
Photon-number superselection and the entangled coherent-state representation

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We introduce the entangled coherent-state representation, which provides a powerful technique for efficiently and elegantly describing and analyzing quantum optics sources and detectors while respecting the photon-number superselection rule that is satisfied by all known quantum optics experiments. We apply the entangled coherent-state representation to elucidate and resolve the long-standing puzzles of the coherence of a laser output field, interference between two number states, and dichotomous interpretations of quantum teleportation of coherent states.

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I. INTRODUCTION

Empirically, quantum optics obeys a photon-number superselection rule (PNSSR) due to the lack of an absolute clock or phase standard at optical frequencies; electromagnetic field sources such as the laser [1], antibunched light sources [2], and the micromaser [3] can be described by incoherent mixtures of number states, and photodetection described by projective measurement in the number state basis. However, coherence is an integral part of quantum optics, and the coherent state [4], which is a coherent superposition of number states, explicitly violates this PNSSR. Pure Gaussian states, such as coherent states and squeezed states, are very “convenient fictions” [5]. Despite the PNSSR, the Gaussian state is often attributed ontological significance when describing things such as the “physical” laser output field [1], the atomic Bose-Einstein condensate [6], local oscillators in homodyne detection [7], and continuous-variable quantum teleportation of coherent states [8–10]. The ontological view of Gaussian states is reinforced by optical homodyne tomography, which claims to reconstruct these states [11]. However, such Gaussian states only appear through a commitment of the partition ensemble fallacy whereby the density operator is preferentially decomposed into a mixture of coherent states [5,12].

The reason for the preference shown towards Gaussian states over number states in quantum optics is the coherent state’s usefulness as a representation in interferometry. The essence of its usefulness is that a linear mode coupling (as in frequency conversion, polarizing beam splitters and directional couplers), described by a unitary transformation that conserves the total number of quanta, will transform a product of two coherent states to another such product state [4]. This simple relation for linear mode coupling is responsible for the ease of calculating with coherent states over alternative representations.

Our aim is to introduce a simple method in quantum optics, which is elegant both as a calculational tool and as a conceptual framework, which respects the PNSSR (whereby sources produce incoherent mixtures of number states, and detectors count photons). We apply this technique to the challenges of describing interference by mixing independent number states [13], coherence of a multimode laser output field [14], the role of the local oscillator in homodyne detection, distillable entanglement versus pure entanglement for two-mode squeezed light [14], and the nature of quantum teleportation of coherent states [12,15,33]. These applications demonstrate that our operational approach to quantum optics respecting the PNSSR can quite simply describe all experiments traditionally described using optical coherence.

Interferometric calculations with number states are tedious: for n-mode coupling, the matrix elements for the unitary transformation are given by the SU(n) Wigner d functions [16,17]. Here we show that these calculations are made simple and easy to interpret by representing number states as entangled coherent states [18–21], with the entanglement taking place over a common phase. This entangled coherent-state approach enables easy calculations with number state sources by exploiting the ease of using the coherent-state representation. Moreover, the entanglement is not fragile: whereas one normally regards multipartite entangled coherent states as fragile and challenging to construct [22], the fragility arises due to decoherence with respect to the optical environment. For the entangled coherent states employed here, a decohering mechanism is described by an environment that is phase sensitive and thus would violate the PNSSR obeyed by all sources and measurements.

We begin by reviewing salient points concerning coherent states, discussing linear mode coupling, the coherent-state representation, and the nature of the laser as a source obeying the PNSSR. We then use the techniques introduced to analyze interferometry between independent number states, homodyne detection, squeezed light sources, and continuous variable quantum teleportation.
II. CONCEPTS AND METHODS

A. Coherent states and linear mode coupling

A coherent state $|\alpha\rangle$, $\alpha \in \mathbb{C}$, can be expressed in terms of the Fock states $|n\rangle$ [4] as

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\sqrt{n!}}{n!} e^{n\alpha^*} |n\rangle,$$

(2.1)

where $\alpha$ is expressed in polar coordinates as $\alpha = \sqrt{n} e^{i\phi}$, with amplitude $\sqrt{n}$ (mean photon number $n$) and phase $\phi$. This coherent state has photon-number statistics given by the Poisson distribution

$$\Pi(n) = e^{-\frac{n}{2}} \frac{n^n}{n!},$$

(2.2)

with mean and variance both equal to $n$. The coherent state is an eigenstate of the annihilation operator $\hat{a}$, satisfying the eigenvalue relation

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle.$$  

(2.3)

It is also a minimum uncertainty state with respect to its conjugate quadrature operators $\hat{q} = \hat{q}_0$ and $\hat{p} = \hat{q}_{\pi/2}$ where (choosing units such that $\hbar = 1$)

$$\hat{q} = \frac{1}{\sqrt{2}} (e^{i\phi} \hat{a} + e^{-i\phi} \hat{a}^\dagger).$$

(2.4)

The canonically conjugate operators satisfy the commutator relation $[\hat{q}, \hat{p}] = i\hbar$, and the minimum uncertainty relation is thus $\Delta q \Delta p = 1/2$. The coherent state is a displaced vacuum state, $|\alpha\rangle = D(\alpha) |0\rangle$, for $D(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$.

The properties discussed above are often cited as the key properties of the coherent state, but another property is crucial for calculations in quantum optics. So far we have considered single-mode coherent states; we introduce the two-mode coherent state $|\alpha, \beta\rangle = |\alpha\rangle \otimes |\beta\rangle$, where $a, b$ label the two modes. The Hamiltonian that generates linear mode coupling is given by

$$\hat{H} = i (g \hat{a}^\dagger \hat{b} - g^* \hat{b}^\dagger \hat{a}),$$

(2.5)

with $|g|$ quantifying the coupling strength between the two modes and $\arg(g)$ the relative phase shift between the modes imposed by the coupling. The Hamiltonian (2.5) generates the unitary evolution operator

$$U(\theta, \phi) = \exp(-i\hat{H}t) = \exp(\theta e^{-i\phi} \hat{a}^\dagger \hat{b} - \theta e^{i\phi} \hat{b}^\dagger \hat{a})$$

(2.6)

for $\theta = |g|/t$, $\phi = \arg(g)$, and the interaction time $t$.

As is well known, the linear coupling unitary transformation (2.6) transforms a two-mode product coherent state to a two-mode product coherent state [4]. The easiest way to establish this property is first to note that the annihilation operators transform according to

$$U^\dagger (\theta, \phi) \hat{a} U(\theta, \phi) = M(\theta, \phi) \hat{a},$$

(2.7)

for

$$M(\theta, \phi) = \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix}.$$  

(2.8)

If the input state is the two-mode coherent state, the output is the eigenstate of the output annihilation operators in Eq. (2.7), namely, the two-mode coherent state,

$$U(\theta, \phi) |\alpha, \beta\rangle = |\alpha \cos \theta + \beta e^{-i\phi} \sin \theta, -\alpha e^{i\phi} \sin \theta + \beta \cos \theta\rangle.$$  

(2.9)

The condition for 50-50 (or 3 dB) splitting is met if $\theta = \pi/4$.

Another important aspect of coherent states is that they constitute an overcomplete basis of the Hilbert space for the harmonic oscillator, giving

$$\int \frac{d^2 \alpha}{2\pi} |\alpha\rangle \langle \alpha| = 1,$$

(2.10)

with 1 the identity operator. An arbitrary density operator can be expressed as

$$\hat{\rho} = \int \frac{d^2 \alpha}{2\pi} P(\alpha) |\alpha\rangle \langle \alpha|,$$

(2.11)

with $P(\alpha)$ the Glauber-Sudarshan $P$ representation [4, 23]. Density operators are said to be nonclassical if and only if $P(\alpha)$ does not satisfy the axioms of a true probability density; if it does, the field density operator is “semiclassical.”

B. Photon-number superselection rule

Quantum optics empirically obeys a PNNSR. Operationally, a superselection rule can be expressed as an invariance of all states and operations (unitary transformations, measurements, dissipation, etc.) by a group action [24]. For a PNNSR, this group is the $U(1)$ group of unitary phase shifts, with the unitary phase-shift operator given by

$$\mathcal{P}(\Delta) = \exp(i \Delta \hat{a}^\dagger \hat{a}),$$

(2.12)

$\Delta \in [0, 2\pi)$, which transforms the coherent state according to

$$\mathcal{P}(\Delta) |\alpha\rangle = |\alpha e^{i\Delta}\rangle.$$  

(2.13)

The PNNSR ensures that density operators for quantum optics sources are $U(1)$ invariant:

$$\mathcal{P}(\Delta) \hat{\rho} \mathcal{P}^{-1}(\Delta) = \hat{\rho}, \quad \Delta \in U(1).$$

(2.14)

Expressing the integration measure as $d^2 \alpha/\pi = d\varphi d\sqrt{n}$, where we use the “slash notation” for the differential operator $d = d/2\pi$, the independence of the density operator on phase shifts (2.14) implies that $P(\alpha)$ is axisymmetric over the complex-$\alpha$ plane:

$$P(\alpha) = P(\sqrt{n}).$$  

(2.15)

This constraint on the representation is quite strong. The constraint (2.15) allows the arbitrary density operator (2.11) to be expressed as

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Generalization to multimode Fock states is straightforward. and with

\[ p_n = 2 \int_0^\infty d\bar{n} \Pi_n(\bar{n}) P(\sqrt{\bar{n}}) \]  \hspace{1cm} (2.17)

and \( \Pi_n(\bar{n}) \) the Poisson distribution defined by Eq. (2.2).

We see that a consequence of the PNSSR is that any optical source can be regarded in two equivalent ways: as a source of coherent states with quasiprobability distribution \( P(\alpha) = P(\sqrt{\bar{n}}) \) that is uniform in phase, or as a source of number states with the photon-number distribution given by Eq. (2.17). Each interpretation is compatible with experimental results; to ascribe ontological significance to one description over the other is a fallacy.

C. Entangled coherent-state representation

We have established above that sources satisfying the PNSSR can be regarded as mixtures of number states. The challenge of using number states for interferometric calculations is that matrix elements of the linear coupling unitary transformation (2.6) in the number state basis are the SU(\( n \)) Wigner \( \rho \) functions, for example, \( d_m(n)(\theta) \) for two-mode coupling; tools for efficiently calculating SU(2) and SU(3) Wigner \( \rho \) functions are available including asymptotic techniques [16,17], but in the following we establish an easier formalism for quantum optics calculations that employ a coherent-state representation.

The coherent states form a basis, and thus we can represent any number state in this basis as a superposition of coherent states. In doing so, there exists an ambiguity due to the overcompleteness of the coherent-state basis. Our preference here is to represent the number state as a superposition of coherent states over a circle in the complex-\( \alpha \) phase space [25].

\[ |n\rangle = [\Pi_n(m)]^{-1/2} \int d\varphi e^{-in\varphi} |\sqrt{m} e^{i\varphi}\rangle, \]  \hspace{1cm} (2.18)

which is valid for any integer \( m > 0 \). We choose to fix \( m = n \) so that the number state is presented as a superposition of all coherent states on the circle with radius \( \sqrt{n} \).

The natural extension of Eq. (2.18) to a two-mode Fock state is given by

\[ |n,n\rangle = [\Pi_n(n)\Pi_n(n')^{-1/2}] \int d\varphi d\varphi' e^{-i(n\varphi+n'\varphi')} |\sqrt{\bar{n}} e^{i\varphi},\sqrt{\bar{n}}' e^{i\varphi'} \rangle, \]  \hspace{1cm} (2.19)

with \( |\sqrt{\bar{n}} e^{i\varphi},\sqrt{\bar{n}}' e^{i\varphi'} \rangle \) a two-mode coherent state. Although at first glance the right-hand side of Eq. (2.19) appears to be a two-mode entangled coherent state [18–21], it is a product state and hence not actually entangled. However, the state becomes a genuine entangled coherent state subsequent to linear coupling by Eq. (2.6) of the two modes. The entangled coherent-state representation is a great advantage in studying linear coupling of number states, as shown in the following.

Consider the linear mode coupling transformation of an input state consisting of \( n \) photons in one mode and no photons (the vacuum state \( |0\rangle \)) in the other mode. In the entangled coherent-state representation we can write

\[ |n,0\rangle = [\Pi_n(n)]^{-1/2} \int_0^{2\pi} d\varphi e^{-in\varphi} |\sqrt{n} e^{i\varphi},0\rangle. \]  \hspace{1cm} (2.20)

The output state, following the transformation (2.6), is

\[ U(\theta,\phi)|n,0\rangle = [\Pi_n(n)]^{-1/2} \int_0^{2\pi} d\varphi e^{-in\varphi} \]

\[ \times |\sqrt{\bar{n}} e^{i\varphi} \cos \theta - e^{i(\varphi+\phi)} \sqrt{\bar{n}} \sin \theta,0\rangle, \]  \hspace{1cm} (2.21)

where we have used the results derived in Eq. (2.9). This output state (2.21) is an entangled coherent state [18–21], with the entanglement over optical phase; this entanglement is robust against any decoherence mechanism involving linear coupling to an environment that also obeys the PNSSR. Only a decoherence mechanism that breaks the PNSSR can destroy this entanglement.

The general two-mode Fock state (2.19) transforms via linear coupling to the entangled coherent state

\[ U(\theta,\phi)|n,n\rangle = [\Pi_n(n)\Pi_n(n')]^{-1/2} \int d\varphi d\varphi' e^{-i(n\varphi+n'\varphi')} |\sqrt{\bar{n}} e^{i\varphi} \cos \theta + \sqrt{\bar{n}}' e^{i(\varphi'-\phi)} \sin \theta, -\sqrt{\bar{n}}' e^{i(\varphi+\phi)} \sin \theta \]

\[ + \sqrt{\bar{n}}' e^{i\varphi'} \cos \theta \rangle, \]  \hspace{1cm} (2.22)

with the entanglement over two optical phases \( \varphi \) and \( \varphi' \). Generalization to multimode Fock states is straightforward.

III. SOURCES: THE LASER FIELD

An important application of this theory is to the laser output field. There are standard theories that describe the formation of the intracavity laser field, which is necessarily diagonal in the number state representation [1]. Nevertheless, the field emitted from the cavity exhibits multimode coherence, and it is tempting to regard the multimode laser output as being in a multimode coherent state. A number state in the cavity appears to lead to a highly entangled multimode output whereas the intracavity coherent state leads very nicely
to a product coherent state in the multimode extracavity field.

The preference for coherent states is highlighted in a recent discussion of the ideal laser and its output field by van Enk and Fuchs (vEF) [14]. They express a preference for treating the laser in terms of coherent states, a view that was originally championed by Glauber [4]. However, the ease of using the coherent-state representation should not be regarded as a justification for a commitment of the partition ensemble fallacy and thus regarding number states as less physical. The formalism developed here clarifies why a number state in the cavity can equally well lead to a coherent multimode output.

As the intracavity field is described by an axisymmetric density matrix of the type (2.16), it is equally valid to describe the source as a distribution of number states or as a distribution of coherent states. With the entangled coherent state formalism, we show that the output field may be regarded as an entangled coherent-state with the entanglement over the optical phase variable of the laser. This entangled state can be expressed as a superposition of product coherent states, which exposes the multimode coherence of the output field. However, the reduction to a multimode coherent state, which is what vEF yearn for in describing their “complete measurement” that would collapse the wave function into a particular overall phase requires that one breaks the PNSSR. We argue in the following that there is no need and no justification for postulating such a decoherence process. We do not argue that such a complete measurement is not possible in principle, only that no process of this type is present in current quantum optics experiments and would require an absolute clock or phase standard optical frequencies. Without such a complete measurement, the number state and coherent-state sources are equally valid physically, and the entangled coherent-state representation clarifies that a number state in the cavity produces exactly the desired multimode coherence.

Specifically, the multimode laser output can be described by employing multiple spectral components, a sequence of pulses, spatial modes, or other possibilities. The actual nature of the output modes is not important to this analysis; only the fact that the coupling between the single-mode intracavity field and the multimode output field is via a linear coupling mechanism. For simplicity we assume that the laser is ideal with Poissonian photon statistics according to the distribution (2.2), and assume that the density operator for the single-mode field in the laser cavity is

\[ \hat{\rho}_L(n) = \int d\varphi \sqrt{n} e^{i\varphi} \langle \sqrt{n} e^{i\varphi} \rangle = \sum_{n=0}^{\infty} \Pi_{m}(n)(n) \langle n \rangle, \]

(3.1)

which is a mixture of coherent states with amplitude \( \sqrt{n} \) in the cavity, uniformly distributed over the optical phase \( \varphi \), and is also a Poissonian mixture of number states with \( n \) the mean number of photons.

The laser field output is related to the input field by linear coupling of the form (2.6), with the annihilation operator \( \hat{b} \) given by a linear combination of annihilation operators \( \hat{b}_k \) for each of output field mode. If we consider, for example, a continuous-wave (cw) output field, the multimode output is described by a sequence of overlapping pulses (spread over both time and frequency) that together constitute the nearly monochromatic output field. This case is the one considered by vEF. The appeal of employing coherent states is that the intracavity state \( |\sqrt{n} e^{i\varphi}\rangle \) can produce the \( N \)-mode product state

\[ |\sqrt{n}N e^{i\varphi}, \ldots, \sqrt{n}N e^{i\varphi}\rangle = \prod_{k=1}^{N} |\sqrt{n}N e^{i\varphi}\rangle_k, \]

(3.2)

describing a state for which the photons have been split equally between the \( N \) modes. The state (3.2) is one possible description of the laser output field: an initial density that is diagonal in the number state representation must yield an output density that is also diagonal in this representation [26] unless the PNSSR is broken, which is certainly not the case for linear coupling.

We now show how a source of number states yields equivalent results. In analogy to the linear coupling Hamiltonian and initial conditions that yield the product state (3.2), we can also consider a number state \( |m\rangle \) in one mode, the vacuum in the other \( N-1 \) modes, and the same linear coupling transformation. The input state of \( m \) photons in the first of \( N \) modes and all other modes in the vacuum state to an equal distribution of photons in all \( N \) modes, as for Eq. (3.2), is given by

\[ \left[ \Pi_m(m) \right]^{-1/2} \int d\varphi e^{im\varphi} |\sqrt{m} e^{i\varphi}, 0, \ldots, 0\rangle \]

\[ \rightarrow \left[ \Pi_m(m) \right]^{-1/2} \int d\varphi e^{im\varphi} |\sqrt{m}N e^{i\varphi}, \ldots, \sqrt{m}N e^{i\varphi}\rangle. \]

(3.3)

This entangled coherent state is a superposition of product coherent states that are identical in amplitude and phase, with coefficients of the superposition distributed uniformly over the phase \( \varphi \).

The entangled coherent state represents the output of the laser field for an \( m \)-photon number state prepared in the single-mode intracavity field. Expression (3.3) is as valid as expression (3.2) in describing the output field. Although the product coherent state has been championed [14], avoiding the partition ensemble fallacy requires each decomposition to be equally acceptable.

The laser’s coherence time or length can be easily described within the entangled coherent-state representation of the number state (3.3) by including a random walk in the phase. For the product coherent state in the following expression representing the amplitudes of successive overlapping pulses, the ideal laser output field can be expressed as

\[ \left[ \Pi_m(m) \right]^{-1/2} \int d\varphi e^{im\varphi} |\sqrt{m}N e^{i\varphi(t_1)}, \ldots, \sqrt{m}N e^{i\varphi(t_N)}\rangle, \]

(3.4)
where $\varphi(t)$ is determined for times $\{t_k\}$ by a stochastic sequence. The sequence can be regarded as a random walk, and correlations are calculated from the above multimode entangled coherent state, averaged over all realizations of this random walk in phase.

With the above expression, the role of vEF’s “complete measurement” [14] is clear. This measurement would ideally measure the phase of one state either in the product state (3.2), or, equivalently, in the entangled coherent state (3.3) and yield a result for the phase. For the case of the intracavity field described by a number state, the result is that the entangled coherent state “collapses” to a product of $n-1$ identical coherent states, regardless of the fact that the intracavity field initiated as a number state. Thus, there is no physical preference for the coherent states as a decomposition of the density operator. Moreover, their “complete measurement” must break the PNSSR, which would require an ancilla state such as atoms in a superposition of different energy eigenstates [26]. This requirement of a superposition of energy eigenstates simply shifts the burden by allowing phase localization to occur by using a source wherein phase localization is available.

IV. INTERFERENCE OF NUMBER STATES

The analysis of interferometry with number states becomes straightforward when using the entangled coherent-state representation, because an interferometer is a linear mode coupler. For an interferometer with $N$ modes, the unitary transformations are elements of the Lie group SU($N-1$) [27]. Transformations can be calculated from matrix elements of the unitary linear coupling transformation, but the calculations, which involve Wigner $d$ functions, are complicated (although solutions are known for small $N$ [16,17]). The entangled coherent-state formalism offers an elegant alternative.

We now use this formalism to examine the remarkable result that interference can be observed between independently generated number states. Consider an initial state of two modes $a$ and $b$ of the light field that takes the form of a product of Fock states $|n_1\rangle_a|n_2\rangle_b$; the modes are subsequently combined at a beam splitter, followed by photodetection at both output modes.

It is well accepted that if the initial states of the two modes were coherent states, then an interference pattern will be recorded at the two detectors. This interference pattern could be passively observed as a function of time (if the two modes were at slightly different frequencies), or as a function of some actively varied phase shift $\theta$ introduced in one of the modes just prior to the beam splitter (see Fig. 1).

It is often stated, however, that, since the first-order correlation function $g^{(1)}$ vanishes for the state $|n_1\rangle_a|n_2\rangle_b$, no interference will be observed in this case. (“This mixing of number states yields a zero correlation function and thus no fringes are obtained.”—Ref. [28], p. 38.) Such arguments are sometimes then applied to the Pfleegor-Mandel experiments [29], in which interference patterns are observed between the outputs of two different lasers, in order to claim that the laser output is necessarily a coherent state.

As we now show, these arguments are erroneous—they ultimately arise from a misconception about the role of correlation functions in determining operationally observable properties of the electromagnetic field. Mølmer has shown [5,30], through intensive calculations and numerical simulations, how two independent Fock states can interfere. We employ the entangled coherent-state representation to show this result analytically through a much simpler analysis. Although our results are phrased in terms of interference between photons, they apply equally well to other bosonic modes such as Bose-Einstein condensates. In fact, by our technique we can reproduce the celebrated result of Javanainen and Yoo [13], again by a simpler analysis.

Consider the case that two spatial modes $a,b$ each contain the same definite number of photons $n$, at the same frequency. Following Eq. (2.19), the initial state of the two cavities can be expressed in the entangled coherent-state representation as

$$|\psi\rangle = |n\rangle_a|n\rangle_b = \frac{1}{\Pi(n)} \int d\varphi d\varphi' e^{-i(n+\varphi)}|\sqrt{n}e^{i\varphi}\rangle_a|\sqrt{n}e^{i\varphi'}\rangle_b.$$  

(4.1)

The field emission from the cavity is described by a linear output coupling. After some time, let $a_1$ ($b_1$) represent the extracavity output fields and $a_2$ ($b_2$) represent the intracavity fields; see Fig. 1. The extracavity modes $a_1, b_1$ now contain some fraction $\epsilon$ of the total light in the mode. The state of the two spatial modes is

$$|\psi\rangle = \frac{1}{\Pi(n)} \int d\varphi d\varphi' e^{-i(n+\varphi)}|\sqrt{n}\epsilon e^{i\varphi}\rangle_{a_1}|\sqrt{n}\epsilon e^{i\varphi'}\rangle_{b_1} \otimes |(1-\epsilon)ne^{i\varphi}\rangle_{a_2}|(1-\epsilon)ne^{i\varphi'}\rangle_{b_2}.$$  

(4.2)

Note that the linear coupling does not maintain a Fock state in the cavity: an indefinite number of photons are leaked out, determined by the coupling parameter $\epsilon$. The output modes $a_1, b_1$ are then combined on the beam splitter, and the resulting state $|\psi'\rangle = U(\pi/4,0)|\psi\rangle$ is
$$\begin{align*}
|\psi'| &= \frac{1}{\Pi_{a}(n)} \int d\phi d\phi' e^{-in(\phi+\phi')} \\
&\times \sqrt{\frac{e}{2}} |a|^{n}(e^{i\phi} + e^{i\phi'})|_{a} \\
&\times \sqrt{\frac{e}{2}} |a|^{n}(-e^{i\phi} + e^{i\phi'})|_{b},
\end{align*}$$

(4.3)

After the beam splitter, photodetection is performed on each mode. Consider the result where A photons are detected in mode $a_{1}$ and B photons are detected in mode $b_{1}$ after the beam splitter. The consequence of this measurement is that the state $|\psi'|$ is collapsed to $|\psi''\rangle = |(A|B)\rangle|\psi'angle$, which we write (ignoring normalization now)

$$\begin{align*}
|\psi''\rangle &= \int d\phi d\phi' e^{-in(\phi+\phi')} C_{A,B}(\phi,\phi') \\
&\times \sqrt{|(1-e)n e^{i\phi}\rangle_{a}} \sqrt{|(1-e)n e^{i\phi}\rangle_{b}},
\end{align*}$$

(4.4)

where

$$C_{A,B}(\phi,\phi') = (A\sqrt{N/2}(e^{i\phi} + e^{i\phi'})) \\
\times (B\sqrt{N/2}(-e^{i\phi} + e^{i\phi'})).$$

(4.5)

The effect of the collapse is that the distribution over $\phi,\phi'$ is no longer uniform, as captured by the function $C_{A,B}(\phi,\phi')$. Ignoring factors that are independent of $\phi,\phi'$ and which are removed by normalization, we have

$$C_{A,B}(\phi,\phi') \approx e^{-i(A+B)(\phi+\phi')/2} |\cos \Delta|^{A} |\sin \Delta|^{B},$$

(4.6)

where $\Delta :=(\phi-\phi')/2$, and where we have used the expansion (2.1) of coherent states in terms of number states. Note that the presence of the factors $e^{-i(\phi+\phi')}$ and $e^{-i(A+B)(\phi+\phi')}$ ensure that Eq. (4.4) is still a state of definite photon number. Moreover, it is a highly entangled state, and as mentioned above such entanglement will be highly robust—to destroy this entanglement requires a violation of the PNSSR. The robustness of such entanglement was first noted and investigated numerically by Mølmer [30].

The distribution $|C_{A,B}(\phi,\phi')\rangle$ is peaked at two values:

$$\Delta = \pm \arctan(\sqrt{B/A}),$$

(4.7)

within the range $[-\pi/2, \pi/2]$. Thus, photodetection collapses the joint state of the cavities into one with correlations in the phase. Moreover, the width of the distribution over $\Delta$ at each peak becomes narrower the greater the total number of photons ($N=A+B$) detected. In terms of the difference $\Delta - \bar{\Delta}$ from each of the maximum values, the relation [17]

$$|\cos \Delta|^{A} |\sin \Delta|^{B} = \sqrt{\frac{A^{A} B^{B}}{N^{N}} |\cos(\Delta - \bar{\Delta})|^{2N}},$$

(4.8)

valid for large $N$, gives an expansion for Eq. (4.6) in terms of this difference for large $N$ as

![FIG. 2. The magnitude $|C_{A,B}(\phi,\phi')\rangle$ of the function in Eq. (4.9), normalized to have unit magnitude at its peak, is plotted as a function of $|\Delta\rangle = |\phi - \phi'|/2$ for equal photocounts $A = B = 1$ (dotted line), $A = B = 4$ (dashed line), and $A = B = 64$ (solid line).](image-url)
The validity and convenience of assuming an independent local oscillator in a coherent state is made evident by the above equation. If the source were a coherent state, the presignal and local oscillator are in a product state and can be considered independent. With the number state approach, the local oscillator is not independent but is rather entangled with the source of the signal state. This approach reveals that the nature of homodyne detection is interferometric: it can be used to characterize a process (given by the unitary $V$ in this case) rather than a state. In particular, reconstruction of the state of the signal mode through optical homodyne tomography [11] relies on the belief that the local oscillator is in a coherent state. Our analysis reveals that this belief is unfounded; however, such tomography can be used more appropriately to reconstruct information about the process regardless of the nature of the common source.

Of course the above analysis is somewhat simplified, and more general signal field states can certainly be considered—such as homodyne detection of one mode of a two-mode state, decoherence and losses included in the transformation of the signal mode, entanglement with ancilla modes, and so on. However, the conclusions for these cases remain unaffected.

In summary, quantum optics sources satisfy the invariance condition (2.14), and, therefore, independent local oscillators with a specified optical phase are not available. The reason that we assume independent local oscillators is that the local oscillator and the signal field are phase locked, for example, by originating from the same coherent source. A decomposition of the density operator in the coherent-state basis makes this clear but has also led to the misconception that coherent states are the “actual physical” states. Here we have shown how the same result occurs by assuming that the source produces number states instead of coherent states and demonstrated that the entangled coherent-state representation yields, in a transparent way, an interpretation of homodyne detection as taking place on an entanglement of product states, one for the signal and the other for the local oscillator, with the entanglement being over the optical phase.

VI. SQUEEZE LIGHT

The generation of two-mode squeezed light is described by the interaction Hamiltonian [28]

$$\hat{H}_{sq}(\xi) = i(\xi^* \hat{c}^\dagger \hat{a} - \xi \hat{c} \hat{a}^\dagger),$$

with $\hat{c}$ the annihilation operator for the pump field, $\hat{a}$ the annihilation operator for the signal field, and $\hat{b}$ the annihilation operator for the idler field. One pump photon is annihilated via this process to produce a pair of signal and idler photons that are correlated in momentum, energy, time of creation, and joint quadrature phase measurements. The unitary evolution generated by the squeezing Hamiltonian is given by

$$U_{sq}(\xi t) = \exp(\xi^* \hat{c}^\dagger \hat{b} - \xi \hat{c} \hat{a}^\dagger),$$

with $t$ the time of evolution. Calculations with these three-mode operators are cumbersome and are generally done either in the low-$\langle \xi^2 \rangle$ limit or by replacing the pump field annihilation operator $\hat{c}$ by a $c$ number.

This $c$-number replacement is employed in investigating squeezed light, such as that generated by a second-order non-
linear optical process in the below-threshold regime. Ideal two-mode squeezing is then obtained if the pump field is treated as a classical coherent pump field with a definite phase. In this case, we replace \( \hat{c} \) by \( \gamma \), with \( \text{arg}(\gamma) \) the phase of the pump field, and let \( \chi = \zeta \gamma^{\ast} t \). Then the idealized squeezing unitary evolution is given by

\[ U_{sq}(\chi) = \exp(\chi^{\ast} \hat{a} \hat{b} - \chi^{\ast} \hat{b}^{\dagger} \hat{a}^{\dagger}) \]  

(6.3)

Thus, by treating the pump as a classical field with a definite phase, the effect of this transformation on the vacuum state for modes \( a \) and \( b \) is the two-mode squeezed vacuum,

\[ |\chi\rangle_{ab} = U_{sq}(\chi)|0,0\rangle_{ab} \]  

(6.4)

It should be noted that two-mode squeezing, as described by the unitary evolution operator (6.3), can equally well be generated by two single-mode squeezers mixed at a beam splitter [8–10], where the same pump field is used for both squeezers and has definite phase (that is transferred to the squeezing orientation); the ideal single-mode squeezing Hamiltonian is given by

\[ H = \chi \hat{a}^{\dagger} \hat{a} + \chi^{\ast} \hat{a}^{\dagger} \hat{b}^{\dagger} \]  

(6.5)

However, we discuss only the direct generation of two-mode squeezing; the principles elucidated here apply just as simply to the case of two-mode squeezing generated by mixing two single-mode squeezed fields.

Consider now squeezing where the pump field is in a number state \(|n\rangle\). Again, expressing this number state in our coherent-state representation, the squeezing transformation (6.3) gives

\[ U_{sq}(\chi)|n\rangle_{c} = |0,0\rangle_{ab} \]

\[ = \frac{1}{\sqrt{\Pi_{n}(n)}} \int d\varphi e^{-in\varphi} e^{\pi n \hat{a}^{\dagger} \hat{b}} |\sqrt{n} e^{i\varphi}\rangle_{c}|0,0\rangle_{ab} \]  

(6.6)

Care must be taken in making the analog of the classical pump approximation for a coherent-state source. However, if \( n \) is large, it is valid to replace \( \hat{c} \) with the \( \sqrt{n} e^{i\varphi} \) inside the integral. Defining \( \chi(\varphi) = \sqrt{n} \zeta te^{i\varphi} \) and using Eq. (6.4) yields

\[ U_{sq}(\chi)|n\rangle_{c} = \frac{1}{\sqrt{\Pi_{n}(n)}} \int d\varphi e^{-in\varphi} |\sqrt{n} e^{i\varphi}\rangle_{c} |\chi(\varphi)\rangle_{ab} \]  

(6.7)

Thus, the modes \( a \) and \( b \) are in a two-mode squeezed state, entangled via the phase with the state of the pump. This state clearly exhibits the distillable entanglement of van Enk and Fuchs [14]: an appropriate measurement on the pump mode will collapse modes \( a \) and \( b \) into a two-mode entangled state. Note, however, that such a measurement violates the PNSSR.

VII. CONCLUSIONS

The fact that quantum optics operationally obeys a PNSSR ensures that it is equally valid to treat all sources as either distributions of number states or coherent states. Additionally, the coherent-state approach has been standard due to the ease of calculations. Here, we have presented a powerful and useful tool to carry out calculations using number state sources with the ease of coherent states through the entangled coherent-state representation. We have demonstrated that, in many standard concepts and experiments in quantum optics where it appears necessary to employ coherent states, it is equally as valid to describe them using sources of number states. In addition, we have shown how to provide a simple analysis of the interference between two initially independent Fock states of photons.

Considerable debate has occurred over the use of coherent states in continuous-variable quantum teleportation. In quantum teleportation, a quantum state can be transmitted by two parties (referred to as Alice and Bob) who share an entangled resource and a classical communication channel. In the standard nomenclature, Alice is the sender, and she performs a joint measurement on her received quantum state and her portion of the entanglement resource and sends the results of her measurement to Bob. Bob transforms his portion of the entanglement resource into a replica of the original state based on the classical information he receives from Alice.

One experimental approach to quantum teleportation has been the teleportation of coherent states [8–10]. However, as we have shown, coherent states are not physical but rather just a convenient representation. Moreover, a description involving number state sources should be equally valid. The teleportation of coherent states is thus quite interesting because this interpretation is only meaningful if the coherent-state decomposition of the density matrix is adopted. It has been suggested by van Enk and Fuchs [14] that acquiring a technology for complete phase measurements could overcome this hurdle, but as we have discussed, these measurements would break the PNSSR. As our results show, it would be equally valid to carry out the calculations for continuous-variable quantum teleportation for a number state source (using the entangled coherent-state representation). The result would no longer be interpretable as a standard quantum teleportation experiment, because the state to be teleported, the shared squeezed vacuum of Alice and Bob, and the local oscillators used by Alice and Bob for homodyne measurements, displacements, and final verification of quantum teleportation are all entangled via the linear coupling of the common source field [12].

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