $\hat{\sigma}$ is the trivialising unit-norm section of the first factor in the decomposition $V \simeq \mathbb{R} \oplus M$. After possibly reordering the p_i's, φ is of the form $\varphi = v + \sum_{i=1}^{n'} G_{p_i} - \sum_{i=n'+1}^{n} G_{p_i}$ for some $0 \leq n' \leq n$.

To conclude, use Lemmas 2.1.3, 2.1.5 and 2.1.6 to compare the asymptotics of the sum of Dirac monopoles $c_{v,b} + \sum_{i=1}^{n'} c_{p_i} - \sum_{i=n'+1}^{n} c_{p_i}$ with the boundary conditions of Definition 2.2.1: $n' = k$ because the charge at infinity has to be $2k - n = k_\infty$ and $p_1 + \ldots + p_k = \frac{1}{2}\left(\sum_{i=1}^{n} p_i + k_\infty q\right)$ for the terms of order $\frac{1}{r}$ to coincide. □

Remark. A similar result holds for reducible $U(2)$–monopoles. In this case we say that a monopole (A, Φ) is reducible if $E \simeq L_1 \oplus L_2$ and (A, Φ) is induced by abelian monopoles on L_1 and L_2. Suppose that $n \geq k$ and $p_1 + \ldots + p_k = \frac{1}{2}\left(\sum_{i=1}^{n} p_i + k_\infty q\right)$ as above. Recall that when the structure group is taken to be $U(2)$ we defined $e_i \in \{\pm 1\}$ for each singularity $p_i \in S$ by $c_1(E) \cdot [\mathbb{S}^1_{p_i}] = e_i$. For each i define integers $e'_i, e''_i \in \{-1, 0, 1\}$ by

$$\begin{cases} e'_i = \dfrac{e_i + 1}{2} \\ e''_i = \dfrac{e_i - 1}{2} \end{cases} \text{if } i = 1, \ldots, k, \qquad \begin{cases} e'_i = \dfrac{e_i - 1}{2} \\ e''_i = \dfrac{e_i + 1}{2} \end{cases} \text{if } i = k+1, \ldots, n$$

Given parameters v_1, v_2, b_1, b_2 such that $v_1 - v_2 = v$ and $b_1 - b_2 = b$, one can show that

$$\left(c_{v_1,b_1} + \sum_{i=1}^{n} e'_i c_{p_i}\,,\; c_{v_2,b_2} + \sum_{i=1}^{n} e''_i c_{p_i} \right)$$

is the unique reducible $U(2)$–monopole with central part $c_{v_1+v_2,b_1+b_2} + \sum_{i=1}^{n} e_i c_{p_i}$ and trace-less part given by the reducible element of $\mathcal{C}_{\delta_1,\delta_2}$ corresponding to the choice of $\{p_1, \ldots, p_k\} \subset S$ in Lemma 3.4.2.

3.4.2 Fredholm theory for the deformation complex

Let $(A, \Phi) = c + \xi \in \Psi^{-1}(0) \subset \mathcal{C}_{\delta_1,\delta_2}$ be a solution to the Bogomolny equation. Our goal is to show that:

(i) The deformation complex $\Omega^0(X^*, V) \xrightarrow{d_1} \Omega(X^*, V) \xrightarrow{d_2} \Omega^1(X^*, V)$ defines a Fredholm complex $\widetilde{W}^{2,2}_{\delta_1,\delta_2} \to W^{1,2}_{\delta_1-1,\delta_2-1} \to L^2_{\delta_1-2,\delta_2-2}$.

(ii) If (A, Φ) is irreducible, *i.e.* d_1 is injective, then d_2 is surjective.

Remark. By Lemmas 3.2.7 and 3.3.9, weighted Sobolev norms defined using (A, Φ) as a background instead of c are equivalent (with constants depending on ξ).

We will need the following elliptic regularity result.

Lemma 3.4.3. *Let $(A, \Phi) = c + \xi \in \mathcal{C}_{\delta_1,\delta_2}$. Then there exists σ, R and C depending on ξ such that*

$$\|u\|_{\widetilde{W}^{2,2}_{\delta_1,\delta_2}} \leq C \left(\|DD^*u\|_{L^2_{\delta_1-2,\delta_2-2}} + \|u\|_{L^2(K_{\sigma,R})} \right)$$

for all $u \in \Omega(X^*; V)$.

Proof. By Corollaries 3.2.13 and 3.3.13 it is enough to show that for all compact sets $K' \subset K \subset X^*$, there exists $C = C(K, K', \xi)$ such that

$$\|u\|_{W^{2,2}(K')} \leq C \left(\|DD^*u\|_{L^2(K)} + \|u\|_{L^2(K)} \right).$$

Here $W^{2,2}$ is the unweighted covariant Sobolev norm

$$\|u\|^2_{W^{2,2}} = \|u\|^2_{L^2} + \|(\nabla_A u, [\Phi, u])\|^2_{L^2} + \|(\nabla_A(D^*u), [\Phi, D^*u])\|^2_{L^2}.$$

1. Let χ be a cut-off function supported in K and such that $\chi \equiv 1$ on K'. Using the Weitzenböck formula Lemma 1.3.1, we have

$$\int \chi^2 \left(|\nabla_A u|^2 + [\Phi, u]^2 \right) + 2 \int \chi \langle \nabla_A u, \nabla\chi \otimes u \rangle + \int \langle \Psi \cdot u, \chi^2 u \rangle$$
$$= \int \langle DD^*u, \chi^2 u \rangle \leq \|DD^*u\|_{L^2(K)} \|u\|_{L^2(K)}$$

where $\Psi = *F_A - d_A\Phi$. Now use Young's inequality with $\varepsilon > 0$ to estimate

$$\int \chi \langle \nabla_A u, \nabla\chi \otimes u \rangle \leq \varepsilon^2 \int \chi^2 |\nabla_A u|^2 + \frac{1}{\varepsilon^2} \|u\|^2_{L^2(K)}.$$

and, together with Hölder's inequality,

$$\int |\Psi|\, |\chi u|^2 \leq \|\Psi\|_{L^2(K)} \|\chi u\|^{\frac{1}{2}}_{L^2} \|\chi u\|^{\frac{3}{2}}_{L^6} \leq \varepsilon^2 \|\chi u\|^2_{L^6} + C_\varepsilon \|\Psi\|^4_{L^2(K)} \|\chi u\|^2_{L^2}.$$

The Sobolev embedding $W^{1,2} \hookrightarrow L^6$ now implies

$$\int |\Psi|\,|\chi u|^2 \leq \varepsilon^2 \|\chi \nabla_A u\|^2_{L^2} + C_\varepsilon (1 + \|\Psi\|^4_{L^2(K)}) \|u\|^2_{L^2(K)}$$

Choosing ε small enough we obtain

$$\|(\nabla_A u, [\Phi, u])\|^2_{L^2(K')} \leq C \|DD^* u\|^2_{L^2(K)} + C(1 + \|\Psi\|^4_{L^2(K)}) \|u\|^2_{L^2(K)}.$$

2. Pick $K'' \subset K'$ and set $\xi = \chi' D^* u$, where $\chi' \in C^\infty_0(K')$ and $\chi' \equiv 1$ on K''. Observe that

$$\|D\xi\|_{L^2} \leq C \left(\|D^* u\|_{L^2(K')} + \|DD^* u\|_{L^2(K')} \right)$$
$$\|\xi\|_{L^2} \leq \|D^* u\|_{L^2(K')} \leq C \left(\|DD^* u\|_{L^2(K)} + \|u\|_{L^2(K)} \right)$$

by Step 1. The Weitzenböck formula for D^*D, an integration by parts and arguments as in Step 1 yield

$$\|(\nabla_A (D^* u), [\Phi, D^* u])\|^2_{L^2(K'')} \leq C \left(\|DD^* u\|_{L^2(K)} + \|u\|_{L^2(K)} \right)$$

for a constant C depending on $\|d_A \Phi\|_{L^2(K)}$ and $\|\Psi\|_{L^2(K)}$.

To conclude, observe that, since $(A, \Phi) = c + \xi$, with c smooth and $\xi \in W^{1,2}_{\delta_1 - 1, \delta_2 - 1}$ (in particular $\xi \in W^{1,2}_{loc}$), $\|d_A \Phi\|_{L^2(K)}$ and $\|\Psi\|_{L^2(K)}$ are bounded in terms of K, the background pair c and $\|\xi\|_{W^{1,2}_{\delta_1 - 1, \delta_2 - 1}}$. □

Lemma 3.4.4. *The operator $DD^* \colon \widetilde{W}^{2,2}_{\delta_1, \delta_2} \to L^2_{\delta_1 - 2, \delta_2 - 2}$ is Fredholm. If (A, Φ) is irreducible then DD^* is an isomorphism.*

Proof. For all $\varepsilon > 0$ we can find $\sigma, R > 0$ such that $\|\xi|_{B_\sigma(p_i)}\|_{W^{1,2}_{\rho, \delta_1 - 1}} < \varepsilon$ and $\|\xi|_{U_R \times \mathbb{S}^1}\|_{W^{1,2}_{\omega, \delta_2 - 1}} < \varepsilon$. By choosing $\varepsilon > 0$ sufficiently small, Propositions 3.2.12 and 3.3.11 and the continuity of the products in Lemmas 3.2.7 and 3.3.9 imply that the Dirichlet problem for the operator DD^* on $B_\sigma(p_i)$ and $U_R \times \mathbb{S}^1$ is an isomorphism. Thus we obtain inverses of DD^* in a neighbourhood of the singularities and at infinity by solving Dirichlet problems with vanishing boundary conditions. The fact that DD^* is a Fredholm operator now follows by gluing these inverses with a parametrix on the compact set $K_{\sigma, R}$, *cf.* for example Råde's [89, Lemma 3.2].

To show that DD^* is an isomorphism if (A, Φ) is irreducible, we proceed in various steps.

Step 1. By the Weitzenböck formula Lemma 1.3.1, if (A, Φ) is irreducible than DD^* is injective. Indeed,

$$0 = \int \langle \nabla_A^* \nabla_A u - \mathrm{ad}^2(\Phi)u, u \rangle = \int |\nabla_A u|^2 + |[\Phi, u]|^2$$

If δ_1, δ_2 are in the ranges specified the integration by parts can be justified using a sequence of cut-off functions converging to 1. The Lemma follows if we prove that the index of DD^* vanishes.

Step 2. There exists a constant $C > 0$ such that $\|u\|_{\widetilde{W}^{2,2}_{\delta_1,\delta_2}} \leq C\|DD^*u\|_{L^2_{\delta_1-2,\delta_2-2}}$.

Suppose the claim is not true: There exists a sequence u_i with $\|u_i\|_{\widetilde{W}^{2,2}_{\delta_1,\delta_2}} = 1$ and $DD^*u_i \to 0$ in $L^2_{\delta_1-2,\delta_2-2}$. Now u_i is bounded in $W^{1,2}(K_{\sigma,R})$ for any $\sigma, R > 0$ and by Rellich's compactness there exists a converging subsequence $u_{i'}$ in $L^2(K_{\sigma,R})$. Then Lemma 3.4.3 implies that $u_{i'}$ is a Cauchy sequence in $\widetilde{W}^{2,2}_{\delta_1,\delta_2}$ and the limit contradicts the injectivity of DD^*.

Step 3. Let χ be a smooth cut-off function such that $\chi \equiv 1$ on $K_{\sigma,R}$ for some $\sigma, R > 0$ to be fixed later. Define a new pair $c' = (A', \Phi') = (A, \Phi) + (\chi - 1)\xi$, so that c' coincides with the background pair c on $B_{\frac{\sigma}{2}}(p_i)$ and on $U_{2R} \times \mathbb{S}^1$. The corresponding operator $D_{c'}D_{c'}^*$ is of the form $DD^* + T$, where

$$\|Tu\|_{L^2_{\delta_1-2,\delta_2-2}} \leq \|(\chi - 1)\xi\|_{W^{1,2}_{\delta_1-1,\delta_2-2}} \|u\|_{\widetilde{W}^{2,2}_{\delta_1,\delta_2}}$$

Choose σ, R so that $\|(\chi - 1)\xi\|_{W^{1,2}_{\delta_1-1,\delta_2-2}} < \varepsilon$ for small enough $\varepsilon > 0$.

By the compactness of the products in Lemmas 3.2.7 and 3.3.9 the index of $D_{c'}D_{c'}^*$ and DD^* coincide. We are going to show that $D_{c'}D_{c'}^*$ is an isomorphism.

Step 4. $D_{c'}D_{c'}^*$ is injective. Indeed, choose ε so that $C\varepsilon < 1$, where C is the constant in Step 2. Then if $D_{c'}D_{c'}^*u = DD^*u + Tu = 0$, Steps 2 and 3 imply $u = 0$.

Step 5. For notational convenience we are going to drop the subscript $_{c'}$ throughout this step. In order to prove that DD^* is surjective, start considering the map $DD^* \colon W^{2,2}_{\delta_1,\delta_2} \to L^2_{\delta_1-2,\delta_2-2}$. Since δ_1, δ_2 are non-exceptional weights for DD^*, standard theory in weighted Sobolev spaces implies that the cokernel of this map is identified with the kernel of DD^* in $L^2_{-\delta_1-1,-\delta_2}$ (*cf.* for example [84, Theorem 10.2.1]). Denote this finite dimensional space by $\ker(DD^*)_{-\delta_1-1,-\delta_2}$.

We claim that there is an injective map $\ker(DD^*)_{-\delta_1-1,-\delta_2} \to \mathbb{R}^{n+1}$. We need the following facts:

1. By elliptic regularity, if u is an element of $\ker(DD^*)_{-\delta_1-1,-\delta_2}$ then $u \in W^{2,2}_{-\delta_1-1,-\delta_2}$. The elliptic estimate in weighted Sobolev spaces that justifies the claim can be obtained by combining the a priori estimates in the proof of Proposition 3.2.12, Lemma 3.3.6, Proposition 3.3.11 and Lemma 3.4.3.
2. Solve Dirichlet problems on balls $B_\sigma(p_i)$ and on $U_R \times \mathbb{S}^1$ (for some small σ and large R) with u itself as boundary datum to show that

   $$u|_{B_\sigma(p_i)} = \frac{\lambda_i}{\rho}\,\hat{\sigma} + u'_i \qquad u|_{U_R\times\mathbb{S}^1} = \lambda_\infty(\log r)\,\hat{\sigma} + u'_\infty$$

   with $u'_i \in \widetilde{W}^{2,2}_{\rho,\delta_1}$ and $u'_\infty \in \widetilde{W}^{2,2}_{\omega,\delta_2}$. Here $\hat{\sigma}$ stands for the trivialising section of the diagonal factor in the decomposition $V \simeq \underline{\mathbb{R}} \oplus M$, with $M = H_{p_i}$ over $B_\sigma(p_i)$ and $M = L_{v,b} \otimes L_q^{k_\infty}$ on $U_R \times \mathbb{S}^1$.

   To show why this is true, choose σ and R so that (A', Φ') coincides with the model Dirac monopoles on $B_\sigma(p_i)$ and $U_R \times \mathbb{S}^1$. By Lemma 3.2.11 the growth of harmonic sections of $V|_{B_\sigma(p_i)}$ is explicitly known. Similarly, the growth of $\mathbb{S}^1$–invariant harmonic functions on the big end of X^* is known. On the other hand, by Remark 3.3.12, we can iterate the a priori estimate in the proof of Proposition 3.3.11: Since $\delta_2 > -1$, the oscillatory and off-diagonal components of a solution $u \in W^{2,2}_{-\delta_1-1,-\delta_2}$ to $DD^*u = 0$ decay very rapidly (as fast as any negative power of r, *cf.* Lemma 3.4.6 below).

Now define a map $\ker(DD^*)_{-\delta_1-1,-\delta_2} \to \mathbb{R}^{n+1}$ by $u \mapsto (\lambda_1, \ldots, \lambda_n, \lambda_\infty)$. Its injectivity follows from Step 4.

Finally, since $\widetilde{W}^{2,2}_{\delta_1,\delta_2}$ is an extension of $W^{2,2}_{\delta_1,\delta_2}$ by an $(n+1)$–dimensional space and $D_{c'}D^*_{c'}\colon \widetilde{W}^{2,2}_{\delta_1,\delta_2} \to L^2_{\delta_1-2,\delta_2-2}$ remains injective by Step 4, we conclude that $D_{c'}D^*_{c'}$ is an isomorphism. □

Standard theory [38, Chapter 4] now implies that

$$S_{(A,\Phi),\epsilon} = \Big\{(A,\Phi) + (a,\psi) \mid d_1^*(a,\psi) = 0,\ \|(a,\psi)\|_{W^{1,2}_{\delta_1-1,\delta_2-1}} < \epsilon\Big\}$$

is a local slice for the action of $\mathcal{G}_{\delta_1,\delta_2}$ on $\mathcal{C}_{\delta_1,\delta_2}$.

Lemma 3.4.5. *Let $(A,\Phi) \in \mathcal{C}_{\delta_1,\delta_2}$ be an irreducible solution to the Bogomolny equation. Then $D\colon W^{1,2}_{\delta_1-1,\delta_2-1} \to L^2_{\delta_1-2,\delta_2-2}$ is a Fredholm surjective operator.*

Proof. We make two preliminary observations. First, we fix $\sigma, R > 0$ as small, large as needed and deform (A,Φ) to $c' = (A',\Phi')$ so that it coincides with the model Dirac monopoles on $B_{2\sigma}(p_i)$ and $U_R \times \mathbb{S}^1$; by Lemmas 3.2.7 and 3.3.9 such a modification changes D by a compact operator. Secondly, we can combine the a priori estimates in the proof of Proposition 3.2.12, Lemma 3.3.6, Proposition 3.3.11 and Lemma 3.4.3 to obtain the elliptic estimate

$$\|\eta\|_{W^{1,2}_{\delta_1-1,\delta_2-1}} \leq C\left(\|D\eta\|_{L^2_{\delta_1-2,\delta_2-2}} + \|\eta\|_{L^2_{\delta_1-1,\delta_2-1}}\right) \tag{3.4.1}$$

for a constant C depending on ξ.

Now, in view of Step 4 and 5 in the proof of Lemma 3.4.4, in order to prove that $D = D_{c'}$ is Fredholm it remains only to show that $\ker D$ is finite dimensional. As in the proof of [84, Theorem 9.1.1] it is enough to show that there exists a compact set $K \subset X^*$ and a constant $C > 0$ such that

$$\|\eta\|_{L^2_{\delta_1-1,\delta_2-1}} \leq C\|\eta\|_{L^2(K)}$$

for all $\eta \in W^{1,2}_{\delta_1-1,\delta_2-1}$ with $D\eta = 0$. Indeed, assume the estimate holds and that $\ker D$ is infinite dimensional: We can find a sequence $\eta_j \in W^{1,2}_{\delta_1-1,\delta_2-1}$ such that $D\eta_j = 0$, $\int |\eta_j|^2 d\mu = 1$ and $\int \langle \eta_j, \eta_{j'}\rangle d\mu = 0$. Here $d\mu$ is a measure on X^* such that $d\mu = \rho_i^{-2\delta_1-1}\,\mathrm{dvol}$ on $B_\sigma(p_i)$, $d\mu = \omega^{-2\delta_2}\,\mathrm{dvol}$ on $U_R \times \mathbb{S}^1$ and $d\mu$ is uniformly equivalent to dvol on the compact set $K_{\sigma,R}$. In particular the $L^2_{\delta_1-1,\delta_2-1}$–norm is equivalent to the $L^2(X^*, d\mu)$–norm. (3.4.1) and Rellich's compactness guarantee the existence of a converging subsequence in $L^2(X^*, d\mu)$, hence a contradiction.

Fix a cut-off function χ such that $\chi \equiv 1$ in $X \setminus K_{\sigma,2R}$ and $\chi \equiv 0$ in $K_{2\sigma,R}$. Since (A',Φ') is reducible on the support of χ, for any η with $D\eta = 0$ we decompose

$\chi\eta = \chi\eta_D + \chi\eta_T$ and study separately the diagonal and the off-diagonal component.

1. Observe that $\triangle(\chi\eta_D) = DD^*(\chi\eta_D) = (\triangle\chi)\eta_D - 2\langle\nabla\chi, \nabla\eta_D\rangle$. Therefore, applying the interior elliptic estimate in Step 2 of the proof of Lemma 3.4.3, we have
$$\|\triangle(\chi\eta_D)\|_{L^2_{\delta_1-3,\delta_2-3}} \leq C\|\eta\|_{L^2(K_{\frac{\sigma}{4},4R})}.$$
Now, since $\delta_1 - 1$ and $\delta_2 - 1$ are non-exceptional weights for the Laplacian
$$\|\chi\eta_D\|_{L^2_{\delta_1-1,\delta_2-1}} \leq C\|\triangle(\chi\eta_D)\|_{L^2_{\delta_1-3,\delta_2-3}} \leq C\|\eta\|_{L^2(K_{\frac{\sigma}{4},4R})}$$
For a proof of the estimate we refer to [84, Proposition 6.2.1], where it follows combining integration by parts and separation of variables arguments as in the proof of Propositions 3.2.12 and 3.3.11.

2. In order to prove the estimate for the off-diagonal component on the exterior domain $U_R \times \mathbb{S}^1$, we exploit the Bochner formula (3.3.7). We showed in Step 2 of the proof of Proposition 3.3.11 that (3.3.7) implies
$$\int \omega^{-2\delta+2}\left(|\nabla_A\eta|^2 + |[\Phi,\eta]|^2\right) \leq C\left(\int \omega^{-2\delta+2}|D\eta|^2 + \int \omega^{-2\delta}|\eta|^2\right)$$
where we set $\eta = \chi\eta_T$ and $\delta = \delta_2$. The integrations by parts are justified because $\eta \in L^2_{\omega,\delta-1}$. Since $|[\Phi,\eta]| \geq c|\eta|$ by (3.3.1), we can choose R large enough so that $cR^2 > C$ and therefore
$$(cR^2 - C)\int \omega^{-2\delta}|\eta|^2 \leq C\int \omega^{-2\delta+2}|D\eta|^2.$$

3. Since (A', Φ') coincides with an Euclidean Dirac monopole of mass 0 on the ball $B_{2\sigma}(p_i)$, $d_A(\rho\Phi) = 0$ in this region. In particular, the Weitzenböck formulas of Lemma 1.3.1 imply that $D(\rho D^*\eta) = D^*(\rho D\eta)$, where $\eta = \chi\eta_T$. An integration by parts (justified because $\eta \in W^{1,2}_{\rho,\delta_1-1}$) yields
$$\int \rho^{-2\delta+1}|D^*\eta|^2 - \int \rho^{-2\delta+1}|D\eta|^2 = 2\delta\int \rho^{-2\delta}\langle D^*\eta - D\eta, d\rho\cdot\eta\rangle,$$
where we set $\delta = \delta_1$. Now use the algebraic identity $2[\Phi,\eta] = D\eta - D^*\eta$:
$$4|[\Phi,\eta]|^2 = |D^*\eta|^2 - |D\eta|^2 + 4\langle D\eta, [\Phi,\eta]\rangle$$

and therefore

$$\int \rho^{-2\delta+1}|[\Phi,\eta]|^2 = -\delta \int \rho^{-2\delta}\langle[\Phi,\eta], d\rho\cdot\eta\rangle + \int \rho^{-2\delta+1}\langle D\eta, [\Phi,\eta]\rangle.$$

Finally, by the Cauchy–Schwarz inequality

$$\int \rho^{-2\delta+1}|[\Phi,\eta]|^2 \leq \delta^2 \int \rho^{-2\delta-1}|\eta|^2 + \int \rho^{-2\delta+1}|D\eta|^2.$$

Conclude using $\delta < \frac{1}{2}$ and $|[\Phi,\eta]| = \frac{1}{2}\rho^{-1}|\eta|$ because $\eta = \chi\eta_T$.

Putting together the estimates in 1, 2 and 3 and using $D(\chi\eta) = d\chi\cdot\eta$, we proved

$$\|\eta|_{X^*\setminus K'}\|_{L^2_{\delta_1-1,\delta_2-1}} \leq C\|\eta\|_{L^2(K)}$$

for compact sets $K' \subset K \subset X^*$ and all $\eta \in W^{1,2}_{\delta_1-1,\delta_2-1}$ with $D\eta = 0$.

We conclude that D is a Fredholm operator; it is surjective by Lemma 3.4.4. □

In view of the two lemmas above and the discussion of irreducibility in Lemma 3.4.2, standard theory [38, Chapter 4] implies that $\mathcal{M}_{\delta_1,\delta_2} = \Psi^{-1}(0)/\mathcal{G}_{\delta_1,\delta_2}$ is a smooth manifold for generic choices of $p_1,\ldots,p_n,q \in X$.

More work is required to bring the L^2–metric into the discussion. We need the following preliminary lemma on the decay at infinity of monopoles in $\mathcal{C}_{\delta_1,\delta_2}$.

Lemma 3.4.6. *Let $(A,\Phi) = c+\xi \in \mathcal{C}_{\delta_1,\delta_2}$ be an irreducible solution to the Bogomolny equation. Then there exist $R>0$ and $g \in \mathcal{G}_{\delta_1,\delta_2}$ such that on $U_R\times\mathbb{S}^1$ $g(A,\Phi) = c+\xi'$ with $\xi' \in W^{1,2}_{\omega,\delta_2-1}$ and $\xi'_D = O(r^{\delta_2-1})$, $\xi'_T = O(r^\mu)$ for all $\mu\in\mathbb{R}$.*

Proof. The line of proof follows [17, Lemma 5.3].

Step 1. First we put (A,Φ) in "Coulomb gauge" with respect to the background c near infinity. Fix $R_0>0$ and a cut-off function $\chi_{R_0}\equiv 1$ on $B_{R_0}\times\mathbb{S}^1$ and $\chi_{R_0}\equiv 0$ on $U_{2R_0}\times\mathbb{S}^1$. Define a new pair $c' = (A',\Phi') = c+\chi_{R_0}\xi$. Then $c'\equiv c$ on $U_{2R_0}\times\mathbb{S}^1$. As in Lemma 3.4.4, we can choose R_0 sufficiently large so that $d_1^*d_1\colon \widetilde{W}^{2,2}_{\delta_1,\delta_2} \to L^2_{\delta_1-2,\delta_2-2}$ remains invertible. Here d_1 is the linearisation at c' of the action of $\mathcal{G}_{\delta_1,\delta_2}$ on $\mathcal{C}_{\delta_1,\delta_2}$.

Now for all $R>R_0$ consider the pair $c'+\xi_R$ defined by $\xi_R = (1-\chi_R)\xi$. Here χ_R is a cut-off function with the same properties of χ_{R_0} but with R in place of R_0. The Implicit Function Theorem implies that, choosing R large so that $\|\xi_R\|_{W^{1,2}_{\delta_1-1,\delta_2-1}}$ is sufficiently small, there exists $g \in \mathcal{G}_{\delta_1,\delta_2}$ such that $g(c'+\xi_R) = c'+\xi'$ with $\xi' \in W^{1,2}_{\delta_1-1,\delta_2-1}$ and $d_1^*\xi' = 0$.

Since $c'+\xi_R = c+\xi$ on $U_{2R}\times\mathbb{S}^1$, restricting to this exterior region, ξ' is a solution to $D\xi' + \xi'\cdot\xi' = 0$. Here D is the Dirac operator 1.3.3 twisted by the background pair c. Recall that c is a solution to the Bogomolny equation on $U_R\times\mathbb{S}^1$.

Step 2. Renaming $\xi = \xi'$, $c = (A,\Phi)$ and setting $\delta = \delta_2$, we reduced the problem to study the decay of solutions $\xi \in W^{1,2}_{\omega,\delta-1}$ to $D\xi = -\xi\cdot\xi$. We start by proving an initial decay $\xi = O(r^\delta)$ and then improve to the required rate.

Apply D^* to the equation and use the Weitzenböck formula Lemma 1.3.1 to derive the differential inequality

$$d^*d(|\xi|) \lesssim |d_A\Phi|\,|\xi| + \left(|\nabla_A\xi| + |[\Phi,\xi]|\right)|\xi|$$

This follows from the Bochner-type formula (3.3.7) and Kato's inequality. Hence $|\xi| \in W^{1,2}$ is a subsolution to

$$dd^*u \leq (A_1 + A_2)u$$

where $A_1 = |d_A\Phi| \in L^\infty$ and $A_2 = |\nabla_A\xi| + |[\Phi,\xi]| \in L^2$. Then Moser iteration on a 3–ball $B_1(p)$ centred at any point $p \in U_{3R}\times\mathbb{S}^1$ as in [48, Theorem 8.17] yields

$$\sup_{B_{\frac{1}{2}}(p)} |\xi| \leq C\|\xi\|_{L^2(B_1(p))} \leq Cr^\delta\|\xi\|_{L^2_{\omega,\delta-1}}$$

for a constant C depending on the L^∞–norm of A_1 and $\|A_2\|_{L^2}$. Here we used that $\omega\sim\omega(p)\sim r$ in $B_1(p)$. Hence $|\xi| \leq Cr^\delta$ on $U_{3R}\times\mathbb{S}^1$ for a constant C depending on the background c, R and $\|\xi\|_{W^{1,2}_{\omega,\delta-1}}$.

Step 3. Recall that the background pair c is abelian on $U_R\times\mathbb{S}^1$. We decompose $\xi = \xi_D + \xi_T$ into diagonal and off-diagonal part and exploit the fact that $\xi \in W^{1,2}_{\omega,\delta-1} \Rightarrow \omega^{-\delta+1}\xi_T \in W^{1,2}$ to improve the decay of ξ_T—first in an integral sense, then as a pointwise statement.

In order to justify the integrations by parts it is necessary to introduce a sequence of cut-off functions χ_i vanishing in a neighbourhood of infinity, such that $|d\chi_i| \leq \frac{2}{r}$ and converging to 1 as $i \to \infty$. Set $\xi_i = \chi_i \, \xi_T$; then $D\xi_i = d\chi_i \cdot \xi_T - \xi \cdot \xi_i$.

If $\xi_T \in W^{1,2}_{\omega,\mu-1}$ then $D\xi_i \in L^2_{\omega,\mu-2+\delta}$ since $\omega^{-\mu+1}\xi_i, \omega^{-\mu+1}\xi_T, \omega^{-\delta}\xi \in W^{1,2}$ and $\delta > -1$. Moreover, $\xi_i \in L^2_{\omega,\mu-1+\delta}$ because $\delta > -1$. The a priori estimate of Proposition 3.3.11 now implies $\xi_i \in W^{1,2}_{\omega,\mu-1+\delta}$—an improvement because $\delta < 0$. By iterating and letting $i \to \infty$, we conclude that $\xi_T \in W^{1,2}_{\omega,\mu-1}$ for all $\mu \in (-\infty, \delta]$.

Step 4. We repeat the argument of Step 2 with the equation $D\xi_T = -\xi \cdot \xi_T$. We have a differential inequality

$$d^*d(|\xi_T|) \lesssim |d_A\Phi| \, |\xi_T| + (|\nabla_A\xi| + |[\Phi,\xi]|) \, |\xi_T| + (|\nabla_A\xi_T| + |[\Phi,\xi_T]|) \, |\xi|$$

of the form $d^*du \lesssim A_1u + A_2u + f$, where $u = |\xi_T| \in W^{1,2}$, $A_1 = |d_A\Phi| \in L^\infty$, $A_2 = |\nabla_A\xi| + |[\Phi,\xi]| \in L^2$ and $f = (|\nabla_A\xi_T| + |[\Phi,\xi_T]|) \, |\xi| \lesssim |\nabla_A\xi_T| + |[\Phi,\xi_T]| \in L^2$ by Step 2. Moser iteration and Step 3 yield $|\xi_T| = O(r^\mu)$ on $U_{4R} \times \mathbb{S}^1$ for all $\mu \in \mathbb{R}$.

Step 5. The diagonal part $\xi_D \in W^{1,2}_{\omega,\delta-1}$ is a solution to the equation

$$\triangle\xi_D = D^*(\xi_T \cdot \xi_T) \in L^2_{\omega,\mu-2}$$

for all $\mu \in \mathbb{R}$. By elliptic regularity $\xi_D \in W^{2,2}_{\omega,\delta-1}$ and an argument analogous to the proof of Lemma 3.3.9.(ii) yields the weighted Sobolev embedding $W^{2,2}_{\omega,\delta-1} \hookrightarrow \omega^{\delta-1}C^0$. □

Remark 3.4.7. (i) In fact we can say a bit more: $|\xi_D| = O(r^{-2})$, the rate of decay of L^2_{ω,δ_2-1}–harmonic functions on X.

(ii) A simplified argument yields the same decay for solutions to $D\xi = 0$.

Theorem 3.4.8. *Choose data $v, b, k_\infty, p_1, \ldots, p_n, q$ defining the boundary conditions of Definition 2.2.1. Fix $\delta_1 \in (0, \frac{1}{2})$, $\delta_2 \in (\delta_0, 0)$ and suppose that the parameters $k_\infty, p_1, \ldots, p_n, q$ are chosen so that all pairs $(A,\Phi) \in \Psi^{-1}(0) \subset \mathcal{C}_{\delta_1,\delta_2}$ are irreducible. Then the moduli space $\mathcal{M}_{\delta_1,\delta_2}$ of $SO(3)$ periodic monopoles with* *non-abelian charge $k = \frac{k_\infty + n}{2}$, centre q and singularities at $p_1, \dots, p_n$ is a smooth manifold, provided it is non-empty.*

Moreover, the tangent space of $\mathcal{M}_{\delta_1,\delta_2}$ at a point $[(A,\Phi)]$ is identified with the L^2–kernel of D and the L^2–metric is a hyperkähler metric on $\mathcal{M}_{\delta_1,\delta_2}$.

Proof. In view of Lemma 3.4.5, only the last two statements need justification.

For the first, by (3.4.1) it is enough to prove that if $\xi \in L^2$ satisfies $D\xi = 0$ then $\xi \in L^2_{\delta_1-1,\delta_2-1}$.

(i) On a small ball $B_\sigma(p_i)$, let $\hat{\xi}$ be the lift of ξ to 4 dimensions as in Definition 3.1.3. Then $\hat{\xi}$ is a solution to $\hat{D}\hat{\xi} = 0$, where $\hat{D}$ is the Dirac operator twisted by the smooth connection $\hat{A}$ obtained from (A,Φ) as in (3.1.1). By elliptic regularity $|\hat{\xi}| = \sqrt{\rho}|\xi|$ is bounded.

(ii) Near infinity we use Lemma 3.4.6 to write $(A,\Phi) = c + \eta$ with $\eta = O(r^{\delta_2 - 1})$. Then ξ is a solution to $D\xi + \eta \cdot \xi = 0$, where D is the Dirac operator (1.3.3) twisted by the background pair c. It follows that $D\xi \in L^2_{\omega,\delta_2-2}$ on $U_R \times \mathbb{S}^1$ for some R large enough. By Proposition 3.3.11 we can write $\xi = \xi' + D^* u$, where $u \in \widetilde{W}^{2,2}_{\omega,\delta_2}$ and $\xi' \in L^2$ with $D\xi' = 0$. Since c coincides with the model periodic Dirac monopole on $U_R \times \mathbb{S}^1$, the diagonal component of ξ' is an L^2 harmonic function on X and therefore $\xi'_D = O(r^{-2})$. On the other hand, by Remark 3.4.7.(ii) $\xi'_T = O(r^\mu)$ for all $\mu \in \mathbb{R}$.

Finally, in order to prove that the L^2–metric is hyperkähler, in view of the discussion of Section 1.4.2 we need only check that the equality

$$\langle \xi, d_1 u \rangle_{L^2} = \langle d_1^* \xi, u \rangle_{L^2}$$

holds for $\xi \in W^{1,2}_{\delta_1-1,\delta_2-1}$ and $u \in \widetilde{W}^{2,2}_{\delta_1,\delta_2}$. This can be verified by using a sequence of cut-off functions on X^* converging to 1. □

The next open question is to compute the index of the Dirac operator D, *i.e.* the dimension of the moduli space. In [34] Cherkis and Kapustin argue that the dimension of the moduli space of $SO(3)$–monopoles with n singularities and non-abelian charge k is $4k - 4$. A proof using a Callias-type index theorem (*cf.* [63] where Kottke computes the dimension of the moduli space of monopoles without singularities on asymptotically conical 3–manifolds), the excision principle and the equivariant index theorem as in Pauly [87] seems feasible.

Conjecture 3.4.9. *The index of the Dirac operator D twisted by a periodic monopole with non-abelian charge k is $4k - 4$.*

Finally, motivated by physical considerations Cherkis and Kapustin [29] make the following conjecture.

Conjecture 3.4.10. *Choose $p_1, \ldots, p_n, q$ as in Theorem 3.4.8 so that $\mathcal{M}_{\delta_1,\delta_2}$ is a smooth manifold and assume it is non-empty. The L^2–metric on $\mathcal{M}_{\delta_1,\delta_2}$ is complete.*

Chapter 4

A gluing construction

4.1 Introduction

The goal of this chapter is to construct solutions (with singularities) to the Bogomolny equation on $X = \mathbb{R}^2 \times \mathbb{S}^1$ satisfying the boundary conditions of Definition 2.2.1. Following a suggestion of Biquard, we tackle this existence problem via gluing methods.

Monopoles on $\mathbb{R}^3$ (with structure group $SU(2)$ and without singularities) were themselves constructed via gluing methods in a seminal work by Taubes [60, Theorem 1.1 §IV.1]. On the other hand, Cherkis and Kapustin's physically-motivated computation of the asymptotics of the L^2–metric on the moduli space of periodic monopoles [33] is obtained by thinking of a charge k monopole as a superposition of particle-like charge 1 components.

The main steps and ingredients of Taubes's original gluing construction for Euclidean monopoles are:

- Charge 1 monopoles on $\mathbb{R}^3$ are completely explicit. Up to translations and scaling there exists a unique solution, localised around the origin in $\mathbb{R}^3$.
- Given k points far apart in $\mathbb{R}^3$, Taubes constructs an approximate solution to (1.2.1) patching together k charge 1 monopoles each localised around one of these points; the choice of gluing maps accounts for a further $k-1$ parameters.
- The approximate solution is deformed to a genuine monopole by an application of the implicit function theorem.

Taubes's existence theorem can be understood as a converse to a compactness statement for smooth monopoles in $\mathbb{R}^3$, *cf.* Atiyah–Hitchin [7, Proposition 3.8]. Uhlenbeck Compactness [105] yields a description of the end of the moduli space in terms of monopoles of charge k breaking into monopoles of lower charges receding from each other along definite directions. Taubes's result is a converse to this process close to the top dimensional stratum of the boundary where all monopole clusters have charge 1.

As periodic monopoles have infinite energy, no compactness result is readily available. In analogy with the Euclidean case, however, one can imagine that the end of the moduli space consists of charge k monopoles obtained as a non-linear superposition of well-separated monopoles of lower charges.

The first difficulty we will have to face is that not even charge 1 periodic monopoles are explicitly known. In fact, numerical experiments of Ward [107] show that a very different behaviour should be expected depending on the sign of the mass: When v is positive and large, charge 1 periodic monopoles are concentrated in an almost spherical region around their centre. When the mass is negative and large in absolute value, the monopoles are instead localised in a slab containing *two* maxima of the energy density. At the moment no clear interpretation of this phenomenon exists and no detailed quantitative estimates are available.

As a consequence, the construction of a charge k periodic monopole as a superposition of k charge 1 monopoles can be carried out only when the charge 1 constituents have large positive mass. There are two ways of arranging this. On one side one can consider periodic monopoles with large mass v. By scaling, the large mass limit $v \to +\infty$ is equivalent to the large radius limit $\mathbb{R}^2 \times \mathbb{R}/2\pi v\mathbb{Z} \to \mathbb{R}^3$. Observe that nothing is special to the case $X = \mathbb{R}^2 \times \mathbb{S}^1$ and it is conceivable that large mass monopoles exist on any 3–manifold satisfying appropriate conditions. More interestingly, the fact that the Green's function of X grows logarithmically at infinity is at the origin of a new phenomenon, that of "bubbling" at infinity. Here when using the term "bubbling" we have in mind the following phenomenon. In Section 3 we will see that a charge 1 $SU(2)$ monopole on $\mathbb{R}^3$ has a definite size, in the sense that half of its energy is concentrated in a ball of radius proportional to $\frac{1}{\lambda}$ around its centre. Here λ is the mass of the monopole and as $\lambda \to \infty$ the $SU(2)$ monopole converges to a Dirac one.

Finite energy solutions to the Bogomolny equation cannot "bubble" as a consequence of Uhlenbeck Compactness [105]: If (A, Φ) is a monopole on $\mathbb{R}^3$ then $\hat{A} = A - \Phi ds$ is ASD on $\mathbb{R}^4$. The L^2–norm of $F_{\hat{A}}$ on a set $B_r \times (-r, r)$ is controlled by $r \int_{B_r} |d_A\Phi|^2 \leq r \int_{\mathbb{R}^3} |d_A\Phi|^2$. Now if (A_i, Φ_i) is a sequence of monopoles with uniformly bounded energy, taking r sufficiently small depending only on the total energy of (A_i, Φ_i), Uhlenbeck's result yields the existence of a converging subsequence as in [38, Proposition 4.4.9].

When the energy is infinite, "bubbling" cannot be excluded so easily. On one hand, we expect that the L^2–metric on the moduli space of periodic monopoles is complete, *i.e.* a curve of *finite* length in the moduli space has a limit point. On the other hand, "bubbling" could still occur along a curve of *infinite* length.

In order to translate these heuristic expectations into an existence result, we proceed as follows. In Section 2 we fix the positions of the singularities $p_1, \dots, p_n$ and the parameters specifying the boundary conditions of Definition 2.2.1. Choosing k additional well-separated points $q_1, \dots, q_k$, we construct a reducible solution to the Bogomolny equation on X^* by taking a sum of periodic Dirac monopoles. This exact solution has the correct boundary behaviour but is singular at $q_1, \dots, q_k$. We want to think of $q_1, \dots, q_k$ as the centres of highly concentrated charge 1 monopoles. We will see that it is necessary to assume that either

(A) the mass v is sufficiently large, or

(B) when the number of singularities n is less than $2(k-1)$, $q_1, \dots, q_k$ are sufficiently far away from each other and from the singularities $p_1, \dots, p_n$.

In Section 3 we collect known facts about charge 1 Euclidean monopoles and in Section 4 we construct initial approximate solutions to the Bogomolny equation by gluing scaled Euclidean charge 1 monopoles in a neighbourhood of $q_1, \dots, q_k$ to resolve the singularities of the sum of periodic Dirac monopoles. By varying the centres and phases (thought of as fixing the choice of gluing maps) of the glued-in charge 1 monopoles, we obtain a $4(k-1)$–dimensional family of inequivalent approximate solutions satisfying the boundary conditions of Definition 2.2.1. In the final subsection we estimate the error and some of the geometric quantities (curvature, size of the Higgs field) of these approximate solutions.

The next step of the construction is to deform the initial approximate solutions into genuine monopoles by means of the Implicit Function Theorem. The crucial step is to study the linearised equation. A first difficulty arises from the fact that, if one fixes the boundary conditions (*i.e.* works with weighted Sobolev spaces forcing certain decay), there is a 3–dimensional space of obstructions to the solvability of the linearised equation. There are two ways to proceed:

(i) Enlarge the Banach spaces in which to solve the Bogomolny equation by allowing the appropriate changes of asymptotics;

(ii) Consider the centre of mass of the centres of the glued-in Euclidean charge 1 monopoles as a free parameter to be fixed only at the end of the construction.

We follow this second approach. In Section 5 we study the linearised equation separately for the building blocks, the charge 1 Euclidean monopoles and the sum of periodic Dirac monopoles. In the former case, there are no obstructions to the solvability of the linearised equation and the use of weighted Sobolev spaces allows to obtain uniform estimates for the norm of a right inverse. In the latter case, we can solve the linearised equation in the chosen weighted Sobolev spaces only modulo obstructions. Furthermore, for technical reasons we have to distinguish between case (A) and (B) above:

(A) When the points $q_1, \dots, q_k$ are contained in a fixed compact set of X and we assume that the mass v is sufficiently large, we can easily adapt the analysis of Chapter 3 and some care is only needed to check that the norm of the right inverse of the linearised operator remains uniformly bounded;

(B) When the points $q_1, \dots, q_k$ move off to infinity, instead, an additional technical difficulty arises from the following fact: It is well-known that for all $f \in C_0^\infty(\mathbb{R}^2)$ with mean value zero there exists a bounded solution u, unique up to the addition of a constant, to $\triangle u = f$ with $\|\nabla u\|_{L^2} < \infty$. However, if f is supported on the union of two balls $B_1(z_1) \cup B_1(z_2)$, say, with non-zero mean value on each of them, then $\|\nabla u\|^2_{L^2} \geq C \log|z_1 - z_2|$.

In Section 6 we patch together the local right inverses obtained in Section 5 and by a simple iteration solve the linearised equation globally modulo obstructions. Finally, in Section 7 we conclude the construction of a $4(k-1)$–parameter family of solutions to the Bogomolny equation satisfying the boundary conditions of Definition 2.2.1; conjecturally the construction yields an open set of the moduli space.

In the final Section 8, we return to the questions posed at the end of Chapter 1. We discuss the existence theorem obtained in Section 7 in the case of monopoles of low charge $k = 1$ and $k = 2$, explain how the gluing construction is a first step in the programme of understanding Cherkis and Kapustin's predictions and discuss briefly broader perspectives on related directions of future work.

Throughout the chapter we restrict to the case of periodic monopoles with structure group $SO(3)$, only indicating how to modify the construction when passing to structure group $U(2)$.

4.2 Sum of periodic Dirac monopoles

We are going to construct a singular reducible solution to the Bogomolny equation. It is determined by the following data.

- A *vacuum background*, *i.e.* constants $v \in \mathbb{R}$ and $b \in \mathbb{R}/\mathbb{Z}$ corresponding to the flat line bundle $L_{v,b}$ on X with constant Higgs field iv and flat connection $ib\, dt$.

- The *singularities*: A collection S of n distinct points $p_i = (m_i, a_i) \in X$ for $i = 1, \dots, n$. Set $X^* = X \setminus S$.

- The *centres of non-abelian monopoles*: Further k points, pair-wise distinct and distinct from the p_i's, which we denote by $q_j = (z_j, t_j)$ for $j = 1, \dots, k$.

We pick an origin in $\mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z}$ so that $z_1 + \ldots + z_k = 0$ and $t_1 + \ldots + t_k = 0$ modulo 2π. Let (μ, α) be defined by $\mu = m_1 + \ldots + m_n$ and $\alpha = a_1 + \ldots + a_n$ modulo 2π. Denote by d the *minimum distance*:

$$d = \min \{ |z_j - z_h|, |z_j - m_i| \text{ for all } j, h = 1, \dots k, j \neq h, \text{ and all } i = 1, \dots n \} \tag{4.2.1}$$

We assume that $d \geq 5$.

Notation. Throughout the chapter constants are allowed to depend on a lower bound for d and on the set S.

The Higgs field. Consider the harmonic function

$$\Phi_{\text{ext}} = v + 2 \sum_{j=1}^{k} G_{q_j} - \sum_{i=1}^{n} G_{p_i} \tag{4.2.2}$$

on $X^* \setminus \{q_1, \dots, q_k\}$, where G_p is the Green's function (2.1.1) of X with singularity at p. Notice that G_{q_j} and G_{p_i} appear in the sum with a different coefficient.

By Lemma 2.1.3 and 2.1.6 for large $|z|$

$$\Phi_{\text{ext}} = v + \frac{2k-n}{2\pi} \log|z| + \frac{1}{2\pi} \text{Re}\,\left(\frac{\mu}{z}\right) + O(|z|^{-2}), \tag{4.2.3}$$

while close to the singularity p_i

$$\Phi_{\text{ext}} = \text{const} + \frac{1}{2\rho_i} + O(\rho_i). \tag{4.2.4}$$

Here $\rho_i = \text{dist}(p_i, \cdot)$ and the constant term is defined by

$$\sum_{m \neq i} G_{p_m}(p_i) - \frac{a_0}{2} + v + \frac{1}{\pi} \sum_{j=1}^{k} \log|z_j - m_i| + O\left(e^{-d}\right). \tag{4.2.5}$$

Lemma 2.1.3 and 2.1.6 give similar expansions also for the derivatives of Φ_{ext}.

Finally, in the ball $B_{\frac{\pi}{2}}(q_j)$

$$\Phi_{\text{ext}} = \lambda_j - \frac{1}{\rho_j} + O\left(\frac{\rho_j}{d} + \rho_j^2\right) \tag{4.2.6}$$

where the constant λ_j is defined by:

$$\lambda_j = v + a_0 + \frac{1}{\pi} \sum_{h=1, h\neq j}^{k} \log|z_h - z_j| - \frac{1}{2\pi} \sum_{i=1}^{n} \log|m_i - z_j| + O\left(e^{-d}\right) \tag{4.2.7}$$

We will refer to λ_j as the *mass* attached to the point q_j.

Definition 4.2.1. Given $\lambda_0, K > 1$ and $d_0 \geq 5$ we say that $v \in \mathbb{R}$ and the points $p_1, \dots, p_n, q_1, \dots, q_k$ are (λ_0, d_0, K)–*admissible* if:

(i) The minimum distance d (4.2.1) satisfies $d \geq d_0$;

(ii) $\lambda := \min_j \lambda_j > \lambda_0$;

(iii) $\overline{\lambda} := \max_j \lambda_j \leq K\lambda$;

(iv) $v > 0$ if $n = 2k$.

Which sets of data are admissible? A first possibility is to fix the points $p_1, \dots, p_n$, $q_1, \dots, q_k$ so that (i) in Definition 4.2.1 is satisfied and then pick v sufficiently large.

For example, we will be able to construct charge $k = 1$ periodic monopoles with large mass $v > 0$.

More interestingly, consider the limit $d \to +\infty$ and assume that

$$\overline{d} = \max\{|z_j - z_h|, |z_j - m_i| \text{ for all } j, h = 1, \dots k, j \neq h, i = 1, \dots n\} \leq K'd$$

for some $K' > 1$. Then

$$\lambda_j \sim \frac{1}{\pi}\left(k - 1 - \frac{n}{2}\right)\log d \tag{4.2.8}$$

for all $j = 1, \dots, k$. Thus if $n < 2(k-1)$ we can fix $v, p_1, \dots, p_n$ *arbitrarily* and then pick the additional k points $q_1, \dots, q_k$ so that $d \geq d_0$ for d_0 sufficiently large. The limit $d \to \infty$ is interesting because, in analogy with Taubes's result [60] for Euclidean monopoles, we expect that it corresponds to (an open subset of) the end of the moduli space.

The next lemma states that the zeros of Φ_{ext} are localised around the points $q_1, \dots, q_k$ if λ_0 is sufficiently large.

Lemma 4.2.2. *Fix $d_0 = 5$ and suppose that $v > 1$ if $n = 2k$. There exists λ_0 such that the following holds.*

Suppose that the initial data are $(\lambda_0, 5, K)$–admissible. Then for all $j = 1, \dots, k$ there exists $\frac{1}{\lambda_j} < \delta(\lambda_j) \leq \frac{2}{\lambda_j}$ such that $\Phi_{ext} \geq 1$ on $X^* \setminus \bigcup_{j=1}^k B_{\delta(\lambda_j)}(q_j)$.

Proof. By Lemma 2.1.3 and 2.1.6 there exists a constant C such that in the annulus $\delta \leq \rho_j \leq \frac{\pi}{2}$

$$\Phi_{\text{ext}} \geq \lambda_j - \frac{1}{\delta} - C.$$

Define λ_0 by $\lambda_0 = 1 + \frac{2}{\pi} + C$ and

$$\delta = \delta(\lambda_j) = (\lambda_j - 1 - C)^{-1}.$$

By choosing λ_0 larger if necessary, we can assume that $\lambda_0 \geq 2(1 + C)$, so that $\delta(\lambda_j) \leq \frac{2}{\lambda_j}$.

Now if $\lambda_j > \lambda_0$, $\delta < \frac{\pi}{2}$ and $\Phi_{\text{ext}} \geq 1$ in the annulus $\delta \leq \rho_j \leq \frac{\pi}{2}$. On the other hand, $\Phi_{\text{ext}} \geq 1$ in a neighbourhood of the singularities $p_1, \dots, p_n$ and at infinity. Here we need to use the hypothesis $v > 1$ if $n = 2k$. The Lemma follows from the minimum principle. □

The bundle and connection. The form $*d\Phi_{\text{ext}}$ is the curvature of the line bundle $M \to X^* \setminus \{q_1, \dots, q_k\}$

$$M = L_{v,b} \otimes \bigotimes_{j=1}^{k} L_{q_j}^2 \otimes \bigotimes_{i=1}^{n} L_{p_i}^{-1}, \tag{4.2.9}$$

where L_p is the line bundle on $X \setminus \{p\}$ associated to the periodic Dirac monopole of Definition 2.1.2.

Consider the reducible $SO(3)$–bundle $\underline{\mathbb{R}} \oplus M$. Its isomorphism class is easy to calculate: $w_2(\underline{\mathbb{R}} \oplus M) = c_1(M)$ modulo 2, so that w_2 evaluated on the torus at infinity is $2k - n$ (mod 2), 1 on a small sphere enclosing one of the n singularities and it vanishes on spheres enclosing each of the k points.

As usual we denote by $\hat{\sigma}$ the trivialising section of the first factor in $\underline{\mathbb{R}} \oplus M$. Multiplying by $\hat{\sigma}$, Φ_{ext} (4.2.2) defines a Higgs field on $\underline{\mathbb{R}} \oplus M$. Fix a connection

$$A_{\text{ext}} = \left(b\, dt + 2 \sum_{j=1}^{k} A_{q_j} - \sum_{i=1}^{n} A_{p_i} \right) \hat{\sigma} \tag{4.2.10}$$

on $\underline{\mathbb{R}} \oplus M$, where A_p is the connection on L_p of Definition 2.1.2. Lemma 2.1.5 and 2.1.6 yield asymptotic expansions of representatives of A_{ext} close to p_i and q_j and at infinity similar to (4.2.3), (4.2.4) and (4.2.6). For example, as $|z| \to \infty$ A_{ext} is gauge equivalent to:

$$A_{\text{ext}} = \left((2k - n) A^\infty + b\, dt + \frac{1}{2\pi} \big(n\pi - \alpha \big) d\theta + \frac{1}{2\pi} \text{Im} \left(\frac{\mu}{z} \right) dt + O(|z|^{-2}) \right) \hat{\sigma}$$

Now, given data v, b, S as above we define boundary conditions as in Definition 2.2.1: Let V be the $SO(3)$–bundle on X^* with $w_2(V) \cdot [\mathbb{S}^2_{p_i}] = 1$ for each $p_i \in S$. Notice that $V \simeq \underline{\mathbb{R}} \oplus M$ over $X^* \setminus \{q_1, \dots, q_k\}$. Define the space $\mathcal{C} = \mathcal{C}(p_1, \dots, p_n, k_\infty, v, b, q)$ of smooth pairs of a connection and a Higgs field as in Definition 2.2.1, where $k_\infty = 2k - n$ and $k_\infty q = -(\mu, \alpha)$. In view of Lemma 3.4.2, we require there be no subset of S of length k with centre of mass at the origin. Thus there are no reducible monopoles in $\mathcal{C}$.

The reducible pair $c_{\text{ext}} := (A_{\text{ext}}, \Phi_{\text{ext}})$ satisfies the Bogomolny equation (1.2.1) and has the correct boundary behaviour at infinity and around points of S, meaning that it satisfies the same boundary conditions as pairs in $\mathcal{C}$. However, it has "conical singularities" at the additional k points $q_1, \dots, q_k$. Our next task is to desingularise

c_{ext} so to obtain a smooth configuration $c = (A, \Phi) \in \mathcal{C}$. There are no topological obstructions because $w_2(\mathbb{R} \oplus M)$ vanishes when evaluated on a small 2–sphere centred at the point q_j.

Remark. We can define a $U(2)$–bundle $M_1 \oplus M_2$ on $X^* \setminus \{q_1, \dots, q_k\}$ endowed with a reducible monopole in a similar way. For each $i = 1, \dots, n$ choose $e_i \in \{\pm 1\}$. By relabelling the p_i's we can assume that $e_i = -1$ for $i = 1, \dots, n_-$. Choose $v_1, v_2 \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}/\mathbb{Z}$ such that $v_1 - v_2 = v$ and $b_1 - b_2 = b$ modulo $\mathbb{Z}$. Set

$$M_1 = L_{v_1,b_1} \otimes \bigotimes_{j=1}^{k} L_{q_j} \otimes \bigotimes_{i=1}^{n_-} L_{p_i}^{-1} \qquad M_2 = L_{v_2,b_2} \otimes \bigotimes_{j=1}^{k} L_{q_j}^{-1} \otimes \bigotimes_{i=n_-+1}^{n} L_{p_i} \tag{4.2.11}$$

Since $M_1 \otimes M_2 = L_{v_1+v_2, b_1+b_2} \otimes \bigotimes_{i=1}^{n} L_{p_i}^{e_i}$ and $M_1 \otimes M_2^{-1} = M$, the reducible $SO(3)$–pair $(A_{\text{ext}}, \Phi_{\text{ext}})$ and

$$\left((b_1 + b_2)\, dt + \sum_{i=1}^{n} e_i A_{p_i}, v_1 + v_2 + \sum_{i=1}^{n} e_i G_{p_i} \right) \text{id} \tag{4.2.12}$$

define the trace-free and central part of a reducible monopole over $X^* \setminus \{q_1, \dots, q_k\}$.

4.3 Charge 1 monopoles on $\mathbb{R}^3$

In this section we introduce the "asymptotically conical" solutions to the Bogomolny equation (1.2.1) that will be used to resolve the singularities of $(A_{\text{ext}}, \Phi_{\text{ext}})$.

In 1975 Prasad and Sommerfield [88] found an explicit smooth finite energy solution to the Bogomolny equation on $\mathbb{R}^3$ with structure group $SU(2)$. By translations and scaling, this explicit solution accounts for all $SU(2)$ Euclidean monopoles of charge 1. We collect the main properties of the Prasad–Sommerfield monopole following Atiyah–Hitchin [7], Taubes [60] and Shnir [92].

The Prasad–Sommerfield monopole. We are going to use the notation $\rho = |x|$, $\hat{x} = \rho^{-1} x \in \mathbb{S}^2 \subset \mathbb{R}^3$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3 \otimes \mathfrak{su}_2$, where $\{\sigma_1, \sigma_2, \sigma_3\}$ is the orthonormal basis of $\mathfrak{su}_2$ defined in (1.1.1).

The Prasad–Sommerfield (PS) monopole $c_{\text{PS}} = (A_{\text{PS}}, \Phi_{\text{PS}})$ is given by the explicit formula, *cf.* [60, IV.1, Equation 1.15]:

$$\Phi_{\mathrm{PS}}(x) = \left(\frac{1}{\tanh(\rho)} - \frac{1}{\rho}\right) \hat{x}\cdot\sigma \qquad A_{\mathrm{PS}}(x) = \left(\frac{1}{\sinh(\rho)} - \frac{1}{\rho}\right) (\hat{x}\times\sigma)\cdot dx \tag{4.3.1}$$

Here $\cdot$ and $\times$ are the scalar and vector product in $\mathbb{R}^3$, respectively. To simplify the notation we will often drop the subscript PS throughout this section.

In formula (4.3.1) we fixed the mass $v = 1$. A monopole with arbitrary mass $v > 0$ is obtained by scaling. The following properties of the PS monopole follow directly from (4.3.1), *cf.* [60, §IV.1].

Lemma 4.3.1. *The pair (A_{PS}, Φ_{PS}) is a solution of the Bogomolny equation (1.2.1) with finite energy.*

(i) (A_{PS}, Φ_{PS}) *has charge $k = 1$ and centre $q = 0$. The charge and the centre of an $SU(2)$ Euclidean monopole were defined in Section 1.2 and 2.2, respectively.*

(ii) Φ *has exactly one zero,* $\Phi(0) = 0$.

(iii) $|\Phi(x)| < 1$.

(iv) $1 - |\Phi(x)| = \frac{1}{\rho} + O(e^{-2\rho})$.

(v) By (ii), over $\mathbb{R}^3 \setminus \{0\}$ we can decompose each $\mathfrak{su}_2$–valued form u into diagonal and off-diagonal part $u = u_D + u_T$, where $u_D = |\Phi|^{-2}\langle u, \Phi\rangle\Phi$. Then $|(d_A\Phi)_D| = O(\rho^{-2})$ and $|(d_A\Phi)_T| = O(e^{-\rho})$.

The asymptotically abelian gauge. As in Lemma 4.3.1.(v), over $\mathbb{R}^3 \setminus \{0\}$ the (trivial) rank 2 complex vector bundle E splits as a sum $H \oplus H^{-1}$ of eigenspaces of Φ. By Lemma 4.3.1.(i) H is the radial extension of the inverse of the Hopf line bundle. The adjoint bundle $(\mathbb{R}^3 \setminus \{0\}) \times \mathfrak{su}_2$ splits as a sum $\underline{\mathbb{R}} \oplus H^2$. We refer to such a gauge over $\mathbb{R}^3 \setminus \{0\}$ as to the *asymptotically abelian gauge* because it yields an asymptotic isomorphism between the PS monopole and a charge 1 Euclidean Dirac monopole.

We can describe such an isomorphism $\eta\colon\ E \to H \oplus H^{-1}$ explicitly, *cf.* [60, §IV.7, 7.1 and 7.2]: Let $U_\pm$ be the standard cover of $\mathbb{S}^2$, $U_\pm = \mathbb{S}^2 \setminus \{(0, 0, \mp 1)\}$. In spherical coordinates centred at the origin, define $\eta_\pm\colon\ (0,+\infty)\times U_\pm \to SU(2)$ by

$$\eta_+(\rho,\phi,\theta)=\begin{pmatrix} \cos\frac{\phi}{2} & e^{-i\theta}\sin\frac{\phi}{2} \\ -e^{i\theta}\sin\frac{\phi}{2} & \cos\frac{\phi}{2}\end{pmatrix}$$

and $\eta_- = e^{\theta\sigma_3}\eta_+$. In this gauge $\eta(A,\Phi)=(\eta A\eta^{-1}-(d\eta)\eta^{-1},\eta\Phi\eta^{-1})$ is

$$\begin{aligned}\eta_+(A) &= (1-\cos\phi)\sigma_3\,d\theta+\frac{1}{2\sinh(\rho)}\left[\begin{pmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0\end{pmatrix}\rho\,d\phi+\begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0\end{pmatrix}\rho\sin\phi\,d\theta\right]\\ \eta_+\Phi\eta_+^{-1} &= \left(\text{cotanh}\,(\rho)-\frac{1}{\rho}\right)\sigma_3\end{aligned} \tag{4.3.2}$$

and similar expressions hold when η_- replaces η_+.

Let (A^0,Φ^0) be the Euclidean Dirac monopole on H of Definition 2.1.1.

Lemma 4.3.2. *Let (A,Φ) be the PS monopole defined in (4.3.1).*

(i) There exists an isomorphism $\eta: E\to H\oplus H^{-1}$ over $\mathbb{R}^3\setminus\{0\}$, given by $\eta_\pm$ above, such that $\eta(A,\Phi)=(A^0,\Phi^0)\,\sigma_3+(a,\psi)$, with a and ψ a 1 and 0–form with values in the $SO(3)$–bundle $\underline{\mathbb{R}}\oplus H^2$.

(ii) (a,ψ) satisfies $d^*_{A^0\sigma_3}a=0=[\Phi^0\sigma_3,\psi]$ and $\partial_\rho \lrcorner\, a=0$. Moreover, as $\rho\to\infty$:

$$|a|+|\psi|+|d_{A^0\sigma_3}a|+|[\Phi^0\sigma_3,a]|+|d_{A^0\sigma_3}\psi|=O(e^{-\rho})$$

Therefore η puts (A,Φ) in "Coulomb gauge" with respect to $(A^0,\Phi^0)\,\sigma_3$. The second statement of the Lemma follows by a direct computation.

Without altering the properties stated in Lemma 4.3.2, we have the freedom to change η by composing with an element in the stabiliser of $(A^0,\Phi^0)\,\sigma_3$, *i.e.* a constant diagonal gauge transformation $e^{2\tau\sigma_3}$, $\tau\in\mathbb{R}/2\pi\mathbb{Z}$. By abuse of notation, we won't distinguish between η and the induced isomorphism of $SO(3)$–bundles $(\mathbb{R}^3\setminus\{0\})\times\mathfrak{su}_2\simeq\underline{\mathbb{R}}\oplus H^2$. As an isomorphism of $SO(3)$–bundles, we have the freedom to compose η with an element of $U(1)/\pm$, where $U(1)\to U(1)/\pm$ is induced by the adjoint representation $SU(2)\to SO(3)$.

The moduli space of charge 1 monopoles on $\mathbb{R}^3$. In Section 1.2, 1.3 and 1.4.2 we discussed the deformation theory of $SU(2)$ monopoles on $\mathbb{R}^3$. The situation is particularly simple when the charge $k=1$.

Let $D = D_{c_{PS}}$ be the Dirac operator (1.3.3) twisted by the PS monopole. The L^2–kernel of D is 4–dimensional, spanned over $\mathbb{H}$ by the vector $(d_A\Phi, 0)$, *i.e.*

$$\ker D = \langle\, (d_A\Phi, 0), \gamma(dx_h)\,(d_A\Phi, 0), h = 1, 2, 3\,\rangle_{\mathbb{R}},$$

where $\gamma(dx_h)$ denotes the Clifford multiplication.

We can explicitly integrate these infinitesimal deformations. Choose $x_0 \in \mathbb{R}^3$ and let T_{x_0} be the translation $x \mapsto x - x_0$. Then $T^*_{x_0} c_{\text{PS}}$ is still a solution to the Bogomolny equation. The corresponding infinitesimal deformation is $-\gamma(x_0)\,(d_A\Phi, 0) = -x_0 \lrcorner\,(F_A, d_A\Phi)$. On the other hand, $(d_A\Phi, 0)$ is the infinitesimal action of the gauge transformation $\exp(-\Phi)$.

In the next lemma, we put $T^*_{x_0} c_{\text{PS}}$ in "Coulomb gauge" with respect to c_{PS} and derive some useful estimates.

Lemma 4.3.3. *There exists κ, C and $\rho_0 > 0$ such that the following holds. For any $x_0 \in \mathbb{R}^3$ with $|x_0| < \kappa$ there exists a solution (A_{x_0}, Φ_{x_0}) to the Bogomolny equation which can be written*

$$(A_{x_0}, \Phi_{x_0}) = (A, \Phi) - x_0 \lrcorner\,(F_A, d_A\Phi) + (a_{x_0}, \psi_{x_0}),$$

where $d_1^*(a_{x_0}, \psi_{x_0}) = 0$. Here d_1 is the linearisation at $c_{PS} = (A, \Phi)$ of the action of gauge transformations. Moreover, $|(a_{x_0}, \psi_{x_0})| \leq C\frac{|x_0|^2}{\rho^3}$ for all $\rho \geq \rho_0$.

Proof. We give a sketch of the proof; it seems to us that the statement should be well-known.

Step 1. We will prove later, *cf.* Lemma 4.5.5, that the operator $DD^*\colon\ W^{2,2}_{w,\delta} \to L^2_{w,\delta-2}$ is an isomorphism for all $\delta \in (-1, 0)$. Here the spaces $W^{m,2}_{w,\delta}$ are defined as in Definition 3.2.1 replacing the distance ρ with the weight function $w = \sqrt{1+\rho^2}$, *cf.* Definition 4.5.4.

Using Lemma 4.3.1.(v) to estimate the size of $d_A\Phi$ in $W^{1,2}_{w,\delta-1}$ one then shows that $DD^* - 2x_0 \lrcorner\,(F_A, d_A\Phi) \cdot D^*$ remains an isomorphism if κ is sufficiently small. The Implicit Function Theorem (*cf.* Lemma 4.7.1) implies that there exists a unique solution $u \in W^{2,2}_{w,\delta}$ to the equation

$$DD^*u + \big(-x_0 \lrcorner\,(F_A, d_A\Phi) + D^*u\big) \cdot \big(-x_0 \lrcorner\,(F_A, d_A\Phi) + D^*u\big) = 0,$$

i.e. such that $(A, \Phi) - x_0 \lrcorner (F_A, d_A\Phi) + D^*u$ satisfies the Bogomolny equation. Moreover, one has $\|D^*u\|_{W^{1,2}_{w,\delta-1}} \leq C|x_0|^2$ and the map $x_0 \mapsto u \in W^{2,2}_{w,\delta}$ is smooth.

Step 2. It remains to show that $D^*u = O(\rho^{-3})$. Consider the equation $D\xi + \xi\cdot\xi = 0$ for $\xi \in W^{1,2}_{w,\delta-1}$.

By Lemma 4.3.2, in the asymptotically abelian gauge we write

$$D_0\xi + 2(a,\psi)\cdot\xi + \xi\cdot\xi = 0,$$

where (a,ψ) is exponentially decaying and D_0 is the Dirac operator twisted by the Dirac monopole (A^0, Φ^0). One can argue as in Lemma 3.4.6 to show that $\xi_T = O(\rho^{-\mu}) = \triangle\xi_D$ for all $\mu \in \mathbb{R}$.

Now apply the result to $\xi = -x_0 \lrcorner (F_A, d_A\Phi) + D^*u$. It follows that we can write

$$D^*u = x_0' \lrcorner (F_A, d_A\Phi) + \tau\,(d_A\Phi, 0) + O(\rho^{-3})$$

for some $x_0' \in \mathbb{R}^3$ and $\tau \in \mathbb{R}$. However, since $u \in W^{2,2}_{w,\delta}$ and $\delta \in (-1,0)$, an integration by parts shows that D^*u is L^2–orthogonal to ker D and therefore $x_0' = 0 = \tau$. □

Remark. There must exist a gauge transformation such that $g(A_{x_0}, \Phi_{x_0}) = T^*_{x_0}(A,\Phi)$. Indeed, there must exists $x_0' \in \mathbb{R}^3$ and $g \in \mathcal{G}$ such that $g(A_{x_0}, \Phi_{x_0}) = T^*_{x_0'}(A,\Phi)$. On the other hand, comparing $|\Phi_{x_0}|$ with $|T^*_{x_0'}\Phi|$, one concludes that $x_0' = x_0$.

4.4 The initial approximate solution

Having introduced the necessary background, we devote this section to the construction of an initial approximate solution to the Bogomolny equation (1.2.1) on X^*. We will desingularise the reducible solution c_{ext} of Section 4.2 by gluing rescaled PS monopoles in small balls centred at the k points $q_1, \ldots, q_k$. When the PS monopoles are centred at $q_1, \ldots, q_k$ themselves the construction is straightforward and involves the choice of a phase factor for each $j = 1, \ldots, k$ determining a gluing map. Since the singular solution c_{ext} is reducible, its stabiliser $\Gamma \simeq U(1)/\pm$ acts on the space of such gluing maps and any two choices yield gauge equivalent pairs if and only if they lie in the same Γ–orbit. In a second step we will also vary the centres of the PS monopoles to obtain a family of configurations $c(x_0,\tau)$ on X^*. In this case some care is needed to arrange for the initial error to be sufficiently small. Moreover, the given boundary conditions are satisfied only if the centres of the PS monopoles have a fixed barycentre $\zeta = 0$. Hence we will obtain a $(4k-1)$–parameter family of inequivalent smooth pairs $c(x_0,\tau) \in \mathcal{C}$, in agreement with the expected dimension of the moduli space, *cf.* Conjecture 3.4.9. In fact, in the next section we will see that, since we are fixing the decay of the solutions to the Bogomolny equation, the gluing problem is obstructed. As usual in such cases, it will then be necessary to consider ζ as a free parameter to be fixed only at the end of the construction. The final part of the section is devoted to the estimate of the error and other geometric quantities related to the pairs $c(x_0,\tau)$.

By (4.2.6), in a neighbourhood of q_j the pair $c_{\text{ext}} = (A_{\text{ext}}, \Phi_{\text{ext}})$ coincides at leading order with a Dirac monopole of charge 2 and mass λ_j (4.2.7). More precisely, in the ball $B_{\frac{\pi}{2}}(q_j)$ we write

$$c_{\text{ext}} = c_0^j + (a_{\text{ext}}^j, \psi_{\text{ext}}^j), \tag{4.4.1}$$

where:

(i) $c_0^j(x) = \big(A^0, \lambda_j \Phi^0\big)(\lambda_j x) \otimes 2\hat{\sigma}$. Here, (A^0,Φ^0) is an Euclidean Dirac monopole of charge 1 and mass $\frac{1}{2}$.

(ii) $|(a_{\text{ext}}^j, \psi_{\text{ext}}^j)| = O(d^{-1}\rho_j + \rho_j^2)$ and $(\nabla a_{\text{ext}}^j, \nabla \psi_{\text{ext}}^j) = O(d^{-1} + \rho_j)$.

We assume that we have chosen $(\lambda_0, 5, K)$–admissible data as in Definition 4.2.1 with λ_0 large enough so that Lemma 4.2.2 holds.

We define *gluing regions* and adapted cut-off functions.

Definition 4.4.1. 1. Let $N > 2$ be a number to be fixed later and set $\delta_j = \lambda_j^{-\frac{1}{2}}$. Taking λ_0 larger if necessary, we assume that $2N\delta_j < \frac{1}{2}$ and $\frac{\delta_j}{2N} > \frac{2}{\lambda_j}$ for all $j = 1, \dots, k$.

Write X^* as the union of open sets

$$U_j = B_{N\delta_j}(q_j) \text{ for } j = 1,\dots,k \qquad U_{\text{ext}} = X^* \setminus \left(\bigcup_{j=1}^k B_{N^{-1}\delta_j}(q_j) \right).$$

Finally, let $A_j, A_{j,\text{ext}}, A_{j,\text{int}}$ be the annuli $U_j \cap U_{\text{ext}}$, $B_{2N\delta_j}(q_j) \setminus B_{N\delta_j}(q_j)$ and $B_{N^{-1}\delta_j}(q_j) \setminus B_{(2N)^{-1}\delta_j}(q_j)$, respectively.

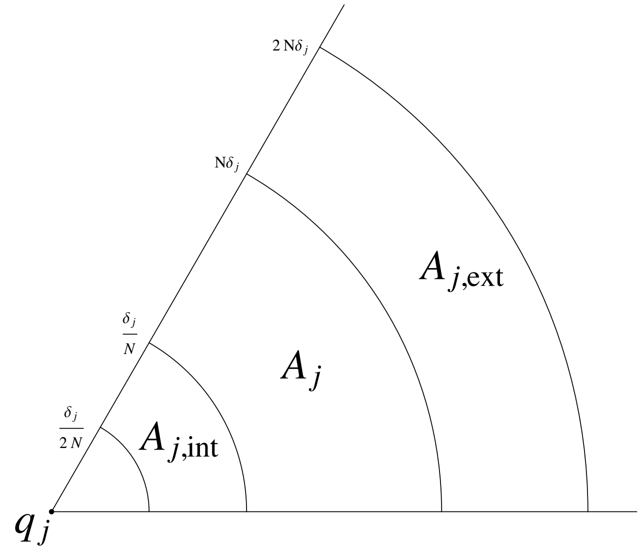


Figure 4.1: Gluing regions

2. For each $j = 1, \dots, k$, fix smooth cut-off functions $\chi^j_{\text{int}}, \chi^j_{\text{ext}}$ such that

$$\chi^j_{\text{int}} = \begin{cases} 1 & \text{in } B_{(2N)^{-1}\delta_j}(q_j) \\ 0 & \text{in } X^* \setminus B_{N^{-1}\delta_j}(q_j) \end{cases} \qquad \chi^j_{\text{ext}} = \begin{cases} 0 & \text{in } B_{N\delta_j}(q_j) \\ 1 & \text{in } X^* \setminus B_{2N\delta_j}(q_j) \end{cases}$$

We assume that $\chi^j_{\text{int}}, \chi^j_{\text{ext}}$ are radial and satisfy $|\nabla \chi^j_{\text{int}}| \leq \frac{8N}{\delta_j}$, $|\nabla \chi^j_{\text{ext}}| \leq \frac{4}{N\delta_j}$.

Now for all $j = 1, \dots, k$ let h_j and η_j be defined as follows:

$$h_j \colon\; X^* \supset B_{\frac{\pi}{2}}(q_j) \longrightarrow B_{\frac{\lambda_j \pi}{2}}(0) \subset \mathbb{R}^3 \tag{4.4.2}$$

is the homothety $q_j + x \mapsto \lambda_j x$;

$$\eta_j \colon\; A_j \times \mathfrak{su}(2) \to \underline{\mathbb{R}} \oplus H^2 \simeq \underline{\mathbb{R}} \oplus M \tag{4.4.3}$$

is the bundle isomorphism obtained by composing the gauge transformation η of Lemma 4.3.2 with a fixed isomorphism $H^2 \simeq M$. We have the freedom to compose η_j with a constant diagonal gauge transformation $e^{\tau_j \hat{\sigma}}$.

We have all the ingredients for the definition of a desingularisation of c_{ext}. First, the bundle: Given a k–tuple $\tau = (e^{\tau_1 \hat{\sigma}}, \dots, e^{\tau_k \hat{\sigma}}) \in (U(1)/\pm)^k$, define an $SO(3)$–bundle $V = V(\tau)$ over X^* by identifying $(U_j, U_j \times \mathfrak{su}(2))$ and $(U_{\text{ext}}, \underline{\mathbb{R}} \oplus M)$ over A_j using $\exp(\tau_j \hat{\sigma}) \circ \eta_j$. Notice that $w_2(V) \cdot [\mathbb{S}^2_{p_i}] \equiv 1$ and the isomorphism class of $V(\tau)$ does not depend on the choice of τ. Next we define a pair $c(\tau)$ on $V(\tau)$:

- Working on the reducible bundle $\underline{\mathbb{R}} \oplus M$ over U_{ext}, modify c_{ext} to

$$c'_{ext} = c_0^j + \chi^j_{\text{ext}}(a^j_{\text{ext}}, \psi^j_{\text{ext}})$$

on each annulus $A_j \cup A_{j,\text{ext}}$.

- Set $c_j = h_j^*(A_{\text{PS}}, \Phi_{\text{PS}})$. On the annulus $A_{j,\text{int}} \cup A_j$ we write $e^{\tau_j \hat{\sigma}} \eta_j(c_j) = c_0^j + (a^j_{\text{int}}, \psi^j_{\text{int}})$ as in Lemma 4.3.2.

  Define a pair c'_j on $U_j \times \mathfrak{su}_2$ by

$$e^{\tau_j \hat{\sigma}} \eta_j(c'_j) = c_0^j + \chi^j_{\text{int}}(a^j_{\text{int}}, \psi^j_{\text{int}}).$$

  Then $c'_j = c_j$ on $B_{(2N)^{-1}\delta_j}(q_j)$ and $e^{\tau_j} \eta_j(c'_j) = c_0^j$ over the annulus A_j.

Now define a smooth pair $c(\tau) = \big(A(\tau), \Phi(\tau)\big)$ on $V(\tau)$ by

$$c(\tau) = \begin{cases} c'_j & \text{on } U_j, \\ c'_{\text{ext}} & \text{on } U_{\text{ext}}. \end{cases} \tag{4.4.4}$$

It will be more convenient to fix a base point $\tau_0 = (\text{id}, \dots, \text{id})$ and regard the configurations $c(\tau)$ as a family of configurations on the fixed $SO(3)$–bundle $V = V(\tau_0)$. As in [38, Lemma 7.2.46], we are going to define a pair on V gauge equivalent to $c(\tau)$. Let $\gamma_1, \dots, \gamma_k, \gamma_{\text{ext}}$ be a partition of unity subordinate to the cover $U_1, \dots, U_k, U_{\text{ext}}$ of X^*. Define gauge transformations g_j on U_j by $\eta_j \circ g_j \circ \eta_j^{-1} = \exp(\tau_j \gamma_{\text{ext}} \hat{\sigma})$ and g_{ext} on U_{ext} by $g_{\text{ext}} = \exp(-\tau_j \gamma_j \hat{\sigma})$ on A_j and $g_{\text{ext}} \equiv 1$ on the complement of $A_1 \cup \ldots \cup A_k$. Then $\eta_j\, g_j\, \eta_j^{-1}\, g_{\text{ext}}^{-1} = e^{\tau_j \hat{\sigma}}$ over A_j and therefore $\eta_j\, g_j(c'_j) = g_{\text{ext}}(c'_{\text{ext}})$. We define a pair on V by

$$\begin{cases} g_j(c'_j) & \text{on } U_j, \\ g_{\text{ext}}(c'_{\text{ext}}) & \text{on } U_{\text{ext}}. \end{cases}$$

Then $(g_1, \dots, g_k, g_{\text{ext}})$ defines an isomorphism $g\colon V(\tau) \xrightarrow{\sim} V$ such that $g\big(c(\tau)\big)$ coincides with the newly defined pair.

Let Γ be the stabiliser of c_{ext}, *i.e.* Γ is the group of constant diagonal gauge transformations of $\underline{\mathbb{R}} \oplus M$. Γ acts diagonally on τ by composition on the left. Since the Prasad–Sommerfield monopole is irreducible it is not difficult to prove that $c(\tau)$

and $c(\tau')$ are gauge equivalent if and only if τ' belongs to the Γ orbit of τ.

The pre-gluing map. The fact that c_{ext} is reducible has another important consequence. In view of the proof of Lemma 3.4.4, the linearisation of the Bogomolny equation d_2 at c_{ext} has a 3–dimensional cokernel as an operator between the weighted Sobolev spaces of Section 3.4. This is due to the parabolicity of X: If $\triangle u = f \in C_0^\infty(X)$, then u grows logarithmically at infinity unless f has mean value zero. There are two ways of solving this problem:

(i) We can enlarge the Banach spaces in which we solve the Bogomolny equation by allowing the appropriate changes of the asymptotics at infinity. More precisely, we have to allow the possibility to vary the centre q in Definition 2.2.1. If one's unique goal were to prove that the moduli spaces of periodic monopoles (with singularities) are non-empty, this would seem the most straightforward way to proceed.

(ii) If instead one wants to construct a family of solutions to the Bogomolny equation with the aim to construct coordinates on the end of a fixed moduli space, a change of the asymptotics is not allowed. We regard the gluing problem as obstructed and in order to compensate for the obstructions we have to introduce a family of initial approximate solutions depending on parameters: It is necessary to vary the centre of mass of the points $q_1, \dots, q_k$.

We are going to follow the second approach. We begin with the definition of the space $\mathcal{P}$ of *gluing parameters*.

Definition 4.4.2. Fix $\kappa \in (0,1)$ so that Lemma 4.3.3 holds. Consider the product of k balls $B_\kappa(0) \subset \mathbb{R}^3$. By the homothethy h_j (4.4.2) identify the jth copy of $B_\kappa(0)$ with $B_{\lambda_j^{-1}\kappa}(q_j) \subset X$. Let $\mathcal{P} = \mathcal{P}_\kappa$ be the trivial $\mathbb{T}^k$–bundle over $B_\kappa(0) \times \ldots \times B_\kappa(0)$. We denote points in $\mathcal{P}$ by k–tuples (x_0, τ) of points $(x_0^j, e^{\tau^j \hat{\sigma}}) \in B_\kappa(0) \times U(1)/\pm$.

We think of $x_0^j \in B_\kappa(0)$ as parametrising the charge 1 monopole $(A_{x_0^j}, \Phi_{x_0^j})$ on $\mathbb{R}^3$ of Lemma 4.3.3, while the k–tuple τ corresponds to the choice of isomorphisms $e^{\tau_j \hat{\sigma}} \circ \eta_j \colon\; A_j \times \mathfrak{su}_2 \to \underline{\mathbb{R}} \oplus M$, $j = 1, \dots, k$.

Remark. In order to work with structure group $U(2)$ it is necessary to introduce the finite cover $\widetilde{\mathcal{P}}$ of $\mathcal{P}$ defined by taking the double cover $U(1) \to U(1)/\pm$ of each phase factor. A point in $\widetilde{\mathcal{P}}$ corresponds to the choice of k charge 1 monopoles on $\mathbb{R}^3$ together with lifts of $\exp(\tau^j\hat{\sigma})\circ\eta_j$ to isomorphisms of $U(2)$–bundles $A_j\times\mathbb{C}^2\simeq M_1\oplus M_2$, where M_1, M_2 are defined in (4.2.11).

Given $(x_0,\tau)\in\mathcal{P}$, we want to define a smooth configuration $c(x_0,\tau)\in\mathcal{C}$ on the $SO(3)$–bundle $V=V(\tau_0)\rightarrow X^*$: We are going to replace c_j with $c_j(x_0)=h_j^*(A_{x_0^j},\Phi_{x_0^j})$ in (4.4.4), but some care is needed to implement the construction.

First, notice that

$$-x_0\lrcorner\,(F_{A_{\text{PS}}},d_{A_{\text{PS}}}\Phi_{\text{PS}})=-\left(\frac{\langle x\times x_0,dx\rangle}{\rho^3},\frac{\langle x,x_0\rangle}{\rho^3}\right)\langle\hat{x},\sigma\rangle+O\left(\frac{|x_0|^2}{\rho^3}\right)$$

in the notation of (4.3.1). When we rescale and introduce cut-off functions, such a term yields a contribution to the initial error of order $\frac{1}{\lambda_j\rho_j^3}$: With our choice $\delta_j=\lambda_j^{-\frac{1}{2}}$, this term blows up as $\lambda_j\rightarrow+\infty$. Because of the obstructions, choosing $\delta_j=\lambda_j^{-\alpha}$ for some $0<\alpha<\frac{1}{2}$ would still not be a solution. It is necessary to refine the construction of $c(x_0,\tau)$ so that c'_{ext} and $c'_j(x_0)$ match at a higher order.

There is a simple way of achieving this: Given $(x_0,\tau)\in\mathcal{P}$, we modify c_{ext} by

$$c_{\text{ext}}(x_0)=c_{\text{ext}}-2\sum_{j=1}^{k}\frac{x_0^j}{\lambda_j}\lrcorner\left(*dG_{q_j},dG_{q_j}\right)\otimes\hat{\sigma}. \tag{4.4.5}$$

By Lemma 2.1.3 and 2.1.6, on $B_{\frac{\pi}{2}}(q_j)$ we have:

$$c_{\text{ext}}(x_0)=c_0^j-\left(\frac{\langle x\times x_0^j,dx\rangle}{\lambda_j\rho_j^3},\frac{\langle x,x_0^j\rangle}{\lambda_j\rho_j^3}\right)\langle\hat{x},\sigma\rangle+O\left(\frac{\rho_j}{d}+\rho_j^2+\frac{1}{\lambda_j}\right) \tag{4.4.6}$$

Now proceed with the construction (4.4.4) replacing the Dirac monopole c_0^j with

$$c_0^j(x_0)=c_0^j-\left(\frac{\langle x\times x_0^j,dx\rangle}{\lambda_j\rho_j^3},\frac{\langle x,x_0^j\rangle}{\lambda_j\rho_j^3}\right)\langle\hat{x},\sigma\rangle. \tag{4.4.7}$$

In other words, by using cut-off functions, we define pairs $c'_j(x_0)$ and $c'_{\text{ext}}(x_0)$ such that $\eta_j\big(c'_j(x_0)\big)$ and $c'_{\text{ext}}(x_0)$ both coincide with $c_0^j(x_0)$ over A_j. Notice that the choice $\delta_j=\lambda_j^{-\frac{1}{2}}$ now minimises the size of $\eta_j\big(c'_j(x_0)\big)-c_0^j(x_0)$ and $c_{\text{ext}}(x_0)-c_0^j(x_0)$.

Remark. In fact, if $n<2(k-1)$ and we let $d\rightarrow\infty$ there is margin of improvement: A better choice would be $\delta_j=\lambda_j^{-\frac{2}{5}}$. This is because the term of order $d^{-1}\rho_j$ in $c_{\text{ext}}(x_0)-c_0^j(x_0)$ is exponential decaying in λ_j by (4.2.8).

We denote by $c(x_0,\tau)$ the smooth pair on V obtained in this way. By Lemmas 2.1.3, 2.1.5 and 2.1.6 the pair $c(x_0,\tau)\notin\mathcal{C}$, in the sense that the boundary conditions at infinity are not met, unless (x_0,τ) satisfies the "balancing" condition

$$\sum_{j=1}^{k}\frac{x_0^j}{\lambda_j}=0. \tag{4.4.8}$$

Notice that the necessity of this constraint and the action of the stabiliser Γ of $c_{\text{ext}}(x_0)$ on the family $\{c(x_0,\tau)\,|\,(x_0,\tau)\in\mathcal{P}\}$ agree with Conjecture 3.4.9 on the dimension of the moduli space.

Since we are fixing the asymptotic behaviour at infinity and therefore expect the presence of obstructions, however, we have to allow the freedom not to satisfy (4.4.8) at the beginning and fix the centre of mass of $\frac{x_0^1}{\lambda_1},\dots,\frac{x_0^k}{\lambda_k}$ only at the end of the construction. We make the following definition.

Definition 4.4.3. For $h=1,2,3$ define

$$o_h=-\frac{1}{2\pi k}\sum_{j=1}^{k}\gamma(dx_h)\left(\chi^j_{\text{ext}}\,dG_{q_j},0\right)\otimes\hat{\sigma}\qquad o_4=-\frac{1}{2\pi k}\sum_{j=1}^{k}\left(\chi^j_{\text{ext}}\,dG_{q_j},0\right)\otimes\hat{\sigma},$$

where $\left(dG_{q_j},0\right)\in\Omega(\underline{\mathbb{R}}\oplus M)$, $dx_1=dx$, $dx_2=dy$, $dx_3=dt$ and, as usual, $\gamma(dx_h)\,(a,0)=\partial_{x_h}\lrcorner\,(*a,a)$ for every 1–form a .

In order to motivate this definition, we mention that d_2o_h, $h=1,2,3$, define a lift of coker d_2 to $\Omega^1(\underline{\mathbb{R}}\oplus M)$, *cf.* Lemma 4.6.1. Another useful way to understand Definition 4.4.3 is to observe that o_h is the derivative of c_{ext} with respect to the translation $q_j\mapsto q_j+\partial_{x_h}$ for all $j=1,\dots,k$. Notice that by Lemmas 2.1.3 and 2.1.6 and Definition 4.4.1.(2) there exists a constant $C>0$ such that:

$$\begin{cases}|o_h|\leq C\rho_j^{-2} & \text{in } B_1(q_j)\setminus B_{N\delta_j}(q_j)\\ |o_h|\leq C & \text{on } X^*\setminus\bigcup_{j=1}^k B_{\frac{1}{2}}(q_j)\end{cases}\qquad\begin{cases}|\nabla o_h|\leq C\rho_j^{-3} & \text{in } B_1(q_j)\setminus B_{N\delta_j}(q_j)\\ |\nabla o_h|\leq C & \text{on } X^*\setminus\bigcup_{j=1}^k B_{\frac{1}{2}}(q_j)\end{cases} \tag{4.4.9}$$

Now, for all $(x_0,\tau)\in\mathcal{P}$ we replace

$$c(x_0,\tau)\rightsquigarrow c(x_0,\tau)+4\pi k\sum_{h=1}^{3}\zeta_h\,o_h \tag{4.4.10}$$

where ζ is the centre of mass of $\frac{x_0^1}{\lambda_1}, \dots, \frac{x_0^k}{\lambda_k}$, *i.e.*

$$\zeta = -\sum_{j=1}^{k} \frac{x_0^j}{\lambda_j}. \tag{4.4.11}$$

By Lemmas 2.1.3, 2.1.5 and 2.1.6 this final modification guarantees that $c(x_0, \tau) \in \mathcal{C}$ for all $(x_0, \tau) \in \mathcal{P}$. By abuse of notation, we take (4.4.10) as the definition of $c(x_0, \tau)$.

Fix a base point $(0, \tau_0) \in \mathcal{P}$ and let $(A, \Phi) = c(0, \tau_0)$. As in Section 3.4 we can take (A, Φ) as a background pair for the definition of a space $\mathcal{C}_{\delta_1,\delta_2}$ of pairs on $V = V(\tau_0)$. Here we choose $\delta_1 \in (0, \frac{1}{2})$ and $\delta_2 \in (\delta_0, 0)$, where δ_0 is given by Proposition 3.3.11.

Lemma 4.4.4. *There exists λ_0 and κ such that the following holds. Suppose that $v, S, q_1, \dots, q_k$ are $(\lambda_0, 5, K)$–admissible and let $\mathcal{P} = \mathcal{P}_\kappa$.*

The construction of $c(x_0, \tau)$ in (4.4.10) *defines a smooth map, the* pre-gluing map*,*

$$c\colon \mathcal{P} \to \mathcal{C}_{\delta_1,\delta_2}.$$

Moreover, $\Gamma \simeq U(1)/\pm$ acts on $\mathcal{P}$ and $c(x_0, \tau)$ and $c(x_0', \tau')$ are gauge equivalent if and only if (x_0', τ') belongs to the Γ–orbit of (x_0, τ).

Estimate of the error and the geometry of $c(x_0, \tau)$. Fix a point $(x_0, \tau) \in \mathcal{P}$ and let $c = (A, \Phi) = c(x_0, \tau)$. We want to estimate how far c is from a solution to the Bogomolny equation, *i.e.* we want to control $\Psi(x_0, \tau) = *F_A - d_A\Phi$. For later use it is also necessary to estimate the size of Φ and of the "curvature" $d_A\Phi = *F_A - \Psi(x_0, \tau)$.

Lemma 4.4.5. *We extend $c'_{ext}(x_0)$ to $X^* \setminus \{q_1, \dots, q_k\}$ as the reducible pair on $\underline{\mathbb{R}} \oplus M$ defined by:*

$$c'_{ext}(x_0) = \begin{cases} c(x_0, \tau) & \text{over } U_{ext} \\ c_0^j(x_0) & \text{over } U_j \setminus \{q_j\} \end{cases} \tag{4.4.12}$$

Taking λ_0 larger and κ smaller if necessary, we can make sure that there exists $\frac{1}{\lambda_j} < \delta(\lambda_j) < \frac{\sqrt{2}}{\lambda_j}$ such that

$$\langle \Phi'_{ext}(x_0), \hat{\sigma} \rangle \geq \frac{1}{2} \tag{4.4.13}$$

over $X^* \setminus \bigcup_{j=1}^k B_{\delta(\lambda_j)}(q_j)$.

Proof. First, from the definition (4.4.5) of $c_{\text{ext}}(x_0)$ it follows that $\langle \Phi_{\text{ext}}(x_0), \hat{\sigma} \rangle$ is a harmonic function on $X^* \setminus \{q_1, \ldots, q_k\}$ for all x_0. Then one can use (4.4.6) and $|x_0| < \kappa$ to argue as in Lemma 4.2.2: There exists $\lambda_0 > 0$ such that if $\lambda_j > \lambda_0$ then $\langle \Phi_{\text{ext}}(x_0), \hat{\sigma} \rangle \geq 1$ on $X^* \setminus \bigcup_{j=1}^k B_{\delta(\lambda_j)}(q_j)$, with $\frac{1}{\lambda_j} < \delta(\lambda_j) \leq \frac{1+\sqrt{1+\kappa}}{2\lambda_j}$.

Secondly, picking λ_0 even larger if necessary (depending on N), one can make sure that the term of order $O\left(\frac{\rho_j}{d} + \rho_j^2 + \frac{1}{\lambda_j}\right)$ that is multiplied by the cut-off function χ^j_{ext} in the definition of $c(x_0, \tau)$ is no bigger than $\frac{1}{4}$ in absolute value.

Finally, by (4.4.9), Definition 4.2.1.(iii) and the fact that $|\zeta| \leq \frac{C\kappa}{\lambda}$, we deduce that we can choose κ small enough (depending on K of Definition 4.2.1) so that

$$\left| 4\pi k \sum_{h=1}^{3} \zeta_h \, o_h \right| \leq \frac{1}{4}. \qquad \square$$

Lemma 4.4.6. *There exist constants* $C_1, C_2 > 0$ *such that if* $\lambda_0 \geq C_1 N^2$ *then*

$$\left(\lambda_j^{-2} + \rho_j^2\right) |d_A \Phi| \leq C_2 \tag{4.4.14}$$

over $B_1(q_j)$ *for all* $j = 1, \ldots, k$.

Proof. We derive the estimate separately in different regions.

- In the ball $\rho_j \leq \frac{\delta_j}{2N}$, $c(x_0, \tau)$ is gauge equivalent to $T^*_{x_0^j} c_{\text{PS}}$ rescaled by λ_j. Since $(1 + \rho^2)|d_A \Phi|$ is a scale invariant quantity, we can use Lemma 4.3.1 and the fact that $|x_0^j| < 1$ to deduce the statement.

- On the annulus A_j, $c(x_0, \tau) = c_0^j(x_0)$. By a direct calculation using (4.4.7)
$$\left(\lambda_j^{-2} + \rho_j^2\right) |d_A \Phi| \leq C \left(1 + N \lambda_j^{-\frac{1}{2}}\right).$$

- We deduce the estimate on the annulus $A_{j,\text{int}}$ from the previous two and Lemma 4.3.3. We write $c(x_0, \tau) = c_0^j(x_0) + \chi_j^{\text{int}}(a, \psi)$, where
$$(a, \psi) = c_j(x_0) - c_0^j(x_0) = O\left(\lambda_j^{-2} \rho_j^{-3}\right).$$

  In Lemma 4.3.3 we didn't calculate the decay of the covariant derivative of

(a, ψ), but we can argue as follows. Observe that

$$d_{A+\chi a}\,(\Phi + \chi\psi) = d_A\Phi + \chi\, d_{A+a}(\Phi + \psi) + d\chi \wedge \psi + \chi(\chi - 1)[a, \psi]$$

where $(A, \Phi) = c_0^j(x_0)$ and $\chi = \chi^j_{\text{int}}$. Since $(A + a, \Phi + \psi)$ is a translation of the Prasad–Sommerfield monopole, we deduce as above that

$$\rho_j^2|d_A\Phi| \leq C\big(1 + N^2\lambda_j^{-1} + N^4\lambda_j^{-2}\big).$$

- Finally, on the annulus $B_1(q_j) \setminus B_{N\delta_j}$ write

$$c(x_0, \tau) = c_0^j(x_0) + \chi^j_{\text{ext}}(a, \psi) + 4\pi k \sum_{h=1}^{3} \zeta_h\, o_h,$$

where $(a, \psi) = O(\rho_j + \lambda_j^{-1})$ and $(\nabla a, \nabla\psi) = O(1)$. A direct computation using (4.4.9) yields $\rho_j^2|d_A\Phi| = O(1 + \lambda_j^{-\frac{1}{2}})$. $\square$

The error $\Psi(x_0, \tau)$ is supported in the annuli $A_{j,\text{int}}$ and $A_{j,\text{ext}}$. We mentioned that the span of $d_2 o_h$, $h = 1, 2, 3$, will play the role of the space of obstructions. It is then natural to set

$$\Psi_\zeta = 4\pi k\, d_2 \left(\sum_{h=1}^{3} \zeta_h\, o_h \right) \tag{4.4.15}$$

and estimate $\Psi(x_0, \tau) - \Psi_\zeta$. Now observe that:

$$\begin{cases} c(x_0, \tau) = c_0^j(x_0) + \chi^j_{\text{int}}\, O(\lambda_j^{-2}\rho_j^{-3}) & \text{over } A_{j,\text{int}} \\ c(x_0, \tau) = c_0^j(x_0) + \chi^j_{\text{ext}}\, O(\lambda_j^{-1} + \rho_j) + 4\pi k \displaystyle\sum_{h=1}^{3} \zeta_h\, o_h & \text{over } A_{j,\text{ext}} \end{cases}$$

We make use of the following remark: If (A, Φ) and $(A, \Phi) + (a, \psi)$ both solve the Bogomolny equation and χ is a smooth function, then

$$*F_{A+\chi a} - d_{A+\chi a}(\Phi + \chi\psi) = *(d\chi \wedge a) - (d\chi)\psi + \chi(\chi - 1)(*[a, a] - [a, \psi]).$$

A direct computation using Definition 4.4.1.(2) yields:

$$\begin{cases} |\Psi(x_0, \tau)| \leq CN^4 & \text{over } A_{j,\text{int}} \\ |\Psi(x_0, \tau) - \Psi_\zeta| \leq C & \text{over } A_{j,\text{ext}}. \end{cases} \tag{4.4.16}$$

We summarise some of the properties of the family $c(x_0, \tau)$ of Lemma 4.4.4.

Proposition 4.4.7. *There exists λ_0 and κ such that the following holds. Suppose that $v, S, q_1, \dots, q_k$ are $(\lambda_0, 5, K)$–admissible and let $\mathcal{P} = \mathcal{P}_\kappa$.*

Then there exists $C > 0$ with the following significance. Let $c\colon \mathcal{P} \to \mathcal{C}_{\delta_1,\delta_2}$ be the pre-gluing map of Lemma 4.4.4. Then:

(i) Given $(x_0, \tau) \in \mathcal{P}$ let ζ be defined by (4.4.11) *and set $(A, \Phi) = c(x_0, \tau)$. Then $\Psi = *F_A - d_A\Phi = (\Psi - \Psi_\zeta) + \Psi_\zeta$, where Ψ_ζ is defined by* (4.4.15) *and $|\Psi - \Psi_\zeta| \leq C$.*

(ii) Still denoting $c(x_0, \tau)$ by (A, Φ), $\left(\lambda_j^{-2} + \rho_j^2\right) |d_A\Phi| \leq C$ over $B_1(q_j)$ for all $j = 1, \dots, k$ and $|\Phi| \geq \frac{1}{2}$ over U_{ext}.

4.5 The linearised equation: The local models

We work with the pair $c = c(x_0, \tau)$ on the $SO(3)$–bundle $V = V(\tau_0)$ for some $(x_0, \tau) \in \mathcal{P}$. The goal is to find $\xi = \xi(x_0, \tau) \in \Omega(X^*; V)$ with the appropriate decay at infinity and at the singularities $p_i \in S$ such that $c(x_0, \tau) + \xi$ is a solution to the Bogomolny equation (1.2.1), *i.e.*

$$d_2\xi + \xi \cdot \xi + \Psi(x_0, \tau) = 0. \tag{4.5.1}$$

Here $d_2\colon \Omega(X^*; V) \to \Omega^1(X^*; V)$ is the linearisation of (1.2.1) at $c(x_0, \tau)$.

In this section we begin the study of the linearised equation $d_2\xi = f$. This is the crucial step to solve (4.5.1). The strategy we are going to follow to understand the invertibility properties of d_2 is standard: We first solve the equation on U_j and U_{ext}. In Section 1.3 we explained that, due to the gauge invariance of the Bogomolny equation, its linearisation $d_2\colon \Omega(X^*; V) \to \Omega^1(X^*; V)$ is not elliptic. We will look for a solution of the form $\xi = d_2^*u$ with $u \in \Omega^1(X^*; V)$. From Lemma 1.3.1 we have a Weitzenböck formula

$$d_2 d_2^* u = \nabla_A^* \nabla_A u - \text{ad}^2(\Phi)u + *[\Psi, u], \tag{4.5.2}$$

where $\Psi = \Psi(x_0, \tau)$. We will adapt the weighted analysis of Chapter 3 to the present situation. When studying the equation $d_2 d_2^* u = f$ over U_{ext}, we will have to distinguish the large mass and large distance limit.

Remark 4.5.1. Sometimes it will be convenient to think of a V–valued 1–form u as a section $(u,0) \in \Omega(X^*;V)$. Then $d_2^* u = D^*(u,0)$ and a computation shows that the equation $d_2 d_2^* u = f$ is equivalent to $DD^*(u,0) = (f, *[\Psi, *u])$.

4.5.1 The linearised equation on U_j

Over the region U_j there are no obstructions to the invertibility of the operator $d_2 d_2^*$. The major issue is instead the fact that by (4.4.14) the curvature term $d_A\Phi$ blows up as $\lambda_j \to \infty$. In view of the Weitzenböck formula Lemma 1.3.1 for D^*D, this implies that the norm of the inverse of the operator $d_2 d_2^* \colon W^{2,2} \to L^2$ between standard Sobolev spaces is not uniformly bounded. This is completely analogous to Taubes's existence result for ASD connections on 4–manifolds [97, 99], where the smallness of the initial error guarantees that the equation can still be solved. Exploiting the conformal invariance of the ASD equations, Freed and Uhlenbeck [42] gave a different argument using Sobolev spaces with respect to the conformal cylindrical metric: With respect to this metric, the curvature of the initial approximate solution is uniformly bounded. In situations where the problem is not conformally invariant, a standard approach is to use weighted Sobolev spaces.

For each $j = 1, \ldots, k$ define a weight function $w_j = \sqrt{\lambda_j^{-2} + \rho_j^2}$. By abuse of notation, we won't distinguish between the globally defined function w_j on $\mathbb{R}^3$ and a fixed smooth increasing function on X^* with the properties $w_j \leq 1$ and:

$$w_j = \begin{cases} \sqrt{\lambda_j^{-2} + \rho_j^2} & \text{if } \rho_j \leq \frac{1}{2} \\ 1 & \text{if } \rho_j \geq 1 \end{cases} \tag{4.5.3}$$

By scaling, we will work on $\mathbb{R}^3$ endowed with the weight function $w = \sqrt{1+\rho^2}$. On the trivial $SO(3)$–bundle $\mathbb{R}^3 \times \mathfrak{su}(2)$ we fix a pair (A, Φ) which coincides with the monopole (A_{x_0}, Φ_{x_0}) of Lemma 4.3.3 if $\rho \leq (2N)^{-1}\sqrt{\lambda_j}$ and with the reducible pair induced by a charge 1 Euclidean Dirac monopole of mass 1 when $\rho \geq N^{-1}\sqrt{\lambda_j}$; in other words we work with the pair obtained from $c'_j(x_0)$ by scaling. In fact, the only properties of (A, Φ) that will be used are (4.4.14), *i.e.* $w^2|d_A\Phi|$ is uniformly bounded, and the fact that A is a metric connection.

The following Hardy-type inequality on $\mathbb{R}^3$ is analogous to Lemma 3.2.4; it is proved in a similar way by integration by parts.

Lemma 4.5.2. *For all $\delta \in (-1, 0)$ and $u \in C_0^\infty(\mathbb{R}^3; \mathfrak{su}_2)$*

$$\int w^{-2\delta-3}\,|u|^2 \leq \frac{1}{\delta^2}\int w^{-2\delta-1}\,|\nabla_A u|^2.$$

Proof. By Kato's inequality, the Lemma reduces to the same estimate for scalar functions. As observed in [51, §3.5], a uniform way to obtain these estimates is to use the inequality [73, Lemma 2]

$$\int u^2|\triangle\psi| \leq 4\int |\nabla u|^2\frac{|\nabla\psi|^2}{|\triangle\psi|}$$

valid for any function ψ with no-where vanishing Laplacian and all $u \in C_0^\infty(\mathbb{R}^3)$. Pick $\psi = \log w$ if $\delta = \frac{1}{2}$ and $\psi = w^{-2\delta-1}$ otherwise. $\square$

Remark 4.5.3. By scaling, the Lemma holds with the same constant if w is replaced by w_j.

Definition 4.5.4. For all $\delta \in \mathbb{R}$ and all smooth sections $u \in \Omega(\mathbb{R}^3; \mathfrak{su}_2)$ with compact support define:

$$\|u\|_{L^2_{w,\delta}} = \|w^{-\delta-\frac{3}{2}}u\|_{L^2} \qquad \|u\|^2_{W^{1,2}_{w,\delta}} = \|u\|^2_{L^2_{w,\delta}} + \|\nabla_A u\|^2_{L^2_{w,\delta-1}} + \|[\Phi, u]\|^2_{L^2_{w,\delta-1}}$$

Define spaces $L^2_{w,\delta}$ and $W^{1,2}_{w,\delta}$ as the completion of C_0^∞ with respect to these norms. Finally, we say that $u \in W^{2,2}_{w,\delta}$ if $u \in W^{1,2}_{w,\delta}$ and

$$\|\nabla_A(D^*u)\|_{L^2_{w,\delta-2}} + \|[\Phi, (D^*u)]\|_{L^2_{w,\delta-2}} < \infty.$$

Lemma 4.5.5. *For all $-1 < \delta < 0$ there exist $\varepsilon(\delta) > 0$ and $C = C_\delta > 0$ such that if $\|w\Psi\|_{L^3} < \varepsilon(\delta)$ then the following holds. For all $f \in L^2_{w,\delta-2}$ there exists a unique solution $u \in W^{2,2}_{w,\delta}$ to $d_2 d_2^* u = f$ and moreover*

$$\|u\|_{W^{2,2}_{w,\delta}} \leq C\|f\|_{L^2_{w,\delta-2}}.$$

Proof. We assume wlog that $f \in C_0^\infty$. The solution u can be found by direct minimisation of the functional

$$\frac{1}{2}\int_{B_R} |d_2^* u|^2 - \langle u, f\rangle.$$

Indeed, by (4.5.2) we write $d_2 d_2^* u = \nabla_A^* \nabla_A u - \text{ad}^2(\Phi) u + *[\Psi, u]$. By Hölder's inequality,

$$|\langle *[\Psi, u], u\rangle_{L^2}| \leq \|w\Psi\|_{L^3} \|w^{-1} u\|_{L^2} \|u\|_{L^6}.$$

Lemma 4.5.2 with $\delta = -\frac{1}{2}$ and the Sobolev inequality imply that $\|d_2^* u\|_{L^2}$ is a norm on $W^{1,2}_{w,\frac{1}{2}}$ if $\|w\Psi\|_{L^3} < \frac{1}{2C_{Sob}}$. Since $f \in C_0^\infty$, the functional $\langle f, u\rangle_{L^2}$ is continuous on $W^{1,2}_{w,\frac{1}{2}}$ and a unique solution $u \in W^{1,2}_{w,\frac{1}{2}}$ exists. Moreover, u is a smooth strong solution and we have to prove the estimate in weighted spaces.

Since $u \in W^{1,2}_{w,\frac{1}{2}} \cap C^\infty_{loc}$ and $d_2 d_2^* u = f \in C_0^\infty$, we have $|u| = O(\rho^{-1})$. Indeed, (A, Φ) is reducible outside of a compact set and we can write $u = u_T + u_D$ in such exterior region, where we also assume $f \equiv 0$. Then u_D is harmonic and therefore $|u_D| \leq C\rho^{-1}$. On the other hand, since $|\Phi| \to 1$ as $\rho \to +\infty$, one can argue as in Lemma 3.4.6 (and Remark 3.4.6.ii) to show that $|u_T| = O(\rho^{-\mu})$ for all $\mu > 0$. In particular, the integrations by parts below are justified:

$$\|u\|_{L^2_{w,\delta}} \|f\|_{L^2_{w,\delta-2}} \geq \int \langle f, u\rangle\, w^{-2\delta-1} = \int \langle *[\Psi, u], u\rangle\, w^{-2\delta-1}$$
$$+ \int (|\nabla_A u|^2 + |[\Phi, u]|^2) w^{-2\delta-1} + (1+2\delta)|\delta| \int |u|^2 w^{2\delta-3}$$

As before, by Hölder's inequality

$$\left|\langle *[\Psi, u], u\, w^{-2\delta-1}\rangle_{L^2}\right| \leq \|w\Psi\|_{L^3} \|u\|_{L^2_{w,\delta}} \|w^{-\delta-\frac{1}{2}} u\|_{L^6}.$$

Since $u \in W^{1,2}_{w,\delta-1} \Rightarrow \nabla_A(w^{-\delta-\frac{1}{2}} u) \in L^2$, the Sobolev inequality and Lemma 4.5.2 yield

$$\|u\|_{W^{1,2}_{w,\delta}} \leq C_\delta \|f\|_{L^2_{w,\delta-2}}$$

if $\|w\Psi\|_{L^3}$ is sufficiently small.

Set $\xi = d_2^* u = D^*(u, 0)$. In order to estimate $\|(\nabla_A \xi, [\Phi, \xi])\|_{L^2_{w,\delta-2}}$ we make use of the Weitzenböck formula for $D^* D\xi$ and the Bochner-type identity (3.3.7). Indeed, by Remark 4.5.1 $D\xi = (f, *[\Psi, *u])$. Then

$$\| *[\Psi, *u]\|_{L^2_{w,\delta-2}} \leq \|w\Psi\|_{L^3} \|w^{-\delta-\frac{1}{2}} u\|_{L^6}$$

and the Sobolev inequality together with the control of $\|u\|_{W^{1,2}_{w,\delta}}$ just obtained imply

$$\|D\xi\|_{L^2_{w,\delta-2}} \leq C\|f\|_{L^2_{w,\delta-2}}.$$

Integrating (3.3.7) against $w^{-2\delta+1}$ yields

$$\int \left(|\nabla_A \xi|^2 + |[\Phi,\xi]|^2\right) w^{-2\delta+1} \leq c_1 \int |\xi|^2 w^{-2\delta-1} + c_2 \int \langle D\xi, \xi\rangle\, w^{-2\delta} + \int w^{-2\delta+1}|D\xi|^2 + \int w^{-2\delta+1}|\Psi|\,|\xi|^2 + \int w^{-2\delta+1}|d_A\Phi|\,|\xi|^2,$$

using $|\nabla w| \leq 1$. Estimate the term involving Ψ as before using $\|w\Psi\|_{L^3} < \varepsilon(\delta)$. Finally, observe that $w^2|d_A\Phi|$ is uniformly bounded by (4.4.14). $\square$

Remark. One can pick $\varepsilon(\delta) = \frac{|\delta|}{2\sqrt{2}C_{Sob}}$ and

$$\|u\|_{W^{1,2}_{w,\delta}} \leq \frac{2}{|\delta|(1+\delta)}\|f\|_{L^2_{w,\delta-2}},$$

which shows that $\delta = 0, -1$ are exceptional weights for the Laplacian on $\mathbb{R}^3$.

4.5.2 The linearised equation on U_{ext}: The high mass case

We move on to study the equation $d_2 d_2^* u = f$ over U_{ext}: We work on the reducible $SO(3)$–bundle $\underline{\mathbb{R}} \oplus M$ of (4.2.9) endowed with the pair $c'_{\text{ext}}(x_0)$ of (4.4.12), which is an exact solution to the Bogomolny equation on the complement of $\bigcup_{j=1}^k B_{2N\delta_j}(q_j)$. We will make use of the decomposition $u = \Pi_0 u_D + \Pi_\perp u_D + u_T$ into $\mathbb{S}^1$–invariant, oscillatory and off-diagonal components as in Section 3.3. With respect to the decomposition $\underline{\mathbb{R}} \oplus M$, the equation $d_2 d_2^* u = f$ can be written:

$$\begin{cases} \triangle u_D = f_D \\ \nabla_A^* \nabla_A u_T + |\Phi|^2 u_T + \Psi \cdot u_T = f_T \end{cases}$$

Weight functions and weighted Sobolev spaces. We start with the definition of weight functions and weighted Sobolev spaces. The novelty with respect to the setting of Section 3.3 is the choice of weight spaces on the big end of X^*. In Chapter 3 we defined weighted Sobolev spaces using the weight function $\sqrt{1+|z|^2}$. Now we have to face two distinct situations: If we are constructing monopoles in the large mass limit $v \to +\infty$ and $q_1, \dots, q_k, S$ are contained in a fixed set $B_{R_0} \times \mathbb{S}^1 \subset X^*$, we can take $\omega = \sqrt{1+|z|^2}$ and let all constants depend on R_0 without further notice. Then the analysis of Section 3.3 applies and some care is needed only to check that the constants remain uniformly bounded.

If instead $n \leq 2(k-1)$ and we allow $d \to \infty$ major changes are needed: The error is concentrated around k points $q_1, \ldots, q_k$ moving off to infinity and we would like to replace $\sqrt{1+|z|^2}$ with a weight function which is uniformly bounded above and below in a neighbourhood of each q_j but maintains the same behaviour $O(|z|)$ at large distances.

We begin with the high mass case and in a second step we will explain how to extend the results to the large distance limit.

Set $\omega = \sqrt{1+|z|^2}$ and introduce weight functions $\hat{\rho}_j, \hat{\rho}_i$ in a neighbourhood of the points $q_1, \ldots, q_k$ and $p_i \in S$. $\hat{\rho}_j$ is a fixed smooth increasing function with the properties $\hat{\rho}_j \leq 1$ and:

$$\hat{\rho}_j = \begin{cases} \rho_j & \text{if } \rho_j \leq \frac{1}{2} \\ 1 & \text{if } \rho_j \geq 1 \end{cases} \tag{4.5.4}$$

$\hat{\rho}_i$ is defined in a similar way, but the transition between ρ_i and 1 takes place on the annulus $B_{2\sigma}(p_i) \setminus B_\sigma(p_i)$, where $\sigma > 0$ is chosen so that the balls $B_{2\sigma}(p_i)$ are all disjoint. Constants will be allowed to depend on σ without further notice.

Definition 4.5.6. Given a triple $(\delta_1, \delta_2, \delta_3) \in \mathbb{R}^3$ and a smooth compactly supported section $u \in \Omega(\underline{\mathbb{R}} \oplus M)$ define $\|u\|_{L^2_{(\delta_1,\delta_2,\delta_3)}}$ as the maximum of the semi-norms:

$$\left\|\omega^{-\delta_1-1}u\right\|_{L^2(\Omega_\sigma)} \qquad \left\|\hat{\rho}_i^{-\delta_2-\frac{3}{2}}u\right\|_{L^2(B_{2\sigma}(p_i))} \qquad \left\|\hat{\rho}_j^{-\delta_3-\frac{3}{2}}u\right\|_{L^2(B_1(q_j))}$$

Here $\Omega_\sigma = X \setminus \bigcup_{i=1}^n B_\sigma(p_i) \cup \bigcup_{j=1}^k B_{\frac{1}{2}}(q_j)$.

Given $\delta > 0$, set $\underline{\delta} = (-\delta, \delta, -\delta)$ and for each $m \in \mathbb{Z}$ let $\underline{\delta} - m$ denote the triple $\underline{\delta} - (m,m,m)$. For smooth compactly supported $f, \xi, u \in \Omega(\underline{\mathbb{R}} \oplus M)$ we say that

1. $f \in L^2_{\underline{\delta}-2}$ if the corresponding norm is finite;
2. $\xi \in W^{1,2}_{\underline{\delta}-1}$ if $\xi \in L^2_{\underline{\delta}-1}$ and $\nabla_A \xi, [\Phi, \xi] \in L^2_{\underline{\delta}-2}$;
3. $u \in W^{2,2}_{\underline{\delta}}$ if $D^* u \in W^{1,2}_{\underline{\delta}-1}$ and $u \in L^2_{(\delta,-\delta,-\delta)}$.

Finally, define spaces $W^{m,2}_{\underline{\delta}-2+m}$ as the completions of C^∞_0 with respect to the corresponding norm.

Remark. As in Definition 3.2.5 and 3.3.7, if $u \in W^{2,2}_{\underline{\delta}}$ then $u_T \in L^2_{\underline{\delta}}$. The reason for the odd definition of $W^{2,2}_{\underline{\delta}}$ is to include diagonal constant sections.

Existence of weak solutions. Recall that by (4.4.13) there exists $\delta(\lambda_j)$ such that $2|\Phi| \geq 1$ outside of $\bigcup_{j=1}^k B_{\delta(\lambda_j)}(q_j)$.

We have the following Poincaré inequality.

Lemma 4.5.7. *Given $\delta > 0$ there exists a constant $C = C_\delta > 0$ such that*

$$\|u\|^2_{L^2_{(\delta,-\frac{1}{2},-\frac{1}{2})}} \leq C \int |\nabla_A u|^2 + |[\Phi, u]|^2$$

for all $u \in C_0^\infty(\underline{\mathbb{R}} \oplus M)$ satisfying $\int \langle u, \hat{\sigma}\rangle\, \omega^{-2(\delta+1)} = 0$.

Proof. First set $B = B_1(q_j)$ and let χ be a smooth cut-off function supported in B with $\chi \equiv 1$ in $\frac{1}{2}B$. Applying Lemma 3.2.4 to χu with $\delta = -\frac{1}{2}$ we obtain

$$\|\hat{\rho}_j^{-1} u\|_{L^2(\frac{1}{2}B)} \leq C \left(\|\nabla_A u\|_{L^2} + \|u\|_{L^2(B \setminus \frac{1}{2}B)} \right)$$

with a uniform constant $C > 0$. A similar estimate holds for $\|\hat{\rho}_i^{-1} u|_{B_\sigma(p_i)}\|_{L^2}$ with a constant $C = C(\sigma) > 0$. Hence we reduced to prove that

$$\int \omega^{-2(\delta+1)} |u|^2 \leq C \int |\nabla_A u|^2 + |[\Phi, u]|^2.$$

Since $\omega \geq 1$, the estimate for $\Pi_\perp u_D$ follows from (3.3.3). It is also easy to prove the estimate for u_T: Write $u_T = \chi u_T + (1-\chi) u_T$, where χ is a smooth cut-off function with $\chi \equiv 1$ in $B_1(q_j)$, say, so that $\nabla\chi$ is supported in the region where $2|\Phi| \geq 1$. Then, by the Poincaré inequality for compactly supported functions, Kato's inequality and (4.4.13),

$$\|u_T\|_{L^2} \leq C \left(\|\nabla_A (\chi\, u_T)\|_{L^2} + \|[\Phi, u]\|_{L^2} \right) \leq C \left(\|\nabla_A u\|_{L^2} + \|[\Phi, u]\|_{L^2} \right).$$

Finally, if $u = \Pi_0 u_D \in C_0^\infty(\mathbb{R}^2)$ and $\int u\, \omega^{-2(\delta+1)} = 0$ apply Lemma 3.3.3. □

As a corollary we obtain the existence of weak solutions to the inhomogeneous equation $d_2 d_2^* u = f$ under suitable conditions on the RHS. Recall that the error $\Psi = \Psi(x_0, \tau)$ is supported on $\bigcup_{j=1}^k B_{2N\delta_j}(q_j)$.

Lemma 4.5.8. *For all $\frac{1}{2} \geq \delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ and $C = C_\delta$ such that the following holds.*

Suppose that $\|\hat{\rho}_j \Psi\|_{L^3} < \varepsilon$ for all $j = 1, \ldots, k$. Let $f \in L^2_{\underline{\delta}-2}$ be a $(\underline{\mathbb{R}} \oplus M)$–valued 1–form satisfying $\int \langle f, \hat{\sigma} \otimes dx_h \rangle = 0$ for $h = 1, 2, 3$. Then there exists a *unique weak solution u to $d_2 d_2^* u = f$ with $\int \langle u, \hat\sigma \otimes dx_h\rangle\, \omega^{-2(\delta+1)} = 0$ and*

$$\|u\|_{L^2_{(\delta,-\frac{1}{2},-\frac{1}{2})}} + \|d_2^* u\|_{L^2} \leq \|f\|_{L^2_{\underline{\delta}-2}}.$$

Proof. First notice that $L^2_{\underline{\delta}-2} \hookrightarrow L^1$ (it would be enough $\delta \in (0,1)$ for this), so the quantity $\int \langle f, \hat\sigma \otimes dx_h\rangle$ is well defined. A second consequence of Hölder's inequality is that $L^2_{\underline{\delta}-2}$ is contained in the dual of $L^2_{(\delta,-\frac{1}{2},-\frac{1}{2})}$ (because $\delta \leq \frac{1}{2}$).

The solution u can be found by direct minimisation of the functional

$$\frac{1}{2}\int |d_2^* u|^2 - \langle u, f\rangle_{L^2}.$$

To justify this claim, introduce the Hilbert space H defined as the closure of C^∞_0 with respect to the norm $\|u\|_{L^2_{(\delta,-\frac{1}{2},-\frac{1}{2})}} + \|d_2^* u\|_{L^2}$. Arguing as in the proof of Lemma 4.5.5, we deduce from Lemma 4.5.7 that $\|d_2^* u\|_{L^2}$ is an equivalent norm on the $L^2(\omega^{-2(\delta+1)}\,\mathrm{dvol})$–orthogonal complement H_0 of the linear span of $\hat\sigma \otimes dx_h$, $h = 1,2,3$, provided $\|\hat\rho_j \Psi\|_{L^3}$ is sufficiently small. More precisely, we require $\|\hat\rho_j \Psi\|_{L^3} < \epsilon$ for an $\epsilon > 0$ which depends only on the Sobolev constant of $\mathbb{R}^3$ and the constant C of Lemma 4.5.7. Since $\int \langle f, \hat\sigma \otimes dx_h\rangle = 0$ and $d_2^*(dx_h \otimes \hat\sigma) = 0$, a weak solution $u \in H_0$ to $d_2 d_2^* u = f$ exists by direct minimisation of the functional. □

A priori estimates. With the next three lemmas we prove uniform a priori estimates for solutions to $d_2 d_2^* u = f$ with $f \in L^2_{\underline{\delta}-2}$. Using a partition of unity, the three lemmas imply that the weak solution of Lemma 4.5.8 actually lies in $W^{2,2}_{\underline{\delta}}$. The implication $d_2 d_2^* u \in L^2_{\underline{\delta}-2} \Rightarrow u \in W^{2,2}_{\underline{\delta}}$ follows from the analysis introduced in Chapter 3, but we have to make sure that the estimates hold with uniform constants.

Lemma 4.5.9. *For all $0 < \delta < 1$ there exists $\varepsilon(\delta) > 0$ and $C = C_\delta$ such that if $\|\hat\rho_j \Psi\|_{L^3} < \varepsilon(\delta)$ then for all $u \in C^\infty$ compactly supported in $B_1(q_j)$*

$$\|u\|_{W^{2,2}_{\underline{\delta}}} \leq C \|d_2 d_2^* u\|_{L^2_{\underline{\delta}-2}}.$$

Proof. The Lemma is proved in the same way as Lemma 4.5.5, using (4.4.14) to show that $\hat\rho_j^2 |d_A \Phi|$ is uniformly bounded. □

The reader will have noticed that in Definition 4.5.6 we used different powers of the weight function in a neighbourhood of the singularities and close to the points q_j. In particular, $u \in W^{2,2}_{\underline{\delta}}$ is forced to have stronger decay at each singularity p_i. This will be necessary to estimate the quadratic term of (4.5.1). Stronger estimates require a bit more work.

Lemma 4.5.10. *For all* $0 < \delta < \frac{1}{2}$ *there exists* $0 < \sigma_0 < \sigma$ *and* $C = C_\delta > 0$ *such that*

$$\|u\|_{W^{2,2}_{\underline{\delta}}} \leq C \|d_2 d_2^* u\|_{L^2_{\underline{\delta}-2}}.$$

for all $u \in C^\infty$ *compactly supported in* $B_{2\sigma_0}(p_i)$.

Proof. In a neighbourhood of p_i, the $W^{2,2}_{\underline{\delta}}$–norm is equivalent to the $\widetilde{W}^{2,2}_{\rho,\underline{\delta}}$–norm of Definition 3.2.5. The statement follows from Corollary 3.2.13 for some $\sigma_0, C > 0$ depending on the size of the deviation of $c'_{\text{ext}}(x_0)$ from an Euclidean Dirac monopole of charge 1 and mass 0. By (4.2.4) and the fact that the modification $c_{\text{ext}} \rightsquigarrow c'_{\text{ext}}(x_0)$ introduces only bounded terms in a neighbourhood of p_i, in $B_{2\sigma}(p_i)$ we write

$$\Phi'_{\text{ext}} = \left(\lambda'_i + \frac{1}{2\rho_i} + v + \frac{1}{2\pi} \sum_{j=1}^{k} \log|z_j - m_i| \right) \hat{\sigma} + \psi, \qquad A'_{\text{ext}} = A^0 \, \hat{\sigma} + a$$

where λ'_i depends only on S and $|(a,\psi)|, |(\nabla a, \nabla \psi)| \leq C$. This is not uniformly bounded because of the large positive constant $v + \frac{1}{2\pi}\sum_{j=1}^{k} \log|z_j - m_i|$. However, we are going to show that this term does not actually influence the estimates.

Since the pair $c'_{\text{ext}}(x_0)$ is reducible, the Lemma holds (for any $\sigma_0 > 0$) on the diagonal component by Proposition 3.2.12. Therefore assume that $u = u_T$.

As in the proof of Lemma 4.5.5 we are going to derive a priori estimates by integrations by parts. First,

$$\int \langle d_2 d_2^* u, u \rangle \, \hat{\rho}_i^{-2\delta-1} = \int (|\nabla_A u|^2 + |[\Phi, u]|^2) \hat{\rho}_i^{-2\delta-1} - (1+2\delta)\delta \int |u|^2 \hat{\rho}_i^{-2\delta-3} \tag{4.5.5}$$

Set $\Phi^0 = \frac{1}{2\rho_i}$. By Lemmas 3.2.4 and 3.2.11

$$\int \left(|\nabla_{A^0} u|^2 + |[\Phi^0, u]|^2 \right) \hat{\rho}_i^{-2\delta-1} \geq \left(\delta^2 + \frac{3}{4} \right) \int |u|^2 \hat{\rho}_i^{-2\delta-3}.$$

Using $|(a,\psi)| \leq C$ and $\langle \Phi'_{\text{ext}}(x_0), \hat{\sigma} \rangle \geq \Phi^0 + \lambda'_i + \psi$, we deduce

$$\int \left(|\nabla_A u|^2 + |[\Phi, u]|^2 \right) \hat{\rho}_i^{-2\delta-1} \geq \left(\delta^2 + \frac{3}{4} - C\sigma_0^2 \right) \int |u|^2 \hat{\rho}_i^{-2\delta-3}$$

Provided σ_0 is sufficiently small, plug this estimate back into (4.5.5):

$$\|\hat{\rho}_i^{\delta-\frac{3}{2}}u\|_{L^2} + \|\hat{\rho}_i^{\delta-\frac{1}{2}}d_2^*u\|_{L^2} \leq C\|\hat{\rho}_i^{\delta+\frac{1}{2}}d_2d_2^*u\|_{L^2}$$

for a constant $C = C_\delta > 0$.

Finally, to estimate the $W^{1,2}_{\underline{\delta}-1}$–norm of d_2^*u, integrate the Bochner identity (3.3.7) against $\hat{\rho}_i^{-2\delta+1}$ as in the proof of Lemma 4.5.5: The error $\Psi \equiv 0$ on the support of d_2^*u and the curvature term $\hat{\rho}_i^2|d_A\Phi|$ is bounded. The latter claim follows from the expression for $c'_{\text{ext}}(x_0)$ on $B_{2\sigma}(p_i)$ given above. □

Lemma 4.5.11. *For all $\delta \in (0,1)$ there exists $C = C_\delta > 0$ such that if $u \in C_0^\infty$ is supported on the complement $\Omega_{\sigma_0} \subset X$ of $\bigcup_{j=1}^k B_{\frac{1}{2}}(q_j) \cup \bigcup_{i=1}^n B_{\sigma_0}(p_i)$, then*

$$\|u\|_{W^{2,2}_{\underline{\delta}}} \leq C\left(\|d_2d_2^*u\|_{L^2_{\underline{\delta}-2}} + \|\omega^{-(\delta+1)}u\|_{L^2} + \|d_2^*u\|_{L^2}\right).$$

Proof. On the diagonal component, the Lemma can be deduced combining Lemma 3.3.6 and standard elliptic regularity on a compact set.

On the off-diagonal component, the estimate can be shown by integrations by parts as in Proposition 3.3.11 using the fact that $2|\Phi| > 1$ on the support of u and that $d_2^*u \in L^2$. Indeed, from the equality

$$\int \omega^{2\delta}\langle d_2d_2^*u, u\rangle = \int \omega^{2\delta}|d_2^*u|^2 + 2\delta\int \omega^{2\delta-1}\langle d_2^*u, d\omega\cdot u\rangle$$

we deduce

$$\|\omega^\delta d_2^*u\|_{L^2} \leq C\left(\|\omega^{\delta+1}d_2d_2^*u\|_{L^2} + \|\omega^{\delta-1}u\|_{L^2}\right)$$

Now notice that $\|\omega^{\delta-1}u\|_{L^2} \leq \|u\|_{L^2} \leq 2\|d_2^*u\|_{L^2}$ because $\delta < 1$ and $u = u_T$.

As many times before, the second order estimates follows by integrating by parts the Bochner identity (3.3.7), provided $\omega|d_A\Phi|$ is uniformly bounded on the support of u. Now, $\Phi'_{\text{ext}}(x_0)$ is a sum of Green's functions and their derivatives. By Lemma 2.1.3.(ii), for any $p = (z_0, t_0) \in X$

$$|\nabla G_p| + |z - z_0|\,|\nabla^2 G_p| \leq \frac{C}{|z - z_0|}$$

for all (z,t) such that $|z - z_0| > 2$. Therefore there exists a constant C depending on σ_0 and R_0 such that $\omega|d_A\Phi| \leq C$. □

Corollary 4.5.12. *For all* $0 < \delta < \frac{1}{2}$ *there exists* $\varepsilon = \varepsilon(\delta)$ *and* $C = C_\delta$ *with the following significance.*

Suppose that $\|\hat{\rho}_j \Psi\|_{L^3} < \varepsilon$ *for all* $j = 1, \ldots, k$. *Then for all* $f \in L^2_{\underline{\delta}-2}$ *such that* $\int \langle f, \hat{\sigma} \otimes dx_h \rangle = 0$ *for* $h = 1, 2, 3$ *there exists a unique solution* $u \in W^{2,2}_{\underline{\delta}}$ *to* $d_2 d_2^* u = f$ *with* $\int \langle u, \hat{\sigma} \otimes dx_h \rangle \, \omega^{-2(\delta+1)} = 0$. *Moreover,*

$$\|u\|_{W^{2,2}_{\underline{\delta}}} \leq C \|f\|_{L^2_{\underline{\delta}-2}}.$$

Proof. Lemma 4.5.8 yields the existence of a weak solution u with

$$\|u\|_{L^2_{(\delta, -\frac{1}{2}, -\frac{1}{2})}} + \|d_2^* u\|_{L^2} \leq C \|f\|_{L^2_{\underline{\delta}-2}}.$$

Choose a partition of unity subordinate to the cover $B_{2\sigma_0}(p_i), B_1(q_j)$ and Ω_{σ_0}. The derivatives of the cut-off functions are supported in regions where the weight functions are uniformly bounded above and below. Thus from Lemmas 4.5.9, 4.5.10 and 4.5.11 we deduce

$$\|u\|_{W^{2,2}_{\underline{\delta}}} \leq C \left(\|f\|_{L^2_{\underline{\delta}-2}} + \|u\|_{L^2_{(\delta, -\frac{1}{2}, -\frac{1}{2})}} + \|d_2^* u\|_{L^2} \right)$$

for a constant $C = C(\delta) > 0$ and the Lemma is proved. □

4.5.3 The linearised equation on U_{ext}: The large distance case

We come to the task of adapting the analysis to deal with the situation in which the points $q_1, \ldots, q_k$ move off to infinity.

By the assumption $d > d_0 = 5$ of Definition 4.2.1, the set $B_2(z_j) \times \mathbb{S}^1$ does not contain any of the points $q_1, \ldots, q_k, p_1, \ldots, p_n$ other than q_j. By taking d_0 larger, we can also assume that there exists $R_0 > 0$ such that that the ball $B_{R_0}(0) \subset \mathbb{R}^2$ is disjoint from $B_2(z_j)$ for all $j = 1, \ldots, k$ and $S \subset B_{R_0} \times \mathbb{S}^1$. Set $z_0 = 0$.

We are going to make the following additional assumption.

Assumption 4.5.13. *There exists* $K' > 0$ *such that*

$$\overline{d} = \max \{ |z_j - z_h|, |z_j - m_i| \text{ for all } j, h = 1, \ldots k, j \neq h, i = 1, \ldots n \} \leq K' d.$$

Now, fix a cover $\{\Omega_j\}_{j=0}^k$ of X such that Ω_j is an open neighbourhood of the set

$$\{(z,t) \in X \text{ such that } |z - z_j| \leq |z - z_h| \text{ for all } h = 0, \dots, k\} \tag{4.5.6}$$

for all $j = 0, \dots, k$. Let $\chi_0, \dots, \chi_k$ be a partition of unity subordinate to this cover.

Set $\omega_j(z,t) = \sqrt{1 + |z - z_j|^2}$ for all $j = 0, 1, \dots, k$. It will be convenient to define a smooth global weight function ω with the following properties:

$$\frac{1}{C_1}\,\omega_j \leq \omega \leq C_1\,\omega_j \quad \text{on } \Omega_j, \qquad \omega \leq C_1\omega_j \quad \text{everywhere} \tag{4.5.7a}$$

$$|\nabla\omega| \leq C_2, \qquad |\omega\,\triangle\omega| \leq C_3 \tag{4.5.7b}$$

In order to justify the existence of the constants C_2 and C_3, consider the following procedure to define ω. Given $z_1, \dots, z_k \in \mathbb{C}$, rescale by d around $z_0 = 0$. By Assumption 4.5.13 $z_1, \dots, z_k$ get mapped to a collection of k points in $\mathbb{C}$ such that the maximum and the minimum of the mutual distances are uniformly bounded above and below. Fix a function $\tilde{r}(z)$ which is a smoothing of $\min_{j=0,\dots,k}\{|z - d^{-1}z_j|\}$ outside of $z_0, \dots, z_k$. Since the distance function on $\mathbb{R}^2$ satisfies $|\nabla r| = 1$ and $r\triangle r = -1$ outside of the origin, $\|\nabla\tilde{r}\|_{L^\infty}$ and $\|\tilde{r}\triangle\tilde{r}\|_{L^\infty}$ are bounded. Now define

$$\omega(z,t) = \sqrt{1 + d^2\,\tilde{r}^2(d^{-1}z)}$$

and check that (4.5.7b) holds with constants depending on $\|\nabla\tilde{r}\|_{L^\infty}$ and $\|\tilde{r}\triangle\tilde{r}\|_{L^\infty}$.

Now proceed to define weighted Sobolev spaces $W^{m,2}_{\underline{\delta}-2+m}$, $m = 0, 1, 2$, as in Definition 4.5.6 using weight functions $\hat{\rho}_i, \hat{\rho}_j$ and ω. Notice that if $f \in L^2_{\underline{\delta}-2}$ then $\chi_j f \in L^2_{\underline{\delta}-2,j}$ by (4.5.7a). Here the subscript $_{,j}$ indicates that we use the weight function ω_j instead of ω in the definition of the norm.

Analysis on the off-diagonal component. We study the equation $d_2 d_2^* u = f$ in these newly defined spaces. When we restrict to the off-diagonal component, only minor modifications to the proof of Corollary 4.5.12 are necessary to show that $d_2 d_2^*\colon W^{2,2}_{\underline{\delta}} \to L^2_{\underline{\delta}-2}$ is an isomorphism.

Lemma 4.5.14. *For all $0 < \delta < \frac{1}{2}$ there exists $\varepsilon = \varepsilon(\delta)$ and $C = C_\delta$ with the following significance.*

Suppose that $\|\hat{\rho}_j\Psi\|_{L^3} < \varepsilon$ for all $j = 1, \dots, k$. Then for all $f = f_T \in L^2_{\underline{\delta}-2}$

there exists a unique solution $u = u_T \in W^{2,2}_{\underline{\delta}}$ *to* $d_2 d_2^* u = f$*. Moreover,*

$$\|u\|_{W^{2,2}_{\underline{\delta}}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}.$$

Proof. Lemma 4.5.8 yields the existence of a weak solution u with

$$\|u\|_{L^2_{(\delta,-\frac{1}{2},-\frac{1}{2})}} + \|d_2^* u\|_{L^2} \leq C\|f\|_{L^2_{\underline{\delta}-2}}.$$

Indeed, since $2|\Phi| > 1$ outside of $\bigcup_{j=1}^k B_{\delta(\lambda_j)}(q_j)$ and $\omega \geq 1$, the weight function ω doesn't play any role at this stage.

The Lemma follows once we prove uniform weighted elliptic estimates. In a neighbourhood of p_i and q_j these are given by Lemma 4.5.9 and 4.5.10.

As in Lemma 4.5.11, on the set $\Omega_{\sigma_0} = X \setminus \bigcup_{j=1}^k B_{\frac{1}{2}}(q_j) \cup \bigcup_{i=1}^n B_{\sigma_0}(p_i)$ the estimate is obtained by integrations by parts. One has to use (4.5.7b) to control terms involving the derivatives of the weight function ω. Moreover, the constant in the estimate depends on $\|\omega d_A \Phi\|_{L^\infty}$. In order to show that this quantity is uniformly bounded, observe that by Lemma 2.1.3.(ii) and (4.5.7a)

$$\omega\left(|\nabla G_{q_j}| + |\nabla^2 G_{q_j}|\right) \leq C\frac{\omega_j}{|z - z_j|} \leq C$$
$$\omega|dG_{p_i}| \leq C\frac{\omega_0}{|z - m_i|} \leq C_{m_i}$$

if $|z - z_j| > 2$ and $|z - m_i| > 2$, respectively, for a constant C_{m_i} depending on $|m_i|$. On the other hand, if $|z - z_j| \leq 2$ and $\rho_j \geq \frac{1}{2}$ or $|z - m_i| \leq 2$ and $\rho_i \geq \sigma_0$, then both ω and $d_A \Phi$ are bounded (depending on σ_0 and R_0). □

Analysis on the diagonal component. On the diagonal component there is an additional technical difficulty. Consider the following definition.

Definition 4.5.15. For all $j = 0, 1, \ldots, k$ define $v_j = -\frac{1}{4\pi^2}\psi_j \log|z - z_j|$, where ψ_j is a smooth cut-off function with $\psi_j \equiv 0$ if $|z - z_j| \leq 1$ and $\psi_j \equiv 1$ if $|z - z_j| \geq 2$.

Important properties of v_j are:

(i) There exists a constant $C > 0$ such that $\|\nabla v_j\|_{L^\infty} + \|\nabla^2 v_j\|_{L^\infty} \leq C$;

(ii) $\int_X \triangle v_j = 1$.

The special role of the functions v_j is explained by the following fact. Given $h \neq j$, set $u = v_j - v_h$. Then $\triangle u$ has mean value zero and $\|\triangle u\|_{L^2_{\underline{\delta}-2}} \leq C$ for a uniform constant C. However, restricting to the annulus $2 \leq |z - z_j| \leq \frac{1}{2}|z_j - z_h|$,

$$\int |\nabla u|^2 \geq c_1 \log |z_j - z_h| - \frac{c_2}{|z_j - z_h|^2} \xrightarrow{d \to \infty} \infty$$

for constants $c_1, c_2 > 0$.

Definition 4.5.16. (i) Let W be the finite dimensional subspace of 1–forms with values in $\underline{\mathbb{R}} \oplus M$ defined by:

$$W = \left\{ \sum_{h=1}^{3} \sum_{j=0}^{k} \alpha_{h,j}\, d_2^* \left(v_j\, \hat{\sigma} \otimes dx_h \right) \text{ such that } \sum_{j=0}^{k} \alpha_{h,j} = 0 \text{ for all } h = 1, 2, 3 \right\}$$

Define a norm on W by declaring $d_2^* \left(v_j\, \hat{\sigma} \otimes dx_h \right)$ an orthonormal system.

(ii) Given $f \in L^2_{\underline{\delta}-2}$ with $\int \langle f, \hat{\sigma} \otimes dx_h \rangle = 0$, denote by $\alpha(f)$ the element of W defined by $\alpha_{h,j} = \int \langle \chi_j f, \hat{\sigma} \otimes dx_h \rangle$ for all $h = 1, 2, 3$ and $j = 0, \ldots, k$. Here χ_j is the partition of unity subordinate to the covering $\{\Omega_j\}_{j=0}^{k}$ of (4.5.6).

Notice that since $0 < \delta < \frac{1}{2}$ the inclusion $L^2_{\underline{\delta}-2} \hookrightarrow L^1$ is continuous. For this, use (4.5.7a) to deduce

$$\int \omega^{-2(\delta+1)} \leq C_1 \sum_{j=0}^{k} \int \omega_j^{-2(\delta+1)} < +\infty.$$

Thus there exists a constant $C > 0$ such that

$$|\alpha(f)| \leq C \|f\|_{L^2_{\underline{\delta}-2}}. \tag{4.5.8}$$

Lemma 4.5.17. *For all $0 < \delta < \frac{1}{2}$ there exists $\varepsilon = \varepsilon(\delta)$ and $C = C_\delta$ with the following significance.*

Suppose that $\|\hat{\rho}_j \Psi\|_{L^3} < \varepsilon$ for all $j = 1, \ldots, k$. Then for all $f = f_D \in L^2_{\underline{\delta}-2}$ such that $\int \langle f, \hat{\sigma} \otimes dx_h \rangle = 0$ for $h = 1, 2, 3$ there exists $\xi = \xi_D \in W^{1,2}_{\underline{\delta}-1}$ such that $d_2 \xi = f - d_2 \alpha(f)$. Moreover,

$$\|\xi\|_{W^{1,2}_{\underline{\delta}-1}} \leq C \|f\|_{L^2_{\underline{\delta}-2}}.$$

Proof. Write $f = \sum_{j=0}^{k} \chi_j f$ using the partition of unity subordinate to the covering $\{\Omega_j\}_{j=0}^{k}$. Set $\alpha_{h,j} = \int \langle \chi_j f, \hat{\sigma} \otimes dx_h \rangle$; $|\alpha_{h,j}| \leq C\|f\|_{L^2_{\underline{\delta}-2}}$ by (4.5.8).

Now $f_j = \chi_j f - \sum_{h=1}^{3} \alpha_{h,j}\, d_2 d_2^* (v_j \otimes dx_h)$ satisfies $\int \langle f_j, \hat{\sigma} \otimes dx_h \rangle = 0$ for all $h = 1, 2, 3$. Moreover, $\|f_j\|_{L^2_{\underline{\delta}-2,j}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}$ by (4.5.7a).

Now apply Corollary 4.5.12: There exists u_j, unique up to the addition of a constant, with the following properties:

(i) u_j is defined on $X^* = X \setminus S$ if $j = 0$ and on $X \setminus \{q_j\}$ otherwise;

(ii) $d_2 d_2^* u_j = f_j$;

(iii) For all $j = 0, \dots, k$

$$\|\omega_j^{\delta} d_2^* u_j\|_{L^2} + \|\omega_j^{\delta+1} \nabla (d_2^* u_j)\|_{L^2} \leq C\|f\|_{L^2_{\underline{\delta}-2}}$$

(iv) If $j = 0$

$$\|u_j|_{B_{2\sigma}(p_i)}\|_{W^{2,2}_{\underline{\delta}}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}$$

for all $i = 1, \dots, n$ and otherwise

$$\|u_j|_{B_1(q_j)}\|_{W^{2,2}_{\underline{\delta}}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}$$

Set $\xi = \sum_{j=0}^{k} d_2^* u_j$. It remains to show that $\xi \in W^{1,2}_{\underline{\delta}-1}$.

First, on the exterior domain $\Omega_\sigma = X \setminus \bigcup_{j=1}^{k} B_{\frac{1}{2}}(q_j) \cup \bigcup_{i=1}^{n} B_\sigma(p_i)$, we use the fact that $\omega \leq C_1 \omega_j$ for all $j = 0, \dots, k$ and (iii) above.

In particular, $d_2^* u_h|_{B_1(q_j)} \in W^{1,2}$ if $h \neq j \neq 0$ and $d_2^* u_h|_{B_{2\sigma}(p_i)} \in W^{1,2}$ if $h \neq j = 0$ because $\omega \geq 1$. Then the Sobolev embedding $W^{1,2} \hookrightarrow L^6$ yields:

$$\|d_2^* u_h|_{B_1(q_j)}\|_{W^{1,2}_{\underline{\delta}-1}} \leq \|\hat{\rho}_j^{\delta-\frac{1}{2}}\|_{L^3}\, \|d_2^* u_h|_{B_1(q_j)}\|_{L^6} + \|\hat{\rho}_j^{\delta+\frac{1}{2}}\|_{L^\infty}\, \|\nabla(d_2^* u_h)|_{B_1(q_j)}\|_{L^2}$$

A similar estimate holds on $B_{2\sigma}(p_i)$ changing δ into $-\delta$. The norms of the weight functions are uniformly bounded because $0 < \delta < \frac{1}{2}$. □

4.6 Solving the linearised equation modulo obstructions

With these technical details out of the way, we combine Lemma 4.5.5, Corollary 4.5.12 and Lemmas 4.5.14 and 4.5.17 to solve the equation $d_2 \xi = f$ modulo obstructions.

Fix $0 < \delta < \frac{1}{2}$. We define weighted Sobolev spaces as in Definition 4.5.6, with the difference that over $B_1(q_j)$ we replace $\hat{\rho}_j$ of (4.5.4) with the smooth weight function w_j of (4.5.3). Moreover, the weight function ω is defined differently in the two situations:

(A) $S, \{q_1, \dots, q_k\} \subset B_{R_0} \times \mathbb{S}^1$ for some $R_0 > 0$;

(B) $d \to \infty$ and $S, q_1, \dots, q_k$ satisfy Assumption 4.5.13 for some $K' > 0$.

With these modifications and distinctions understood, the $W^{m,2}_{\underline{\delta}+m-2}$–norm coincides with the $W^{m,2}_{w,\delta}$–norm of Definition 4.5.4 over U_j; over U_{ext}, the spaces $W^{m,2}_{\underline{\delta}+m-2}$ are equivalent to the ones used in Section 4.5.2 in case (A) and to those introduced in Section 4.5.3 in case (B). In the latter case, it is also necessary to consider the finite dimensional space W of Definition 4.5.16.

Cut-off functions. We are going to define cut-off functions $\gamma_j, \gamma_{\text{ext}}, \beta_j, \beta_{\text{ext}}$ with some specific properties. As in [38, Lemma 7.2.10], the freedom to choose the parameter N in Definition 4.4.2 plays an important role.

Let γ_j be a smooth function supported in $B_{2\delta_j}(q_j)$ and such that $\gamma_j \equiv 1$ on $B_{\frac{\delta_j}{2}}(q_j)$. We require that $|\nabla\gamma_j| \leq \frac{2}{\delta_j}$. Define γ_{ext} by

$$\gamma_{\text{ext}} = \begin{cases} 1 - \gamma_j & \text{if } \rho \leq 2\delta_j, \\ 1 & \text{otherwise.} \end{cases}$$

The definition of β_j requires more care. Suppose that $N > 2$. Let β be a smooth function $\mathbb{R} \to \mathbb{R}$ such that $\beta \equiv 1$ on $(-\infty, 0]$ and $\beta \equiv 0$ on $[1, +\infty)$. Define $\beta_j(\rho_j) = \beta(a + b\log\rho_j)$ if $2\delta_j \leq \rho_j \leq N\delta_j$, where

$$\begin{cases} a = -\dfrac{\log 2\delta_j}{\log N - \log 2} \\ b = \dfrac{1}{\log N - \log 2} \end{cases}$$

and $\beta_j \equiv 1$ on $B_{2\delta_j}(q_j)$, $\beta_j \equiv 0$ if $\rho_j \geq N\delta_j$. Then

$$|\nabla\beta_j| \leq \frac{C}{\log\frac{N}{2}}\frac{1}{\rho_j} \tag{4.6.1}$$

for a constant C depending on $\|\beta\|_{C^1}$. In a similar way, define a cut-off function β_{ext} such that $\beta_{\text{ext}} \equiv 1$ on the complement of $\bigcup_{j=1}^k B_{\frac{\delta_j}{2}}(q_j)$, $\beta_{\text{ext}} \equiv 0$ on $B_{N^{-1}\delta_j}(q_j)$ and

$$|\nabla\beta_{\text{ext}}| \leq \frac{C}{\log\frac{N}{2}}\frac{1}{\rho_j}.$$

Obstructions. In Definition 4.4.3 we distinguished smooth sections $o_h \in \Omega(V)$. They are supported on U_{ext} and under the identification $V|_{U_{\text{ext}}} \simeq \underline{\mathbb{R}} \oplus M$ have only diagonal component. The crucial property of o_h, $h = 1, 2, 3$, is given by the following lemma.

Lemma 4.6.1. *For all* $h, l = 1, 2, 3$, $\langle d_2 o_h, \hat{\sigma} \otimes dx_l\rangle_{L^2} = \delta_{hl}$.

Proof. Since $c(x_0, \tau)$ is abelian in U_{ext}, $d_2 o_h \oplus d_1^* o_h = \not{D} o_h$, where $\not{D}$ is the Dirac operator of X. Since the Clifford multiplication by dx_l commutes with $\not{D}$, it is enough to prove that

$$\langle \not{D} o_4, \hat{\sigma} \otimes dx_h\rangle_{L^2} = 0, \qquad \langle \not{D} o_4, \hat{\sigma}\rangle_{L^2} = 1.$$

Recall that $o_4 = -\frac{1}{2\pi k}\sum_{j=1}^k \big(\chi_{\text{ext}}^j\, dG_{q_j}, 0\big)\hat{\sigma}$. Drop the index j and calculate:

$$-\langle \not{D}(\chi\, dG, 0), \hat{\sigma}\rangle_{L^2} = \int_B d(\chi * dG) = \int_{\partial B} *dG = 2\pi$$
$$\langle \not{D}(\chi\, dG, 0), \hat{\sigma} \otimes dx_h\rangle_{L^2} = \int_B d(\chi dG) \wedge dx_h = \int_{\partial B} dG \wedge dx_h = 0$$

Here $B = B_{2N\delta_j}(q_j)$. □

Definition 4.6.2. We refer to $\operatorname{span}\{d_2 o_h \,|\, h = 1, 2, 3\} \subset L^2_{\underline{\delta}-2}$ as the space of *obstructions*. Define a map $\pi\colon L^2_{\underline{\delta}-2} \to L^2_{\underline{\delta}-2}$ by

$$\pi(f) = f - \sum_{h=1}^{4} \langle f, \gamma_{\text{ext}}\, \hat{\sigma} \otimes dx_h\rangle_{L^2}\, d_2 o_h. \tag{4.6.2}$$

Lemma 4.6.3. *There exists a constant* C *such that*

$$\|\pi(f)\|_{L^2_{\underline{\delta}-2}} \leq C\left(N^{-2}\lambda\right)^{\frac{1-\delta}{2}} \|f\|_{L^2_{\underline{\delta}-2}}.$$

If f *is supported on the union of the annuli* $A_{j,int} \cup A_j \cup A_{j,ext}$ *for* $j = 1, \ldots, k$ *then* *the estimate can be improved to*

$$\|\pi(f)\|_{L^2_{\underline{\delta}-2}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}.$$

Proof. We have already observed (just before (4.5.8)) that $L^2_{\underline{\delta}-2} \hookrightarrow L^1$ is continuous. Moreover, if f is supported in $\bigcup_{j=1}^{k}\left(A_{j,\text{int}} \cup A_j \cup A_{j\text{ext}}\right)$ then

$$\|f\|_{L^1} \leq C\left(N^{-2}\lambda\right)^{-\frac{1-\delta}{2}}\|f\|_{L^2_{\underline{\delta}-2}}$$

It is enough to estimate $\|d_2 o_h\|_{L^2_{\underline{\delta}-2}}$. From Definition 4.4.3

$$|d_2 o_h| \leq C\sum_{j=1}^{k}\rho_j^{-2}|\nabla\chi^j_{\text{ext}}|.$$

Since $\nabla\chi^j_{\text{ext}}$ is supported in the region where the $w_j \sim \rho_j$ uniformly we conclude

$$\|d_2 o_h\|_{L^2_{\underline{\delta}-2}} \leq C\left(N^{-2}\lambda\right)^{\frac{1-\delta}{2}}. \tag{4.6.3}$$

□

Solving the linearised equation. We can finally solve the inhomogeneous equation $d_2\xi = f$ modulo obstructions.

Proposition 4.6.4. *Fix* $0 < \delta < \frac{1}{2}$.

(A) Fix $R_0 > 0$ *such that* $S, \{q_1, \dots, q_k\} \subset B_{R_0} \times \mathbb{S}^1$. *There exist* $\varepsilon = \varepsilon(\delta) > 0$, $N_0 > 2$ *and* $C = C_\delta > 0$ *with the following significance.*

Suppose that $\|w_j\Psi\|_{L^3} < \varepsilon(\delta)$ *for all* $j = 1, \dots, k$ *and* $N > N_0$. *Then there exists a map* $Q\colon$ *im* $\pi \subset L^2_{\underline{\delta}-2} \to W^{1,2}_{\underline{\delta}-1}$ *such that* $\pi \circ d_2 Q(f) = f$ *and*

$$\|Qf\|_{W^{1,2}_{\underline{\delta}-1}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}.$$

(B) In the large distance case, suppose that $S, q_1, \dots, q_k$ *satisfy Assumption 4.5.13 for some* $K' > 0$. *There exist* $\varepsilon = \varepsilon(\delta) > 0$, $N_0 > 2$ *and* $C = C_\delta > 0$ *with the following significance.*

Suppose that $\|w_j\Psi\|_{L^3} < \varepsilon(\delta)$ *for all* $j = 1, \dots, k$ *and* $N > N_0$. *Then there exist a map* $Q = (Q_1, Q_2)$, *where* $Q_1\colon$ *im* $\pi \subset L^2_{\underline{\delta}-2} \to W^{1,2}_{\underline{\delta}-1}$ *and*

$Q_2\colon$ *im* $\pi \subset L^2_{\underline{\delta}-2} \to W$, *such that* $\pi \circ d_2 Q_1(f) + \pi \circ d_2\big(\beta_{ext}\, Q_2(f)\big) = f$. *Moreover,*

$$\|Q_1(f)\|_{W^{1,2}_{\underline{\delta}-1}} + |Q_2(f)| \leq C\|f\|_{L^2_{\underline{\delta}-2}}.$$

Proof. We prove the statement in (B). The statement in (A) follows in a similar way using Corollary 4.5.12 instead of Lemmas 4.5.14 and 4.5.17.

By abuse of notation, regard d_2 as the operator $d_2\colon\ W^{1,2}_{\underline{\delta}-1} \oplus W \to L^2_{\underline{\delta}-2}$

$$d_2(\xi,\eta) = d_2(\xi + \beta_{\text{ext}}\,\eta).$$

Given $f \in L^2_{\underline{\delta}-2}$ such that $f = \pi(f)$, write $f = \sum_{j=1}^k \gamma_j f + \gamma_{\text{ext}} f$ and define maps $Q_1'\colon\ \text{im}\,\pi \to W^{1,2}_{\underline{\delta}-1}$ and $Q_2'\colon\ \text{im}\,\pi \to W$ as follows: $Q_2'(f) = \alpha(\gamma_{\text{ext}}\, f)$, where α is the map of Definition 4.5.16.(ii).

$$Q_1'(f) = \sum_{j=1}^k \beta_j\,\xi_j + \beta_{\text{ext}}\,\xi_{\text{ext}},$$

where $\xi_j = d_2^* u_j$ for the solution u_j to $d_2 d_2^* u_j = \gamma_j f$ of Lemma 4.5.5 and ξ_{ext} is the solution to $d_2\xi_{\text{ext}} = \gamma_{\text{ext}} f - \alpha(\gamma_{\text{ext}} f)$ obtained combining Lemmas 4.5.14 and 4.5.17. From the estimates in those lemmas, (4.5.8) and (4.6.1) we deduce the existence of $C > 0$ such that

$$\|Q_1'(f)\|_{W^{1,2}_{\underline{\delta}-1}} + |Q_2'(f)| \leq C\|f\|_{L^2_{\underline{\delta}-2}}.$$

Moreover, if we set $Q' = (Q_1', Q_2')$,

$$f - d_2 Q'(f) = \sum_{j=1}^k \nabla\beta_j \cdot \xi_j + \nabla\beta_{\text{ext}} \cdot \xi_{\text{ext}} + \nabla\beta_{\text{ext}} \cdot \alpha(\gamma_{\text{ext}} f).$$

It follows from (4.6.1) and Hölder's inequality that

$$\|f - d_2 Q'(f)\|_{L^2_{\underline{\delta}-2}} \leq \frac{C}{\log\frac{N}{2}}\|Q_1'(f)\|_{W^{1,2}_{\underline{\delta}-1}} + C\frac{\lambda^{-\frac{1+\delta}{2}}}{\log\frac{N}{2}}\|Q_2'(f)\|_{L^\infty} \leq \frac{C}{\log\frac{N}{2}}\|f\|_{L^2_{\underline{\delta}-2}}$$

because $\|\eta\|_{L^\infty} \leq C|\eta|$ for all $\eta \in W$. Since $f - d_2 Q'(f)$ is supported in $\bigcup_{j=1}^k A_j$, Lemma 4.6.3 yields

$$\|f - \pi \circ d_2 Q'(f)\|_{L^2_{\underline{\delta}-2}} \leq \frac{C}{\log\frac{N}{2}}\|f\|_{L^2_{\underline{\delta}-2}}.$$

If N is sufficiently large we can iterate:

$$Q := Q' \circ \sum_{m=0}^{\infty} (\mathrm{id} - \pi \circ d_2 \circ Q')^m \qquad \square$$

4.7 Deformation

In this section we are going to complete the construction of a family of solutions to the non-linear equation (4.5.1). Using the projection π of Definition 4.6.2, (4.5.1) can be split into an infinite dimensional and a finite dimensional equation. We solve the infinite dimensional equation by means of the Implicit Function Theorem. We state a version adapted to the statement of Proposition 4.6.4 in case (B). The statement in case (A) is obtained by setting $W = \{0\}$.

Lemma 4.7.1. *Given $c = c(x_0, \tau)$, let $\Psi\colon W^{1,2}_{\underline{\delta}-1} \oplus W \to L^2_{\underline{\delta}-2}$ be the smooth map*

$$\Psi(\xi, \eta) = d_2(\xi + \beta_{ext}\eta) + (\xi + \beta_{ext}\eta) \cdot (\xi + \beta_{ext}\eta) + \Psi(x_0, \tau).$$

Suppose that the following conditions hold:

(i) There exists a projection $\pi\colon L^2_{\underline{\delta}-2} \to L^2_{\underline{\delta}-2}$ such that the map

$$\pi \circ d_2\colon W^{1,2}_{\underline{\delta}-1} \oplus W \to \mathit{im}\,\pi \subset L^2_{\underline{\delta}-2}$$

admits a right inverse $Q = (Q_1, Q_2)$ with

$$\|Q_1(f)\|_{W^{1,2}_{\underline{\delta}-1}} + |Q_2(f)| \le C\|f\|_{L^2_{\underline{\delta}-2}}$$

for all $f \in L^2_{\underline{\delta}-2}$ with $\pi(f) = f$.

(ii) There exists $q > 0$ such that:

$$\|\pi\big((\xi,\eta)\cdot(\xi,\eta) - (\xi',\eta')\cdot(\xi',\eta')\big)\|_{L^2_{\underline{\delta}-2}} \le \\ q\left(\|\xi + \xi'\|_{W^{1,2}_{\underline{\delta}-1}} + |\eta + \eta'|\right)\left(\|\xi - \xi'\|_{W^{1,2}_{\underline{\delta}-1}} + |\eta - \eta'|\right)$$

Here $(\xi,\eta)\cdot(\xi,\eta) = (\xi + \beta_{ext}\eta)\cdot(\xi + \beta_{ext}\eta)$.

(iii) $\|\pi\big(\Psi(x_0,\tau)\big)\|_{L^2_{\underline{\delta}-2}} \le \frac{1}{8qC^2}$.

Then there exists a unique $(\xi,\eta) \in \text{im}\, Q \subset W^{1,2}_{\underline{\delta}-1} \oplus W$*, with*

$$\|\xi\|_{W^{1,2}_{\underline{\delta}-1}} + |\eta| \leq 2C \|\pi\big(\Psi(x_0,\tau)\big)\|_{L^2_{\underline{\delta}-2}}$$

and such that $\pi\big(\Psi(\xi,\eta)\big) = 0$.

Proof. The Lemma is an immediate consequence of the contraction mapping principle. □

Proposition 4.6.4 shows that the condition in (i) is satisfied if $\Psi = \Psi(x_0,\tau)$ is sufficiently small, in the sense that $\|w_j \Psi(x_0,\tau)\|_{L^3} < \varepsilon$ for all $j = 1,\dots,k$. We are going to show that the conditions (ii) and (iii) are also satisfied if λ is sufficiently large.

The quadratic term. As a preliminary remark, observe that the product $\cdot$ is commutative and therefore $(\xi,\eta)\cdot(\xi,\eta) - (\xi',\eta')\cdot(\xi',\eta') = (\xi+\xi',\eta+\eta')\cdot(\xi-\xi',\eta-\eta')$. In view of Lemma 4.6.3, in order to verify condition (ii) it is enough to show that the product $\cdot$ defines a continuous map $\Big(W^{1,2}_{\underline{\delta}-1} \oplus W\Big) \times \Big(W^{1,2}_{\underline{\delta}-1} \oplus W\Big) \to L^2_{\underline{\delta}-2}$. Notice that this property also justifies the claim that the map Ψ of Lemma 4.7.1 is smooth.

Since the product $\cdot$ is induced by the Lie bracket on $\mathfrak{su}(2)$, with respect to the decomposition $V \simeq \underline{\mathbb{R}} \oplus M$ over U_{ext} there is no $\xi_D \cdot \xi'_D$ term in the product. Moreover, recall that $\beta_{\text{ext}}\eta$ for $\eta \in W$ has only diagonal component.

As in Lemma 3.2.7 and 3.3.9 the statement follows from the Hölder inequality and the Sobolev embedding $W^{1,2} \hookrightarrow L^6$:

- On the ball $B_1(q_j)$, if $\xi \in W^{1,2}_{\underline{\delta}-1}$ and $\eta \in W$

$$\int_{B_1} w_j^{2\delta+1} |\xi \cdot \beta_{\text{ext}}\eta|^2 \leq C \|w_j \eta\|^2_{L^\infty} \int_{B_1} w_j^{2\delta-1} |\xi|^2 \leq C|\eta|^2 \|\xi\|_{W^{1,2}_{\underline{\delta}-1}}$$

  because $w_j \leq 1$ and $\|\eta\|_{L^\infty} \leq C|\eta|$ by Definition 4.5.16.

  On the other hand, we have a continuous embedding $W^{1,2}_{\underline{\delta}-1} \hookrightarrow w_j^{-\delta-\frac{1}{2}} L^6\big(B_1(q_j)\big)$. In order to estimate the norm $\|\xi \cdot \xi'\|_{L^2_{\underline{\delta}-2}}$ for $\xi, \xi' \in W^{1,2}_{\underline{\delta}-1}$, by the Hölder inequality it is enough to observe that

$$\int_{B_1} w_j^{2\delta+1} |\xi|^4 \leq \lambda_j^{2\delta} \, \|w_j^{\delta-\frac{1}{2}} \xi\|_{L^2} \, \|w_j^{\delta+\frac{1}{2}} \xi\|^3_{L^6} \leq C \lambda_j^{2\delta} \|\xi\|^4_{W^{1,2}_{\underline{\delta}-1}},$$

The factor of λ_j is due to $w_j \geq \lambda_j^{-1}$.

- On the ball $B_{2\sigma}(p_i)$ the same calculations with $-\delta$ in place of δ yield

$$\|(\xi,\eta)\cdot(\xi',\eta')\|_{L^2_{\underline{\delta}-2}(B_{2\sigma})} \leq C\|(\xi,\eta)\|_{W^{1,2}_{\underline{\delta}-1}\oplus W}\|(\xi',\eta')\|_{W^{1,2}_{\underline{\delta}-1}\oplus W}$$

with a uniform constant C.

- Finally, on the set $\Omega_\sigma = X \setminus \bigcup_{j=1}^k B_{\frac{1}{2}}(q_j) \cup \bigcup_{i=1}^n B_\sigma(p_i)$ write $\xi = \xi_D + \xi_T$ with respect to the decomposition $V \simeq \underline{\mathbb{R}} \oplus M$.

  Because of the properties (4.5.7) of the weight function ω, if $\xi \in W^{1,2}_{\underline{\delta}-1}$ then $\omega^\delta \xi_D, \omega^{\delta+1}\xi_T \in W^{1,2} \hookrightarrow L^p$ for all $2 \leq p \leq 6$. Therefore

$$\|\omega^{\delta+1}(\xi\cdot\xi')\|_{L^2} \leq \|\xi\|_{L^3}\|\omega^{\delta+1}\xi'_T\|_{L^6} + \|\xi'\|_{L^3}\|\omega^{\delta+1}\xi_T\|_{L^6}$$
$$\|\omega^{\delta+1}(\xi\cdot\eta)\|_{L^2} \leq \|\eta\|_{L^\infty}\|\omega^{\delta+1}\xi_T\|_{L^2}.$$

In conclusion, combining these estimates and using Definition 4.2.1.(iii), we proved that there exists a uniform constant $C > 0$ such that

$$\|(\xi,\eta)\cdot(\xi',\eta')\|_{L^2_{\underline{\delta}-2}} \leq C\lambda^\delta\|(\xi,\eta)\|_{W^{1,2}_{\underline{\delta}-1}\oplus W}\|(\xi',\eta')\|_{W^{1,2}_{\underline{\delta}-1}\oplus W}. \tag{4.7.1}$$

Together with Lemma 4.6.3 this implies that condition (ii) of Lemma 4.7.1 holds with a constant

$$q = C\lambda^{\frac{1+\delta}{2}}. \tag{4.7.2}$$

The error. It remains to show that the error $\Psi(x_0,\tau)$ can be made as small as required by taking λ sufficiently large, uniformly for all $(x_0,\tau) \in \mathcal{P}$.

By Definition 4.4.3 and Lemma 4.6.1 $\pi(\Psi_\zeta) = 0$. Here Ψ_ζ is the component of $\Psi(x_0,\tau)$ defined by (4.4.15). We need to show that:

(a) $\|w_j\Psi\|_{L^3} < \varepsilon(\delta)$, where $\varepsilon(\delta) > 0$ is given in Proposition 4.6.4.

(b) $\pi(\Psi - \Psi_\zeta)$ satisfies the smallness condition in Lemma 4.7.1.(iii).

- By (4.4.16)

$$\|\Psi(x_0,\tau) - \Psi_\zeta\|_{L^2_{\underline{\delta}-2}} \leq C\lambda^{-1-\frac{\delta}{2}}. \tag{4.7.3}$$

Now use the last statement in Lemma 4.6.3 to deduce

$$\|\pi\big(\Psi(x_0,\tau)\big)\|_{L^2_{\underline{\delta}-2}} = O(\lambda^{-1-\frac{\delta}{2}}). \tag{4.7.4}$$

- Similarly,

$$\|w_j\left(\Psi(x_0,\tau)-\Psi_\zeta\right)\|_{L^3} \leq C\lambda^{-1}.$$

On the other hand, the definition (4.4.15) of Ψ_ζ, the fact that $\zeta = -\sum_{j=1}^k \frac{x_0^j}{\lambda_j}$ and $|d_2 o_h| \leq \sum_{j=1}^k \rho_j^{-2}|\nabla\chi^j_{\text{ext}}|$ yield

$$\|w_j\Psi_\zeta\|_{L^3} \leq C\lambda^{-\frac{1}{2}}.$$

Existence results. We have all the ingredients to prove the main result of this chapter, an existence theorem for periodic monopoles (with singularities). We begin with a rewriting of the pregluing map of Lemma 4.4.4.

Fix $d_0 \geq 5$, $K > 1$, parameters v, b, the set S of singularities and the centres of non-abelian monopoles $q_1,\dots,q_k$. For all $N > 2$ we fix $\lambda_0(N)$ sufficiently large and assume that $v, S, q_1,\dots,q_k$ are (λ_0, d_0, K)–admissible. Moreover, we assume that either:

(A) There exists $R_0 > 0$ such that $S, \{q_1,\dots,q_k\} \subset B_{R_0}\times\mathbb{S}^1$; or

(B) There exist $R_0, K' > 0$ such that $S \subset B_{R_0}\times\mathbb{S}^1$, $q_1,\dots,q_k \in X\setminus(B_{R_0}\times\mathbb{S}^1)$ and Assumption 4.5.13 is satisfied.

Finally, for $\kappa \in (0,1)$ sufficiently small, let $\mathcal{P} = \mathcal{P}_\kappa$ be the set of gluing data of Definition 4.4.2.

Consider the family $c(x_0,\tau)$ of Lemma 4.4.4. If κ is sufficiently small and $\lambda_0(N)$ sufficiently large, we can assume that (4.4.13), (4.4.14) and (4.4.16) are satisfied, uniformly for all $(x_0,\tau)\in\mathcal{P}$.

Fix a base point $(0,\tau_0)\in\mathcal{P}$ and $\delta > 0$ sufficiently small. Then we can consider the affine space $\mathcal{C}_{\underline{\delta}} = c(0,\tau_0) + \left(W^{1,2}_{\underline{\delta}-1}\oplus W\right)$. For notational convenience, here and in the rest of the section we set $W=\{0\}$ when condition (A) above holds. Notice that, once $q_1,\dots,q_k$ are fixed, the weighted Sobolev norms used to define $\mathcal{C}_{\underline{\delta}}$ are equivalent to those used in Section 3.4 to construct the affine space $\mathcal{C}_{\delta_1,\delta_2}$ with $\delta_1 = \delta = -\delta_2$. Then the pregluing map of Lemma 4.4.4 can be considered as a smooth map

$$c\colon \mathcal{P} \to \mathcal{C}_{\underline{\delta}}.$$

Smoothness follows from Lemma 4.3.3 and the explicit construction of $c(x_0, \tau)$.

The group $\Gamma \simeq U(1)/\pm$ acts on $\mathcal{P}$ by $e^{2is} \cdot (x_0, \tau) = (x_0, \tau + s)$ and on $\mathcal{C}_{\underline{\delta}}$ as the gauge transformation $\exp(s\gamma_{\text{ext}}\hat{\sigma})$; the map c is Γ–equivariant.

Theorem 4.7.2. *Fix data as above and $N > N_0$, where N_0 is given by Proposition 4.6.4. Then there exists $\lambda_0' \geq \lambda_0(N)$ such that if $v, S, q_1, \ldots, q_k$ are (λ_0', d_0, K)–admissible then the following holds.*

(i) There exists a smooth Γ–equivariant map

$$c_1\colon \mathcal{P} \to \mathcal{C}_{\underline{\delta}}$$

of the form $c_1(x_0, \tau) = c(x_0, \tau) + \xi(x_0, \tau)$, where $\xi(x_0, \tau) \in W^{1,2}_{\underline{\delta}-1} \oplus W$ and

$$\|\xi(x_0, \tau)\|_{W^{1,2}_{\underline{\delta}-1} \oplus W} \leq C\lambda^{-1-\frac{\delta}{2}}.$$

(ii) There exist smooth Γ–invariant maps $H, h\colon \mathcal{P} \to \mathbb{R}^3$ with

$$H(x_0, \tau) = -\sum_{j=0}^{k} \frac{x_0^j}{\lambda_j}$$

and $|h(x_0, \tau)| = O(\lambda^{-\frac{3}{2}})$, such that $c_1(x_0, \tau)$ is a solution to the Bogomolny equation if and only if $H(x_0, \tau) + h(x_0, \tau) = 0$.

(iii) Given $(x_0, \tau) \in \mathcal{P}$ and $\zeta \in \mathbb{R}^3$, let $x_0 + \zeta$ denote the k–tuple $x_0 + (\zeta, \ldots, \zeta)$. For all $(x_0, \tau) \in \mathcal{P}_{\frac{\kappa}{2}}$ such that $H(x_0, \tau) = 0$ there exists $\zeta \in \mathbb{R}^3$ such that $|\zeta| = O(\lambda^{-\frac{1}{2}})$ and

$$H(x_0 + \zeta, \tau) + h(x_0 + \zeta, \tau) = 0.$$

Proof. (i) First of all, the map

$$\pi \circ \Psi\colon \mathcal{P} \times \left(W^{1,2}_{\underline{\delta}-1} \oplus W\right) \to L^2_{\underline{\delta}-2}$$

defined by

$$\pi \circ \Psi(x_0, \tau, \xi) = \pi\big(d_2\xi + \xi \cdot \xi + \Psi(x_0, \tau)\big)$$

is smooth because of the smoothness of $c\colon \mathcal{P} \to \mathcal{C}_{\underline{\delta}}$, Lemma 4.6.3 and the continuity of the product $\Big(W^{1,2}_{\underline{\delta}-1} \oplus W\Big) \times \Big(W^{1,2}_{\underline{\delta}-1} \oplus W\Big) \to L^2_{\underline{\delta}-2}$.

Choosing $\lambda_0(N)$ larger if necessary, we can assume that $\|w_j \Psi(x_0, \tau)\|_{L^3} = O(\lambda^{-\frac{1}{2}}) < \varepsilon$, where ε is given by Proposition 4.6.4. Now fix $N > N_0$ so that Proposition 4.6.4 holds. Finally, we can choose $\lambda_0' \geq \lambda_0(N)$ so that

$$\|\pi\big(\Psi(x_0, \tau)\big)\|_{L^2_{\underline{\delta}-2}} = O(\lambda^{-1-\frac{\delta}{2}}) \leq \frac{1}{2qC^2} = O(\lambda^{-\frac{1}{2}-\frac{\delta}{2}})$$

whenever $\lambda > \lambda_0'$. Here q and C are given by (4.7.2) and Proposition 4.6.4, respectively, and we used (4.7.4).

Lemma 4.7.1 now yields the existence of the map c_1. The fact that c_1 is smooth follows from the fact that the family of right inverses Q of Proposition 4.6.4 depends smoothly on (x_0, τ).

(ii) We are left with the three equations

$$\langle d_2^* \xi(x_0, \tau) + \xi(x_0, \tau) \cdot \xi(x_0, \tau) + \Psi(x_0, \tau), \gamma_{\text{ext}}\, \hat{\sigma} \otimes dx_h \rangle_{L^2} = 0,$$

for $h = 1, 2, 3$. We write $\xi(x_0, \tau) = \xi' + \beta_{\text{ext}}\eta$, with $\xi' \in W^{1,2}_{\underline{\delta}-1}$ and $\eta \in W$.

- $\langle d_2\xi', \gamma_{\text{ext}}\, \hat{\sigma} \otimes dx_h \rangle_{L^2} = O(\lambda^{-\frac{3}{2}})$. Indeed, integrating by parts

$$|\langle d_2\xi', \gamma_{\text{ext}}\, \hat{\sigma} \otimes dx_h \rangle_{L^2}| \leq \|\xi'\|_{W^{1,2}_{\underline{\delta}-1}} \|w_j^{-\delta+\frac{1}{2}} \nabla\gamma_{\text{ext}}\|_{L^2} = O(\lambda^{-\frac{3}{2}}).$$

- $\langle d_2(\beta_{\text{ext}}\eta), \gamma_{\text{ext}}\, \hat{\sigma} \otimes dx_h \rangle_{L^2} = 0$, because $\gamma_{\text{ext}} \equiv 0$ on the support of $\nabla\beta_{\text{ext}}$, $\beta_{\text{ext}} \equiv 1 \equiv \gamma_{\text{ext}}$ on the support of $d_2\eta$ and $\langle d_2\eta, \hat{\sigma} \otimes dx_h \rangle_{L^2} = 0$ by the definition of W.

- By the continuity of the embedding $L^2_{\underline{\delta}-2} \hookrightarrow L^1$ and (4.7.1)

$$|\langle \xi(x_0, \tau) \cdot \xi(x_0, \tau), \gamma_{\text{ext}}\, \hat{\sigma} \otimes dx_h \rangle_{L^2}| \leq C\lambda^{\delta} \|\xi(x_0, \tau)\|^2_{W^{1,2}_{\underline{\delta}-1} \oplus W} = O(\lambda^{-2}).$$

– Finally, as in the proof of Lemma 4.6.3,

$$|\langle \Psi(x_0,\tau) - \Psi_\zeta, \gamma_{\text{ext}}\hat{\sigma} \otimes dx_h \rangle_{L^2}| \leq C\lambda^{-\frac{1-\delta}{2}} \|\Psi(x_0,\tau) - \Psi_\zeta\|_{L^2_{\underline{\delta}-2}} = O(\lambda^{-\frac{3}{2}})$$

because $\Psi(x_0,\tau) - \Psi_\zeta$ is supported on $\bigcup_{j=1}^k (A_{j,\text{int}} \cup A_j \cup A_{j,\text{ext}})$.

On the other hand, by Lemma 4.6.1

$$\langle \Psi_\zeta, \gamma_{\text{ext}}\hat{\sigma} \otimes dx_h \rangle_{L^2} = -\sum_{j=1}^k \frac{x_0^j}{\lambda_j}$$

(iii) The claim follows from Brouwer's Fixed Point Theorem by writing

$$\zeta = -\frac{h(x_0+\zeta,\tau)}{\sum_{j=1}^k \lambda_j^{-1}} = O(\lambda^{-\frac{1}{2}}).$$

In the last equality we used Definition 4.2.1.(iii). □

4.8 Directions of future work

In this final section we return to the questions posed at the end of Chapter 1. We are not yet in the position to answer those questions definitively, but we want to discuss in what sense the results of this thesis represent a step in the direction of a rigorous understanding of Cherkis and Kapustin's predictions of [29, 33, 34]. We also want to present broader perspectives on related future directions of work.

Existence. Theorems 3.4.8 and 4.7.2 show that, for generic choices of parameters and if the mass v is sufficiently large when $n \geq 2(k-1)$, the moduli space $\mathcal{M}_{n,k}$ of charge k $SO(3)$ periodic monopoles with n singularities is a non-empty smooth hyperkähler manifold. Questions that remain open are a rigorous calculation of the dimension of $\mathcal{M}_{n,k}$, an understanding of the compactness properties of sequences of periodic monopoles and the related question of the completeness of the L^2–metric on $\mathcal{M}_{n,k}$.

It is interesting to specialise the existence result of Theorem 4.7.2 to low charge $k=1$ and $k=2$. Note that in these cases both the condition in Definition 4.2.1.(iii) and Assumption 4.5.13 are automatically satisfied.

In the charge 1 case, Theorem 4.7.2 yields the existence of periodic monopoles with $n = 0, 1$ or 2 singularities and large mass. More precisely, if $(m_i, a_i) \in X$, $i = 1, 2$, are two distinct points and $(z_0, t_0) \in X$ satisfies $|z_0 - m_i| \geq 5$, there exists $v_0 > 0$ such that a charge 1 $SO(3)$ periodic monopole with n singularities, mass v and centre (z_0, t_0) exists, provided $v > v_0$ if $n = 0$, $v - \frac{1}{2\pi} \log |z_0 - m_1| > v_0$ when $n = 1$ or $v - \frac{1}{2\pi} \log |z_0 - m_1| - \frac{1}{2\pi} \log |z_0 - m_2| > v_0$ if $n = 2$.

It would be interesting to have a direct proof of the existence of charge 1 periodic monopoles for an arbitrary value of the mass v. So far, the only argument for the existence of charge 1 periodic monopoles for any $v \in \mathbb{R}$ goes through the Nahm Transform (see below). It should be noted that an explicit formula is probably difficult to obtain, because no continuous symmetries are expected.

Moreover, one would like to obtain as much information as possible on the behaviour of periodic charge 1 monopoles as the mass varies. In particular, it seems interesting to complement the numerical simulations of Ward [107] with detailed quantitative estimates. If enough information is obtained, one can imagine extending the gluing construction to arbitrary configurations of well-separated points, with no restriction on the mass in the case $2(k-1) \leq n \leq 2k$.

Asymptotics of the metric. Theorem 4.7.2 yields the existence of charge 2 periodic monopoles in three different cases:

(i) If $n = 0$ or 1 then we can choose the mass $v \in \mathbb{R}$ arbitrarily and pick two points q_1, q_2 such that $d \geq d_0$ for d_0 sufficiently large.

(ii) If $n = 2$ we can still let $d \to \infty$. However, by (4.2.8) it is necessary to assume that the mass $v > v_0$ for v_0 sufficiently large in order to guarantee that the data are admissible.

(iii) In view of (4.2.8), when $n = 3$ or 4 we have to constrain the points q_1, q_2 to lie in a fixed compact set of X. Then Theorem 4.7.2 implies that there exists $v_0 > 0$ such that charge 2 periodic monopoles localised around q_1 and q_2 exist whenever the mass $v > v_0$.

Conjecture 4.8.1. *For generic choices of parameters, the moduli space of charge* 2 $SO(3)$ *periodic monopoles with* n *singularities is a smooth complete hyperkähler* 4*–manifold with only one end corresponding to the region in the moduli space in which charge* 2 *monopoles break into charge* 1 *components.*

Since X is parabolic the mass of these charge 1 components varies as their centres move off to infinity. In case (i) and (ii) above (under the assumption that the mass is sufficiently large in the latter case), the charge 1 components are close to the high mass limit in the "moduli space" of charge 1 monopoles, *i.e.* they are close to "bubbling off" as $d \to \infty$. Therefore in these cases we expect that Theorem 4.7.2 could be enhanced to give a complete description of the end of the moduli space. Notice that, at a rough heuristic level, the expected geometry agrees with Cherkis and Kapustin's predictions: The end of the moduli space admits a double cover (monopoles are indistinguishable) which is a circle bundle (the fibres correspond to the choice of gluing map) over an exterior domain in $\mathbb{R}^2 \times \mathbb{S}^1$, in agreement with (1.6.3). Part of the work to be done is not only to prove the surjectivity of the gluing construction, but also to understand the exact topology of this circle bundle depending on the number of singularities.

Finally, with additional work one could hope to derive a rigorous asymptotic formula for the L^2–metric on the moduli space in the coordinates provided by the gluing construction. It is perhaps instructive to recall what is known in the case of Euclidean monopoles beyond the explicit Atiyah–Hitchin metric. In [44] Gibbons and Manton conjectured that, in the region of the moduli space where charge k monopoles break into k charge 1 components, the L^2–metric on the moduli space of uncentred charge k $SU(2)$ monopoles on $\mathbb{R}^3$ without singularities is exponentially close to an explicit $\mathbb{T}^k$–invariant hyperkähler metric on a $\mathbb{T}^k$–bundle over the configuration space of k well-separated points in $\mathbb{R}^3$. The asymptotic metric on the moduli space of centred monopoles is then obtained as the hyperkähler quotient by the diagonal fibre-wise action of the circle. In [13] Bielawski confirmed this conjecture, using the Nahm Transform to pass to moduli spaces of solutions to Nahm's equations and construct the Gibbons–Manton metric in terms of these and twistor theory. It seems likely that the result can be recovered directly from Taubes's gluing construction: Gibbons–Manton's $\mathbb{T}^k$–bundle over the configuration space of k well-separated points in $\mathbb{R}^3$ is naturally identified with the space of gluing parameters.

Rational elliptic surfaces and the Hitchin–Kobayashi correspondence. In [29, 34] Cherkis and Kapustin show how to adapt Hitchin's definition of the scattering map [55] for Euclidean monopoles to the periodic (singular) case. In Donaldson's Theorem 1.6.1 on the identification of monopoles with rational maps one has to choose a direction in $\mathbb{R}^3$, corresponding to the choice of a complex structure on the moduli space. In the periodic case, we have a preferred direction corresponding to the circle factor: Cherkis and Kapustin consider the holonomy h of the (non-unitary) connection $A_t - i\Phi\, dt$ along circles $\gamma_z(t) = (z, e^{it}) \in \mathbb{C} \times \mathbb{S}^1$.

To describe the construction in slightly more details, it is convenient to pick structure group $U(2)$. Then if E is the rank 2 complex vector bundle on which the monopole (A, Φ) is defined, set $\mathcal{E} = E|_{\mathbb{C}\times\{0\}}$, assuming that none of the singularities $p_i = (m_i, a_i)$ has $a_i \equiv 0$ (mod 2π). We consider $\mathcal{E}$ as a holomorphic bundle endowed with the complex structure induced by the connection A. Then the Bogomolny equation implies that the holonomy h is a holomorphic section of $\mathrm{End}(\mathcal{E})$ which is an isomorphism outside of $m_1, \dots, m_n \in \mathbb{C}$. In the simple case of a periodic Dirac monopole of charge k with singularity at (z_0, t_0) and twisted by the flat abelian monopole $(ib\, dt, iv)$, one computes $h(z) = \frac{1}{e^{2\pi(v+ib)}(z-z_0)^k}$. In the non-abelian case, the z–coordinates of the singularities and of the centre of the monopole determine the asymptotic behaviour of the eigenvalues of h at infinity and at the singularities. The $\mathbb{R}/2\pi\mathbb{Z}$–valued t–coordinates of the singularities and the centre should instead be used to define a stability condition. The Hitchin–Kobayashi correspondence for periodic monopoles should then identify the moduli space of periodic monopoles (with singularities) with that of stable pairs $(\mathcal{E}, h)$.

Charbonneau and Hurtubise [27] studied the analogous question for singular monopoles on the product of a Riemann surface with a circle. Working on a compact base space, they were able to appeal to existent analytic results of Simpson [94] for the construction of the map $(\mathcal{E}, h) \mapsto (A, \Phi)$. In comparison with Charbonneau and Hurtubise's result, not only are periodic monopoles defined over a non-compact base manifold, but they also have infinite energy. In particular, it is not immediately clear how to formulate the stability condition for pairs $(\mathcal{E}, h)$ corresponding to periodic monopoles.

Assuming that a form of the Hitchin–Kobayashi correspondence holds for periodic monopoles, the connection with rational elliptic surfaces goes through the spectral data: By a standard construction, to the pair $(\mathcal{E}, h)$ one associates the spectral curve S of equation $\det\big(w\,\mathrm{id} - h(z)\big) = 0$, $(z, w) \in \mathbb{C} \times \mathbb{P}^1$, together with a line bundle (in the generic case when S is smooth) $L \to S$. For example, the spectral curve of a charge 2 $SU(2)$ periodic monopole with no singularities has the form

$$S_u : w^2 - e^{2\pi(v+ib)} \left((z-\mu)^2 - u\right) w + 1 = 0,$$

where $v, b, \mu \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C}$ are the parameters defining the boundary conditions satisfied by the monopole and $u \in \mathbb{C}$ is a free parameter. It is easy to see that S_u is a curve of genus 1, a double cover of $\mathbb{P}^1_z$ branched at 4 points, smooth except when $u = \pm\frac{2}{\lambda}$ (in this case 2 branch points coalesce). The family $\{S_u \mid u \in \mathbb{P}^1\}$ is a pencil of cubics such that the corresponding rational elliptic surface has a singular fibre of Kodaira type I_4^* over $u = \infty$.

The Nahm Transform and Higgs bundles on $\mathbb{P}^1$ with punctures. In [29, 34] Cherkis and Kapustin show (modulo analytic technicalities) that there exists a bijection between moduli spaces of periodic monopoles (with singularities) and those of solutions to Hitchin's equations on $\mathbb{C}^*$ (with punctures). The correspondence is an incarnation of the Nahm Transform, a non-linear Fourier transform for anti-self-dual connections on $\mathbb{R}^4$ invariant under a group of translations. Hitchin's equations

$$\begin{cases} F_B + [\phi, \phi^*] = 0, \\ \overline{\partial}_B \phi = 0, \end{cases} \tag{4.8.1}$$

where B is a unitary connection and ϕ, the Higgs field, a $(1,0)$–form with values in the complexified adjoint bundle, are the dimensional reduction of the anti-self-duality equations on a Riemann surface [57]. The solutions to Hitchin's equations corresponding to periodic monopoles have both logarithmic (or parabolic) and irregular singularities, meaning that the Higgs field has simple or multiple poles at the punctures. The framework to study the former case was introduced by Simpson in [95]. Under certain conditions on the irregular part of the fields, moduli spaces of solutions to Hitchin's equations with irregular singularities on a compact Riemann surface are studied by Biquard and Boalch in [17]. When smooth, these moduli spaces are shown to carry complete hyperkähler metrics.

In analogy with other known cases (*e.g.* Braam–van Baal [24] for ASD connections on $\mathbb{T}^4$ and Nakajima [82] in the case of Euclidean monopoles without singularities), it is expected but not yet proved that the Nahm transform is an isometry between moduli spaces of periodic monopoles (with singularities) and those of solutions to Hitchin's equations on the cylinder.

In fact, one can think of $\mathbb{C}^*$ as $\mathbb{P}^1$ minus two points and consider solutions to Hitchin's equations on the Riemann sphere with more general configurations of singularities, both in terms of number of punctures and pole order. When the moduli space of solutions to Hitchin's equations is 4–dimensional, the natural hyperkähler L^2–metric is expected to be of type ALG. The main reason for this is that in the complex structure in which the moduli space parametrises Higgs bundles the Hitchin map exhibits the moduli space as an elliptic fibration over $\mathbb{C}$. Here, in the context of the Hitchin–Kobayashi correspondence, one splits Hitchin's equations into a complex and real one: Higgs bundles are solutions to the complex equation, *i.e.* to the second equation in (4.8.1).

In [20] Boalch proposes a realisation of each rational elliptic surface as a moduli space of solutions to Hitchin's equations on $\mathbb{P}^1$ with certain rank and configuration of singularities (not all of these satisfying the hypothesis of [17]). The cases best understood from the algebraic point of view are those in which the singularities are all parabolic: Let E be an elliptic curve with $\mathbb{Z}_r$, $r = 2, 3, 4$ or 6, acting by automorphisms. Then we can regard $\mathbb{P}^1$ with 4 (if $r = 2$) or 3 (otherwise) punctures as the orbifold $E/\mathbb{Z}_r$. Groechenig [49] showed that the moduli space of stable parabolic Higgs bundles on $E/\mathbb{Z}_r$ is isomorphic to the $\mathbb{Z}_r$–Hilbert scheme of T^*E (the fixed locus of the $\mathbb{Z}_r$–action on the Hilbert scheme of r points on T^*E). In particular, it is a crepant resolution of $T^*E/\mathbb{Z}_r$ and an isotrivial rational elliptic surface. However, nothing is known about the hyperkähler L^2–metric on the moduli space.

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