# A Computational Model for Multi-Variable Differential Calculus☆

Abbas Edalat*

*Department of Computing, Imperial College London, UK*

André Lieutier

*Dassault Systemes Provence, Aix-en-Provence, France*

Dirk Pattinson

*Department of Computing, Imperial College London, UK*

---

## Abstract

*We develop a domain-theoretic computational model for multi-variable differential calculus, which for the first time gives rise to data types for piecewise differentiable or more generally Lipschitz functions, by constructing an effectively given continuous Scott domain for real-valued Lipschitz functions on finite dimensional Euclidean spaces. The model for real-valued Lipschitz functions of $n$ variables is built as a sub-domain of the product of two domains by tupling together consistent information about locally Lipschitz functions and their differential properties as given by their L-derivative or equivalently Clarke gradient, which has values given by non-empty, convex and compact subsets of $\mathbb{R}^n$. To obtain a computationally practical framework, the derivative information is approximated by the best fit compact hyper-rectangles in $\mathbb{R}^n$. In this case, we show that consistency of the function and derivative information can be decided by reducing it to a linear programming problem. This provides an algorithm to check consistency on the rational basis elements of the domain, implying that the domain can be equipped with an effective structure and giving a computable framework for multi-variable differential calculus. We also develop a domain-theoretic, interval-valued, notion of line integral and show that if a Scott continuous function, representing a non-empty, convex and compact valued vector field, is integrable, then its interval-valued integral over any closed piecewise $C^1$ path contains zero. In the case that the derivative information is given in terms of compact hyper-rectangles, we use techniques from the theory of minimal surfaces to deduce the converse result: a hyper-rectangular valued vector field is integrable if its interval-valued line integral over any piecewise $C^1$ path contains zero. This gives a domain-theoretic extension of the fundamental theorem of path integration. Finally, we construct the least and the greatest piecewise linear functions obtained from a pair of function and hype-rectangular derivative information.*

---

☆A preliminary version of this article appeared in the Proceedings of the Eight International Conference on Foundations of Software Science and Computational Structures, FOSSACS 2005, LNCS, vol. 3441, Springer, pp 505-519.

*Corresponding author

*Email addresses:* a.edalat@ic.ac.uk (Abbas Edalat), Andre.LIEUTIER@3ds.com (André Lieutier), d.pattinson@ic.ac.uk (Dirk Pattinson)

*When the pair is consistent, this provides the least and greatest maps to witness consistency.*

## 1. Introduction

We develop a domain-theoretic computational model for multi-variable differential calculus, which for the first time gives rise to data types for real-valued Lipschitz or piecewise differentiable maps on finite dimensional Euclidean spaces. This extends the corresponding result in [14] for dimension $n = 1$ to higher dimensions $n > 1$. While many of the properties of the domain of Lipschitz functions on $\mathbb{R}$ have been extended, as in [11], even to infinite dimensional Banach spaces, constructing an effective structure for the domain in the finite dimensional case $n > 1$ has been a challenge.

The model is a continuous Scott domain for Lipschitz functions of $n$ variables. It allows us to deal with Lipschitz or piecewise differentiable functions in a recursion theoretic setting, and is thus fundamental for applications in computational geometry, geometric modelling, ordinary and partial differential equations and other fields of computational mathematics. The overall aim of the framework is to synthesize differential calculus and computer science, which are two major pillars of modern science and technology.

The set of real-valued Lipschitz functions defined in a region in $\mathbb{R}^n$ has very useful closure and convergence properties which make it a suitable set of maps for computation. Lipschitz maps are in particular closed under taking $\min$, $\max$ and absolute value which are essential operations in nearly all areas of scientific computation and CAD. We also have developed a notion of finitary derivative for Lipschitz maps which leads to a practical model of computation. As in dimension one, the basic idea of the model, for a finite dimensional Euclidean space $\mathbb{R}^n$ or for an infinite dimensional Banach space $X$, is to collect together the local differential properties of the function by developing a generalization of the concept of Lipschitz constant to a non-empty, convex and compact set-valued Lipschitz constant in $\mathbb{R}^n$ in the finite dimensional case and a non-empty, convex and weak* compact set-valued Lipschitz constant in the dual $X^*$ for the infinite dimensional case. The collection of these local differential properties are then used to define the Lipschitz or *L-derivative* of a function.

The L-derivative is in fact closely related to Clarke's gradient, which is a key tool in nonsmooth analysis, control theory and optimisation theory [8, 9] and is defined by using the generalized directional derivative based on taking the limit superior of the rate of change of the function along a given direction. It has been shown in [11] that the L-derivative and the Clarke's gradient coincide in finite dimensions. The Clarke gradient extends the classical derivative to Lipschitz maps, in the sense that the Clarke gradient of any $C^1$ map coincides with the classical derivative. In dimension one for example, the Clarke gradient of the absolute value function at zero is the interval $[-1, 1]$, and similarly one can derive the Clarke gradient of a piecewise $C^1$ function at a point of non-differentiability as the compact interval obtained by taking the left and right limits of the derivative at that point.

Using the collection of the local differential properties that define the L-derivative, we also obtain the set of primitives of a Scott continuous, non-empty, convex and compact (respectively, weak* compact) set-valued vector field in $\mathbb{R}^n$ (respectively, in $X^*$). This leads to an extended Fundamental Theorem of Calculus for set-valued derivatives, which was shown first for dimension one [14] and then for infinite dimensions [11]. The Fundamental Theorem is used here, in the finite dimensional case, to construct the domain of Lipschitz functions as a sub-domain of the product of the space of interval-valued functions of $n$ variables and the space of L-derivatives of Lipschitz functions of $n$ variables which take non-empty,

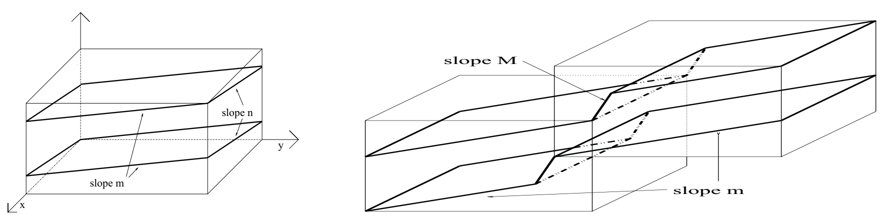


**Figure 1. Two examples of consistent function and derivative approximations**

convex compact sets as values. Geometrically, the first component of such a pair serves as an approximation of the function value, and the second component simultaneously approximates the L-derivative or all $n$ partial derivatives. We call such a pair consistent if there exists a Lipschitz function which is approximated by the first component, and whose L-derivative or derivative information is approximated by the second component of the pair.

Each step function that approximates the function value is represented by a finite set of pairs $(a_i, b_i)_{i\in I}$ where $a_i \subseteq \mathbb{R}^n$ is a rational hyper-rectangle and $b_i \subseteq \mathbb{R}$ is a compact interval such that $b_i$ and $b_j$ have non-empty intersection whenever this is the case for the interiors of $a_i$ and $a_j$. Similarly, approximations of the $n$ partial derivatives are given as finite sets of pairs $(a_i, b_i)_{i\in I}$ where the $a_i$ are as above but the $b_j$ are now compact rational polyhedra.

We specifically focus on the special but important case where the derivative information is always a compact hyper-rectangle with sides parallel to the coordinate planes. In fact, we can always approximate the derivative information in this setting by replacing the non-empty compact convex set with the smallest such hyper-rectangle that contains it. This approach is consistent with interval analysis [24] in that each partial derivative would be a compact interval. It will result in some loss of information but will greatly simplify the framework and provides a practical setting for implementation.

In fact, in the case that the derivative information is given by compact hyper-rectangles with faces parallel to the coordinate planes for a consistent pair, as we will show in this paper, there is always a piecewise linear witness for consistency. In addition, there are a least and a greatest piecewise linear functions which satisfy the function and the partial derivative constraints, both witness to consistency.

Figure 1 shows two examples of consistent tuples for $n = 2$ and in each case the least and greatest functions consistent with the derivative constraints are drawn. In the first case, on the left, there is a single hyper-rectangle for function approximation and the derivative approximations in the $x$ and $y$ directions over the whole domain of the function are given respectively by the constant intervals $[n, N]$ and $[m, M]$ with $n, m > 0$. In the second case, on the right, there are two intersecting hyper-rectangles for the function approximation and the derivative approximations are the constant intervals $[0, 0]$ and $[m, M]$ with $m > 0$.

A main question now is whether consistency of a pair of step functions containing function and derivative information is actually decidable. This problem can be addressed directly or alternatively broken in two parts: (i) first decide whether the derivative information is integrable, and, in case the answer is positive, (ii) decide whether there is an integral which is consistent with the function information. For $n = 1$, where, as in the classical setting, the derivative information is always integrable, it was shown in [14] that consistency is decidable and in [13], a linear algorithm was presented (linear in the number of pairs in the two step functions) which decides the consistency in this case.

For $n \geq 2$, as in classical multi-variable calculus, a Scott continuous function of type $\mathbb{R}^n \to \mathbf{C}\mathbb{R}^n$, where $\mathbf{C}\mathbb{R}^n$ is the domain of non-empty, compact and convex subsets of $\mathbb{R}^n$ ordered by reverse inclusion, may fail to be integrable.

The first main result of the paper is to tackle directly, i.e. without breaking the problem into two as above, the question whether consistency is decidable on basis elements. Given any convex compact polygon $C \subset \mathbb{R}^n$, we define a finitely generated associated cone centred at the origin in $\mathbb{R}^{n+1}$ which has the following property: the graph of any function of type $\mathbb{R}^n \to \mathbb{R}$ that vanishes at the origin, and whose L-derivative is locally bounded by $C$, will be locally contained in the associated cone.

We then show that consistency of function and derivative approximations when the latter is provided by compact hyper-rectangles can be established algebraically by reducing the problem of consistency to a linear programming problem: we impose a hyperrectangular grid on the domain of definition of the function where in each grid sub-hyperrectangle both the function and the derivative approximations are constant and then use the cones associated with the constant derivative values to check consistency of function and derivative information along all coordinate axes. This reduces the question of consistency of the function and hyper-rectangular derivative approximation to whether a finite collection of rational semi-hyperplanes in $\mathbb{R}^{n+1}$ has non-empty intersection. Since the latter question is equivalent to the consistency of the constraints in a linear programming problem, it follows that consistency of basis elements in our domain is decidable in this case. Given that this check succeeds, one can construct a witness for consistency by linearly interpolating between solutions of the linear programming problem. As a special case, this also shows that the question of integrability of a hyper-rectangular-valued step function is decidable.

The decidability of consistency of basis elements in this case leads to an effective structure for the domain of real-valued Lipschitz functions of several variables allowing us to enumerate the countable set of computable Lipschitz maps and that of computable functionals defined on this domain. It also enables us to construct increasingly finer approximations for a desired multivariable Lipschitz function by ensuring that at each stage our construction is sound, in the sense that it is a possible approximation to the desired function. Such a situation arises in CAD where one seeks to use piecewise linear surfaces to approximate a given map.

In the next part of the paper, we derive a domain-theoretic generalization of Green's theorem, also called the fundamental theorem of path integration, for a vector field to be a gradient, i.e., to be an exact differential. Recall that, in the classical setting, a continuous vector field is the gradient of a differentiable function if and only if the path integral of the vector field over any closed smooth path vanishes [22, pages 286-291].

With this goal in mind, we first introduce a domain-theoretic notion of line integration, developed here for the general framework, which defines the line integral as an interval valued function. Then we prove that if a Scott continuous compact convex valued function is integrable, i.e., if there is a Lipschitz map whose Clarke gradient is bounded by the Scott continuous function, then its path integral over any closed piecewise $C^1$ path is an interval that contains zero.

The converse is far more conceptually and technically involved to establish. We use techniques from the theory of minimal surfaces to show that if the line integral of a Scott continuous hyper-rectangular valued function over any piecewise $C^1$ closed path contains zero then the Scott continuous function is integrable with two canonical witnesses (up to a constant) that respectively maximise the lower line integral and minimise the upper line integral from a given point. Thus, we have a domain-theoretic generalization of Green's theorem: a necessary and sufficient condition for a Scott continuous hyper-rectangular valued function to be integrable is that zero must be contained in the line integral of the function with respect to any closed piecewise $C^1$ path in its domain of definition.

Finally, we construct the least and the greatest piecewise linear functions obtained from a pair of hyper-rectangular valued step functions, representing function and derivative approximation in which the derivative part is assumed to be integrable. These surfaces are obtained respectively by patching together, at a finite number of points given by the function information, the minimal and the maximal witnesses for the integrability of the derivative information. Furthermore, the least and greatest surfaces can be effectively constructed. If consistency holds, then any witness for consistency will lie between the least and the greatest surface.

### 1.1. Related Work

The domain for real-valued Lipschitz functions has led to applications in solving initial value problems [13, 19, 15, 17] and in developing a denotational semantics for hybrid automata [18]. The domain for the Lipschitz functions on finite dimensional Euclidean spaces has been used to develop domain-theoretic inverse and implicit function theorems for Lipschitz functions [16]. The L-derivative has enabled us to define the weak topology on locally Lipschitz maps which has been shown to be coarser than the Lipschitz norm topology [12]. The L-derivative has also been used to develop a typed Lambda calculus, an extension of PCF with real numbers, equipped with a derivative operator [10], which gives a denotational semantics for algorithmic differentiation [21].

We have already pointed out the related work of Clarke [8] and the equivalence of the L-derivative and the Clarke in finite dimensional Euclidean spaces. In a series of papers, Borwein and his collaborators have studied various properties of the Clarke gradient and developed new related notions [4, 5, 6]. In particular, given a weak* upper semi-continuous map $g$ that is non-empty, convex and weak* compact set-valued from a Banach space to the space of subsets of its dual, a *$g$-Lipschitz map* is defined as one whose Clarke gradient at every point is contained in the set value of $g$ at that point. In finite dimensions, the set of $g$-Lipschitz maps is precisely the set of primitives of $g$, a result which is a direct consequence of the equivalence of the L-derivative and Clarke gradient. Whereas in the domain-theoretic setting the generalized differential properties are used to develop the notion of primitives and the extended Fundamental Theorem of Calculus is then deduced as a proposition, in the work of Borwein et al. the $g$-Lipschitz maps are defined precisely by using the relation that expresses the Fundamental Theorem of Calculus. In addition, we also mention the necessary and sufficient condition given by Borwein et al. [6] for the integrability of a Scott continuous function defined on a connected open set, a condition which is based on the existence of a measurable subset and a measurable selection.

In computable analysis, Pour-El and Richards [25] relate the computability of a function with the computability of its derivative. The scheme employed in particular by Weihrauch [29] leads to partially defined representations, but there is no general result on decidability. Interval analysis [24] also provides a framework for verified numerical computation. In this context, differentiation is performed by symbolic techniques [26] in contrast to our sequence of approximations of the functions.

In [7], a more recent application of domain theory in differential calculus, in the context of viscosity solutions of Hamiltonian equations, is introduced which uses the *strong derivative*. This notion is not directly related to our work here since, as shown in [23, Proposition 1.9], there are Lipschitz functions defined on the unit interval which have a non-point valued Clarke gradient, equivalently, L-derivative at every point and are not strongly differentiable at any point.

### 1.2. Notations and terminology

We assume the reader is familiar with elements of domain theory and multivariable differential calculus. Some basic knowledge of linear programming is also used to deduce the decidability of consistency. We use the standard notions of domain theory as in [1, 20]. We write $\mathbb{R}$ for the set of real numbers and $\mathbf{I}\mathbb{R} = \{[a,b] \mid a \leq b \in \mathbb{R}\} \cup \{\mathbb{R}\}$ for the interval domain, i.e. the set of compact, nonempty intervals, equipped with a least element $\perp = \mathbb{R}$, ordered by reverse inclusion. It has a canonical basis consisting of all compact intervals with rational end points augmented with $\perp$. We write a non-bottom element $v \in \mathbf{I}\mathbb{R}$ as $v = [v^-, v^+]$. As usual, we identify any real number $x \in \mathbb{R}$ with the singleton $\{x\} \subset \mathbb{R}$ so that we identify the set of maximal elements of $\mathbf{I}\mathbb{R}$ as $\mathbb{R}$.

We will also consider the (smash) product domain $\mathbf{I}\mathbb{R}^n$ consisting of all non-empty compact hyper-rectangles with faces parallel to the standard coordinate planes ordered with reverse inclusion and augmented with the whole space $\mathbb{R}^n$ as the bottom element. It has a canonical basis consisting of all its rational (compact) hyper-rectangles and the bottom element. We denote the continuous Scott domain of the nonempty, compact and convex subsets of $\mathbb{R}^n$, taken together with $\mathbb{R}^n$ as the bottom element and ordered by reverse inclusion, by $\mathbf{C}\mathbb{R}^n$. We will use a canonical basis of $\mathbf{C}\mathbb{R}^n$, consisting of rational convex compact polyhedra together with the set $\mathbb{R}^n$ as the bottom element.

For an open subset $U \subset \mathbb{R}^n$, let $C^0(U)$ be the function space of all continuous functions of type $U \to \mathbb{R}$. We will also use domains of function spaces of the form $(U \to D)$ where $D$ is a countably based continuous dcpo, which is either $\mathbf{I}\mathbb{R}$, $\mathbf{I}\mathbb{R}^n$ or $\mathbf{C}\mathbb{R}^n$ in this paper. Thus, $(U \to D)$ consists of Scott continuous functions partially ordered pointwise by the order inherited from $D$. For convenience, we write $D^0(U) = U \to \mathbf{I}\mathbb{R}$. A function $f \in D^0(U)$ is given by a pair of respectively lower and upper semi-continuous functions $f^-, f^+ : U \to \mathbb{R}$ with $f(x) = [f^-(x), f^+(x)]$ when $f(x) \neq \perp$ for all $x \in U$. Recall that given an open subset $a \subset U$ and an element $b \in D$, the *single step function* $b\chi_a : X \to D$ is defined as $(b\chi_a)(x) = b$ if $x \in a$ and $\perp$ otherwise, where we have used the notation in [20]. Single-step functions are continuous with respect to the Scott topology. Any finite set of single-step functions that are bounded in the function space $U \to D$ has a least upper bound, called a *step function*; the set of step functions provides a basis for the continuous Scott domain $U \to D$. This basis in turn gives a countable and canonical basis of *rational step functions* for $U \to D$, where $D = \mathbf{I}\mathbb{R}$, $\mathbf{I}\mathbb{R}^n$ or $\mathbf{C}\mathbb{R}^n$, generated by single-step functions of the form $b\chi_a$ where $a$ is a rational open hyper-rectangle with faces parallel to the coordinate hyper-planes of $\mathbb{R}^n$ and $b$ is a rational interval for $D = \mathbf{I}\mathbb{R}$, a rational hyper-rectangle for $D = \mathbf{I}\mathbb{R}^n$ and a rational compact convex polyhedron in $\mathbb{R}^n$ for $D = \mathbf{C}\mathbb{R}^n$. Finally, in our list of domain-theoretic terminology, the set of elements above an element $c$ in a domain is denoted by $\uparrow c$.

We use standard operations of interval arithmetic [24], which extend the usual operations such as addition and multiplication of numbers by pointwise application to sets of points. There are two such operations we specifically use in this paper. Let $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ be the standard Euclidean norm of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Then the Euclidean norm is extended pointwise to $b \in \mathbf{C}\mathbb{R}^n$ by $\|b\| = \max\{\|x\| : x \in b\}$. We will also consider the extension $- \cdot - : \mathbf{C}\mathbb{R}^n \times \mathbb{R}^n \to \mathbf{I}\mathbb{R}$ of the scalar product which is defined pointwise $b \cdot x = \{y \cdot x : y \in b\}$.

Recall that the directional derivative of a map $f : U \subset \mathbb{R}^n \to \mathbb{R}$ at $x \in U$, in the direction $u \neq 0$ when it exists, is defined as

$$f'(x; u) = \lim_{h \to 0^+} \frac{f(x + hu) - f(x)}{h}.$$

Recall also that the derivative of $f$ at $x \in U$, when it exists, is defined as the linear map $T : \mathbb{R}^n \to \mathbb{R}$

with

$$\lim_{\|x-y\|\to 0} \frac{|f(x)-f(y)-T(x-y)|}{\|x-y\|} = 0.$$

The linear map $T$ is denoted by $f'(x)$. Let $\nabla f$ denote the gradient of $f$, when it exists, i.e.,

$$(\nabla f)_i(x) = \frac{\partial f}{\partial x_i}(x) =$$

$$\lim_{x_i' \to x_i} \frac{f(x_1,\dots,x_i,\dots,x_n) - f(x_1,\dots,x_i',\dots,x_n)}{x_i - x_i'},$$

for $1 \leq i \leq n$. Recall that if the derivative exists at a point then the gradient also exists at that point and has the same value. We will reserve the notation $\mathcal{L}f$ exclusively in this paper for the L-derivative of $f$ which will be introduced later. The interior of a set $A \subset \mathbb{R}^n$ is denoted by $A^\circ$ and its closure by $\mathrm{cl}(A)$.

We next define the generalized (Clarke) gradient of a function [8, Chapter 2] and explain its properties. Let $U \subset X$ be an open subset of a Banach space $X$ and let $f : U \to \mathbb{R}$ be Lipschitz near $x \in U$ and $v \in X$. The *generalized directional derivative* of $f$ at $x$ in the direction of $v$ is

$$f^\circ(x; v) = \limsup_{y\to x \ t\downarrow 0} \frac{f(y+tv)-f(y)}{t}.$$

Let us denote by $X^*$ the dual of $X$, i.e. the set of real-valued continuous linear functions on $X$. We consider $X^*$ with its weak* topology, i.e., the weakest topology on $X^*$ in which for any $x \in X$ the map $f \mapsto f(x) : X^* \to \mathbb{R}$ is continuous.

The *generalized gradient* of $f$ at $x$, denoted by $\partial f(x)$ is the subset of $X^*$ given by

$$\{A \in X^* : f^\circ(x; v) \geq A(v) \text{ for all } v \in X\}.$$

It is shown in [8, page 27] that

- $\partial f(x)$ is a non-empty, convex, weak* compact subset of $X^*$.
- For $v \in X$, we have:
$$f^\circ(x; v) = \max\{A(v) : A \in \partial f(x)\}.$$

When $X$ is finite dimensional, say $X = \mathbb{R}^n$, there is a simpler characterization of the generalized gradient. In this case, by Rademacher's theorem [9, page 148], a locally Lipschitz map $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is Fréchet differentiable almost everywhere with respect to the Lebesgue measure. If $\Omega_f$ is the nullset where $f$ fails to be differentiable then:

$$\partial f(x) = \mathrm{Co}\{\lim f'(x_i) : x_i \to x,\ x_i \notin \Omega_f\}, \tag{1}$$

where $\mathrm{Co}(S)$ is the convex hull of a subset $S \subset \mathbb{R}^n$ [8, page 63]. The above expression is interpreted as follows. Consider all sequences $(x_i)_{i\geq 0}$, with $x_i \notin \Omega_f$, for $i \geq 0$, which converge to $x$ such that $\lim_{i\to\infty} f'(x_i)$ exists. Then the generalized gradient is the convex hull of all such limits. Note that, in the above definition, since $f$ is locally Lipschitz at $x$, it is differentiable almost everywhere in a neighbourhood of $x$ and thus there are plenty of sequences $(x_m)_{m\geq 0}$ such that $\lim_{m\to\infty} x_m = x$ and $\lim_{m\to\infty} f'(x_m)$ exists.

## 2. Domain for Lipschitz Functions

This section reviews the necessary background on the $L$-derivative and the domain of Lipschitz maps, specialised to finite dimensions, from [11] to which we also refer for all proofs. The local differential properties of a function are formalised in the domain-theoretic framework by the notion of a set-valued Lipschitz constant. Assume $U \subset \mathbb{R}^n$ is an open subset.

**Definition 2.1.** The continuous function $f : U \to \mathbb{R}$ has *a non-empty, convex and compact set-valued Lipschitz constant* $b \in \mathbf{C}\mathbb{R}^n$ in an open subset $a \subset U$ if for all $x, y \in a$ we have: $b\cdot(x-y) \sqsubseteq f(x)-f(y)$. The *single-step tie* $\delta(a,b) \subseteq C^0(U)$ of $a$ with $b$ is the collection of all partial functions $f$ on $U$ with $a \subset \text{dom}(f) \subset U$ in $C^0(U)$ which have $b$ as a non-empty convex compact set-valued Lipschitz constant in $a$.

Note that, as stated in the introduction, we have used the extension of the scalar product to subsets of $\mathbb{R}^n$ and identified in the above definition the real number $f(x)-f(y)$ with the singleton $\{f(x)-f(y)\}$. For example, if $n=2$ and $b = b_1 \times b_2 \subseteq \mathbb{R}^2$, the information relation above reduces to $b_1(x_1-y_1) + b_2(x_2-y_2) \sqsubseteq f(x)-f(y)$. For a single-step tie $\delta(a,b)$, one can think of $b$ as the non-empty compact-set Lipschitz constant for the family of functions in $\delta(a,b)$. The classical Lipschitz constant for $f$ would simply be $k = \|b\| \geq 0$. By generalizing the concept of a Lipschitz constant in this way, one is able to obtain essential information about the differential properties of the function. In particular, if $f \in \delta(a,b)$ for $a \neq \emptyset$ and $b \neq \bot$, then $f(x)$ is maximal for each $x \in a$ and the induced function $f : a \to \mathbb{R}$ is Lipschitz: for all $x, y \in a$ we have $|f(x)-f(y)| \leq \|b\|\|x-y\|$. For $f \in C^1(U)$, the following three conditions are shown to be equivalent in [11]: (i) $f \in \delta(a,b)$, (ii) $\forall z \in a.\ f'(z) \in b$ and (iii) $a \searrow b \sqsubseteq f'$. For the rest of this section, we assume we are in dimension $n \geq 2$ and for convenience we write $C^0$ for $C^0(U)$.

**Definition 2.2.** A *step tie* of $C^0$ is any finite intersection $\bigcap_{i\in I} \delta(a_i, b_i) \subset C^0$, where $I$ is a finite indexing set. A *tie* of $C^0$ is any intersection $\Delta = \bigcap_{i\in I} \delta(a_i,b_i) \subset C^0$, for an arbitrary indexing set $I$. The *domain* of a non-empty tie $\Delta$ is defined as $dom(\Delta) = \bigcup_{i\in I}\{a_i \mid b_i \neq \bot\}$.

A non-empty step tie with rational intervals gives us a family of functions with a *finite* set of consistent differential properties, and a non-empty general tie gives a family of functions with a consistent set of differential properties. Recall that a function $f : U \to \mathbb{R}$ defined on the open set $U \subseteq \mathbb{R}^n$ is *locally Lipschitz* if it is Lipschitz in a neighbourhood of any point in $U$. If $\Delta \subset C^0$ is a tie and $f \in \Delta$, then $f(x)$ is maximal for $x \in dom(\Delta)$ and $f$ is locally Lipschitz on $dom(\Delta)$.

We now collect some simple properties of step ties, which we will use later and refer to [11] for proofs. For any indexing set $I$, the family of step functions $(b_i\chi_{a_i})_{i\in I}$ is consistent if $\bigcap_{i\in I}\delta(a_i,b_i) \neq \emptyset$. One important corollary of this is that consistency of a family of step functions can be determined from the associated ties in a finitary manner: The family $(b_i\chi_{a_i})_{i\in I}$ is consistent if for any finite subfamily $J \subseteq I$ we have $\bigcap_{i\in J}\delta(a_i,b_i) \neq \emptyset$.

Let $(T^1(U), \supseteq)$ be the dcpo of ties of $C^0$ ordered by reverse inclusion. We are finally in a position to define the set of primitives of a Scott continuous function; in fact now we can do more and define a continuous functional as follows:

**Definition 2.3.** The *primitive map* $\int : (U \to \mathbf{C}\mathbb{R}^n) \to T^1(U)$ is defined by $\int(g) = \bigcap_{i\in I}\delta(a_i,b_i)$, where $g = \sup_{i\in I} b_i\chi_{a_i}$. We usually write $\int(f)$ as $\int f$ and call it the set of *primitives* of $f$.

The primitive map is well-defined, onto and continuous. For $n \geq 2$, as we are assuming here, the primitive map will have the empty tie in its range, a situation which does not occur for $n = 1$.

**Example 2.4.** Let $g \in [0,1]^2 \to \mathbf{C}\mathbb{R}^2$ be given by

$$g = (g_1, g_2) = (\lambda x_1.\lambda x_2.1, \lambda x_1.\lambda x_2.x_1).$$

Then $\frac{\partial g_1}{\partial x_2} = 0 \neq 1 = \frac{\partial g_2}{\partial x_1}$, and it will follow that $\int g = \emptyset$.

Therefore, we have the following important notion in dimensions $n \geq 2$.

**Definition 2.5.** A map $g \in U \to \mathbf{C}\mathbb{R}^n$ is said to be *integrable* if $\int g \neq \emptyset$.

Given a continuous function $f : U \to \mathbb{R}$, the relation $f \in \delta(a, b)$ provides, as we have seen, finitary information about the local differential properties of $f$. By collecting all such local information, we obtain the complete differential properties of $f$, namely its derivative.

**Definition 2.6.** The *derivative* of a continuous function $f : U \to \mathbb{R}$ is the map

$$\mathcal{L}f = \bigsqcup_{f \in \delta(a,b)} b\chi_a : U \to \mathbf{C}\mathbb{R}^n.$$

We have the following properties, which are established in [11] for the case of arbitrary (possibly infinite) dimension.

**Theorem 2.7.** (i) *$\mathcal{L}f$ is well-defined and Scott continuous.*

(ii) *If $f \in C^1(U)$ then $\mathcal{L}f = f'$.*

(iii) *$f \in \delta(a, b)$ iff $b\chi_a \sqsubseteq \mathcal{L}f$.*

(iv) *If $f$ is differentiable at $x \in U$, then $f'(x) \in \mathcal{L}f(x)$.*

We also obtain the generalization of Theorem 2.7(iii) to ties, which provides a duality between the domain-theoretic derivative and integral and can be considered as a variant of the Fundamental Theorem of Calculus:

**Corollary 2.8.** *$f \in \int g$ iff $g \sqsubseteq \mathcal{L}f$.*

Moreover, we have the fundamental result:

**Theorem 2.9.** *[11, Corollary 8.2] In finite dimensional Euclidean spaces, the L-derivative coincides with the Clarke gradient.*

The set of primitive maps of $g$ is closely related to the notion of $g$-Lipschitz functions due to Borwein et al. [6] defined as follows; we restrict to finite dimensions. Let $g \in (U \to \mathbf{C}\mathbb{R}^n)$ be Scott continuous. Then, the set of *$g$-Lipschitz maps* is defined in terms of the Clarke gradient $\partial f$ of locally Lipschitz functions as

$$\chi_g = \{f : U \to \mathbb{R} : f \text{ is locally Lipschitz and } \partial f(x) \subset g(x) \text{ for all } x \in U\}.$$

By the equivalence of the Clarke gradient and the L-derivative in finite dimensions (Theorem 2.9), it follows immediately from Corollary 2.8 that $\chi_g = \int g$.

A domain for locally Lipschitz functions and for $C^1(U)$ is constructed as follows. The idea is to use $D^0(U)$ to represent the function and $U \to \mathbf{C}\mathbb{R}^n$ to represent the differential properties (partial derivatives) of the function. Consider the *consistency* relation

$$\mathsf{Cons} \subset D^0(U) \times (U \to \mathbf{C}\mathbb{R}^n),$$

defined by $(f, g) \in \mathsf{Cons}$ if $\uparrow f \cap \int g \neq \emptyset$. For a consistent $(f, g)$, we think of $f$ as the *function part* or the *function approximation* and $g$ as the *derivative part* or the *derivative approximation*. We will show that the consistency relation is Scott closed.

**Proposition 2.10.** *Let $g \in U \to \mathbf{C}\mathbb{R}^n$ and $(f_i)_{i\in I}$ be a non-empty family of functions $f_i : dom(g) \to \mathbb{R}$ with $f_i \in \int g$ for all $i \in I$. If $h_1 = \inf_{i\in I} f_i$ is real-valued (i.e., if the family $\{f_i : i \in I\}$ is non-empty and bounded below) then $h_1 \in \int g$. Similarly, if $h_2 = \sup_{i\in I} f_i$ is real-valued (i.e., if the family is non-empty and bounded above), then $h_2 \in \int g$.*

In later sections, we will consider piecewise linear paths in $dom(g)$; it is convenient to work with a connected component $O$ of $dom(g)$ as we will do in the following. Let $R(U)$ be the set of partial maps of $U$ into the extended real line $\mathbb{R} \cup \{\infty, -\infty\}$. Consider the two dcpos $(R(U), \leq)$ and $(R(U), \geq)$ with pointwise ordering inherited from the extended real line. Define the maps $s : D^0(O) \times (U \to \mathbf{C}\mathbb{R}^n) \to (R(U), \leq)$ and $t : D^0(O) \times (U \to \mathbf{C}\mathbb{R}^n) \to (R(U), \geq)$ by

$$s : (f, g) \mapsto \inf\{h : dom(g) \to \mathbb{R} \,|\, h \in \int g \;\&\; h \geq f^-\}$$

$$t : (f, g) \mapsto \sup\{h : dom(g) \to \mathbb{R} \,|\, h \in \int g \;\&\; h \leq f^+\}.$$

We use the convention that the infimum and the supremum of the empty set are $\infty$ and $-\infty$, respectively. Note that if $O \cap dom(f) = \emptyset$, then $s(f, g)(x) = -\infty$ and $t(f, g)(x) = \infty$ for $x \in O$. In words, $s(f, g)$ is the least primitive map of $g$ that is greater than the lower part of $f$, whereas $t(f, g)$ is greatest primitive map of $g$ less that the upper part of $f$. It then follows that the following three conditions are equivalent: (i) $(f, g) \in \mathsf{Cons}$, (ii) $s(f, g) \leq t(f, g)$ and (iii) There exists a locally Lipschitz function $h : dom(g) \to \mathbb{R}$ with $g \sqsubseteq \mathcal{L}h$ and $f \sqsubseteq h$ on $dom(g)$.

Moreover, the maps $s$ and $t$ are Scott continuous and the relation Cons is Scott closed. We can sum up the situation for a consistent pair of function and derivative information.

**Corollary 2.11.** *Let $(f, g) \in \mathsf{Cons}$. Then in each connected component $O$ of the domain of definition of $g$ which intersects the domain of definition of $f$, there exist two locally Lipschitz functions $s(f, g) : O \to \mathbb{R}$ and $t(f, g) : O \to \mathbb{R}$ such that $s(f, g), t(f, g) \in \uparrow f \cap \int g$ and for each $u \in \uparrow f \cap \int g$, we have $s(f, g)(x) \leq u(x) \leq t(f, g)(x)$ for all $x \in O$.*

The central notion of this paper is now presented as follows:

**Definition 2.12.** The *domain of locally Lipschitz functions on $U$ with non-empty, convex and compact derivatives* is given by

$$D^1(U) = \{(f, g) \in D^0(U) \times (U \to \mathbf{C}\mathbb{R}^n) : (f, g) \in \mathsf{Cons}\}$$

and the *domain of locally Lipschitz functions on $U$ with rectangular derivative* is the space

$$D^1_R(U) = \{(f,g) \in D^0(U) \times (U \to \mathbf{I}\mathbb{R}^n) : (f,g) \in \mathsf{Cons}\}$$

where $U \subset \mathbb{R}^n$ is an arbitrary open subset.

Both posets $D^1(U)$ and $D^1_R(U)$ are continuous Scott domains, i.e. bounded complete countably based continuous dcpos and the inclusion $i : D^1_R(U) \to D^1(U)$ is Scott-continuous. If $L(U)$ and $C^1(U)$ denote the collection of real-valued locally Lipschitz functions and continuously differentiable functions on $U$ respectively, we have the maps

$$\Gamma : L(U) \to D^1_R(U) \text{ defined by } \Gamma(f) = (f, \mathcal{L}f)$$

$$\Gamma_1 : C^1(U) \to D^1_R(U) \text{ defined by } \Gamma_1(f) = (f, f').$$

We note that $\Gamma_1$ and $i \circ \Gamma_1$ are continuous injections of $C^1(U)$, equipped with the $C^1$ norm topology, into the set of maximal elements of $D^1_R(U)$ and $D^1(U)$, respectively. In [12], the weakest topology on $L(U)$ is defined as the coarsest topology that makes the map $\Gamma$ continuous. With respect to this topology on $L(U)$, the map $\Gamma$ and $i \circ \Gamma$ are embeddings (where $\mathcal{L}f \neq \bot$ for all $x$).

The difference between $D^1(U)$ and $D^1_R(U)$ is mainly one of taste and the problem at hand. Theoretically, $D^1(U)$ provides a much more fine grained approximation of values of the L-derivative. However, working with $D^1_R(U)$, which entails some loss of information, is conceptually easier and computationally more practical as derivative values are represented by compact rectangles rather than non-empty, convex and compact sets. We now proceed to show that $D^1_R(U)$ can be equipped with an effective structure, thus paving the way for a computational and domain theoretic analysis of differentiable functions.

## 3. Effective Structure for $D^1_R(U)$

We show in this section that $D^1_R(U)$, which uses hyper-rectangles as derivative information, can be equipped with an effective structure by proving that consistency of the derivative and function information is decidable for it. We will however present a general framework, allowing us to deal with convex and compact polyhedra as derivative information, which can be used to study the consistency in $D^1(U)$ as well. This we think is justified in particular because the general setting is not more conceptually or technically involved and thus we do not gain much by restricting it to hyper-rectangles. The general setting is also used to show, by providing a counter-example, why our solution for decidability of consistency in the hyper-rectangular case does not work in the more general domain $D^1(U)$.

We assume in the sequel that $U = (0,1)^n$ is the open unit cube which permits us to focus on effective structures for differentiable functions (we can also take $U$ to be the closed unit cube $[0,1]^n$). As explained in the introduction, we take the canonical countable bases of $\mathbf{I}\mathbb{R}$, $\mathbf{I}\mathbb{R}^n$ and $\mathbf{C}\mathbb{R}^n$ consisting of non-empty compact rational intervals, compact hyper-rectangles and non-empty rational compact convex polyhedra respectively. These three bases respectively generate canonical bases of $(U \to \mathbf{I}\mathbb{R})$, $(U \to \mathbf{I}\mathbb{R}^n)$ and $(U \to \mathbf{C}\mathbb{R}^n)$ consisting of step functions made up of rational single-step functions with values of the three different kinds above and defined on rational open hyper-rectangles in $U$ as open sets.

We now introduce the main concept required in our framework for investigating consistency of function and derivative information. Recall that a set $C \subseteq \mathbb{R}^n$ is called a *cone* with vertex at the origin if $x \in C$ implies $rx \in C$ for all $r \geq 0$. The set $-C = \{-x : x \in C\}$ is the *mirror cone* of $C$. For any

$w \in \mathbb{R}^n$, the set $C + w = \{x + w : x \in C\}$ is the cone $C$ *transported* to $w$. A cone $C \subset \mathbb{R}^n$ is *finitely generated* by the vectors $k_1, k_2, \cdots, k_m$ if

$$C = \{\sum_{j=1}^{m} r_j k_j : r_j \geq 0, j = 1, \ldots, m\}.$$

Let $b \in \mathbf{C}\mathbb{R}^n$ be any non-empty convex compact subset of $\mathbb{R}^n$.

**Definition 3.1.** The *upper* $b$ *surface* through the origin is defined by $U_b : \mathbb{R}^n \to \mathbb{R}$ with $U_b(x) = (b \cdot x)^+$. Similarly, the *lower* $b$ *surface* through the origin is defined as $L_b : \mathbb{R}^n \to \mathbb{R}$ with $L_b(x) = (b \cdot x)^-$, for any vector $x \in \mathbb{R}^n$. The $b$-cone centred at the origin is the cone $C_b$ in $\mathbb{R}^{n+1}$ bounded by the lower and upper $b$-surfaces through the origin.

In fact, $U_b$ is the *support function* of the compact convex set $b$ [27, 3] in a finite dimensional Euclidean space and is a bounded convex function; it is thus Lipschitz [8, Proposition 2.2.6]. Note that $L_b(x) = -U_b(-x)$; see Figure 2 in which $x$ has been chosen as a unit vector so that $v \cdot x$ is simply the projection of $v$ along the vector $x$, i.e., the distance from the origin of orthogonal projection of $v$ into $x$. For example, for the unit $n$-dimensional closed unit disc, $b = \mathsf{D}_1(0)$, we have $U_{\mathsf{D}_1(0)}(x) = \|x\|$ whereas $L_{\mathsf{D}_1(0)}(x) = -\|x\|$. The $\mathsf{D}_1(0)$-cone $C_{\mathsf{D}_1(0)}$ at the origin is then given by

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : -\|x\| \leq y \leq \|x\|\}.$$

For convex functions, the directional derivative exists at each point for each direction and coincides with the corresponding generalised directional derivative. This implies that the Clarke gradient at each point is the convex closure of all the directional derivatives at that point. [8, 2.3.4 and 2.3.6]. Moreover, the directional derivative of the support function $U_b$ at a point $x$ in the direction of $u \neq 0$ is given by

$$U_b'(x; u) = U_{b \cap H(x)}(u), \tag{2}$$

where $H(x) = \{y \in \mathbb{R}^n : y \cdot x = U_b(x)\}$ [28, page 40]. This means that $U_b$ is differentiable at $x$ if and only if $b \cap H(x)$ is a singleton. It is now straightforward to conclude the following results.

**Proposition 3.2.** *For all* $x \in \mathbb{R}^n$ *we have* $(\mathcal{L}L_b)(x) \sqsupseteq b$ *and* $(\mathcal{L}U_b)(x) \sqsupseteq b$*, and* $(\mathcal{L}L_b)(0) = (\mathcal{L}U_b)(0) = b$*.*

From now on, assume that $b$ is a non-empty convex compact polyhedron in $\mathbb{R}^n$. Let $v$ be a vertex of $b$ formed by the intersection of, say, $i$ faces $D_1, D_2, \cdots, D_i$ of $b$ and let $k_j$ be the outer unit normal to $D_j$ for $1 \leq j \leq i$. The vertex $v \in b$ induces an *associated cone* $P_v(b)$ in $\mathbb{R}^n$ centered at the origin and finitely generated by the $i$ vectors $k_j$ for $1 \leq j \leq i$. When the polyhedron $b$ is clear from the context, we write $P_v(b)$ simply as $P_v$. For any two different vertices $v$ and $w$ of $b$, the interiors of the two cones $P_v(b)$ and $P_w(b)$ are disjoint and the $n$-dimensional space $\mathbb{R}^n$ is the union of the cones associated to all the vertices of $b$. It follows from Equation (2) that $U_b$ is differentiable for any $x \in P_v^\circ$ with $U_b'(x) = v$, which implies $U_b(x) = v \cdot x$. Figure 2 illustrates an example in $\mathbb{R}^2$ with $b$ a triangle. The three vertices $v_i$ of $b$ induce the three associated cones $P_{v_i}$, for $1 \leq i \leq 3$. On the left, two horizontal cross sections of the lower surface $L_b$ of the cone $C_b$ have been depicted as the two dashed triangles with $L_b(x) = c_1$ and $L_b(x) = c_2$ for two constants $c_1 > c_2 > 0$.

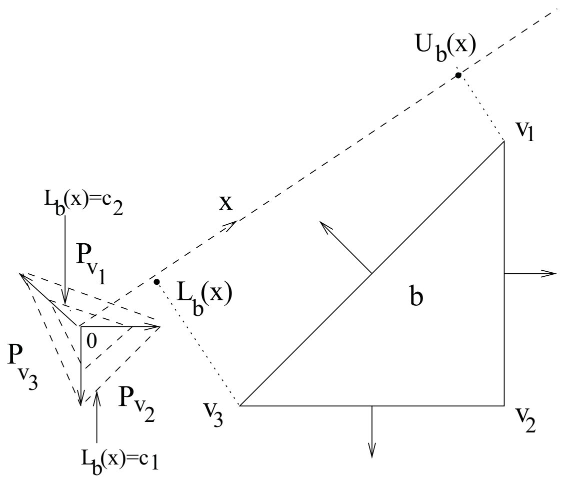


**Figure 2. The associated cones of the three vertices of a triangle $b$ and a unit vector $x$. Two horizontal sections of the lower surface $L_b$ are depicted on the left.**

In the language of convex analysis, the faces $D_j$ for $1 \leq j \leq i$ are the extremal support hyper-planes for the boundary point $v$ of the convex set $b$. Furthermore, the mirror cone $-P_v$ is finitely generated by the vectors $-k_j$ for $1 \leq j \leq i$ and $\mathbb{R}^n$ is the union, with disjoint interiors, of all cones $-P_v$ for all vertices $v$ of $b$. For any $x \in -P_v$ we have $(b \cdot x)^- = v \cdot x$. Summarising we have:

**Proposition 3.3.** *We have $U_b(x) = v \cdot x$ for $x \in P_v$ and $L_b(x) = v \cdot x$ for $x \in -P_v$.*

Consider now the two restrictions $U_{b\restriction P_v} : P_v \to \mathbb{R}$ and $L_{b\restriction -P_v} : -P_v \to \mathbb{R}$ of the upper and lower surfaces of $C_b$ through the origin respectively.

**Proposition 3.4.** *The graph of the restrictions $U_{b\restriction P_v}$ and $L_{b\restriction -P_v}$ are finitely generated cones and the two graphs are mirror cones.*

*Proof.* We claim that the graph of $U_{b\restriction P_v}$ is finitely generated by the $i$ vectors

$$(k_j, U_b(k_j)) = (k_j, v \cdot k_j) \in \mathbb{R}^n \times \mathbb{R}$$

for $j = 1, \ldots, i$. In fact, any vector in $P_v$ is of the form $\sum_{j=1}^i r_j k_j$. Since $U_b(\sum_{j=1}^i r_j k_j) = v \cdot (\sum_{j=1}^i r_j k_j)$, any vector in the graph of $U_{b\restriction P_v}$ is of the form

$$(\sum_{j=1}^{i} r_j k_j, U_b(\sum_{j=1}^{i} r_j k_j)) = (\sum_{j=1}^{i} r_j k_j, (v \cdot \sum_{j=1}^{i} r_j k_j)) = \sum_{j=1}^{i} r_j (k_j, v \cdot k_j)$$

as required. A dual argument shows that the graph of $L_{b\restriction -P_v}$ is finitely generated by the $i$ vectors $(-k_j, -v \cdot k_j) \in \mathbb{R}^n \times \mathbb{R}$ for $j = 1, \ldots, i$. Since for any $x \in P_v$ we have $-x \in -P_v$ with $(x, U_b(x)) = (x, v \cdot x) = -(-x, -v \cdot x) = -(-x, L_b(-x))$, it follows that the graphs of $U_{b\restriction P_v}$ and $L_{b\restriction -P_v}$ are mirror cones. $\square$

Putting the graphs of $U_{b\upharpoonright P_v}$ (respectively $U_{b\upharpoonright -P_v}$) together for all vertices $v$ of $b$ we obtain:

**Corollary 3.5.** *The graph of $U_b$ and $L_b$ are mirror cones of each other and each is the union of finitely generated cones with generators that are computed from $b$ and the normal vectors to the faces of $b$.*

It now follows that if $b$ is a rational polyhedron and $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ is a point with rational coordinates, then the inequalities $L_b(x) \leq y \leq U_b(x)$ are decidable. Since we have $C_b = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : L_b(x) \leq y \leq U_b(x)\}$, we obtain:

**Corollary 3.6.** *For a rational polyhedron $b$, the membership predicate of the $b$-cone is decidable for rational points.*

We now explain the framework to study consistency when $n = 2$ as the general case for $n > 2$ is similar. We impose a grid $(p_0, \ldots, p_k) \times (q_0, \ldots, q_l)$ on the unit square $U$ such that the function approximation given by the step function $f : U \to \mathbf{I}\mathbb{R}$ and the derivative approximation given by the step function $g : U \to \mathbf{C}\mathbb{R}^n$ are constant respectively with values $c_{ij} \in \mathbf{I}\mathbb{R}$ and $b_{ij} \in \mathbf{C}\mathbb{R}^n$ inside every subrectangle $(p_i, p_{i+1}) \times (q_j, q_{j+1})$, for $i = 0, \ldots, k-1$ and $j = 0, \ldots, l-1$, defined by adjacent grid points. Note that if $c_{ij} = \perp$ or $b_{ij} = \perp$ then $f$ or $g$ are undefined in the subrectangle and their contribution can be ignored in the following analysis.

If the pair of step functions $(f, g) \in (U \to \mathbf{I}\mathbb{R}) \times (U \to \mathbf{C}\mathbb{R}^n)$ is consistent then we can find values $h_{i,j} \in \mathbb{R}$ at all the grid points $(p_i, q_j)$ for $i = 0, \ldots, k$ and $j = 0, \ldots, l$ such that for $i = 0, \ldots, k-1$ and $j = 0, \ldots, l-1$ they satisfy the following two conditions:

C(i) $c_{ij}^- \leq h_{s,t} \leq c_{ij}^+$ for $s = i, i+1$ and $t = j, j+1$,

C(ii) the cone $C_{b_{ij}}$ transported to any of the four points

$$((p_i, q_j), h_{ij}),\ ((p_{i+1}, q_j), h_{i+1,j}),\ ((p_i, q_{j+1}), h_{i,j+1}),\ ((p_{i+1}, q_{j+1}), h_{i+1,j+1}),$$

contains all the other three points as well (i.e., each of the four cones must contain all the four points).

In fact, if $f$ and $g$ are consistent with a Lipschitz witness $h : U \to \mathbb{R}$, then we put $h_{i,j} = h((p_i, q_j))$ which indeed must satisfy condition C(i) above. Furthermore, since $b_{ij} \sqsubseteq (\mathcal{L}h)(x)$ for all $x \in (p_i, p_{i+1}) \times (q_j, q_{j+1})$, it follows that $((p_s, q_t), h_{s,t}) \in C_{b_{ij}} + ((p_u, q_v), h_{u,v})$ for $s, u = i, i+1$ and $t, v = j, j+1$, since $h$ goes through all the four values at the corners of the subrectangle.

Next we look at the converse of the above result. Suppose there exist real numbers $h_{ij}$, for $i = 0, \ldots, k$ and $j = 0, \ldots, l$, such that conditions C(i) and C(ii) above hold. Consider the piecewise linear function

$$h : U \to \mathbb{R} \tag{3}$$

defined as follows:

W(i) In the triangle with vertices $(p_i, q_j)$, $(p_i, q_{j+1})$ and $(p_{i+1}, q_j)$, the map $h$ linearly interpolates between the values $h_{ij}$, $h_{i,j+1}$ and $h_{i+1,j}$ at these vertices respectively.

W(ii) In the triangle with vertices $(p_{i+1}, q_{j+1})$, $(p_i, q_{j+1})$ and $(p_{i+1}, q_j)$, the map $h$ linearly interpolates between the values $h_{i+1,j+1}$, $h_{i,j+1}$ and $h_{i+1,j}$ at these vertices respectively.

**Proposition 3.7.** *Given a pair of step functions* $(f, g) \in (U \to \mathbf{I}\mathbb{R}) \times (U \to \mathbf{I}\mathbb{R}^n)$, *if there exist real numbers* $h_{ij}$, *for* $i = 0, \dots, k$ *and* $j = 0, \dots, l$, *such that conditions* $\mathsf{C}$*(i) and* $\mathsf{C}$*(ii) above hold, then the piecewise linear map* $h$ *defined in Equation (3) satisfying* $\mathsf{W}$*(i) and* $\mathsf{W}$*(ii) is a witness of consistency of* $(f, g)$.

*Proof.* By $\mathsf{C}$(i), the map $h$ satisfies the function information. To check that it also satisfies the derivative information, let $b_{ij} = b^1_{ij} \times b^2_{ij}$ for compact intervals $b^1_{ij}, b^2_{ij} \in \mathbf{I}\mathbb{R}$. Since by $\mathsf{C}$(ii), the cone $C_{b_{ij}}$ transported to $((p_i, q_j), h_{ij})$ contains $((p_{i+1}, q_j), h_{i+1,j})$ and $((p_i, q_{j+1}), h_{i,j+1})$, we have:

$$h_{i+1,j} - h_{i,j} \in b \cdot ((p_{i+1}, q_j) - (p_i, q_j)) = b^1_{ij}(p_{i+1} - p_i),$$

$$h_{i,j+1} - h_{i,j} \in b \cdot ((p_i, q_{j+1}) - (p_i, q_j)) = b^2_{ij}(q_{j+1} - q_j).$$

By division we obtain:

$$c := \left( \frac{h_{i+1,j} - h_{i,j}}{p_{i+1} - p_i}, \frac{h_{i,j+1} - h_{i,j}}{q_{j+1} - q_j} \right) \in b^1_{ij} \times b^2_{ij} = b_{ij}.$$

Since for $h$ restricted to the interior of the triangle with vertices $(p_i, q_j)$, $(p_i, q_{j+1})$ and $(p_{i+1}, q_j)$, we have $h' = c$, it follows that $\mathcal{L}h(x) = \{h'(x)\} = \{c\} \subset b_{ij}$ for $x$ in the interior of this triangle. Similarly, $\mathcal{L}h(x) = \{h'(x)\} = \{c\} \subset b_{ij}$ in the interior of the the triangle with vertices $(p_{i+1}, q_{j+1})$, $(p_i, q_{j+1})$ and $(p_{i+1}, q_j)$. By Scott continuity of $\mathcal{L}h$ we have $\mathcal{L}h(x) \sqsupseteq g(x)$ for all $x \in U$, including at the boundary points of the triangles.

□

Note that the witness function $h$ can alternatively be constructed by interpolating the values given at the vertices of the two other triangles, i.e., one with vertices $(p_i, q_j)$, $(p_i, q_{j+1})$ and $(p_{i+1}, q_{j+1})$, and the other with vertices $(p_i, q_j)$, $(p_{i+1}, q_j)$ and $(p_{i+1}, q_{j+1})$.

**Corollary 3.8.** *A pair* $(f, g) \in (U \to \mathbf{I}\mathbb{R}) \times (U \to \mathbf{I}\mathbb{R}^n)$ *is consistent iff we can find values* $h_{i,j} \in \mathbb{R}$ *at grid points such that* $\mathsf{C}$*(i) and* $\mathsf{C}$*(ii) ae satisfied.*

Since as basis elements, $f$ and $g$ are given in terms of rational numbers, the question of consistency is then reduced to solving a finite set of inequalities with rational coefficients for the $(k + 1) \times (l + 1)$ unknowns $h_{i,j}$ for $i = 0, \dots, k$ and $j = 0, \dots, l$, i.e., the intersection of a finite set of half-spaces, which is decidable; it in fact represents the set of constraints for a linear programming problem.

For $n > 2$, the same result holds by considering a similar grid in the unit cube $U$ in $\mathbb{R}^n$ with sub-hyperrectangles in which the $f$ and $g$ values are constant. In order to find the witness for consistency when $2^n$ real values are given for the vertices of each sub-hyperrectangle satisfying the containment condition for the corresponding transported cone, we choose any partition of each sub-hyperrectangle into the union of $n$-simplexes with disjoint interiors.

**Corollary 3.9.** *The predicate* $\mathsf{Cons}$ *is decidable on the basis elements of* $(U \to \mathbf{I}\mathbb{R}) \times (U \to \mathbf{I}\mathbb{R}^n)$.

We have therefore proved:

**Corollary 3.10.** *The domain* $D^1_R(U)$ *can be given an effective structure.*

We also note that given a rational step function $g : U \to \mathbf{I}\mathbb{R}^n$, we can determine if $g$ is integrable by dropping condition C(i) and deciding if there exists $h : U \to \mathbb{R}$ satisfying condition C(ii). Thus we have:

**Corollary 3.11.** *The integrability of a basis element of* $(U \to \mathbf{I}\mathbb{R}^n)$ *is decidable.*

Finally, we note that in the more general case when $(f, g) \in (U \to \mathbf{I}\mathbb{R}) \times (U \to \mathbf{C}\mathbb{R}^n)$ and the two conditions C(i) and C(ii) for some $h_{ij}$ values on the grid points are satisfied, the map $h$ defined in Equation (3) with W(i) and W(ii) does not necessarily give a witness for consistency. For example take $b = [(0, 0), (2, 1)]$, the closed line segment from the origin to the point $(2, 1)$ in the plane, and consider the three points $(0, 0, 0)$, $(1, 0, 1)$ and $(0, 1, 0)$. Then we can easily check that the $b$-cone transported to each of the three points contains the other two. However, the plane $f$ through the three points $(0, 0, 0)$, $(1, 0, 1)$ and $(0, 1, 0)$ is given by $f(x, y) = x$ with $f' = (1, 0) \notin b = [(0, 0), (2, 1)]$. Thus, the question of decidability of consistency remains open in this more general framework.

## 4. Interval-valued Line Integration

In this section, we extend the classical theory of line integration to interval-valued vector fields and use it to derive, in this section and the next, an extension of the fundamental theorem of path integration to the domain-theoretic setting. We will use it to show in Section 6 that for step functions $f \in D^0(U)$ and $g \in U \to \mathbf{I}\mathbb{R}^n$, the maps $s(f, g)$ and $t(f, g)$, as in Corollary 2.11, will be piecewise linear, which can be effectively obtained when $f$ and $g$ are rational step functions.

Let $g \in U \to \mathbf{C}\mathbb{R}^n$ be a rational step function. Since $\mathbf{I}\mathbb{R}^n \subset \mathbf{C}\mathbb{R}^n$, we will use the following notions also for $g \in U \to \mathbf{I}\mathbb{R}^n$. Recall that a *crescent* is the intersection of an open set and a closed set. The domain $\mathrm{dom}(g)$ of $g$ is partitioned into a finite set of disjoint crescents $\{C_j : j \in I\}$, in each of which the value of $g$ is constant as a non-empty compact and convex set; they are called the *associated crescents*, or simply the *crescents* of $g$, which play a main part in the framework for deciding integrability as we will see later in this section. Each associated crescent has boundaries parallel to the coordinate planes and these boundaries intersect at points, which are called the *corners* of the crescent.

In Figure 3, an example of a step function $g$ is given with its associated crescents, the interval in each crescent gives the value of $g$ in that crescent. A solid line on the boundary of a crescent indicates that the boundary is in the crescent, whereas a broken line indicates that it is not.

**Remark 4.1.** We have the *adjacency property* of the values of a step function $g$ as follows. By the Scott continuity of $g$ at a point $x$ of the boundary of any number of crescents, it follows that $g(x) \sqsubseteq g(y)$ for $y$ in any of of the neighbouring crescents of $x$. If fact, for any $b \ll g(x)$ there exists a neighbourhood $N$ of $x$ for which $b \ll g(y)$ for $y \in N$ and our claim follows. Since $g(x)$ is itself the value of $g$ at the crescent to which $x$ belongs, it follows that $g(x) = \sqcap_{i \in I} c_i$ where $\{c_i : i \in I\}$ is the set of values of $g$ in the neighbouring crescents of $x$. In particular, if $x$ is on the boundary of precisely two crescents, then the constant values $c_1$ and $c_2$ of the two crescents satisfy $c_1 \sqsubseteq c_2$ or $c_2 \sqsubseteq c_1$.

A path in a connected region $O \subset \mathbb{R}^n$ is a continuous map $p : [0, 1] \to O$ with endpoints $p(0)$ and $p(1)$. We say $p$ is piecewise $C^1$, if $p'$ exists and is continuous except for a finite number of points at which the left and right derivatives of $p$ exist and are limits of $p'$ from left and right respectively. The space $P(O)$ of piecewise $C^1$ paths in the region $O \subset \mathbb{R}^n$ is equipped with the $C^1$ norm:

$$\|p\| = \max\{\max_{r \in [0,1]} \|p(r)\|, \sup_{p'(r) \text{ exists}} \|p'(r)\|\}. \tag{4}$$

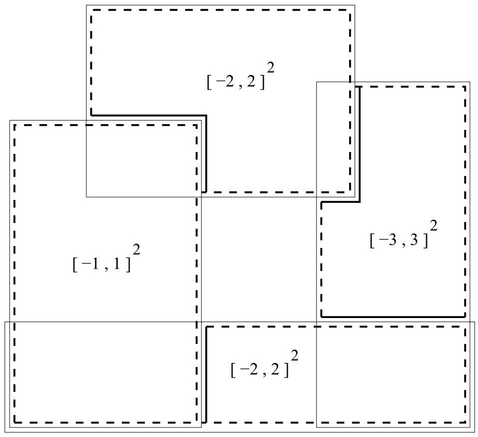


**Figure 3. Crescents of a step function**

A path $p$ is *non-self-intersecting* if $p(r) = p(r')$ for $r < r'$ implies $r = 0$ and $r' = 1$. We will be mainly concerned with the subset $P_0(O) \subset P(O)$ of piecewise linear paths in this paper. For these paths, there exists a strictly increasing finite sequence of points $(r_i)_{0\leq i\leq k}$ for some $k \in \mathbb{N}$ with $0 = r_0 < r_1 < \cdots r_{k-1} < r_k = 1$ such that $p$ is linear in $[r_i, r_{i+1}]$ for $0 \leq i \leq k-1$. The points $p(r_i)$ for $i = 0, \cdots, k$, are said to be the *nodes* of $p$; the nodes $p(r_i)$ for $i = 1, \cdots, k-1$ are called the *inner* nodes. The line segment $\{p(r) : r_i \leq r \leq r_{i+1}\}$ is denoted by $p([r_i, r_{i+1}])$. If $p(0) = p(1)$, the path is said to be *closed*. We also use the notation $p_i := p(r_i)$ for the nodes of the path $p$ in the rest of the paper. A *simple* path in a region $O \subset \mathbb{R}^n$ is a non-self-intersecting piecewise $C^1$ map.

Recall that given a vector field $F : O \to \mathbb{R}^n$ in a region $O \subset \mathbb{R}^n$ and a piecewise $C^1$ path $p : [0, 1] \to O$, the line integral of $F$ with respect to $p$ from 0 to $w \in [0, 1]$ is defined as $\int_0^w F(p(r)) \cdot p'(r)\, dr$, when the integral exists. Here, $u \cdot v = \sum_{i=1}^n u_i v_i$ denotes the usual scalar product of two vectors $u, v \in \mathbb{R}^n$.

Before introducing the interval-valued line integral, we derive some technical properties. For any $A \in \mathbf{C}\mathbb{R}^n$ and $\delta > 0$, we consider the open set $A_\delta = \{x \in \mathbb{R}^n : d(x, A) < \delta\} \subset \mathbb{R}^n$, where $d(x, A)$ is the minimum distance from the point $x$ to $A$.

**Lemma 4.2.** *The map* $- \cdot - : \mathbb{R}^n \times \mathbf{C}\mathbb{R}^n \to \mathbf{I}\mathbb{R}$ *given by* $r \cdot A = \{r \cdot x : x \in A\}$ *is Scott continuous.*

*Proof.* Assume $r \in \mathbb{R}^n$ and $A \in \mathbf{C}\mathbb{R}^n$. Suppose $\epsilon > 0$ is given and consider the open ball $B(r, \delta)$ centered at $r$ and of radius $\delta = \min(1, \epsilon/2(M+1))$ with $M = \sup\{\|r\| : r \in A\}$. Let $B \in \mathbf{C}\mathbb{R}^n$ with $B \subset A_{\epsilon/2(\|r\|+1)}$. For any $x \in B$, take $y \in A$ with $|x - y| \leq \epsilon/2(\|r\| + 1)$. Then, for any $s \in B(r, \delta)$, we have: $s \cdot x = s \cdot x - s \cdot y + s \cdot y - r \cdot y + r \cdot y = s \cdot (x - y) + (s - r) \cdot y + r \cdot y < \epsilon/2 + \epsilon/2 + r \cdot y$ and thus $s \cdot x < r \cdot y + \epsilon \leq (r \cdot A)^+ + \epsilon$ and it follows that $(s \cdot B)^+ < (r \cdot A)^+ + \epsilon$. Similarly, we have: $(s \cdot B)^- > (r \cdot A)^- - \epsilon$. □

**Corollary 4.3.** *For a Scott continuous* $g : U \to \mathbf{C}\mathbb{R}^n$*, where* $U \subset \mathbb{R}^n$ *is an open set, and a piecewise* $C^1$ *path* $p : [0, 1] \to U$*, the map* $t \mapsto g(p(t)) \cdot p'(t) : [0, 1] \to \mathbf{I}\mathbb{R}$ *is Scott continuous.*

**Corollary 4.4.** *For any piecewise* $C^1$ *path* $p \in P(U)$*, the map* $g \mapsto \lambda t.g(p(t)) \cdot p'(t) : (U \to \mathbf{C}\mathbb{R}^n) \to D^0(U)$ *is Scott continuous.*

Note that in Corollaries 4.3 and 4.4 we have $g(p(t)) \cdot p'(t) = \perp$ at the finite number of points $t$ where $p'(t)$ is undefined. We now define the notion of line integral of the compact-convex polyhedron valued Scott continuous function $g \in U \to \mathbf{C}\mathbb{R}^n$ with respect to any piecewise $C^1$ path from $y$ to $x$ in a connected component of $\mathrm{dom}(g)$.

**Definition 4.5.** Given $g \in U \to \mathbf{C}\mathbb{R}^n$ and a piecewise $C^1$ path $p \in P(U)$ with image in the domain of $g$ the *line integral* of $g$ over $p$ is defined as:

$$\int_{p[0,1]} g(r)\,dr = \left[\mathsf{L}\int_{p[0,1]} g(r)\,dr, \mathsf{U}\int_{p[0,1]} g(r)\,dr\right] \tag{5}$$

where the *lower integral* and the *upper integral* of $g$ over $p$ are respectively given by

$$\mathsf{L}\int_{p[0,1]} g(r)\,dr = \int_0^1 (g(p(r)) \cdot p'(r))^-\,dr,$$

$$\mathsf{U}\int_{p[0,1]} g(r)\,dr = \int_0^1 (g(p(r)) \cdot p'(r))^+\,dr.$$

Note that since $\lambda t.(g(p(t)) \cdot p'(t))^+$ and $\lambda t.(g(p(t)) \cdot p'(t))^-$ are, by Corollary 4.3, respectively upper and lower semi-continuous functions, the Lebesgue integrals in the above definition exist. In dealing with line integrals as above, we always assume implicitly that the piecewise $C^1$ path $p$ lies in the domain of the function $g$. We sometimes write $\int_p g(r)\,dr$ for $\int_{p[0,1]} g(r)\,dr$. Furthermore, when the path $p$ from $p(0) = y$ to $p(1) = x$ is clear from the context, we sometimes write

$$\int_{p[0,1]} g(r)\,dr = \int_y^x g(r)\,dr,$$

to emphasise the dependence of the integral on $y$ and $x$ for the given path. In addition, we sometimes write the path $p$ from $y$ to $x$ as $p_y$ or $p^x$ to emphasise its initial or end points.

**Proposition 4.6.** *Given $p \in P(U)$, the map $\int_{p[0,1]} : (U \to \mathbf{C}\mathbb{R}^n) \to \mathbf{I}\mathbb{R}$ is Scott continuous.*

*Proof.* It is easy to see that $\int_{p[0,1]}$ is monotonic. If $g = \sup_{i\geq 0} g_i$ is the supremum of an increasing chain of maps $g_i \in (U \to \mathbf{C}\mathbb{R}^n)$ for $i \geq 0$ and $p$ lies in the domain of $g_i$ for all $i \geq 0$, then by Corollary 4.4 and Lebesgue's monotone convergence theorem it follows that $\sup_{i\geq 0} \mathsf{L}\int_{p[0,1]} g_i(r)\,dr = \mathsf{L}\int_{p[0,1]} g(r)\,dr$ and $\inf_{i\geq 0} \mathsf{U}\int_{p[0,1]} g_i(r)\,dr = \mathsf{U}\int_{p[0,1]} g(r)\,dr$. □

**Remark 4.7.** The interval-valued line integral $\int_{p[0,1]} g(p(t)) \cdot p'(t)\,dt$ for a step function $g \in (U \to \mathbf{C}\mathbb{R}^n)$ and a piecewise linear path $p$ is easy to compute. Consider a straight line segment $p : [a,b] \to C$ with $p(t) = p(a) + (t-a)(p(b) - p(a))/(b-a)$, and thus $p'(t) = (p(b) - p(a))/(b-a) \in \mathbb{R}^n$, for $t \in (a,b) \subset [0,1]$, contained in an associated crescent $C$ of $g$ with value $K \in \mathbf{C}\mathbb{R}^n$, say. We have:

$$\mathsf{L}\int_{p[a,b]} g(t)\,dt = \int_a^b (p'(t) \cdot K)^-\,dt = \frac{(b-a)((p(b)-p(a)) \cdot K)^-}{b-a} = ((p(b)-p(a)) \cdot K)^-.$$

$$\mathsf{U}\int_{p[a,b]} g(t)\,dt = \int_a^b (p'(t) \cdot K)^+\,dt = \frac{(b-a)((p(b)-p(a)) \cdot K)^+}{b-a} = ((p(b)-p(a)) \cdot K)^+.$$

Thus we conclude:

$$\int_{p[a,b]} g(t)\,dt = (p(b)-p(a))\cdot K = \{(p(b)-p(a))\cdot x : x\in K\} = [m,M],$$

with $m$ and $M$ respectively the least and greatest values of $(p(b)-p(a))\cdot x$ for $x\in K$:

$$m = \min_{x\in K}\{(p(b)-p(a))\cdot x\} = (p(b)-p(a))\cdot u \qquad M = \max_{x\in K}\{(p(b)-p(a))\cdot x\} = (p(b)-p(a))\cdot v,$$

where $u$ and $v$ are two points on the boundary of $K$. In fact, $u$ can be taken to be any point of the boundary of $K$ where the hyper-plane with outer normal $p(b)-p(a)$ touches $K$ whereas $v$ can be taken to be any point of the boundary of $K$ where the hyper-plane with outer normal $-(p(b)-p(a))$ touches $K$. In the case that $K$ is a compact convex polyhedron we have $p(b)-p(a)\in -P_u$ and $p(b)-p(a)\in P_v$ as in Proposition 3.3.

A simple property of the lower and upper integrals is given in the following.

**Proposition 4.8.** *Given a single-step function $g = b\chi_a \in U \to \mathbf{C}\mathbb{R}^n$ and a straight line segment $p\in P_0(a)$ from the fixed point $p(0)=y$ to the point $p(1)=x$, the L-derivatives with respect to $x$ of the lower and upper integrals satisfy:*

$$g(x)\sqsubseteq \mathcal{L}\mathsf{L}(\int_y^x g(r)\,dr), \qquad g(x)\sqsubseteq \mathcal{L}\mathsf{U}(\int_y^x g(r)\,dr)$$

*Proof.* We shall prove the property for the upper integral as the case for the lower integral is entirely similar. We can assume $b$ is a convex, compact polyhedron as the general case follows from this by Proposition 4.6 since any convex, compact subset is the intersection of a shrinking sequence of convex, compact polyhedra. Then, for fixed $y$, we have:

$$\begin{array}{rcl} f(x) & := & \mathsf{U}(\int_y^x g(r)\,dr) \\ & = & ((x-y)\cdot b)^+ \\ & = & (x-y)\cdot v \end{array}$$

where $(x-y)\in P_v$, by Proposition 3.3, for some vertex $v\in b$. Assume now that $x-y$ is not perpendicular to any of the faces of $b$. Then, for $u$ in a small neighbourhood of $x$ we have: $f'(u)=v$. It follows from Equation (1) and the equivalence of the L-derivative and the Clarke gradient in finite dimensional Euclidean spaces (Theorem 2.9), that $\mathcal{L}f(x)=\{v\}\sqsupseteq b$. By Scott continuity it follows that $\mathcal{L}f(x)\sqsupseteq b$ for any $x\in a$. □

## 5. Fundamental Theorem of Path Integration

Recall the fundamental theorem of path integration, a classical result in multi-variable differential calculus:

**Theorem 5.1.** *[22, 2] Let $O$ be an open, connected set in $\mathbb{R}^n$, and let $G : O\to\mathbb{R}^n$ be a continuous vector field. Then the following two conditions are equivalent.*

- *There is a differentiable function $F : O\to\mathbb{R}$ such that $G=F'$.*

- *The line integrals of $G$ over closed, piecewise $C^1$ curves in $O$ are zero.*

We note that one can replace $C^1$ curves with piecewise linear curves in the above theorem, since every $C^1$ curve can be obtained as the $C^1$ limit of a sequence of piecewise linear curves. In this section, we will develop a domain-theoretic extension of the fundamental theorem of path integration, but first we will point out a related result in this area.

Borwein et al. [6] give a necessary and sufficient condition for a Scott continuous function $g : O \to \mathbf{C}\mathbb{R}^n$ where $O \subset \mathbb{R}^n$ is a non-empty open connected subset to be integrable (see also [5, Theorem 8]). We will now explain their condition.

The *line integral* of a measurable map $f : O \to \mathbb{R}^n$ on the line segment $[a, b] \subset O$ is given by the Lebesgue integral:

$$\int_{[a,b]} f(z) \cdot dz := \int_0^1 f(tb + (1-t)a) \cdot (b - a)\, dt.$$

The line integral of $f$ on a piecewise linear path $P$ in $U$ is simply the sum of its line integrals on the line segments of $P$. For any fixed $\epsilon > 0$, an ordered collection of line segments $P(\epsilon) = \{[a_i, b_i] : 1 \leq i \leq n\}$ is an $\epsilon$-path from $a$ to $b$ provided:

$$\|a - a_1\| + \sum_{i=1}^{n-1} \|a_{i+1} - b_i\| + \|b_n - b\| < \epsilon.$$

Such a path is closed if $a = b$. For a Borel subset $E \subset O$, an $\epsilon$-path $P(\epsilon)$ is an $E$-admissible $\epsilon$-path from $a$ to $b$ if $\lambda(\{t \in [0, 1] : tb_i + (1-t)a_i \notin E\}) = 0$ for $1 \leq i \leq n-1$, where $\lambda$ is the Lebesgue measure. Line integrals on an $\epsilon$-path are defined similarly as above.

**Theorem 5.2.** *[6, Theorem 8] Let $U$ be a non-empty open connected subset of $\mathbb{R}^n$ and let $g : U \to \mathbf{C}\mathbb{R}^n$ be a bounded Scott continuous map. Then $g$ is integrable if and only if there exists a Borel set $E \subset U$ with $\lambda(U \setminus E) = 0$ and a measurable selection $f : E \to \mathbb{R}^n$ of $g$ so that $\lim_{\epsilon \to 0^+} \int_{P(\epsilon)} f(z)\, dz = 0$, where $P(\epsilon)$ is any closed $E$-admissible $\epsilon$-path in $U$.*

The existence of a measurable selection as above is in general non-decidable. We have already shown in Corollary 3.9 that for a rational step function $g \in (U \to \mathbf{I}\mathbb{R}^n)$ integrability is a decidable predicate. In this section, we will derive an alternative necessary and sufficient condition for the integrability of a Scott continuous function $g : U \to \mathbf{I}\mathbb{R}^n$, which is the domain-theoretic counterpart for the classical condition that the line integral over closed piecewise $C^1$ paths be zero.

We now introduce the main concept in the domain-theoretic generalization of the fundamental theorem of path integration for the integrability of a vector field.

**Definition 5.3.** Given $g \in (U \to \mathbf{C}\mathbb{R}^n)$ and a closed simple path $p$ in a connected component of dom$(g)$, we say that $g$ satisfies the *zero-containment loop condition for* $p$ if

$$0 \in \int_{p[0,1]} g(r)\, dr.$$

We say that $g \in (U \to \mathbf{C}\mathbb{R}^n)$ satisfies the *zero-containment loop condition* if it satisfies the zero-containment loop condition for any closed simple path $p$ in any connected component of dom$(g)$.

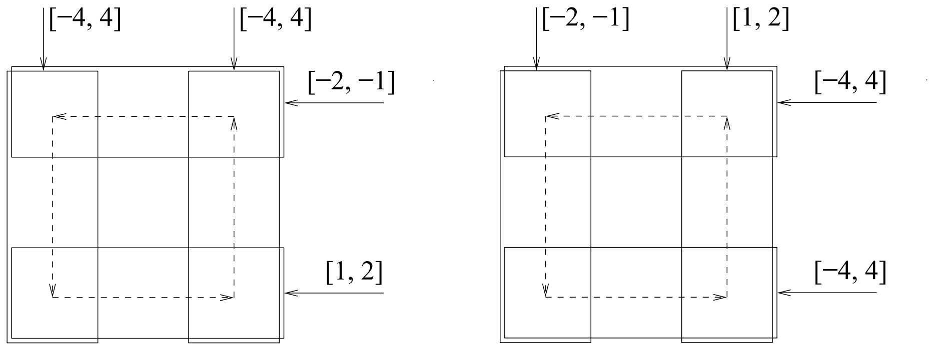


**Figure 4. Failure of zero-containment loop condition: $g_1$ (left) and $g_2$ (right)**

We note that if $g$ is a step function then in the zero-containment loop condition above, it suffices to consider piecewise linear closed simple paths. In fact, it can be easily shown that the integral of a step function with respect to a piecewise $C^1$ path is the limit of integrals of the step function over a sequence of piecewise linear paths that converge in the $C^1$ norm (Equation (4)) to the piecewise $C^1$ path.

If $g$ only takes point (maximal) values, then the zero-containment loop condition is simply the standard condition for $g$ to be a gradient i.e., the line integral of $g$ vanishes on any closed path. Figure 4 gives an example of rectangular valued step function $g = (g_1, g_2) \in U \to \mathbf{I}\mathbb{R}^2$, with $\mathrm{dom}(g) = ((0,3) \times (0,3)) \setminus ([1,2] \times [1,2])$ which does not satisfy the zero-containment loop condition. The values of $g_1$ (left) and $g_2$ (right) are given for each of the four single-step functions. Denote the dashed path by $p$; it has nodes at $p(0) = p(1) = (1/2, 1/2)$, $p(1/4) = (5/2, 1/2)$, $p(1/2) = (5/2, 5/2)$ and $p(3/4) = (1/2, 5/2)$. A simple calculation shows that

$$p'(r) = \begin{cases} (8,0) & 0 < r < 1/4 \\ (0,8) & 1/4 < r < 1/2 \\ (-8,0) & 1/2 < r < 1/4 \\ (0,-8) & 3/4 < r < 1 \end{cases}$$

Thus, evaluating the line integral for the four parts of the path, we obtain:

$$\mathsf{L}\int_{p[0,\frac{1}{4}]} g(r)dr = \mathsf{L}\int_{p[\frac{1}{4},\frac{1}{2}]} g(r)dr = \mathsf{L}\int_{p[\frac{1}{2},\frac{3}{4}]} g(r)dr = \mathsf{L}\int_{p[\frac{3}{4},1]} g(r)dr = 2$$

Summing the contributions from the four parts above, the lower line integral of $g$ over $p$ then gives a strictly positive value:

$$\mathsf{L}\int_{p[0,1]} g(r)dr = 8.$$

In the proof of the following theorem we invoke the mean value theorem for the Clarke gradient due to Lebourg, which we state here for finite dimensional Euclidean spaces where the Clarke gradient and the L-derivative coincide:

**Theorem 5.4.** *[8, Theorem 2.37] Let $x, y \in \mathbb{R}^n$ and assume the line segment between them is in the domain of a real-valued Lipschitz function $h$. Then there exists a point $u$ in the open interval between $x$ and $y$ such that $h(y) - h(x) \in (\mathcal{L}h(u)) \cdot (y - x)$.*

**Theorem 5.5.** *Suppose $g \sqsubseteq \mathcal{L}h$ for a Scott continuous function $g \in (U \to \mathbf{C}\mathbb{R}^n)$ and a Lipschitz map $h : \mathrm{dom}(g) \to \mathbb{R}$. Then for any piecewise $C^1$ path $p$ from $y$ to $x$ in a connected component of dom$(g)$ we have $h(x) - h(y) \in \int_{p[0,1]} g(r)\, dr$.*

*Proof.* By Proposition 4.6, we only need to prove the result for a step function $g$, in which case we can also take the path $p$ to be piecewise linear. By introducing additional inner nodes if required, we can also assume that the interior of each line segment $p([r_{i-1}, r_i])$ of $p$, with $1 \leq i \leq k-1$, lies in a crescent of $g$ with a constant value $K_i$ say. By Lebourg's mean value property (Theorem 5.4), there exists $u_i$ in the interior of $p([r_{i-1}, r_i])$ such that

$$h(p(r_i)) - h(p(r_{i-1})) \in (\mathcal{L}h(u_i))\cdot(p(r_i) - p(r_{i-1})) \sqsupseteq g(u_i)\cdot(p(r_i) - p(r_{i-1})) = K_i\cdot(p(r_i) - p(r_{i-1})) = \int_{r_{i-1}}^{r_i} g(r)\, dr.$$

Adding the above relations for $i = 1, \cdots, k$, we obtain the result. □

Recall that $g \in (U \to \mathbf{C}\mathbb{R}^n)$ is called integrable if $\int g \neq \emptyset$. The following corollary is an extension of Green's Theorem also called the Gradient Theorem in classical differential calculus [22, 2].

**Corollary 5.6.** *Suppose $g \in (U \to \mathbf{C}\mathbb{R}^n)$ is an integrable function. Then $g$ satisfies the zero-containment loop condition.*

*Proof.* Assume $h \in \int g$ and thus, by Corollary 2.8,

$$g \sqsubseteq \mathcal{L}h. \tag{6}$$

Take any closed piecewise linear path $p$ in a connected component $O$ of dom$(g)$ and put $x = y = p(0)$ in Theorem 5.5. □

We are now ready to introduce a key concept of this paper. Note that any step function $g$ can be extended to the boundary of dom$(g)$ by the lower and upper semi continuity of $g_i^-$ and $g_i^+$ respectively for $1 \leq i \leq n$. For a step function $g \in (U \to \mathbf{C}\mathbb{R}^n)$ and $x, y \in$ cl$(O)$, where $y$ is regarded as the parameter and $x$ as the variable, we put

$$V_g(x, y) = \sup\{\mathsf{L}\int_{p[0,1]} g(r)\, dr : p \text{ a piecewise linear path in cl}(O) \text{ from } y \text{ to } x\},$$

$$W_g(x, y) = \inf\{\mathsf{U}\int_{p[0,1]} g(r)\, dr : p \text{ a piecewise linear path in cl}(O) \text{ from } y \text{ to } x\}.$$

**Corollary 5.7.** *Suppose $g \sqsubseteq \mathcal{L}h$ for a step function $g \in (U \to \mathbf{C}\mathbb{R}^n)$ and a Lipschitz map $h : \mathrm{dom}(g) \to \mathbb{R}$. Then for any piecewise linear path from $y$ to $x$ in a connected component of $g$ we have:*

$$V_g(x, y) \leq h(x) - h(y) \leq W_g(x, y).$$

*Proof.* This follows immediately from Theorem 5.5 by taking the supremum, respectively the infimum, of the values of the lower integrals, respectively the upper integrals, over all piecewise linear paths from $y$ to $x$. □

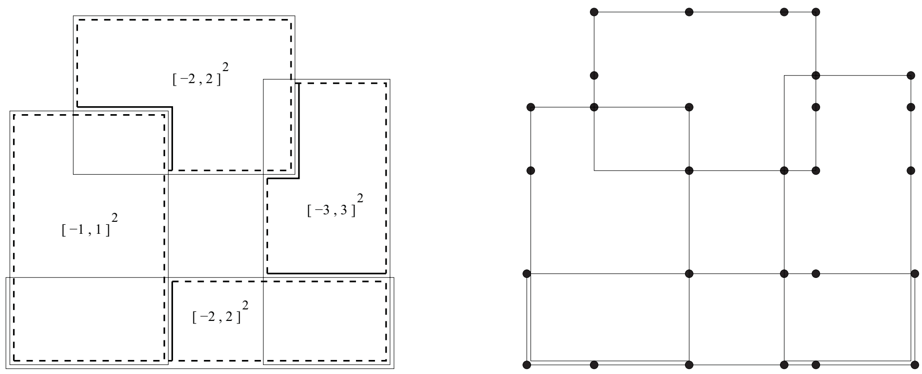


**Figure 5. Crescents of a step function (left); the corners and their coaxial points (right)**

From now on, i.e., for the rest of this paper, we will restrict ourselves to hyper-rectangular valued derivative information. We will show that if

$$g = (g_1, \ldots, g_n) \in (U \to \mathbf{I}\mathbb{R}^n)$$

satisfies the zero-containment loop condition, then it is integrable. Let $O$ be a connected component of dom$(g)$ for the rest of this paper. We adopt the following convention.

**Remark 5.8.** If two crescents have a common boundary, we consider their common boundary as infinitesimally separated so that they have distinct boundaries. This means that a line segment of a piecewise linear simple path on a common boundary of two different crescents is always regarded as the limit of a sequence of parallel segments contained on one side of this boundary. In the two dimensional case $n = 2$, this means that the same edge on the boundary of two crescents can represent two different line segments, i.e., one in each crescent.

**Definition 5.9.** Let $O$ be a connected component of the domain of a step function $g \in (U \to \mathbf{I}\mathbb{R}^n)$. We say that any orthogonal projection of a point $x \in$ cl$(O)$ to any $m$-dimensional boundary of a crescent in $O$ (for $1 \leq m \leq n-1$) is a *coaxial point* of $x$.

Clearly, each point has a finite number of coaxial points. In Figure 5, the coaxial points of the corners of the crescents of the step function in Figure 3, reproduced on the left, are illustrated on the picture on the right. We collect a few technical results before we are able to prove the equivalence of integrability and the zero-containment loop condition. Our first challenge is to show that there are actually paths which respectively attain the lower and the upper integrals of $g$ from $y$ to $x$.

We consider $O$ as the disjoint union of a set of crescents generated by open hyper-rectangles in $\mathbb{R}^n$. In this section, we take this set of crescents precisely as the crescents on which $g$ has constant value. (In the next section we will refine these crescents with those of dom$(f)$.) Let $E_O$ denote the set of vertices of the crescents of $O$. Given a finite set of points $x_1, \ldots, x_t$, we denote by $H_O(x_1, \ldots, x_t)$, the coarsest hyper-rectangular partition that includes as vertices the points in $E_O \cup \{x_1, \ldots, x_t\}$ and all their coaxial points. Let $E_O(x_1, \ldots, x_t)$ denote the set of all corners of the hyper-rectangles in $H_O(x_1, \ldots, x_t)$. Denote the hyper-rectangles in $H_O(x, y)$ by $R_i$ where $i \in I$ for some finite indexing set $I$. Suppose that $K_i \in \mathbf{I}\mathbb{R}^n$ is the (constant) value of $g$ on $R_i$ for $i \in I$.

Next, consider a vector $z \in \mathbb{R}^n$. Then for each coordinate index $k = 1, \cdots, n$ we have $z_k \gtrless 0$ where $\gtrless$ stands for $\geq$ or $\leq$. This gives at most $2^n$ possible *direction types* for $z$, each of which we can represent by the finite sequence $s = s_1 s_2 \cdots s_n$ where $s_k \in \{-, +\}$ according to the sign of $z_k$. Note that by our formulation, for $z_k = 0$ we have both $s_k = +$ and $s_k = -$. Also observe that if $z$ is direction type $s$ then $-z$ has the direction type $-s$ where $(-s)_k = -s_k$ with the usual multiplication of signs $\pm$.

If $K \in \mathbf{I}\mathbb{R}^n$, we say a vertex $v$ of $K$ has type $s \in \{+, -\}^n$ if $z := v - y$ has type $s$ for some $y \in K$, and we write $v = K(s)$. If the interior of $K$ is non-empty then each vertex of $K$ will have a unique *corner type*. It is now easy to see the following simple proposition which gives, in the case of rectangular derivative, the complete description of the more general results in Proposition 3.3.

**Proposition 5.10.** *If $z$ has direction type $s$ then the two vertices, $K(s)$ and $K(-s)$, of $K$ of corner types $s$ and $-s$ respectively satisfy: $(K \cdot z)^+ = K(s) \cdot z$ and $(K \cdot z)^- = K(-s) \cdot z$.*

The first lemma shows that if $g$ satisfies the zero-containment loop condition, then, given $x, y \in \mathrm{cl}(O)$, the supremum of lower path integrals from $x$ to $y$ is always attained for a simple piecewise linear path whose nodes are in the set $E_O(x, y)$.

**Lemma 5.11.** *Suppose the step function $g$ satisfies the zero-containment loop condition. If $x, y \in \mathrm{cl}(O)$, then*

$$\sup\{\mathsf{L}\int_q g(r)dr \mid q \textit{ piecewise linear path in } O \textit{ from } x \textit{ to } y\}$$

*is attained for a simple piecewise linear path with nodes in the set $E_O(x, y)$. The dual property holds for the infimum of upper integrals.*

*Proof.* We show that every piecewise linear path $p$ with nodes

$$p_0, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_l$$

from $x = p_0$ to $y = p_l$ can be modified to a piecewise linear path $q$, also from $x$ to $y$ such that

(i) $q$ is simple (i.e., non-intersecting);

(ii) the nodes of $q$ are in $E_O(x, y)$;

(iii) $\mathsf{L}\int_q g(r)dr \geq \mathsf{L}\int_p g(r)dr$.

First note that by the zero-containment loop condition, if we remove any loop from $p$ then the lower integral will increase, i.e., condition (iii) will be satisfied. We can therefore remove all loops from $p$, which means that we can assume our path is simple, i.e., (i) is established. Next, suppose that $p$ is a simple piecewise linear path in $O$ with nodes $x = p_0, \ldots, p_l = y$. We may assume without loss of generality that each open line segment $(p_{j-1}, p_j)$ lies within a single hyper-rectangle $R_{i_j} \subseteq O$ of $H_O(x, y)$; note that $R_{i_j}$ for $j = 1, \ldots, l$ are not necessarily distinct. By Remark 4.7, we have:

$$\mathsf{L}\int_p g(r)dr = \sum_{j=1}^{l} (K_{i_j} \cdot (p_j - p_{j-1}))^- = \sum_{j=1}^{l} \min_{x \in K_{i_j}} x \cdot (p_j - p_{j-1}). \qquad (7)$$

We consider two steps to ensure that condition (ii) is met. First we remove all inner nodes that are in the interior of the hyper-rectangles $R_i$ for $i \in I$. Suppose that for some $j$ with $1 \leq j \leq l-1$, the node

$p_j$ is in the interior of the hyper-rectangle $R_{i_j}$. By our assumption above, it follows that $(p_{j-1}, p_j)$ and $(p_j, p_{j+1})$ are both contained in $R_{i_j}$. Since for any two vectors $a, b \in \mathbb{R}^n$ we have $(K_{i_j} \cdot a) + (K_{i_j} \cdot b) \supset K_{i_j} \cdot (a+b)$, it follows that $(K_{i_j} \cdot (p_j - p_{j-1})) + (K_{i_j} \cdot (p_{j+1} - p_j)) \supset K_{i_j} \cdot (p_{j+1} - p_{j-1})$, which implies that $(K_{i_j} \cdot (p_{j+1} - p_{j-1}))^- + (K_{i_j} \cdot (p_{j+1} - p_j)))^- \leq (K_{i_j} \cdot (p_{j+1} - p_{j-1}))^-$. Thus, by removing the node $p_j$ from our simple piecewise linear path $p$, we obtain a new simple piecewise linear path $q$ with consecutive nodes $x = p(0) = p_0, \ldots, p_{j-1}, p_{j+1}, \ldots, p_l = p(1) = y$ with $\mathsf{L}\int_q g(r)dr \geq \mathsf{L}\int_p g(r)dr$. By repeatedly removing inner nodes that are in the interior of the hyper-rectangles $R_i$ for $i \in I$, we can assume that for any simple piecewise linear path $p$ there is a simple piecewise linear path $q$ that satisfies (iii) and has its inner nodes at the boundaries of the hyper-rectangles of $H_O(x,y)$.

Our final step is then to show that a simple piecewise linear path whose nodes are at the boundaries of $R_i$'s can be replaced with one whose nodes are all in $E_O(x,y)$, i.e., at the vertices of $R_i$ for $i \in I$. Assume therefore that $p$ has its nodes $p_j$ for $1 \leq j \leq l$ on the boundaries of the hyper-rectangles $R_i$. For any node $p_j$ on a co-dimension one boundary hyper-plane of some $R_i$, the coordinates $p_{jk}$ with $1 \leq k \leq n$ have maximum and minimum values as determined by the given boundary hyper-plane of $R_i$. Consider the segment of $p$ from $p_{j-1}$ to $p_j$ which by our assumption lies in $R_{i_j}$. Let $s_j \in \{-,+\}^n$ be the direction type of $p_j - p_{j-1}$. Then, by Proposition 5.10, we have:

$$\min_{x \in K_{i_j}} x \cdot (p_j - p_{j-1}) = K_{i_j}(-s_j) \cdot (p_j - p_{j-1}).$$

Thus, from Equation (7), we can write the lower integral for path $p$ as:

$$\mathsf{L}\int_p g(r)dr =$$

$$\sum_{j=1}^{l} \min_{x \in K_{i_j}} x \cdot (p_j - p_{j-1}) = \sum_{j=1}^{l} K_{i_j}(-s_j) \cdot (p_j - p_{j-1}) =$$

$$-K_{i_1}(-s_1) \cdot p_0 + K_{i_l}(-s_l) \cdot p_l + \sum_{j=1}^{l-1} (K_{i_j}(-s_j) - K_{i_{j+1}}(-s_{j+1})) \cdot p_j.$$

We consider the lower integral $\mathsf{L}\int_p g(r)dr$ over the path $p$ above as the objective function of a linear programming problem, with variables $p_j$ $(j = 1, \ldots l-1)$ and constraints given by the position of each $p_j$ on a given boundary of $R_{i_j}$ together with the minimum and maximum values this boundary imposes on the coordinates of $p_j$. We note that the vertices of the convex polytope that represents the domain of the objective function are given by the points in $E_O(x,y)$. In fact, the conditions $p_{(j-1)k} - p_{jk} \gtrless 0$ do not give rise to new extremal points of the domain of the objective function. For, if at least one of the coordinates $p_{(j-1)k}$ and $p_{jk}$, for some $k = 1, \cdots, n$, is constant for the two boundaries of $R_{i_j}$ on which $p_{(j-1)k}$ and $p_{jk}$ lie, then one or the other of the inequalities $p_{(j-1)k} - p_{jk} \gtrless 0$ is satisfied, otherwise if both $p_{(j-1)k}$ and $p_{jk}$ are non-constant for the two boundaries, then $p_{(j-1)k}$ and $p_{jk}$ will have the same range of values $[m, M]$ say and the two inequalities $p_{(j-1)k} - p_{jk} \gtrless 0$ would correspond to the two right angled triangles with vertices $(m,m), (m,M), (M,M)$ and $(m,m), (M,m), (M,M)$, which correspond to the $k$th coordinates of two corners of $R_i$. Thus, for a given set of values $s_j \in \{-,+\}^n$ with $1 \leq j \leq l$ for $l$ sequences of $\pm$, each of length $n$, the supremum of the lower integral is attained, by basic linear programming, for a value of each $p_j$ at a vertex of the hyper-rectangle $R_{i_j}$ for $1 \leq j \leq l$. Since there are a finite number (in fact $2^{nl}$) of such sets of sequences of length $n$, it follows that the supremum of the lower integral is attained for a path going through the vertices of the hyper-rectangles in $H_O(x, y)$, which completes the proof.

□

We note that the proof of Lemma 5.11 crucially uses the fact that the set of coaxial points of a given point is finite and is closed under taking coaxial points. This clearly holds under the assumption we have in this section, i.e., the values of the derivative information $g$ are hyper-rectangles with faces parallel to the coordinate planes. In the case that $g$ takes values as non-empty compact and convex rational polyhedra, we do not know if in general a given point gives rise to a finite set closed under taking coaxial points.

**Corollary 5.12.** *Suppose the path $p \in P_0(U)$ from $y$ to $x$ satisfies $V_g(x, y) = \mathsf{L} \int_{p[0,1]} g(r)\, dr$. Then for any point $z = p(r_0)$ with $r_0 \in [0, 1]$ we have*

$$V_g(z, y) = \mathsf{L} \int_{p[0,r_0]} g(r)\, dr, \qquad V_g(x, z) = \mathsf{L} \int_{p[r_0,1]} g(r)\, dr, \qquad V_g(x, y) = V_g(x, z) + V_g(z, y) \quad (8)$$

*A similar result holds for $W_g$.*

*Proof.* Suppose the first equality in Equation 8 does not hold. Then if $q$ is the path from $y$ to $z$ that attains the lower integral of $g$, we can concatenate $q$ with the second segment of $p$, i.e., from $r = r_0$ to $r = 1$, to get a new path from $y$ to $x$ which has a greater value for its lower integral than $p : [0, 1] \to U$. This gives a contradiction and hence the first equality holds. Similarly the second equality holds. The third is simply the additivity of the line integral. □

**Corollary 5.13.** *Let $O$ be a connected component of step function $g \in (U \to \mathbf{I}\mathbb{R}^n)$. If $g \sqsubseteq g_0$, then for any $x, y \in cl(O)$ we have $V_g(x, y) \le V_{g_0}(x, y) \le W_{g_0}(x, y) \le W_g(x, y)$.*

*Proof.* Suppose $p$ attains $\mathsf{L} \int_{p[0,1]} g\, dr = V_g(x, y)$. By the definition of $V_{g_0}(x, y)$ and monotonicity of path integration (Proposition 4.4), we obtain: $V_g(x, y) \le V_{g_0}(x, y)$. Similarly for the upper integrals. □

We now need a technical lemma to proceed. Fix $y \in cl(O)$ and let $x$ and $z$ be in the interior $R^\circ$ of the same hyper-rectangle $R$ of $H_O(y)$ (which is a coarser partition than $H_O(x, y)$). Consider any path $p^x$ from $y$ to $x$ with $l + 1$ nodes $y = p^x_0, p^x_1, \ldots, p^x_l = x$ in $E_O(x, y)$. Then, there exists a path $p^z$ from $y$ to $z$ also with $l + 1$ nodes $y = p^z_0, p^z_1, \ldots, p^z_l = z$ in $E_O(z, y)$ such that

- for each $i = 1, \ldots, l - 1$ either $p^x_i = p^z_i \in E_O(y)$, or $p^x_i$ and $p^z_i$ are coaxial points of $x$ and $z$ respectively that lie on the same face of a hyper-rectangle in $H_O(y)$, and
- for $0 \le i \le l$ the two line segments $p^x_i, p^x_{i+1}$ and $p^z_i, p^z_{i+1}$ belong to the same hyper-rectangle in $H_O(y)$.

The second condition ensures that in case $p^x_i, p^x_{i+1}$ and $p^z_i, p^z_{i+1}$ are both on the boundary of some crescents, then they would be considered infinitesimally contained in the same crescent of $g$. We say that the two paths $p^x$ and $p^z$ have the same *type* for $x, z \in R^\circ$.

**Lemma 5.14.** *For a step function $g$, the line integral $\int_{p^x} g(r)\, dr$ depends linearly on the coordinates of $x \in R^\circ$ when the type of $p^x$ is unchanged.*

*Proof.* Suppose the two paths $p^x$ and $p^z$, with $x, z \in R^\circ$, have the same type each having $l+1$ nodes. Then their $j$th segments from $p_{j-1}$ to $p_j$ (for each $j = 1, \ldots, l$) lie in the same hyperrectangle in $H_O(y)$ of constant $g$ value $K_{i_j}$ say. We have,

$$\mathsf{L}\int_{p^x} g(r)dr = \sum_{j=1}^{l}(K_{i_j}\cdot(p_j^x - p_{j-1}^x))^- = \sum_{j=1}^{l}\min_{u\in K_{i_j}} u\cdot(p_j^x - p_{j-1}^x).$$

$$\mathsf{L}\int_{p^z} g(r)dr = \sum_{j=1}^{l}(K_{i_j}\cdot(p_j^z - p_{j-1}^z))^- = \sum_{j=1}^{l}\min_{u\in K_{i_j}} u\cdot(p_j^z - p_{j-1}^z).$$

Since $p_j^x - p_{j-1}^x$ and $p_j^z - p_{j-1}^z$ belong to the same orthant in $\mathbb{R}^n$, we obtain:

$$\mathsf{L}\int_{p^x} g(r)dr = \sum_{j=1}^{l} u_j\cdot(p_j^x - p_{j-1}^x).$$

$$\mathsf{L}\int_{p^z} g(r)dr = \sum_{j=1}^{l} u_j\cdot(p_j^z - p_{j-1}^z),$$

where $-(p_j^x - p_{j-1}^x), -(p_j^z - p_{j-1}^z) \in P_{u_j}$, and thus,

$$\mathsf{L}\int_{p^z} g(r)dr - \mathsf{L}\int_{p^x} g(r)dr = (z-x)\cdot u_l + \sum_{j=1}^{l-1}(u_j - u_{j+1})\cdot(p_j^z - p_j^x). \tag{9}$$

Since the two paths are of the same type, for $j = 1, \ldots, l-1$ either $p_j^x = p_j^z \in E_O(y)$, thus making no contribution to Equation (9) or $p_i^x$ and $p_i^z$ are coaxial points of $x$ and $z$ respectively, which depend linearly on $n-1$ coordinates of $x$ and $z$ respectively. It follows that $\int_{p^x} g\,dr$ depends linearly on $x$ as this point varies in $R^\circ$ while the type of the path is unchanged. □

For a given step function $g \in (U \to \mathbf{I}\mathbb{R}^n)$ with a connected component $O$, the maps $\lambda p \in P_0(O).\mathsf{L}\int_p g(r)\,dr$ and $\lambda p \in P_0(O).\mathsf{U}\int_p g(r)\,dr$ are not in general continuous with respect respect to the $C^1$ norm on $P_0(O)$, as it is easy to verify by considering a path consisting of a single line segment that coincides with the boundary of two crescents with different values. However, the maps $V_g(\cdot, y), W_g(\cdot, y)$ giving the lower and upper integrals from the point $y$ respectively are continuous:

**Proposition 5.15.** *If the step function $g$ satisfies the zero-containment loop condition, then, for all $y \in \mathrm{cl}(O)$, the two maps given by $V_g(\cdot, y), W_g(\cdot, y)$ : cl$(O) \to \mathbb{R}$ are continuous, piecewise linear and satisfy $V_g(y, y) = W_g(y, y) = 0$.*

*Proof.* Fix $y \in \mathrm{cl}(O)$, and consider a path $p^x$ from $y$ to $x$ as this point varies in the interior of a hyperrectangle $R$ of $H_O(y)$, while the type of $p^x$ remains unchanged. By Lemma 5.14, the integral $\mathsf{L}\int_{p^x} g(r)\,dr$ depends linearly on the coordinates of $x \in R^\circ$. Thus, we obtain for $x \in R^\circ$ a finite set of paths $p^x$ with $p^x(0) = y$ and $p^x(1) = x$, of all possible types, such that the mapping $\lambda x.\mathsf{L}\int_{p^x} g(r)\,dr$ is a continuous piecewise linear surface. Therefore the map $V_g(\cdot, y)$, which is locally the maximum of a finite number of continuous piecewise linear local surfaces is itself a continuous piecewise linear surface.

In order to show that $V_g(y, y) = 0$, we note that the trivial constant path $p$ with constant value $y$ is a piecewise linear simple path from $y$ to $y$ with $\mathsf{L}\int_{p[0,1]} g(r)\, dr = 0$. By the zero-containment loop condition, any other closed piecewise linear simple path $q$ from $y$ to $y$ satisfies $\mathsf{L}\int_q g(r)\, dr \leq 0$ and thus $V_g(y, y) = \mathsf{L}\int_{p[0,1]} g(r)\, dr = 0$. The statement for the upper line integral is entirely dual. □

The importance of the maps $V_g(\cdot, y)$ and $W_g(\cdot, y)$, for a fixed $y \in \mathrm{cl}(O)$, lies in the fact that their derivatives provide a refinement of $g$.

**Proposition 5.16.** *If the step function $g$ satisfies the zero-containment loop condition, then the maps $V_g(\cdot, y)$ and $W_g(\cdot, y)$ satisfy*

$$g(x) \sqsubseteq \mathcal{L}V_g(x, y) \qquad \textit{and} \qquad g(x) \sqsubseteq \mathcal{L}W_g(x, y)$$

*for all $y \in \mathrm{cl}(O)$.*

*Proof.* We prove that $g \sqsubseteq \mathcal{L}V_g(\cdot, y)$, as the statement concerning the upper line integral is entirely analogous. We can assume that $x$ is in the interior of a crescent $C$ of $g$ with constant value $K$, since the general case then follows by Scott continuity. Thus, there exists a small $a > 0$ such that the closed $n$-dimensional open ball $\mathsf{B}_a(x)$ of radius $a$ and center $x$ lies in the interior $C^\circ$. Denote the boundary of this disk by $\mathsf{S}_a(x)$. For any $z \in \mathsf{S}_a(x)$ and $u \in \mathsf{B}_a(x)$, we have: $V_g(u, y) \geq V_g(z, y) + V_g(u, z)$ since any pair of piecewise linear paths $p_1$ (from $y$ to $z$) and $p_2$ (from $z$ to $u$) gives rise, by concatenation, to a piecewise linear path ($p_1$ followed by $p_2$) from $y$ to $u$. Moreover,

$$V_g(u, y) = \sup\{V_g(z, y) + V_g(u, z) : z \in \mathsf{S}_a(x)\}, \tag{10}$$

since $V_g(u, y)$ is the maximum value of the lower path integral over all piecewise linear paths from $y$ to $u$ and any path $p_0$ as such will intersect $\mathsf{S}_a(x)$ at some point $z$ and thus gives rise to a piecewise linear path $p_1$ from $y$ to $z$ and a piecewise linear path $p_2$ from $z$ to $u$. Now for fixed $y$ and fixed $z \in \mathsf{S}_a(x)$, the map $\lambda u.V_g(z, y) + V_g(u, z)$, by Proposition 4.8, satisfies

$$g \sqsubseteq \mathcal{L}(\lambda u.V_g(z, y) + V_g(u, z))$$

since $V_g(z, y)$ is a constant and $u \in \mathsf{B}_a(x)$. Thus by Proposition 2.10 applied to the family of maps on the right hand side of Equation (10), we have $g(u) \sqsubseteq \lambda u \in \mathsf{B}_a(x).\mathcal{L}V_g(u, y)$. Since $u \in \mathsf{B}_a(x)$ is an arbitrary point, by letting $u = x$ we obtain $g \sqsubseteq \mathcal{L}V_g(\cdot, y)$ and the proof is complete. □

**Corollary 5.17.** *A step function $g \in (U \to \mathbf{I}\mathbb{R}^n)$ is integrable iff it satisfies the zero-containment loop condition.*

*Proof.* Corollary 5.6 gives the left to right implication. In the other direction, if $g$ satisfies the zero-containment loop condition, then Propositions 5.15 and 5.16 imply that in any connected component $O \subset \mathrm{dom}(g)$ the maps $V_g(\cdot, y)$ and $W_g(\cdot, y)$, for any $y \in O$, are both witness to the integrability of $g$. □

**Corollary 5.18.** *The zero-containment loop condition is decidable for a basis element of $(U \to \mathbf{I}\mathbb{R}^n)$.*

*Proof.* This follows from Corollary 3.11 and Corollary 5.17. □

We can also extend Corollary 5.17 to the following result, which can be regarded as the extension of the fundamental theorem of path integration to Scott continuous rectangular-valued vector fields.

**Theorem 5.19.** *A function $g \in (U \to \mathbf{I}\mathbb{R}^n)$ is integrable iff it satisfies the zero-containment loop condition.*

*Proof.* The left to right direction was shown in Corollary 5.6. Suppose that $g$ satisfies the zero-containment loop condition. Since $U \to \mathbf{I}\mathbb{R}^n$ is a continuous Scott domain, we have $g = \sup_{i\geq 0} g_i$ where $(g_i)_{i\geq 0}$ is an increasing sequence of step functions. Then by the monotonicity of line integration each $g_i$ also satisfies the zero-containment loop condition and is thus integrable by Corollary 5.17. Since $(g_i)_{i\geq 0}$ is an increasing sequence of integrable functions, by Corollary 5.13 we have for any fixed $y \in \text{cl}(O)$:

$$\cdots, V_{g_i}(\cdot, y) \leq V_{g_{i+1}}(\cdot, y) \leq \cdots \leq W_{g_{i+1}}(\cdot, y) \leq W_{g_i}(\cdot, y) \leq \cdots,$$

whenever $y \in \text{cl}(\text{dom}(g_i))$ for $i \geq 0$. Moreover, we have:

$$g_i \sqsubseteq \mathcal{L}V_{g_i}(\cdot, y) \qquad \text{and} \qquad g_i \sqsubseteq \mathcal{L}W_{g_i}(\cdot, y).$$

Let $h_y = \sup_{i\geq 0} V_{g_i}(\cdot, y)$. Then $h_y : O \to \mathbb{R}$ is real-valued and thus by Proposition 2.10, we have $g_i \sqsubseteq \mathcal{L}h_y$ for each $i \geq 0$. It follows that $g \sqsubseteq \mathcal{L}h_y$ and thus $g$ is integrable.

□

# 6. Construction of Least and Greatest Consistency Witness

In this section, we will explicitly construct $s(f, g)$ and $t(f, g)$, for step functions $f \in (U \to \mathbf{I}\mathbb{R})$ and $g \in (U \to \mathbf{I}\mathbb{R}^n)$, which will be piecewise linear functions and would respectively be the least and the greatest witness for consistency when $(f, g)$ is indeed consistent. These results extend those in [14] to higher dimensions. Let $x$ and $y$ be in the same connected component $O$ of $\text{dom}(g)$ with $O \cap \text{dom}(f) \neq \emptyset$.

**Theorem 6.1.** *The maps $V_g(\cdot, y), W_g(\cdot, y) : cl(O) \to \mathbb{R}$ are respectively the least and the greatest continuous maps $L, G : cl(O) \to \mathbb{R}$ with $L(y) = 0$ and $G(y) = 0$ such that $g \sqsubseteq \mathcal{L}L$ and $g \sqsubseteq \mathcal{L}G$.*

*Proof.* Consider any function $F : \text{cl}(O) \to \mathbb{R}$ with $F(y) = 0$ and $g \sqsubseteq \mathcal{L}F$. By Corollary 5.7, we obtain $F(x) - F(y) \geq V_g(x, y)$, i.e., $F(x) \geq V_g(x, y)$ for all $x \in \text{cl}(O) \to \mathbb{R}$. We conclude that $L = V_g(\cdot, y)$. The case for $G$ is similar. □

Let $S_{(f,g)}(x, y) = V_g(x, y) + \overline{\lim} f^-(y)$.

**Corollary 6.2.** *Let $O$ be a connected component of dom$(g)$ with non-empty intersection with dom$(f)$. For $x \in O$, we have:*

$$s(f, g)(x) = \sup_{y \in O \cap dom(f)} S_{(f,g)}(x, y). \tag{11}$$

*Proof.* By Theorem 6.1, the map $h_y = \lambda x.S_{(f,g)}(x, y)$ is the least function with $h_y(y) = \overline{\lim} f^-(y)$ such that $g \sqsubseteq \mathcal{L}h_y$. By definition, $s(f, g)$ is precisely the upper envelope of $h_y$ for $y \in O \cap \text{dom}(f)$. □

**Proposition 6.3.** *Let $O$ be a connected component of dom$(g)$ with non-empty intersection with dom$(f)$. There exist a finite number of points $y_0, y_1, \ldots, y_i \in$ cl$(O \cap$ dom(f)$)$ with*

$$s(f,g)(x) = \max\{S_{(f,g)}(x, y_j) : j = 0, 1, \ldots, i\}$$

*for $x \in O$.*

*Proof.* Consider $O$ as the disjoint union of the crescents in $O \setminus \mathrm{dom}(f)$ and the intersection of crescents of $O$ and $\mathrm{dom}(f)$. For a fixed $x \in O$, consider any piecewise linear path $p_y$ from $y$ to $x \in O$ with nodes at points in the set $E_O(x, y)$. Suppose $y$ belongs to a hyper-rectangle $R$ of the coarser partition $H_O(x)$. Since the integral $\int_{p_y} g(r)\, dr$ depends, for a fixed type of $p_y$, continuously on $y$ and since, by Lemma 5.14, it depends linearly on the coordinates of $y \in R^\circ$, it follows that the maximum value of the path integral will be reached for $y$ at a corner of $R$. On the other hand, for fixed $x$, the value of $V_g(x, y)$ is attained by a piecewise linear path $p_y$ from $y$ to $x$ with nodes at points in the set $E_O(x, y)$. Thus, the maximum value of $V_g(x, y)$ for $y \in R$ is reached for $y$ in a corner of $R$ and thus some point in $\underline{E_O}(x)$. Since $f^-$ is constant in the interior of each crescent in $O$, it also follows that the maximum value of $\overline{\lim} f^-$ in $R$ is always attained at the corners of $R$. Therefore, $s(f, g)(x) = \max\{S(f, g)(x, y) : y \in E_O(x)\}$ which completes the proof. ∎

Results dual to those above are obtained for $t(f, g)$ as follows. We put $T_{(f,g)}(x, y) = W_g(x, y) + \underline{\lim} f^+(y)$. Then, for $x \in O$, we have

$$t(f,g)(x) = \inf_{y \in O \cap dom(f)} T_{(f,g)}(x, y),$$

and there exist $y_0, y_1, \ldots, y_i \in \mathrm{cl}(O \cap \mathrm{dom(f)})$ with

$$t(f,g)(x) = \min\{T_{(f,g)}(x, y_j) : j = 0, 1, \ldots, i\}$$

We can therefore conclude:

**Corollary 6.4.** *If $O$ is a connected component of dom$(g)$ with non-empty intersection with dom$(f)$, then $s(f, g)$ and $t(f, g)$ are piecewise linear maps in $O$, which are respectively the least and the greatest consistency witnesses in $O$ when $(f, g)$ is a consistent pair.*

## 7. Concluding remarks

We have proved the decidability of consistency in the construction of the domain of real-valued multi-variable Lipschitz maps with hyper-rectangular valued L-derivative or Clarke gradient which ensures that this domain can be given an effective structure. We have also extended the celebrated Gradient theorem of differential calculus to the domain-theoretic setting in the case of Lipschitz maps with hyper-rectangular valued L-derivative and used it to obtain the least and the greatest maps that satisfy the constraints of a consistent pair of step functions.

The question still remains if consistency is decidable for a pair of rational step functions $(f, g) \in (U \to \mathbf{I}\mathbb{R}) \times (U \to \mathbf{C}\mathbb{R}^n)$ and if the results on the zero-containment loop condition as a necessary and sufficient condition for integrability, and the construction of the least and the greatest witness for consistency can be extended from the rectangular valued derivative $g : O \to \mathbf{I}\mathbb{R}^n$, to the more general case of compact, convex valued derivative of the form $g : O \to \mathbf{C}\mathbb{R}^n$. These will be addressed in future work.

The results in this paper lay the groundwork for developing domain-theoretic computational models in vector calculus, differential geometry and differential topology as well as in complex analysis, all based on the class of locally Lipschitz maps which have appropriate closure and differential properties.

## References


[1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3. Clarendon Press, 1994.

[2] T. Apostol. *Calculus*, volume 2. Wiley, 1969.

[3] T. Bonnesen and W. Fenchel. *Theory of Convex Bodies*. BCS Associates, 1988.

[4] J. M. Borwein and W. B. Moors. Essentially smooth Lipschitz functions. *Journal of Functional Analysis*, 149(2), 1997.

[5] J. M. Borwein, W. B. Moors, and Y. Shao. Subgradient representation of multifunctions. *Journal of the Australian Mathematical Society-Series B*, 40:301–313, 1998.

[6] J. M. Borwein, W. B. Moors, and X. Wang. Generalized subdifferentials: A Baire categorical approach. *Transactions of the American Mathematical Society*, 353(10), 2001.

[7] R. Cazacu and J. D. Lawson. Quasicontinuous functions, domains, extended calculus, and viscosity solutions. *Applied General Topology*, 8:1–33, 2007.

[8] F. H. Clarke. *Optimization and Nonsmooth Analysis*. Wiley, 1983.

[9] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth Analysis and Control Theory*. Springer, 1998.

[10] P. Di Gianantonio and A. Edalat. A language for differentiable functions. In *Proceedings of the 16th International Conference on Foundations of Software Science and Computation Structures (FoSSaCS)*, 2013.

[11] A. Edalat. A continuous derivative for real-valued functions. In S. B. Cooper, B. Löwe, and A. Sorbi, editors, *New Computational Paradigms, Changing Conceptions of What is Computable*, pages 493–519. Springer, 2008.

[12] A. Edalat. A differential operator and weak topology for Lipschitz maps. *Topology and its Applications*, 157,(9):1629–1650, June 2010.

[13] A. Edalat, M. Krznarić, and A. Lieutier. Domain-theoretic solution of differential equations (scalar fields). In *Proceedings of MFPS XIX*, volume 83 of *Electronic Notes in Theoretical Computer Science*, 2003. `www.entcs.org/files/mfps19/mfps19.html`, full paper in `www.doc.ic.ac.uk/~ae/papers/scalar.ps`.

[14] A. Edalat and A. Lieutier. Domain theory and differential calculus (Functions of one variable). *Mathematical Structures in Computer Science*, 14(6):771–802, December 2004.

[15] A. Edalat and D. Pattinson. Domain theoretic solutions of initial value problems for unbounded vector fields. In M. Escardó, editor, *Proc. MFPS XXI*, volume 155 of *ENTCS*, pages 565–581, 2005.

[16] A. Edalat and D. Pattinson. Inverse and implicit functions in domain theory. In P. Panangaden, editor, *Proc. 20th IEEE Symposium on Logic in Computer Science (LICS 2005)*, pages 417–426, 2005.

[17] A. Edalat and D. Pattinson. A domain theoretic account of Euler's method for solving initial value problems. In J. Dongarra, K. Madsen, and J. Wasniewski, editors, *Proc. PARA 2004*, volume 3732 of *Lecture Notes in Comp. Sci.*, pages 112–121, 2006.

[18] A. Edalat and D. Pattinson. Denotational semantics of hybrid automata. *Journal of Logic and Algebraic Programming*, 73:3–21, 2007.

[19] A. Edalat and D. Pattinson. A domain-theoretic account of Picard's theorem. *LMS Journal of Computation and Mathematics*, 10:83–118, 2007.

[20] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains*. Cambridge University Press, 2003.

[21] A. Griewank and A. Walther. *Evaluating Derivatives*. Siam, second edition, 2008.

[22] W. Kaplan. *Advanced Calculus*. Addison-Wesley, 1991.

[23] G. Lebourg. Generic differentiability of Lipschitzian functions. *Transaction of AMS*, 256:125–144, 1979.

[24] R. E. Moore. *Interval Analysis*. Prentice-Hall, Englewood Cliffs, 1966.

[25] M. B. Pour-El and J. I. Richards. *Computability in Analysis and Physics*. Springer-Verlag, 1988.

[26] Louis B. Rall and George F. Corliss. Automatic differentiation: Point and interval AD. In P. M. Pardalos and C. A. Floudas, editors, *Encyclopedia of Optimization*. Kluwer, 1999.

[27] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton Landmarks in Mathematics. Princeton, 1997.

[28] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*. Cambridge University Press, 1993.

[29] K. Weihrauch. *Computable Analysis (An Introduction)*. Springer, 2000.