# ALMOST INVARIANT SUBSPACES
# AND
# GENERALIZED LINEAR SYSTEMS

by

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# ABSTRACT

For the linear system (C,A,B), the properties of almost controlled and almost conditionally invariant subspaces are reviewed and new properties are obtained for sliding subspaces.

These concepts are used to give :

i) A transparent geometric interpretation for the infinite-zeros of $G(s) = C(sI-A)^{-1}B$.

ii) A condition so that the orders of the asymptotes (as a scalar $g \to \infty$) of $(A+gBC)$ coincide with the orders of the infinite-zeros of G(s).

iii) A new method for the choice of an output fedback map R which assigns the asymptotes of (A+gBRC).

The concept of a nonsingular proportional-derivative (P.D.) law, $u = F_1x + F_2\dot{x}$ is introduced. Properties of a linear system (A,B) under such a law are described. Under the solvability condition given by Willems, it is shown that the almost disturbance decoupling can be achieved by a nonsingular P.D. law involving finite maps.

The notion a regular P.D. law is also introduced and it is applied to the theory of almost controlled invariant subspaces. Such a law is then used to solve the disturbance decoupling problem in a situation where a state feedback law does not do it.

The thesis also contains a concise geometric theory of the

regular pencil (sE-A) which includes a geometric criterion for the regularity of the pencil.

The generalized linear system $E\dot{x} = Ax+Bu$; $y = Cx$, where E is a singular map, is studied using geometric tools and the following contributions are given:

i) Necessary and sufficient conditions for the controllability and observability of the infinite-zeros of the pencil (sE-A).

ii) Proof that controllability of the infinite-zeros is equivalent to the existence of a state feedback map which assigns those zeros to arbitrary positions in the complex plane.

iii) An interpretation in terms of invariant subspaces for the controllability and observability of the finite-zeros of (sE-A).

iv) A method for the choice of a state feedback map F which assigns the zeros of the pencil (sE - A-BF).

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# PREFACE

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*To my great friends who have always supported me,
Emília, my mother (who I miss so much) and
Augusto, my father.*

# NOTATION AND PRELIMINARIES

This section contains some basic concepts of linear algebra and the notation employed in the thesis.

We shall denote throughout vectors by roman lower case letters, time functions and distributions by script underlined lower case leters, matrices (linear operators) by roman capitals and subspaces by script capitals.

We shall often consider families of subspaces of a given finite dimensional vector space $X$. Let $\underline{L}$ be such a family, we shall say that $\underline{L}$ is closed under addition if $L_1, L_2 \in \underline{L} \Rightarrow L_1 + L_2 \in \underline{L}$ and closed under intersection if $L_1, L_2 \in \underline{L} \Rightarrow L_1 \cap L_2 \in \underline{L}$. The subspace sup $\underline{L}$ denotes the smallest subspace which contains every element of $\underline{L}$ while inf $\underline{L}$ denotes the largest subspace contained in every element of $\underline{L}$. In general sup $\underline{L}$ and inf $\underline{L}$ do not belong to $\underline{L}$. However, the following case yields a result which is important for the geometric approach to linear systems theory.

<u>Lemma 0</u>: If $\underline{L}$ is closed under addition then sup $\underline{L} \in \underline{L}$ and if $\underline{L}$ is closed under intersection then inf $\underline{L} \in \underline{L}$.

If n is a positive integer, then $\underline{n}$ stands for the set of integers $\{1,2,\ldots,n\}$.

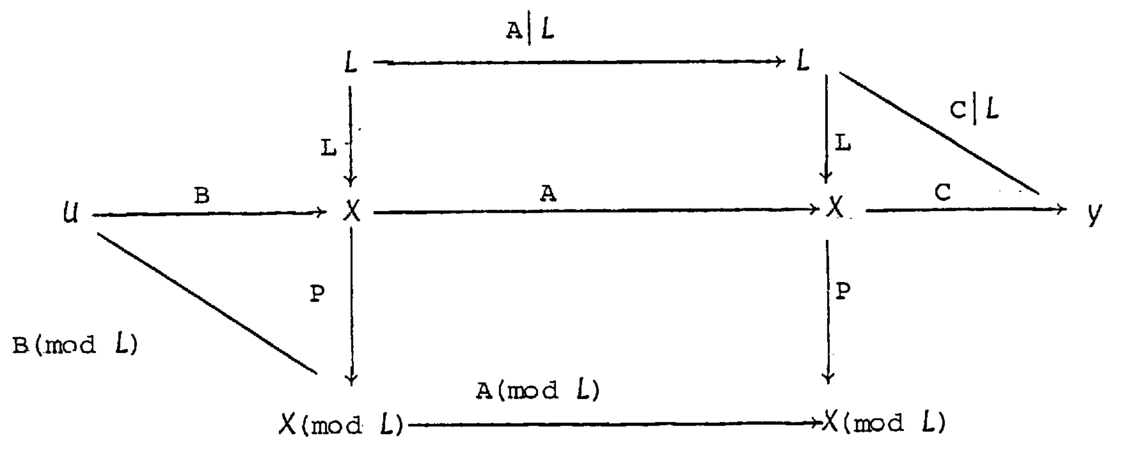
If $\{L_i\}$, $i \in \underline{n}$, is such that $L \supset L_1 \supset \ldots \supset L_n$ then we say that $\{L_i\}$ is a chain *in* $L$ while if $L \subset L_1 \subset \ldots \subset L_n$ then we say that $\{L_i\}$ is a chain *around* $L$.

The orthogonal complement of $L$ is denoted by $L^\perp$.

The dimension of a subspace $L \subset X$ is denoted by dim $L$ and codim denotes codimension, codim $L$ = dim $X$ - dim $L$ = dim $L^\perp$.

For a given subspace $\mathcal{L} \subset \mathcal{X}$ we represent the associated quotient space $\{x+\mathcal{L} \mid x \in \mathcal{X}\}$ by $\mathcal{X}/\mathcal{L}$ or $\mathcal{X}(\text{mod } \mathcal{L})$ with $\dim \mathcal{X}/\mathcal{L} = \dim \mathcal{X} - \dim \mathcal{L}$.

Let $A : \mathcal{X} \to \mathcal{X}$ be a linear operator and $\mathcal{L} \subset \mathcal{X}$ be such that $A\mathcal{L} \subset \mathcal{L}$ then we say that $\mathcal{L}$ is A-invariant. Let $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ be finite dimensional vector spaces and $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ be linear. The maps $A|\mathcal{L}$, $A(\text{mod } \mathcal{L})$, $B(\text{mod } \mathcal{L})$ and $C|\mathcal{L}$ are then defined by the following commutative diagram




where L is the canonical injection and P is the canonical projection, a map such that ImP : $\mathcal{X}(\text{mod } \mathcal{L})$ and ker P = $\mathcal{L}$.

Im and ker denote image (range) and kernel (null space), respectively.

Let $\mathcal{L}, \mathcal{K} \subset \mathcal{X}$. Then we shall say that $\mathcal{L}$ is $A(\text{mod } \mathcal{K})$-invariant if $A\mathcal{L} \subset \mathcal{L} + \mathcal{K}$ and that $\mathcal{L}$ is $A|\mathcal{K}$ invariant if $A(\mathcal{L} \cap \mathcal{K}) \subset \mathcal{L}$.

Furthermore, $\langle A|\mathcal{L}\rangle := \mathcal{L} + A\mathcal{L} + \ldots + A^{n-1}\mathcal{L}$ and $\langle \mathcal{L}|A\rangle := \mathcal{L} \cap A^{-1}\mathcal{L} \cap \ldots A^{-n+1}\mathcal{L}$ denote respectively, the smallest A-invariant subspace containing $\mathcal{L}$ and the largest A-invariant subspace contained in $\mathcal{L}$.

In terms of the linear system $\sum : \dot{x} = Ax + Bu;\ y = Cx$, $\mathcal{R} :=$ $\langle A|\mathrm{Im}\, B\rangle$ is the reachable subspace while $\mathcal{N} := \langle \ker C|A\rangle$ is the unobservable subspace.

$\mathcal{B}$ stands for Im B and $A_F := A+BF$ denotes the closed-loop map obtained by the state feedback control law $u = Fx$ on $\sum$.

If $A : \mathcal{X} \to \mathcal{X}$ is a linear operator and $\mathcal{L}, \mathcal{K} \subset \mathcal{X}$ are such that $A\mathcal{L} \subset \mathcal{K}$ we then denote the restriction of A to $\mathcal{L}$ with codomain $\mathcal{K}$ by $\mathcal{K}|A|\mathcal{L}$.

Let $\mathcal{X}$ be a normed vector space and consider any measurable function $\underline{\ell} : [0,\infty) \to \mathcal{X}$. We then say that $\underline{\ell}$ belongs to the $L_p$-space if $\|\underline{\ell}\|_{L_p} < \infty$, where

$$\|\underline{\ell}\|_{L_p} := \begin{cases} \left(\int_0^\infty \|\underline{\ell}(t)\|^p\right)^{\frac{1}{p}} & , \quad 1 \leq p < \infty \\ \sup\limits_{t \geq 0} \|\underline{\ell}(t)\| & , \quad p = \infty \end{cases} .$$

The following list shows the symbols often found throughout the thesis together with their usage and(or) meaning.

| Symbol | | Usage/Meaning |
|---|---|---|
| 1) | $:=$ | $x := y$, x is defined as y. |
| 2) | $\oplus$ | $\mathcal{L} \oplus \mathcal{K}$, direct sum |
| 3) | $\cup$ | $\Lambda_1 \cup \Lambda_2$, list combination |
| 4) | $\mathbb{R}$ | the real line |
| 5) | $\mathbb{R}^+$ | the nonnegative interval $[0,\infty)$ on the real line |
| 6) | $\mathbb{C}$ | complex plane |

| | | |
|---|---|---|
| 7) | $\mathbb{C}^-$ | open left-half complex plane |
| 8) | * | complex conjugate |
| 9) | dim $X$ | dimension of $X$ |
| 10) | $\sigma(A)$ | the spectrum (i.e. the set of eigenvalues, counting multiplicities) of the square matrix A |
| 11) | Re | real part |
| 12) | Mat A | matrix of A(i.e. the representation of the map A in a certain basis) |
| 13) | $X'$ | dual vector space |
| 14) | $A^T$ | transpose of A(the dual map to A) |
| 15) | $\underline{n}$ | the set of positive integers $\{1,2,...,n\}$ |
| 16) | $\mathbb{R}(s)$ | field of the rationals over $\mathbb{R}$ |
| 17) | $\mathbb{R}[s]$ | ring of the polynomials over $\mathbb{R}$ |
| 18) | $\mathbb{R}_+(s)$ | $\{f \in \mathbb{R}(s) \mid f = q/p,\ p,\ q \in \mathbb{R}[s]$ and degree p > degree q$\}$ |

formulation of relaxed versions of control synthesis problems. For example, consider the linear system

$$\dot{x} = Ax + Bu + Gd \qquad (0.1)$$
$$z = Dx$$

where d is a vector of disturbances and z denotes the to-be-controlled outputs.

We can then ask : does these exist a control law u = Fx such that in the closed loop system the influence of d on z is arbitrarily small in some precise mathematical sense? Such a problem is called the almost disturbance decoupling problem (ADDP).

Willems [1.14] has given a necessary and sufficient condition for the solution of (ADDP) in terms of a certain type of almost controlled invariant subspace which we denote in this introduction by $V_b$. Once the condition is satisfied, namely the problem is solvable, then in order to obtain an arbitrarily small influence of d on z we must have a sequence of controlled invariant subspaces $V_\epsilon$ [1.12] such that $V_\epsilon$ approaches $V_b$ as $\epsilon \to 0$. Moreover, in the approximation process the feedback maps $F_\epsilon$ for which $(A+BF_\epsilon)V_\epsilon \subset V_\epsilon$ are such that $F_\epsilon \to \infty$ as $\epsilon \to 0$.

It is in this context that we introduce the notion of a non-singular proportional-derivative (P.D) state feedback law, $u = F_1 + F_2\dot{x}$, which is a law such that the map $(I-BF_2)$ is nonsingular. The terminology stems from the fact that a PD law gives rise to the following closed loop system.

$$\dot{x} = (I-BF_2)^{-1}(A+BF_1)x + (I-BF_2)^{-1}Gd$$
$$z = Dx.$$

The interesting fact is that under the same solvability condition for (ADDP), as given by Willems, we can also obtain an arbitrarily small influence of $d$ on $z$ by means of a nonsingular PD law involving finite feedback maps. More explicitly, it is shown that the subspaces $V_\epsilon$ can be made invariant under the operators $(I-BF_{2\epsilon})^{-1}(A+BF_1)$ where $F_1$ is a finite map and $F_{2\epsilon} \xrightarrow[\epsilon\to 0]{} F_2$ is a map such that $(I-BF_2)$ is singular.

Such a result illustrates the usefulness of the derivative feedback in the context of (ADDP) and it can be interpreted in the following way : a trajectory of $\Sigma$ which remains in a controlled invariant subspace that is close to an almost controlled invariant subspace is characterized by high velocities. Consequently the more relevant information to be fedback is the derivative $\dot{x}$. The high gain state feedback task is transferred to the derivative which accomplishes it with finite gain.

It is also shown that a nonsingular P.D. law preserves all the important features of a pair (A,B) such as the controllable subspace and the controllability indices.

Another significant point about the use of derivative feedback is that, under the same solvability condition for (ADDP), we can achieve *exact* disturbance decoupling (influence of $d$ on $z$ is zero) by means of a law with the from

$$u = F_1 x + F_2\dot{x} + F_3 d \tag{0.2}$$

which yields the following closed loop system

$$(I-BF_2)\dot{x} = (A+BF_1)x + (G+BF_3)d \qquad (0.3)$$

$$z = Dx.$$

The maps $F_1$ and $F_2$ involved in the solution are such that $(I-BF_2)$ is singular and the pencil $s(I-BF_2) - (A+BF_1)$ is regular [4.7]. A PD law, represented by the pair $(F_1,F_2)$, with such features will be called a regular PD law.

There are interesting connections between regular PD laws and almost controlled invariant subspaces. For example, there exists a kind of an almost controlled invariant subspace, called a sliding subspace, which cannot be made invariant under state feedback. However a regular P.D. law can be used to establish a type of invariance for such a subspace. It is shown that for a sliding subspace $\mathcal{R}_s$ there exists a regular P.D. law $(F_1,F_2)$ such that $(I-BF_2)\mathcal{R}_s \subset (A+BF_1)\mathcal{R}_s$.

Note from (0.3) that with $d = 0$ we obtain an autonomous linear system of the type

$$E\dot{x} = Ax \qquad (0.4)$$

where $E$ is a singular map. This kind of system will be called a generalized linear system to distinquish it from the ordinary (or regular) linear system when $E$ is nonsingular.

The theoretical development which leads to the exact solution of the disturbance decoupling by the law (0.2) depends upon some geometric properties of a regular pencil (sE-A).

Although the main features of a regular pencil are well known [4.1, 4.3, 4.7, 4.16] it seems that there does not exist in the literature geometric conditions for the regularity of a pencil which is a point required for the definition of the law (0.2). For this reason we have included a compact geometric theory on regular pencils which unifies some existing results and also provides solutions to questions not answered before. The presentation of the theory emphasizes dynamical interpretations with respect to the associated generalized linear system (0.4).

A regular P.D. law can also be introduced as a motivation for the study of structural properties of a generalized linear system described by

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad (0.5)$$

where $E$ is a singular map.

Several properties of this kind of system have already been described in [4.3, 4.15]. It is well known that if we allow arbitrary initial conditions for the autonomous generalized linear system (0.4) we then have in general a response characterized by exponential and impulsive modes. This is due to the fact that a pencil (sE-A) has in general finite and infinite-zeros.

Verghese [4.15, 4.16] has pointed out that the controllability and observability criteria for the infinite-zeros (relative to the system(0.5)) given by Resenbrock were valid only in a special case. He has then introduced tests to check for the controllability and observability of such zeros which are applicable in any situation. Based on his tests we have been able to provide correct necessary and sufficient conditions for the controllability and observability of the infinite-zeros in terms of the maps E, A, B and C. We also present an interpretation for the controllability and observability of the finite-zeros in terms of invariant subspaces associated with the pencil (sE-A).

Cobb [4.3] has given a necessary and sufficient condition for the existence of a state feedback law u = Fx for (0.5) which brings the infinite-zeros to finite positions of the complex plane without pre-specifying those positions.

We show here that his condition is in fact the controllability of the infinite-zeros and we also obtain the stronger result that controllability of the infinite-zeros is equivalent to the existence of a state feedback law which assigns pre-specified complex numbers to those zeros. The result may be considered as an extension of the celebrated result on pole placement by Wonham [3.17] for linear systems for which E is the identity map. A new method for the assignment of all zeros (finite and infinite) by state feedback is also presented.

Other topics considered in this thesis are those of infinite-zeros and root-loci for multivariable linear systems.

Consider the linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

and its associated transfer matrix $G(s) = C(sI-A)^{-1}B$.

Commault and Dion [2.2] have given a geometric definition for the infinite-zero structure of G(s) by relating it to the notions of almost controlled invariant subspaces. It turns out that infinite-zeros and almost invariant subspaces play an important role in the root-loci analysis of invertible linear systems which are those for which G(s) is invertible over the field of the rationals (note that this implies that the number of inputs is equal to the number of outputs).

From a state space point of view, the use of almost controlled invariant subspaces in the study of root-loci properties is adequate for the simple reason that the concept of an almost invariant subspace is intimately connected with high gain feedback.

By exploiting properties of representations of a sliding subspace, it is shown how to construct an output feedback law, $u = gRy$, where $g$ is a real scalar, $g \to \infty$, such that the eigenvalues that go to infinity in the closed loop map $(A+gBRC)$ approach pre-assigned asymptotes.

The assignment procedure suggested has the following features :

- the orders of the assigned asymptotes coincide with the orders of the infinite-zeros of the transfer matrix of the invertible system.

- asymptotes of distinct orders are assigned independently .
- limit eigenvectors are also assigned.

Our method differs from that presented in [2.11] with some advantage in that the asymptotes can be assigned without the knowledge of the asymptotic structure of the closed loop map (A+gBC), $g \to \infty$. It is well known that the asymptotes of this last map may not have the same orders as the orders of the infinite-zeros of $C(sI-A)^{-1}B$. In this respect we show a condition derived directly from the maps B and C which ensures that both entities, infinite-zeros and asymptotes, have the same orders. The condition is constructive in the sense that the magnitude of the asymptotes can also be computed from it.

Finally the thesis contains a summary of the main known properties of almost invariant subspaces together with some new ones.

Main Contributions of the Thesis

We list below the main contributions and we refer to the theorems and propositions which are related to them:

- the description of properties of sliding subspaces: Theorems 1.4, 1.5 and 1.6.
- an extension of a result due to Trentelman [1.12] regarding the spectrum of a controlled invariant subspace which complements a $L_p$-almost controlled invariant subspace : Theorem 1.8.
- a derivation of the prime subsystem in Morse's canonical form in terms of almost controllability subspaces (Theorem 2.1 Item d) together with a detailed geometric interpretation for the infinite-zeros of the transfer matrix $C(sI-A)^{-1}B$.
- a new procedure for the construction of a high gain output feedback map for a linear system with invertible transfer matrix so that the asymptotes of the closed loop system take on pre-assigned values : Proposition 2.2.
- a condition which ensures that the asymptotes of a linear scalar gain output feedback have the same orders as the orders of the infinite-zeros of the transfer matrix : Proposition 2.3.

Regarding the generalized linear system $E\dot{x} = Ax + Bu; \quad y = Cx$ we have given :

- necessary and sufficient conditions for the controllability and observability of the infinite-zeros of the pencil (sE-A) : Theorems 3.3 and 3.5 (see also Theorems 3.2, 3.4 and Propositions 3.3, 3.4).
- the proof that controllability of the infinite-zeros is equivalent to the existence of a state feedback map which assigns those zeros to pre-specified complex numbers : Theorem 3.7.
- a method to assign all the zeros (finite and infinite) by state feedback : Proposition 3.6.
- the proof that controllability and observability of the infinite-zeros is equivalent to the existence of an output feedback map which converts those zeros into finite ones : Theorem 3.8.
- a procedure for zero assignment via observers for systems described by $J\dot{x} = x + Bu$; $y = Cx$ where $J$ is a nilpotent map (see Section 3.5).

Relative to the geometric theory of a pencil (sE-A) the following contributions have been given:

- the identification of the minimal column indices of a singular pencil with certain indices derived from a sequence of subspaces and the determination of the map whose eigenvalues are the finite-zeros of such a pencil : Theorem 4.1.
- a necessary and sufficient condition for a pencil to be regular : Theorem 4.2.

- miscellaneous geometric properties of a regular pencil stated in Proposition 4.1, Corollaries 4.1, 4.2, Lemma 4.1 and Theorem 4.3.

Finally, it is believed that all the results in Chapter IV concerning properties of a linear system under a regular and a nonsingular PD law and the use of such laws in the exact and almost disturbance decoupling problems are new. The construction of PID observers in the context of almost conditionally invariant subspaces also seems to be original.

# CHAPTER I

# ALMOST INVARIANT SUBSPACES

## I.1 INTRODUCTION

Linear systems theory has evolved extraordinarily since the introduction of (A-B) and (A-C) invariant subspaces [1.2-3, 1.17]. These concepts proved to be fundamental in the structural analysis of linear systems and also served as an excellent framework for solving several control synthesis problems such as the disturbance decoupling problem, tracking, regulation, the model following problem, the synthesis of noninteracting controllers, etc. . The methodology that employs the above concepts to deal with analysis and synthesis of linear systems has been labelled as the "geometric" approach in the important book by Wonham [1.17].

In this thesis we shall adopt the nomenclature suggested in [1.2, 1.14-5]. (A-B) invariant subspaces will be termed A(mod ImB) or controlled invariant subspaces and (A-C) invariant subspaces will be called $A|\ker C$ invariant subspaces or conditionally invariant subspaces. Such a terminology is, of course, related to the basic features of such subspaces which are presented in the following sections.

The concepts of almost controlled and almost conditionally invariant subspaces have been recently introduced by Willems [1.13-5] and they can, undoubtedly, be considered as one of the most important new developments in the "geometric" approach to linear systems theory.

The names "almost controlled invariant subspaces and almost conditionally invariant subspaces" stem from the fact that such subspaces are limit subspaces of sequences of controlled invariant and conditionally invariant subspaces.

The new notions have proved to be fundamental to deal with high gain feedback questions. As an example, the almost disturbance decoupling problem, by state feedback and by measurement feedback, has been solved in [1.14-5]. As another application we shall see in Chapter II that almost invariant subspaces provide an excellent framework for a state space analysis of the root-loci problem for multivariable linear systems.

Almost invariant subspaces have also played a role in the structural analysis of linear systems : the geometric definition of infinite-zeros given in [1.4] builds on them.

The concept of invariant subspaces is rich not only because it plays a fundamental role in linear systems theory but also because it is the kind of concept which has its counterpart in nonlinear systems theory. It is also expected that the concept of almost invariant subspaces should be generalized in the context of nonlinear systems. For potential applications we refer to [1.14-5].

The objective of this chapter is twofold : firstly, to describe known concepts which are essential for the subsequent development of the thesis and secondly, to present some new properties of almost invariant subspaces.

## I.2 ALMOST CONTROLLED INVARIANT SUBSPACES

### 1.2.1 Basic Concepts

The material of this section is known and can be found in

[1.17, 1.13-14]. It consists of a collection of definitions and results which are needed for the development of the thesis. The exposition style here follows closely that one of [1.14]. For the reader well acquainted with the concept of almost controlled invariant subspaces we suggest to examine this section for the notation and proceed directly to the next section.

Consider the linear system

$$\sum : \dot{x} = Ax + Bu \tag{1.1}$$

where $x \in \mathcal{X} := \mathbb{R}^n$ ; $u \in \mathcal{U} := \mathbb{R}^m$ ; $\dim \mathcal{B} = m$

and let $\sum_{\chi}$ denote all possible state trajectories generated by $\sum$.

<u>Definition 1.1</u>: A subspace $\mathcal{V} \subset \mathcal{X}$ is said to be a controlled invariant subspace if $\forall\, x_0 \in \mathcal{V}$, $\exists\, \underset{\sim}{\chi} \in \sum_{\chi}$ such that $\underset{\sim}{\chi}(0) = x_0$ and $\underset{\sim}{\chi}(t) \in \mathcal{V}$, $\forall t$. A subspace $\mathcal{V}_a \subset \mathcal{X}$ is said to be an almost controlled invariant subspace if $\forall x_0 \in \mathcal{V}_a$ and $\epsilon > 0$, $\exists\, \underset{\sim}{\chi} \in \sum_{\chi}$ such that $\underset{\sim}{\chi}(0) = x_0$ and

$$d(\underset{\sim}{\chi}(t), \mathcal{V}_a) := \inf_{x \in \mathcal{V}_a} \| \underset{\sim}{\chi}(t) - x \| \leq \epsilon \,, \ \forall t. \tag{1.2}$$

Note that an almost invariant subspace is characterized by the fact that there exists a trajectory of $\sum_{\chi}$ that remains arbitrarily close to it.

<u>Definition 1.2</u>: A subspace $\mathcal{R} \subset \mathcal{X}$ is said to be a controllability subspace if $\forall x_0$ , $x_1 \in \mathcal{R}$, $\exists T > 0$ and $\underset{\sim}{\chi} \in \sum_{\chi}$ such that $\underset{\sim}{\chi}(0) = x_0$, $\underset{\sim}{\chi}(T) = x_1$ and $\underset{\sim}{x}(t) \in \mathcal{R}$, $\forall t$. A subspace $\mathcal{R}_a$ is said to be an almost controllability subspace if $\forall x_0$, $x_1 \in \mathcal{R}_a$, $\exists T > 0$ such that $\forall \epsilon > 0$,

$\exists \underline{x} \in \sum_{X}$ with the properties that $\underline{x}(0) = x_0$, $\underline{x}(T) = x_1$ and $d(\underline{x}(t), R_a) \leq \epsilon$, $\forall t$.

The above definition shows that (almost) controllability subspaces are (almost) controlled invariant subspaces which possess a reachability property.

Let $\underline{V}$, $\underline{R}$, $\underline{V}_a$, $\underline{R}_a$ denote the set of all controlled invariant subspaces, etc. and $\underline{V}(K)$, $\underline{R}(K)$, $\underline{V}_a(K)$ and $\underline{R}_a(K)$ those contained in a given subspace $K \subset X$. From definitions 1.1 and 1.2 it follows that $\underline{R} \subset \underline{V} \subset \underline{V}_a$ and $\underline{R} \subset \underline{R}_a \subset \underline{V}_a$.

The first property of the subspaces introduced in definitions 1.1 and 1.2 is extremely important in applications.

Theorem 1.1: $\underline{V}$, $\underline{R}$, $\underline{V}_a$ and $\underline{R}_a$ are closed under subspace addition. Consequently,

$$\sup \underline{V}(K) := V^*_K \in \underline{V} \qquad \sup \underline{R}(K) := R^*_K \in \underline{R}$$

$$\sup \underline{V}_a(K) := V^*_{a,K} \in \underline{V}_a \qquad \sup \underline{R}_a(K) := R^*_{a,K} \in \underline{R}_a \quad .$$

The subspaces introduced in definition 1.1 and 1.2 also admit "state feedback" characterizations which are shown in the next theorem.

Theorem 1.2:

a) $\{V \in \underline{V}\} \iff \{\exists F \text{ such that } A_F V \subset V\} \iff \{AV \subset V + B\}$ .

b) $\{R \in \underline{R}\} \iff \{\exists F \text{ and } B_1 \subset B \text{ such that } R = \langle A_F | B_1 \rangle\}$.

c) $\underline{V}_a = \underline{V} + \underline{R}_a$, i.e. $\{V_a \in \underline{V}_a\} \iff \{\exists V \in \underline{V} \text{ and } R_a \in \underline{R}_a \text{ such that } V_a = V + R_a\}$.

d) $\{R_a \in \underline{R}_a\} \iff \{\exists F$ and a chain $\{B_i\}$ *in* $B$ such that $R_a = B_1 + A_F B_2 + \ldots + A_F^{n-1} B_n\}$ .

The notation $F(V)$, $F(R)$ and $F(R_a)$ will be used to denote the sets of maps F which describe, respectively $V$, $R$ and $R_a$.

It has been proven in [1.14] that the set $\underline{V}_a$ is the closure of the set $\underline{V}$. In other words, arbitrarily close to any almost controlled invariant subspace $V_a$ there exists a controlled invariant subspace $V$ and in the approximation process, the feedback gain of the approximating controlled invariant subspace goes to infinity. Conversely, if a given subspace $K$ can be approximated arbitrarily close by a controlled invariant subspace, then $K \in \underline{V}_a$. Such a characterization of $V_a$ establishes the link with high gain feedback.

Another important characterization of an almost controlled invariant subspace $V_a$ is its equivalence to a controlled invariant subspace when distributional inputs are allowed in (1.1). In fact, there is a type of distribution which when used as input yields a "trajectory" which remains in $V_a$ if one starts in $V_a (x_0 \in V_a)$. Such a distribution is described in the next definition.

<u>Definition 1.3</u>: A distribution $\underline{\delta}$ with support on $\mathbb{R}^+$ is said to be of Bohl type if there exist vectors $f_i$ and matrices F, G, H such that

$$\underline{\delta} = \sum_{i=0}^{N} f_i \, \delta^{(i)} + \underline{\delta}'$$

where $\underline{\delta}' : t \to He^{Ft}G$ and $\delta^{(i)}$ is the $i^{th}$ distributional derivative of the delta functional. Equivalently, the distribution $\underline{\delta}$ is Bohl if its Laplace transform is rational.

The above characterization establishes a distinction between a controlled invariant and an almost controlled invariant subspace with respect to the type of open-loop inputs needed to hold a trajectory in either of such subspaces. It is well known [1.2] that controlled invariant require piecewise continuous functions as inputs, whereas almost controlled invariant subspaces require distributions.

It is also possible to hold a "trajectory" in $V_a$, if one starts there, by a distributional state feedback law $u : x \to Fx + \sum_{i=0}^{n} \delta^{(i)} F_i x$, for certain maps $F$ and $F_i$, $i \in \underline{n}$ (see [1.14]).

We shall show in Chapter IV that it is possible to keep a "trajectory" in $V_a$ by making use of a proportional-derivative state feedback law of the kind $u : x \to F_1 x + F_2 \dot{x}$ for certain maps $F_1$ and $F_2$.

In the following we review the algorithms which yield the supremal subspaces defined in Theorem 1.1.

The subspace $V_K^*$ is the limit of the nonincreasing sequence, i.e.

$$V_K^* := V^{\dim K+1} \; ; \; V^u = K \cap A^{-1}(V^{u-1} + B) \; ; \quad V^0 = X. \qquad (1.3)$$

It has been shown by Willems [1.13] that the subspace $R_{a,K}^*$ is the limit of the following monotone nondecreasing sequence.

$$R_{a,K}^* := R_a^{\dim K} \; ; \; R_a^u = K \cap (AR_a^{u-1} + B) \quad ; \quad R_a^0 = 0. \qquad (1.4)$$

The above sequence has been used in [1.17] in order to compute $R_K^*$, which is given by

$$R_K^* = V_K^* \cap R_{a,K}^* \qquad (1.5)$$

From Theorem 1.2-c, it follows that

$$\mathcal{V}^*_{a,\mathcal{K}} = \mathcal{V}^*_{\mathcal{K}} + \mathcal{R}^*_{a,\mathcal{K}} \quad . \tag{1.6}$$

It is clear from Definition 1.1 that $\mathcal{V}^*_{a,\mathcal{K}}$ consists of all those points $x_0 \in \mathcal{K}$, for which there exists a trajectory that starts in $x_0$ and remains arbitrarily close to $\mathcal{K}$ in the sense of the norm (1.2). Such a norm does not constitute the only way of measuring the distance of a trajectory through a point $x_0 \in \mathcal{X}$ to the subspace $\mathcal{K}$. In fact, there are many ways to measure (integrated) pointwise distance to $\mathcal{K}$ which are shown in the next definition.

Definition 1.4: The $\Sigma$- distance in the $L_p$ - sense from a point $x_0 \in \mathcal{X}$ to a subspace $\mathcal{K} \subset \mathcal{X}$ is defined by

$$d_p(x_0,\mathcal{K}) := \inf_{\substack{\underline{\chi}\in\Sigma_\chi \\ \underline{\chi}(0)=x_0}} \| \underline{d}(\underline{\chi},\mathcal{K}) \|_{L_p} \quad , \; 1 \le p \le \infty$$

where

$$\| d(\underline{\chi},\mathcal{K}) \|_{L_p} := \left( \int_{-\infty}^{+\infty} d^p(\underline{\chi}(t),\mathcal{K})\,dt \right)^{\frac{1}{p}} \quad , \; 1 \le p < \infty$$

and

$$\| d(\underline{\chi},\mathcal{K}) \|_{L_\infty} := \sup_t \| d(\underline{\chi}(t),\mathcal{K}) \| \; .$$

The points $x_0$ of interest are, of course, those for which $x_0$ is zero distance from $\mathcal{K}$. Moreover, from the linearity of $\Sigma_\chi$, it follows that such points constitute a subspace of $\mathcal{X}$. This has led Willems to the following definition.

<u>Definition 1.5</u>: $\mathcal{V}^*_{p,\mathcal{K}} := \{x_0 \in \mathcal{X} | d_p(x_0,\mathcal{K}) = 0\}$ will be called the supremal $L_p$ - almost controlled invariant subspace "contained" in $\mathcal{K}$ and

$$\mathcal{R}^*_{p,\mathcal{K}} := \mathcal{R}^*_{a,\mathcal{V}^*_{p,\mathcal{K}}}$$

will be termed the supremal $L_p$-almost controllability subspace "contained" in $\mathcal{K}$.

The subspaces above introduced are directly connected to the subspace $\mathcal{V}^*_{a,\mathcal{K}}$ and $\mathcal{R}^*_{a,\mathcal{K}}$ as the next theorem shows.

<u>Theorem 1.3</u>:

a) $\mathcal{R}^*_{\infty,\mathcal{K}} = \mathcal{R}^*_{a,\mathcal{K}}$ and $\mathcal{V}^*_{\infty,\mathcal{K}} = \mathcal{V}^*_{a,\mathcal{K}}$

b) for $1 \leq p < \infty$ : $\mathcal{R}^*_{p,\mathcal{K}} = A\mathcal{R}^*_{a,\mathcal{K}} + \mathcal{B}$ and $\mathcal{V}^*_{p,\mathcal{K}} = \mathcal{R}^*_{p,\mathcal{K}} + \mathcal{V}^*_{\mathcal{K}}$ .

It is clear from this theorem that $\mathcal{V}^*_{\infty,\mathcal{K}} \subset \mathcal{K}$, but $\mathcal{V}^*_{p,\mathcal{K}}$ need not be a subspace of $\mathcal{K}$. Part b of the above theorem shows that $\mathcal{R}^*_{p,\mathcal{K}}$ does not depend on $1 \leq p < \infty$ and for this reason the following notation is adopted.

$$\mathcal{R}^*_{b,\mathcal{K}} := A\mathcal{R}^*_{a,\mathcal{K}} + \mathcal{B} \tag{1.7}$$

and

$$\mathcal{V}^*_{b,\mathcal{K}} := \mathcal{R}^*_{b,\mathcal{K}} + \mathcal{V}^*_{\mathcal{K}} = A\mathcal{V}^*_{a,\mathcal{K}} + \mathcal{B} + \mathcal{V}^*_{a,\mathcal{K}} = A\mathcal{V}^*_{a,\mathcal{K}} + \mathcal{B} + \mathcal{V}^*_{\mathcal{K}} \quad \ldots \tag{1.8}$$

The subspace $R^*_{b,K}$ can be computed through the following monotone nondecreasing sequence.

$$R^*_{b,K} := S^{\dim K+1} \;;\; S^u = A(S^{u-1} \cap K) + B; \quad S^0 = 0. \qquad (1.9)$$

Trentelman [1.12] has shown how to construct sequences of controlled invariant subspaces which approach $R^*_{a,K}$ and $R^*_{b,K}$.

### I.2.2 Decomposition of an Almost Controlled Invariant Subspace and Some Geometric Relationships Among the Various Subspaces

We start this section by showing some relationships among the sequences of subspaces defined in (1.3-4), (1.9) and the monotone nondecreasing sequence which yields $R^*_K$ given by

$$R^*_K := R^{\dim V^*_K} \;;\; R^u = V^*_K \cap (AR^{u-1} + B) \;;\; R^0 = 0 \qquad (1.10)$$

Lemma 1.1:

a) $S^u = AR^{u-1}_a + B$

b) $S^u \cap V^*_K = R^u$

c) $R^u_a \cap V^*_K = R^u$

d) $S^u \cap K = R^u_a$ .

Proof: a) is proven in [1.14] ; b) is proven in [1.10], c) and d) follow immediately from a) and b) and have been shown in [1.5].

□

We introduce now two classes of subspaces which are extremely important in the geometric definition of finite-zeros, infinite-zeros and synthesis of control systems.

Definition 1.6: A subspace $\mathcal{C} \in \underline{\mathcal{V}}$ is said to be a coasting subspace if $\mathcal{R}^*_{\mathcal{C}} = 0$. A subspace $\mathcal{R}_s \in \underline{\mathcal{V}}_a$ is said to be a sliding subspace if $\mathcal{V}^*_{\mathcal{R}_s} = 0$.

A coasting subspace does not possess the reachability property or equivalently $\sigma[(A+BF)|\mathcal{C}]$ is fixed for any $F \in F(\mathcal{C})$ (see [1.14] and thm. 5.7 in [1.17]).

In a sliding subspace there are no trajectories generated by $\sum$ other than the null trajectories. All "trajectories" in that subspace are generated by distributional inputs.

We shall show next that sliding subspaces appear in the following way : let $\mathcal{R}_a$ be an almost controllability subspace; then $\mathcal{R}_a = \mathcal{R}^*_{\mathcal{R}_a} \oplus \mathcal{R}_s$ , i.e., sliding subspaces show up as complements of $\mathcal{R}^*_{\mathcal{R}_a}$ to $\mathcal{R}_a$. To show this we introduce two more sequences of subspaces which will be used throughout the thesis.

Let $\bar{\mathcal{B}}$ be any subspace such that

$$\mathcal{B} = \mathcal{B} \cap \mathcal{V}^*_K \oplus \bar{\mathcal{B}} \tag{1.11}$$

and consider the subspaces $\bar{\mathcal{R}}_{b,K}$ and $\bar{\mathcal{R}}_{a,K}$ defined by the following sequences of subspaces.

$$\bar{R}_{b,K} := \bar{S}^{\dim K+1}; \quad \bar{S}^{u} = A(\bar{S}^{u-1} \cap K) + \bar{B} \; ; \quad \bar{S}^{0} = 0. \tag{1.12}$$

$$\bar{R}_{a,K} := \bar{R}_{a}^{\dim K} \; ; \; \bar{R}_{a}^{u} = K \cap (A\bar{R}_{a}^{u-1} + \bar{B}) \quad ; \quad \bar{R}_{a}^{0} = 0 \, . \tag{1.13}$$

We then obtain the following relationships among the sequences defined in (1.3-4), (1.9-10) and (1.12-13).

Lemma 1.2:

a) $S^{u} = \bar{S}^{u} \oplus R^{u}$

b) $\bar{S}^{u} \cap V_{K}^{*} = 0$

c) $\bar{S}^{u} = A\bar{R}_{a}^{u-1} \oplus \bar{B}$

d) $\bar{R}_{a}^{u} = K \cap \bar{S}^{u}$

e) $R_{a}^{u} = \bar{R}_{a}^{u} \oplus R^{u}$

f) $\bar{R}_{a}^{u} \cap V_{K}^{*} = 0$

Proof: a) and b) are proven in [1.10] ; c) is analogous to part a) in Lemma 1.1 ; (d-f) follow immediately from (a-c) and are shown in [1.5].

□

From Lemma 1.2(e-f), it follows that

$$\mathcal{R}^*_{a,\mathcal{K}} = \mathcal{R}^*_{\mathcal{K}} \oplus \bar{\mathcal{R}}_{a,\mathcal{K}} \tag{1.14}$$

with $\bar{\mathcal{R}}_{a,\mathcal{K}}$ , a sliding subspace. Since $\mathcal{R}^*_{a,\mathcal{K}}$ is the supremal almost controllability subspace in $\mathcal{K}$, it follows that the sequence (1.13) yields a sliding subspace of maximal dimension in $\mathcal{K}$. Note that distinct sliding subspaces are obtained in (1.13) for distinct complements of $\mathcal{B} \cap \mathcal{V}^*_{\mathcal{K}}$ chosen in (1.11).

From (1.14) with $\mathcal{K} := \mathcal{V}_a$ and Theorem 1.2c we have

$$\mathcal{V}_a = \mathcal{C} \oplus \mathcal{R}^*_{\mathcal{V}_a} \oplus \bar{\mathcal{R}}_{a,\mathcal{V}_a} \tag{1.15}$$

where $\mathcal{C}$ is a coasting subspace such that $\mathcal{C} \oplus \mathcal{R}^*_{\mathcal{V}_a} = \mathcal{V}^*_{\mathcal{V}_a}$ . The decomposition (1.15) has been obtained in [1.14] and it describes the structural features of an almost controlled invariant subspace. The only subspace which is uniquely defined in (1.15) is $\mathcal{R}^*_{\mathcal{V}_a}$ , which is a subspace such that for any symmetric set $\Lambda$ of dim $\mathcal{R}^*_{\mathcal{V}_a}$ ,complex numbers there exists $F \in F(\mathcal{V}^*_{\mathcal{V}_a})$ such that $\sigma[(A+BF)|\mathcal{R}^*_{\mathcal{V}_a}] = \Lambda$. To conclude this section we refer to [1.7] for a matrix pencil characterization of almost controlled invariant subspaces.

### I.2.3 Properties of Sliding Subspaces

The decomposition (1.15) of an almost controlled invariant subspace has shown that sliding subspaces constitute one of the key concepts in the new theory developed by Willems. This motive alone would be sufficient to justify the study of properties of such subspaces. But this is not the only reason. We shall see in Chapter II that some of the properties described here are important to obtain a good formulation for the assignment of root-loci asymptotes and to analyse Morse's canonical form in the light of invariant and almost invariant subspaces.

Let $\mathcal{R}_s$ be a sliding subspace. The first property is a simple one and it is direct consequence of the fact that $\mathcal{V}^*_{\mathcal{R}_s} = 0$.

Proposition 1.1: Let $\mathcal{R}_s$ be a sliding subspace. Then

a) $\mathcal{R}_s \cap A_F^{-1}\mathcal{B} = 0 \quad , \quad \forall F : \mathcal{X} \to \mathcal{U}$

b) $\max \dim \mathcal{R}_s = n-m$

Proof: Consider the algorithm 1.3 with $\mathcal{K} := \mathcal{R}_s$. It is straightforward to see that the subspace $\mathcal{V}_u$ is invariant under the transformation $A \to A+BF$, $\forall F : \mathcal{X} \to \mathcal{U}$. Since $\mathcal{R}_s$ is a sliding subspace, it follows by the stopping rule of the algorithm that for some $u \in \underline{n}$, $\mathcal{R}_s \cap A_F^{-1} \mathcal{B} = 0$ and a) follows. To show b) note that a) implies $A_F \mathcal{R}_s \cap \mathcal{B} = 0$ and $\ker A_F \cap \mathcal{R}_s = 0$. This and $\dim \mathcal{B} = m$ imply b).

We already know that a sliding subspace is an almost controllability subspace and therefore it admits a feedback representation as shown in Theorem 1.2d. What we want is to obtain some properties of the feedback map that describes a sliding subspace and to analyse in more detail the algorithm (1.13) which is slightly different from the algorithm (1.4). In (1.13) we have discarded the subspace $\mathcal{B} \cap \mathcal{V}^*_{\mathcal{K}}$ which gives rise to the supremal controllability subspace in $\mathcal{K}$.

Theorem 1.4:

a) The sequence $\{\bar{R}_a^u\}$ is monotone nondecreasing ; moreover

$$\bar{R}_a^{\dim K} = \bar{R}_a^{\infty} := \lim_{u\to\infty} \bar{R}_a^u \text{ and } \bar{R}_a^u = \bar{R}_a^{u-1} \Rightarrow \bar{R}_a^{u-1} = \bar{R}_a^{\infty}.$$

b) $\bar{R}_a^{u-1} = \sup\{J \subset K \mid \exists F$ and a chain $\{B_i\}$ *in* $\bar{B}$ such that $J = B_1 + A_F B_2 + \ldots + A_F^{u-2} B_{u-1}$ with $BF\ \bar{R}_a^{u-2} \subset \bar{B}\}$.

c) Let $k \in \underline{n}$ be such that $\bar{R}_a^{k-1} = \bar{R}_a^{\infty}$. Then

$$\bar{R}_a^{k-1} = B_1 \oplus A_F B_2 \oplus \ldots \oplus A_F^{k-2} B_{k-1}$$

for some chain $\{B_i\}$ in $\bar{B}$. The subspace $\bar{R}_a^{k-1}$ can also be written as

$$\bar{R}_a^{k-1} = L_2 \oplus L_3 \oplus \ldots \oplus L_k \tag{1.16}$$

where

$$L_{u+1} = B'_{u+1} \oplus A_F B'_{u+1} \oplus \ldots \oplus A_F^{u-1} B'_{u+1}$$

with

$$B'_{u+1} \subset B_u \quad , \quad u \in \{1,2,\ldots,k-2\}$$

and

$$L_k = B_{k-1} \oplus A_F B_{k-1} \oplus \ldots \oplus A_F^{k-2} B_{k-1} .$$

Furthermore, the map $F$ can be chosen so that

$$BF\ L_2 \subset B'_1 \tag{1.17}$$

and

$$BF\mathcal{L}_{u+1} \subset \mathcal{B}'_1 + \mathcal{B}'_2 + \ldots + \mathcal{B}'_{u-1},\ u \in \{2,3,\ldots,k-1\} \qquad (1.18)$$

Proof: The proof of a) is absolutely identical to that in [1.17 page 106]. The result b) is easily proven by induction on u. To this end suppose that

$$\hat{\mathcal{B}}_1 + A_F\, \hat{\mathcal{B}}_2 + \ldots + A_F^{u-2}\, \hat{\mathcal{B}}_{u-1} \subset \bar{\mathcal{R}}_a^{u-1}$$

for some chain $\{\hat{\mathcal{B}}_i\}$ in $\bar{\mathcal{B}}$. Hence

$$\hat{\mathcal{B}}_1 + A_F\, \hat{\mathcal{B}}_2 + \ldots + A_F^{u-1}\, \hat{\mathcal{B}}_u \subset K \cap (\hat{\mathcal{B}}_1 + A_F\, \bar{\mathcal{R}}_a^{u-1}) \subset K \cap (A\bar{\mathcal{R}}_a^{u-1} + \bar{\mathcal{B}}) = \bar{\mathcal{R}}_a^u .$$

We now show that the subspace $\bar{\mathcal{R}}_a^u$ can itself be written as

$$\bar{\mathcal{R}}_a^u = \mathcal{B}_1 \oplus A_F\mathcal{B}_2 \oplus \ldots \oplus A_F^{u-1}\mathcal{B}_u .$$

The proof is inductive and it also yields the results described in c). Some of the steps involved in the proof are similar to those used in the proof of the second theorem in [1.13].

Consider the sequence (1.13) for u = 1 and let $\bar{\mathcal{R}}_a^1 = \mathcal{B}_1 :=$ $K \cap \bar{\mathcal{B}}$. Let $\mathcal{B}'_1$ be any subspace such that

$$\mathcal{B}'_1 \oplus \mathcal{B}_1 = \mathcal{B}_0 := \bar{\mathcal{B}}$$

and let $\{b_i\}$ , $i \in \underline{\ell}_1$, be a set of linearly independent vectors which span $\mathcal{B}_1$. Write

$$x_{i,1} := b_i \quad , \quad i \in \underline{\ell}_1 .$$

For $u = 2$, $\bar{R}_a^2 = K \cap (A\bar{R}_a^1 + \bar{\mathcal{B}}) = K \cap (\bar{R}_a^1 + A\mathcal{B}_1 + \mathcal{B}_1')$. Since $\bar{R}_a^1 \subset K$, there exist linearly independent vectors $x_{i,2}$, $i \in \underline{\ell}_2$ such that $\bar{R}_a^2 = \bar{R}_a^1 \oplus \mathrm{span}\{x_{i,2}\}$. Since $K \cap \mathcal{B}_1' = 0$, the vectors $x_{i,2}$ have the form

$$x_{i,2} = A\hat{b}_i + b'_{i,1} \quad , \; i \in \underline{\ell}_2 \tag{1.19}$$

where the $\hat{b}_i \in \mathcal{B}_1$, $i \in \underline{\ell}_2$, are linearly independent and $b'_{i,1} \in \mathcal{B}_1'$.

Let

$$\mathcal{B}_2 := \mathrm{span}\{\hat{b}_i\} \; , \quad \mathcal{B}_2 \subset \mathcal{B}_1$$

and define $F : \mathcal{B}_2 \to \mathcal{U}$ such that

$$BF\,\hat{b}_i = b'_{i,1} \; . \tag{1.20}$$

Then from (1.19)

$$x_{i,2} = (A+BF)\hat{b}_i \; .$$

Let $\mathcal{B}_2'$ be any subspace such that

$$\mathcal{B}_2' \oplus \mathcal{B}_2 = \mathcal{B}_1 .$$

Thus

$$\bar{R}_a^2 = \mathcal{B}_1 \oplus A_F\mathcal{B}_2 = L_2 \oplus L_3$$

with

$$L_2 := \mathcal{B}_2'$$

$$L_3 := \mathcal{B}_2 \oplus A_F\mathcal{B}_2.$$

Note from (1.20) that $BF\mathcal{B}_2 \subset \mathcal{B}_1'$ and by the independence of $L_2$ and $L_3$ we can define $F$ on $L_2$ and on span $\{x_{i,2}\} = A_F\mathcal{B}_2$ so that

$$BFL_2 \subset \mathcal{B}_1'$$

and

$$BFL_3 \subset \mathcal{B}_1'$$

which proves (1.17) and (1.18) for u = 2.

Now assume that for u - 1

$$\bar{R}_a^{u-1} = \mathcal{B}_1 \oplus A_F\mathcal{B}_2 \oplus \dots \oplus A_F^{u-2}\mathcal{B}_{u-1} \qquad (1.21)$$

$$= L_2 \oplus L_3 \oplus \dots \oplus \hat{L}_u$$

with

$$L_{j+1} = \mathcal{B}_{j+1}' \oplus A_F\mathcal{B}_{j+1}' \oplus \dots \oplus A_F^{j-1}\mathcal{B}_{j+1}', \; j \in \{1,2,\dots,u-2\} \qquad (1.22)$$

$$\hat{L}_u = \mathcal{B}_{u-1} \oplus A_F\mathcal{B}_{u-1} \oplus \dots \oplus A_F^{u-2}\mathcal{B}_{u-1} \qquad (1.23)$$

$$BFL_2 \subset \mathcal{B}_1' \qquad (1.24)$$

$$BFL_{j+1} \subset \mathcal{B}_1' + \mathcal{B}_2' + \dots + \mathcal{B}_{j-1}', \; j \in \{2,3,\dots,u-1\} \qquad (1.25)$$

$$\mathcal{B}_j' \oplus \mathcal{B}_j = \mathcal{B}_{j-1} \ , \ j \in \{1,2,\ldots,u-1\}. \tag{1.26}$$

Now

$$\bar{\mathcal{R}}_a^u = \mathcal{K} \cap (A\bar{\mathcal{R}}_a^{u-1} + \bar{\mathcal{B}}) \tag{1.27}$$

and since $\bar{\mathcal{R}}_a^{u-1} \subset \bar{\mathcal{R}}_a^{k-1}$ and $\bar{\mathcal{R}}_a^{k-1}$ is a sliding subspace there follows by Proposition 1.1a that

$$\dim(A_{\hat{F}} \bar{\mathcal{R}}_a^{u-1} \oplus \bar{\mathcal{B}}) = \dim(A\bar{\mathcal{R}}_a^{u-1} \oplus \bar{\mathcal{B}}) \ , \ \forall \ \hat{F} : \mathcal{X} \to \mathcal{U} \tag{1.28}$$

But from (1.24-6)

$$BF\bar{\mathcal{R}}_a^{u-1} \subset \bar{\mathcal{B}} \quad .$$

Thus

$$A_F\bar{\mathcal{R}}_a^{u-1} \oplus \bar{\mathcal{B}} \subset A\bar{\mathcal{R}}_a^{u-1} \oplus \bar{\mathcal{B}} \ .$$

Taking into account (1.28) it follows that

$$A\bar{\mathcal{R}}_a^{u-1} \oplus \bar{\mathcal{B}} = A_F\bar{\mathcal{R}}_a^{u-1} \oplus \bar{\mathcal{B}} \ . \tag{1.29}$$

Hence (1.27) can be rewritten as

$$\begin{aligned} \bar{\mathcal{R}}_a^u &= \mathcal{K} \cap (A_F\bar{\mathcal{R}}_a^{u-1} \oplus \bar{\mathcal{B}}) \\ &= \mathcal{K} \cap (\bar{\mathcal{R}}_a^{u-1} + \mathcal{B}_1' + A_F\mathcal{B}_2' + \ldots + A_F^{u-2}\mathcal{B}_{u-1}' + A_F^{u-1}\ \mathcal{B}_{u-1}) . \end{aligned}$$

Since $\bar{R}_a^{u-1} \subset K$, there exist linearly independent vectors $x_{i,u}$, $i \in \underline{\ell}_u$ such that $\bar{R}_a^u = \bar{R}_a^{u-1} \oplus \text{span}\{x_{i,u}\}$ .

Since

$$K \cap (B_1' + A_F B_2' + \ldots + A_F^{u-2} B_{u-1}') = 0$$

the vectors $x_{i,u}$ have the form

$$x_{i,u} = A_F^{u-1} \tilde{b}_i + \sum_{j=1}^{u-1} A_F^{j-1} b_{i,j}' \quad , \; i \in \underline{\ell}_u$$

where the vectors $\tilde{b}_i \in B_{u-1}$ are linearly independent and $b_{i,j}' \in B_j'$.

Define the following vectors

$$x_{i,1} = \tilde{b}_i$$

$$x_{i,2} = b_{i,u-1}' + A_F \tilde{b}_i$$

$$\vdots$$

$$x_{i,u-1} = b_{i,2}' + A_F b_{i,3}' + \ldots + A_F^{u-2} \tilde{b}_i$$

$$x_{i,u} = b_{i,1}' + A_F b_{i,2}' + \ldots + A_F^{u-2} b_{i,u-1}' + A_F^{u-1} \tilde{b}_i$$

for $i \in \underline{\ell}_u$ and note that

$$x_{i,j} = A_F x_{i,j-1} + b_{i,u-j+1}' \; , \; i \in \underline{\ell}_u, \; j \in \underline{u} \quad .$$

Define F" on $\{x_{i,j}\}$, $i \in \underline{\ell}_u$, $j \in \{1,2,\ldots,u-1\}$ so that

$$BF'' x_{i,j} = b_{i,u-j+1}' \quad . \tag{1.30}$$

Thus

$$x_{i,j} = (A_F + BF")^{j-1} \tilde{b}_i \quad , \; i \in \underline{\ell}_u , \; j \in \underline{u} \; .$$

Let

$$\mathcal{B}_u := \text{span}\{\tilde{b}_i\} \quad , \; \mathcal{B}_u \subset \mathcal{B}_{u-1}$$

and let $\mathcal{B}'_u$ be any subspace such that

$$\mathcal{B}'_u \oplus \mathcal{B}_u = \mathcal{B}_{u-1} \quad .$$

Define F" on span $\{x_{i,u}\}$ , $i \in \underline{\ell}_u$ so that

$$BF" \; x_{i,u} \subset \mathcal{B}'_{u-1} \tag{1.31}$$

and finally define $F' : \bar{\mathcal{R}}^u_a \rightarrow \mathcal{U}$ by

$$F' := F + F" \text{ on span}\{x_{i,j}\} \; , \; i \in \underline{\ell}_u , \; j \in \{1,2,\ldots,u-1\}$$

$$F' := F" \text{ on span}\{x_{i,u}\} \; , \; i \in \underline{\ell}_u$$

$$F' := F \text{ on } L_2 \oplus L_3 \oplus \ldots \oplus L_{u-1} \oplus L_u$$

where

$$L_u := \mathcal{B}'_u \oplus A_F \mathcal{B}'_u \oplus \ldots \oplus A_F^{u-2} \mathcal{B}'_u .$$

It now follows that

$$\bar{R}_a^u = \mathcal{B}_1 \oplus A_{F'} \mathcal{B}_2 \oplus \ldots \oplus A_{F'}^{u-1} \mathcal{B}_u \tag{1.32}$$

$$= \qquad L_2 \oplus \ldots \oplus L_u \oplus L_{u+1}$$

with

$$L_{u+1} := \mathcal{B}_u \oplus A_{F'} \mathcal{B}_u \oplus \ldots \oplus A_{F'}^{u-1} \mathcal{B}_u \ .$$

It is not difficult to see from the inductive hypothesis (1.24-5), (1.30-1) and the definition of F', that

$$BF' L_2 \subset \mathcal{B}_1'$$

and

$$BF' L_{j+1} \subset \mathcal{B}_1' + \mathcal{B}_2' + \ldots + \mathcal{B}_{j-1}', \ j \in \{2,3,\ldots,u\}$$

which proves part c) of the theorem.

□

Corollary 1.1: Let $R_s \subset K$ be a sliding subspace of maximal dimension described by

$$R_s = \mathcal{B}_1 \oplus A_F \mathcal{B}_2 \oplus \ldots \oplus A_F^{u-1} \mathcal{B}_u$$

for some $F : X \to U$ and some chain $\mathcal{B}_i$ in $\bar{\mathcal{B}}$ , where $\bar{\mathcal{B}}$ is as in (1.11) . Then necessarily

$$BF(\mathcal{B}_1 \oplus A_F \mathcal{B}_2 \oplus \ldots \oplus A_F^{u-2} \mathcal{B}_{u-1}) \subset \bar{\mathcal{B}} \tag{1.33}$$

Proof: Just note that at the step u of the algorithm (1.13), (1.33) must hold in order to have the equality (1.28) and then write $\bar{\mathcal{R}}_a^u$ as in (1.32).

□

Remarks 1.1:

a) Note from (1.28) and Lemma 1.2e that

$$\begin{aligned} \bar{S}^u &= A_F \bar{\mathcal{R}}_a^{u-1} \oplus \bar{\mathcal{B}} \\ &= \bar{\mathcal{B}} \oplus A_F \mathcal{B}_1 \oplus \ldots \oplus A_F^{u-1} \mathcal{B}_{u-1} \end{aligned} \tag{1.34}$$

which shows that $\bar{S}^u$ also admits a state feedback representation.

b) Also note that we have defined F" on span $x_{i,u}$ (see (1.31)) to obtain the extra property (1.25) which is to be used in Chapter II. However, we need not have done this. From (1.30) it follows that with F" defined on $\{x_{i,j}\}$ , $i \in \underline{\ell}_u$, $j \in \{1,2,\ldots,u-1\}$ we have automatically

$$x_{i,u} = (A_F + BF")^{u-1} \tilde{b}_i .$$

Thus unless we wish some extra property, the conclusion is that to obtain the representation

$$\bar{\mathcal{R}}_a^u = \mathcal{B}_1 \oplus A_F \mathcal{B}_2 + \ldots + A_F^{u-1} \mathcal{B}_u$$

the map F need not be defined on $A_F^{u-1} \mathcal{B}_u = \text{span}\{x_{i,u}\}$.

Our next result is simple and concerns the existence of sliding subspaces of maximal dimension.

Theorem 1.5: There exists a sliding subspace of maximal dimension (n-m) if and only if the pair (A,B) is controllable.

Proof

=>) Let $\mathcal{R}_s$ be a sliding subspace of dimension n-m. Then by Proposition 1.1a, it follows that

$$A_F\mathcal{R}_s \oplus \mathcal{B} = \mathcal{X}$$

where

$$\mathcal{R}_s = \mathcal{B}_1 \oplus A_F\mathcal{B}_2 \oplus \dots \oplus A_F^{k-1}\mathcal{B}_k$$

for some chain $\{\mathcal{B}_i\}$ in $\mathcal{B}$, some $F : \mathcal{X} \to \mathcal{U}$ and some $k \in \underline{n}$. Since the controllable subspace is invariant under state feedback, it follows that the pair (A,B) is controllable.

<=) If the pair (A,B) is controllable, then the space $\mathcal{X}$ can be decomposed into a direct sum of m controllability subspaces $\mathcal{R}_i$, $\dim(\mathcal{R}_i) = k_i$, where the $k_i$, $i \in \underline{m}$, are the controllability indices of the pair (A,B). It is easy to see that if

$$\mathcal{X} = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \dots \oplus \mathcal{R}_m \tag{1.35}$$

where the

$$\mathcal{R}_i = b_i + A_F b_i + \dots + A_F^{k_i-1} b_i \quad , \; i \in \underline{m}$$

are controllability subspaces, then

$$\mathcal{R}_s = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots \oplus \mathcal{L}_m \tag{1.36a}$$

where

$$\mathcal{L}_i = b_i + A_F b_i + \ldots + A_F^{k_i - 2} b_i, \quad i \in \underline{m} \tag{1.36b}$$

is a sliding subspace of dimension n - m.

□

In the following we shall discuss some features of sliding subspaces of maximal dimension.

Let $\mathcal{R}_s$ be a sliding subspace of dimension n - m, i.e.

$$\mathcal{R}_s = \mathcal{B}_1 \oplus A_F \mathcal{B}_2 \oplus \ldots \oplus A_F^{k-1} \mathcal{B}_k \tag{1.37}$$

for some chain $\{\mathcal{B}_i\}$ in $\mathcal{B}$, some $F : \mathcal{X} \to \mathcal{U}$ and some $k \in \underline{n}$ .

It should be clear from Theorem 1.4c that $\mathcal{R}_s$ can also be written as

$$\mathcal{R}_s = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots \oplus \mathcal{L}_p \quad , \quad p = \dim \mathcal{B}_1 \tag{1.38a}$$

where

$$\mathcal{L}_i = b_i + A_F b_i + \ldots + A_F^{n_i - 2} b_i, \quad i \in \underline{p} \tag{1.38b}$$

for some set $\{n_i\}$ , $i \in \underline{p}$.

It is clear that

$$A_F \mathcal{R}_s \oplus \mathcal{B} = \mathcal{X} \tag{1.39}$$

$$= \mathcal{B} \oplus A_F \mathcal{B}_1 \oplus A_F^2 \mathcal{B}_2 \oplus \ldots \oplus A_F^k \mathcal{B}_k$$

$$= \mathcal{R}_{a_1} \oplus \mathcal{R}_{a_2} \oplus \ldots \oplus \mathcal{R}_{a_m} \tag{1.40}$$

where the

$$\mathcal{R}_{a_i} = b_i + A_F b_i + \ldots + A_F^{n_i-1} b_i, \quad i \in \underline{m} \tag{1.41}$$

are almost controllability subspaces. A precise definition of the set $\{n_i\}$, $i \in \underline{m}$, will be given in a moment.

By comparing (1.36) and (1.38) we see that we have two types of sliding subspaces of dimension n-m. The first one in (1.36) gives rise to a decomposition of the state space into m controllability subspaces. We call this type of sliding subspace a prime sliding subspace in connection with the definition of a prime system in [1.10]. The second type in (1.38) originates a decomposition of the state space into m almost controllability subspaces and is termed here a irreducible sliding subspace in connection with the definition of irreducible systems in [1.5].

Our next objective is to compare the sets $\{k_i\}$ and $\{n_i\}$, $i \in \underline{m}$. For this consider the algorithms (1.4) and (1.9) with $\mathcal{K} := \mathcal{R}_s$. Then from Remark 1.1, it follows that if at the step u-1 of the algorithm (1.4)

$$\mathcal{R}_a^{u-1} = \mathcal{B}_1 \oplus A_F \mathcal{B}_2 \oplus \ldots \oplus A_F^{u-2} \mathcal{B}_{u-1}$$

then at the step u of the algorithm (1.9)

$$\mathcal{S}^u = \mathcal{B} \oplus A_F \mathcal{B}_1 \oplus A_F^2 \mathcal{B}_2 \oplus \ldots \oplus A_F^{u-1} \mathcal{B}_{u-1} .$$

Let

$$\phi_0 := m, \quad \phi_u = \dim \left(\frac{S^{u+1}}{S^u}\right) \quad u \in \{1, 2, \ldots, n-1\} \ .$$

We have defined u up to n-1 to keep symmetry with the definition of the set $\{k_i\}$ which is to come.

Since $\mathcal{B} \supset \mathcal{B}_1 \supset \ldots \supset \mathcal{B}_k$, it follows that

$$\phi_0 \geq \phi_1 \geq \cdots \geq \phi_{n-1} \geq 0$$

and by (1.39)

$$\phi_0 + \phi_1 + \ldots + \phi_{n-1} = n.$$

Let

$n_i$ := number of integers in the set

$\{\phi_0, \phi_1, \ldots, \phi_{n-1}\}$ which are $\geq i$.

Then

$$n_1 \geq n_2 \geq \quad \ldots \geq n_m \geq 1$$

and

$$n_1 + n_2 + \ldots + n_m = n. \tag{1.42}$$

Note that the set $\{n_i\}$ , $i \in \underline{m}$ above defined is the same as that of (1.41).

We now review the way that the set of controllability indices $\{k_i\}$ , $i \in \underline{m}$ is defined [1.17]. For this consider the following sequence of subspaces

$$\psi^u = \mathcal{B} + A\mathcal{B} + \dots + A^u\mathcal{B}, \quad u \in \{1,2,\dots,n-1\} \text{ and let}$$

$$\rho_0 := m \qquad \rho_u = \dim\left(\frac{\psi^u}{\psi_{u-1}}\right), \; u \in \{1,2,\dots,n-1\}.$$

It is well known that

$$\rho_0 \geq \rho_1 \geq \dots \geq \rho_{n-1} \geq 0$$

and

$$\rho_0 + \rho_1 + \dots + \rho_{n-1} = n.$$

Let

$k_i$ := number of integers in the set

$\{\rho_0, \rho_1, \dots, \rho_{n-1}\}$ which are $\geq i$ .

Thus

$$k_1 \geq k_2 \geq \dots \geq k_m$$

with

$$k_1 + k_2 + \dots + k_m = n. \tag{1.43}$$

We are finally in a position to state our next result.

<u>Theorem 1.6</u>: The sets $\{k_i\}$ and $\{n_i\}$ are related by the following inequalities

$$\sum_{i=1}^{j} k_{m-i+1} \geq \sum_{i=1}^{j} n_{m-i+1} \quad , \quad j \in \underline{m} \tag{1.44}$$

with equality holding for j = m.

<u>Proof</u>: The equality at j = m follows from (1.42-3). Now, note that

$$\mathcal{S}^2 = A(\mathcal{B} \cap \mathcal{R}_s) + \mathcal{B} \subset A\mathcal{B} + \mathcal{B} = \psi^1$$

and if $\mathcal{S}^u \subset \psi^{u-1}$, then

$$\mathcal{S}^{u+1} = A(\mathcal{S}^u \cap \mathcal{R}_s) + \mathcal{B} \subset A(\psi^{u-1} \cap \mathcal{R}_s) + \mathcal{B} \subset A\psi^{u-1} + \mathcal{B} = \psi^u .$$

Therefore

$$\sum_{i=0}^{\ell} \phi_i \leq \sum_{i=0}^{\ell} \rho_i \quad , \quad \ell \in \{1,2,\ldots,n-1\} . \tag{1.45}$$

As a direct consequence of the definitions of $k_i$ and $n_i$, $i \in \underline{m}$, it follows that the integers $\rho_u$ and $\phi_u$ can be written in the following form

$$\rho_u = \begin{cases} m, u = 0,1,\ldots,k_m-1 \\ m-1, u = k_m,\ldots,k_{m-1}-1 \\ \vdots \\ 1, \ u = k_2,\ldots,k_1-1 \\ 0, \ u = k_2,\ldots,n-1 \end{cases} \qquad \phi_u = \begin{cases} m, \ u = 0,1,\ldots,n_m-1 \\ m-1, u = n_m,\ldots,n_{m-1}-1 \\ \vdots \\ 1, \ u = n_2,\ldots,n_1-1 \\ 0, \ u = n_1,\ldots,n-1 \end{cases} \tag{1.46}$$

It follows immediately from the lists (1.46) and (1.45) that $k_m \geq n_m$. The proof now goes by induction. Thus, assume that (1.44) is true up to $j = m-\ell$, i.e., up to the inequality

$$k_m + k_{m-1} + \ldots + k_{\ell+1} \geq n_m + n_{m-1} + \ldots + n_{\ell+1} . \qquad (1.47)$$

We then have two cases :

i) $k_\ell \geq n_\ell$

Hence by (1.47)

$$k_m + k_{m-1} + \ldots + k_{\ell+1} + k_\ell \geq n_m + n_{m-1} + \ldots + n_{\ell+1} + k_\ell$$
$$\geq n_m + m_{m-1} + \ldots + n_{\ell+1} + n_\ell .$$

ii) $k_\ell < n_\ell$

Suppose that

$$k_m + k_{m-1} + \ldots + k_\ell < n_m + n_{m-1} + \ldots + n_\ell .$$

This implies

$$k_m + k_{m-1} + \ldots + k_\ell + (\ell-1) n_\ell < n_m + n_{m-1} + \ldots + \ell n_\ell$$

which is equivalent to

$$mk_m + (m-1)(k_{m-1}-k_m) + (m-2)(k_{m-2} - k_{m-1}) + \ldots + \ell(k_\ell - k_{\ell+1})$$
$$+ (\ell-1)(n_\ell - k_\ell)$$

$$< m\, n_m + (m-1)(n_{m-1}-n_m) + (m-2)(n_{m-2}-n_{m-1}) + \ldots + \ell(n_\ell - n_{\ell+1}).$$

The last inequality implies by the lists (1.46) that

$$\sum_{u=0}^{n_\ell-1} \rho_u < \sum_{u=0}^{n_\ell-1} \phi_u$$

which contradicts (1.45). The desired result then follows.

□

Comments: It is interesting to note that the set of inequalities (1.44) shows up in [1.11, Chapter 5, Theorem 4.2]. The set $\{n_i\}$ $i \in \underline{m}$, there, refers to the possible degrees of the invariant polynomials of the map $A + BF$, $\forall F : \mathcal{X} \to \mathcal{U}$, when the pair $(A,B)$ is controllable.

It will become clear from the results of Chapter II that if $C : \mathcal{X} \to \mathcal{Y}$ is a map with $\ker C = \mathcal{R}_s$ and $\mathcal{Y}$ is an arbitrary space of dimension $m$, then the set $\{n_i\}$, $i \in \underline{m}$ constitutes the set of infinite-zeros of the transfer matrix $C(sI-A)^{-1}B$.

Another way of presenting the set of inequalities (1.44) is as in [1.8]. It can be easily seen that such a set is equivalent to the set

$$\sum_{i=1}^{j} n_i \geq \sum_{i=1}^{j} k_i \quad , j \in \underline{m}$$

with equality at $j = m$.

### I.2.4 The System $\sum_\chi (\text{mod } \mathcal{R}_s)$

In this section we describe some features of the quotient space

$X/R_s$, where $\underset{\sim}{R}_s$ is, as usual, a sliding subspace.

Let $\sum_{\chi}$ be the set of trajectories generated by $\sum$ which are infinitely differentiable, i.e. $\tilde{\sum}_{\chi} := \{\underline{x} : R \to X | \underline{x} \in C^{\infty}$ and $\dot{\underline{x}}(t) - A\underline{x}(t) \in B, \forall t\}$.

Let $N$ be a subspace of $X$ and consider $\tilde{\sum}_{\chi}(\text{mod } N)$, i.e. $\{\underline{x}' \in \tilde{\sum}_{\chi} \ (\text{mod } N)\} \iff \{\exists \underline{x} \in \tilde{\sum}_{\chi} | \underline{x}'(t) = \underline{x}(t) \ (\text{mod } N), \forall t\}$.

Theorem A in [1.14] states that there exists a pair of maps $(\hat{A},\hat{B})$ such that $\tilde{\sum}_{\chi}(\hat{A},\hat{B}) = \tilde{\sum}_{\chi} \ (\text{mod } N)$ if and only if $N \in V_a$.

The map $\hat{B}$ is the insertion map in $X$ of the subspace

$$\hat{B} = (AN + B) \ (\text{mod}N). \tag{1.48a}$$

To describe the map $\hat{A}$, consider any subspace $W$ of $X$ such that $W \oplus N = X$. Then $\hat{A} : W \to W$ with

$$\hat{A}w := A(w,0) \ \text{mod}(N). \tag{1.48b}$$

It is well known [1.17] that if $V \in \underline{V}$, then there exists a pair $(\bar{A},\bar{B})$ such that $\sum_{\chi}(\bar{A},\bar{B}) = \sum_{\chi}(\text{mod } V)$ with

$$\bar{A} := A_F \ \text{mod}(V) \quad , \quad \forall F \in F(V)$$

and

$$\bar{B} = B(\text{mod}V).$$

Moreover, if the pair (A,B) is controllable, then the quotient space $\bar{X} := X/V$ is controllable, i.e. $\langle\bar{A}|\bar{B}\rangle = \bar{X}$.

One of the aims of this section is to show that if the pair (A,B) is controllable and $\mathcal{N} := \mathcal{R}_s$ , then the quotient space $\hat{\mathcal{X}} := \mathcal{X}/\mathcal{R}_s$ is controllable, i.e. $<\hat{A}|\hat{\mathcal{B}}> = \hat{\mathcal{X}}$.

First note that

$$\mathcal{S} := A\mathcal{R}_s \oplus \mathcal{B} \supset \mathcal{R}_s \ .$$

To see this, let $\mathcal{K} = \mathcal{R}_s$ in algorithm (1.4). Then, obviously $\mathcal{R}^*_{a,\mathcal{K}} = \mathcal{R}_s$ and the claim follows on noting that by Lemma 1.1d

$$\mathcal{S} \cap \mathcal{R}_s = \mathcal{R}_s \ .$$

Now let $\tilde{\mathcal{S}} \subset \mathcal{S}$ be any subspace such that

$$\mathcal{S} = \tilde{\mathcal{S}} \oplus \mathcal{R}_s \tag{1.49}$$

and let $\tilde{S} : \tilde{\mathcal{S}} \to \mathcal{X}$ and $R_s : \mathcal{R}_s \to \mathcal{X}$ be the insertion maps of $\tilde{\mathcal{S}}$ and $\mathcal{R}_s$ in $\mathcal{X}$.

<u>Proposition 1.2</u>: Let (A,B) be a controllable pair, let $\mathcal{N} = \mathcal{R}_s$ be a sliding subspace and consider the pair $(\hat{A},\hat{B})$ defined in (1.48). Then :

a) $<\hat{A}|\hat{\mathcal{B}}> = \hat{\mathcal{X}}$

b) Consider any subspace $\mathcal{W}$ such that $\mathcal{W} \oplus \mathcal{R}_s = \mathcal{X}$. Then, there exist maps $Z_1$ and $Z_2$ such that

$$(A + \tilde{S}Z_1 + R_s\ Z_2)\mathcal{W} \subset \mathcal{W}$$

and

$$\sigma[(A + \tilde{S}Z_1 + R_s Z_2)|W] = \Lambda$$

where $\Lambda$ is a set of dim $\mathcal{W}$ symmetric complex numbers.

Proof:

a) The proof of this part is similar to the proof of Lemma 3.5 in [1.17]. Let $\mathcal{W}$ be as described above and let $P : \mathcal{X} \rightarrow \mathcal{X}$ be a projection on $\mathcal{W}$ along $\mathcal{R}_s$. We shall show that

$$<PA|P(A\mathcal{R}_s \oplus \mathcal{B})> = \mathcal{W}.$$

For this it is enough to verify that

$$x^T(\mathcal{R}_s \oplus <PA|P(A\mathcal{R}_s \oplus \mathcal{B})>) = 0$$

implies $x^T = 0$, for all $x^T \in \mathcal{X}'$.

Now

$$x^T \mathcal{R}_s = 0 \Rightarrow x^T(I-P) = 0 \Rightarrow x^T = x^TP$$

$$x^TP(A\mathcal{R}_s \oplus \mathcal{B}) = 0 \Rightarrow x^T(A\mathcal{R}_s \oplus \mathcal{B}) = 0 \Rightarrow \left\{ \begin{array}{l} x^T\mathcal{B} = 0 \\ x^TA\mathcal{R}_s = 0 \end{array} \right.$$

which implies $x^TAP = x^TA$.

Similarly

$$x^T PAP(A\mathcal{R}_s \oplus \mathcal{B}) = 0 \Rightarrow x^T A(A\mathcal{R}_s \oplus \mathcal{B}) = 0 \Rightarrow \left.\begin{array}{l} x^T A\mathcal{B} = 0 \\ x^T A^2 \mathcal{R}_s = 0 \end{array}\right.$$

By induction on $i$ we have $x^T A^{i-1}\mathcal{B} = 0$, $i \in \underline{n}$, i.e. $x^T\langle A|\mathcal{B}\rangle = 0$ and hence $x^T = 0$. The result follows on identifying $PA|\mathcal{W}$ and $P(A\mathcal{R}_s + \mathcal{B})$ with $\hat{A}$ and $\hat{\mathcal{B}}$.

b) First note by (1.49) that

$$\hat{\mathcal{B}} = P(A\mathcal{R}_s \oplus \mathcal{B}) = P\mathcal{S} = P\tilde{\mathcal{S}}$$

From part a), the pair $(PAP, P\tilde{S})$ is controllable. Thus there exists a map $Z_1 : \mathcal{W} \to \tilde{S}$ such that

$$\sigma[(PAP + P\tilde{S}Z_1)|\mathcal{W}] = \Lambda . \qquad (*)$$

Let $M : \mathcal{X} \to \mathcal{R}_s$ be the projection on $\mathcal{R}_s$ along $\mathcal{W}$. It is clear that $MR_s$ is nonsingular : thus there exists a map $Z_2$ such that

$$MAP + M\tilde{S}Z_1 + MR_sZ_2 = 0 \qquad (1.50a)$$

and note that since $PR_s = 0$, $(*)$ is equivalent to

$$\sigma[P(AP + \tilde{S}Z_1 + R_s Z_2)|\mathcal{W}] = \Lambda . \qquad (1.50b)$$

From (1.50) the result b) follows.

□

### I.2.5 Finding a Controlled Invariant Subspace With Pre-Assigned Spectrum as a Complement of an Almost Invariant Subspace

Consider any given almost controllability subspace $\mathcal{R}_a$. Trentelman [1.12] proved a very interesting and important result concerning the existence of a controlled invariant subspace with arbitrary spectrum that complements $\mathcal{R}_a$. The statement of his result is as follows.

Theorem 1.7: Let (A,B) be controllable and let $\mathcal{R}_a$ be an almost controllability subspace. Suppose $\Lambda$ is a symmetric set of n - dim $\mathcal{R}_a$ complex numbers. There exists $\mathcal{V} \in \underline{\mathcal{V}}$ and $F \in F(\mathcal{V})$ such that

$$\mathcal{V} \oplus \mathcal{R}_a = \mathcal{X}$$

and

$$\sigma[(A+BF)\,|\,\mathcal{V}] = \Lambda \quad .$$

It can be shown that the above result also holds for the $L_p$-almost controllability subspace $\mathcal{R}_b = A\mathcal{R}_a + \mathcal{B}$. The statement relative to $\mathcal{R}_b$ is identical to that of Theorem 1.7 when we replace $\mathcal{R}_a$ by $\mathcal{R}_b$.

The above results can be further extended with a minor modification for the case of an almost invariant subspace $\mathcal{V}_a$ and the corresponding $L_p$-almost invariant subspace $\mathcal{V}_b = A\mathcal{V}_a + \mathcal{V}_a + \mathcal{B}$.

For this, let $\Lambda_z$ be the set of complex numbers given by (see Theorem 5.7 in [1.17]).

$$\Lambda_z := \sigma[((A+BF)|V^*_{V_a})(\text{mod } R^*_{V_a})] , \quad \forall F \in F(V^*_{V_a}) .$$

The above set can be considered as the set of finite-zeros of the transfer matrix of the system

$$\dot{x} = Ax + Bu$$

$$y = x(\text{mod } V_a) .$$

We then obtain the following result.

Theorem 1.8: Let (A,B) be controllable and consider the subspaces $V_a$ and $V_b$ as described above. Let $\Lambda$ be a symmetric set of $n - \dim V_a(V_b)$ complex numbers such that $\Lambda \cap \Lambda_z = \phi$. There exist $V \in \underline{V}$ and $F \in F(V)$ such that

$$V \oplus V_a(V_b) = X$$

and

$$\sigma[(A+BF)|V] = \Lambda$$

Proof: We shall prove the statement concerning the subspace $V_b$. The proof for $V_a$ follows similar steps. The technique of proof follows that one of [1.12] and it makes use of two intermediate lemmas.

We first describe a decomposition of $V_b$. From (1.8)

$$V_b = R^*_{b,V_a} + V^*_{V_a} .$$

Using Lemma 1.2a,b we obtain

$$V_b = \bar{R}_{b,V_a} \oplus V^*_{V_a}$$

where $\bar{R}_{b,V_a}$ is given by the sequence (1.12) with $K := V_a$ and by Lemma 1.2c

$$\bar{R}_{b,V_a} = A\bar{R}_{a,V_a} \oplus \bar{B}$$

where $\bar{R}_{a,V_a}$ is given by the limit of the sequence (1.13) and $\bar{B}$ is as in (1.11), i.e.

$$B = \bar{B} \oplus B \cap V^*_{V_a} \quad .$$

From Remark 1.1 we also have that

$$\bar{R}_{b,V_a} = A_F \bar{R}_{a,V_a} \oplus \bar{B} \qquad (1.51)$$

where F is a map as given by Theorem 1.4.

Let $\bar{R}_{a,V_a} := \bar{R}^{k-1}_{a,V_a}$ for some $k \in \underline{n}$ in the sequence (1.13). Then from Theorem 1.4c and

$$\bar{R}_{a,V_a} = B_1 \oplus A_F B_2 \oplus \ldots \oplus A_F^{k-1} B_k$$

where $B_i$ is a chain in $\bar{B}$. Consider subspaces $B'_i$ as in (1.26), i.e.

$$B'_i \oplus B_i = B_{i-1}; \; B_0 := \bar{B}, \; i \in \underline{k} \quad .$$

Let $B'_{k+1} := B_k$ . Then by using (1.51) and (1.16) we obtain

$$\bar{R}_{b,V_a} = M_1 \oplus M_2 \oplus \ldots \oplus M_k \tag{1.52}$$

where

$$M_i = B_i' \oplus A_F B_i' \oplus \ldots \oplus A_F^{i-1} B_i' \quad , \quad i \in$$

Since $\bar{R}_{a,V_a} \cap V_{V_a}^* = 0$ it follows that the map $F$ can also be taken from the set $F(V_{V_a}^*)$. We can now state the following lemma.

Lemma 1.3: Let A, B, $V_b$, $\Lambda$ and F as above. Then there exists $D_0 \subset X$ such that

$$D_0 \oplus V_b = X$$

and

$$\sigma[P_{D_0} A_F | D_0] = \Lambda$$

where $P_{D_0} : X \to X$ is the projection on $D_0$ along $V_b$.

Proof: Let $D \subset X$ be any subspace such that $D \oplus V_b = X$ and consider a map $Q : V_b \to D$ defined by $v \to P_D A_F v$ , where $P_D$ is the projection on $D$ along $V_b$.

Consider the pair $(P_D A_F, Q)$ and suppose it is not controllable. Then there exists a subspace $W \subset D$ , $W \neq D$ such that

$$P_D A_F W \subset W \quad , \quad W \supset \text{Im}Q$$

which implies

$$A_F W \subset W \oplus V_b \qquad A_F V_b \subset W \oplus V_b$$

whence

$$A_F(W \oplus V_b) \subset W \oplus V_b$$

with

$$\mathcal{W} \oplus \mathcal{V}_b \neq \mathcal{X} \text{ and } \mathcal{W} \oplus \mathcal{V}_b \supset \mathcal{B}$$

which in turn implies that the pair (A,B) is not controllable which is a contradiction. Thus the pair $(P_{\mathcal{D}} A_F, Q)$ is controllable and there exists $Z : \mathcal{D} \to \mathcal{V}_b$ such that

$$\sigma[P_{\mathcal{D}} A_F|\mathcal{D} + QZ] = \Lambda.$$

Let $S : \mathcal{X} \to \mathcal{X}$ be a map defined by

$$S|\mathcal{D} = I_{\mathcal{D}} + P_{\mathcal{V}_b} Z \quad ; \quad S|\mathcal{V}_b = I_{\mathcal{V}_b}$$

where $P_{\mathcal{V}_b}$ is the projection on $\mathcal{V}_b$ along $\mathcal{D}$.

Define $\mathcal{D}_0 := S\mathcal{D}$ and note that $\mathcal{D}_0 \oplus \mathcal{V}_b = \mathcal{X}$. Let $P_{\mathcal{D}_0}$ be the projection on $\mathcal{D}_0$ along $\mathcal{V}_b$. Then since $P_{\mathcal{D}_0} = SP_{\mathcal{D}}$, it follows that the diagram below commutes.

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{P_{\mathcal{D}} A_F + P_{\mathcal{D}} A_F P_{\mathcal{V}_b} Z} & \mathcal{D} \\
S \downarrow & & \downarrow S \\
\mathcal{D}_0 & \xrightarrow{P_{\mathcal{D}_0} A_F|\mathcal{D}_0} & \mathcal{D}_0
\end{array}$$

The lemma follows on noting that $S$ is an isomorphism between $\mathcal{D}$ and $\mathcal{D}_0$. □

The proof of the theorem still requires a further lemma.

Lemma 1.4: There exists a map T and a map $F_1$ such that with $\mathcal{D}_1 := T\mathcal{D}_0$ the following relations hold :

a) $\mathcal{X} = \mathcal{D}_1 \oplus \mathcal{V}_b$

b) $\sigma[P_{\mathcal{D}_1}(A_F+BF_1)|\mathcal{D}_1] = \Lambda$, where $P_{\mathcal{D}_1}$ is the projection on $\mathcal{D}_1$ along $\mathcal{V}_b$.

c) $(A_F+BF_1)\mathcal{D}_1 \subset \mathcal{D}_1 \oplus \mathcal{V}^*_{\mathcal{V}_a}$

d) $(A_F+BF_1)|\mathcal{V}^*_{\mathcal{V}_a} = A_F|\mathcal{V}^*_{\mathcal{V}_a}$

Proof: The proof of the above lemma is absolutely analogous to the proof of Lemma 5.2 in [1.12] on considering the decomposition (1.52). □

Let $\Lambda_r$ be a symmetric set of dim $\mathcal{R}^*_{\mathcal{V}_a}$ complex numbers such that $\Lambda_r \cap \Lambda = 0$. Then [1.17] there exists $F_v \in F(\mathcal{V}^*_{\mathcal{V}_a})$, $F_v : \mathcal{X} \to \mathcal{V}$ such that

$$\sigma[(A_F+BF_v)|\mathcal{V}^*_{\mathcal{V}_a}] = \Lambda_r \cup \Lambda_z.$$

Since $(\Lambda_r \cup \Lambda_z) \cap \Lambda = 0$ it follows that there exists a map $J : \mathcal{D}_1 \to \mathcal{V}^*_{\mathcal{V}_a}$ which solves the following Silvester's equation (see [1.17]).

$$(A_F+BF_v)|\mathcal{V}^*_{\mathcal{V}_a} J - J P_{\mathcal{D}_1}(A_F+BF_1)|\mathcal{D}_1 - P_{\mathcal{V}^*}(A_F+BF_1)|\mathcal{D}_1 = 0$$

where $P_{\mathcal{V}^*}$ is the projection on $\mathcal{V}^*_{\mathcal{V}_a}$ along $\mathcal{D}_1$.

Define $F_2 : \mathcal{X} \to \mathcal{U}$ by $F_2|\mathcal{D}_1 = 0$ ; $F_2|\mathcal{V}^*_{\mathcal{V}_a} = F_v|\mathcal{V}^*_{\mathcal{V}_a}$ ; $F_2|\ddot{\mathcal{R}}_{b,\mathcal{V}_a} = 0$.

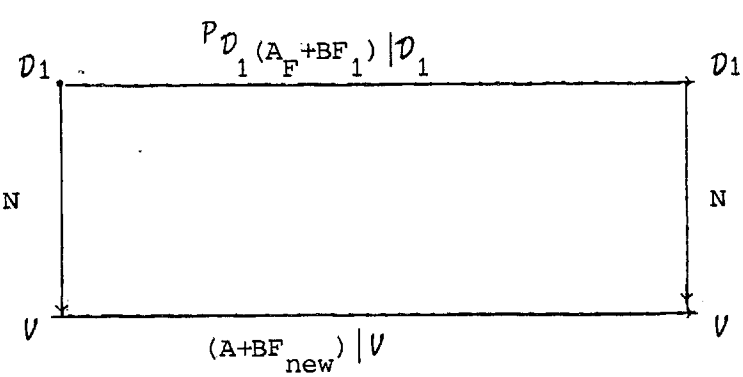
Also consider the map N defined by

$$N|\mathcal{D}_1 := I_{\mathcal{D}_1} - J \qquad , \qquad N|\mathcal{V}_b = I_{\mathcal{V}_b} \quad .$$

Finally take $\mathcal{V} = N\mathcal{D}_1$ and $F_{new} := F + F_1 + F_2$. Then clearly

$$\mathcal{X} = \mathcal{V} \oplus \mathcal{V}_b \quad .$$

Using Lemma 1.4c it is not difficult to see that the following diagram commutes .




It now follows that $(A+BF_{new})\,\mathcal{V} \subset \mathcal{V}$ and since N is an isomorphism between $\mathcal{D}_1$ and $\mathcal{V}$ , the conclusion is that $\sigma[A+BF_{new})|\mathcal{V}] = \Lambda$.

□

The result proved by Trentelman (Theorem 1.7) and the extension shown here (Theorem 1.8) are important because if we are required to approximate an almost invariant subspace (say $\dot{\mathcal{R}}_a$) by a sequence of controlled invariant subspaces, then we may fix $F|\mathcal{V}$, where $\mathcal{V}$ is as in Theorem 1.7, and a symmetric set $\Lambda$ of n-dim $\mathcal{R}_a$ complex numbers. It has been shown in [1.12] that there exists a sequence of subspaces $\mathcal{V}_\epsilon$ and a sequence of map $F_\epsilon : \mathcal{V}_\epsilon \to \mathcal{U}$ such that $\mathcal{V}_\epsilon \oplus \mathcal{V} = \mathcal{X}$, $(A+BF_\epsilon)\,\mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon$ and $\mathcal{V}_\epsilon \xrightarrow[\epsilon \to 0]{} \mathcal{R}_a$.

Moreover, in the approximation process the elements of the set $\Lambda_\epsilon := \sigma[(A+BF_\epsilon|V_\epsilon]$ go to infinity and the magnitude of the asymptotes can be chosen freely.

To conclude this section we point out that the existence of a controlled invariant subspace with pre-assigned spectrum that complements a subspace $J$ is generic in the following sense.

<u>Proposition 1.3</u>: Let $J$ be any subspace of $X$ and let the pair (A,B) be controllable. Consider a symmetric set $\Lambda$ of n-dim $J$ complex numbers. Then there exists a controlled invariant subspace $V$ and a map $F : X \to U$ such that

$$J \oplus V = X \text{ and } \sigma[(A+BF)|V] = \Lambda_\epsilon$$

where $\Lambda_\epsilon$ is a symmetric set of n-dim J complex numbers whose elements are arbitrarily close to the elements of the pre-specified set $\Lambda$.

<u>Proof</u>: Let $Z$ be an arbitrary space of dimension $\ell :=$ n-dim $J$ and define a map $H : X \to Z$ such that ker H = J and rank H = $\ell$.

We may assume that the pair (A,b) is controllable for some $b \in B$. Otherwise, there exists a map $F_0 : X \to U$ such that $(A+BF_0,b)$ is controllable for any $b \in B$ [1.17 Lemma 2.2].

Let $B\theta = b$, for some $\theta$. Then from the proof of the main result in [1.6], there exists a map $R^* : Z \to U$, with $R^* := \theta R$ for some $1 \times \ell$ vector R, and a subspace $V$ such that

$$(A+BR^*H)V \subset V \quad , \quad V \oplus \ker H = X$$

and

$$\sigma[(A+BR^*H)|V] = \Lambda_\epsilon \quad .$$

Set $F := R^*H$ and the result follows.

□

## I.2.6 Properties Of Coasting Subspaces

Coasting subspaces also play an important role in the theory of almost controlled invariant subspaces. In the proof of Theorem 6 in [1.14] is shown that a sliding subspace can be approximated by a sequence of coasting subspaces.

The next proposition consists of a collection of basic properties of coasting subspaces.

Proposition 1.4: Let $\underline{C}$ be the set of coasting subspaces. Then

a) $\max \dim C \in \underline{C} = n-m$

b) $(F_1-F_2)|C = 0, \quad \forall F_1, F_2 \in F(C), \quad \forall C \in \underline{C}$.

Now let the pair (A,B) be controllable. Then

c) For any symmetric set $\Lambda$ of $p$ complex numbers, $p \leq n-m$, there exists $C \in \underline{C}$ and a map $F : X \to U$ such that

$$\sigma[(A+BF)|C] = \Lambda$$

d) For any $C \in \underline{C}$ and for any $F \in F(C)$, the pair $(\overline{A+BF}, \bar{B})$ induced in $X/C$ is controllable. The controllability indices $\bar{k}_i$, $i \in \underline{m}$, of $(\overline{A+BF}, \bar{B})$ are the same for all $F \in F(C)$ and $\bar{k}_i \leq k_i$, $i \in \underline{m}$, where the $k_i$ are the controllability indices of the pair (A,B).

Proof:

a) From Definition 1.6, $\mathcal{R}_{\mathcal{C}}^{*} = 0$, which implies $\mathcal{C} \cap \mathcal{B} = 0$ and the result follows.

b) See [1.17, page 88].

c) Let $\Lambda_1$ be a symmetric set of n-m complex numbers such that

$$\Lambda_1 = \Lambda \cup \Lambda_2$$

where $\Lambda_2$ is also symmetric set of complex numbers with

$$\Lambda \cap \Lambda_2 = \phi \ .$$

By Lemma 3.5 in [1.17], there exists $\mathcal{C}_1 \in \underset{\sim}{\mathcal{C}}$ and $F : \mathcal{X} \to \mathcal{U}$ such that

$$\mathcal{C}_1 \oplus \mathcal{B} = \mathcal{X}$$

$$(A+BF)\mathcal{C}_1 \subset \mathcal{C}_1$$

and

$$\sigma[(A+BF)|\mathcal{C}_1] = \Lambda_1 \ .$$

Let C be the span of the generalized eigenvectors associated with $\Lambda$. Then

$$(A+BF)\mathcal{C} \subset \mathcal{C}$$

and

$$\sigma[(A+BF)|\mathcal{C}] = \Lambda$$

d) For the first statement of this part, see Proposition 4.1 in [1.17].

Since $\mathcal{C} \cap \mathcal{B} = 0$, it follows that dim $\bar{\mathcal{B}} = m$. The controllability indices of the pair $(\overline{A+BF}, \bar{B})$ are determined from the numbers (see Section 2.3).

$$\bar{\rho}_0 := m \ , \ \bar{\rho}_u = \dim \left(\frac{\bar{\phi}^u}{\bar{\phi}^{u-1}}\right) \quad , \ u \in \{1,2,\ldots,n-1\} \tag{1.53}$$

where

$$\bar{\phi}^u = \bar{\mathcal{B}} + (\overline{A+BF})\bar{\mathcal{B}} + \ldots + (\overline{A+BF})^u\bar{\mathcal{B}} .$$

Let $P : X/\mathcal{C}$ be the canonical projection. Since $(\overline{A+BF})P = P(A+BF)$ and $\bar{\mathcal{B}} = P\mathcal{B}$ , it follows that

$$\bar{\phi}^u = P\phi^u$$

where

$$\phi^u = \mathcal{B} + (A+BF)\mathcal{B} + \ldots + (A+BF)^u\mathcal{B}$$
$$= \mathcal{B} + A\mathcal{B} + \ldots + A^u\mathcal{B}.$$

Thus

$$\bar{\rho}_u = \rho_u - \dim\left(\frac{\phi^u \cap \mathcal{C}}{\phi^{u-1} \cap \mathcal{C}}\right) \tag{1.54}$$

where

$$\rho_0 := m \quad \rho_u = \dim\left(\frac{\phi^u}{\phi^{u-1}}\right), \ u \in \{1,2,\ldots,n-1\}$$

Let $\bar{k}_i$ be the number of integers in the set $\{\bar{\rho}_0, \bar{\rho}_1, \ldots, \bar{\rho}_{n-1}\}$ which are $\geq i$. Thus $\bar{k}_1 \geq \bar{k}_2 \geq \ldots \geq \bar{k}_m \geq 1$ and $\sum_{i=1}^{m} \bar{k}_i = \dim X/C$.

From (1.53) it follows that $\bar{\rho}_u \leq \rho_u$, which implies $\bar{k}_i \leq k_i$, $i \in \underline{m}$.

□

Part d) of the above proposition obviously holds with a minor alteration if we replace $\underline{C}$ by the set $\underline{V}$. Instead of having $\bar{\rho}_0 = m$ in (1.53) we shall have $\bar{\rho}_0 = p$, where $p := \dim \frac{V+B}{B} \leq m$. (1.54) still holds and the conclusion is $\bar{k}_i \leq k_i$, $i \in \underline{p}$.

Now let the pair (A,B) be controllable again and consider any $V \in \underline{V}$. Let $(\overline{A+BF})$ and $\bar{B}$ be the maps induced in $X/V$ by A+BF and B, $\forall F \in F(V)$. A natural question then arises : what is the nature of the subspace that complements $V$ to $X$ so as to have $\overline{A+BF}$ and $\bar{B}$ in Brunovsky canonical form? The answer is provided by the following proposition.

<u>Proposition 1.5:</u> Let $V$ be a controlled invariant subspace and the pair (A,B) be controllable. Let $p := \dim \bar{B}$. Then there are almost controllability subspaces $R_{a_i}$, $i \in \underline{p}$, given by

$$R_{a_i} = b_i + A_F b_i + \ldots + A_F^{\bar{k}_i - 1} b_i$$

such that

$$X = V \oplus R_{a_1} \oplus \ldots \oplus R_{a_p}$$

and the pair $(\overline{A+BF}, \bar{B})$ is in Brunovsky canonical form.

Proof: We shall just sketch the proof which is based on [1.1]. Let $\tilde{\mathcal{B}}$ be any subspace such that

$$\mathcal{B} = \tilde{\mathcal{B}} \oplus \mathcal{B} \cap \mathcal{V}.$$

Since $A^i(\mathcal{B} \cap \mathcal{V}) \subset \mathcal{V} + \tilde{\mathcal{B}} + A\tilde{\mathcal{B}} +\ldots+ A^{i-1}\tilde{\mathcal{B}}$ , $i \in \underline{n}$, it follows that

$$\mathcal{X} = \mathcal{V} + \langle A|\mathcal{B}\rangle = \mathcal{V} + \langle A|\tilde{\mathcal{B}}\rangle.$$

Now construct a basis as in Brunovisky canonical form see[1.17, page 120]. For this let $\{\tilde{b}_i\}$, $i\in\underline{p}$ be a basis for $\tilde{\mathcal{B}}$ and, $\{v_i\}$, $i \in \underline{r}$, be a basis for $\mathcal{V}$, with $r := \dim\mathcal{V}$. Then write down the list

$$\{v_1,\ldots,v_r;\ \tilde{b}_1,\ldots,\tilde{b}_p;\ldots;\ A^{n-1}\tilde{b}_1,\ldots,A^{n-1}\tilde{b}_p\}$$

Proceeding as in [1.17] it is then possible to obtain subspaces $\mathcal{R}_{a_i}$, $i\in\underline{p}$ given by

$$\mathcal{R}_{a_i} = b_i + A_F b_i +\ldots+ A_F^{\bar{k}_i-1} b_i \qquad , \qquad b_i \in \tilde{\mathcal{B}}$$

$$\mathcal{X} = \mathcal{V} \oplus \mathcal{R}_{a_1} +\ldots\oplus \mathcal{R}_{a_p} \tag{1.55}$$

where the map $F$ is defined on $\mathcal{R}_{a_i}$ and has the property that

$$A_F^{\bar{k}_i} b_i \in \mathcal{V}. \tag{1.56}$$

From (1.55), it follows that $F$ can be taken from $F(\mathcal{V})$. By taking (1.55) as basis for $\mathcal{X}$ and using (1.56), it follows that the pair $(\overline{A+BF},\bar{B})$ is in Brunovsky canonical form and therefore the integers $\bar{k}_i$, $i \in \underline{p}$, are the controllability indices of this pair.

□

## I.3 ALMOST CONDITIONALLY INVARIANT SUBSPACES

### 1.3.1 Basic Concepts

In this Section we present a summary of the main concepts introduced in [1.15], which will be used in Chapter IV in connection with the construction of observers which make use of differentiators.

The notion of a conditionally invariant subspace is dual to that of a controlled invariant subspace and it can be introduced this way [1.10, 1.17]. However, as remarked in [1.15], it is more natural from the linear systems theory point of view to introduce it in the context of observer design.

To this end, consider the system

$$\Sigma' : \dot{x} = Ax \; ; \; y = Cx \tag{1.57}$$

where

$$x \in \mathcal{X} := \mathbb{R}^n \; ; \; y \in \mathcal{Y} := \mathbb{R}^r$$

and observers with the form

$$\Sigma^{obs} : \dot{w} = Kw + Ly \tag{1.58}$$

with

$$w \in \mathcal{W} := \mathbb{R}^{\ell} .$$

Definition 1.7: A subspace $\mathcal{S} \subset \mathcal{X}$ is said to be conditionally invariant if there exist matrices K, L such that $\underline{w}(0) = \underline{x}(0) \pmod{S}$ yields $\underline{w}(t) = \underline{x}(t) \pmod{\mathcal{S}}$, $t \in \mathbb{R}$.

The above definition shows that associated with a conditionally invariant subspace there is an observer which reconstructs $x \pmod{\mathcal{S}}$ from the observations y.

The following proposition establishes the connection between conditional invariance and $A|\ker C$ invariance [1.15-16]. Let $A^{L'}$ denote $A+L'C$, i.e., the result of output injection $L'y$ in $\Sigma'$.

Proposition 1.6: The following statements are equivalent :

1) $\mathcal{S}$ is a conditionally invariant subspace.

2) $\mathcal{S}$ is $A|\ker C$ invariant (i.e., $A(\mathcal{S} \cap \ker C) \subset \mathcal{S}$).

3) There exists $L' : \mathcal{Y} \to \mathcal{X}$ such that $A^{L'}\mathcal{S} \subset \mathcal{S}$.

Let $\mathcal{S}$ be a conditionally invariant subspace and let L' be as in part 3 of the above proposition. Let $P : \mathcal{X} \to \mathcal{X}/\mathcal{S}$ be the canonical projection and consider $A^{L'} \pmod{\mathcal{S}}$ which is the unique map such that

$$A^{L'} \pmod{\mathcal{S}} P = PA^{L'} .$$

Hence

$$P\dot{x} = PAx$$
$$= P(A+L'C)x - PL'y$$

and then

$$\dot{x}(\text{mod } S) = A^{L'}(\text{mod } S)\ x(\text{mod } S) - L'(\text{mod } S)y \qquad (1.59)$$

where $PL' = L'(\text{mod } S) : Y \to X/S$.

Consider now the observer

$$\dot{w} = A^{L'}(\text{mod } S)w - L'(\text{mod } S)y \qquad (1.60)$$

for x(mod $S$) and define e := w - x(mod $S$). Then from (1.59) and (1.60) it follows that

$$\dot{e} = A^{L'}(\text{mod } S)e \qquad (1.61)$$

and if $\underline{w}(0) = \underline{x}(0)(\text{mod } S)$, then $\underline{e} = 0$, i.e., $\underline{w}(t) = \underline{x}(t)(\text{mod } S)$, $\forall t \in \mathbb{R}$.

The equation (1.60) shows that the observer for x(mod $S$) is completely specified by defining

$$K := A^{L'}(\text{mod } S), L := -L'(\text{mod } S) .$$

The almost version of definition 1.7 is as follows.

<u>Definition 1.8</u>: A subspace $S_a \subset X$ is said to be almost conditionally invariant if $\forall \underline{x}(0) \in S_a$ and $\epsilon > 0$ there exist K,L such that $\underline{w}(0) = 0$ yields $\| \underline{w}(t) - \underline{x}(t)(\text{mod } S_a) \| \leq \epsilon$ for $t \in \mathbb{R}^+$.

A very useful way of expressing the duality between (almost) conditionally invariant subspaces and (almost) controlled invariant subspaces is as follows.

Let $\mathcal{V}$ be a controlled invariant subspace and $V:\mathcal{V} \to \mathcal{X}$ be the canonical injection. Also let $Q_V : \mathcal{X} \to \mathcal{X}/\mathcal{V}$. Then the controlled invariance property can be expressed by the equation

$$Q_V(sI-A)^{-1}BU(s) = Q_V(sI-A)^{-1}V \tag{1.62}$$

which should be solved for U(s). From [1.17] we know that (1.62) has solution for $U(s) \in \mathbb{R}_+(s)$ if and only if $\mathcal{V}$ is controlled invariant.

Now let $\mathcal{S}$ be a conditionally invariant subspace and let $X(s) := (sI-K)^{-1}L$ be the transfer matrix of the observer (1.58). Let $S : \mathcal{S} \to \mathcal{X}$ be the canonical injection and $Q_S : \mathcal{X} \to \mathcal{X}/\mathcal{S}$ be the canonical projection.

Then with $\underline{w}(0) = 0$, the conditional invariance can be expressed by the following equation

$$X(s)C(sI-A)^{-1}S = Q_S(sI-A)^{-1}S \tag{1.63}$$

or

$$S^T(sI-A^T)^{-1}C^TX^T(s) = S^T(sI-A^T)^{-1}Q_S^T \tag{1.64}$$

where $S^T : \mathcal{X} \to \mathcal{X}/\mathcal{S}^\perp$ is the canonical projection and $Q_S^T : \mathcal{S}^\perp \to \mathcal{X}$ is the canonical injection.

We recognize immediately that equation (1.64) expresses the controlled invariance of the subspace $\mathcal{S}^\perp$ with respect to the system

$$\sum^{*} : \dot{x} = A^T x + C^T v .$$

Thus $S$ is a conditionally invariant relative to $\sum'$ if and only if $S^{\perp}$ is controlled invariant relative to $\sum^{*}$, a fact easily deducible from the equivalence $A(\ker C \cap S) \subset S \Leftrightarrow A^T S^{\perp} \subset S^{\perp} + \operatorname{Im} C^T$.

In this sense we consider the equation (1.63) as dual to the equation (1.62) and also by duality it follows that (1.63) has a solution $X(s) \in \mathbb{R}_+(s)$ if and only $S$ is a conditionally invariant subspace.

The same duality principle is applicable to the almost version [1.15] and the observer for $X/S_a$ which achieves

$$\| (sI-K)^{-1} L\, C(sI-A)^{-1} S_a - Q_{S_a} (sI-A)^{-1} S_a \| \leq \epsilon, \quad \forall\, \epsilon > 0$$

is a high-gain observer.

If we now allow X(s) to belong to $\mathbb{R}(s)$ we then obtain the following definition.

<u>Definition 1.9</u>: A subspace $S_D \subset X$ is said to be a distributionally conditionally invariant subspace if there exist K, $L_0, L_1, \ldots, L_n$ such that

$$\sum_1^{obs} : \dot{v} = Kv+Ly , \; w = v+L_0 y + L_1 y^{(1)} + \ldots + L_n y^{(n)}$$

with $x(0) = 0$ and $\underline{x}(0-) = 0$ yields for all $\underline{x}(0) \in S_D$, $\underline{w}(t) = \underline{x}(t)$ (mod $S_D$), $t \in \mathbb{R}^+$.

Note that the observer $\sum_1^{obs}$ is a P.I.D. type since it operates proportionally to the input y and also makes use of the integral of y and its derivatives.

Equation (1.61) shows that the error dynamics (for $\underline{e}(0) \neq 0$) are governed by the spectrum of $A^{L'}$ (mod $S$). It is thus important to identify the sub-class of the class of the conditionally invariant subspaces for which such spectrum is freely assignable.

Definition 1.10: A conditionally invariant subspace $N$ is said to be a complementary observability subspace if for any symmetric set of complex numbers $\Lambda$ there exist K,L as in Definition 1.7 such that $\sigma(K) = \Lambda$.

The corresponding notion for the almost case is described in the following definition.

Definition 1.11: An almost conditionally invariant subspace $N_a$ is said to be an almost complementary observability subspace if $\forall \underline{x}(0) \in N_a$, $\lambda \in \mathbb{R}$ and $\epsilon > 0$ there exist K, L such that $e := w - x$ (mod $N_a$) is of the form $\underline{e}(t) = e^{Kt}\underline{e}(0) + \underline{d}(t)$ with Re $\sigma(K) \leq \lambda$ and $\| \underline{d}(t) \| \leq \epsilon$ , $t \in \mathbb{R}^+$.

If we now allow X(s) in (1.63) to belong to R[s] we then get an extremely fast observer which is related to the following class of subspaces.

Definition 1.12: A distributionally conditionally invariant subspace $N_D$ is said to be a distributionally complementary observability subspace if there exist $L_0, L_1, \ldots, L_n$ such that $\Sigma_2^{obs} : w = L_0 y + L_1 y^{(1)} + \ldots + L_n y^{(n)}$ and $\underline{x}(0) \in N_D$ yields $\underline{w}(t) = \underline{x}(t)$ (mod $N_D$) for $t \in \mathbb{R}^+$, i.e, the estimation error is zero for $t \geq 0$.

In the following we denote $\underline{S}$, $\underline{S}_a$, $\underline{S}_D$, $\underline{N}$, $\underline{N}_a$ and $\underline{N}_D$ as the sets of conditionally invariant, almost conditionally invariant, etc., subspaces and $\underline{S}(L)$, $\underline{S}_a(L)$, $\underline{S}_D(L)$, $\underline{N}(L)$, $\underline{N}_a(L)$ and $\underline{N}_D(L)$ those containing the subspace $L \subset X$.

By dualization of Theorem 5 in [1.14] we have that $\underline{S}_a = \underline{S}_D$ and $\underline{N}_a = \underline{N}_D$.

In the sequel we state some dual results to those obtained for almost controlled invariant subspaces.

<u>Theorem 1.1'</u> (dual of Theorem 1.1): $\underline{S}$, $\underline{N}$, $\underline{S}_a$ and $\underline{N}_a$ are closed under subspace intersection. Consequently

$$\inf \underline{S}(L) := S_L^* \in \underline{S} \qquad \inf \underline{N}(L) := N_L^* \in \underline{N}$$

$$\inf \underline{S}_a(L) := S_{a,L}^* \in \underline{S}_a \qquad \inf \underline{N}_a(L) := N_{a,L}^* \in \underline{N}_a \quad .$$

Let $K := \ker C$ and consider $A^L := A+LC$ for some $L : Y \to X$. The next theorem gives the output injection characterizations of (almost) conditionally invariant subspaces.

<u>Theorem 1.2'</u> (dual of Theorem 1.2):

a) $\{S \in \underline{S}\} \iff \{\exists L \text{ such that } A^L S \subset S\} \iff \{A(K \cap S) \subset S\}$.

b) $\{N \in \underline{N}\} \iff \{\exists L \text{ and } K' \supset K \text{ such that } N = \langle K' | A^L \rangle\}$.

c) $\underline{S}_a = \underline{S} \cap \underline{N}_a$ , i.e. $\{S_a \in \underline{S}_a\} \iff \{\exists S \in \underline{S} \text{ and } N_a \in \underline{N}_a \text{ such that } S_a = S \cap N_a\}$

d) $\{N_a \in \underline{N}_a\} \iff \{\exists L$ and a chain $\{K_i\}$ *around* $K$ such that $N_a = K_1 \cap (A^L)^{-1} K_2 \cap \ldots \cap (A^L)^{-n+1} K_n\}$.

We shall sometimes use the notation $L(S)$ to denote the set of maps $L$ for which $A^L S \subset S$.

In the sequel we describe the sequences which yield the infimal subspaces of Theorem 1.1'.

$$S_L^* := S^{\text{codim } L+1};\ S^u = L + A(S^{u-1} \cap K),\ S^0 = 0 \qquad (1.65)$$

$$N_{a,L}^* := N_a^{\text{codim } L+1};\ N_a^u = L + (A^{-1}N_a^{u-1}) \cap K;\ N_a^0 = X \qquad (1.66)$$

$$N_L^* = S_L^* + N_{a,L}^* \qquad (1.67)$$

$$S_{a,L}^* = S_L^* \cap N_{a,L}^* \quad . \qquad (1.68)$$

The sequences (1.65-6) and the relations (1.67-8) are dual to (1.3-6) and can be derived from the duality principle explained previously. For example, $N_{a,L^\perp}^{*\perp}$ is the supremal almost controllability subspace relative to $\sum^*$ which is contained in $L^\perp$.

Now let H : $X \to X/L$ and consider an equation as (1.63) given by

$$X(s)C(sI-A)^{-1}x_0 = H(sI-A)^{-1}x_0$$

where $\quad x_0 \in X.$

The following question may be raised : for which points $x_0$ can we obtain a $L_p$- approximate solution $X_\epsilon(s) \in \mathbb{F}_+(s)$, $1 \le p \le \infty$? More formally we want to identify the points $x_0$ for which we can achieve

$$\| X_\epsilon(s)C(sI-A)^{-1}x_0 - H(sI-A)^{-1}x_0 \|_{L_p} \le \epsilon$$

for any $\epsilon > 0$, where $\| \quad \|_{L_p}$ denotes the $L_p$-norm.

The above consideration gives the motivation for the introduction of the analogs of the $L_p$-almost controlled invariant subspaces.

<u>Definition 1.13</u>: The $\sum'$-observation distance in the $L_p$-sense from a point $x_0 \in X$ to a subspace $L$ is given by $d'(x_0, L) := \inf_{K,L} \| \underline{w} - \underline{x}(\text{mod } L) \|_{(0,\infty)}$ where $\underline{w}(0) = 0$. The set $S^*_{p,L} := \{x_0 \in X | d'(x_0, L) = 0\}$ will be called the infimal $L_p$ - almost conditional invariant subspace "containing" $L$ and $N^*_{p,L} := N^*_{a,S^*_{p,L}}$ will be called the infimal $L_p$- almost complementary observability subspace "containing" $L$.

The subspace $S^*_{p,L}$ characterizes all the outputs $x(\text{mod } L)$ which are arbitrarily accurately, in the $L_p$ sense, reconstructible from y.

The relationship of the above subspaces with the subspace $S^*_{a,L}$ and $N^*_{a,L}$ is displayed in the next theorem.

<u>Theorem 1.3'</u> (dual of Theorem 1.3) :

a) $N^*_{\infty,L} = N^*_{a,L}$ and $S^*_{\infty,L} = S^*_{a,L}$

b) for $1 \leq p < \infty$ : $N^*_{p,L} = (A^{-1}N^*_{a,L}) \cap K$ and $S^*_{p,L} = N^*_{p,L} \cap S^*_L$ .

It should be noted that $S^*_{p,L}$ and $N^*_{p,L}$ need not contain $L$ and similarly to Theorem 1.3 we define

$$N^*_{b,L} := (A^{-1}N^*_{a,L}) \cap K \tag{1.69}$$

and

$$S^*_{b,L} := N^*_{b,L} \cap S^*_L = (A^{-1}S^*_{a,L}) \cap K \cap S^*_{a,L} = (A^{-1}S^*_{a,L}) \cap K \cap S^*_L \quad \ldots \quad (1.70)$$

The subspace $N^*_{b,L}$ is given by the following sequence,

$$N^*_{b,L} := V^{\mathrm{codim}L+1} \; ; \; V^u = K \cap A^{-1}(V^{u-1}+L) ; \; V^0 = X \, . \qquad (1.71)$$

We remark that for $L := B$ we obtain from (1.9) and (1.65), (1.3) and (1.70) that

$$R^*_{b,K} = S^*_B \qquad (1.72)$$

and

$$V^*_K = N^*_{b,B} \, . \qquad (1.73)$$

From (1.70), (1.72-3) and Lemma 1.1b it follows that

$$S^*_{b,B} = V^*_K \cap R^*_{b,K} = R^*_K \, . \qquad (1.74)$$

From (1.8) and (1.72-3) we also obtain

$$V^*_{b,K} = S^*_B + N^*_{b,B} = N^*_B \qquad (1.75)$$

where the last equality follows by Lemma 1.1' which is stated in the next section.

The relations (1.72-5) have appeared in the work by Malabre [1.9].

It should be clear from this section that (almost) controlled invariant subspaces and (almost) conditionally invariant subspaces play an important role in the solution of equations as (1.62-3). For a more detailed exposition on the object we refer to [1.15].

### I.3.2 Properties Of Almost Conditionally Invariant Subspaces

Consider the monotone nonincreasing sequence which yields $N_L^*$ [1.9].

$$N_L^* := N^{\mathrm{codim} S_L^* + 1} ; N^u = S_L^* + (A^{-1} N^{u-1}) \cap K ; N^0 = X . \qquad (1.76)$$

We then have the following relationships among the sequences (1.65-6), (1.71) and (1.76).

Lemma 1.1' (dual of Lemma 1.1) :

a) $V^u = (A^{-1} N_a^{u-1}) \cap K$

b) $V^u + S_L^* = N^u$

c) $N_a^u + S_L^* = N^u$

d) $V^u + L = N_a^u$ .

The dual concepts of coasting and sliding subspaces are described in the following definition.

Definition 1.14: A subspace $L_i \in \underline{S}$ is said to yield a locked-in observer if $N^*_{L_i} = X$. A subspace $I \in \underline{N}_a$ is said to yield an instantaneously acting observer if $S^*_I = X$.

For a subspace $L_i$ as in Definition 1.14 there exist unique K and L|Im C such that $\dot{w} = Kw + Ly$ is an observer for $X/L_i$, which implies that the error dynamics in (1.61) cannot be altered (the eigenvalues of $A^L(\text{mod } L_i)$ are fixed for all $L \in L(L_i)$).

Note that a subspace $I$ as in Definition 1.14 cannot be covered by any proper conditionally invariant subspace and as pointed out in [1.15] the instantaneously acting observer degenerates into a bank of differentiators when the degree of approximation of $X(\text{mod } I)$ becomes very tight. In terms of the equation (1.63) this simply means that

$$X(s)C(sI-A)^{-1}I = Q_i(sI-A)^{-1}I$$

in solvable for $X(s) \in \mathbb{R}[s]$, where $I : I \to X$ and $Q_i : X \to X/I$ are, respectively, the canonical injection and the canonical projection.

By dualizing Proposition 1.1a we have that if $I$ is a subspace as above then $I + A^L \ker C = X$, $\forall L : Y \to X$ whereas if $L_i$ yields a locked-in observer, then from a dual argument used in the proof of Proposition 1.3a we obtain $L_i + \ker C = X$. If rank $C = r$, then $CL_i = Y$ and min dim $I$ = min dim $L_i = r$.

Following a dual procedure to that which has led to Lemma 1.2 we can generate a subspace $I$ in the following way. Let $\bar{K} \supset K$ be any subspace such that

$$K = \bar{K} \cap (K+S^*_L) = K+\bar{K}\cap S^*_L \text{ , i.e., } \bar{K}\cap S^*_L \subset K \qquad (1.77)$$

with the independence relation

$$\bar{K} + S_L^* = X$$

and consider the subspaces $\bar{N}_{b,L}$ and $\bar{N}_{a,L}$ obtained by means of the following sequences.

$$\bar{N}_{b,L} := \bar{V}^{\text{codim}L+1};\ \bar{V}^u = \bar{K} \cap A^{-1}(\bar{V}^{u-1}+L)\ ;\ \bar{V}^0 = X \tag{1.78}$$

$$\bar{N}_{a,L} := \bar{N}_a^{\text{codim}L+1};\bar{N}_a^u = L + (A^{-1}\bar{N}_a^{u-1}) \cap \bar{K}:\ \bar{N}_a^0 = X\ . \tag{1.79}$$

Since $V^*_{\bar{N}_{a,L}^{\perp}} = 0 \iff S^*_{\bar{N}_{a,L}} = X$, then $\bar{N}_{a,L}$ is indeed a subspace which yields an instantaneously acting observer.

The following relationships among the sequences (1.65-6), (1.71), (1.76) and (1.78-9) may then be established.

<u>Lemma 1.2</u>'(dual of Lemma 1.2):

a) $V^u = \bar{V}^u \cap N^u$ with $\bar{V}^u + N^u = X$

b) $\bar{V}^u + S_L^* = X$

c) $\bar{V}^u = (A^{-1}\bar{N}_a^{u-1}) \cap \bar{K}$ with $A^{-1}\bar{N}_a^{u-1} + \bar{K} = X$

d) $\bar{N}_a^u = L + \bar{V}^u$

e) $N_a^u = \bar{N}_a^u \cap N^u$ with $\bar{N}_a^u + N^u = X$

f) $\bar{N}_a^u + S_L^* = X.$

In the following we show a decomposition of an almost conditionally invariant subspace $S_a$. Let $L := S_a$. Then from Lemma 1.2'e.

$$N^*_{a,S_a} = \bar{N}_{a,S_a} \cap N^*_{S_a} \quad \text{with } \bar{N}_{a,S_a} + N^*_{S_a} = X. \tag{1.80a}$$

By direct dualization we may also write

$$S^*_{S_a} = L_i \cap N^*_{S_a} \qquad \text{with } L_i + N^*_{S_a} = X \tag{1.80b}$$

and

$$L_i + \bar{N}_{a,S_a} = X \tag{1.80c}$$

for some subspace $L_i$ which yields a locked-in observer.

Then from Theorem 1.2'e and (1.80) we obtain

$$S_a = L_i \cap N^*_{S_a} \cap \bar{N}_{a,S_a} \tag{1.81}$$

with the independence relations

$$L_i + N^*_{S_a} = L_i + \bar{N}_{a,S_a} = N^*_{S_a} + \bar{N}_{a,S_a} = X\ . \tag{1.82}$$

The above decomposition shows that the estimation of $x(\text{mod } S_a)$ requires three observers with distinct characteristics.

Lemma 1.2'e will be used in Chapter IV when we construct a P.I.D. observer for $x(\text{mod } S^*_{b,L})$.

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# CHAPTER II

## INFINITE ZEROS AND ROOT-LOCI FOR MULTIVARIABLE LINEAR SYSTEMS

This chapter is divided in two parts as the title suggests. In the first part we deal with the infinite-zeros issue and in the second part some aspects of the root-loci theory for multivariable linear systems are tackled from a state space point of view. By "root-loci" we mean the analysis of asymptotic properties for invertible linear systems under high scalar gain output feedback.

## II.1 INFINITE-ZEROS

### II.1.1 Introduction

Infinite-zeros show up naturally in the study of a rational matrix G(s), where s is the complex variable. In Section 1.2 we review quickly the definition of infinite-zeros via the Smith-McMillan form of G(s)[2.18, 2.23] together with their dynamical interpretation [2.24].

In particular, when $G(s) := C(sI-A)^{-1}B$, the transfer matrix associated with a multivariable linear system represented by the triple (C,A,B), an important connection can be established between the infinite-zero structure of G(s) and the infinite-zero structure of the system or Rosenbrock matrix P(s) given by

$$P(s) = \begin{pmatrix} sI-A & -B \\ C & 0 \end{pmatrix} . \qquad (2.1)$$

In Section 1.3 we show that the infinite-zero structure of G(s) and P(s) are isomorphic. It is then easy to show that the number of infinite-zeros and their orders are given by the list $I_4$ in [2.14] which characterizes a prime subsystem derived from a triple (C,A,B).

In Section 1.4 we turn our attention to the role played by a sliding subspace $\bar{R}_a$[2.26] of maximal dimension in ker C, which can be obtained from the sequence (1.13). Starting from a decomposition of $\bar{R}_a$ into singly-generated subspaces $\bar{R}_{a_i}$ we then obtain a new derivation of Morse's canonical form concerning the prime subsystem and we show that the dimensions of the subspaces $\bar{R}_{a_i}$ determine the orders of the infinite-zeros for orders higher than one. Commault and Dion [2.2-3] have been the first authors to give a geometric interpretation for the infinite-zeros by relating them to the notions of almost controlled invariant subspaces. The geometric definition given in Section 1.4 is, of course, equivalent to that in [2.2-3] but in our opinion, our exposition is more detailed and shows more transparently the structure of the geometric sources (almost controllability subspaces) of the infinite-zeros.

## II.1.2 The Smith-McMillan Form at Infinity

Let T(s) be an arbitrary rational matrix of rank $r$ and let $\omega := \frac{1}{s}$. The following definition is well known [2.17-8, 2.23].

Definition 2.1: The rational matrix T(s) is said to have an infinite-zero of order $k$ when $\omega = 0$ is a finite-zero of order $k$ for $T\left(\frac{1}{\omega}\right)$.

The infinite-zero structure of a rational matrix $T(s)$ can be determined from the Smith-McMillan form of $T\left(\frac{1}{\omega}\right)$ which is given by

$$D(\omega) = U_1(\omega)T\left(\frac{1}{\omega}\right)U_2(\omega) \tag{2.2}$$

where $U_1(\omega)$ and $U_2(\omega)$ are unimodular matrices and $D(\omega)$ is a $r\times r$ diagonal matrix with diagonal elements $d_i(\omega)$, $i \in \underline{r}$, given by

$$d_i(\omega) = \frac{\epsilon_i(\omega)}{\psi_i(\omega)} \quad , \quad i \in \underline{r}, \quad \epsilon_i(\omega), \psi_i(\omega) \in \mathbb{R}[s].$$

A unimodular matrix is a nonsingular polynomial matrix in $\omega$ with a polynomial inverse, or equivalently, a polynomial matrix with a constant non-zero determinant. It follows that the matrices $U_1(\omega)$ and $U_2(\omega)$ have their poles and zeros at $\omega = \infty$, and thus (2.2) is a valid decomposition at $\omega = 0$ in the sense that the structure of $T\left(\frac{1}{\omega}\right)$ at $\omega = 0$ is isomorphic to that of $D(\omega)$ at $\omega = 0$ [2.24].

Write

$$\epsilon_i(\omega) = \omega^{k_i}\hat{\epsilon}_i(\omega) \quad , \quad i \in \underline{r}$$

with $\omega$ and $\hat{\epsilon}_i(\omega)$ coprime.

This leads to the following definition [2.23].

<u>Definition 2.2</u>: The set of nonnegative integers $\{k_i\}$, $i \in \underline{r}$ is termed the infinite-zero structure of $T(s)$ and the positive $k_i$'s are the orders of the infinite-zeros of $T(s)$.

A very nice dynamical interpretation for poles and zeros of a rational matrix has been given in [2.24]. As an example, for the case of infinite-zeros, consider the equation

$$z(s) = D(s)\ v(s)$$

with

$$D(s) = \begin{pmatrix} \frac{1}{s^2} & 0 \\ 0 & \frac{1}{s^3} \end{pmatrix} .$$

It is clear that to the input vectors $v_1^T(s) = [1 \quad 0]^T$ and $v_2^T(s) = [s \quad 0]^T$ there correspond the output vectors $z_1^T(s) = z_1^T(s) = [\frac{1}{s^2} \quad 0]^T$ and $z_2^T(s) = [\frac{1}{s} \quad 0]^T$.

This shows that the polynomial components $v_1^T(s)$ and $v_2^T(s)$ are absorbed by the rational matrix D(s), i.e. they disappear from the system outputs $z_1^T(s)$ and $z_2^T(s)$ which are strictly proper rational vectors. Since $v_1^T(s)$ and $v_2^T(s)$ are linearly independent over the field of the real numbers, then the matrix D(s) has an infinite-zero of order two.

In general, to an infinite-zero of order k of a rational matrix T(s) there correspond k linearly independent polynomial input vectors.

### II.1.3 The Infinite-Zeros of the System Matrix P(s)

Consider the linear system

$$\dot{x} = Ax + Bu \qquad (2.3)$$

$$y = Cx$$

where

$$x \in X := \mathbb{R}^n ; \quad u \in U := \mathbb{R}^m ; \quad y \in Y := \mathbb{R}^\rho$$

$$\text{rank } B = m ; \qquad \text{rank } C = \rho$$

and its associated transfer matrix $G(s) = C(sI-A)^{-1}B$.

In this section we shall show in a simple way that the infinite-zero structure of G(s) is isomorphic to the infinite-zero structure of P(s) in (2.1), which in turn coincides with the list $I_4$, in [2.14].

We need the following lemma of [2.25].

<u>Lemma 2.1</u>: Use a constant non-singular transformation on the left of the pencil sK-L to bring it to the form

$$s\begin{bmatrix} K_1 \\ 0 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

where $K_1$, has full row rank. Then the zero structure of sK-L at infinity is isomorphic to the zero structure of

$$\begin{bmatrix} K_1 - L_1\omega \\ - L_2 \end{bmatrix}$$

at $\omega = 0$.

We can now establish the desired isomorphism.

<u>Lemma 2.2</u>: The infinite-zero structure of G(s) is isomorphic to the infinite-zero structure of P(s).

Proof: Let $\omega = \frac{1}{s}$. Then a realization of $G\left(\frac{1}{\omega}\right)$ is given by

$$P_R(\omega) = \begin{pmatrix} I-\omega A & -\omega B \\ C & 0 \end{pmatrix}. \qquad (2.4)$$

Moreover the above realization is controllable and observable at $\omega = 0$. It then follows from [2.18-19] (see [2.12] for an interesting discussion on finite-zeros) that the zero structure of $G\left(\frac{1}{\omega}\right)$ at $\omega = 0$ is isomorphic to the zero structure of $P_R(\omega)$ (defined from its Smith form) at $\omega = 0$. The result now follows by applying Lemma 2.1 to (2.4).

□

Remarks 2.1:

a) Lemma 2.2 remains valid if G(s) is replaced by $\hat{G}(s) := C(sI-A)^{-1}B+D$ and P(s) by $\hat{P}(s)$, with

$$\hat{P}(s) := \begin{pmatrix} sI-A & -B \\ C & D \end{pmatrix}$$

b) A much stronger result has been proved by Verghese [2.24, Theorem 3.9] and is as follows. Consider the rational matrix

$$\tilde{G}(s) = CR^{-1}(s)B$$

where R(s) is a polynomial matrix.

If the matrices

$$[R(s) \quad -B] \quad \text{and} \quad \begin{pmatrix} R(s) \\ C \end{pmatrix}$$

have no infinite-zeros, then the infinite-zero structure of $\tilde{G}(s)$ is isomorphic to the infinite-zero structure of

$$P(s) = \begin{pmatrix} R(s) & -B \\ C & 0 \end{pmatrix}$$

Since the matrices $[sI-A \quad -B]$ and $\begin{pmatrix} sI-A \\ C \end{pmatrix}$ have no infinite-zeros, Lemma 2.2 follows. The reason for the previous proof of Lemma 2.2, without using the stronger result by Verghese, is that it is simple and exploits the form of a very simple polynomial matrix R(s), which in our case is given by (sI-A).

The next proposition gives the result claimed at the beginning of this section.

Proposition 2.1: The infinite-zero (i.z.) structure of G(s) coincides with the list $I_4$ given in [2.14].

Proof: Let H, T, G be automorphisms of $\mathcal{Y}$, $\mathcal{X}$ and $\mathcal{U}$, respectively. Let $F : \mathcal{X} \to \mathcal{U}$ and $L : \mathcal{Y} \to \mathcal{X}$ be arbitrary maps. Then the matrices

$$M := \begin{pmatrix} T & -TL \\ 0 & H \end{pmatrix} \qquad N := \begin{pmatrix} T^{-1} & 0 \\ FT^{-1} & G \end{pmatrix} \tag{2.5}$$

are nonsingular. By Lemma 2.2

$$\text{i.z. } G(s) = \text{i.z. } P(s) = \text{i.z. } MP(s)N = \text{i.z. } \bar{P}(s)$$

where

$$\bar{P}(s) := \begin{pmatrix} sI - T(A+BF+LC)T^{-1} & -TBG \\ HCT^{-1} & 0 \end{pmatrix} \tag{2.6}$$

It follows from [2.14] that the maps H, G, T, F and K can be chosen so that

$$\bar{P}(s) = \begin{bmatrix} P_1(s) & 0 & 0 & 0 \\ 0 & P_2(s) & 0 & 0 \\ 0 & 0 & P_3(s) & 0 \\ 0 & 0 & 0 & P_4(s) \end{bmatrix} \tag{2.7}$$

with

$$P_1(s) = sI-A_1 \ ; \ P_2(s) = [sI-A_2 -B_2] \ ; \ P_3 = \begin{bmatrix} sI-A_3 \\ C_3 \end{bmatrix} \tag{2.8}$$

and

$$P_4(s) = \begin{bmatrix} sI-A_4 & -B_4 \\ C_4 & 0 \end{bmatrix} . \tag{2.9}$$

The pairs $(A_2,B_2)$ and $(C_3,A_3)$ in (2.8) are controllable and observable, respectively. Therefore the pencils $P_2(s)$ and $P_3(s)$ do not have finite zeros. Obviously, $P_2(s)$ and $P_3(s)$ do not possess infinite-zeros as well.

By Lemma 3 in [2.4], the invariant polynomials of P(s) are given by the invariant polynomials of $A_1$.

The pencil (2.9) in turn, is a square pencil with

$$A_4 = \text{diag}[\hat{A}_i] \ , \ B_4 = \text{diag}[\hat{b}_i], \ \ C_4 = \text{diag}[\hat{c}_i], \ i \in \underline{r}$$

and

$$\hat{A}_i = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & 1 \\ 0 & . & . & . & . & . & 0 \end{bmatrix}_{k_i \times k_i} \qquad \hat{b}_i = \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ 1 \end{bmatrix}_{k_i \times 1} , \; i \in \underline{r}$$

$$\hat{c}_i = [1 \quad 0 \quad . \; . \; . \; . \quad 0]_{1 \times k_i} \qquad \ldots \qquad (2.10)$$

The set $\{k_i\}$, $i \in \underline{r}$, constitutes Morse's list $I_4$.

It follows from (2.10) that det $P_4(s) = 1$, i.e. the pencil $P_4(s)$ is regular [2.5]. We also have from (2.10)

$$C_4(sI-A_4)^{-1}B_4 = \text{diag}\left(\frac{1}{s^{k_i}}\right) , \; i \in \underline{r} \qquad (2.11)$$

From (2.11) and Lemma 2.2 it follows that the infinite-zero structure of $P_4(s)$ is given by

$$\text{i.z. } P_4(s) = \{k_i\} \qquad , \; i \in \underline{r} \; .$$

Since all the infinite-zeros of $\bar{P}(s)$ and thus $P(s)$ are concentrated on the pencil $P_4(s)$ we must have

$$\text{i.z. } G(s) = \text{i.z. } P(s) = \text{i.z. } \bar{P}(s) = \text{i.z. } P_4(s) = \{k_i\} \; , \; i \in \underline{r}$$

with $r = \text{rank } C(sI-A)^{-1}B$ [2.4]. □

Remark 2.2:

1) We have proved that the infinite-zero structure of G(s) coincides with Morse's list $I_4$, without resorting to the notions of almost controlled invariant subspaces as in [2.2-3].

2) The decomposition of the pencil P(s) in (2.7) corresponds to the classical decomposition of a singular pencil described by Gantmacher [2.5]. The controllability indices of the pairs $(A_2, B_2)$ and $(A_3^T, C_3^T)$ correspond to the minimal column indices and minimal row columns, respectively.

3) The finite-zeros of P(s), represented by the eigenvalues of $A_1$, constitute the transmission zeros of the triple (C,A,B). From (2.5-6), it follows that the transmission-zeros and the infinite-zeros are invariant under state feedback and output injection, which are represented by the maps F and L, respectively.

## II.1.4 Sliding Subspaces and Infinite-Zeros

In this section we show that a prime subsystem can be constructed from any sliding subspace $\bar{\mathcal{R}}_a$ of maximal dimension in ker C. It is then easy to see the connection between the list $I_4$ (equivalent to the infinite-zero structure) defined in the previous section and the structure of $\bar{\mathcal{R}}_a$.

We first recall the concept of a prime system [2.14]. For this, let $\mathcal{R}$ be called a prime controllability subspace if :

a) $\dim(\mathcal{R} \cap \mathcal{B}) = 1$

b) there exists some $F : \mathcal{X} \to \mathcal{U}$ and $b \in \mathcal{R} \cap \mathcal{B}$ such that

$$\mathcal{R} = b + A_F b + \ldots + A_F^{k-1} b \ , \quad k := \dim \mathcal{R}$$

$$CA_F^{i-1} b = 0, \ i \in \{1,2,\ldots,k-1\}$$

$$CA_F^{k-1} b \neq 0.$$

Definition 2.3: A controllable system, represented by its triple (C,A,B) is called prime if there exist prime controllability subspace $\mathcal{R}_i$, $i \in \underline{m}$, satisfying

$$\mathcal{X} = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \ldots \oplus \mathcal{R}_m$$

and

$$\mathcal{Y} = C\mathcal{R}_1 \oplus C\mathcal{R}_2 \oplus \ldots \oplus C\mathcal{R}_m \ .$$

Remark 2.3:

a) A prime system has the same number of inputs and outputs.

b) The notation used for the triple of Definition 2.3 should not confuse the reader : the only property required for the triple (C,A,B) in (2.3) is that rank B = m, rank C = $\rho$.

Let $G^*$ be a group transformation [2.14], with an element of $G^*$ given by (H,L,T,F,G), where $K : \mathcal{Y} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{U}$ are arbitrary. The maps H,T and G are automorphisms of $\mathcal{Y}$, $\mathcal{X}$ and $\mathcal{U}$, respectively. The action of $G^*$ on the triple (C,A,B) is defined by

$$(C,A,B) \rightarrow (HCT^{-1}, T(A+BF+LC)^{-1}, TBG) \ . \tag{2.12}$$

The above transformation has been used by Morse to identify important invariants associated with a triple (C,A,B). The information about such invariants is contained in the pencil (2.7).

Let $\mathcal{K}$ := ker C in algorithms (1.3-4) and (1.9-10) of Chapter I. It can be easily shown that the subspaces $\mathcal{V}^u$, $\mathcal{R}_a^u$, $\mathcal{S}^u$ and $\mathcal{R}^u$ in those algorithms are all invariant under the transformation $G^*$.

In the sequel we describe decompositions of some subspaces introduced in the previous chapter in order to display structural features of the linear system (2.3).

a) Decomposition of $\mathcal{V}^*_{b,\mathcal{K}}$ .

Consider $\mathcal{V}^*_{\mathcal{K}}$ and define a map F on $\mathcal{V}^*_{\mathcal{K}}$ so that $A_F\mathcal{V}^*_{\mathcal{K}} \subset$ $\subset \mathcal{V}^*_{\mathcal{K}}$ and so that the minimal polynomial of $A_F|\mathcal{R}^*_{\mathcal{K}}$ is coprime with $\alpha(\lambda)$, the minimal polynomial of $A_F|\mathcal{V}^*_{\mathcal{K}}$ (mod $\mathcal{R}^*_{\mathcal{K}}$).

Then, as in [2.14] define

$$\mathcal{X}_1 := \mathcal{V}^*_{\mathcal{K}} \cap \ker \alpha(A_F) . \tag{2.13}$$

Hence

$$\mathcal{V}^*_{\mathcal{K}} = \mathcal{X}_1 \oplus \mathcal{R}^*_{\mathcal{K}} \tag{2.14}$$

and

$$A_F X_1 \subset X_1 \ .$$

Note that $X_1$ is a coasting subspace. The invariant polynomials of $A_F|X_1$ constitute the set of transmission polynomials of the triple (C,A,B).

Let $\bar{R}_{a,K}$ be a sliding subspace of maximal dimension given by the sequence (1.13). From Lemma 1.2f we have that $\bar{R}_{a,K} \cap V_K^* = 0$ which implies that F can be defined on $\bar{R}_{a,K}$ as in Theorem 1.4c.

Hence by (1.8) and Lemma 1.2 a-b

$$V_{b,K}^* = R_{b,K}^* + V_K^* = \bar{R}_{b,K} \oplus V_K^* \ .$$

By using (2.14) and noting Remark 1.1a, it now follows that

$$V_{b,K}^* = X_1 \oplus R_K^* \oplus \bar{R}_{b,K} \qquad (2.15a)$$

where

$$\bar{R}_{b,K} = A_F \bar{R}_{a,K} \oplus \bar{B} \qquad (2.15b)$$

and F is the map above defined.

b) Decomposition of $\bar{R}_{b,K}$ and the construction of a prime subsystem

Our next step is to obtain a decomposition of the subspace $\bar{R}_{b,K}$ into a direct sum of singly-generated subspaces $M_i$, with the form

$$M_i = b_i + A_F b_i + \ldots + A_F^{n_i} b_i, \text{ for some } n_i,\ i \in \underline{q}$$

where q := dim $\bar{\mathcal{B}}$ and $\bar{\mathcal{B}}$ is as in (1.11).

The set $\{n_i\}$ , $i \in \underline{q}$, could be defined in an analogous way to that of Section 2.3 in Chapter I. However, to emphasize an aspect on infinite-zeros later, we shall define such a set through a slightly different way.

For this, let $\not{p} := \dim(\bar{\mathcal{B}} \cap \mathcal{K})$ and let $\mathcal{B}^*$ be any subspace such that

$$\bar{\mathcal{B}} = \mathcal{B}^* \oplus \bar{\mathcal{B}} \cap \mathcal{K} . \tag{2.16}$$

Let

$$r_u := \dim \left( \frac{\bar{\mathcal{R}}_a^u}{\bar{\mathcal{R}}_a^{u-1}} \right) \qquad , \quad u \in \underline{n} \tag{2.17}$$

where the subspaces $\bar{\mathcal{R}}_a^u$ are given by (1.13).

From the proof of Theorem 1.4 we have that

$$\bar{\mathcal{R}}_{a,\mathcal{K}} = \mathcal{B}_1 \oplus A_F \mathcal{B}_2 \oplus \ldots \oplus A_F^{k-1} \mathcal{B}_k$$

for some $k \in \underline{n}$ and $\mathcal{B}_1 := \bar{\mathcal{B}} \cap \mathcal{K}$ .

Since $\mathcal{B}_1 \supset \mathcal{B}_2 \supset \ldots \supset \mathcal{B}_u$, it follows that

$$r_1 \geq r_2 \geq \ldots \geq r_n \geq 0$$

and

$$r_1 + r_2 + \ldots + r_n = \dim \bar{\mathcal{R}}_{a,\mathcal{K}} .$$

Let

$$n_i := \text{number of integers in the set } \{r_1, r_2, \ldots, r_n\} \text{ which are } \geq i \,. \tag{2.18}$$

Then

$$n_1 \geq n_2 \geq \ldots \geq n_p \geq 1 \tag{2.19a}$$

and

$$n_1 + n_2 + \ldots + n_p = \dim \bar{\mathcal{R}}_{a,K} \,. \tag{2.19b}$$

Proceeding with the definition of the set $\{n_i\}$, $i \in \underline{q}$, let

$\hat{n}_{p+i}$ := number of integers in $\{q-p\}$ (a set of a single element) which are $\geq i$.

Then

$$\hat{n}_{p+1} = \hat{n}_{p+2} = \ldots = \hat{n}_q = 1. \tag{2.20}$$

and write

$$n_{p+i} := \hat{n}_{p+1} - 1 = 0, \; i \in \{1,2,\ldots,q-p\} \,. \tag{2.21}$$

Since $\dim \mathcal{B}^* = q-p$, it follows from (2.17-8) and (2.21) that

$$\dim(\mathcal{B}^* \oplus \mathcal{B}_1 \oplus A_F \bar{\mathcal{R}}_{a,K}) = \dim \bar{\mathcal{R}}_{b,K} = \sum_{i=1}^{q} n_i + 1 = q-p + \sum_{i=1}^{p} n_i + 1.$$

By (2.17-8), the subspace $\bar{\mathcal{R}}_{a,K}$ can be decomposed into the following direct sum

$$\bar{\mathcal{R}}_{a,K} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots \oplus \mathcal{L}_p \tag{2.22}$$

with

$$\mathcal{L}_i = b_i + A_F b_i + \ldots + A_F^{n_i - 1} b_i \;, \qquad i \in \underline{p}$$

and span $\{b_1, b_2, \ldots, b_p\} = \mathcal{B}_1$.

From Lemma 1.2e

$$\mathcal{R}_a^u = \bar{\mathcal{R}}_a^u \oplus \mathcal{R}^u \quad , \quad u \in \underline{n} \; .$$

Since $\dim(\mathcal{R}_a^u)$ and $\dim(\bar{\mathcal{R}}_a^u)$ are invariant under the group transformation $G^*$, it follows that dim $\bar{\mathcal{R}}_a^u$ is also invariant. Therefore, the indices $n_i$, $i \in \underline{p}$, are invariant under $G^*$ and also do not depend on the sliding subspace chosen. Recall that the sliding subspace $\bar{\mathcal{R}}_{a,K}$ depends on the subspace $\bar{\mathcal{B}}$ chosen.

From (2.15b) it is now clear that

$$\bar{\mathcal{R}}_{b,K} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \ldots \oplus \mathcal{M}_q \tag{2.23}$$

where

$$\mathcal{M}_i = b_i + A_F b_i + \ldots + A_F^{n_i} b_i, \qquad i \in \underline{p}$$

and

$$\mathcal{M}_i = b_i \quad , \; b_i \in \mathcal{B}^*, \; i \in \{p+1, \ldots, q\}$$

with

$$b_i + A_F b_i + \ldots + A_F^{n_i - 1} b_i = \mathcal{L}_i \subset K, \; i \in \underline{p} \; . \tag{2.24}$$

From Lemma 1.2d, $\bar{\mathcal{R}}_{a,K} = K \cap \bar{\mathcal{R}}_{b,K}$. Thus

$$CA_F^{n_i} b_i \neq 0 \quad , \; i \in \underline{p} \tag{2.25a}$$

and obviously

$$Cb_i \neq 0, \; b_i \in \mathcal{B}^*, \; i \in \{p+1,\ldots,q\}. \tag{2.25b}$$

It is also clear that the set (2.25) constitutes a basis for the space $C\bar{\mathcal{R}}_{b,\mathcal{K}}$. Thus, we can define a map $L : C\bar{\mathcal{R}}_{b,\mathcal{K}} \to \mathcal{X}$ in the following way.

$$LCA_F^{n_i} b_i = -A_F\, A_F^{n_i} b_i \quad , \; i \in \underline{p}$$

and

$$LCb_i = -A_F b_i$$

whence

$$(A_F+LC)^{n_i}\, A_F^{n_i} b_i = 0 \qquad , \; i \in \underline{p} \tag{2.26a}$$

and

$$(A_F+LC)\, b_i = 0 \qquad , \; i \in \{p+1,\ldots,q\} . \tag{2.26b}$$

Since $b_i \in \mathcal{K}$, $i \in \underline{p}$, it follows that

$$(A_F+LC)\, b_i = A_F b_i .$$

If it is true that $(A_F+LC)^{\ell-1} b_i = A_F^{\ell-1} b_i$, for $2 \leq \ell \leq n_i$, then

$$(A_F+LC)^{\ell} b_i = (A_F+LC)\, A_F^{\ell-1} b_i = A_F^{\ell} b_i \tag{2.27}$$

since $A_F^{\ell-1} b_i \in \mathcal{K}$, for $2 \leq \ell \leq n_i$.

Therefore, the subspaces $\mathcal{M}_i$ are not altered by the output injection map $L$ above defined and hence

$$\bar{R}_{b,K} = M_1 \oplus M_2 \oplus \ldots \oplus M_q \tag{2.28}$$

where

$$M_i = b_i + (A_F+LC)b_i + \ldots + (A_F+LC)^{n_i} b_i \quad , \; i \in \underline{p}$$

and

$$M_i = b_i , \quad i \in \{p+1, \ldots, q\} .$$

By (2.26-7), it follows that the subspaces $M_i$, $i \in q$, are controllability subspaces with respect to the pair (A+LC,B), Then

$$(A+BF+LC)\bar{R}_{b,K} \subset \bar{R}_{b,K} \; . \tag{2.29}$$

Let $\bar{B}$ be the insertion map of $\bar{B} = B \cap \bar{R}_{b,K}$ in $\bar{R}_{b,K}$, $\bar{C} := C|\bar{R}_{b,K}$ and $\bar{A} := (A+BF+LC)|\bar{R}_{b,K}$. It is clear from (2.23) and (2.25) that the triple $(\bar{C},\bar{A},\bar{B})$ so defined has the same number of inputs and outputs.

It now follows from definition 2.3, remark 2.3a, (2.24-5) and (2.29) that the triple $(\bar{C},\bar{A},\bar{B})$ is prime.

c) Decomposition of $X$

We just sketch here the derivation of the subspace $Z$ which complements $V^*_{b,K}$ to $X$ and to which there corresponds another list of invariants.

From (1.75) we have that $V^*_{b,K} = N^*_B$, i.e., $V^*_{b,K}$ is the infimal complementary **observability** subspace which covers $B$.

This means that there exists a map $L_1 : Y \to X$ such that

$$(A + BF + L_1C)\mathcal{V}^*_{b,K} \subset \mathcal{V}^*_{b,K} \tag{2.30}$$

and

$$\sigma[A + BF + L_1C) (\mathrm{mod}\ \mathcal{V}^*_{b,K})] = \Lambda \tag{2.31}$$

where $\Lambda$ is a symmetric set of n-dim $\mathcal{V}^*_{b,K}$ complex numbers.

The property (2.31) is equivalent to the observability of the subsystem $(\bar{C}_1,\bar{A}_1)$ where

$$\bar{C}_1 := C : \mathcal{X} \to \mathcal{Y}\ (\mathrm{mod}\ C\mathcal{V}^*_{b,K})\ .$$

and

$$\bar{A}_1 := (A+BF+L_1C)(\mathrm{mod}\ \mathcal{V}^*_{b,K})$$

The list $I_3$ in Morse's canonical form corresponds to the controllability indices of the pair $(\bar{A}_1^T, \bar{C}_1^T)$, which as shown there, are invariant under $G^*$ as well.

Following the above ideas it is possible to define maps $F_1$, $L_1$ and a subspace $\mathcal{Z}$ such that

$$\mathcal{X} = \mathcal{V}^*_{b,K} \oplus \mathcal{Z}$$
$$\mathcal{Y} = C\mathcal{R}^*_{b,K} \oplus C\mathcal{Z}$$
$$(A+BF_1+L_1C)\mathcal{Z} \subset \mathcal{Z}$$
$$F_1|\mathcal{V}^*_{b,K} = F|\mathcal{V}^*_{b,K}$$
$$L_1|C\mathcal{R}^*_{b,K} = L_1|C\,\bar{\mathcal{R}}_{b,K} = L|C\,\bar{\mathcal{R}}_{b,K}\ .$$

We summarize below all the results obtained thus far in a theorem, which is in fact, Theorem 4.1 in [2.14]. The differences between the proofs will be commented on after the statement of the theorem.

Let

$$\mathcal{X}_2 := \mathcal{R}_K^* \ , \ \mathcal{X}_3 := \mathcal{Z} \ , \qquad \mathcal{X}_4 := \bar{\mathcal{R}}_{b,K}$$

$$\mathcal{Y}_1 := C\mathcal{Z} \ , \ \mathcal{Y}_2 := C\mathcal{R}_{b,K}^*$$

and define maps $C_j : \mathcal{X} \to \mathcal{Y}_j$ , $j \in \underline{2}$ , by

$$C_1|\mathcal{Z} = C|\mathcal{Z}$$

$$C_1|\mathcal{V}_{b,K}^* = 0$$

$$C_2|\mathcal{Z} = 0$$

$$C_2|\mathcal{V}_{b,K}^* = C|\mathcal{R}_{b,K}^*$$

<u>Theorem 2.1</u> (Morse) : Let (C,A,B) be a fixed triple. There exist subspaces $\mathcal{X}_i$, $i \in \underline{4}$ , $\mathcal{Y}_j$, $j \in \underline{2}$ and maps $F_1 : \mathcal{X} \to \mathcal{U}$, $L_1 : \mathcal{Y} \to \mathcal{X}$, $C_j : \mathcal{X} \to \mathcal{Y}_j$, $j \in \underline{2}$ , for which the following conditions hold:

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$$

$$(A+BF_1+L_1C)\mathcal{X}_i \subset \mathcal{X}_i \quad , \quad i \in \underline{4}$$

$$\mathcal{B} = \mathcal{B} \cap \mathcal{X}_2 \oplus \mathcal{B} \cap \mathcal{X}_4$$

$$\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$$

$$C = C_1 \oplus C_2$$

$$\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \subset \ker C_2$$

$$\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4 \subset \ker C_1$$

$$\mathcal{X}_4 = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \dots \oplus \mathcal{M}_q$$

where the $\mathcal{M}_i$, $i \in \underline{q}$, are controllability subspaces with respect to the pair $(A+L_1C,B)$ and $q := \dim \mathcal{B} \cap \mathcal{X}_4$.

Write $B_i$ for the insection maps of $\mathcal{B} \cap \mathcal{X}_i$ in $\mathcal{X}_i$, $i \in \{2,4\}$, $A_i = (A+BF_1+L_1C)|\mathcal{X}_i$, $i \in \underline{4}$, $C_i = C_{i-2}|\mathcal{X}_i$, $i \in \{3,4\}$. Then

a) the invariant polynomials of $A_1$ coincide with the transmission polynomials of $(C,A,B)$, which are invariant under $G^*$.

b) $(A_2,B_2)$ is controllable, with controllability indices invariant under $G^*$.

c) $(C_3,A_3)$ is observable and the controllability indices of $(A_3^T,C_3^T)$ are invariant under $G^*$.

d) $(C_4, A_4, B_4)$ is prime and the dimensions of the controllability subspaces $\mathcal{M}_i$, $i \in \underline{q}$, are invariant under $G^*$.

□

Remark 2.4: The subsystems defined in a-d are the same as those in (2.8-9).

Comments:

The difference between the proof here and the proof in [2.14] concerning the prime subsystem lies in the description of the complement of $\mathcal{R}_K^*$ to $\mathcal{R}_{b,K}^*$.

From Lemma 1.2a

$$\mathcal{R}_{b,K}^* = \bar{\mathcal{R}}_{b,K} \oplus \mathcal{R}_K^* \tag{2.32}$$

and $\bar{\mathcal{R}}_{b,K}$ is described in [2.14] by means of an output injection map $L : \mathcal{Y} \to \mathcal{X}$, i.e.

$$\bar{R}_{b,K} = \langle A+LC | \bar{B} \rangle .$$

In our derivation, by Theorem 1.4 and Remark 1.1a,

$$\bar{R}_{b,K} = A_F \bar{R}_{a,K} \oplus \bar{B} \tag{2.33}$$

i.e., $\bar{R}_{b,K}$ is described by a state feedback map $F : X \to U$.

In our opinion the prime subsystem shows up more naturally by the use of description (2.33). The identification of $V^*_{b,K}$ with $N^*_{\bar{B}}$ also provides more insight in the derivation of the subspace $Z$, which has been obtained by Morse by pure duality arguments.

In bases provided by the subspaces $X_i$, $i \in \underline{4}$, the maps $(A+BF_1+L_1C)$, $B$ and $C$ can be represented as

$$\text{Mat}(A+BF_1+L_1C) = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} \quad \text{Mat } B = \begin{pmatrix} 0 & 0 \\ B_2 & 0 \\ 0 & 0 \\ 0 & B_4 \end{pmatrix}$$

$$\text{Mat } C = \begin{pmatrix} 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{pmatrix} \qquad \ldots \quad (2.34)$$

with

$$A_4 = \mathrm{diag}[\hat{A}_1,\ldots,\hat{A}_p,\ \hat{A}_{p+1}\ ,\ldots,\ \hat{A}_q]$$
$$B_4 = \mathrm{diag}[\hat{b}_1,\ldots,\ \hat{b}_p,\ \hat{b}_{p+1},\ldots,\ \hat{b}_q] \qquad (2.35)$$
$$C_4 = \mathrm{diag}[\hat{c}_1,\ldots,\ \hat{c}_p,\ \hat{c}_{p+1},\ldots,\ \hat{c}_q]$$

$$\hat{A}_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & \cdots & 1 \\ 0 & \cdot & \cdots & 0 \end{bmatrix}_{(n_i+1)\times(n_i+1)} \qquad \hat{b}_i = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}_{(n_i+1)\times 1}$$

$$\hat{c}_i = [1 \quad 0 \quad \cdots \quad 0]_{(1\times(n_i+1)}$$

for $i \in \underline{p}$ and

$$\hat{A}_i = 0, \quad \hat{b}_i = 1, \quad \hat{c}_i = 1$$

for $i \in \{p+1,\ldots,q\}$.

It is clear from the above representation and Proposition 2.1 that $n_i+1 = k_i$, $i \in \underline{q}$, and $q = r$, i.e., the set $\{n_i+1\}$, $i \in \underline{q}$ describes the infinite-zero structure of $G(s) = C(sI-A)^{-1}B$.

According to the subspace $\bar{\mathcal{B}}$ chosen in (1.11) a different subspace $\bar{\mathcal{R}}_{b,K}$ is obtained as complement of $\mathcal{R}_K^*$ to $\mathcal{R}_{b,K}^*$ in (2.32). For this reason the infinite-zero structure is associated with the quotient space $\mathcal{R}_{b,K}^*/\mathcal{R}_K^*$ or equivalently, it can also be obtained from the quotient space $\mathcal{V}_{b,K}^*/\mathcal{V}_K^*$ as in [2.2-3].

Based on the above discussion, the following "geometric" definition may be adopted for the infinite-zero structure of $G(s)= C(sI-A)^{-1}B$.

Definition 2.4: Let $\dim \frac{B+V_K^*}{V_K^*} = q$ and let $n_i$, $i \in \underline{q}$ be defined as in (2.18) and (2.21). Then G(s) is said to have q infinite-zeros of respective orders $n_i+1$.

Remarks 2.5:

a) From the above definition and (2.19a), it follows that the orders of the infinite-zeros which are higher than one, are determined from the dimensions of the sliding subspaces in the decomposition (2.22). Any sliding subspace of maximal dimension in $K$ admits a decomposition as in (2.22), where the decomposing subspaces have always dimensions $n_i$, $i \in \underline{p}$. Thus the important entity for the infinite-zeros of order higher than one is $R_{\alpha,K}^*/R_K^*$.

b) Definition 2.3, expression (2.21) and the nature of the subspace $B^*$ defined in (2.16), imply that the quotient space $\frac{B+K}{K}$ contributes only to first order infinite-zeros.

c) It is remarkable the richness provided by the geometric approach in theoretical terms. It might be argued that the Smith-McMillan decomposition of G(s) at infinity or the structure of a column reduced G(s) (see [2.23-4]) yield the infinite-zero structure. However such an approach does not provide as clear a picture of the structure of the triple (C,A,B) as the geometric approach does. Moreover, the geometric definitions of finite and infinite-zeros have been and will be very important in the search for solutions of control synthesis problems.

A dynamical interpretation for the infinite-zeros can be easily derived. For this, consider the transfer matrix of (2.34) which is clearly given by

$$C_4(sI-A_4)^{-1}B_4 = \text{diag}\ \frac{1}{s^{n_i+1}} \quad , i \in \underline{q} .$$

Consider the equation

$$y(s) = C_4(sI-A_4)^{-1}B_4\ u(s) + C_4(sI-A_4)^{-1}x_0$$

where $x_0 \in \bar{R}_{b,K}$ is a given initial condition.

Suppose that a control u(s) is required so that y(s) = 0. From the representation of $(C_4, A_4, B_4)$ in (2.35), it follows that $y(s) \equiv 0$ implies $y_i(s) = 0$, $i \in \underline{q}$, which in turn implies

$$u_i(s) = -[s^{n_i} \quad , \quad s^{n_i-1},\dots,s,1]x_i, \quad i \in \underline{q} \tag{2.36}$$

where

$$x_0 = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_q \end{pmatrix} \quad , \quad x_i \in M_i \ , \quad \dim x_i = n_i + 1, \quad i \in \underline{q} \ .$$

Hence from (2.35-6)

$$\begin{aligned} u_i &= -[\delta^{n_i},\delta^{n_i-1},\dots,\delta^{(1)}, \delta]x_i \ , \quad i \in \underline{p} \\ u_i &= -\delta x_i \quad , \quad i \in \{p+1,\dots,q\} \end{aligned} \tag{2.37}$$

where $\delta$ is the delta functional and $\delta^{(j)}$, $j \in \underline{n}_i$ is the $j^{th}$ derivative of $\delta$ in the distributional sense.

This shows that an impulsive control type is necessary to drive $x_0 \in \bar{\mathcal{R}}_{b,K}$ to ker C.

To conclude, if $x_0 \in \mathcal{R}^*_{b,K}$, but $x_0 \notin \mathcal{R}^*_K$, then an impulsive control is needed to achieve $y(s) = 0$. The quotient space $\mathcal{R}^*_{b,K}/\mathcal{R}^*_K$ is related to the infinite-zero structure.

On the other hand, if $x_0 \in \mathcal{V}^*_K$, then there exists a piece-wise continuous control so that $y(s) = 0$. The quotient space $\mathcal{V}^*_K/\mathcal{R}^*_K$ is related to the finite-zero structure [2.27].

## II.2 ROOT-LOCI FOR MULTIVARIABLE LINEAR SYSTEMS : A STATE-SPACE APPROACH

### II.2.1 Introduction

Consider the linear system

$$\dot{x} = Ax + Bu \tag{2.38}$$

$$y = Cx \tag{2.39}$$

with

$$\dim \mathcal{U} = \dim \mathcal{Y} = m$$

$$\text{rank } B = \text{rank } C = m.$$

The control law considered here is given by

$$u = g\, R\, y \tag{2.40}$$

for some map $R : \mathcal{Y} \to \mathcal{U}$ and a real scalar parameter $g$.

Classical root-loci theory is concerned with asymptotic properties of the closed-loop map

$$(A+gBRC) \tag{2.41}$$

obtained by application of (2.40) to (2.38) as $g \to \infty$.

The first results for multivariable linear systems appeared in [2.8, 2.9] and from there on a great attention has been paid to the subject, especially on the part of the Cambridge school [2.6, 2.10-11] which has developed the theory mainly in the frequency domain.

Apart from the work by Young et al [2.28] which have used the technique of singular perturbation to study the asymptotic behaviour of (2.41) (for R = I, the identity map) in the special case rank CB = m, little has been done in the state-space domain. The reason for this is very simple. We had not had, until the work by Willems [2.26], high gain concepts in the state-space to allow a development of the root-loci theory in this domain.

Our aim is to give a contribution to the state-space theory of root-loci by making use of the notions of almost controlled invariant subspaces introduced by Willems.

We shall restrict our attention to linear systems described by (2.38-9) and for which the transfer matrix $G(s) = C(sI-A)^{-1}B$ is invertible over the field of the rationals. Accordingly, we shall sometimes refer to an invertible linear system.

The limit behaviour of those eigenvalues of (2.41) which go to infinity is closely related to the subject of infinite-zeros.

In Section 2.4 we give a procedure for the assignment of the asymptotes which, as far as we know, differs from those existing in the literature and has some advantages. We shall compute a map R : $\mathcal{X} \to \mathcal{U}$ such that the asymptotes of (2.41) are assigned and such that they have the same orders as the infinite-zeros. Moreover, the assignment of asymptotes of distinct orders is done independently and the corresponding limit eigenvectors can also be assigned.

When R = I, the identity map, the asymptotes of the closed-loop map (2.41) may not have the same orders as the infinite-zeros. A condition has been given by Owens [2.15] to ensure that both entities (asymptotes and infinite-zeros) have the same orders. His condition is derived from automorphisms G and H of $\mathcal{U}$ and $\mathcal{Y}$, respectively, used to obtain Morse's canonical form (see Section 1.4). In Section 2.5, we show a condition derived directly from the maps B and C. The condition gives simultaneously a way to compute the value of the asymptotes, which is not very clear in [2.15].

Part of the material presented in Sections 2.2-2.6 is based on [2.1].

## II.2.2 Properties of Invertible Linear Systems

We describe in the following, properties of invertible linear systems, i.e., systems represented by (2.38-9) with an invertible transfer matrix $G(s) = C(sI-A)^{-1}B$. The properties mentioned here are fundamental for the root-locus study in state-space.

Let again $\mathcal{K}$ := ker C in the algorithms of Section 2.1, Chapter I.

A necessary and sufficient geometric condition for G(s) to be invertible is that [2.27].

$$V_K^* + R_{b,K}^* = X \qquad (2.42)$$

and

$$V_K^* \cap R_{b,K}^* = 0 \quad . \qquad (2.43)$$

From (2.43) and Lemma 1.1b, it follows that $R_K^* = 0$. Thus a multivariable invertible linear system can be considered as an extension of a single-input, single-output linear system when we restrict our attention to $K$. Recall that controllable single-input systems do not possess controllability subspaces other than 0 and $X$.

Since $R_K^* = 0$, it now follows from Lemma 1.2e, that $R_{a,K}^*$, the supremal almost controllability subspace in $K$, is a sliding subspace.

In fact, we can show that

$$K = V_K^* \oplus R_{a,K}^* \quad . \qquad (2.44)$$

Just note from (2.42-3) that

$$V_K^* \oplus R_{b,K}^* = X \; . \qquad (2.45)$$

By Lemma 1.1a and Proposition 1.1.

$$\mathcal{R}^*_{b,K} = A\mathcal{R}^*_{a,K} \oplus \mathcal{B}$$

with (2.46)

$$\dim A\mathcal{R}^*_{a,K} = \dim \mathcal{R}^*_{a,K} \quad .$$

Thus, from (2.45-6) we have that $\dim(\mathcal{R}^*_{a,K} \oplus \mathcal{V}^*_K) = n-m$ and since rank $C = m$, (2.44) follows.

In the sequel we obtain a decomposition of the subspace $\mathcal{R}^*_{b,K}$. For this, let k-1 be a nonnegative integer such that $\mathcal{R}^*_{a,K} = \mathcal{R}^{k-1}_a$ . Then from Theorem 1.4c

$$\mathcal{R}^{k-1}_a = \mathcal{B}_1 \oplus A_F\mathcal{B}_2 \oplus \ldots \oplus A_F^{k-2}\, \mathcal{B}_{k-1}$$

for some chain $\{\mathcal{B}_i\}$ in $\mathcal{B}$ and some $F$.

Let $\mathcal{B}_i'$, $i \in \{1,2,\ldots,k-1\}$ be subspaces such that

$$\mathcal{B}_i' \oplus \mathcal{B}_i = \mathcal{B}_{i-1} \; ; \qquad \mathcal{B}_0 := \mathcal{B} \quad . \tag{2.47}$$

Hence, from Theorem 1.4c

$$\mathcal{R}^{k-1}_a = \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \ldots \oplus \mathcal{L}_k \tag{2.48}$$

with

$$\mathcal{L}_{i+1} = \mathcal{B}'_{i+1} \oplus A_F\mathcal{B}'_{i+1} \oplus \ldots \oplus A_F^{i-1}\mathcal{B}'_{i+1}, \quad i \in \{1,\ldots,k-2\}$$

and

$$\mathcal{L}_k = \mathcal{B}_{k-1} \oplus A_F\mathcal{B}_{k-1} \oplus \ldots \oplus A_F^{k-2}\mathcal{B}_{k-1}$$

for some $F$ which obeys (1.17-8).

To facilitate the notation let $\mathcal{B}'_k := \mathcal{B}_{k-1}$. It is then not difficult to see from (2.46-7) that

$$\mathcal{S}^k := \mathcal{R}^*_{b,K} = A_F \mathcal{R}^*_{a,K} \oplus \mathcal{B}$$
$$= \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \ldots \oplus \mathcal{M}_k \qquad (2.49)$$

where

$$\mathcal{M}_i = \mathcal{B}'_i \oplus A_F \mathcal{B}'_i \oplus \ldots \oplus A_F^{i-1} \mathcal{B}'_i \;, \quad i \in \underline{k}$$

Note that $\mathcal{S}^k \supset \mathcal{B}$ and that from (2.47)

$$\mathcal{B} = \mathcal{B}'_1 \oplus \mathcal{B}'_2 \oplus \ldots \oplus \mathcal{B}'_k \qquad (2.50a)$$

which yields the decomposition

$$\mathcal{U} = \mathcal{U}'_1 \oplus \mathcal{U}'_2 \oplus \ldots \oplus \mathcal{U}'_k \qquad (2.50b)$$

where

$$B\mathcal{U}'_i = \mathcal{B}'_i \quad , \quad i \in \underline{k} \; .$$

Since $\mathcal{R}^{k-1}_a$ is the supremal subspace with the form (2.48) in $\mathcal{K}$ and since $\mathcal{K} \cap \mathcal{B}'_1 = 0$, it follows from (2.45) and (2.49) that

$$\mathcal{Y} = C\mathcal{X} = C\mathcal{S}^k$$

and

$$\mathcal{Y} = C\mathcal{X} = \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \ldots \oplus \mathcal{Y}_k \qquad (2.51)$$

where

$$\mathcal{Y}_i = CA_F^{i-1}\mathcal{B}'_i \quad , \quad i \in \underline{k} \; .$$

Let $d_i := \dim \mathcal{B}_i'$, $i \in \underline{k}$. It is then clear from (2.23) and Definition 2.3 that the transfer matrix G(s) of (2.38-9) has $\{d_i\}$ $i^{th}$ order infinite-zeros, $i \in \underline{k}$.

The decomposition (2.49) is obviously not unique, but any decomposition will display the infinite-zero structure. If, for example, G(s) has no second order infinite-zeros, then the resulting decomposition will show no subspace of the $\mathcal{M}_2$ type (equivalently $\mathcal{B}_2' = 0$).

The decompositions (2.49, 2.50-1) will be used in Section 2.4 to obtain representations for the maps A, B and C.

## II.2.3 The Case Rank CB = m

This section reviews the limit behaviour of the closed-loop eigenvalues of (A+gBC), $g \to \infty$, for the special case rank CB = m. Such a case has already been analysed in [2.8] by the use of the spectral decomposition of the map BC and in [2.28] through the singular perturbation technique.

It is our opinion that from a state space viewpoint, neither of the above approaches provide a deeper insight. For this reason we could not resist the temptation of presenting this simple case, before tackling the more general case, i.e. rank CB < m. It is our objective to show that the use of suitable concepts, i.e., notions of (almost) controlled invariant subspaces, makes the analysis of this special case to be trivial.

The consequences of rank CB = m are as follows :

a) $\mathcal{K} \cap \mathcal{B} = 0$

b) Since dim $\mathcal{B}$ = m and dim $\mathcal{K}$ = n-m, there follows by a) that $\mathcal{K} \oplus \mathcal{B} = \mathcal{X}$.

c) $A\mathcal{K} \subset \mathcal{X} = \mathcal{K} \oplus \mathcal{B}$. Thus $\mathcal{V}_K^* = \mathcal{K}$.

d) $A(\mathcal{K} \cap \mathcal{B}) \subset \mathcal{B}$, which implies $\mathcal{R}_{b,K}^* = \mathcal{B}$.

Since $\mathcal{V}_K^* \oplus \mathcal{R}_{b,K}^* = \mathcal{X}$, it follows from (2.42-3) that rank CB = m implies that G(s) is invertible.

Let then

$$\mathcal{X} = \mathcal{K} \oplus \mathcal{B} \tag{2.52}$$

and consider $Q : \mathcal{X} \to \mathcal{X}$, the projection on $\mathcal{K}$ along $\mathcal{B}$.

We claim that $\sigma(QA|\mathcal{K}) = \sigma[(A+BF)|\mathcal{K}]$, $\forall F \in F(\mathcal{K})$.

This is easily shown by using matrix arguments. In the decomposition (2.52)

$$\text{Mat } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \text{Mat } B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{2.53}$$

$$\text{Mat } C = [\,0 \quad C_2\,]$$

where $A_{11} = \text{Mat } QA|\mathcal{K}$ and $B_2$, $C_2$ are nonsingular matrices of dimension m.

Let $F = [F_1 \quad F_2]$ be a compatible partition of a map $F : \mathcal{X} \to \mathcal{U}$. Then from (2.53)

$$\text{Mat } A{+}BF = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}+B_2F_1 & A_{22}+B_2F_2 \end{bmatrix} \tag{2.54}$$

and any $F \in \mathbf{F}(\mathcal{K})$ is such that $F_1 = -B_2^{-1}A_{21}$, which verifies the claim.

It follows from [2.4] that $\sigma[QA|\mathcal{K}]$ constitute the set of transmission zeros of the triple (C,A,B).

From (2.53) we have that the closed-loop map A + g BC admits the following representation

$$\text{Mat } A{+}gBC = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}+gB_2C_2 \end{bmatrix}. \tag{2.55}$$

Since $B_2C_2$ is nonsingular, it follows from Lemma A.1 in the Appendix that as $g \to \infty$

$$\sigma(A_{22}+gB_2C_2) \to g\,\sigma(B_2C_2).$$

Therefore, the closed-loop map is characterized by two coupled "subsystems", one with finite eigenvalues and the other one with eigenvalues which tend to infinity. It is then easy to show [2.13, 2.28] that one eigensubspace of (2.55) with dimension n-m approaches $\mathcal{K}$ so that n-m eigenvalues tend to $\sigma(A_{11}) = \sigma(QA|\mathcal{K})$, as $g \to \infty$. The remaining m eigenvalues go to infinity with asymptotes given by $\sigma(B_2C_2)$ and the corresponding eigensubspace approaches $\mathcal{B}$.

The closed-loop map (A+gBC), $g \to \infty$, is then said to have m first order asymptotes in the sense that m eigenvalues go to infinity with power one in g. It is also to be noted that rank CB = m implies that the infinite-zero structure of G(s) is given by m first order infinite-zeros.

An interpretation of the above results is described in the following.

The class of maps $L(\mathcal{B})$ can be easily characterized. For this let $\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$ be a compatible partition of a map $L: \mathcal{Y} \to \mathcal{X}$. Hence from (2.53).

$$A+LC = \begin{pmatrix} A_{11} & A_{12}+L_1C_2 \\ A_{21} & A_{22}+L_2C_2 \end{pmatrix}$$

and any $L \in L(\mathcal{B})$ is such that $L_1 = A_{12}C_2^{-1}$.

Define the following maps:

a) $L \in L(\mathcal{B})$ such that $(A+LC)\mathcal{B} = 0$, i.e. set $L_2 = -A_{22}C_2^{-1}$ .

b) $F \in F(K)$ such that $F|B = 0$, i.e. set $F_2 = 0$ in (2.54).

Then

$$(A+BF+LC)K = (A+BF)K \subset K$$

and

$$(A+BF+LC)B = (A+LC)B = 0.$$

The map (A+BF+LC+gBC) admits the following representation.

$$\text{Mat}(A+BF+LC+gBC) = \begin{pmatrix} A_{11} & 0 \\ 0 & gB_2C_2 \end{pmatrix} \tag{2.56}$$

and we can see that the limit behaviour of (2.55) coincides with the behaviour of (2.56). The explanation for this is simple. Just note that as $g \to \infty$

$$\frac{\| A_{21} \|}{\| A_{22}+gB_2C_2 \|} \to 0 \quad \text{and} \quad \frac{\| A_{12} \|}{\| A_{22}+gB_2C_2 \|} \to 0 .$$

The above discussion has shown that a suitable state feedback map F and a suitable output injection map L can help in the comprehension of limit properties. This observation will be extended in Section 2.5 for systems with rank $CB < m$.

## II.2.4 Independent Assignment of Asymptotes of Distinct Integer Orders

This section deals with the assignment of asymptotes by output feedback for an invertible system described by (2.38-9). The assignment procedure suggested here is applicable to all invertible systems, i.e., rank CB can be any integer between zero and m.

If $\lambda$ is an eigenvalue of (A+gBC), $g \to \infty$ and its behaviour is given by

$$\lambda^{\ell} \to g\alpha \ , \ g \to \infty, \ \alpha \in \mathbb{C}, \ \ell \in \mathbb{R}$$

then $\alpha$ is said to be an asymptote of order $\ell$.

To begin with our constructive method , we first identify the transmission zeros of (C,A,B) as the eigenvalues of a certain map. For this, let $\mathcal{W}$ be any subspace such that $\mathcal{S}^k = \mathcal{B} \oplus \mathcal{W}$, where $\mathcal{S}^k$ is as in (2.49).

Let $Q_v : \mathcal{X} \to \mathcal{X}$ be the projection on $\mathcal{V}_K^*$ along $\mathcal{S}^k$. Analogously to the previous section we claim that $\sigma(QA|\mathcal{V}_K^*) = \sigma[(A+BF)|\mathcal{V}_K^*]$ , $\forall F \in F(\mathcal{V}_K^*)$.

To see this consider the decomposition

$$\mathcal{X} = \mathcal{V}_K^* \oplus \mathcal{B} \oplus \mathcal{W} \ .$$

Then in some basis provided by the above decomposition we obtain

$$\text{Mat } A = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{pmatrix} \qquad \text{Mat } B = \begin{pmatrix} 0 \\ B_1 \\ 0 \end{pmatrix} \tag{2.57}$$

$$\text{Mat } C = [\,0 \quad C_1 \quad C_2\,]$$

where $A_{00} = \text{Mat } QA|\mathcal{V}_K^*$ and $B_1$ is a nonsingular matrix.

Let $F : \mathcal{X} \to \mathcal{U}$ be any map and write $F = [F_0, F_1, F_2]$, according to the partitioning of Mat B in (2.57). Then

$$\text{Mat } A{+}BF = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10}+B_1F_0 & A_{11}+B_1F_1 & A_{12}+B_1F_2 \\ 0 & A_{21} & A_{22} \end{pmatrix}.$$

It is clear that any $F \in F(\mathcal{V}_K^*)$ is such that $F_0 = -B_1^{-1} A_{10}$. This shows that $A_{00}$ is a representation of $(A+BF)|\mathcal{V}_K^*$, $\forall F \in F(\mathcal{V}_K^*)$ and consequently $\sigma(A_{00}) = \sigma(QA|\mathcal{V}_K^*)$ constitute the set of transmission zeros of the triple (C,A,B) [2.4].

We now use the decompositions (2.49, 2.50-1) and Theorem 1.4c to obtain suitable representations for the maps A, B and C.

Let G and H be automorphisms in $\mathcal{U}$ and $\mathcal{Y}$, respectively, such that

$$BG = \text{diag}(B_1, \ldots, B_k) \tag{2.58}$$

where

$$\text{Im } B_i = \mathcal{B}_i', \quad i \in \underline{k}$$

and

$$HC = \mathrm{diag}(C_1, \ldots, C_k) \tag{2.59}$$

where

$$\mathrm{Im}\ C_i = CA_F^{i-1}\, \mathcal{B}_i', \quad i \in \underline{k}.$$

Note that the matrices $B_i$ and $C_i$ are nonsingular and of dimension $d_i$, $i \in \underline{k}$.

For simplicity of presentation, suppose that $\mathcal{R}^*_{b,K} = \mathcal{S}^3$ in (2.49). We shall see that there is no loss of generality in doing this. Hence

$$\mathcal{X} = \mathcal{V}_K^* \oplus \mathcal{S}^3$$

and using (2.49)

$$\mathcal{X} = \mathcal{V}_K^* \oplus \mathcal{B}_1' \oplus \mathcal{B}_2' \oplus A_F\mathcal{B}_2' \oplus \mathcal{B}_3' \oplus A_F\mathcal{B}_3' \oplus A_F^2\mathcal{B}_3' \ . \tag{2.60}$$

From (1.16-8), it follows that

$$A\, A_F^{\ell-1} B_{i+1}' = A_F^{\ell}\mathcal{B}_{i+1}' - BFA_F^{\ell-1}\mathcal{B}_{i+1}' \subset A_F^{\ell}\mathcal{B}_{i+1}' + \mathcal{B}_1' + \ldots + \mathcal{B}_{i-1}' \tag{2.61}$$

for $\ell \in \underline{i}$, $i \in \underline{k}$ and $\mathcal{B}_0' := 0$.

From (2.50-1) and (2.61), it follows that the maps B, C and A admit the following representations.

$$\text{Mat } A = \left[\begin{array}{c|c|cc|ccc} A_{00} & x & 0 & x & 0 & 0 & x \\ \hline x & x & x & x & x & x & x \\ \hline x & x & 0 & x & 0 & 0 & x \\ 0 & x & I_{d_2} & x & 0 & 0 & x \\ \hline x & x & 0 & x & 0 & 0 & x \\ 0 & x & 0 & x & I_{d_3} & 0 & x \\ 0 & x & 0 & x & 0 & I_{d_3} & x \end{array}\right] \quad \text{Mat } BG = \begin{bmatrix} 0 & 0 & 0 \\ B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.62}$$

$$\text{Mat } HC = \begin{bmatrix} 0 & C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_3 \end{bmatrix} .$$

The symbols $x$ denote matrices which are possibly nonzero and $I_{d_i}$ denotes the identity map of dimension $d_i$. It is readily seen that in general, Mat A will have $k$ diagonal blocks $A_i$ with the form

$$A_i = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & x \\ I_{d_i} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & x \\ 0 & I_{d_i} & \cdot & \cdot & \cdot & \cdot & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & I_{d_i} & x \end{bmatrix}_{id_i \times id_i} , \quad i \in \underline{k} . \tag{2.63}$$

Note that $A_i$ has the structure of a block companion matrix.

Consider the following maps

$$A_s := Q_s A|S^k \; ; \; B_s := Q_s BG \; ; \; C_s := HC|S^k$$

where $Q_s : X \to X$ is the projection on $S^k$ along $V_K^{*-}$.

Hence, the characteristic matrix of A+gBC on $S^k$ has the following representation

$$\text{Mat}(A_s+gB_sC_s-\lambda I) = \left[\begin{array}{c|cc|ccc} x+gB_1C_1-\lambda I_{d_1} & x & x & x & x & x \\ \hline x & -\lambda I_{d_2} & x+gB_2C_2 & 0 & 0 & x \\ x & I_{d_2} & x-\lambda I_{d_2} & 0 & 0 & x \\ \hline x & 0 & x & -\lambda I_{d_3} & 0 & x+gB_3C_3 \\ x & 0 & x & I_{d_3} & -\lambda I_{d_3} & x \\ x & 0 & x & 0 & I_{d_3} & x-\lambda I_{d_3} \end{array}\right]$$

(2.64)

It will be shown next that the eigenvalues of $A_s + gB_sC_s$, as $g \to \infty$, tend to the asymptotes which are determined from

$$|\lambda^i I_{d_i} - B_iC_i| = 0, \quad i \in \underline{k}. \tag{2.65}$$

From (2.65), it follows that there are $d_i$ $i^{th}$ order asymptotes. This implies that the structure of the asymptotes is isomorphic to the infinite-zero structure (see Section 2.2).

Let the $i^{th}$ diagonal block of $A_s + gB_sC_s$ be denoted by $N_i$, $i \in \underline{k}$, where

$$N_i = \begin{bmatrix} -\lambda I_{d_i} & 0 & \cdot & \cdot & \cdot & \cdot & 0 & X_{ii}+gB_iC_i \\ I_{d_i} & -\lambda I_{d_i} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & I_{d_i} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\lambda I_{d_i} & X_{2i} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & I_{d_i} & X_{1i}-\lambda I_{d_i} \end{bmatrix} \quad (2.66)$$

It can be easily seen that

$$|N_i| = |\lambda^i I_{d_i} - \lambda^{i-1}X_{1i} - \lambda^{i-2}X_{2i} - \ldots - X_{ii} - gB_iC_i|$$

where the matrices $X_{ji}$, $j \in \underline{i}$ , denote the matrices symbolized by x in (2.63).

Hence, by Lemma A.1 in the Appendix, it follows that the eigenvalues $\lambda_j^i$ , $j \in \underline{d_i}$, of a block $N_i$ approach the asymptotes given by (2.65), as $g \to \infty$.

Note that the eigenvalues of distinct diagonal blocks go to infinity with distinct rates, where the rate associated with an eigenvalues of the block $N_i$ is $g^{\frac{1}{i}}$ . Also note that the norm of the off-diagonal matrices are finite (see the example in (2.64)).

It follows that the Gerschgorin sets [2.20] associated with any two distinct diagonal blocks $N_i$ and $N_j$, $i \neq j$, are disjoint as $g \to \infty$. Further, the distance between any two sets i and j, $i \neq j$, tends to $g^{|\frac{1}{i} - \frac{1}{j}|}$, as $g \to \infty$.

This shows that the eigenvalues of $A_s + gB_sC_s$ approach the asymptotes given by (2.65), as $g \to \infty$. The $d_i$ $i^{th}$ order asymptotes are determined from the eigenvalues of the nonsingular matrix $B_iC_i$.

Since all the eigenvalues of $A_s + gB_sC_s$ go to infinity, as $g \to \infty$, it follows by the same arguments used in [2.9, 2.13] that the finite eigenvalues of A + gBC tend to the eigenvalues of $A_{00}$ (the transmission zeros), whereas the infinite eigenvalues tend to the asymptotes given by (2.65).

A simple procedure to assign the asymptotes of all orders is described next. The method used here is similar to the one used by Kimura [2.7] for the assignment of the first order asymptotes in the case rank CB = m.

Since $C_iB_i$ is nonsingular, there are vectors $v_{qi}$, $q \in \underline{d}_i$, $i \in \underline{k}$, so that the matrix

$$T_i = [C_iB_iv_{1i}, \ldots, C_iB_iv_{d_ii}], \; i \in \underline{k} \tag{2.67}$$

is nonsingular.

Let

$$\{\gamma_{qi}\}, \; q \in \underline{d}_i \quad , \quad i \in \underline{k} \tag{2.68}$$

be a set of complex numbers such that for each $i \in \underline{k}$, the set $\{\gamma_{qi}\}$ is symmetric with $d_i$ distinct complex numbers.

Let

$$Z_i := [v_{1i}, \ldots, v_{d_ii}]\mathrm{diag}[\gamma_{1i}, \ldots, \gamma_{d_ii}]T_i^{-1} \; , \; i \in \underline{k} \tag{2.69a}$$

Hence

$$B_iZ_iC_iB_iv_{qi} = \gamma_{qi}B_iv_{qi}, \; q \in \underline{d}_i, \quad i \in \underline{k} \tag{2.69b}$$

which means that $(\gamma_{qi}, B_i v_{qi})$ is an eigenvalue-eigenvector pair of $B_i Z_i C_i$.

Define

$$Z := \text{diag}(Z_1, \ldots, Z_k)$$

and

$$R := GZH.$$

Then the closed-loop map $(A+gBRC)$, $g \to \infty$, will possess $d_i$ $i^{th}$ order asymptotes given by $\gamma_{qi}$, $q \in \underline{d}_i$, $i \in \underline{k}$.

In the sequel, we discuss some properties of the eigenvectors associated with the asymptotes. Since the eigenvalues of any two blocks $N_i$ and $N_j$, $i \neq j$, go to infinity with distinct rates, it follows that as $g \to \infty$, $\sigma(N_i) \cap \sigma(N_j) = \phi$, and thus the eigenvectors of a block $N_i$ are eigenvectors of $A + gBRC$, as $g \to \infty$. Also from the structure of $N_i$ in (2.66) and Lemma A.1, it can be concluded that the limit behaviour of the eigenvectors associated with $i^{th}$ order asymptotes can be extracted from the following characteristic equation

$$\begin{pmatrix} -\lambda I_{d_i} & \cdot & \cdot & \cdot & 0 & 0 & gB_iZ_iC_i \\ I_{d_i} & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & I_{d_i} & -\lambda I_{d_i} \end{pmatrix} \begin{pmatrix} x_{1,i} \\ x_{2,i} \\ \cdot \\ \cdot \\ x_{d_i,i} \end{pmatrix} = 0 \qquad (2.70)$$

Let $((g\gamma_{qi})^{\frac{1}{i}}, x)$ be a pair eigenvalue-eigenvector in (2.70), where

$$x^T = (x_{1,i}^T, \ldots, x_{d_i,i}^T)^T .$$

Then, from (2.70)

$$(-g\gamma_{qi} I + gB_i Z_i C_i) x_{d_i,i} = 0 \tag{2.71}$$

$$x_{d_i-1,i} = (g\gamma_{qi})^{\frac{1}{i}} x_{d_i,i}$$

$$\cdot$$

$$\cdot$$

$$x_{1,i} = (g\gamma_{qi})^{\frac{i-1}{i}} x_{2,i} . \tag{2.72}$$

From (2.71), it can be concluded that the vector $x_{d_i,i}$ corresponds to the eigenvector $B_i\ v_{qi}$ in (2.69b).

From (2.72) we have that as $g \to \infty$

$$\frac{\| x_{j,i} \|}{\| x_{1,i} \|} \to 0 \quad j \in \{2,3,\ldots,d_i\}$$

which shows that the eigenvector $x$ approaches Im $B_i\ v_{qi}$, since $x_{1,i} \in \mathcal{B}_i'$ according to the basis chosen for the representation of the map A.

Another conclusion to be drawn from (2.72) is that the $i$ eigenvectors associated with the $i$ eigenvalues $\lambda_q^i \to g\gamma_q$, $q \in \underline{d}_i$, converge to the direction given by Im $B_i\ v_{qi}$.

We summarize below all the results obtained so far in a proposition.

Proposition 2.2: Consider an invertible system described by (2.38-9) with infinite-zero structure $\{d_i\}$, $i \in \underline{k}$, shown in the decomposition (2.49). Let $\{\gamma_{qi}\}$, $q \in \underline{d}_i$, $i \in \underline{k}$ be a set of complex numbers as described in (2.68).

Then, there exists a map $R : \mathcal{Y} \to \mathcal{V}$ such that the closed-loop map $(A+gBRC), g \to \infty$, has $d_i$ $i^{th}$ order asymptotes given by $\{\gamma_{qi}\}$, $q \in \underline{d}_i$, $i \in \underline{k}$.

Moreover, the i eigenvectors associated with the i eigenvalues $\lambda_q^i \to g\gamma_{qi}$, $q \in \underline{d}_i, i \in \underline{k}$ approach Im $B_i$ $v_{qi} \subset \mathcal{B}_i'$, where the $v_{qi}$, $q \in \underline{d}_i$, are chosen so that $T_i$ in (2.67) is nonsingular, $i \in \underline{k}$.

□

Comments:

1) In our opinion, the method for assignment of asymptotes suggested here has two advantanges when compared to the approach in [2.11], namely :

a) the procedure proposed in [2.11] requires the knowledge of all asymptotes of A+gBC, $g \to \infty$, and the eigenvectors of the so-called Markov parameters. Moreover, such procedure is valid only in case that the orders of the asymptotes of A+gBC, $g \to \infty$, coincide with the orders of the infinite-zeros.

The method proposed here does not have such a limitation, i.e., we assign the asymptotes (with the same orders as the infinite-zero orders) without previous knowledge of the asymptotic structure of A + gBC, $g \to \infty$.

b) The assignment of asymptotes of different integer orders can be done independently, i.e. the map $Z_i$, $i \in \underline{k}$, which assigns the asymptotes of order $i$, can be computed separately from $Z_j$, $i \neq j$.

2) From a state space viewpoint, the assignment of asymptotes via (almost) controlled invariant subspaces is the most natural, for a simple reason : the eigensubspace associated with finite eigenvalues tends to $\mathcal{V}^*_K$ and the eigenvectors associated with infinite eigenvalues converge to subspaces of $\mathcal{B}$ whose structure is determined from $\mathcal{R}^*_{a,K}$.

The conclusion that the eigenvectors coresponding to the infinite eigenvalues approach $\mathcal{B}$ had already been obtained in [2.22]. The advantage of the approach adopted here is that we have been able to obtain a much richer information. It follows from (2.47) that the limit eigensubspace associated with the asymptotes of $i^{th}$ order is related to the quotient space $\mathcal{B}_i/\mathcal{B}_{i-1}$, $i \in \{0,1,2,...,k-1\}$, which is completely determined from the structure of $\mathcal{R}^*_{b,K}$. In other words, any subspace $\mathcal{B}'_i$ which yields a direct sum in (2.47) can be chosen as a limit eigen-subspace. Such a flexibility may be important, for example, in connection with the assignment of "pivots" [2.11], which are the points of radiation of asymptotes.

3) If the system (2.38-9) is not invertible, then the state space can be decomposed as

$$\mathcal{X} = \mathcal{V}^*_{b,K} \oplus \mathcal{X}_c$$

where $\mathcal{X}_c$ is any subspace that complements $\mathcal{V}^*_{b,K}$ and

$$\mathcal{V}^*_{b,K} = \mathcal{X}_1 \oplus \mathcal{R}^*_K \oplus \bar{\mathcal{R}}_{b,K}$$

where the above subspaces are as in (2.15a).

It can be shown in a similar way as we did before that a map R : $\mathcal{Y} \to \mathcal{U}$ can be defined so that a set of dim $\bar{\mathcal{R}}_{b,K}$ eigenvalues approach pre-specified asymptotes with the same orders as the infinite-zero orders.

It can also be shown that a set of dim $\mathcal{X}_1$ eigenvalues approach the transmission zeros of the triple (C,A,B).

But the more important point to be emphasized here is that now $K$ has a controllability subspace and that $\mathcal{V}^*_{b,K}$ is a complementary observability subspace. This means that there exist F : $\mathcal{X} \to \mathcal{U}$ and L : $\mathcal{Y} \to \mathcal{X}$ such that

$$\sigma[(A+BF)\,|\,\mathcal{R}^*_K\,] = \Lambda_1$$

and

$$\sigma[(A+LC)(\text{mod } \mathcal{V}^*_{b,K}\,)] = \Lambda_2$$

where $\Lambda_1$ and $\Lambda_2$ are symmetric sets of dim $\mathcal{R}^*_K$ and n-dim $\mathcal{V}^*_{b,K}$ complex numbers, respectively.

Unfortunately, it is not true, in general, that the subspaces $\mathcal{R}^*_K$ and $\mathcal{V}^*_{b,K}$ are (A+BRC)-invariant for some R : $\mathcal{Y} \to \mathcal{U}$, i.e. that they are simultaneously A(mod$\mathcal{B}$) and A|ker C invariant subspaces.

This poses a great difficulty as to how to choose R : $\mathcal{Y} \to \mathcal{U}$ that not only assigns the asymptotes but which also ensures pre-specified complex numbers for a set of dim ($\mathcal{X}_c \oplus \mathcal{R}^*_K$ ) eigenvalues, as

$g \to \infty$. As far as we know there has been no progress in this area.

## II.2.5 Some Asymptotic Properties

Consider the invertible system (2.38-9). We shall give here a sufficient condition for the asymptotes of the closed-loop map $(A+gBC)$, $g \to \infty$, to have the same orders as the infinite-zeros orders of $G(s)$. The main difference of the analysis here when compared to that of the previous section is that we shall not make use of the automorphisms G and H in (2.58-9).

Consider again the case $k = 3$ and the decomposition (2.60). In such a decomposition the map A is represented as in (2.62) and the maps B and C admit the following representations.

$$\text{Mat } B = \begin{pmatrix} 0 \\ B_1 \\ B_2 \\ 0 \\ B_3 \\ 0 \\ 0 \end{pmatrix} \qquad \text{Mat } C = [0 \quad C_1 \quad 0 \quad C_2 \quad 0 \quad 0 \quad C_3]$$

Let $Q_s : \mathcal{X} \to \mathcal{X}$ be the projection on $\mathcal{S}^k$ along $\mathcal{V}_K^*$ and consider the following maps

$$A_s := Q_s A|\mathcal{S}^k \; ; \; \hat{B}_s := Q_s B \qquad \hat{C}_s := C|\mathcal{S}^k \, .$$

Hence, the characteristic matric of A + gBC on $S^k$ has the following form.

$$\mathrm{Mat}(A_s+g\hat{B}_s\hat{C}_s-\lambda I) = \left[\begin{array}{c|ccc|ccc} x+gB_1C_1-\lambda I_{d_1} & x & x+gB_1C_2 & & x & x & x+gB_1C_3 \\ \hline x+gB_2C_1 & -\lambda I_{d_2} & x+gB_2C_2 & & 0 & 0 & x+gB_2C_3 \\ x & I_{d_2} & x-\lambda I_{d_2} & & 0 & 0 & x \\ \hline x+gB_3C_1 & 0 & x+gB_3C_2 & & -\lambda I_{d_3} & 0 & x+gB_3C_3 \\ x & 0 & x & & I_{d_3} & -\lambda I_{d_3} & x \\ x & 0 & x & & 0 & I_{d_3} & x-\lambda I_{d_3} \end{array}\right] \quad \ldots (2.73)$$

By comparing the matrices in (2.64) and (2.73), we see that the closed-loop matrix in (2.73) displays couplings among the input maps $B_i$ and the output maps $C_j$, $i \neq j$.

Form the nonsingular matrix $\Gamma$

$$\Gamma := \begin{pmatrix} B_1C_1 & B_1C_2 & \cdot & \cdot & \cdot & B_1C_k \\ B_2C_1 & \cdot & \cdot & \cdot & \cdot & B_2C_k \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_kC_1 & B_kC_2 & \cdot & \cdot & \cdot & B_kC_k \end{pmatrix} .$$

Assume that the matrices $\Gamma_i$

$$\Gamma_i := \begin{pmatrix} B_1C_1 & B_1C_2 & \cdot & \cdot & \cdot & B_1C_i \\ B_2C_1 & \cdot & \cdot & \cdot & \cdot & B_2C_i \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_iC_1 & B_iC_2 & \cdot & \cdot & \cdot & B_iC_i \end{pmatrix} \quad i \in \{1,2,\ldots,k-1\} \qquad (2.74)$$

are nonsingular.

Let the Gauss generalized algorithm [2.5] be applied to $\Gamma$, where the pivots are square matrices of dimension $d_i$, $i \in \{1,2,\ldots,k-1\}$. The assumption (2.74) ensures that the pivots obtained during the application of the algorithm are nonsingular. Hence, the block triangular matrix $\bar{\Gamma}$ obtained is

$$\bar{\Gamma} = \begin{bmatrix} F_{11} & F_{12} \cdots & F_{1k} \\ 0 & F_{22} \cdots & F_{2k} \\ \cdot & \cdot & \cdot \\ 0 & 0 & F_{kk} \end{bmatrix} \tag{2.75}$$

where

$$F_{11} = B_1C_1$$

$$F_{22} = B_2C_2 - B_2C_1F_{11}^{-1}B_1C_2$$

$$F_{33} = (B_3C_3 - B_3C_1F_{11}^{-1}B_1C_3) - (B_3C_1F_{11}^{-1}B_1C_2)F_{22}^{-1}(B_2C_3-B_2C_1F_{11}^{-1}B_1C_3)$$

$$\vdots$$

and so on.

The matrices $F_{ij}$, $i \neq j$, which result from the application of the algorithm do not play any role here. Note that the matrices $F_{ii}$, $i \in \underline{k}$, (the pivots) are nonsingular.

It will be shown next that the $i^{th}$ order asymptotes are determined from

$$|\lambda^i I_{d_i} - F_{ii}| = 0 \ , \quad i \in \underline{k} \quad . \tag{2.76}$$

Such a result is expected since we are replacing the nonsingular matrices $B_i C_i$, $i \in \underline{k}$, in (2.64) by the nonsingular matrices $F_{ii}$, $i \in \underline{k}$, which ensures the existence of $d_i$ $i^{th}$ order asymptotes.

To illustrate the derivation of (2.76), consider the case k = 2. From (2.73)

$$\mathrm{Mat}(\hat{A}_s+g\hat{B}_s C_s-\lambda I) = \left[\begin{array}{c|cc} x+gB_1C_1-\lambda I_{d_1} & x & x+gB_1C_2 \\ \hline x+gB_2C_1 & -\lambda I_{d_2} & x+gB_2C_2 \\ x & I_{d_2} & x-\lambda I_{d_2} \end{array}\right] . \quad (2.77)$$

By Lemma A.2 in the Appendix, it follows that the diagonal block of dimension $2d_2$ cannot have asymptotes of order less than two. Therefore by Lemma A.1 all the first order asymptotes are given by

$$|\lambda I_{d_1} - B_1 C_1| = 0.$$

Consider now the application of the Gauss generalized algorithm to (2.77) and take the matrices $(gB_1C_1-\lambda I_{d_1})$ and $-\lambda I_{d_2}$ as pivots. Hence, for $\frac{\lambda}{g} \to 0$ , it follows that

$$(gB_1C_1-\lambda I_{d_1})^{-1} \to g^{-1}(B_1C_1)^{-1}.$$

Hence, after the pivoting and consideration of $g \to \infty$, the matrix in (2.77) is transformed to the following matrix

$$\begin{bmatrix} gB_1C_1 - I_{d_1} & x & x \\ \hline 0 & -\lambda I_{d_2} + H_1 & gF_{22} \\ 0 & I_{d_2} & -\lambda I_{d_2} + H_2 \end{bmatrix}$$

where $H_1$ and $H_2$ denote matrices obtained in the pivoting process and $F_{22}$ is as in (2.75).

The remaining asymptotes are then given by the limit of the roots of the determinant of the diagonal block of dimension $2d_2$, which is

$$|\lambda^2 I_{d_2} - \lambda(H_1 + H_2) + H_1 H_2 - gF_{22}| = 0. \qquad (2.78)$$

By Lemma A.1, it now follows that the roots of (2.78) tend to second order asymptotes given by

$$|\lambda^2 I - F_{22}| = 0.$$

The successive application of the Gauss generalized algorithm together with Lemmas A.1, A.2 and the hypothesis (2.74) lead to (2.76). The above result is stated in the next proposition.

<u>Proposition 2.3</u>: Consider an invertible system decribed by (2.38-9) with infinite-zero structure $\{d_i\}$, $i \in \underline{k}$, as shown in the decomposition (2.49). If the condition (2.74) holds, then the closed-loop map (A+gBC), $g \to \infty$ has $d_i$ $i^{th}$ order asymptotes, $i \in \underline{k}$, given by (2.76).

□

Comments:

1) The condition (2.74) is similar to that one given by Owens [2.15], except that our condition is expressed directly in terms of the map B and C.

2) The expressions for the matrices $F_{ii}$ show that there is a nested influence from the faster "subsystems" to the slower ones. For example, the first order asymptotes influence all the asymptotes of higher orders. This influence has been avoided in the assignment of the asymptotes by making use of suitable automorphisms G and H.

Note that if the transfer matrix G(s) has no infinite-zeros of order i, $i \in \underline{k}$, then $B_i = C_i = 0$ and the condition (2.74) is not required at i. This simply implies $F_{ii} = 0$, which shows that no asymptotes of $i^{th}$ order are present. We point out this fact because Owens [2.16] has shown that the results in [2.8] were not valid in case of absence of infinite-zeros of order $\ell$, $1 < \ell < k$.

3) A decomposition of the spaces $\mathcal{U}$ and $\mathcal{Y}$ has been obtained in [2.6]. The decomposition is related to the way in which the system (2.37) responds to the delta functional and its derivatives and it has been obtained by applying the techniques of singular value decomposition to the so-called Markov parameters.

We have seen that the concept of almost controllability subspaces has led to very simple and natural decompositions described in (2.50-1). The point we want to make is as follows : geometric concepts such as (almost) invariant subspaces provide a clear theoretical picture which should be distinguished from theoretical aspects obtained by using a numerical technique such as the singular value decomposition.

For single input-single systems we have that [2.26]

$$\mathcal{R}^*_{a,K} = b + Ab + \ldots + A^{k-1}b, \text{ for some } k \in \underline{n}.$$

Hence from (2.46)

$$\mathcal{R}^*_{b,K} = b + Ab + \ldots + A^k b.$$

In the decomposition

$$X = \mathcal{V}^*_K \oplus \mathcal{R}^*_{b,K}$$

the maps A, b and c admit the following representations

$$\text{Mat } A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \qquad \text{Mat } b = \begin{pmatrix} 0 \\ b_1 \end{pmatrix}$$

$$\text{Mat } c = [\, 0 \quad c_1 \,]$$

where $A_{00} = \text{Mat } Q_v \, A|\mathcal{V}^*_K$ has as its eigenvalues the transmission zeros of the triple (c,A,b) and

$$\text{Mat } A_{11} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & x \\ 1 & 0 & \cdot & \cdot & 0 & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & x \end{bmatrix}_{(k+1)\times(k+1)}$$

$$\text{Mat } b_1 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{bmatrix}_{(k+1)\times 1} \qquad \text{Mat } c_1 = [0 \quad 0 \quad . \quad . \quad 0 \quad \hat{c}]_{1\times(k+1)} .$$

where $\hat{c} = cA^k b \neq 0$.

By using Lemma 13.2 in [2.27] it follows that the asymptotes of $A_{11}+gb_1c_1$ and hence (analogously to the analysis of (2.55)) the asymptotes of A+gbc, $g \to \infty$, are given by the roots of

$$\lambda^{k+1} + 1 = 0 \qquad (2.79)$$

where obviously, $k+1 = n - \dim \mathcal{V}_K^*$.

Equation (2.79) is a classical one for the root-locus of single-input, single-output systems. It shows that such systems have fixed asymptotes which cannot be altered. This is a striking contrast with multi-input, multi-output linear systems, where it has been shown that there exists freedom to assign all the asymptotes of orders coinciding with the infinite-zero orders.

In the sequel we discuss a property concerning the invariance of the asymptotic behaviour under state feedback and output injection for an invertible system.

From Theorem 2.1a we know that the transmission zeros are invariant under state feedback and output injection. In other words, the map $Q_V(A+BF+LC)|V_K^*$ is fixed for any $F : X \to U$ and $L : Y \to X$. $A_{00}$ in (2.57) is just a representation of this map.

We have mentioned in Section 1.4 that the subspaces $R_a^u$ in the algorithm (1.4) are invariant under state feedback and output injection and from Theorem 2.1 we know that the orders of the infinite-zeros are also invariant under such transformations. These facts imply that under condition (2.74), the asymptotes have their orders and magnitudes invariant under state feedback and output injection, since the magnitude depend only on a structural partition of the maps B and C, which is determined from $R_a^u$.

Due to the above invariance, it follows that we may obtain the asymptotic behaviour of (A+gBC), $g \to \infty$, from another map (A+BF+LC+gBC), for suitable maps F and L. This fact has also been pointed out in [2.15].

Since $L_i \subset K$, $i \in \{2,3,\ldots,k\}$, we can proceed as in Section 1.4 and define a map $L : Y \to X$ so that

$$M_i = B_i' \oplus (A_F+LC)B_i' \oplus \ldots \oplus (A_F+LC)^{i-1}B_i', \quad i \in \underline{k}$$

with

$$(A_F+LC)^i B_i' = 0 \qquad , \quad i \in \underline{k} \ .$$

As a consequence, the subspace $S^k$ gets decomposed into controllability subspaces $M_i$, $i \in \underline{k}$, with respect to the pair (A+LC,B). Also, since $R_{a,K}^* \cap V_K^* = 0$, the map F can be chosen so that $A_F V_K^* \subset V_K^*$. Thus $(A_F+LC)V_K^* \subset V_K^*$ and for the case k = 3, the map (A+BF+LC+gBC) admits the representation below.

$$\left[\begin{array}{c|c|cc|ccc}
A_{00} & 0 & & 0 & 0 & 0 & 0 \\
\hline
0 & gB_1C_1 & 0 & gB_1C_2 & 0 & 0 & gB_1C_3 \\
\hline
0 & gB_2C_1 & 0 & gB_2C_2 & 0 & 0 & gB_2C_3 \\
0 & 0 & I_{d_2} & 0 & 0 & 0 & 0 \\
\hline
0 & gB_3C_1 & 0 & gB_3C_2 & 0 & 0 & gB_3C_3 \\
0 & 0 & 0 & 0 & I_{d_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{d_3} & 0
\end{array}\right]$$

Note that the above representation displays only the necessary information to extract limit properties and it corresponds to the extension promised in Section 2.3.

A nice interpretation for the asymptotes assigned in Section 2.4 is that the set $\{\gamma_{qi}\}$ , $q \in \underline{d}_i$, $i \in \underline{k}$, in Proposition 2.2, corresponds to a set of $\{id_i\}$, $i \in \underline{k}$, eigenvalues assigned to the controllability subspaces $\mathcal{M}_i$ with respect to the pair (A+LC,B), i.e.

$$\sigma[(A + BF + LC + BRC)\,|\,\mathcal{M}_i] = \{\gamma_{q_i}^{\frac{1}{i}}\},\ q \in \underline{d}_i,\ i \in \underline{k}\ .$$

To conclude this section, we would like to point out that the analysis of the closed-loop system

$$\dot{x} = (A + gBC)x,\ g \to \infty$$

is equivalent to the analysis of a generalized singularly perturbed system

$$E(\epsilon)\dot{x} = A(\epsilon)x, \qquad \epsilon \to 0 , \qquad \epsilon = \frac{1}{g}$$

where in the case k = 2

$$E(\epsilon) = \left(\begin{array}{c|c|cc} I_{d_0} & 0 & 0 & 0 \\ \hline 0 & \epsilon I_{d_1} & 0 & 0 \\ \hline 0 & 0 & \epsilon I_{d_2} & 0 \\ 0 & 0 & 0 & I_{d_2} \end{array}\right) \qquad A(\epsilon) = \left(\begin{array}{c|c|cc} A_{00} & x & 0 & x \\ \hline \epsilon x & \epsilon x + B_1C_1 & \epsilon x & \epsilon x + B_1C_2 \\ \hline \epsilon x & \epsilon x + B_2C_1 & 0 & \epsilon x + B_2C_2 \\ 0 & x & x & x \end{array}\right)$$

Here $d_0 = \dim \mathcal{V}_K^*$ and the matrices $E(\epsilon)$ and $A(\epsilon)$ are obtained from (2.73). This way of visualizing the closed-loop system may have some significance in connection with recent studies of generalized singularly perturbed systems, mainly for the case where condition (2.74) does not hold. For a discussion about the case where the asymptotes do not have the same orders as the infinite-zeros, see [2.10].

## II.2.6 An Example

The following system is taken from an example in [2.15]. The example is simple but it serves the purpose of illustrating what has been revealed so far.

Consider the invertible system defined by the triple (C,A,B)

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

where $\mathcal{V}_K^* = 0$.

It can be shown that $\mathcal{B} \cap K = \text{span}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \mathcal{B}_1$.

The subspace $\mathcal{B}_1'$ is chosen as $\text{span}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

So

$$R_a^1 = \mathcal{B}_1$$

$$R_a^2 = K \cap (A\mathcal{B}_1 + \mathcal{B}) = K \cap (\mathcal{B}_1 + A\mathcal{B}_1 + \mathcal{B}_1')$$

where

$$A\mathcal{B}_1 = \text{span}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Choose

$$Ab_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in A\mathcal{B}_1 \text{ and } b_1' = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{B}_1'.$$

Hence

$$R_a^2 = \text{span}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = K$$

Now, $Bu = b_1'$, implies $u_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Define $F : \mathbb{R}^4 \to \mathbb{R}^2$ so that

$$BFb_1 = b_1'$$

and

$$BF(Ab_1 + b_1') = b_1' .$$

Thus

$$F = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the subspace span $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ has been chosen arbitrarily

as well as the action of $F$ on this subspace.

Hence

$$F = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$A_F = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_F\mathcal{B}_1 = \text{span}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A_F^2 B_1 = \text{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} .$$

The representation of the maps C, A, B in the basis

$$\{-b_1', b_1, A_F b_1, A_F^2 b_1\} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.80}$$

is given by

$$\text{Mat } C = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 \end{pmatrix} \qquad \text{Mat } A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix}$$

$$\text{Mat } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} . \tag{2.81}$$

Thus

$$\lambda I - (A - gBC) = \left(\begin{array}{c|ccc} \lambda-(1-g) & -1 & -1 & -1+g \\ \hline -g & \lambda & 0 & 2g \\ 0 & -1 & \lambda & 0 \\ -1 & 0 & -1 & \lambda+1 \end{array}\right) .$$

The characteristic polynomial $\Delta(\lambda)$ of $(A-gBC)$ is given by

$$\Delta(\lambda) = \lambda^4+g\lambda^3-g\lambda^2+2g\lambda+g^2 = (\lambda^3+g)(\lambda+g) - g\lambda^2 + g\lambda$$

Hence

$$\frac{\Delta(\lambda)}{\lambda^3+g} = \lambda+g-\frac{g(\lambda^2-\lambda)}{\lambda^3+g} = \psi(\lambda). \tag{2.82}$$

Set $\lambda = gt$ in (2.82) and divide by $g$. Then

$$\frac{\psi(gt)}{g} = t+1 - \frac{gt^2-t}{g^2t^3+1} \quad . \tag{2.83}$$

From (2.83) it follows that

$$\frac{\psi(gt)}{g} \to t+1 \text{ as } g \to \infty, \text{ uniformly for } \frac{1}{2} \leq |t| \leq \frac{3}{2} \quad .$$

Hence

$$\psi(\lambda) \to \lambda+g, g \to \infty \; .$$

It follows that $\Delta(\lambda)$ has a first order asymptote in $-1$ and a third order asymptote in $-1$.

Note that the term $-g\lambda^2$ does not play any role in the determination of the asymptotes.

Define $L$ so that

$$LCb_1' = -A_F b_1' \text{ and } LCA_F^2 b_1 = -A_F^3 b_1$$

whence

$$L = \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the map A + BF + LC in the same basis (2.80) is represented as

$$\text{Mat}(A+BF+LC) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence

$$\det(\lambda I-A-BF-LC+gBC) = \lambda^4 + g\lambda^3 + 2g\lambda + g^2$$

and we can note that the term $-g\lambda^2$ has been eliminated.

The asymptotes can also be determined from the process described in Section 2.3. Just note from (2.81) that

$$\begin{pmatrix} B_1C_1 & B_1C_3 \\ B_3C_1 & B_3C_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

It follows that

$$\lambda_1 = -B_1C_1 = -1$$

and

$$\lambda_2^3 = -(B_3C_3 - B_3C_1(B_1C_1)^{-1}B_1C_3) = -1.$$

To assign the asymptotes consider

$$H = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

so that

$$HC = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence the matrix $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$

assigns one first order asymptote in $-z_1$ and one third order asymptote in $-z_2$ to the closed-loop map (A-gBZHC), $g \to \infty$.

## APPENDIX

Lemma A.1: Let $L(\lambda)$ be a matrix polynomial given by

$$L(\lambda) = \lambda^i I + \lambda^{i-1} X_1 + \ldots + \lambda X_{i-1} + X_i + gX_{i+1}$$

where $I$ is the identity matrix of dimension $n$, $X_j$ are square matrices of dimension $n$, $j \in \{1,2,\ldots,i+1\}$, $g$ is a scalar and $X_{i+1}$ is nonsingular.

Then as $g \to \infty$, all the roots of $|L(\lambda)| = 0$ go to infinity and they tend asymptotically to the roots of $|\lambda^i I + gX_{i+1}| = 0$.

Proof: $|L(\lambda)| = 0$ implies $|X_{i+1}^{-1} L(\lambda)| = 0$. Hence

$$|X_{i+1}^{-1} L(\lambda)| = |gI + A(\lambda)|$$

where

$$A(\lambda)=\lambda^i X_{i+1}^{-1} + \lambda^{i-1} X_{i+1}^{-1} X_1 +\ldots+ X_{i+1}^{-1} X_i .$$

But, [2.5]

$$|gI + A(\lambda)| = g^n + \sum_{k=1}^{n} \gamma_k g^{n-k} \tag{A1}$$

where $\gamma_k$ is the sum of the principal minors of order $k$ of $A(\lambda)$.

Let $\beta_k$, $k \in \underline{n}$, be the sum of the principal minors of order $k$ of $X_{i+1}^{-1}$. Hence, on dividing (A1) by $g^n$ and setting $\frac{\lambda^i}{g} = t$, the result is

$$\psi(t,g) := 1 + \sum_{k=1}^{n} \beta_k t^k + 0\left(\frac{1}{g^{\frac{1}{i}}}\right)$$

as $g \to \infty$, uniformly for $0 \leq -M + |\min \sigma(X_{i+1})| \leq |t| \leq |\max \sigma(X_{i+1})|+M$ where $M$ is some positive constant.

Let $\phi(t) := 1 + \sum_{k=1}^{n} \beta_k t^k$ and let $t^*$ be a root of $\phi(t)$ of multiplicity $\sigma$. Then for $\delta > 0$ small and fixed, there exists $g_0 > 0$ such that

$$|\psi(t,g) - \phi(t)| < |\phi(t)|$$

for all $t$ such that $|t-t^*| = \delta$ and all $g \geq g_0$. By Rouché's theorem [2.21], $\psi(t,g)$ has exactly $\sigma$ roots in $|t-t^*| < \delta$. Let $t^*(g)$ be one of these roots. Then

$$\frac{\lambda^{*i}}{g} := t^*(g)$$

satisfies

$$\psi(\frac{\lambda^{*i}}{g}, g) = 0$$

and

$$|t^*(g) - t^*| = |\frac{\lambda^{*i}}{g} - t^*| < \delta.$$

Thus, in the limit all the roots of $\psi(t,g)$ approach the roots of $\phi(t)$. The result follows on noting that

$$\phi(t) = 0 \iff \phi\left(\frac{\lambda^i}{g}\right) = 1 + \sum_{k=1}^{n} \beta_k \left(\frac{\lambda^i}{g}\right)^k = 0$$

$$\iff |gI + \lambda^i X_{i+1}^{-1}| = 0 \iff |\lambda^i I + gX_{i+1}| = 0.$$

□

<u>Lemma A.2</u>: Let $L(\lambda)$ be a matrix as in Lemma A.1, where now $X_{i+1}$ is an arbitrary square matrix of dimension n. Let $\hat{\lambda}$ be a root of $|L(\lambda)| = 0$ which goes to infinity as $g \to \infty$. Then, there is no real number $r \in (0,i)$ such that $\lim_{g\to\infty} \frac{\hat{\lambda}^r}{g} = \alpha$, $\alpha \neq 0$, $\alpha \in \mathbb{C}$.

<u>Proof</u>: Suppose that there is a $\hat{\lambda} \to \infty$ with $\lim_{g\to\infty} \frac{\hat{\lambda}^r}{g} = \alpha$, $\alpha \neq 0$, $r \in (0,i)$.

Then $|L(\hat{\lambda})| = 0$ implies $|\frac{1}{g} L(\hat{\lambda})| = 0$. This and $\lim_{g\to\infty} \frac{\hat{\lambda}}{g} = \alpha$ imply $\lim_{g\to\infty} |\tilde{L}(\alpha,g)| = 0$, where

$$\tilde{L}(\alpha,g) = \alpha^{\frac{i}{r}} g^{\frac{i-r}{r}} I + \alpha^{\frac{i-1}{r}} g^{\frac{i-r-1}{r}} X_1 + \ldots + g^{-1} X_i + X_{i+1}.$$

Since $\alpha^{-\frac{i}{r}} \neq 0$ and $g^{\frac{r-i}{r}} \to 0$, we then have

$$\lim_{g\to\infty} \left|\alpha^{-\frac{i}{r}} g^{\frac{r-i}{r}} \tilde{L}(\alpha,g)\right| = \lim_{g\to\infty} \left(\alpha^{-\frac{i}{r}} g^{\frac{r-i}{r}}\right)^n \left|\tilde{L}(\alpha,g)\right| = 0.$$

Hence

$$\lim_{g\to\infty} \left|I + A(\alpha,g)\right| \to 0 \tag{A2}$$

with

$$A(\alpha,g) = \alpha^{-\frac{1}{r}} g^{-\frac{1}{r}} X_1 + \alpha^{-\frac{2}{r}} g^{-\frac{2}{r}} X_2 + \ldots + g^{-\frac{i}{r}} X_i + g^{\frac{r-1}{r}} X_{i+1} .$$

By the Gerschgorin theorem [2.20], it follows that the eigenvalues of $I + A(\alpha,g)$ are located in one of the discs $D_j$, $j \in \underline{n}$, given by

$$D_j = \{\gamma| \ |\gamma-(1+a_{jj})| \leq \sum_{\substack{k=1\\k\neq j}}^{n} |a_{jk}|\}, \quad j \in \underline{n}$$

where $a_{jk}$ is the $jk^{th}$ element of $A(\alpha,g)$.

From the form of $A(\alpha,g)$, it follows that as $g \to \infty$, the centres of the discs tend to one while their radii go to zero. Hence $I + A(\alpha, g)$ is nonsingular for $g \to \infty$, which contradicts (A2). Therefore $\frac{\hat{\lambda}^r}{g}$ cannot converge to $\alpha \neq 0$, for $r \in (0,i)$.

□

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# CHAPTER III

## GENERALIZED LINEAR SYSTEMS

### III.1 INTRODUCTION

Consider a time-invariant system described by

$$E\dot{x} = Ax + Bu \qquad (3.1)$$
$$y = Cx$$

where

$x \in \mathcal{X} := \mathbb{R}^n$ ; $u \in \mathcal{U} := \mathbb{R}^m$; $y \in \mathcal{Y} := \mathbb{R}^r$

rank $B = m$;   rank $C = r$

and $E$ is a singular map.

A system described by (3.1) is termed a generalized state space system, a singular system, or even a descriptor system, [3.13, 3.3] . The term "generalized linear system" is adopted here to recall that (3.1) is a linear system and it is "generalized" by the fact that $E$ is singular.

It has been suggested in [3.13] that the equation (3.1) can be used to describe the behaviour of systems in which a sudden change in structure or, parameter values occurs. The reasoning used is as follows : let $\underline{x}(t)$ be the state of a system, not necessarily described by (3.1), and let $\underline{x}(t) \to x_0$ as $t \to 0^-$. Suppose that at $t = 0$ a switching occurs and the system is now modelled by (3.1). Then $x_0$ may be considered a initial condition for (3.1) which together with $\underline{u}(t)$ will determine its response fot $t \geq 0$. It is clear that $x_0$ may assume any value since nothing has been said about the system structure for $t < 0$.

It has been shown in [3.8], that when an arbitrary initial condition is allowed then the solution of (3.1) belongs to the class of distributions and that we should instead consider the equation

$$\begin{aligned} E\dot{x} &= Ax + \delta E x_0 + Bu \\ y &= Cx \end{aligned} \tag{3.2}$$

where $\delta$ is the delta functional (see [3.18] , for example). It can be concluded from the results in [3.13] that with $u := 0$, the solution of (3.2) is a distribution of Bohl type (see Definition 1.3) given by

$$\underline{x} = \sum_{i=0}^{n} x_i \delta^{(i)} + \underline{x}' \tag{3.3}$$

where $\underline{x}' : t \to e^{Lt}\tilde{x}$, $t \geq 0$. Here $x_i$, $i \in \underline{n}$, and $\tilde{x}$ are vectors and $L$ is a matrix.

In this chapter and in the next one we shall adopt the formulation (3.2), for we are chiefly interested in properties such as controllability and observability for the system (3.1) when subject to an arbitrary initial condition $\underline{x}(0^-) = x_0$. To facilitate the notation and exposition we shall sometimes refer to the system (3.1), which should then be understood in the sense of (3.2).

Assume that the pencil (sE-A) is regular [3.9, vol II], i.e. (sE-A) is invertible over the field of the rationals. Then by taking the Laplace transform of (3.2), it follows that its unique solution, for a given $x_0$ and a given $\underline{u}(t)$, $t \geq 0$, is given by

$$x(s) = (sE-A)^{-1}[Ex_0 + Bu(s)] \qquad (3.4)$$
$$y(s) = Cx(s)$$

where s denotes the complex variable.

The main features of a generalized linear system are :

a) The number of zeros of the pencil, which determines the free response of (3.2), is given by $h := \text{rank } E < n$.

b) The transfer matrix of (3.4), $G(s) = C(sE-A)^{-1}B$, consists of the sum of a strictly proper rational matrix and a polynomial matrix.

c) The degree of $\det(sE-A) := r \leq h < n$, and if $r < h$, then the regular pencil (sE-A) has h-r infinite-zeros which correspond to h-r impulsive motions in the free-response.

The above features of a generalized linear system have been discussed by Verghese et. al [3.13] who have also established tests for the controllability and observability of the infinite-zeros. The tests are then used in a method, based on the Jordan canonical form, to extract a controllable and observable subsystem relative to the controllable and observable infinite-zeros.

One of the aims of this chapter is to present the concepts of controllability and observability of the infinite-zeros from a geometric point of view. Such an approach has then led us to obtain necessary and sufficient conditions for the controllability and observability of those zeros in terms of the maps E, A, B and C. The conditions obtained here correct erroneous ones in the literature.

The geometric language used here also provides an alternative way of obtaining a controllable and observable subsystem without resorting to a canonical form as in [3.13] and in our opinion it gives a clearer picture of the controllable and unobservable infinite-zeros.

We also give an interpretation for the controllability and observability of the finite-zeros which is based on the concept of invariant subspaces associated with the pencil (sE-A). For this reason, and because invariant subspaces play a major role in Chapter IV, a detailed study of the subject is carried out here.

A benefit of the geometric approach is that it facilitates new results in zero placement (we shall refer to zero placement instead of pole placement) by state feedback and output feedback. In [3.3], a necessary and sufficient condition has been given so that the infinite-zeros can be converted into finite ones by state feedback, without pre-specification of the resulting finite-zeros. We show here in a simple way that such a condition corresponds to the controllability of the infinite-zeros. We also obtain the stronger result that controllability of the infinite-zeros implies the existence of a state feedback map that assigns pre-specified complex values to those zeros. This is simply an extension of the celebrated result by Wonham [3.17].

It is also shown here that controllability and observability of the infinite-zeros are necessary and sufficient conditions for :

a) the existence of an output feedback map which converts those zeros into finite ones.

b) the assignment of the infinite-zeros to pre-specified complex values via an observer.

The chapter is organized as follows. In Section 2, some properties of invariant subspaces associated with the pencil (sE-A) are discussed. Section 3 describes the concepts of controllability and observability of the infinite-zeros from a geometric point of view. In Section 4, the issue of zero placement by state feedback is studied, while Section 5 deals with the assignment of the infinite-zeros by means of an observer.

## III.2 INVARIANT SUBSPACES

In order to obtain invariant subspaces associated with the regular pencil (sE-A), we decompose it into two pencils such that one of them has only finite-zeros and the other one has infinite-zeros.

Let $\sigma(E,A) := \{\lambda_i\}$ , $i \in \underline{k}$, denote the set of finite-zeros of the pencil (sE-A). Write

$$\det(sE-A) = \phi \prod_{i=1}^{k} (s-\lambda_i)^{n_i} \tag{3.5}$$

where $0 \neq \phi \in \mathbb{R}$, $\lambda_i \neq \lambda_j$ for $i \neq j$ and $n_i$ is the multiplicity of $\lambda_i$.

Let $\alpha \in \mathbb{C} - \sigma(E,A)$. Then $(\alpha E-A)$ is invertible. Define

$$\mathcal{V}_s := \bigoplus_{i=1}^{k} \ker\left((\alpha E-A)^{-1}E - \frac{1}{\alpha-\lambda_i}I\right)^{n_i} \tag{3.6}$$

and

$$\mathcal{W}_f := \ker[(\alpha E-A)^{-1}E]^{n-r} \tag{3.7}$$

where

$$r = \sum_{i=1}^{k} n_i = \text{degree of } \det(sE-A).$$

The following theorem yields the desired decomposition. It has been obtained by Cobb [3.3] and its proof is based on the theory of regular pencils described by Gantmacher [3.9].

Theorem 3.1

1) $\mathcal{V}_s \oplus \mathcal{W}_f = \mathcal{X}$, $\dim \mathcal{V}_s = r$

2) There exists a nonsingular map $M : \mathcal{X} \to \mathcal{X}$ such that :

a) $\mathcal{V}_s$ and $\mathcal{W}_f$ are ME and MA-invariant

b) $ME|\mathcal{V}_s = I$, $MA|\mathcal{W}_f = I$

c) $J := ME|\mathcal{W}_f$ is nilpotent.

d) $L := MA|\mathcal{V}_s$ is such that $\det(sI-L) = \prod_{i=1}^{k} (s-\lambda_i)^{n_i}$

We describe now how the above map M has been obtained by Cobb and we shall show later that it can be easily interpreted with the help of invariant subspaces. Let

$$M_1 := (\alpha E-A)^{-1} E|\mathcal{V}_s$$

$$M_2 := (\alpha E-A)^{-1} E|\mathcal{W}_f \tag{3.8}$$

Define $\bar{M} : \mathcal{X} \to \mathcal{X}$ as

$$\bar{M}x = \begin{cases} M_1^{-1}x & , \quad x \in \mathcal{V}_s \\ (\alpha M_2 - I)^{-1}, & x \in \mathcal{W}_f \end{cases} \tag{3.9}$$

and finally,

$$M := \bar{M}(\alpha E-A)^{-1} . \tag{3.10}$$

Hence, if the pencil (sE-A) is pre-multiplied by M, its representation in the basis $\mathcal{V}_s \oplus \mathcal{W}_f$ is given by

$$\text{Mat}(sME-MA) = \begin{pmatrix} sI-L & 0 \\ 0 & sJ-I \end{pmatrix} . \tag{3.11}$$

The representation (3.11) corresponds to the decomposition of the pencil (sE-A) into the pencil (sI-L) which has only finite-zeros and the pencil (sJ-I) characterized by infinite-zeros only [3.14] .

Cobb [3.4] has also shown that the subspaces $\mathcal{V}_s$, $\mathcal{W}_f$ and the map M are independent of $\alpha \in \mathbb{C} - \sigma(E,A)$. Consequently, the maps L and J are independent of $\alpha$ as well.

Corollary 3.1

a) ker J = ker E

b) ker L = ker A

c) $\mathcal{W}_f \supset \ker E$

d) $\mathcal{V}_s \supset \ker A$

e) rank $J$ = rank $E-r$

Proof:

a) $\ker E = \ker ME = \ker ME|\mathcal{V}_s \oplus \mathcal{W}_f = \ker ME|\mathcal{W}_f = \ker J$

b) $\ker A = \ker MA = \ker MA|\mathcal{V}_s \oplus \mathcal{W}_f = \ker MA|\mathcal{V}_s = \ker L.$

The items c) and d) follow immediately from the proofs of a) and b).

For the item e) just note that rank $J$ = rank $ME - r$ and the result follows. □

We are now in a position to start the study about invariant subspaces. Let $x = x_s + x_f$ where $x_s \in \mathcal{V}_s$ and $x_f \in \mathcal{W}_f$. Let $\lambda_i$ be an eigenvalue of $L$, or equivalently a finite-zero of $(sE-A)$. Then from (3.11)

$$\begin{pmatrix} \lambda_i I-L & 0 \\ 0 & \lambda_i J-I \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} = 0$$

implies $x_f = 0$, since $J$ is nilpotent.

Now let $\gamma = \infty$ be an infinite-zero. Then

$$\begin{pmatrix} \gamma I-L & 0 \\ 0 & \gamma J-I \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} = 0$$

implies $x_s = 0$, since $\gamma I-L$ is nonsingular and $\alpha J-I$ is singular [3.14].

This shows that eigenvectors associated with finite-zeros are contained in $\mathcal{V}_s$ and that eigenvectors corresponding to infinite-zeros are contained in $\mathcal{W}_f$.

In fact, the subspace $\mathcal{V}_s$ is spanned by the generalized eigenvectors of the map L. A typical chain of k eigenvectors associated with a finite-zero $\lambda_i$ is given by

$$(L-\lambda_i I)v_j = v_{j-1}, \; v_0 = 0, \; j \in \underline{k} \qquad (3.12)$$

or equivalently by theorem 3.1

$$(A-\lambda_i E)v_j = Ev_{j-1} \;, \quad v_0 = 0, \quad j \in \underline{k} \;. \qquad (3.13)$$

Note that the subspace $\mathcal{V}$ spanned by $v_j$, $j \in \underline{k}$ is characterized by

$$A\mathcal{V} \subset E\mathcal{V}$$

Based on the above considerations the next definition is very natural.

<u>Definition 3.1</u>: A subspace $\mathcal{V}$ is said to be (A,E) invariant if $A\mathcal{V} \subset E\mathcal{V}$.

Since the class of the (A,E) invariant subspaces is closed under addition then it possesses a supremal given by

$$\mathcal{V}^{\square} = \sup\{\mathcal{V} \,|\, A\mathcal{V} \subset E\mathcal{V}\} \;. \qquad (3.14)$$

It has been shown in [3.1] that for a regular pencil $\mathcal{V}^{\square} = \mathcal{V}_s$ and an algorithm is proposed for the computation of $\mathcal{V}^{\square}$. Note that for a regular pencil $\mathcal{V}^{\square} \cap \ker E = 0$.

From Theorem (3.1) we then have

$$L\mathcal{V}_s \subset \mathcal{V}_s \iff MA\mathcal{V}_s \subset ME\mathcal{V}_s \iff A\mathcal{V}_s \subset E\mathcal{V}_s \quad . \tag{3.15}$$

It has been shown in [3.14] that to each Jordan block of dimension $\ell$ in the map J, there corresponds an infinite-zero of multiplicity $\ell-1$. We can therefore associate with such infinite-zero a chain of $\ell$ eigenvectors given by

$$Jw_i = w_{i-1}, \quad w_0 = 0, \quad i \in \underline{\ell} \quad . \tag{3.16}$$

Since $w_i \in \mathcal{W}_f$, $i \in \underline{\ell}$, it follows from Theorem 3.1 and (3.16) that

$$\begin{aligned} &Ew_1 = 0 \\ &Ew_i = Aw_{i-1}, \quad i \in \{2,3,\ldots,\ell\} \ . \end{aligned} \tag{3.17}$$

Observe that the subspace $\mathcal{W}$ spanned by $w_i$, $i \in \underline{\ell}$, is characterized by

$$E\mathcal{W} \subset A\mathcal{W}, \quad \mathcal{W} \cap \ker E \neq 0 \tag{3.18}$$

which leads to the following definition.

<u>Definition 3.2</u>: A subspace $\mathcal{W}$ is said to be (E,A) invariant if $E\mathcal{W} \subset A\mathcal{W}$ and $\mathcal{W} \cap \ker E \neq 0$.

In Chapter IV we shall define a family of subspaces closely related to that described in the above definition. It will then be possible to obtain $\mathcal{W}_f$ as the limit of a sequence of subspaces.

From Theorem (3.1) we also obtain that

$$J\mathcal{W}_f \subset \mathcal{W}_f \Longleftrightarrow ME\mathcal{W}_f \subset MA\mathcal{W}_f \Longleftrightarrow E\mathcal{W}_f \subset A\mathcal{W}_f \quad . \tag{3.19}$$

We can now describe the structure of the map M in Theorem 3.1.

<u>Proposition 3.1</u>:

$$M^{-1}x = \begin{cases} Ex \quad , & x \in \mathcal{V}_s \\ Ax \quad , & x \in \mathcal{W}_f \end{cases} \tag{3.20}$$

<u>Proof</u>: We have first to show that $E\mathcal{V}_s \oplus A\mathcal{W}_f = \mathcal{X}$

Suppose that Ev = Aw, for some $v \in \mathcal{V}_s$ and $w \in \mathcal{W}_f$. Then by Theorem 3.1, it follows that

$$MEv = MAw \Longleftrightarrow v = w$$

which is not possible since $\mathcal{V}_s \cap \mathcal{W}_f = 0$.

By Corollary 3.1 c,d we have that

$$\dim E\mathcal{V}_s = \dim \mathcal{V}_s \text{ and } \dim A\mathcal{W}_f = \dim \mathcal{W}_f$$

and thus $E\mathcal{V}_s \oplus A\mathcal{W}_f = \mathcal{X}$ .

Now let

$$\mathcal{X} = \mathcal{V}_s \oplus \mathcal{W}_f \tag{3.21}$$

and let Ex and Ax be represented in the basis

$$\mathcal{X} = E\mathcal{V}_s \oplus A\mathcal{W}_f \quad . \tag{3.22}$$

By (3.15) and (3.16), it follows that in the above bases the maps E and A admit the following representations.

$$\text{Mat } E = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \qquad \text{Mat } A = \begin{pmatrix} L & 0 \\ 0 & I \end{pmatrix} \tag{3.23}$$

The expression (3.20) follows from (3.22).

□

Remark 3.1: We could have obtained (3.20) directly from Theorem 3.1. Just note that

$$MEx = x \iff M^{-1}x = Ex, \quad x \in \mathcal{V}_s$$

and

$$MAx = x \iff M^{-1}x = Ax, \quad x \in \mathcal{W}_f \ .$$

However, the purpose of Proposition 3.1 has been to show that there is no need for the complex number $\alpha$ used in the definition of M (see (3.8 - 10)). M is just the inverse of a map whose columns span $E\mathcal{V}_s$ and $A\mathcal{W}_f$ (see (3.22)).

Let

$$Q_s : \mathcal{X} \rightarrow \mathcal{V}_s \text{ be the projection on } \mathcal{V}_s \text{ along } \mathcal{W}_f$$

and (3.24)

$$Q_f : \mathcal{X} \rightarrow \mathcal{W}_f \text{ be the projection on } \mathcal{W}_f \text{ along } \mathcal{V}_s .$$

Consider the free-system

$$E\dot{x} = Ax \ . \qquad (3.25)$$

The pre-multiplication of (3.25) by $Q_sM$ and by $Q_fM$ and the use of Theorem 3.1a results in the following decomposition

$$\dot{x}_s = Lx_s \qquad (3.26)$$

and

$$J\dot{x}_f = x_f \qquad (3.27)$$

with

$$x_s = Q_sx \qquad \text{and } x_f = Q_fx.$$

The dynamical interpretation for a initial condition $\underline{x}_s(0^-) \in \mathcal{V} \subset \mathcal{V}_s$, where $A\mathcal{V} \subset E\mathcal{V}$ is obvious : it implies $\underline{x}(t) \in \mathcal{V}$, $t \geq 0$.

The "dynamical" interpretation for a initial condition $\underline{x}_f(0^-) := x_{f0} \in \mathcal{W} \subset \mathcal{W}_f$, where $J\mathcal{W} \subset \mathcal{W}$ is as follows. Consider the equation

$$J\dot{x}_f = x_f + \delta Jx_{f0} \ . \qquad (3.28)$$

It has been shown in [3.13] that the solution for (3.28) is the distribution

$$\underline{x}_f = - \sum_{i=1}^{q-1} \delta^{i-1} J^i x_{f0} \tag{3.29}$$

where $q$ is the index of nilpotency of $J(J^q = 0)$.

From (3.29), it follows that if $x_{f0} \in \mathcal{W}$ with $J\mathcal{W} \subset \mathcal{W}$ then $\underline{x}_f \in \mathcal{W}$, i.e. the distribution also lies in $\mathcal{W}$.

Note from (3.26) and (3.27) that the free-response of (3.25) subject to an arbitrary initial condition $x(0^-) = x_s(0^-) + x_f(0^-)$, $\underline{x}_s(0^-) \in \mathcal{V}_s$, $\underline{x}_f(0^-) \in \mathcal{W}_f$, is given by

$$\underline{x} = \underline{x}_s + \underline{x}_f \tag{3.30}$$

with

$\underline{x}_s : t \in \mathbb{R}^+ \to e^{Lt} \underline{x}_s(0^-)$ and $\underline{x}_f$ given by (3.29).

The solution (3.30) represents the distribution mentioned in the introduction of this chapter.

Cobb [3.5] has obtained some very interesting results concerning the solution (3.29) as the limit solution of a singularly perturbed system

$$J_n \dot{x}_f = x_f \qquad , \; x_{f0} \text{ given} \tag{3.31}$$

where $J_n \to J$, for integer $n$ and $n \to \infty$.

He proves that the limiting solution of (3.31) is unique and is given by (3.29).

This section is closed with a discussion about the eigensubspaces which appear in (3.6). It can be easily shown that

$$\ker\left((\alpha E-A)^{-1}E - \frac{1}{\alpha-\lambda_i} I\right)^{n_i} = \ker(\lambda_i E-A)^{n_i}$$

holds for $n_i = 1$.

Since $V_s$ is independent of $\alpha$, the above fact makes one wonder whether the equality remains for $n_i > 1$. Unfortunately this is not true in general. The next proposition gives a sufficient condition for the equality to hold with $n_i > 1$.

Proposition 3.2: If AE = EA then

$$\ker\left((\alpha E-A)^{-1}E - \frac{1}{\alpha-\lambda_i} I\right)^{n_i} = \ker(\lambda_i E-A)^{n_i} \quad , n_i > 1.$$

Proof:

$$\ker(\lambda_i E-A)^{n_i} = \ker\left((\alpha E-A)((\alpha E-A)^{-1}E - \frac{1}{\alpha-\lambda_i} I)\right)^{n_i} .$$

It is clear that if $(\alpha E-A)$ and $((\alpha E-A)^{-1}E - \frac{1}{\alpha-\lambda_i} I)$ commute, then the desired equality holds. It can be shown that if AE = EA, then the above maps commute.

□

The above proposition shows that although $V_s$ does not depend on $\alpha \in \mathbb{C} - \sigma(E,A)$, it has in general, to be computed with the help of $\alpha$, which is rather awkward. The same thing holds for $W_f$.

We have already identified the structure of the map M, eliminating the need of $\alpha$ for its computation. In Chapter IV we shall do the same for the subspaces $\mathcal{V}_s$ and $\mathcal{W}_f$, by describing them as limits of suitable algorithms. Once they are computed, the maps L and J can be determined and the finite-zeros can be computed as eigenvalues of L. This avoids the computation of the finite-zeros as roots of det(sE-A).

A final remark should be made regarding the regularity of the pencil(sE-A). It is clear that if (sE-A) is regular then

$$\ker E \cap \ker A = 0 \qquad (3.32)$$

and

$$\mathrm{Im}E + \mathrm{Im}A = \mathcal{X} \qquad . \qquad (3.33)$$

A necessary and sufficient condition for the regularity of the pencil(sE-A) will be given in Chapter IV.

## III.3 A GEOMETRIC STUDY OF CONTROLLABILITY AND OBSERVABILITY

Let M be a map as described in Proposition 3.1 (which is the same as the one in Theorem 3.1). Also consider $Q_s : \mathcal{X} \to \mathcal{V}_s$ and $Q_f : \mathcal{X} \to \mathcal{W}_f$, the maps described in (3.24).

Pre-multiplying (3.1) by $Q_sM$ and $Q_fM$, it follows from Theorem 3.1 that the system (3.1) is decomposed as

$$\dot{x}_s = Lx_s + B_su \qquad (3.34)$$

and

$$J\dot{x}_f = x_f + B_fu \qquad (3.35)$$

$$y = [C_s \quad C_f]\begin{pmatrix} x_s \\ x_f \end{pmatrix} \tag{3.36}$$

where

$$x_s = Q_s x \quad \text{and} \quad x_f = Q_f x$$

$$B_s = Q_s MB \qquad B_f = Q_f MB$$

$$C_s = C|V_s \qquad C_f = C|W_f \quad .$$

From (3.34-6) it follows that

$$G(s) = C(sME - MA)^{-1}MB = R(s) + D(s)$$

where

$$R(s) = C_s(sI-L)^{-1}B_s$$

and

$$D(s) = C_f(sJ-I)^{-1}B_f \ .$$

Thus, as pointed out in [3.13], the transfer matrix G(s) is decomposed into a sum of a strictly proper matrix R(s) and a polynomial matrix D(s).

If the control $\underline{u}$ belongs to the class of the q-1 differentiable functions, where q is the index of nilpotency of J, then the unique solution of (3.34) in the class of functions is given by [3.2]

$$\underline{x}_f(t) = -\sum_{i=0}^{q-1} J^i B_f \, \underline{u}^i(t) \quad , \quad t \geq 0 \quad . \tag{3.37}$$

It is also shown in [3.2, 3.4] that the solution of (3.1) is given by

$$\underline{x}(t) = \underline{x}_s(t) + \underline{x}_f(t)$$

with $\underline{x}_f(t)$ given by (3.37) and $\underline{x}_s(t)$ given by

$$\underline{x}_s(t) = e^{Lt}\,\underline{x}_s(0^-) + \int_0^t e^{L(t-\xi)} B_s \underline{u}(\xi)\,d\xi \ . \tag{3.38}$$

It can be easily seen from (3.37) and (3.38) that the reachable subspaces in $V_s$ and $W_f$ are, respectively

$$R_s = \langle L|B_s\rangle$$

and

$$R_f = \langle J|B_f\rangle \ .$$

It is shown next that we may have a situation where $R_f \subset W_f$ and yet all the infinite-zeros are controllable. Henceforth controllability will mean modal controllability.

### III.3.1 Infinite-Zeros Controllability

We shall show that controllability is equivalent to reachability of certain quotient spaces. For this let $W_1 \subset W_f$ be any subspace such that

$$W_1 \oplus \ker J \cap \mathrm{Im} J = \ker J \ . \tag{3.39}$$

Note that $W_1$ provides only simple eigenvectors to the map J, in the sense that if $\{w_i\}$, $i \in \underline{t}$, $t := \dim W_1$, is a basis for

$\mathcal{W}_1$, then $Jw_i = 0$, $i \in \underline{t}$, and there are no generalized eigenvectors starting from $w_i$, $i \in \underline{t}$. In other words, the subspace $\mathcal{W}_1$ is associated with all the simple elementary divisors of $J$.

Let $\bar{\mathcal{W}} = \mathcal{W}_f/\mathcal{W}_1$ and let $P : \mathcal{W}_f \to \bar{\mathcal{W}}$ be the canonical projection. Let $\bar{J}$ be the unique map induced in $\bar{\mathcal{W}}$ such that $\bar{J}P = PJ$. define

$$\bar{B}_f := B_f(\mathrm{mod}\ \mathcal{W}_1) \ : \ \mathcal{U} \to \bar{\mathcal{W}}$$

and (3.40)

$$\bar{C}_f := C_f(\mathrm{mod}\ \mathcal{W}_1) \ : \ \bar{\mathcal{W}} \to C\mathcal{W}_f/C\mathcal{W}_1 \ .$$

Matrix representations for $\bar{J}$, $\bar{B}_f$ and $\bar{C}_f$ can be readily obtained. For this, write

$$\begin{aligned} \mathcal{W}_f &= \mathcal{W}_1 \oplus \mathcal{W}_2 \\ \mathcal{B}_f &= \mathcal{B}_f \cap \mathcal{W}_1 \oplus \mathcal{B}_2 \\ C\mathcal{W}_f &= C\mathcal{W}_1 \oplus \mathcal{Y}_2 \end{aligned} \qquad (3.41)$$

where $\mathcal{W}_2$, $\mathcal{B}_2$ and $\mathcal{Y}_2$ are any subspaces which yield a direct sum, respectively, for $\mathcal{W}_f$, $\mathcal{B}_f$ and $C\mathcal{W}_f$.

In the bases provided by the subspaces in (3.41) it follows that

$$\mathrm{Mat}\ J = \begin{pmatrix} 0 & J_{12} \\ 0 & J_{22} \end{pmatrix} \qquad \mathrm{Mat}\ B_f = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

$$\mathrm{Mat}\ C_f = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix} \ . \qquad (3.42)$$

In the above representation

$$\text{Mat } \bar{J} = J_{22}, \text{ Mat } \bar{B}_f = B_{22}, \text{ Mat } \bar{C}_f = C_{22} .$$

Note from (3.39) that $J\mathcal{W}_1 = 0 \subset \mathcal{W}_1$ and there is a subspace $\hat{\mathcal{W}}_2$ such that $J\hat{\mathcal{W}}_2 \subset \hat{\mathcal{W}}_2$ and $\mathcal{W}_1 \oplus \hat{\mathcal{W}}_2 = \mathcal{W}_f$. The last statement can be verified quickly by thinking of the eigenvector chains of $J$ and (3.39). Thus by Proposition 0.5 in [3.16 ], it follows that the elementary divisors of $J|\mathcal{W}_1$, with those of $\bar{J}$ in $\bar{\mathcal{W}}$, together, give all the elementary divisors of $J$. Hence $\bar{J}$ is a map which possesses all the elementary divisors of order greater than one of $J$.

Let $\hat{\mathcal{W}}_2$ be a subspace as described above. Then, by using as basis $\mathcal{W}_1 \oplus \hat{\mathcal{W}}_2$, it follows that in this basis the subsystem (3.35) can be represented as

$$x_{f1} = -B_{f1}u \qquad (3.43)$$

and

$$J_{22}\dot{x}_{f2} = x_{f2} + B_{f2}u \qquad (3.44)$$

where $x_{f1} \in \mathcal{W}_1$ and $x_{f2} \in \hat{\mathcal{W}}_2$. Here $J_{22}$ and $B_{f2}$ are representations of $\bar{J}$ and $\bar{B}_f$, respectively.

Note that the subsystem (3.43) is "static", in the sense that at each instant t, $\underline{x}_{f1}(t)$ is a linear combination of the control variables $\underline{u}(t)$. So, all the dynamics are concentrated in the subsystem (3.44).

Consider the system determined by the pair $(\bar{J},\bar{B}_f)$. Then this system is reachable if and only if

$$<\bar{J}|\bar{B}_f> = \bar{\mathcal{B}}_f + \bar{J}\bar{\mathcal{B}}_f + \ldots + \bar{J}^{q-1}\,\bar{B}_f = \bar{\mathcal{W}} \quad . \tag{3.45}$$

Test 4.2 in [3.13] describes a necessary and sufficient condition for the infinite-zeros to be observable. The dual of this test gives a necessary and sufficient condition for controllability of the infinite-zeros. The test is described in the following.

Test 3.1: Apply nonsingular transformations on the right of the pencil (sE-A B) so as to bring it to the form

$$(sE_1 - A_1 \quad A_2 \quad B)$$

with $E_1$ of full column rank and $A = [A_1 \quad A_2]$ . Then the system is controllable at infinity if and only if

$$(E_1 \quad A_2 \quad B) \tag{3.46}$$

has full row rank, or $\mathrm{Im}\,E_1 + \mathrm{Im}A_2 + \mathcal{B} = \mathcal{X}$.

We are now in a position to show the link between (3.45) and (3.46).

Theorem 3.2: $<\bar{J}|\bar{\mathcal{B}}_f> = \bar{\mathcal{W}}$ if and only if the system

$$\bar{J}\,\dot{\bar{x}}_f = \bar{x}_f + \bar{B}_f u$$

is controllable (at infinity).

Proof: It is well known [3.16] that $<\bar{J}|\bar{\mathcal{B}}_f> = \bar{\mathcal{W}}$ if and only if

$$\operatorname{Im}(\lambda I-\bar{J}) + \bar{\mathcal{B}}_f = \bar{\mathcal{W}} \quad , \forall\lambda \in \mathbb{C} \ . \tag{3.47}$$

Since $\bar{J}$ is nilpotent it is clear that (3.47) holds for $\lambda \neq 0$. Therefore

$$\langle \bar{J} | \bar{\mathcal{B}}_f \rangle = \bar{\mathcal{W}} \iff \operatorname{Im} \bar{J} + \bar{\mathcal{B}}_f = \bar{\mathcal{W}} \ . \tag{3.48}$$

Applying test 3.1 to the pencil $(s\bar{J}-I \quad \bar{B}_f)$ and taking $\bar{J}$ in Jordan canonical form it is readily seen that the condition (3.46) reduces to (3.48).

□

Based on the result of Theorem 3.2, a procedure can be given to obtain the controllable subsystem of a system described by (3.35).

<u>Procedure 3.1</u>:

1. Choose any subspace $\mathcal{W}_1$ according to (3.39) and choose any subspace $\mathcal{W}_2$ such that $\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{W}_f$ .

2. Compute the subsystem induced in $\mathcal{W}_2$ (see (3.42)).

3. Compute the least $J_{22}$-invariant subspace $\mathcal{W}_c$ which contains $\mathcal{B}_{22}$, i.e., $\mathcal{W}_c = \langle J_{22} | \mathcal{B}_{22} \rangle$ . Taking $\mathcal{W}_c$ as part of a basis for $\mathcal{W}_2$, the following representation is then obtained.

$$\operatorname{Mat} J_{22} = \begin{pmatrix} \hat{J}_{11} & \hat{J}_{12} \\ 0 & \hat{J}_{22} \end{pmatrix} \qquad \operatorname{Mat} B_{22} = \begin{pmatrix} \hat{B}_1 \\ 0 \end{pmatrix} \tag{3.49}$$

so that the subsystem

$$\hat{J}_{22}\,\dot{\hat{x}}_2 = \hat{x}_2$$

is uncontrollable and the subsystem

$$\hat{J}_{11}\,\dot{\hat{x}}_1 = \hat{x}_1 + \hat{B}_1 u \tag{3.50}$$

is controllable.

Remark 3.2

1) According to (3.39) there are many subspaces $\mathcal{W}_1$ which can complement $\ker J \cap \mathrm{Im}J$. Therefore, controllability of the infinite-zeros is associated with reachability of any quotient space defined according to the choice of $\mathcal{W}_1$.

2) Procedure 3.1 described above is not based on the Jordan canonical form as in [3.13] and it is an alternative way to obtain a controllable subsystem.

3) In general, the subspace $\mathcal{W}_c = \langle J_{22}|\mathcal{B}_{22}\rangle$ does not decompose $\mathcal{W}_2$, i.e. the union of the elementary divisors of $\hat{J}_{11}$ and $\hat{J}_{22}$ in (3.49) does not yield the set of the elementary divisors of $J_{22}$. Consequently, although $J_{22}$ has only elementary divisors of order $\geq 2$, it is possible that $\hat{J}_{11}$ has some simple elementary divisors (order one). This follows from Lemma 1 in [3. 6]. If this is the case, the controllable subsystem (3.50) will have some static variables which are induced by the process of extraction of the uncontrollable subsystem.

The ensuing theorem provides a test for controllability of the infinite-zeros in terms of the maps A, E and B.

Theorem 3.3: The infinite-zeros of system (3.1) are controllable if and only if

$$\mathrm{Im}E + A\ \ker E + \mathcal{B} = \mathcal{X} \tag{3.51}$$

Proof: Let $\mathcal{W}_1$ be a subspace as in (3.39) and consider the canonical projection $P : \mathcal{W} \to \mathcal{W}/\mathcal{W}_1 = \bar{\mathcal{W}}$. Let $\bar{J}$ and $\bar{B}_f$ be the maps induced by J and $B_f$ in $\bar{\mathcal{W}}$ (see (3.40)).

Consider a subspace $\hat{\mathcal{W}}_2 \subset \mathcal{W}_f$ such that $\mathcal{W}_1 \oplus \hat{\mathcal{W}}_2 = \mathcal{W}_f$ and $J\hat{\mathcal{W}}_2 \subset \hat{\mathcal{W}}_2$. The representations for $\bar{J}$ and $\bar{B}_f$ are given by $J_{22}$ and $B_{22}$ in (3.42). Note that since $J\hat{\mathcal{W}}_2 \subset \hat{\mathcal{W}}_2$, then $J_{12} = 0$ in (3.42).

The space $\mathcal{X}$ is now decomposed as

$$\mathcal{X} = \mathcal{V}_s \oplus \mathcal{W}_1 \oplus \hat{\mathcal{W}}_2 \tag{3.52}$$

Let M be a map as in Theorem 3.1. Hence, since $\mathrm{Im}E = E\mathcal{X}$ and $\mathcal{W}_1 \subset \ker J = \ker E \subset \mathcal{W}_f$, it follows that pre-multiplication of (3.51) by M results in

$$ME(\mathcal{V}_s \oplus \hat{\mathcal{W}}_2) + MA\ \ker E + M\mathcal{B}.$$

From Theorem 3.1, (3.39) and (3.42) the above expression is equal to

$$\mathcal{V}_s + \mathrm{Im}J_{22} + (\mathcal{W}_1 \oplus \ker J \cap \mathrm{Im}J) + \mathrm{Im}B_{22} \quad . \tag{3.53}$$

But $\ker J \cap \mathrm{Im}J = \ker J_{22}$ and since $J_{22}$ is nilpotent it follows that $\ker J_{22} \subset \mathrm{Im}\, J_{22}$.

From (3.52), it follows that (3.53) can be rewritten as

$$\mathcal{V}_s \oplus \mathcal{W}_1 \oplus (\mathrm{Im}\, J_{22} + \mathrm{Im}\, B_{22}) \ .$$

Hence

$$M(\mathrm{Im}E + A \ker E + \mathcal{B}) = \mathcal{X}$$

if and only if $\mathrm{Im}\, J_{22} + \mathrm{Im}\, B_{22} = \hat{\mathcal{W}}_2$, or by (3.48) if and only if the infinite-zeros are controllable.

□

<u>Remark 3.3</u>:

1) Corollary 4.1 of Chapter IV shows that the regular pencil (sE−A) has no static variables if and only if $A \ker E \subset \mathrm{Im}E$.

In this case condition (3.51) reduces to

$$\mathrm{Im}E + \mathcal{B} = \mathcal{X} = \langle J|\mathcal{B}_f\rangle \oplus \mathcal{V}_s \qquad (3.54)$$

which is the condition given by Rosenbrock [3.11] and Cobb [3.3]. This shows that their condition is correct if no static variables are present.

2) From (3.51) and Theorem (3.1) it follows that

$$\text{Im}J + \ker J + \mathcal{B}_f = \mathcal{W}_f \quad . \tag{3.55}$$

Thus, conditions (3.48), (3.51) and (3.55) are equivalent necessary and sufficient conditions for controllability of the infinite-zeros.

It is possible to give a characterization of the reachable subspace in terms of (E,A) invariant subspaces for the case $\mathcal{W}_f = \bar{\mathcal{W}}$, i.e. the map J in (3.35) has all elementary divisors of order greater than one (no static variables).

Proposition 3.3: If the map J has all elementary divisors of order greater than one, then the reachable subspace associated with the controllable infinite-zeros is the least subspace $\mathcal{W}_c \subset \mathcal{W}_f$ such that

$$E\mathcal{W}_c \subset A\mathcal{W}_c \text{ and } A\mathcal{W}_c + E\mathcal{V}_s \supset \mathcal{B} \; .$$

Proof: By hypothesis, the subspace $\mathcal{W}_1 = 0$ in procedure 3.1. Hence, by (3.45) the reachable subspace $\mathcal{W}_c$ is the least J-invariant subspace $\mathcal{W}_c$ which contain $\mathcal{B}_f$, i.e.

$$J\mathcal{W}_c \subset \mathcal{W}_c \tag{3.56}$$

and

$$\mathcal{W}_c \supset Q_f M\mathcal{B} \quad . \tag{3.57}$$

But (3.56) is equivalent to $E\mathcal{W}_c \subset A\mathcal{W}_c$ and (3.57) implies by Theorem 3.1

$$Q_f M\mathcal{B} \subset Q_f MA\mathcal{W}_c \iff \mathcal{B} \subset (Q_f M)^{-1} Q_f MA\mathcal{W}_c = A\mathcal{W}_c + M^{-1}\mathcal{V}_s .$$

But by (3.20) $M^{-1}\mathcal{V}_s = E\mathcal{V}_s$ and the result follows.

□

## III.3.2 Infinite-Zeros Observability

Consider the system

$$J\dot{x}_f = x_f + \delta J \underline{x}_f(0^-) \qquad (3.58)$$
$$y = C_f\, x_f \quad .$$

Write $\mathcal{W}_f = \mathcal{W}_1 \oplus \hat{\mathcal{W}}_2$ where $\mathcal{W}_1$ is as in (3.39) and $\hat{\mathcal{W}}_2$ is such that $J\hat{\mathcal{W}}_2 \subset \hat{\mathcal{W}}_2$. Then in the basis $\mathcal{W}_1 \oplus \hat{\mathcal{W}}_2$ the above system can be represented as

$$\begin{pmatrix} 0 & 0 \\ 0 & J_{22} \end{pmatrix} \begin{pmatrix} \dot{x}_{f1} \\ \dot{x}_{f2} \end{pmatrix} = \begin{pmatrix} x_{f1} \\ x_{f2} \end{pmatrix} + \begin{pmatrix} 0 \\ \delta\, J_{22}\, \underline{x}_{f2}(0^-) \end{pmatrix}$$

$$y = [C_{f1} \quad C_{f2}] \begin{pmatrix} x_{f1} \\ x_{f2} \end{pmatrix} .$$

Hence $x_{f1} = 0$ and $y = C_{f2}\, x_{f2}$. That is, the variables $x_{f1}$ are nondynamic (there are no impulsive motions in $\mathcal{W}_1$) and the infinite-zero observability is then related to $\bar{\mathcal{W}}$ and to the pair $(C_{f2}, \bar{J})$ induced in $\bar{\mathcal{W}}$.

The system (3.58) is said to be observable if there are no impulsive motions in $\bar{\mathcal{W}}$ which are simultaneously in ker $C_{f2}$. This is a simple extension of the observability of finite-zeros.

<u>Theorem 3.4</u>: $\langle \ker C_{f2} | \bar{J} \rangle = 0$ if and only if the system (3.58) is observable (at infinity).

Proof: Identical to the proof of Theorem 3.2, on considering the pair $(\bar{J}^T, C^T_{f2})$ and test (4.2) in [3.13] .

□

Theorem 3.3 leads to a procedure to obtain an observable system from (3.58).

Procedure 3.2: The first two steps are identical to those in Procedure 3.1. The third one can be described as follows : compute the largest $J_{22}$-invariant subspace $\mathcal{W}_u$ contained in ker $C_{f2}$, $\mathcal{W}_u =$ $= \langle \ker C_{f2} | J_{22} \rangle$. By taking $\mathcal{W}_u$ as part of a basis for $\mathcal{W}_2$, the following representation is then obtained

$$\text{Mat } J_{22} = \begin{pmatrix} \hat{J}_{11} & \hat{J}_{12} \\ 0 & \hat{J}_{22} \end{pmatrix}$$

$$\text{Mat } C_{f2} = [\, 0 \quad \hat{C}_{22} \,]$$

so that the subsystem

$$\hat{J}_{22}\dot{\hat{x}} = \hat{x}_2 + \delta\hat{J}_{22}\,\hat{x}_2(0^-)$$
$$y = \hat{C}_{22}\,\hat{x}_2$$

is observable and the subsystem

$$\hat{J}_{11}\,\dot{\hat{x}}_1 = \hat{x}_1 + \delta\hat{J}_{11}\,\hat{x}_1(0^-) \tag{3.59}$$

is unobservable.

It should be noted that the unobservable impulsive motions in (3.58) will in general consists of delta functionals and their derivatives. This is emphasized here because the definition of unobservable impulsive motions in [3.13] seems to indicate that they consist only of delta functionals. This is so because the definition in [3.13] is based on a Jordan canonical form which does not occur here.

The following theorem shoes how observability of the infinite-zeros can be tested directly from the maps E,A and C. Its proof follows from Theorem 3.4 and the result is a simple dualization of (3.51).

Theorem 3.5: The infinite-zeros of the system (3.1) are observable if and only if

$$\ker E \cap A^{-1}(\mathrm{Im}E) \cap \ker C = 0 \tag{3.60}$$

□

Remark 3.4: Proposition 4.2 in Chapter IV shows that the subspace $\ker E \cap A^{-1}(\mathrm{Im}\,E)$ is the subspace of ker E responsible for the generation of the infinite-zeros in the pencil (sE-A). It is also shown in Corollary 4.2 of Chapter IV that a regular pencil (sE-A) has no infinite-zeros if and only if $\ker E \cap A^{-1}(\mathrm{Im}E) = 0$.

When the regular pencil has no static valuables, i.e. $A \ker E \subset \mathrm{Im}E$, then (3.60) reduces to

$$\ker E \cap \ker C = 0 \quad . \tag{3.61}$$

The dual of Proposition 3.3 gives the unobservable subspace in terms of an (E,A) invariant subspace.

Proposition 3.4: If the map $J$ has all elementary divisors of order greater than one, then the unobservable subspace associated with those unobservable infinite-zeros is the largest subspace $\mathcal{W}_u \subset \mathcal{W}_f$ such that

$$E\mathcal{W}_u \subset A\mathcal{W}_u \quad \text{and} \quad \mathcal{W}_u \subset \ker C.$$

Proof: Analogous to the proof of Proposition 3.3, by using Theorem 3.4.

□

It should be emphasized that only under the hypothesis of Proposition 3.3 (3.4) there is a unique correspondence between controllable (unobservable) infinite-zeros and a reachable (unobservable) subspace. If $J$ has at least one simple elementary divisor, then there are many reachable (unobservable) subspaces associated with those controllable (unobservable) infinite-zeros.

To conclude this section, a characterization in terms of (A,E) invariant subspaces is given for the reachable and unobservable subspaces relative to the system (3.34).

Proposition 3.5:

a) The reachable subspace of the system (3.34) is the least subspace $\mathcal{V}_c \subset \mathcal{V}_s$ such that

$$A\mathcal{V}_c \subset E\mathcal{V}_c \text{ and } E\mathcal{V}_c + A\mathcal{W}_f \supset \mathcal{B}$$

b) The unobservable subspace of the system (3.34) is the largest subspace $\mathcal{V}_u \subset \mathcal{V}_s$ such that

$$A\mathcal{V}_u \subset E\mathcal{V}_u \text{ and } \mathcal{V}_u \subset \ker C$$

Proof: Analogous to the proof of Proposition 3.3.

□

Note that when $\mathcal{V}_s = \mathcal{X}$, $\mathcal{W}_f = 0$ and $E = I$, then a) and b) simply give the reachable and unobservable subspaces for the system $\dot{x} = Ax + Bu$; $y = Cx$.

## III.4 ZERO ASSIGNMENT BY STATE FEEDBACK

This section will deal with the issue of zero placement by state feedback. It is a tradition in the control literature to term pole placement instead of zero placement. The reason for this is that when the systems (3.34) and (3.35) are controllable then the zero structure of the pencils (sI–L) and (sJ–I) is isomorphic to the pole structure of $R(s) = C_s(sI-L)^{-1}B_s$ and $D(s) = C_f(sJ-I)^{-1}B_f$, respectively [3.12, 3.15] . Thus the matrix D(s) is polynomial and has all its poles at infinity.

We prefer the term "zero placement", because we shall be dealing all the time with the pencil (sE –(A+BF)) resulting from a state feedback map F and it is the zero structure of such transformed pencil which determines the dynamical response of the system

$$E\dot{x} = (A+BF)x$$

The first theorem of this section is concerned with a necessary and sufficient condition to bring all the infinite-zeros to finite positions of the complex plane without pre-specification of those positions. Cobb [3.3] has shown that such a condition is given by (3.55) but he has not recognized that it corresponds to the controllability of the infinite-zeros. Our proof parallels his proof but it contains some modifications to take into account the concept of controllability developed here and in [3.13].

Consider the system (3.35) and the state feedback law $u = Fx_f$. The pencil associated with the closed-loop system is then given by $(sJ-I-B_fF)$.

As mentioned in the introduction of this chapter, the number of zeros of the regular pencil (sE-A) is h = rank E. Since the pencil (sI-L) in (3.11) has r finite-zeros, it follows that the pencil (sJ-I) has $\gamma := h-r$ infinite-zeros. Therefore, the maximum degree of $\det(sJ-I-B_fF)$, $\forall F : \mathcal{W}_f \to \mathcal{U}$ is given by $\gamma = \text{rank } J$ (see Corollary 3.1e).

Let $\mathcal{W}_1$ be a subspace as in (3.39), $t := \dim \mathcal{W}_1$ and let $\bar{\mathcal{W}} = \mathcal{W}/\mathcal{W}_1$ be the associated quotient space. Let $\bar{J}$ and $\bar{B}_f$ be the induced maps by $J$ and $B_f$ in $\bar{\mathcal{W}}$. Clearly rank $\bar{J} = \gamma$.

<u>Theorem 3.6</u>: There exists a map $F : \mathcal{W}_f \to \mathcal{U}$ such that $\deg(sJ-I-B_fF) = \gamma$ if and only if $\text{Im}\bar{J} + \bar{\mathcal{B}}_f = \bar{\mathcal{W}}$.

<u>Proof</u>: We restrict ourselves to an example, where $\dim \bar{J} = 5$. The general case only causes complication of notation and does not yield more insight.

Recall that all elementary divisors of $\bar{J}$ are greater than one.

Thus, in our example they must be $s^3$ and $s^2$.

Let $\mathcal{W}_2$ be any subspace such that $\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{W}_f$. Choose a basis $\{e_{11}, e_{12}, e_{13}; e_{21}, e_{22}\}$ for $\mathcal{W}_2$ such that in this basis $J_{22} = \text{mat}\ \bar{J}$ displays two Jordan blocks, one of dimension three and the other one of dimension two.

Let $\bar{F} : \bar{\mathcal{W}} \to \mathcal{U}$ be an arbitrary map and in the above basis write

$$\text{Mat}(I+\bar{B}_f \bar{F}) = I + B_{22} F_2 = h_{ij}, \quad i, j \in \underline{5}.$$

Note that

$$\text{Im}\ J_{22} = \text{span}\{e_{11}, e_{12}, e_{21}\}$$

$$\ker J_{22} = \text{span}\{e_{11}, e_{21}\}$$

Let $\mathcal{T} := \text{span}\{e_{13}, e_{22}\}$. In fact $\mathcal{T}$ is the span of two any cyclic subspaces associated with $s^3$ and $s^2$ of $\bar{J}$ [3.9, vol I, page 200].

Hence

$$I+B_{22}F_2-sJ_{22} = \begin{pmatrix} h_{11} & h_{12}-s & h_{13} & h_{14} & h_{15} \\ h_{21} & h_{22} & h_{23}-s & h_{24} & h_{25} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45}-s \\ h_{51} & h_{52} & h_{53} & h_{54} & h_{55} \end{pmatrix}.$$

It can be seen by Laplace expansion that in order for $\deg \det(I+B_{22}F_2-sJ_{22}) = \text{rank } J_{22} = \text{rank } J = \gamma$, the following condition must hold

$$\det\begin{pmatrix} h_{31} & h_{34} \\ h_{51} & h_{54} \end{pmatrix} \neq 0 \quad . \tag{3.62}$$

Let $Q_t$ be the projection on $T$ along Im $J_{22}$. Then (3.62) is equivalent to the requirement that the map

$$Q_t(I + B_{22} F_2)|\ker J_{22} \tag{3.63}$$

is invertible. Now

$$Q_t(I+B_{22}F_2)|\ker J_{22} = Q_tB_{22}F_2|\ker J_{22} \tag{3.64}$$

since Im $J_{22} \supset \ker J_{22}$.

Let $\tilde{F}_2 = F_2|\ker J_{22}$. A map $\tilde{F}_2$ exists such that $Q_t B_{22} \tilde{F}_2$ is invertible if and only if

$$\text{Im } Q_t B_{22} = Q_t \text{ Im } B_{22} = T \tag{3.65}$$

which is equivalent to

$$\text{Im } J_{22} + \text{Im } B_{22} = W_2 \ . \tag{3.66}$$

From (3.66) it follows that

$$\mathrm{Im}\ \bar{J} + \bar{\mathcal{B}}_f = \bar{\mathcal{W}} .$$

If (3.66) holds, choose a suitable $F_2$ which renders (3.63) invertible. From the representation (3.42) define $F$ according to the partitioning of Mat $B_f$, i.e.

$$F = \begin{bmatrix} 0 & 0 \\ 0 & F_2 \end{bmatrix}.$$

Note that $F|\mathcal{W}_1 = 0$. Hence

$$\mathrm{Mat}(I+B_fF-sJ) = \begin{bmatrix} I & B_{12}F_2-sJ_{12} \\ 0 & I+B_{22}F_2-sJ_{22} \end{bmatrix}$$

and

$$\deg\det(I+B_fF-sJ) = \gamma$$ □

Remark 3.5: Note that expression (3.65) is equivalent to the requirement that the rows of $B_{22}$ corresponding to the last position of each Jordan block are linearly independent. This is simply the controllability test based on the Jordan form described in [3.13].

The next theorem shows that controllability does not only imply that the infinite-zeros can be moved as stated in Theorem 3.6. Controllability of the infinite-zeros is in fact equivalent to the assignment of those zeros to pre-specified values in the complex plane by a suitable state feedback map.

Let $\Lambda_f = \{\lambda_{f_i}\}$ , $i \in \underline{\gamma}$ be an arbitrary symmetric set of

$\gamma$ complex numbers.

<u>Theorem 3.7</u>: $\langle \bar{J} | \bar{B}_f \rangle = \bar{W}$ if and only if there exists a map $F : W_f \to U$ such that the set of roots of $\det(sJ-I-B_fF) = 0$ is $\Lambda_f$.

<u>Proof</u>:

=>) Let $W_1$ be again a subspace as in (3.39) and consider any subspace $W_2$ such that $W_1 \oplus W_2 = W_f$. Let $J_{22}$ and $B_{22}$ be representations of $\bar{J}$ and $\bar{B}_f$, according to (3.42). Since $\langle J_{22} | \text{Im } B_{22} \rangle = W_2$, the basis constructed by Wonham [3.16] can be used so that $J_{22}$ and $B_{22}$ are represented as

$$\text{Mat } J_{22} = \begin{bmatrix} \hat{J}_{11} & & & & 0 \\ & \hat{J}_{22} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \hat{J}_{pp} \end{bmatrix} \quad \text{Mat } B_{22} = \left[\begin{array}{ccccc|c} b_{11} & \cdot & \cdot & \cdot & b_{1p} & \\ & b_{22} & \cdot & \cdot & b_{2p} & \\ & & \cdot & & & \\ & & & \cdot & & \tilde{B}_{22} \\ & 0 & & & \cdot & \\ & & & & b_{pp} & \end{array}\right] \quad \overbrace{\phantom{\tilde{B}_{22}}}^{d}$$

$$\cdots \cdot \qquad (3.67)$$

where $J_{ii}$ is cyclic with minimal polynomial equal to $\alpha_i$, the $i^{th}$ invariant polynomial of $J$. Since $J$ is nilpotent, (i.e. has all eigenvalues = 0) it follows that $\alpha_i = s^{\ell_i}$, for some $\ell_i$, with $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_p$.

It is also shown in [3.16] that the pair $(\hat{J}_{ii}, b_{ii})$ is controllable. Thus we may assume that $\hat{J}_{ii}$ and $b_{ii}$ are represented as

$$\hat{J}_{ii} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdot & \cdots & \cdots & 0 \end{pmatrix}_{\ell_i \times \ell_i} \qquad b_{ii} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{\ell_i \times 1} .$$

The submatrix $\tilde{B}_{22}$ plays no role in the present analysis.

Define $F : \mathcal{W}_f \rightarrow \mathcal{U}$ as

$$F = \begin{pmatrix} 0 & 0 \\ 0 & F_2 \end{pmatrix} \tag{3.69}$$

where the partitioning follows that one of $B_f$ in (3.42) and $F_2$ is given by

$$F_2 = \left[\begin{array}{cccc} f_1^T & & & \\ & f_2^T & & \bigcirc \\ & & \ddots & \\ \bigcirc & & & f_p^T \\ \hline & & \bigcirc & \end{array}\right] \begin{matrix} \\ \\ \\ \\ \}d \end{matrix} \tag{3.70}$$

with

$$f_i^T = [f_{i1}, f_{i2}, \ldots, f_{i\ell_i} -1]^T \qquad , i \in \underline{p} .$$

From (3.42) and (3.69), it follows that

$$\mathrm{Mat}(I+B_f F - sJ) = \begin{pmatrix} I & B_{12}F_2 - sJ_{12} \\ 0 & I+B_{22}F_2 - sJ_{22} \end{pmatrix} . \tag{3.71}$$

From (3.67), (3.68) and (3.70) it follows that $I + B_{22}F_2 - sJ_{22}$ is upper triangular with matrices $P_i(s), i \in \underline{p}$ in its diagonal given by

$$P_i(s) = \begin{pmatrix} 1 & -s & 0 & . & . & 0 \\ 0 & 1 & -s & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & . & & -s \\ f_{i1} & f_{i2} & & & & f_{i\ell_i} \end{pmatrix}_{\ell_i \times \ell_i} \qquad i \in \underline{p} \quad .$$

It can be readily seen that the expansion by the last row of $P_i(s)$ yields

$$p_i(s) := \det P_i(s) = f_{i1}s^{\ell_i - 1} + f_{i2}s^{\ell_i - 2} + \ldots + f_{i\ell_i}$$

whence

$$\det(I + B_f F - sJ) = \prod_{i=1}^{p} p_i(s) \quad .$$

It is clear that the zeros of $p_i(s)$ can be assigned to $\Lambda_f$ by a suitable choice of the vector $f_i^T$.

$\Leftarrow$) Assume that $\langle \bar{J} | \bar{B}_f \rangle$ is a proper subspace of $\bar{W}$. This implies that $\mathrm{Im}\,\bar{J} + \bar{B}_f$ is a proper subspace of $\bar{W}$, which in turn imply by Theorem 3.6 that there is no $F$ such that $\deg\det(sJ - I - B_f F) = \gamma$, which contradicts the hypothesis.

□

<u>Remark 3.6</u>: As mentioned in Section 2, to each matrix $\hat{J}_{ii}$ there corresponds an infinite-zero of order $\ell_i - 1$. The above theorem

thus shows that each infinite-zero is converted into $\ell_i-1$ finite-zeros, which are the roots of $p_i(s)$.

We describe now a procedure of zero placement including the assignment of the finite and infinite-zeros. Such a procedure is similar to the one in [3.3], but we include the result of Theorem 3.7 and some new interpretations.

Let $u = \hat{F}x + v$ where $v$ is an external variable and $\hat{F}$ is defined by

$$\hat{F}|V_s = 0 \tag{3.72}$$

and

$$\hat{F}|W_f = F$$

where $F$ is given by Theorem 3.7.

It follows that

$$\text{Mat } M(sE-A-B\hat{F}) = \begin{pmatrix} sI-L & -B_sF \\ 0 & sJ-I-B_fF \end{pmatrix} . \tag{3.73}$$

The expression (3.73) shows that the pencil has now $h = \text{rank } E$ finite-zeros which are given by $\sigma(E,A) \cup \Lambda_f$.

Let $\Lambda_f$ be such that

$$\sigma(E,A) \cap \Lambda_f = 0. \tag{3.74}$$

Let $\alpha \in \mathbb{C} - \sigma(E,A) \cup \Lambda_f$ and define

$$\hat{V} = \bigoplus_{i=1}^{\delta} \ker\left((\alpha E-A-B\hat{F})^{-1}E - \frac{1}{\alpha-\lambda_{f_i}} I\right)^{m_i} \tag{3.75}$$

where $m_i$ is the multiplicity of $\lambda_{f_i} \in \Lambda_f$, $\sum_{i=1}^{\delta} m_i = \gamma = \text{rank } E - r$.

From (3.72), it follows that

$$(A+B\hat{F})\,\mathcal{V}_s = A\mathcal{V}_s \subset E\mathcal{V}_s \tag{3.76}$$

so $\mathcal{V}_s$ is also $((A+B\hat{F}),E)$ invariant and by Theorem 5 in [3.3] we have

$$\bigoplus_{i=1}^{k} \ker\left((\alpha E-A-BF)^{-1}E - \frac{1}{\alpha-\lambda_i} I\right)^{n_i} = \mathcal{V}_s \quad . \tag{3.77}$$

From (3.75) and (3.77) it follows that

$$\mathcal{V}_s \cap \hat{\mathcal{V}} = 0$$

with

$$\dim(\mathcal{V}_s \oplus \hat{\mathcal{V}}) = \text{rank } E \ . \tag{3.78}$$

From (3.77) and by the fact that the pencil (3.73) is regular it follows that the space $\mathcal{X}$ is decomposed as

$$\mathcal{X} = \mathcal{V}_s \oplus \hat{\mathcal{V}} \oplus \ker E \quad . \tag{3.78a}$$

Define the new map $\hat{M} : \mathcal{X} \to \mathcal{X}$ from its inverse

$$\hat{M}^{-1}x = \begin{cases} Ex & , \quad x \in \mathcal{V}_s \oplus \hat{\mathcal{V}} \\ (A+B\hat{F})x \ , & \quad x \in \ker E \end{cases} \ .$$

From (3.76) and (3.77) it is clear that

$$\hat{M}(A+BF)\,|\mathcal{V}_s = \hat{M}A|\mathcal{V}_s = L.$$

Thus the closed-loop system

$$\hat{M}E\dot{x} = \hat{M}(A+B\hat{F})x + \hat{M}Bv$$

can be decomposed as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ 0 \end{pmatrix} = \begin{pmatrix} L & 0 & 0 \\ 0 & \hat{L} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} v \qquad (3.79)$$

with $x_1 \in \mathcal{V}_s$, $x_2 \in \hat{\mathcal{V}}$ , $x_3 \in \ker E$

$$\hat{L} := \hat{M}(A+B\hat{F})\,|\hat{\mathcal{V}} \qquad \sigma(\hat{L}) = \Lambda_f$$

$$B_1 = Q_s\ \hat{M}B \qquad \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} = (I-Q_s)\hat{M}B$$

where $Q_s$ is the projection on $\mathcal{V}_s$ along $\hat{\mathcal{V}} \oplus \ker E$.

It has been shown in [3.11, 3.12] that a finite-zero $\lambda_i \in \sigma(E,A)$ is controllable if and only if

$$\mathrm{Im}(\lambda_i E-A) + \mathcal{B} = \mathcal{X} \ . \qquad (3.80)$$

Since

$$\mathrm{Im}(\lambda_i E-A-BF) + \mathcal{B} = \mathrm{Im}(\lambda_i E-A) + \mathcal{B}, \quad \forall F : \mathcal{X} \to \mathcal{U}$$

it follows that state feedback does not alter the controllability property.

It can now be easily shown that the set $\sigma(E,A)$, which coincides with the set of eigenvalues of $L$, is controllable if and only if

$$\mathrm{Im}(\lambda_i I-L) + \mathcal{B}_1 = \mathcal{V}_s \tag{3.81}$$

or equivalently

$$\langle L|\mathcal{B}_1\rangle = \mathcal{V}_s \ .$$

If (3.81) holds define a map $\tilde{F} : \mathcal{X} \to \mathcal{U}$

$$\tilde{F}|\hat{\mathcal{V}} \oplus \ker E = 0$$

and (3.82)

$$\tilde{F}|\mathcal{V}_s = \tilde{F}_1$$

where $\tilde{F}_1$ is such that $\sigma(L+B_1\tilde{F}_1) = \Lambda_s$ and $\Lambda_s$ is a set of pre-specified symmetric complex numbers.

The advantage of having (3.74) satisfied is that the subspace $\mathcal{V}_s$ and the map $L$ are preserved and then a suitable map $\tilde{F}$, as in (3.82), can be defined so as not to alter the zeros already assigned (the eigenvalues of $\hat{L}$) and so as to assign the eigenvalues of $L + B_1 \tilde{F}_1$ to $\Lambda_s$.

The following assignment procedure can be established:

Procedure 3.3:

1) Compute the decomposition of Theorem 3.1 and obtain the systems (3.34) and (3.35). (In Chapter IV we suggest an alternative way to obtain the same decomposition).

2) Assign the zeros to the system (3.35) according to Theorem 3.7.

3) Compute $\hat{M}$ and $B_1$. Assign the zeros to the system $\dot{x} = Lx + B_1 u$.

We proceed with an interesting and amusing study of the form of closed-loop eigenvalues in systems described by (3.35). We shall see that such a form reflects once more the symmetry between systems described by (3.34) and (3.35) which had already shown up through the concepts of (A,E) and (E,A) invariant subspaces.

Suppose that a feedback map $F : \mathcal{W}_f \to \mathcal{U}$ has been defined according to Theorem 3.7 and that $\lambda \neq 0$ is a finite-zero of multiplicity one obtained by means of this feedback map.

Let $w$ be the closed-loop eigenvector corresponding to the finite-zero $\lambda$. Then it satisfies

$$(I + B_f F)w = \lambda J w$$

which implies

$$w \in (\lambda J - I)^{-1} \mathcal{B}_f$$

or

$$w \in \mathcal{B}_f + \lambda J\mathcal{B}_f + \ldots + \lambda^{q-1} J^{q-1}\mathcal{B}_f . \tag{3.83}$$

Since the map $F$ in Theorem 3.7 has been chosen so that the pencil $(sJ-I-B_f F)$ is regular, it follows $w \notin \ker J = \ker E$

Consider a single-input system

$$\bar{J}\dot{\bar{x}}_f = x_f + \bar{b}_f\, u$$

where $\bar{J}$ and $\bar{b}_f$ are the maps induced by $J$ and $b_f$ in $\bar{\mathcal{W}} = \mathcal{W}_f/\mathcal{W}_1$ and $\mathcal{W}_1$ is a subspace as in (3.39).

If the system is controllable, then by Theorem 3.2 $\langle\bar{J}|\bar{b}_f\rangle = \bar{\mathcal{W}}$ and thus $\bar{J}$ is cyclic with $\bar{b}_f$ a cyclic generator. Hence $\bar{J}$ and $\bar{b}_f$ admit the following representations

$$\text{Mat } \bar{J} = \begin{pmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & . & . & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{k\times k} \quad \text{Mat } \bar{b}_f = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_{k\times 1} \qquad \ldots \quad (3.84)$$

since $\bar{J}$ is nilpotent. Here $k = n-r-t$, with $t = \dim \mathcal{W}_1$.

From (3.83) and (3.84)

$$w \in \text{Im} \cdot \begin{pmatrix} \lambda^{k-1} \\ . \\ . \\ \lambda \\ 1 \end{pmatrix} \qquad \text{and } w \cdot \notin \ker E \qquad (3.85)$$

For purpose of illustration suppose that dim L = k in (3.34) and that the pair $(L, b_s)$ is controllable. Take $L$ in companion form. Then any closed-loop eigenvector $v$ with associated closed-loop eigenvalue $\alpha \neq 0$ of multiplicity one is such that

$$v \in \operatorname{Im} \begin{bmatrix} 1 \\ \alpha \\ \cdot \\ \cdot \\ \alpha^{k-1} \end{bmatrix} \text{ and } v \notin \ker A \quad . \qquad (3.86)$$

The symmetry between (3.85) and (3.86) is obvious.

We now turn our attention for an extension of a result in [3.10] concerning simultaneous zero (in our terminology) and eigenvector assignment by state feedback.

Proposition 3.6: Consider a controllable generalized linear system given by (3.1). Let $\{\lambda_i\}$ , $i \in \underline{h}$, $h = \text{rank } E$, be any set of (finite) $h$ distinct complex numbers. Let $\{v_i\}$, $i \in \underline{h}$, be nonzero vectors such that $v_i \in \mathcal{X}$ if $\lambda_i$ is real and $v_i = v_j^*$ if $\lambda_i = \lambda_j^*$. Let $\mathcal{V} := \text{span}\ \{v_i\}$ be such that $\mathcal{V} \cap \ker E = 0$. Then there exists a map $F : \mathcal{X} \to \mathcal{U}$ such that $(A+BF)v_i = \lambda_i Ev_i$ and the pencil $(sE-A-BF)$ is regular if and only if

i) the $v_i$, $i \in \underline{h}$, are linearly independent

ii) $v_i \in \mathcal{X}_{\lambda_i} = (\lambda_i E-A)^{-1}\mathcal{B}$ .

Proof: Conditions i) and ii) are the generalizations of Proposition 1 in [3.10] obtained by replacing $(\lambda_i I-A)$ by $(\lambda_i E-A)$. The requirement $V \cap \ker E = 0$ follows from the regularity of any pencil. Modal controllability is necessary and sufficient for the assignment of all zeros (finite and infinite).

As in [3.10] associate with each number $\lambda \in \mathbb{C}$ the matrix

$$P_\lambda = [\lambda E - A \,\vdots\, -B] \ .$$

From (3.80) it follows that rank $P_\lambda = n$ for any finite $\lambda$.

Let

$$Q_\lambda = \begin{pmatrix} N_\lambda \\ R_\lambda \end{pmatrix}$$

be a compatible partitioned matrix whose columns span ker $P_\lambda$. It can be shown that rank $B = m$ implies that the columns of $N_\lambda$ are linearly independent.

=>) From

$$(A+BF)\, v_i = \lambda_i E v_i \quad , \quad i \in \underline{h} \qquad (3.87)$$

it follows that

$$[\lambda_i E-A) \,\vdots\, -B] \begin{pmatrix} v_i \\ Fv_i \end{pmatrix} = 0 \ .$$

Since the columns of $Q_{\lambda_i}$ form a basis for ker $P_{\lambda_i}$, it follows that $v_i \in \text{span}\, N_{\lambda_i} = X_{\lambda_i}$.

From (3.87) and the fact that the pencil (sE-A-BF) is regular it follows that

$$(\alpha E-A-BF)^{-1}Ev_i = \frac{1}{\alpha-\lambda_i}\, v_i \quad , \quad i \in \underline{h} \tag{3.88}$$

where $\alpha \neq \lambda_i$. Since $(\alpha-\lambda_i) \neq (\alpha-\lambda_j)$ whenever $\lambda_i \neq \lambda_j$, (3.88) implies that the $v_i$, $i \in \underline{h}$, must be linearly independent.

$\Leftarrow$) Since $v_i \in X(\lambda_i) = \text{span}\, N_{\lambda_i}$, $i \in \underline{h}$, then the $v_i$ can be expressed as $v_i = N_{\lambda_i} k_i$ for some $k_i$ which is unique. Hence

$$(\lambda_i E-A)N_{\lambda_i} k_i - B\, R_{\lambda_i} k_i = 0$$

and define $F_0 : V \to U$ by

$$F_0 v_i = -R_{\lambda_i} k_i \quad , \quad i \in \underline{h} .$$

It remains to define a suitable extension to $F_0$ such that the pencil (sE-A-BF) is regular.

Since $\dim V = h$, $\dim \ker E = n-h$ and $V \cap \ker E = 0$, it follows that

$$V \oplus \ker E = X \quad . \tag{3.89}$$

Also note that

$$E\mathcal{V} = E(\mathcal{V} \oplus \ker E) = E\mathcal{X} = \operatorname{Im} E.$$

Let $x \in \mathcal{X}$ be represented in (3.89) and consider any extension $\tilde{F} : \mathcal{X} \to \mathcal{U}$ to $F_0$. Let $Ex$ and $(A+B\tilde{F})x$ be represented in the basis

$$E\mathcal{V} \oplus \hat{\mathcal{X}} \tag{3.90}$$

where $\hat{\mathcal{X}}$ is any subspace of dimension n-h which complements Im E.

In the basis (3.89) and (3.90) the maps $E$ and $A+B\tilde{F}$ can be represented as

$$\text{Mat } E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{Mat } A+B\tilde{F} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where $\dim I = h$ and $\sigma(A_{11}) = \{\lambda_i\}$ , $i \in \underline{h}$.

It is clear that the pencil $(sE-A-B\tilde{F})$ is regular if and only if $A_{22}$ is nonsingular.

Consider $Q_{\hat{X}}$, the natural projection on $\hat{\mathcal{X}}$ along $E\mathcal{V} = \operatorname{Im} E$. Then $A_{22}$ nonsingular is equivalent to the nonsingularity of

$$Q_{\hat{X}}(A+B\tilde{F})\,|\ker E\ . \tag{3.91}$$

Now

$$Q_{\hat{X}}(A+B\tilde{F})\,|\ker E = Q_{\hat{X}}\, A|\ker E + Q_{\hat{X}}\, BF_1$$

with $\quad F_1 = \tilde{F}|\ker E.$

There exists a map $F_1 : \ker E \to \mathcal{U}$ such that (3.91) is nonsingular if and only if the zero eigenvalues of $Q_{\hat{X}} A|\ker E$ are controllable by $Q_{\hat{X}} B$, i.e.

$$Q_{\hat{X}}(A \ker E + \mathcal{B}) = \hat{\mathcal{X}} \quad . \tag{3.92}$$

Since the system (3.1) has no uncontrollable infinite-zeros, ir follows from Theorem 3.3 that

$$\operatorname{Im} E + A \ker E + \mathcal{B} = \mathcal{X}$$

which then implies (3.92). Therefore, an appropriate $F_1$ can be chosen so as to make (3.91) nonsingular. Finally define
$F : \mathcal{X} \to \mathcal{U}$ by

$$F|\mathcal{V} = F_0$$

and

$$F|\ker E = F_1 .$$

The definition of a real map $F$ in case conjugate complex numbers are assigned, can be handled in the same way as that in [3.10] . □

The above proposition constitutes an alternative way of zero assignement to that one described in Procedure 3.3. All the zeros are assigned together with their eigenvectors and no decomposition or special basis (as in Theorem 3.7) are necessary.

## III.5 ZERO ASSIGNMENT VIA OBSERVERS

This section is concerned with zero assignment via an observer in systems described by

$$J\dot{x}_f = x_f + B_f u \qquad (3.93)$$
$$y_f = C_f\, x_f \,.$$

We start the section by showing an expected result, namely, that the infinite-zeros can be brought to finite-positions of the complex plane by output feedback if and only if they are controllable and observable through $\bar{C}_f$.

Theorem 3.8: There exists a map $K : \mathcal{Y} \to \mathcal{U}$ such that deg $\det(sJ-I-B_f K C_f) = \gamma$ only if the system (3.93) is controllable and the pair $(C_{22}, J_{22})$ is observable.

Proof: The proof follows the same lines as those of Theorem 3.6 up to the expression (3.64) where now it is required that the map

$$Q_t\, B_{22}\, K_2\, C_{22}|\ker J_{22} \qquad (3.94)$$

is invertible for some $K_2$, where $B_{22}$ and $C_{22}$ are as in (3.42).

Since the pairs $(J_{22}, B_{22})$ and $(C_{22}, J_{22})$ are controllable and observable, respectively, then

i) $\mathrm{Im}\, Q_t\, B_{22} = \mathcal{T}$ (as in 3.65)), which is the controllability condition and

ii) $\ker C_{22} \cap \ker J_{22} = 0$

or

$$\ker \bar{C}_f \cap \ker \bar{J} = 0$$

which is an observability condition derived from Theorem 3.4.

On the other hand, if i) and ii) above hold then the pairs $(0, \mathcal{Q}_t\ B_{22})$ and $(C_{22}|\ker J_{22}, 0)$ are controllable and observable at the mode $\lambda = 0$. It then follows by Lemma 3 in [3.7] that there exists $K_2$ so that (3.94) is nonsingular and hence define $K : \mathcal{Y} \to \mathcal{U}$ by

$$K = \begin{pmatrix} 0 & 0 \\ 0 & K_2 \end{pmatrix}$$

where the partitioning accords with that one of $B_f$ and $C_f$ in (3.42). Thus

$$\mathrm{Mat}(I+B_f\ K\ C_f-sJ) = \begin{pmatrix} I & B_{12}K_2C_{22}-sJ_{12} \\ 0 & B_{22}\ K_2\ C_{22}-sJ_{22} \end{pmatrix}$$

and the result follows. □

*Remark 3.7 (see page 204).

In the following we shall construct an observer for the system (3.93). We shall show that the new system formed by the combination of (3.93) and the observer has zeros freely assignable if and only if (3.93) is controllable and observable.

Consider the following observer

$$J\dot{z} = Nz - Ky_f + B_f u \qquad (3.95)$$

with $N := I + KC_f$, for some $K : \mathcal{Y} \to \mathcal{Z}$ .

Consider a control law

$$u = Fz \tag{3.96}$$

for some $F : \mathcal{Z} \rightarrow \mathcal{U}$ , and define the error between $x_f$ and $z$ by

$$e = x_f - z \quad . \tag{3.97}$$

From (3.93) and (3.95-7), it follows that

$$J\dot{x}_f = (I+B_fF)x_f - B_fFe \tag{3.98}$$

and

$$J\dot{e} = (I + KC_f)e \tag{3.99}$$

or in matrix form

$$\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} \dot{x}_f \\ \dot{e} \end{pmatrix} = \begin{pmatrix} I+B_fF & -B_fF \\ 0 & I+KC_f \end{pmatrix}\begin{pmatrix} x_f \\ e \end{pmatrix} \quad . \tag{3.100}$$

It is clear that the zeros of (3.100) can be assigned by suitable choice of $F$ and $K$ if and only if the system (3.93) is controllable and observable. The zero assignment to (3.99) can be accomplished by a dual procedure to the one shown in the proof of Theorem 3.7.

Note that observability is equivalent to the fact that the observer (3.95) can be constructed with pre-specified finite-zeros.

Assume that the system (3.93) is observable and let the zeros associated with (3.99) be assigned to $\mathbb{C}^-$. Decompose (3.99) as in Theorem 3.1. Hence

$$\dot{e}_1 = L_e\, e_1 \tag{3.101}$$

$$e_2 = 0 \qquad , \; e_2 \in \ker E \tag{3.102}$$

where $L_e$ is a map such that $\sigma(L_e) \subset \mathbb{C}^-$.

Equation (3.101) shows that the error between the dynamic variables of (3.98) and the dynamic variables of (3.95) goes to zero as $t \to \infty$. Equation (3.102) shows that the observer gives at each $t \geq 0$ the exact value of $\underline{x}_f(t)$ in ker E. However, if the ultimate purpose is to assign zeros via an observer and not to recover the value of $\underline{x}_f(t)$ in ker E, then an observer of reduced dimension equal to dim $\bar{\mathcal{W}}$ can be constructed,(*) Here $\bar{\mathcal{W}} = \mathcal{W}_f/\mathcal{W}_1$, with $\mathcal{W}_1$ given by (3.39).

(*) provided that the pair $(\bar{C}_f, \bar{J})$ is observable.

Let $P : \mathcal{W}_f \to \bar{\mathcal{W}}$ and let $\bar{J}$ be the unique map induced in $\bar{\mathcal{W}}$ such that $PJ = \bar{J}P$. Let $\bar{B}_f = PB_f$ and consider the following observer

$$\bar{J}\dot{z} = \bar{N}z - PK\, y_f + \bar{B}_f u \quad . \tag{3.103}$$

From (3.93) it follows that

$$PJ\, \dot{x}_f = Px_f + \bar{B}_f u \ . \tag{3.104}$$

Let $Px_f = \bar{x}_f$. Then from (3.104) we have

$$\bar{J}\dot{\bar{x}}_f = \bar{x}_f + \bar{B}_f u \quad . \tag{3.105}$$

Define the error as

$$e = \bar{x}_f - z$$

and let $K : \mathcal{Y} \to \mathcal{Z}$ be such that

$$K|C_f \mathcal{W}_1 = 0 \ . \tag{3.106}$$

Then

$$(I + KC_f)\mathcal{W}_1 = \mathcal{W}_1 .$$

Let

$$\bar{N} := \overline{(I+KC_f)} \tag{3.107}$$

i.e., $\bar{N}$ is the map induced by $(I+KC_f)$ in $\bar{\mathcal{W}}$.

From (3.103), (3.105) and (3.107), it follows that

$$\bar{J}\dot{e} = \bar{N}e. \tag{3.108}$$

Consider the control law $u = \bar{F}z$. Then from (3.105)

$$\bar{J}\dot{\bar{x}}_f = (I+\bar{B}_f\bar{F})\bar{x}_f - \bar{B}_f\bar{F}e. \tag{3.109}$$

The combination of (3.108) and (3.109) in matrix form results in

$$\begin{pmatrix} \bar{J} & 0 \\ 0 & \bar{J} \end{pmatrix} \begin{pmatrix} \dot{\bar{x}}_f \\ \dot{e} \end{pmatrix} = \begin{pmatrix} I + \bar{B}_f\bar{F} & -\bar{B}_f \bar{F} \\ 0 & \bar{N} \end{pmatrix} \begin{pmatrix} \bar{x}_f \\ e \end{pmatrix} . \tag{3.110}$$

It suffices now show that all the zeros associated with (3.110) can be arbitrarily assigned by the choice of $\bar{F}$ and K, with $K|C_f\mathcal{W}_1 = 0$.

The proof of Theorem 3.7 shows that there always exists $\bar{F}$ such that the zeros associated with the system

$$\bar{J}\dot{\bar{x}}_f = \bar{x}_f + \bar{B}_f u$$

are assigned to pre-specified values, provided that the pair $(\bar{J},\bar{B}_f)$ is controllable.

Dually, if the pair $(\bar{C}_f, \bar{J})$ is observable then the zeros of the pencil $(\bar{N} - s\bar{J})$ can be assigned.

The above discussion can also be translated into matrical language. Consider the representation (3.42) and define

$$F = \begin{pmatrix} 0 & 0 \\ 0 & F_2 \end{pmatrix} \qquad K = \begin{pmatrix} 0 & 0 \\ 0 & K_2 \end{pmatrix} \tag{3.111}$$

according to the partitioning of $B_f$ and $C_f$.

In (3.111), $F_2 : \mathcal{W}_2 \to \mathcal{U}$, where $\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{W}_f$ and $K_2 : \mathcal{Y}_2 \to \mathcal{W}_f$, where $\mathcal{Y}_2$ is such that $\mathcal{CW}_1 \oplus \mathcal{Y}_2 = \mathcal{CW}_f$

Thus

$$\text{Mat}(I+\bar{B}_f\bar{F} - s\bar{J}) = I + B_{22} F_2 - sJ_{22}$$

and

$$\text{Mat}(\bar{N} - s\bar{J}) = I + K_2 C_{22} - sJ_{22}.$$

Hence

$$I + B_f F - sJ = \begin{pmatrix} I & B_{12} F_2 - sJ_{12} \\ 0 & I + B_{22} F_2 - sJ_{22} \end{pmatrix}$$

and

$$I + KC_f - sJ = \begin{pmatrix} I & -sJ_{12} \\ 0 & I+K_2C_{22}-sJ_{22} \end{pmatrix} .$$

Once the triple $(C_{22}, J_{22}, B_{22})$ is controllable and observable, then maps $F_2$ and $K_2$ can be computed from Theorem 3.7 so as to assign the zeros of $(I + B_{22} F_2 - sJ_{22})$ and $(I + K_2 C_{22} - sJ_{22})$.
There are basically two alternatives to assign zeros in a system described by (3.34) and (3.35) by dynamical output feedback. The first one is suggested by Theorem 3.8 and is as follows : use an output feedback map so as to convert all the infinite-zeros into finite ones.

Then compute the decomposition of the closed loop system as in Theorem 3.1 to obtain a system with the form

$$\dot{x}_s = Lx_s + B_s u$$
$$y = C_s x_s$$

where dim L = rank E.

A standard zero placement by dynamical output feedback may now be carried in the above system.

The second way, which is not worked out here, consists of constructing one observer for the subsystem (3.34) and another one for the subsystem (3.35) according to the procedure described previously. What remains to be researched is a procedure by which the observer constructed for the subsystem (3.34) does not interfere with the zeros already assigned by the observer (3.103) to the subsystem (3.35).

To conclude this section we would like to point out a simple fact about output feedback and the regularity of a pencil (sE-A). Given a singular pencil, it may be desirable to try to modify it by output feedback so as to render it regular. A sufficient condition for this to be possible is that A + BKC is nonsingular for some $K : \mathcal{Y} \to \mathcal{U}$ . Such a map exists if and only if the zero eigenvalues of A are controllable and observable, i.e.

$$\text{Im}\, A + \mathcal{B} = \mathcal{X}$$

and

$$\ker C \cap \ker A = 0.$$

Given that the above conditions hold, then almost any output feedback map will make A + BKC nonsingular [3.7].

*REMARK 3.7: By using analogous reasoning to that in the proof of Theorem 3.8 it can be shown that if J has all elementary divisors of order greater than one (i.e. no static variables) then there exists K such that degree $\det(sJ-I-B_f KC_f) = \gamma$ if and only if the system (3.93) is controllable and observable. Note that in this case $\mathcal{W}_1 = 0$ in (3.39) and then K must be such that $Q_t\, B_f\, K\, C_f|\ker J$ is nonsingular.

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# CHAPTER IV

## THE GEOMETRIC STRUCTURE OF A REGULAR PENCIL AND THE USE OF P.D. LAWS IN THE THEORY OF ALMOST INVARIANT SUBSPACES

### IV.1 INTRODUCTION

This chapter begins with a geometric description of a regular pencil which includes the identification of the subspaces associated, respectively, with the finite-zeros and infinite-zeros, as limits of suitable sequences of subspaces.

We then turn our attention to the study of a proportional-derivative (P.D.) state feedback law, $u = F_1x + F_2\dot{x}$, for the linear system $\dot{x} = Ax + Bu$. We present several geometric properties of the closed-loop pencil $s(I-BF_2) - (A+BF_1)$ and we also show how to choose $F_1$ and $F_2$ so that the distributional response of the closed loop system belongs to a prescribed almost controlled invariant subspace. An application is then made for the disturbance decoupling problem and we also stress the importance of a P.D. law in the solution of the almost disturbance decoupling problem. Finally, we show the use of P.I.D. observers in the context of almost conditionally invariant subspaces.

## IV.2 THE REGULAR PENCIL (sE-A)

### IV.2.1 Introduction

The main features of a regular pencil are nowadays fairly well known [4.1, 4.3, 4.7, 4.10, 4.16]. A pencil (sE-A) is said to be regular if it is invertible over the field of the rationals. This implies that the maps E and A are square. We also assume that E is a singular map.

Associated with a pencil (sE-A) we have an autonomous generalized linear system described by

$$E\dot{x} = Ax \tag{4.1}$$

where $x \in \mathcal{X} := \mathbb{R}^n$.

Because some of the results described here have application in linear systems theory we shall emphasize dynamical interpretations relative to (4.1).

Recall from Section III.1 that if the initial condition $\underline{x}(0^-) := x_0$ for (4.1) is arbitrary then we should consider the distributional differential equation

$$E\dot{x} = Ax + \delta E x_0 \tag{4.2}$$

where $\delta$ denotes the delta functional.

We shall stress geometric features of a regular pencil for this kind of pencil plays an important role in subsequent sections but we shall also pay some attention to singular pencils.

For a regular pencil there exists $\alpha \in \mathbb{C}$ such that $\det(\alpha E-A) \neq 0$ and we have seen in Section III.2 that such a pencil has in general finite and infinite-zeros. We have also shown there a modal decomposition of the state space $X$ obtained by Cobb [4.3]. The decomposition is given by a direct sum of two subspaces

$$X = V_s \oplus W_f$$

such that on $V_s$ there is a dynamical motion of (4.2) characterized by the finite-zeros and on $W_f$ there occur impulsive motions due to the infinite-zeros.

We recall below the expressions for $V_s$ and $W_f$ so as to facilitate the comments that follow.

$$V_s = \bigoplus_{i=1}^{k} \ker\left[(\alpha E-A)^{-1}E - \frac{1}{\alpha-\lambda_i}I\right]^{n_i} \tag{4.3}$$

$$W_f = \ker\left((\alpha E-A)^{-1}E\right)^{n-r} \tag{4.4}$$

where $r = \dim V_s = \deg\det(sE-A)$ and $\{\lambda_i\}$, $i \in \underline{k}$, is the set of finite-zeros ($\lambda_i$ has multiplicity $n_i$) of the regular pencil $(sE-A)$.

Although $V_s$ and $W_f$ do not depend on $\alpha$, so long as $(\alpha E-A)$ is nonsingular, the above expressions do not show clearly how $V_s$ and $W_f$ are related to the original maps $E$ and $A$. Further, the computation of $V_s$ requires the knowledge of the finite-zeros.

Several questions can then be posed:

a) Is it possible to obtain characterizations for $\mathcal{V}_s$ and $\mathcal{W}_f$ from the geometry of the maps E and A and at the same time compute them without the knowledge of the finite-zeros?

b) What is the geometric source of the infinite-zeros?

c) What is the condition for a regular pencil (sE-A) not to have infinite-zeros?

d) What is the condition for a pencil (sE-A) to be regular?

We shall provide answers to the above questions and we shall see in Section IV.3 that the analysis developed here has importance in the study of a proportional-derivative state feedback law. We also believe that the answer to some of the above questions and our presentation constitute an original contribution to the theory of pencils.

## IV.2.2 A Useful Representation for the Maps E and A

We have seen in Section III.2 (see Proposition 3.1) that by a choice of two distinct decompositions in the state space $\mathcal{X}$, the maps E and A admit the following decompositions

$$\text{Mat } E = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \qquad \text{Mat } A = \begin{pmatrix} L & 0 \\ 0 & I \end{pmatrix} \tag{4.5}$$

where J is a nilpotent map whose Jordan decomposition yields the number and order of the infinite-zeros and L is a map whose eigenvalues coincide with the finite-zeros of the regular pencil (sE-A).

In the sequel we shall obtain a representation for the maps E and A which does not require regularity of the pencil (sE-A) and that helps to develop some intuition. For this let

$$\mathcal{E} := \text{Im } E \quad ; \quad \mathcal{N} := \ker E$$

and consider the following decompositions of the space $\mathcal{X}$

$$\mathcal{X} = \mathcal{C}_1 \oplus \mathcal{N} \tag{4.6}$$

and

$$\mathcal{X} = \mathcal{E} \oplus \mathcal{C}_2 \tag{4.7}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are any subspaces which yield a direct sum.

Let $x \in \mathcal{X}$ be represented in the decomposition (4.6) and let Ex and Ax be represented in the decomposition (4.7). Note that $E\mathcal{C}_1 = \mathcal{E}$. Thus if $\{c_i\}$, $i \in \underline{\ell}$, $\ell := \dim \mathcal{C}_1$ is a basis for $\mathcal{C}_1$ we may take $\{Ec_i\}$, $i \in \underline{\ell}$, as a basis for $\mathcal{E}$. This implies that if

$$x = c + v \; ; \quad c \in \mathcal{C}_1, \quad v \in \mathcal{N} \tag{4.8}$$

with

$$c = \sum_{i \in \underline{\ell}} \alpha_i \, c_i \tag{4.9}$$

then

$$Ex = Ec = \sum_{i \in \underline{\ell}} Ec_i \tag{4.10}$$

so that both $c$ and Ex have the same representation

$$[\alpha_1, \alpha_2, \ldots, \alpha_\ell]^T := z^T \tag{4.11}$$

with respect to the bases $\{c_i\}$ and $\{Ec_i\}$, $i \in \underline{\ell}$.

Hereafter, the bases $\{c_i\}$ and $\{Ec_i\}$, $i \in \underline{\ell}$, are fixed so that the vector $z$ can be identified with $c$ and Ex.

It then follows that in the decompositions (4.6-7)

$$\text{Mat } E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{Mat } A = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \tag{4.12}$$

where $I$ is the identity map of dim $\mathcal{C}_1$ = dim $\mathcal{E}$ and the partitioning of Mat A follows that of Mat E.

The representation (4.12) presents some interesting features, namely :

1) $$|sE-A| = \begin{vmatrix} sI-\hat{A} & -\hat{B} \\ -\hat{C} & -\hat{D} \end{vmatrix} = |sI-\hat{A}| \; |-\hat{G}(s)| \; .$$

Thus, the pencil (sE-A) is regular if and only if the proper rational matrix $\hat{G}(s)$ is invertible over the field of the rationals with

$$\hat{G}(s) = \hat{C}(sI-\hat{A})^{-1}\hat{B} + \hat{D} \tag{4.13}$$

2) If $\hat{D}$ is nonsingular, then by using (4.8), (4.11-2), the system (4.1) becomes

$$\dot{z} = (\hat{A} - \hat{B}\hat{D}^{-1}\hat{C})z$$

and

$$v = \hat{D}^{-1}\hat{C}z \ . \tag{4.14}$$

It is easy to see that this situation implies regularity of the pencil (sE-A) and absence of infinite-zeros. The regularity is implied by

$$\lim_{s\to\infty} \hat{G}(s) = \hat{D}$$

so that $\hat{G}(s)$ is invertible, whereas the absence of infinite-zeros follows (see Section III.1) from the fact that the number of zeros of a regular pencil is given by dim $E$ and that (4.14) has dim $E$ finite-zeros, which are the eigenvalues of $(\hat{A}-\hat{B}\hat{D}^{-1}\hat{C})$. For this reason (absence of infinite-zeros) we have used equation (4.1) and not (4.2).

3) The separation of the variables $v \in N$. The initial condition $\underline{v}(0^-)$ does not influence the distributional response of (4.2). By using (4.8), (4.11-2) in (4.2) we obtain

$$\dot{z} = \hat{A}z + \hat{B}z + \delta\underline{z}(0^-) \tag{4.15}$$

$$0 = \hat{C}z + \hat{D}v$$

and it is clear that $\underline{\nu}(0^-)$ is immaterial for the response of (4.15).

It is easy to obtain explicit expressions for z(s) and v(s), the Laplace transforms of the distributions $\underline{z}$ and $\underline{\nu}$ in case the pencil is regular. To see this let

$$(sE-A)^{-1} = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix}$$

where E and A are as in (4.12) and the above partitioning accords with that one of E and A. From (4.2)

$$x(s) = (sE-A)^{-1}E\underline{x}(0^-) \tag{4.16}$$

or

$$\begin{bmatrix} z(s) \\ v(s) \end{bmatrix} = \begin{bmatrix} T_{11}(s) & 0 \\ T_{21}(s) & 0 \end{bmatrix} \begin{bmatrix} z(0^-) \\ \nu(0^-) \end{bmatrix} .$$

By using (4.15) we obtain

$$\begin{aligned} T_{11}(s) &= (sI-\hat{A})^{-1}[I-\hat{B}\hat{G}^{-1}(s) \quad \hat{C}(sI-\hat{A})^{-1}] \\ T_{21}(s) &= -\hat{G}(s) \; \hat{C}(sI-\hat{A})^{-1} . \end{aligned} \tag{4.17}$$

The expressions (4.17) can also be found in the work by Francis [4.6] who has studied the distributional convergence of the singularly perturbed system

$$\begin{aligned} \dot{z} &= \hat{A}(\epsilon)z + \hat{B}(\epsilon)v + \delta\, \underline{z}(0^-) \\ \epsilon\dot{v} &= \hat{C}(\epsilon)z + \hat{D}(\epsilon)v \end{aligned}$$

when $\epsilon$, a positive scalar, goes to zero. When $\epsilon = 0$, the above system becomes identical to that one in (4.15).

We can also see from (4.15) that the variable $v \in \mathcal{N}$ "controls" the variable $z \in \mathcal{E}$ (recall that $z$ determines Ex). Here "controls" means that the Bohl distribution $\nu$ with support on $\mathbb{R}^+$ (see Definition 1.3 and Section III.1) is such that it drives the distribution $\underline{z}$ (also Bohl with support on $\mathcal{R}^+$) so as to obey the constraint

$$\hat{C}\underline{z} + \hat{D}\underline{v} = 0 \ .$$

## IV. 2.3 Geometric Features of the Regular Pencil

We start this section by discussing the distributional solution of the autonomous generalized system

$$J\dot{x}_f = x_f + \delta J x_f(0^-) \qquad , \quad \underline{x}_f \in \mathcal{W}_f \tag{4.18}$$

which results from the modal decomposition (4.5) (see also Section III.2).

From Corollary 3.1a, ker $J = \mathcal{N}$ and let $J$ be taken in Jordan canonical form, i.e.,

$$J = \text{diag}(J_1, \ldots, J_p, 0) \tag{4.19}$$

where

$$J_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & \cdot & \cdot & \dots & 1 \\ 0 & \cdot & \cdot & \dots & 0 \end{bmatrix}_{(n_i+1)\times(n_i+1)} \qquad i \in \underline{p} \; .$$

In (4.19) the map J is decomposed into p ($p \geq 0$) Jordan blocks of size $n_i+1 (n_i \geq 1)$ and a zero block of dimension m so that

$$\sum_{i\in\underline{p}} n_i + m = \dim \mathcal{W}_f \; . \tag{4.20}$$

Let $x_f^T = [\tilde{x}_1,\dots,\tilde{x}_p, \tilde{x}_{p+1}]^T$ with $\dim \tilde{x}_i = \tilde{n}_i+1$, $i\in\underline{p}$, and $\dim \tilde{x}_{p+1} = m$.

The block diagonal structure (4.19) implies that the equation (4.18) becomes decomposed into a set of p equations

$$J_i\dot{\tilde{x}}_i = \tilde{x}_i + \delta J_i \; \underline{\tilde{x}}_i(0^-) \tag{4.21}$$

and a trivial equation

$$\tilde{x}_{p+1} = 0. \tag{4.22}$$

The variables $\tilde{x}_{p+1}$ are called static variables by Verghese [4.15] in the sense that they exhibit no dynamical behaviour. It is also shown in [4.15] that the distributional solution of (4.21) is given by

$$\tilde{\underline{x}}_{i,1} = - \tilde{\underline{x}}_{i,2}(0^-)\delta - \ldots - \tilde{\underline{x}}_{i,n_i+1}(0^-)\delta^{(n_i-1)}$$

$$\tilde{\underline{x}}_{i,n_i} = -\tilde{\underline{x}}_{i,n_i+1}(0^-)\delta \tag{4.23}$$

$$\tilde{\underline{x}}_{i,n_i+1} = 0$$

where $\delta^{(j)}$ is $j^{th}$ distributional derivative of the delta functional.

We have seen in Section III.2 (see (3.16-7)) that to each Jordan block there corresponds a chain of generalized eigenvectors given by

$$Ew_{i,1} = 0 \qquad , i \in \underline{p}$$

$$Ew_{i,j} = Aw_{i,j-1}, \; i \in \underline{p} \; , \; j \in \{2,\ldots,n_i+1\} \tag{4.24}$$

and the regular pencil (sE-A) has $p$ infinite-zeros of respective orders $n_i$, $i \in \underline{p}$. We shall identify in a moment such nonnegative integer $p$.

In the sequel we define maps which are of importance in the geometric analysis of a pencil.

Let

a) $T := A : \mathcal{X} \to \mathcal{X}(\text{mod } \mathcal{E})$, which denotes $x \to Ax(\text{mod } \mathcal{E})$

(4.25)

b) $\bar{D} := A : \mathcal{N} \to \mathcal{X}(\text{mod } \mathcal{E})$, which denotes $x \to Ax(\text{mod } \mathcal{E})$, $x \in \mathcal{N}$ .

In the decompositions (4.6-7) the maps above defined admit the following representations

$$\text{Mat } T = [\hat{C} \quad \hat{D}] \qquad \text{Mat } \bar{D} = \hat{D} \tag{4.26}$$

The ensuing proposition identifies the geometric sources of those variables in $N$ which are static.

Proposition 4.1: Let $\tilde{N} \subset N$ be any subspace such that

$$N \cap \ker T \oplus \tilde{N} = N .$$

Then the regular pencil (sE-A) has m := dim $\tilde{N}$ static variables.

Proof: First note that

$$\ker T = A^{-1}(E)$$

and

$$\hat{N} := N \cap \ker T = N \cap A^{-1}(E) = \ker \bar{D} \quad . \tag{4.27}$$

Consider now the following decompositions for $X$

$$X = C_1 \oplus \hat{N} \oplus \tilde{N} \tag{4.28}$$

and

$$X = E \oplus C_3 \oplus A\tilde{N}. \tag{4.29}$$

Here $C_1$ is as in (4.6) and $C_3$ is any subspace which complements to $X$ in the decomposition (4.29). Note that

$$\dim A\tilde{N} = \dim \tilde{N}$$

since $N \cap \ker A = 0$, which is implied by the regularity of the pencil (sE-A). Also note that

$$A\tilde{N} \cap E = 0$$

and

$$A\hat{N} \subset E.$$

Let $x \in X$ be represented in the decomposition (4.28) and Ex and Ax be represented in the decomposition (4.29). Hence

$$\text{Mat } E = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Mat } A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & 0 & 0 \\ A_{31} & 0 & I \end{bmatrix}. \quad (4.30)$$

The identity matrix in Mat E has dim $E$ and the identity matrix in Mat A has dim m.

Consider the pencil (sE-A) with E and A as in (4.30) and post-multiply it by the following nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -A_{31} & 0 & I \end{bmatrix} .$$

This operation does not alter the finite and infinite-zero structure and the resulting pencil is represented as

$$\left[\begin{array}{cc|c} sI-A_{11} & -A_{12} & 0 \\ -A_{21} & 0 & 0 \\ \hline 0 & 0 & I \end{array}\right] . \tag{4.31}$$

It is clear from the block diagonal structure of (4.31) that the finite and infinite-zero structure of (sE-A) can be obtained from the reduced pencil

$$\begin{bmatrix} sI-A_{11} & -A_{12} \\ -A_{21} & 0 \end{bmatrix} .$$

We have thus far shown that the number of static variables is $\geq$ m. The next proposition shows that the number is indeed m, and it is important enough to be stated separately. □

<u>Proposition 4.2</u>. Let $p := \dim \hat{N}$. Then regular pencil (sE-A) has p infinite-zeros.

<u>Proof:</u> Let $\{w_{i,1}\}$, $i \in \underline{p}$, be a basis for $\hat{N}$. To prove the above statement it is enough to show that there are generalized eigen-vectors starting from $\{w_{i,1}\}$, $i \in \underline{p}$ (see (4.24)).

First recall that $A\hat{N} \subset E$ and $N \cap \ker A = 0$. This implies that the vectors

$$e_i = Aw_{i,1} \qquad , \quad i \in \underline{p}$$

are in $E$ and are linearly independent. This in turn implies that

$$e_i = Ew_{i,2} \quad , \quad i \in \underline{p}$$

for some set $\{w_{i,2}\}$ , $i \in \underline{p}$, of linearly independent vectors.

It remains to show that the sets $\{w_{i,1}\}$ and $\{w_{i,2}\}$ are mutually independent, or equivalently that span $\{w_{i,1}\} \cap$ span $\{w_{i,2}\} = 0$. Suppose that there exists $0 \neq x \in$ span $\{w_{i,1}\} \cap$ span $\{w_{i,2}\}$. Then

$$x = \sum_{i \in \underline{p}} \beta_i w_{i,1} = \sum_{i \in \underline{p}} \gamma_i w_{i,2}$$

for some real scalars $\beta_i$ and $\gamma_i$, where at least one $\beta_i$ and $\gamma_i$ are different from zero. This yields

$$Ex = 0 = \sum_{i \in \underline{p}} \gamma_i e_i$$

which con tradicts the fact that the vectors $e_i$, $i \in \underline{p}$, are $\ell$.i. .

□

<u>Remark 4.1</u>

a) Proposition 4.2 has shown that there are p generalized eigenvectors starting from $\{w_{i,1}\}$ and thus p infinite-zeros. We do not know yet the orders of the infinite-zeros, since the positive integers $n_i$ in (4.19) have not been specified. We shall do it later.

b) Proposition 4.2 has also shown that the number of static variables is exactly m, as defined in Proposition 4.1, since $\hat{N}$

generates dynamic variables which behave impulsively.

Corollary 4.1 : The regular pencil (sE-A) has no static variables if and only if $A\mathcal{N} \subset \mathcal{E}$.

Proof: Immediate from Proposition 4.1. □

Corollary 4.2: The regular pencil (sE-A) has no infinite-zeros if and only if $\bar{D}$ is nonsingular, i.e. $\mathcal{N} \cap A^{-1}(\mathcal{E}) = 0$.

Proof:

$\Rightarrow$) If the pencil has no infinite-zeros then by Proposition 4.2

$$\hat{\mathcal{N}} = \mathcal{N} \cap A^{-1}(\mathcal{E}) = 0$$

$\Leftarrow$) If $\bar{D}$ is nonsingular, then $A\mathcal{N} \cap \mathcal{E} = 0$. By using the decompositions

$$\mathcal{X} = \mathcal{C}_1 \oplus \mathcal{N}$$

and

$$\mathcal{X} = \mathcal{E} \oplus A\mathcal{N}$$

where $\mathcal{C}_1$ is as in (4.6), we obtain

$$\text{Mat } E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{Mat } A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & I \end{pmatrix}$$

so that the regular pencil (sE-A) has dim $\mathcal{E}$ finite-zeros which are the eigenvalues of $A_{11}$.

□

We shall analyse now in detail two families of subspaces similar to those introduced in III.2, namely

$$F_1 = \{\mathcal{V} \subset \mathcal{X} | A\mathcal{V} \subset E\mathcal{V},\ \mathcal{V} \subset \mathcal{M}\} \tag{4.32}$$

$$F_2 = \{\mathcal{W}_a \subset \mathcal{X} |\ \mathcal{W}_a = \mathcal{K} \cap E^{-1}(A\mathcal{W}_a)\} \tag{4.33}$$

where $\mathcal{M}$ and $\mathcal{K}$ are given subspaces.

It is easy to see that the family $F_1$ is closed under addition. This implies that $F_1$ contains a supremal element $\mathcal{V}^{\square} \subset \mathcal{M}$ which can be computed through the following sequence [ 4.1].

$$\mathcal{V}^{\square} := \mathcal{V}^n;\ \mathcal{V}^u = \mathcal{M} \cap A^{-1}(E\mathcal{V}^{u-1}),\ u \in \underline{n};\ \mathcal{V}^0 = \mathcal{X} \tag{4.34}$$

Hereafter we shall assume that $\mathcal{M} := A^{-1}(\mathcal{E})$. Thus $\mathcal{V}^{\square}$ is the supremal subspace of the family $F_1$ which is contained in $\mathcal{X}$.

The family $F_2$ also appears in [4.1] but it is not given the meaning and importance that we show such a family possesses. The next proposition states that $F_2$ has a unique least element.

Proposition 4.3: There is a unique element $\mathcal{W}_a^* \in F_2$ such that $\mathcal{W}_a^* \subset \mathcal{W}_a$ for every $\mathcal{W}_a \in F_2$.

Proof: Define a sequence by

$$\mathcal{W}_a^* := \mathcal{W}_a^n;\ \mathcal{W}_a^u = \mathcal{K} \cap E^{-1}(A\mathcal{W}_a^{u-1}),\ u \in \underline{n};\ \mathcal{W}_a^0 = 0\ . \tag{4.35}$$

First note that the sequence is nondecreasing. We have

$\mathcal{W}_a^1 \supset \mathcal{W}_a^0$ and if $\mathcal{W}_a^u \supset \mathcal{W}_a^{u-1}$ , then $\mathcal{W}_a^{u+1} = K \cap E^{-1}(A\,\mathcal{W}_a^u) \supset K \cap E^{-1}$ $(A\mathcal{W}_a^{u-1}) = \mathcal{W}_a^u$.

Thus there exists $k \in \underline{n}$ such that $\mathcal{W}_a^u = \mathcal{W}_a^k$ , $u \geq k$. Set $\mathcal{W}_a^* := \mathcal{W}_a^k$. Then $\mathcal{W}_a^* \in F_2$. To show that $\mathcal{W}_a^*$ is infimal let $\mathcal{W}_a \in F_2$. Then $\mathcal{W}_a \supset \mathcal{W}_a^0$ and if $\mathcal{W}_a \supset \mathcal{W}_a^u$, we obtain

$$\mathcal{W}_a = K \cap E^{-1}(A\mathcal{W}_a) \supset K \cap E^{-1}(A\mathcal{W}_a^u) = \mathcal{W}_a^{u+1} .$$

Thus $\mathcal{W}_a \supset \mathcal{W}_a^u$ , $\forall u \in \underline{n}$ and hence $\mathcal{W}_a \supset \mathcal{W}_a^*$ . □

From now on we take $K := A^{-1}(E)$ in the sequence (4.35).

In the sequel we consider the subspaces $\mathcal{W}_b^*$ and $\mathcal{T}^*$, which belong to $F_2$ and play an important role in the geometric theory of a pencil. Such subspaces are defined by the following sequences.

$$\mathcal{W}_b^* := \mathcal{W}_b^n \; ; \; \mathcal{W}_b^u = E^{-1}(A\mathcal{W}_b^{u-1}),\; u \in \underline{n};\; \mathcal{W}_b^0 = 0 \qquad (4.36)$$

$$\mathcal{T}^* := \mathcal{T}_n \; ; \; \mathcal{T}^u = \mathcal{V}^{\square} \cap E^{-1}(A\mathcal{T}^{u-1}),\; u \in \underline{n};\; \mathcal{T}^0 = 0 . \qquad (4.37)$$

The next lemma shows properties of the sequences above defined and some relations among them.

<u>Lemma 4.1</u>:

a) $E\mathcal{T}^u = A\mathcal{T}^{u-1} \; ; \; \mathcal{T}^u \subset \mathcal{V}^{\square},\; \mathcal{T}^* \in F_1 \cap F_2$

b) $\mathcal{T}^u = \mathcal{V}^{\square} \cap \mathcal{W}_b^u$

c) $\mathcal{W}_a^u = K \cap \mathcal{W}_b^u$

d) $\mathcal{T}^u = \mathcal{V}^\Box \cap \mathcal{W}_a^u$

e) $E\mathcal{W}_b^u = A\mathcal{W}_a^{u-1}$

<u>Proof</u>:

a) The equality $E\mathcal{T}^u = A\mathcal{T}^{u-1}$ is shown in [4.1]. Thus $E\mathcal{T}^* = A\mathcal{T}^*$ which implies $A\mathcal{T}^* \subset E\mathcal{T}^*$ and $E\mathcal{T}^* \subset A\mathcal{T}^*$ so that $\mathcal{T}^* \in F_1 \cap F_2$.

b) Note that $\mathcal{T}^1 = \mathcal{V}^\Box \cap N = \mathcal{V}^\Box \cap \mathcal{W}_b^1$ and if $\mathcal{T}^{u-1} = \mathcal{V}^\Box \cap \mathcal{W}_b^{u-1}$, then

$$\begin{aligned}
\mathcal{T}^u &= \mathcal{V}^\Box \cap E^{-1}(A(\mathcal{V}^\Box \cap \mathcal{W}_b^{u-1})) = \mathcal{V}^\Box \cap E^{-1}(A(A^{-1}(E\mathcal{V}^\Box)) \cap \mathcal{W}_b^{u-1}) \\
&= \mathcal{V}^\Box \cap E^{-1}(A\mathcal{W}_b^{u-1} \cap E\mathcal{V}^\Box) \\
&= \mathcal{V}^\Box \cap E^{-1}(A\mathcal{W}_b^{u-1}) \cap E^{-1}(E\mathcal{V}^\Box) \\
&= \mathcal{V}^\Box \cap (E^{-1}(A\mathcal{W}_b^{u-1}) \cap (\mathcal{V}^\Box + N)) \\
&= \mathcal{V}^\Box \cap (E^{-1}(A\,\mathcal{W}_b^{u-1}) \cap \mathcal{V}^\Box + N) \\
&= E^{-1}(A\mathcal{W}_b^{u-1}) \cap \mathcal{V}^\Box + \mathcal{V}^\Box \cap N \\
&= E^{-1}(A\mathcal{W}_b^{u-1}) \cap \mathcal{V}^\Box = \mathcal{V}^\Box \cap \mathcal{W}_b^u
\end{aligned}$$

c) The proof is analogous to b) by replacing $\mathcal{V}^\Box$ by $K = A^{-1}(E)$.

d) From b) and c)

$$\mathcal{T}^u = \mathcal{V}^{\square} \cap K \cap \mathcal{W}_b^u = \mathcal{V}^{\square} \cap \mathcal{W}_a^u$$

□

e) Note that $E\mathcal{W}_b^1 = 0 = A\mathcal{W}_a^0$ and if $E\mathcal{W}_b^{u-1} = A\mathcal{W}_a^{u-2}$ , then

$$E\mathcal{W}_b^u = EE^{-1}(A\mathcal{W}_b^{u-1}) = A\mathcal{W}_b^{u-1} \cap E$$

and

$$A\mathcal{W}_a^{u-1} = A(A^{-1}(E) \cap E^{-1}(E\mathcal{W}_b^{u-1}))$$

$$= A(A^{-1}(E) \cap (\mathcal{W}_b^{u-1} + N))$$

$$= A(A^{-1}(E) \cap \mathcal{W}_b^{u-1})$$

$$= A\mathcal{W}_b^{u-1} \cap E \ .$$

□

Let (sE-A) be a singular pencil whose columns are linearly independent over the ring of the polynomials. Let x(s) be a solution for

$$(sE-A)\ x(s) = 0 \qquad (4.38)$$

with

$$x(s) = x_k - sx_{k-1} + s^2x_{k-2} - \ldots + (-1)^k\ s^k x_0 .$$

Substituting this solution in (4.38) and equating the coefficients of the same power in s we obtain

$$Ax_k = 0\ ;\ \ Ex_k = Ax_{k-1}\ ;\ldots;\ Ex_1 = Ax_0\ ;\ Ex_0 = 0\ . \qquad (4.39)$$

It has been shown by Gantmacher [4.7] that the vectors $x_i$,

$i \in \{0,1,\ldots,k\}$ are linearly independent. Let $\mathcal{D} = \text{span}\,\{x_i\}$. Then from (4.39), it follows that $E\mathcal{D} = A\mathcal{D}$, whence $\mathcal{D} \subset \mathcal{V}^{\square}$. This simple observation suggests that the subspace $\mathcal{T}^*$ (recall that $E\mathcal{T}^* = A\mathcal{T}^*$ by Lemma 4.1 a) is the subspace which provides vectors for a polynomial basis for ker(sE-A). In other words, if $x_i(s)$, $i \in \underline{\ell}$, is a basis for ker(sE-A) with

$$x_i(s) = x_{i,k_i} - sx_{i,k_i-1} + s^2x_{i,k_i-2} - \ldots + (-1)^{k_i} s^{k_i}x_{i,0}, \quad i \in \underline{\ell}$$

for some set of nonnegative integers $\{k_i\}$, $i \in \underline{\ell}$, then $\mathcal{T}^* = \text{span}\,\{x_{i,j}\}$, $i \in \underline{\ell}$, $j \in \{0,1,\ldots,k_i\}$.

To show this we proceed with a decomposition of $\mathcal{T}^*$. First, note from Lemma 4.1a that $E\mathcal{T}^* = A\mathcal{T}^*$, which implies $\dim(\mathcal{N} \cap \mathcal{T}^*) = \dim(\ker A \cap \mathcal{T}^*)$. Also from Lemma 4.1b, it follows that $\mathcal{N} \cap \mathcal{T}^u = \mathcal{V}^{\square} \cap \mathcal{N}$, which implies $\dim(\mathcal{N} \cap \mathcal{T}^u)$ = constant.

Let

$$t_u := \dim\left(\frac{\mathcal{T}^u}{\mathcal{T}^{u-1}}\right) \quad , \qquad u \in \underline{n}\ .$$

Since $E\mathcal{T}^u = A\mathcal{T}^{u-1}$, we then obtain

$$\dim \mathcal{T}^u - \dim(\mathcal{N} \cap \mathcal{T}^u) = \dim \mathcal{T}^{u-1} - \dim(\ker A \cap \mathcal{T}^{u-1})$$

whence

$$t_u = \text{constant} - \dim(\ker A \cap \mathcal{T}^{u-1}).$$

Thus

$$t_u = t_{u+1} + \dim \left(\frac{\ker A \cap T^u}{\ker A \cap T^{u-1}}\right)$$

and since $T^u \supset T^{u-1}$ , it follows that $t_u \geq t_{u+1}$, $u \in \underline{n}$.

Now let

$$q_i := \text{number of integers in the set } \{t_1, t_2, \ldots, t_n\} \text{ which are } \geq i.$$

Then

$$q_1 \geq q_2 \geq \ . \ . \ . \ \geq q_\ell \geq 1$$

where $\ell := \dim(N \cap T^*) = \dim(\ker A \cap T^*)$.

Write

$$k_{\ell-i+1} := q_i - 1, \quad i \in \underline{\ell}. \tag{4.40}$$

Then $0 \leq k_1 \leq k_2 \leq \ . \ . \ . \ \leq k_\ell$. It now follows from (4.40) and Lemma 4.1a that there exist a set of linearly independent vectors $\{t_{i,j}\}$, $i \in \underline{\ell}$, $j \in \{0,1,\ldots,k_i\}$ such that $T^* = \text{span}$ $\{t_{i,j}\}$ with

$$\begin{aligned} Et_{i,0} &= 0 \\ Et_{i,j} &= At_{i,j-1} \qquad i \in \underline{\ell},\ j \in \underline{k}_i \\ At_{i,k_i} &= 0 \ . \end{aligned} \tag{4.41}$$

We can now state the following theorem.

Theorem 4.1: Let (sE-A) be a singular pencil. Then:

a) $\ker(sE-A) = \text{span}\{x_i(s)\}, i \in \underline{\ell}$, where

$$\ell = \dim(\mathcal{N} \cap \mathcal{T}^*) = \dim(\ker A \cap \mathcal{T}^*)$$

$$x_i(s) = t_{i,k_i} - st_{i,k_i-1} + s^2 t_{i,k_i-2} - \ldots + (-1)^{k_i} s^{k_i} t_{i,0}, \quad i \in \underline{\ell} .$$

The set $\{k_i\}$ , $i \in \underline{\ell}$, is termed the set of minimal column indices.

b) Let $\bar{A}$ be a map such that

$$\bar{A} : \mathcal{V}^{\square}(\text{mod } \mathcal{T}^*) \to E\mathcal{V}^{\square}(\text{mod } E\mathcal{T}^*) .$$

Then

$$\sigma(\bar{A}) = \{\text{finite-zeros of the singular pencil } (sE-A)\}$$

Proof: Let $t \in \mathcal{T}^*$ be represented in the basis $\{t_{i,j}\}$, $i \in \underline{\ell}$, $j \in \{0,1,\ldots,k_i\}$ and let At and Et be represented in the basis $\{Et_{i,j}\}$ , $i \in \underline{\ell}$, $j \in \underline{k}_i$. Then from (4.41) it follows that

$$\text{Mat}[E\mathcal{T}^* \,|\, (sE-A) \,|\, \mathcal{T}^*] = \text{diag}[P_i(s)] , \quad i \in \underline{\ell}$$

where

$$P_i(s) = \begin{bmatrix} s & 1 & \cdot & \cdot & 0 & 0 \\ 0 & s & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & 0 \\ 0 & 0 & \cdot & \cdot & s & 1 \end{bmatrix}_{k_i \times (k_i+1)} .$$

We now show that $P(s) := \mathrm{diag}[P_i(s)]$ corresponds to the set of canonical blocks associated with minimal column indices in Gantmacher's decomposition of a singular pencil.

For this let $\mathcal{V}_1$ be any subspace such that

$$\mathcal{V}^{\square} = \mathcal{T}^* \oplus \mathcal{V}_1 \tag{4.42}$$

and note that $\mathcal{V}_1 \cap \mathcal{N} = 0$ since $\mathcal{N} \cap \mathcal{V}^{\square} \subset \mathcal{T}^*$.

It can also be shown that $E\mathcal{T}^* \cap E\mathcal{V}_1 = 0$ .

For, suppose that $Et = Ev$ for $t \in \mathcal{T}^*$, $v \in \mathcal{V}_1$. Then $v - t \in \mathcal{N} \cap \mathcal{V}^{\square}$ and thus $v \in \mathcal{T}^*$, which is impossible.

Thus we may consider the direct sum

$$E\mathcal{V}^{\square} = E\mathcal{T}^* \oplus E\mathcal{V}_1 \tag{4.43}$$

so that in the decomposition (4.42-3)

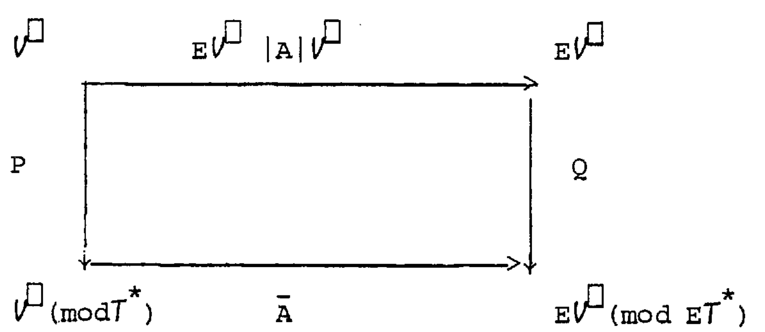
$$\mathrm{Mat}[E\mathcal{V}^{\square} | (sE-A) | \mathcal{V}^{\square}] = \begin{bmatrix} P(s) & -A_{12} \\ 0 & sI-A_{22} \end{bmatrix} \tag{4.44}$$

where $\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \mathrm{Mat}[E\mathcal{V}^{\square} | A | \mathcal{V}_1]$.

Since $(sI-A_{22})$ is a regular pencil it follows that $x_i(s)$, $i \in \underline{\ell}$, as given in the theorem's statement is indeed a basis for $\ker(sE-A)$. The set $\{k_i\}$, $i \in \underline{\ell}$, is the set of minimal column indices due to the uniqueness of the canonical form of singular pencils under strict equivalence transformation [4.7].

Since the eigenvectors associated with the finite-zeros belong to $\mathcal{V}^{\square}$ $(Av = \lambda Ev \Rightarrow v \in \mathcal{V}^{\square})$, it follows that the finite-zeros of the pencil $(sE-A)$ are the eigenvalues of $A_{22}$.

As a matter of fact, $A_{22}$ is the representation of the map $\bar{A} : \mathcal{V}^{\square}(\text{mod } \mathcal{T}^*) \to E\mathcal{V}^{\square}(\text{mod } E\mathcal{T}^*)$. The following diagram shows how $\bar{A}$ is defined.




In the above diagram P and Q are the canonical projections. To show that the diagram commutes, let $\hat{A} := E\mathcal{V}^{\square}|A|\mathcal{V}^{\square}$. Then since $t \in \mathcal{T}^*$ implies $\hat{A}t = E\hat{t}$, for $\hat{t} \in \mathcal{T}^*$, it follows that $\bar{A}P = Q\hat{A}$ and the map $\bar{A}$ is well defined.

□

We turn now our attention to a geometric criterion for regularity of the pencil (sE-A).

Theorem 4.2: The pencil (sE-A) is regular if and only if

$$\mathcal{V}^{\square} \oplus \mathcal{W}_b^* = \mathcal{X} \tag{4.45}$$

Proof:

$\Rightarrow$) If the pencil is regular, then by Theorem 4.1 and Lemma 4.1b we obtain $\mathcal{T}^* = \mathcal{V}^{\square} \cap \mathcal{W}_b^* = 0$. The regularity of the pencil also implies that its number of zeros is dim $\mathcal{E}$.

Now, from Lemma 4.1c, $\mathcal{W}_a^* \subset \mathcal{W}_b^*$. Let $\tilde{\mathcal{W}}_b$ be any subspace such that

$$\mathcal{W}_b^* = \mathcal{W}_a^* \oplus \tilde{\mathcal{W}}_b . \tag{4.46}$$

We show next that dim $\tilde{\mathcal{W}}_b$ = dim $\mathcal{N}$. From Lemma 4.1e, it follows that

$$\dim E\mathcal{W}_b^* = \dim A\mathcal{W}_a^* .$$

Note that $\mathcal{N} \subset \mathcal{W}_b^*$ and ker A $\subset \mathcal{V}^{\square}$. Since $\mathcal{V}^{\square} \cap \mathcal{W}_b^* = 0$ then ker A $\cap\, \mathcal{W}_a^* = 0$. Thus

$$\dim \mathcal{W}_b^* = \dim \mathcal{W}_a^* + \dim \mathcal{N}$$

which confirms our claim.

We shall prove in Theorem 4.3 that the number of zeros of the regular pencil is

$$\dim \mathcal{V}^{\square} + \dim \mathcal{W}_a^* = \dim \mathcal{E}.$$

Therefore from (4.46)

$$\dim \mathcal{V}^{\square} + \dim \mathcal{W}_b^* = \dim \mathcal{V}^{\square} + \dim \mathcal{W}_a^* + \dim \widetilde{\mathcal{W}}_b$$

$$= \dim \mathcal{E} + \dim \mathcal{N} = n$$

so that

$$\mathcal{V}^{\square} \oplus \mathcal{W}_b^* = \mathcal{X}$$

$\Leftarrow$) We first show that $\mathcal{V}^{\square} \oplus \mathcal{W}_b^* = \mathcal{X}$ implies $E\mathcal{V}^{\square} \oplus A\mathcal{W}_b^* = \mathcal{X}$. To see this note that $\mathcal{V}^{\square} \cap \mathcal{W}_b^* = 0$ implies $\mathcal{N} \cap \mathcal{V}^{\square} = 0$ and $\ker A \cap \mathcal{W}_b^* = 0$. Thus $\dim E\mathcal{V}^{\square} = \dim \mathcal{V}^{\square}$ and $\dim A\mathcal{W}_b^* = \dim \mathcal{W}_b^*$.

Now suppose that

$$Ev = Aw, \quad v \in \mathcal{V}^{\square}, \qquad w \in \mathcal{W}_b^* .$$

This implies that $w \in \mathcal{V}^{\square}$ which is not possible since $\mathcal{V}^{\square} \cap \mathcal{W}_b^* = 0$. Thus $w = 0$ and hence $Ev = 0$. Since $\mathcal{N} \subset \mathcal{W}_b^*$ and $\mathcal{V}^{\square} \cap \mathcal{W}_b^* = 0$ it follows that $v = 0$.

Therefore

$$E\mathcal{V}^{\square} \cap A\mathcal{W}_b^* = 0$$

and hence

$$E\mathcal{V}^{\square} \oplus A\mathcal{W}_b^* = \mathcal{X}$$

Now define a map $M : \mathcal{X} \to \mathcal{X}$ as in Proposition 3.1, i.e.

$$M^{-1}x = \begin{cases} Ex \quad , & x \in \mathcal{V}^{\square} \\ Ax \quad , & x \in \mathcal{W}_b^* \end{cases} .$$

Since $A\mathcal{V}^{\square} \subset E\mathcal{V}^{\square}$ and $E\mathcal{W}_b^* \subset A\mathcal{W}_b^*$, it follows that in the decomposition

$$\mathcal{X} = \mathcal{V}^{\square} \oplus \mathcal{W}_b^*$$

$$\text{Mat } M(sE-A) = \begin{pmatrix} sI-L & 0 \\ 0 & sJ-I \end{pmatrix} \tag{4.47}$$

where $L := E\mathcal{V}^{\square}|A|\mathcal{V}^{\square}$ and $J := A\mathcal{W}_b^*|E|\mathcal{W}_b^*$. It will be shown in the next theorem that $J$ is a nilpotent map. Consequently $|sJ-I| = \alpha \neq 0$ and hence from (4.47)

$$\det(sE-A) = \alpha\beta|sI-L| \quad , \quad \beta = |M|^{-1} \neq 0 \tag{4.48}$$

and the pencil is regular.

The next theorem describes the fundamental elements of a regular pencil.

Theorem 4.3: For a regular pencil (sE-A):

a) There are subspaces $\mathcal{V}^{\square}$ and $\mathcal{W}_b^*$ such that

$$\mathcal{V}^{\square} \oplus \mathcal{W}_b^* = \mathcal{X}$$

$$E\mathcal{V}^{\square} \oplus A\mathcal{W}_b^* = \mathcal{X}$$

with

$$A\mathcal{V}^{\square} \subset E\mathcal{V}^{\square} \quad , \qquad \mathcal{V}^{\square} \supset \ker A$$

$$E\mathcal{W}_b^* \subset A\mathcal{W}_b^* \quad , \qquad \mathcal{W}_b^* \supset \ker E$$

b) There are $\dim \mathcal{V}^{\square}$ finite-zeros which are the eigenvalues of the map $L := E\mathcal{V}^{\square}|A|\mathcal{V}^{\square}$.

c) The map $J := A\mathcal{W}_b^*|E|\mathcal{W}_b^*$ is nilpotent and there are $p := (\dim \ker E \cap A^{-1}(\mathcal{E}))$ infinite-zeros of respective orders $n_i$, $i \in \underline{p}$, which are shown to be determined from the sequence (4.35). Further

$$\sum_{i=1}^{p} n_i = \dim \mathcal{W}_a^* \text{ with } \mathcal{W}_a^* = A^{-1}(\mathcal{E}) \cap E^{-1}(A\mathcal{W}_a^*).$$

d) $A^{-1}(\mathcal{E}) = \mathcal{V}^{\square} \oplus \mathcal{W}_a^*$

Proof:

a) See Theorem 3.1 and Section III.2

b) This is clear from (4.47-8).

c) We recall again the notation $\mathcal{K} = A^{-1}(E)$ and $\mathcal{N} = \ker E$. Proposition 4.2 has already shown that there are $\dim(\mathcal{K} \cap \mathcal{N})$ infinite-zeros.

Now define

$$w_u := \dim\left(\frac{\mathcal{W}_a^u}{\mathcal{W}_a^{u-1}}\right) \quad , \; u \in \underline{n} .$$

Since dim $A\mathcal{W}_a^u = \dim \mathcal{W}_a^u$ , it follows from (4.35) that max $w_u \leq p$, as it must be, for we already know that the pencil has p infinite-zeros.

Let

$$n_i := \text{number of integers in the set } \{w_1, w_2, \ldots, w_n\} \text{ which are } \geq i.$$

Then

$$n_1 \geq n_2 \geq \ldots \geq n_p \geq 1$$

with

$$\sum_{i=1}^{p} n_i = \dim \mathcal{W}_a^* .$$

Note that the definition of $n_i$ allows one to identify the size of an eigenvector chain associated with an infinite-zero i, $i \in p$. By using (4.35) it follows from (4.24) that

$$\text{span}\{w_{i,j}\} = \mathcal{W}_a^* \quad , \quad i \in \underline{p}, \quad j \in \underline{n}_i .$$

We may also find the $n_i$'s by using Kalman's crate [4.8] which is a method absolutely equivalent to the above described.

To see that J is nilpotent, suppose $Ew = \lambda Aw$, $\lambda \neq 0$, for some $w \in \mathcal{W}_b^*$. This implies $w \in \mathcal{V}^{\square}$, which is impossible, since $\mathcal{W}_b^* \cap \mathcal{V}^{\square} = 0$ (recall that $\mathcal{V}^{\square}$ is the supremal subspace of the family $F_1$ contained in $\mathcal{X}$).

d) First note that $V^{\square} \oplus W_a^* \subset K$ and let $\tilde{W}_b$ be any subspace such that $W_b^* = W_a^* \oplus \tilde{W}_b$. From Lemma 4.1c $\tilde{W}_b \cap K = 0$. Thus $X = V^{\square} \oplus W_a^* \oplus \tilde{W}_b$ and the result follows. $\square$

Remark 4.4:

a) Note in (4.23) that the components $\tilde{x}_{i,j}$, $j \in \underline{n}_i$ belong to $W_a^*$ and that $\tilde{x}_{i,n_i+1}$ is a component on a subspace such as $\tilde{W}_b$. This shows that the distributional response occurs in $W_a^*$.

b) It is clear from Theorem 4.2 that $V^{\square} = V_s$ and $W_b^* = W_f$, where $V_s$ and $W_f$ are the subspaces given in (4.3-4).

c) There are basically three distinct situations for the initial condition $\underline{x}(0^-)$ in (4.2).

1) $\underline{x}(0^-) \in V^{\square}$. In this case the response $\underline{x}$ in (3.3) is a function and it consists of exponential motions determined by those infinite-zeros which are excited (see equation (3.26)).

2) $\underline{x}(0^-) \in W_b^*$. This is the situation where the response $\underline{x}$ in (3.3) consists only of impulsive motions, namely, the delta functional and its distributional derivatives (see (3.28-9)).

3) $\underline{x}(0^-) \notin V^{\square}$ and $\underline{x}(0^-) \notin W_b^*$. In this case the response $\underline{x}$ in (3.3) consists of impulsive and exponential modes.

The geometric condition for the regularity of a pencil involving the direct sum of two subspaces, as given by Theorem 4.2, is somewhat expected if we recall that pencil is regular if and only if $\hat{G}(s)$ defined in (4.13), is invertible and noting the geometric criterion for this to occur given in (2.42-3). It is however not simple to work with $\hat{G}(s)$ and obtain a condition which displays clearly the geometry of the maps $E$ and $A$.

By introducing suitable sequences of subspaces and analysing their properties we have been able to develop a compact geometric theory of regular pencils. We do not claim that the work developed here is completely new. But we believe to have given a contribution by providing a geometric condition for the regularity of a pencil, by identifying the number and order of infinite-zeros along with their source and by analyzing the origin of static variables.

The Remark 4.a on the distributional response will be shown to be of importance in the ensuing section when we study some properties of a proportional-derivative state feedback for linear systems.

## IV.3 PROPORTIONAL-DERIVATIVE (P.D.) STATE FEEDBACK LAWS AND ALMOST CONTROLLED INVARIANT SUBSPACES

### IV.3.1 Introduction

Consider the multivariable linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{4.49}$$

The solution of several control synthesis problems such as, for example, disturbance decoupling, tracking and regulation involve the use of proportional-integral (P.I.) dynamic compensators. Such compensators operate with the output $y$ and its integral as inputs and produce as output the control $u$ for the system (4.49).

The need for P.I. compensators has stimulated a great deal of research (see [4.12] and the references therein) which has led to a very good understanding of structural properties of such compensators.

It is surprising to note how scarce is the literature on the theory of P.I.D. compensators, namely, compensators which also use the derivative $\dot{y}$ as input. Apart from some results on pole placement [4.13] it seems that little is known about P.I.D. compensators. Our surprise comes from the well known fact that P.I.D. compensators have been employed successfully in some industrial applications.

Most classical control books dedicate a few pages to the subject and explain, usually by means of an example, the qualitative effect caused by the derivative on the closed-loop time response. They also point out the anticipatory feature of a control law which makes use of the derivative. Willems [4.18] has also pointed out the importance of P.I.D. compensators in the synthesis of a controller in the disturbance decoupling problem.

We consider the study undertaken in the following sections as the beginning of a theory for P.I.D. compensators and as such we start by the simplest control law which is given by $u = F_1x + F_2\dot{x}$, for some maps $F_1$ and $F_2$. Such a law is termed a proportional-derivative (P.D.) state feedback law.

We concentrate almost totally our attention on the role of such a law in connection with the theory of almost controlled invariant subspaces.

Willems [4.17] has shown that under a certain condition there exists a high gain state feedback law, $u = Fx$, $F \to \infty$ which solves the *almost* disturbance decoupling problem in the sense that the influence of the disturbance variables d on the regulated variables $z = Dx$ is arbitrarily small.

It is shown here that under the same condition there exists a P.D. law involving *finite* maps which achieves the same objective.

We also solve the *exact* disturbance decoupling problem (the disturbance d has no influence on z) by a law of the type $u = F_1x + F_2\dot{x} + F_3d$. It is important to note that we can solve the problem with such a law in a situation where the law $u = F_1x + F_3d$ cannot do it.

## IV.3.2 Regular P.D. Laws and Almost Controlled Invariant Subspaces

Consider again the linear system

$$\dot{x} = Ax + Bu \tag{4.50}$$

where

$$x \in \mathcal{X} := \mathbb{R}^n \ ; \quad u \in \mathcal{U} := \mathbb{R}^m \ \text{rank } B = m$$

and a P.D. law

$$u = F_1 x + F_2 \dot{x} + v \tag{4.51}$$

for some maps $F_1$, $F_2$ and an external control variable $v$. The resulting closed loop system is then given by

$$(I-BF_2)\dot{x} = (A+BF_1)x + Bv. \tag{4.52}$$

Note that when $(I-BF_2)$ is singular and the pencil $s(I-BF_2) - (A+BF_1)$ is regular, then (4.52) becomes a generalized linear system. Hence the connection among regular pencils, generalized linear systems and P.D. laws. This leads to the following definition.

Definition 4.1: A P.D. law, $u = F_1 x + F_2 \dot{x}$, is termed regular if $(I-BF_2)$ is a singular map and the closed loop pencil $s(I-BF_2) - (A+BF_1)$ is regular for some maps $F_1$ and $F_2$.

Our next theorem establishes the link between regular P.D. laws and almost controlled invariant subspaces. It describes a holdability property of a "trajectory" in an almost controlled invariant subspace $\mathcal{V}_a$ by means of a regular P.D. law.

Theorem 4.4: Let $V_a$ be a given almost controlled invariant subspace and let the pair (A,B) be controllable. Then there exists a regular P.D. law such that the closed loop system

$$(I-BF_2)\dot{x} = (A+BF_1)x$$

has the property that the distributional response $\underline{x}$ with support in $\mathbb{R}^+$ belongs to $V_a$, $\forall \underline{x}(0-) \in V^*_{b,V_a}$.

Proof: The proof is a little long and for this reason we have organized it by steps .

Step 1: Decomposition of the state-space

Consider $V_a$ and its associated $L_p$-almost controlled invariant subspace $V^*_{b,V_a}$. From (1.8) with $K := V_a$, we have

$$V^*_{b,V_a} = R^*_{b,V_a} + V^*_{V_a} \quad . \tag{4.53}$$

Let $\bar{B} \subset B$ be any subspace such that

$$B = B \cap V^*_{V_a} \oplus \bar{B}$$

and consider the subspaces $\bar{R}_{b,V_a}$ and $\bar{R}_{a,V_a}$ obtained through the sequences (1.12-3). Then by Lemma 1.2, it follows that :

a) $$R^*_{b,V_a} = \bar{R}_{b,V_a} \oplus R^*_{V_a}$$

b) $$\bar{\mathcal{R}}_{b,\mathcal{V}_a} \cap \mathcal{V}^*_{\mathcal{V}_a} = 0 \qquad (4.54)$$

c) $$\bar{\mathcal{R}}_{b,\mathcal{V}_a} = A\bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \bar{\mathcal{B}}$$

d) $$\bar{\mathcal{R}}_{a,\mathcal{V}_a} = \mathcal{V}_a \cap \bar{\mathcal{R}}_{b,\mathcal{V}_a}$$

e) $$\mathcal{R}^*_{a,\mathcal{V}_a} = \bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \mathcal{R}^*_{\mathcal{V}_a}$$

f) $$\bar{\mathcal{R}}_{a,\mathcal{V}_a} \cap \mathcal{V}^*_{\mathcal{V}_a} = 0.$$

From the decomposition (1.15) we have that

$$\mathcal{V}_a = \bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \mathcal{V}^*_{\mathcal{V}_a} \qquad (4.55)$$

and from (4.53), (4.54 a,b) it follows that

$$\mathcal{V}^*_{b,\mathcal{V}_a} = \bar{\mathcal{R}}_{b,\mathcal{V}_a} \oplus \mathcal{V}^*_{\mathcal{V}_a} \qquad (4.56)$$

since $\mathcal{R}^*_{\mathcal{V}_a} \subset \mathcal{V}^*_{\mathcal{V}_a}$.

Since the pair (A,B) is controllable then by Theorem 1.8 there exist a controlled invariant subspace $\mathcal{C}$ and a map $F_c \in F(\mathcal{C})$ such that

$$\mathcal{X} = \mathcal{V}^*_{b,\mathcal{V}_a} \oplus \mathcal{C} \qquad (4.57)$$

and

$$\sigma[(A+BF_c)|\mathcal{C}] = \Lambda_c \qquad (4.58)$$

where $\Lambda_c$ is a symmetric set of dim $C$ complex numbers such that $\Lambda_c \cap \Lambda_z = \emptyset$, where $\Lambda_z = \sigma[(A+BF)|V^*_{V_a} \pmod{R^*_{V_a}}]$, $\forall F \in F(V^*_{V_a})$.

From (4.56-7) it now follows that

$$X = V^*_{V_a} \oplus C \oplus \bar{R}_{b,V_a} \tag{4.59}$$

which constitutes our desired decomposition.

Step 2: Definition of the maps $F_1$ and $F_2$.

We now define the maps $F_1$ and $F_2$ on some subspaces above constructed and we shall check later the consistency of the definition with the desired result.

Let $q := \dim \bar{B}$ with $\{b_i\}$, $i \in \underline{q}$, a basis for $\bar{B}$. Then

$$\bar{b}_i = Bu_i, \; i \in \underline{q}$$

for some $u_i$.

Also let $\ell := \dim A\bar{R}_{a,V_a} = \dim\bar{R}_{a,V_a}$ (recall that $\bar{R}_{a,V_a}$ is a sliding subspace) and consider the set $\{\bar{r}_i\}$, $i \in \underline{\ell}$, a basis for $\bar{R}_{a,V_a}$. Thus $\{A\bar{r}_i\}$, $i \in \underline{\ell}$, is a basis for $A\bar{R}_{a,V_a}$.

Define $F_s : \bar{R}_{b,V_a} \to U$ (see (4.54c)) by

$$F_s \bar{b}_i = u_i, \quad i \in \underline{q} \tag{4.60a}$$

and

$$F_s A\bar{r}_i = 0, \quad i \in \underline{\ell} \tag{4.60b}$$

Thus

$$(I-BF_s)\bar{b}_i = 0 \tag{4.61a}$$

and

$$(I-BF_s)A\bar{r}_i = A\bar{r}_i \quad . \tag{4.61b}$$

Define $F_2 : \mathcal{X} \to \mathcal{U}$ by

$$F_2|\bar{\mathcal{R}}_{b,\mathcal{V}_a} = F_s|\bar{\mathcal{R}}_{b,\mathcal{V}_a} \tag{4.62a}$$

$$F_2|(\mathcal{V}^*_{\mathcal{V}_a} \oplus C) = 0 \ . \tag{4.62b}$$

We now proceed towards the definition of $F_1$. For this let $\hat{F}$ be a map specified by

$$\begin{aligned} &\hat{F}|\mathcal{C} = F_c|\mathcal{C} \\ &\hat{F}|\mathcal{V}^*_{\mathcal{V}_a} = F_v|\mathcal{V}^*_{\mathcal{V}_a}, \ F_v \in F(\mathcal{V}^*_{\mathcal{V}_a}) \\ &\hat{F}|\bar{\mathcal{R}}_{b,\mathcal{V}_a} = 0. \end{aligned} \tag{4.63}$$

Let $\mathcal{V}^{\square} := \mathcal{V}^*_{\mathcal{V}_a} \oplus \mathcal{C}$. Then in the decomposition

$$\mathcal{X} = \mathcal{V}^{\square} \oplus \bar{\mathcal{R}}_{b,\mathcal{V}_a} \tag{4.64}$$

we obtain

$$\mathrm{Mat}(A+B\hat{F}) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \tag{4.65}$$

where $A_{11} = \mathrm{Mat}(A+B\hat{F})|\mathcal{V}^{\square}$ and $A_{22} = \mathrm{Mat}(A+B\hat{F})(\mathrm{mod}\ \mathcal{V}^{\square})$.

In the same decomposition (4.64) we obtain from (4.54c) and (4.61-2)

$$\text{Mat}(I-BF_2) = \begin{pmatrix} I & 0 \\ 0 & E_{22} \end{pmatrix} \tag{4.66}$$

where dim I = dim $\mathcal{V}^{\square}$ and $E_{22}$ is a singular map.

From (4.65) and (4.66) we see that the pencil $s(I-BF_2) - (A+B\hat{F})$ admits the following representation in the decomposition (4.64).

$$\text{Mat}[s(I-BF_2)-(A+B\hat{F})] = \begin{pmatrix} sI-A_{11} & -A_{12} \\ 0 & sE_{22}-A_{22} \end{pmatrix} \tag{4.67}$$

The representation (4.67) shows that if $A_{22}$ is nonsingular then the above pencil is regular (we shall show later that the regularity of the pencil, with $\hat{F}$ defined by (4.63), also implies $A_{22}$ nonsingular).

Our next step consists of the definition of $F_1$ such that the map $(A+BF_1)(\text{mod } \mathcal{V}^{\square})$ is nonsingular. For this, note from (4.54b,c) that

$$\bar{\mathcal{R}}_{b,\mathcal{V}_a} = A\bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \bar{\mathcal{B}} = \bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \tilde{\mathcal{R}}_{b,\mathcal{V}_a} \tag{4.68}$$

where $\tilde{\mathcal{R}}_{b,\mathcal{V}_a}$ is any subspace which yields a direct sum.

Consider the decompositions

$$\mathcal{X} = \mathcal{V}^{\square} \oplus \bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \tilde{\mathcal{R}}_{b,\mathcal{V}_a} \tag{4.69a}$$

and

$$\mathcal{X} = \mathcal{V}^{\square} \oplus A\bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \bar{\mathcal{B}} \; . \tag{4.69b}$$

Let $x \in \mathcal{X}$ be represented in (4.69a) and $(A+B\hat{F})x$ be represented in the decomposition (4.69b). Then

$$\mathrm{Mat}(\widehat{A+BF}) = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & I & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} \qquad \mathrm{Mat}\, B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & B_3 \end{pmatrix} \qquad (4.70)$$

where
$$A_{11} = \mathrm{Mat}(\widehat{A+BF}) \mid \mathcal{V}^{\square}$$
$$B_1 = \mathrm{Mat}\mathcal{B} \cap \mathcal{V}^{*}_{\mathcal{V}_a}$$
$$B_3 = \mathrm{Mat}\bar{\mathcal{B}}$$
$$\dim I = \dim \bar{\mathcal{R}}_{a,\mathcal{V}_a}.$$

It is easy to see that if $A_s := \mathrm{Mat}\ (\widehat{A+BF})(\mathrm{mod}\ \mathcal{V}^{\square})$ then

$$A_s = P_2 \begin{pmatrix} I & A_{23} \\ 0 & A_{33} \end{pmatrix} \qquad (4.71)$$

for some nonsingular matrix $P_2$ of dim $\bar{\mathcal{R}}_{b,\mathcal{V}_a}$. For this, let Q and P be nonsingular matrices composed of linearly independent vectors from the decompositions (4.69a) and (4.69b). Note that

$$P = Q \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

for some nonsingular matrices $P_1$ and $P_2$ such that dim $P_1$ = dim $\mathcal{V}_s$ and dim $P_2$ = dim $\bar{\mathcal{R}}_{b,\mathcal{V}_a}$.

Hence from (4.70)

$$(A+B\hat{F})Q = Q \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} A_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_s \end{pmatrix}$$

with

$$\tilde{A}_{12} := [0 \quad A_{13}]$$

$$\tilde{A}_s := \begin{pmatrix} I & A_{23} \\ 0 & A_{33} \end{pmatrix}$$

which establishes the claim (4.71).

Note that $\dim \tilde{\mathcal{R}}_{b,\mathcal{V}_a} = \dim \bar{\mathcal{B}}$ and that $B_3$ is nonsingular. Thus the pair $(A_{33}, B_3)$ is controllable and hence there exists $F_3$ such that $A_{33} + B_3F_3$ is nonsingular. Equivalently, there exists $\tilde{F} : \tilde{\mathcal{R}}_{b,\mathcal{V}_a} \to \mathcal{U}$ such that $(P_{\bar{\mathcal{B}}} A|\tilde{\mathcal{R}}_{b,\mathcal{V}_a} + P_{\bar{\mathcal{B}}} B\tilde{F})$ is nonsingular, where

$P_{\bar{\mathcal{B}}}$ is the projection on $\bar{\mathcal{B}}$ along $\mathcal{V}^{\square} \oplus A\bar{\mathcal{R}}_{a,\mathcal{V}_a}$.

Finally, define

$$F_1 = \hat{F} + \tilde{F}$$

and note that

$$\begin{aligned} F_1|\mathcal{V}^{\square} &= \hat{F}|\mathcal{V}^{\square} \\ F_1|\bar{\mathcal{R}}_{a,\mathcal{V}_a} &= 0 \\ F_1|\tilde{\mathcal{R}}_{b,\mathcal{V}_a} &= \tilde{F}|\tilde{\mathcal{R}}_{b,\mathcal{V}_a} \end{aligned} \qquad (4.72)$$

It follows now from (4.71) that $(A+BF_1)(\mathrm{mod}\ \mathcal{V}^{\square})$ is non-singular and consequently the pencil $s(I-BF_2) - (A+BF_1)$ is regular.

Step 3: Geometric validation

Let

$$\tilde{E} := (I-BF_2) \ ; \ \tilde{A} := (A+BF_1)$$

$$\mathcal{W}_b^* := \bar{\mathcal{R}}_{b,\mathcal{V}_a} \quad , \ \mathcal{V}^{\square} = \mathcal{V}_{\mathcal{V}_a}^* \oplus C \quad .$$

Now, if $(s\tilde{E}-\tilde{A})$ is a regular pencil with a subspace $\mathcal{V}^{\square}$ spanned by eigenvectors associated with the finite-zeros and a subspace $\mathcal{W}_b^*$ which contains the eigenvectors associated with the infinite-zeros, we must have by Theorem 4.3

a) $\quad \mathcal{X} = \mathcal{V}^{\square} \oplus \mathcal{W}_b^*$

b) $\quad \mathcal{X} = \tilde{E}\mathcal{V}^{\square} \oplus \tilde{A}\mathcal{W}_b^* \qquad (4.73)$

such that

c) $\quad \tilde{A}\mathcal{V}^{\square} \subset \tilde{E}\mathcal{V}^{\square} \ , \quad \mathcal{V}^{\square} \supset \ker\tilde{A}$

d) $\quad \tilde{E}\mathcal{W}_b^* \subset \tilde{A}\mathcal{W}_b^* \ , \quad \mathcal{W}_b^* \supset \ker\tilde{E} \quad .$

We proceed with the verification of items a-d.

a) It is true by (4.64) .

b) Note that by (4.62b)

$$(I-BF_2)\mathcal{V}^{\square} = \mathcal{V}^{\square} \quad . \qquad (4.74)$$

Let $(A+BF_1)$ be represented in the decomposition (4.64).

Hence

$$\mathrm{Mat}(A+BF_1) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{4.75}$$

where $A_{11} = \mathrm{Mat}(A+BF_1)|\mathcal{V}^{\square}$ and $A_{22} = \mathrm{Mat}(A+BF_1)(\mathrm{mod}\ \mathcal{V}^{\square})$ which is nonsingular by construction.

Hence for $0 \neq s \in \bar{\mathcal{R}}_{b,\mathcal{V}_a}$, we have

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ s \end{bmatrix} = \begin{bmatrix} A_{12}s \\ A_{22}s \end{bmatrix} \notin \mathcal{V}^{\square} \tag{4.76}$$

since $A_{22}s \neq 0$. Also from (4.76) it follows that $\ker(A+BF_1) \subset \mathcal{V}^{\square}$. Hence from (4.75), (4.76)

$$\mathcal{X} = \mathcal{V}^{\square} \oplus (A+BF_1)\bar{\mathcal{R}}_{b,\mathcal{V}_a}$$

c) From (4.62b) and (4.72)

$$(I-BF_2)\mathcal{V}^{\square} = \mathcal{V}^{\square} \supset (A+BF_1)\mathcal{V}^{\square} .$$

We have already shown in b) that $\mathcal{V}^{\square} \ni \ker(A+BF_1)$.

d) For any $F_2 : \mathcal{X} \to \mathcal{U}$

$$\ker(I-BF_2) \subset \mathcal{B}.$$

But from (4.62b), $F_2|(\mathcal{B} \cap \mathcal{V}^*_{\mathcal{V}_a}) = 0$. Thus from (4.61a)

$$\ker(I-BF_2) = \bar{\mathcal{B}} \subset \bar{\mathcal{R}}_{b,\mathcal{V}_a} \quad .$$

In fact, it can be readily seen from the definition of $F_2$ that

$$\mathrm{Im}(I-BF_2) = A\bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \mathcal{V}^{\square} \quad . \tag{4.77}$$

Furthermore, by (4.54b)

$$(A+BF_1)\bar{\mathcal{R}}_{b,\mathcal{V}_a} \supset (A+BF_1)\bar{\mathcal{R}}_{a,\mathcal{V}_a} = A\bar{\mathcal{R}}_{a,\mathcal{V}_a} = (I-BF_2)\bar{\mathcal{R}}_{b,\mathcal{V}_a}$$

which verifies d).

<u>Step 4</u>: The zeros of the pencil $s(I-BF_2) - (A+BF_1)$.

From (4.77) we have that the number of zeros of the regular pencil $(s\tilde{E}-\tilde{A})$ is equal to $\dim \bar{\mathcal{R}}_{a,\mathcal{V}_a} + \dim \mathcal{V}^{\square}$. We shall check in the following that the number of finite-zeros is $\dim \mathcal{V}^{\square}$ and that the number of infinite-zeros is $\dim \bar{\mathcal{R}}_{a,\mathcal{V}_a}$ . This will confirm that the subspace $\mathcal{V}^{\square}$ is spanned by eigenvectors associated with finite-zeros and that the eigenvectors corresponding to infinite-zeros belong to $\mathcal{W}_b^*$. For this, we proceed by proving that

$$\tilde{A}^{-1}(\tilde{E}) = \bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \mathcal{V}^{\square}$$

i.e.,

$$\bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \mathcal{V}^{\square} = \{x \mid (A+BF_1)x \in A\bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \mathcal{V}^{\square}\} \ .$$

From the definition of $F_1$ we obviously have $\bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \mathcal{V}^{\square} \subset \tilde{A}^{-1}(\tilde{E})$. Now consider the decomposition (4.69a) and let $x = \tilde{r} + \bar{r} + v$ with $\tilde{r} \in \tilde{\mathcal{R}}_{b,\mathcal{V}_a}$, $\bar{r} \in \bar{\mathcal{R}}_{a,\mathcal{V}_a}$, $v \in \mathcal{V}^{\square}$. Then if $x \in \tilde{A}^{-1}(\tilde{E})$ we must have

$$(A+BF_1)x = A\bar{r}_1 + v_1 \tag{4.78}$$

for some $\bar{r}_1 \in \bar{\mathcal{R}}_{a,\mathcal{V}_a}$ and some $v_1 \in \mathcal{V}^{\square}$.

From the definition of $F_1$ we obtain from (4.78)

$$(A+BF_1)\tilde{r} \in A\bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \mathcal{V}^{\square}.$$

Hence

$$(A+BF_1)\tilde{r} = (A+BF_1)\bar{r}_2 + v_2$$

or

$$(A+BF_1)(\tilde{r}-\bar{r}_2) = v_2$$

for some $\bar{r}_2 \in \bar{\mathcal{R}}_{a,\mathcal{V}_a}$ and $v_2 \in \mathcal{V}^{\square}$. By (4.73b)

$$v_2 = 0 \text{ and } \tilde{r} - \bar{r}_2 = 0 \tag{4.79}$$

which implies $\tilde{r} \in \bar{\mathcal{R}}_{a,\mathcal{V}_a}$ and thus $\tilde{A}^{-1}(\tilde{E}) = \bar{\mathcal{R}}_{a,\mathcal{V}_a} \oplus \mathcal{V}^{\square}$.

From (4.73) it follows that the number of finite-zeros of the pencil $(s\tilde{E}-\tilde{A})$ is at least dim $\mathcal{V}^{\square}$. To show that it is exactly dim $\mathcal{V}^{\square}$ suppose that there exists a pair eigenvalue-eigenvector $(\lambda,s)$, $0 \neq s \in \mathcal{W}_b^*$ such that

$$(A+BF_1)s = \lambda(I-BF_2)s.$$

By (4.77) it follows that

$$(A+BF_1)s = \lambda A\bar{r} + \lambda v, \quad \bar{r} \in \bar{R}_{a,V_a}, \; v \in V^{\square} \qquad (4.80)$$

and from the definition of $F_1$

$$(A+BF_1)(s-\lambda\bar{r}) = \lambda v.$$

Thus by (4.73b) $\lambda v = 0$ and $s = \lambda\bar{r}$, so that (4.80) reduces to

$$\lambda A\bar{r} = \lambda A\bar{r}$$

which corresponds to the trivial equality

$$(A+BF_1)\bar{R}_{a,V_a} = (I-BF_2)A\bar{R}_{a,V_a}.$$

We shall now establish that the number of infinite-zeros (counting their orders or multiplicities) is indeed dim $\bar{R}_{a,V_a}$.

By Proposition 4.2 we have that the regular pencil $(s\tilde{E}-\tilde{A})$ has

$$p := \dim[\bar{B}\cap\tilde{A}^{-1}(\tilde{E})] = \dim \bar{B} \cap (\bar{R}_{a,V_a} \oplus V^{\square})]$$

infinite-zeros. (We recognize that this is a little confusing. What we mean is that there are $p$ infinite-zeros of respective orders $n_i$, $i \in \underline{p}$, for positive integers $n_i$).

Let $b^* \in B^*$, where $B^*$ is any subspace such that

$$\bar{B} = \bar{B} \cap \bar{R}_{a,V_a} \oplus B^*.$$

Then as in (4.78-9) we have that $(A+BF_1)b^* \in \text{Im}(I-BF_2)$ implies $b^* = 0$. Thus $\bar{B} \cap \tilde{A}^{-1}(\tilde{E}) = \bar{B} \cap \bar{R}_{a,V_a}$ and consequently the number of infinite-zeros is $p = \dim(\bar{B} \cap \bar{R}_{a,V_a})$. It is not difficult to show that the subspace $\bar{R}_{a,V_a}$ is equal to the subspace $W_a^*$ defined by the sequence (4.35). From the sequence (1.13) we have that $\bar{R}_{a,V_a} = \bar{R}_a^n$ ; $\bar{R}_a^u = V_a \cap (A\bar{R}_a^{u-1} + \bar{B})$; $\bar{R}_a^0 = 0$ and from (4.35) with $K = \tilde{A}^{-1}(\tilde{E}) = \bar{R}_{a,V_a} \oplus V^{\square} = V_a \oplus C$ , $W_a^* = W_a^n$; $W_a^u = (V_a \oplus C) \cap (I-BF_2)^{-1}((A+BF_1)W_a^{u-1})$, $W_a^0 = 0$.

Hence as above

$$\bar{R}_a^1 = V_a \cap \bar{B} = W_a^1$$

and if $W_a^{u-1} = \bar{R}_a^{u-1}$, then since $\ker F_1 \supset \bar{R}_{a,V_a}$

$$\begin{aligned} W_a^u &= (V_a \oplus C) \cap (I-BF_2)^{-1}(A\bar{R}_a^{u-1}) \\ &= (V_a \oplus C) \cap (A\bar{R}_a^{u-1} + \bar{B}) \\ &= V_a \cap (A\bar{R}_a^{u-1} + \bar{B}) = \bar{R}_a^u \quad . \end{aligned} \tag{4.81}$$

Since $\bar{R}_a^u = W_a^u$ , $u \in \underline{n}$, it follows that $\bar{R}_{a,V_a} = W_a^*$ and that the orders of the infinite-zeros of $(s\tilde{E}-\tilde{A})$, $n_i$, $i \in \underline{p}$, as defined by Theorem 4.3 coincide with the positive integers defined by (2.18) with $\sum_{i=1}^{p} n_i = \dim \bar{R}_{a,V_a}$ .

Let $\underline{x}(0-) \in V_a$, $\underline{x}(0-) = \underline{x}_1(0-) + \underline{x}_2(0-)$ with $\underline{x}_1(0-) \in V_{V_a}^*$ and $\underline{x}_2(0-) \in \bar{R}_{b,V_a}$. Then by (3.29) and (3.30) the distributional response of $(I-BF_2)\dot{x} = (A+BF_1)x$ is given by

$$\underline{x} = \underline{x}_f + \underline{x}_s$$

with

$$\underline{x}_f = - \sum_{i=1}^{\alpha-1} \delta^{i-1} J^i \underline{x}_2(0-)$$

$$\underline{x}_s : t \to e^{L_v t} \underline{x}_1(0-), \quad t \in \mathbb{R}^+$$

where $\alpha-1$ is the largest order of the infinite-zeros, $J$ is a nilpotent map and $L_v$ is a map given by

$$L_v = (A+BF_1) | \mathcal{V}^*_{\mathcal{V}_a} .$$

The function $\underline{x}_s(t)$ clearly remains in $\mathcal{V}^*_{\mathcal{V}_a}$ and by Remark 4. a, the distribution $\underline{x}_f$ lies in $\bar{\mathcal{R}}_{a,\mathcal{V}_a}$. Thus $\underline{x} \in \mathcal{V}_a$ □

Remarks 4.3:

1) It should be noted that the controllability hypothesis has been used only to find the controlled invariant subspace $\mathcal{C}$ and therefore it is unnecessary in case $\mathcal{V}^*_{b,\mathcal{V}_a} = \mathcal{X}$.

2) From (4.66) and (4.75) we obtain

$$|s\tilde{E} - \tilde{A}| = |sI - A_{11}| \; |sE_{22} - A_{22}|$$

and since the pencil $(s\tilde{E}-\tilde{A})$ has dim $\mathcal{V}^{\square}$ finite-zeros and is regular we must have $|sE_{22} - A_{22}| = \text{const.} \neq 0$, which implies that the pencil $(sE_{22} - A_{22})$ has only infinite-zeros and therefore $A_{22} = \text{Mat}(A+BF_1)$ (mod $\mathcal{V}^{\square}$) has to be nonsingular. We stress this point to show that the nonsingularity of $(A+BF_1)(\text{mod } V^{\square})$, for $F_1 \in F(V^{\square})$ and $F_2$ as in (4.62), is a necessary and sufficient condition for the regularity of the pencil $s(I-BF_2) - (A+BF_1)$.

3) Let $F \in F(\bar{R}_{a,V_a})$ be a map as in Theorem 1.4c and let $A_F := A+BF$, for such a map F. From (1.29) we have that

$$A\bar{R}_a^{u-1} + \bar{B} = A_F\bar{R}_a^{u-1} + \bar{B} \quad , \quad u \in \underline{n}.$$

Thus the sequence $\bar{R}_a^u$ is not altered by the change of A by $A_F$.

From the above equality and Lemma 1.2c it follows that the sequence $\bar{S}_u$ is also invariant under the above feedback transformation. Moreover, the sequence $V^u$ in (1.3) is invariant under the transformation $A \to A+BF$.

Thus, by the above considerations, we may replace the map A in Theorem 4.3 by $A_F$. This leads to a simple visualization of an eigenvector chain associated with an infinite-zeros of the pencil $s(I-BF_2) - (A_F+BF_1)$.

Note from (2.22) that

$$\bar{R}_{a,V_a} = L_1 \oplus L_2 \oplus \ldots \oplus L_p$$

with

$$L_i = b_i + A_F b_i + \ldots + A_F^{n_i-1} b_i \quad , \quad i \in \underline{p}$$

and $\text{span}\{b_1, b_2, \ldots, b_p\} = \bar{B} \cap \bar{R}_{a,V_a}$.

From the definition of $F_1$ and $F_2$ in the previous theorem we obtain

$$(I-BF_2)b_i = 0$$

$$(I-BF_2)A_F^j b_i = A_F^j b_i = (A_F+BF_1)A_F^{j-1} b_i, \; j\in \underline{n_i} \; i\in \underline{p}$$

so that the vectors $\{A_F^j b_i\}$, $j\in\{0,\ell,\ldots,n_i\}$ constitute an eigenvector chain associated with an infinite-zero of order $n_i$.

4) The result of the above theorem may be interpreted as a way of holding a "trajectory" in a given almost controlled invariant subspace, provided that we admit a initial condition for $(I-BF_2)\dot{x} = (A+BF_1)x$ as specified in the statement of the theorem.

In the sequel we emphasize some aspects which appeared in the proof of the previous theorem and which concern general relationships between almost controlled invariant subspaces and a P.D. law.

Proposition 4.3a: Let $\mathcal{K}$ be a given subspace and let $\bar{\mathcal{R}}_{b,\mathcal{K}} = A\bar{\mathcal{R}}_{a,\mathcal{K}} \oplus \bar{\mathcal{B}}$ $\bar{\mathcal{B}}$, $\bar{\mathcal{R}}_{b,\mathcal{K}}$ and $\bar{\mathcal{R}}_{a,\mathcal{K}}$ be given by (1.11-3). Then there exist maps $F_1 : \mathcal{X} \to \mathcal{U}$ and $F_2 : \mathcal{X} \to \mathcal{U}$ such that

a) $$(I-BF_2)\bar{\mathcal{R}}_{b,\mathcal{K}} \subset (A+BF_1)\bar{\mathcal{R}}_{a,\mathcal{K}}$$

b) $$(I-BF_2)\bar{\mathcal{R}}_{a,\mathcal{K}} \subset (A+BF_1)\bar{\mathcal{R}}_{a,\mathcal{K}} \qquad (4.82)$$

c) $$\bar{\mathcal{R}}_{a,\mathcal{K}} = \mathcal{K} \cap (I-BF_2)^{-1}(A\bar{\mathcal{R}}_{a,\mathcal{K}}).$$

Proof: The construction of the maps $F_1$ and $F_2$ is identical to that shown in (4.62) and (4.72). The proof of (4.82a) is analogous to that of (4.73d). To see (4.82b) note that

$$(I-BF_2)\bar{R}_{a,K} \subset (I-BF_2)\bar{R}_{b,K} = A\bar{R}_{a,K} = (A+BF_1)\bar{R}_{a,K}$$

since $\bar{R}_{b,K} \supset \bar{R}_{a,K}$ and $\ker F_1 \supset \bar{R}_{a,K}$. (4.82c) follows from the definition of $F_2$ and Lemma 1.2d. It is interesting to note that the sliding subspace $\bar{R}_{a,K}$ in (4.82c) is characterized by a pure derivative feedback which is represented by the map $F_2$. □

Let $F_1$ and $F_2$ be given maps and consider the pencil $s(I-BF_2) - (A+BF_1)$. We show now that the invariant subspaces of such a pencil correspond to the controlled and almost controlled invariant subspaces associated with the system $\dot{x} = Ax + Bu$. The statement of the next properties is essentially trivial.

Proposition 4.4: Consider the (not necessarily regular) pencil $s(I-BF_2) - (A+BF_1)$ for certain maps $F_1$ and $F_2$ and let $V$ be a subspace such that $(A+BF_1)V \subset (I-BF_2)V$. Then $V$ is a controlled invariant subspace. Conversely, let $V$ be a given controlled invariant subspace. Then there exist maps $F_1$ and $F_2$ such that $(A+BF_1)V \subset (I-BF_2)V$.

Proof: $(A+BF_1)V \subset (I-BF_2)V$ implies $AV \subset V + B$ and if $V$ is controlled invariant subspace then there exists $F_1$ such that $(A+BF_1)V \subset V$. Define $F_2$ so that $\ker F_2 \supset V$. Hence $(A+BF_1)V \subset$ $\subset (I-BF_2)V$.

□

Now let $F_1$ and $F_2$ be maps such that the pencil $s(I-BF_2) - (A+BF_1)$ is regular and consider the supremal subspace of the family (4.32) which is contained in $K$, i.e.

$$\tilde{V} = \sup\{V \subset K \,|\, (A+BF_1)V \subset (I-BF_2)V\}.$$

We then have

<u>Proposition 4.5</u>: Let $s(I-BF_2) - (A+BF_1)$ be a regular pencil for some maps $F_1$ and $F_2$ and let $K$ be a given subspace. Let $W_a \subset K$ be a subspace such that $(I-BF_2)W_a \subset (A+BF_1)W_a$, $W_a \cap \tilde{V} = 0$. Then $\tilde{V} \subset V_K^*$ and $W_a \subset R_{a,K}^*$.

<u>Proof</u>: By proposition 4.4, $\tilde{V}$ is a controlled invariant subspace and hence $\tilde{V} \subset V_K^*$.

Note that $(I-BF_2)W_a \subset (A+BF_1)W_a$ and $W_a \cap \tilde{V} = 0$ imply by Theorem 4.3 that $W_a \subset W_b^*$, where $W_b^*$ is defined by the sequence (4.36) with respect to $\tilde{E} := (I-BF_2)$ and $\tilde{A} := (A+BF_1)$.

Also note that $\hat{B} := \ker(I-BF_2) \subset B$ and since the pencil is regular $\hat{B} \cap \tilde{V} = 0$.

Since $W_a$ is a subspace associated with infinite-zeros, it follows that $W_a = \text{span}\{w_{i,j}\}$ where the $w_{i,j}$ are linearly independent and satisfy

$$(I-BF_2)w_{i,1} = 0$$
$$(I-BF_2)w_{i,j} = (A+BF_1)w_{i,j-1}$$

where $i \in \underline{\ell}$, $j \in \underline{m}_i$ such that $\sum_{i=1}^{\ell} m_i = \dim W_a$.

Consider the sequence (1.4). Hence, since $\hat{\mathcal{B}} \subset \mathcal{B}$, it follows that $w_{i,1} \in \mathcal{K} \cap \mathcal{B} = \mathcal{R}_a^1$ and if $w_{i,u-1} \in \mathcal{R}_a^{u-1}$, then

$$(I-BF_2)w_{i,u} = (A+BF_1)w_{i,u-1}$$

whence

$$w_{i,u} \in \mathcal{K} \cap (A\mathcal{R}_a^{u-1} + \mathcal{B}) = \mathcal{R}_a^u$$

and thus $\mathcal{W}_a \subset \mathcal{R}^*_{a,\mathcal{K}}$ .

□

IV.3.3 Modal Controllability under Regular P.D. Law

Consider a regular P.D. law as given by (4.51) and the corresponding closed-loop system (4.52). We then obtain the following expected modal controllability property.

Theorem 4.5: Let the pair (A,B) be controllable and consider a regular P.D. state feedback law. Let $F_2 : \mathcal{X} \to \mathcal{U}$ be a map such that $\ker(I-BF_2) = \hat{\mathcal{B}} \subset \mathcal{B}$, with $\dim \hat{\mathcal{B}} = \ell$. Then the resulting $n-\ell$ zeros of the pencil $s(I-BF_2) - (A+BF_1)$ are controllable.

Proof: It is well known [4.11, page 100] that the uncontrollable finite modes of a generalized linear system

$$\tilde{E}\dot{x} = \tilde{A}x + \delta\tilde{E}x_0 + Bu \qquad x_0 := \underline{x}(0-)$$

are the roots of the invariant polynomials (or the finite-zeros) of the pencil $[s\tilde{E}-\tilde{A} \,\vdots\, -B]$.

Now, note that with

$$\tilde{E} := (I-BF_2) \quad ; \quad \tilde{A} := (A+BF_1)$$

we have

$$[s(I-BF_2) - (A+BF_1) \,\vdots\, -B] = [sI-A \,\vdots\, -B] \begin{pmatrix} I & 0 \\ sF_2+F_1 & I \end{pmatrix} \quad \ldots \quad (4.83)$$

Since the last matrix on the right of (4.83) is unimodular it follows that the invariant polynomials of

$$[s(I-BF_2) - (A+BF_1) \,\vdots\, -B]$$

and

$$[sI-A \,\vdots\, -B]$$

coincide [4.7].

But the pair (A,B) is controllable, which implies that the Smith form of $[sI-A \,\vdots\, -B]$ is $[I \,\vdots\, 0]$ [4.11], where $I$ is the identity matrix of dimension $n$. Thus the resulting finite modes associated with the closed-loop system (4.52) are controllable. Note that the above discussion has also shown that if the system (4.52) has uncontrollable modes, then they remain uncontrollable, i.e. they cannot be converted into infinite modes.

It remains to be shown that the infinite-zeros associated with the system (4.52) are also controllable. This follows from Theorem 3.3, i.e.

$$(I-BF_2)\mathcal{X} + (A+BF_1)\hat{\mathcal{B}} + \mathcal{B} = \mathcal{X}$$

□

<u>Remarks 4. :</u>

a) It is interesting to note that we could have applied the sufficient condition for the controllability of the infinite-zeros given by (3.54). Note that

$$(I-BF_2)\mathcal{X} + \mathcal{B} = \mathcal{X}. \tag{4.84}$$

The sufficient condition is applicable in this case due to the special form of the singular map $\tilde{E} = I-BF_2$

b) Consider the subspaces $\mathcal{W}_b^* = \bar{\mathcal{R}}_{b,\mathcal{V}_a}$ and $\mathcal{V}^\square$ of the Theorem 4.3. Let

$$J := (A+BF_1)\mathcal{W}_b^* \,|\, (I-BF_2) \rceil (I-BF_2) \,|\mathcal{W}_b^*$$

$$B_f := Q_f B$$

where $Q_f : \mathcal{X} \to \mathcal{X}$ is the projection on $(A+BF_1)\mathcal{W}_b^*$ along $(I-BF_2)\mathcal{V}^\square$.

It follows from (4.84) and (3.54) that

$$\langle J|\mathcal{B}_f\rangle = \mathcal{W}_b^*.$$

## IV. 3.4 Disturbance Decoupling by a Regular P.D. Law

Consider the linear system

$$\dot{x} = Ax + Bu + Gd \tag{4.85a}$$

$$z = Dx \tag{4.85b}$$

where

$$x \in \mathcal{X} := \mathbb{R}^n;\ u \in \mathcal{U} := \mathbb{R}^m;\ d \in \mathcal{D} := \mathbb{R}^s\ ;\ z \in \mathcal{Z} := \mathbb{R}^\ell\ .$$

The term $d$ in (4.85a) represents an unknown disturbance signal and the output variables $z$ are the regulated variables.

The disturbance decoupling problem (DDP) consists of finding a law $u = Fx$ such that the response of the closed loop system.

$$\dot{x} = (A+BF)x + Gd \quad \underline{x}(0) = 0$$

belongs to $\mathcal{K} := \ker D$, for all signals $d$.

Wonham [4.20] has shown that DDP has solution if and only if

$$\mathcal{G} \subset \mathcal{V}_{\mathcal{K}}^{*} \quad , \quad \mathcal{G} := \text{ImG} \tag{4.86}$$

where $\mathcal{V}_{\mathcal{K}}^{*}$ is the supremal controlled invariant subspace contained in $\mathcal{K}$.

Willems [4.17] has introduced a relaxed version of the above problem in the sense that the regulated variables are only required to have $L_p$-norm arbitrarily small. Such a problem is called the almost disturbance decoupling problem (ADDP) and it consists of finding a law $u = F_\epsilon x$ for (4.85a) such that with $\underline{x}(0)=0$ for the closed loop system, there holds $\| \underline{z} \|_{L_p} \leq \epsilon \| \underline{d} \|$, $\forall \epsilon > 0$, $1 \leq p \leq \infty$. It has been proven in [4.17] that ADDP has solution if and only if

$$G \subset \mathcal{V}^*_{b,K} \tag{4.87}$$

where $\mathcal{V}^*_{b,K}$ is the supremal $L_p$-almost controlled invariant subspace "contained" in $K$.

The solution of ADDP requires high gain feedback and to obtain a small $L_p$-norm for $\underline{z}$ we must approximate $\mathcal{V}^*_{b,K}$ by controlled invariant subspaces [4.14, 4.17].

A natural question then arises : what can be achieved in terms of disturbance decoupling by the use of a regular P.D. law such as that of Theorem 4.4 ? The aim of this section is to examine some aspects of this question.

Let $\mathcal{V}^*_{a,K}$ be the supremal almost controlled invariant subspace contained in $K$ . Then by (1.15) we can write

$$\mathcal{V}^*_{a,K} = \mathcal{V}^*_K \oplus \bar{R}_{a,K}$$

where $\bar{R}_{a,K}$ is a sliding subspace of maximal dimension in $K$.

From (2.15) we can also write

$$\mathcal{V}^*_{b,K} = \mathcal{V}^*_K \oplus \bar{R}_{b,K} \tag{4.88}$$

where

$$\bar{R}_{b,K} = A\bar{R}_{a,K} \oplus \bar{B} \tag{4.89}$$

and $\bar{B}$ is a subspace such that

$$\mathcal{B} = \mathcal{B} \cap \mathcal{V}_K^* \oplus \bar{\mathcal{B}} .$$

Let $u = F_1x + F_2\dot{x}$ be a regular P.D. law as in Theorem 4.4 i.e., the distributional response of

$$(I-BF_1)\dot{x} = (A+BF_1)x$$

belongs to $\mathcal{V}_{a,K}^*$ , $\forall \underline{x}(0-) \in \mathcal{V}_{b,K}^*$.

Also, from the proof of theorem 4.4, we obtain the following decompositions of the state space

$$\mathcal{X} = \mathcal{V}^{\square} \oplus \bar{\mathcal{R}}_{b,K} \tag{4.90}$$

$$\mathcal{X} = \mathcal{V}^{\square} \oplus (A+BF_1)\bar{\mathcal{R}}_{b,K} \tag{4.91}$$

where

$$\mathcal{V}^{\square} = \mathcal{V}_K^* \oplus \mathcal{C}$$

and $\mathcal{C}$ is a controlled invariant subspace.

Moreover, for the maps $F_1$ and $F_2$ defined there, we have that

$$(I-BF_2)\mathcal{V}^{\square} = \mathcal{V}^{\square} \supset (A+BF_1)\mathcal{V}^{\square} \tag{4.92}$$

and

$$(A+BF_1)\bar{\mathcal{R}}_{b,K} \supset (A+BF_1)\bar{\mathcal{R}}_{a,K} = A\bar{\mathcal{R}}_{a,K} = (I-BF_2)\bar{\mathcal{R}}_{b,K}. \tag{4.93}$$

The above regular P.D. law yields the following generalized linear system

$$(I-BF_2)\dot{x} = (A+BF_1)x + Gd \tag{4.94b}$$

$$z = Dx.$$

Hence, as in Chapter III (see (3.20), define the invertible map

$$M^{-1}x = \begin{cases} x & , \; x \in \mathcal{V}^{\square} \\ (A+BF_1)x & , \; x \in \bar{\mathcal{R}}_{b,K} \end{cases} . \qquad (4.94c)$$

Let $Q_v : \mathcal{X} \to \mathcal{V}^{\square}$ be the projection on $\mathcal{V}^{\square}$ along $\bar{\mathcal{R}}_{b,K}$ and $Q_r : \mathcal{X} \to \bar{\mathcal{R}}_{b,K}$ be the projection on $\bar{\mathcal{R}}_{b,K}$ along $\mathcal{V}^{\square}$. Also let $x_1 := Q_v \; x \in \mathcal{V}^{\square}$ and $x_2 := Q_r \; x \in \bar{\mathcal{R}}_{b,K}$. Then by pre-multiplying the system (4.94a) by $Q_vM$ and $Q_rM$, respectively, the following decomposition is obtained.

$$\begin{aligned} \dot{x}_1 &= Lx_1 + G_1d \\ J\dot{x}_2 &= x_2 + G_2d \\ z &= D_1x_1 + D_1x_2 \end{aligned} \qquad (4.95)$$

where

$$L := \text{Mat}(A+BF_1)|\mathcal{V}^{\square}$$

$$J = \text{Mat } M(I-BF_2)\,|\bar{\mathcal{R}}_{b,K}$$

$$G_1 = Q_vMG, \quad G_2 = Q_r\, MG.$$

Since we want $z = 0$, for all possible disturbance signals $\underline{d}$, it follows that the rational transfer function $G(s)$ from d to z must be zero. From (4.95) it follows that

$$G(s) = R(s) + P(s)$$

where

$$R(s) = D_1(sI-L)^{-1}G_1$$

and

$$P(s) = D_2(sJ-I)^{-1}G_2.$$

Since R(s) is strictly proper and P(s) is polynomial, it is easy to see that G(s) = 0 implies R(s) = 0 and P(s) = 0. Thus, we say that the system (4.95) is disturbance decoupled if $D_1x_1 = 0$ and $D_2x_2 = 0$.

Since $\ker F_2 \supset \mathcal{V}^{\square}$, it follows from Proposition 3.5a that the subspace $\mathcal{V}_d \subset \mathcal{V}^{\square}$, which is influenced by the disturbance is the least subspace such that

$$(A+BF_1)\mathcal{V}_d \subset \mathcal{V}_d \text{ and } \mathcal{V}_d \oplus (A+BF_1)\bar{\mathcal{R}}_{b,K} \supset \mathcal{G} \tag{4.96}$$

and since $Q_v M\mathcal{V}_d = \mathcal{V}_d$ and $Q_vM(A+BF_1)\bar{\mathcal{R}}_{b,K} = 0$, then (4.96) is equivalent to

$$\mathcal{V}_d = \langle A+BF_1 | \mathcal{G}_1 \rangle \tag{4.97}$$

which is clearly a controlled invariant subspace. Thus, as in the proof of Theorem 4.2 in [4.20], we have that $D_1x_1 = 0$ if and only if

$$\mathcal{G}_1 \subset \mathcal{V}_K^*. \tag{4.98}$$

At this point we establish a connection with the solvability criterion for the ADDP as given by (4.87).

Unfortunately, we cannot ensure that $G_1 \subset V_K^*$ when $G \subset V_{b,K}^*$ which implies that we cannot guarantee that $D_1 x_1 = 0$ in (4.95).

To see this let $g \in G \subset V_{b,K}^*$. Then from (4.88) and since $\ker F_1 \supset \bar{R}_{a,K}$, it follows that

$$g = v + A\bar{r} + \bar{b} = (I-BF_2)v + (A+BF_1)\bar{r} + \bar{b}, \tag{4.99}$$

where

$$v \in V_K^*,\ \bar{r} \in \bar{R}_{a,K} \text{ and } \bar{b} \in \bar{B}.$$

Hence

$$Mg = v + \bar{r} + M\bar{b}$$

and

$$Q_v Mg = v + Q_v M\bar{b} \in V^{\square} \tag{4.100}$$

where $M$ is as in (4.94c).

From (4.100) we see that $Q_v M\bar{b}$ may not belong to $V_K^*$, i.e. the component of $G$ on $\bar{B}$ gives rise to a certain difficulty.

The above discussion is the basis for the next theorem which gives a condition to achieve *exact* disturbance decoupling by a regular P.D. law.

<u>Theorem 4.6</u>: Let $G \subset V_K^* + A\bar{R}_{a,K}$ for some sliding subspace $\bar{R}_{a,K}$ of maximal dimension in $K$ and let the pair (A,B) be controllable. Then there exists a P.D. law $u = F_1 x + F_2 \dot{x}$ such that the response $\underline{x}(t)$ of the closed-loop system

$$(I-BF_2)\dot{x} = (A+BF_1)x + Gd \qquad ; \qquad \underline{x}(0-) = 0$$

belongs to $K$, $t \in \mathbb{R}^+$.

Proof: Consider the decomposition (4.90) and let $F_1$ and $F_2$ be as in Theorem 4.4. Then as in (4.99)

$$g = v + A\bar{r} = (I-BF_2)v + (A+BF_1)\bar{r}, \quad v \in V_K^*, \ \bar{r} \in \bar{R}_{a,K} \ .$$

Hence

$$Mg = v + \bar{r}$$

and since $\bar{r} \in \bar{R}_{b,K}$ we obtain

$$G_1 = Q_v \ MG \ \subset V_K^* \tag{4.101}$$

and

$$G_2 = Q_r \ MG \subset \bar{R}_{a,K} \ .$$

From (4.93) and Theorem 3.1 b,c, it follows that

$$M(I-BF_2)\bar{R}_{a,K} \ \subset M(A+BF_1)\bar{R}_{a,K}$$

and

$$J\bar{R}_{a,K} \subset \bar{R}_{a,K} \tag{4.102}$$

where $J$ is as in (4.95).

The inclusion (4.102) now implies that the response of the subsystem $J\dot{x}_2 = x_2 + G_2 d$, belongs to $\bar{R}_{a,K}$ (see (3.37)). Also from (4.105) we have that the response of the subsystem $\dot{x}_1 = Lx + G_1 d$, belongs to $V_K^*$. Consequently the total response is contained in $V_K^* \oplus \bar{R}_{a,K} = V_{a,K}^* \subset K$.

□

The condition $G \subset V_K^* + A\bar{R}_{a,K}$ is not easily checked because there are plenty of sliding subspaces of maximal dimension in $K$, unless the system represented by the triple (D,A,B) is invertible (see II.2.2), in which case $R_{a,K}^*$, the supremal almost controllability subspace in $K$, is a sliding subspace. This suggests that for a non-invertible system it is better to start the examination of the condition by testing if $G \subset V_K^* + AR_{a,K}^*$ and $G \cap AR_K^* = 0$, since $R_{a,K}^* = R_K^* \oplus \bar{R}_{a,K}$, where $R_K^*$ is the supremal controllability subspace in $K$. Even so, it does not seem trivial to continue with the checking of the condition.

Despite the difficulty above mentioned it is interesting to note that by the use of a regular P.D. law we have achieved *exact* disturbance decoupling in a situation where by the use of a law $u = Fx$ we can only obtain almost disturbance decoupling involving high gain feedback.

The reader is probably wondering if there is a control law involving the derivative of the state which corresponds to the condition (4.87). This is the object of the next theorem.

Theorem 4.7: Let the pair (A,B) be controllable. Then $G \subset V_{b,K}^*$ if and only if there exists a law $u = F_1x + F_2\dot{x} + F_3d$ such that the response $\underline{x}(t)$ of the closed-loop system

$$(I-BF_2)\dot{x} = (A+BF_1)x + (BF_3+G)d, \quad \underline{x}(0-) = 0$$

belongs to $K$, $t \in \mathbb{R}^+$ and the pencil $s(I-BF_2) - (A+BF_1)$ is regular.

Proof: =>) Consider again the decomposition (4.90) and the maps $F_1$ and $F_2$ of Theorem 4.4. Since $\mathcal{G} \subset \mathcal{V}_K^* + A\bar{\mathcal{R}}_{a,K} + \bar{\mathcal{B}}$,

$$Gd_i = v_i + A\bar{r}_i + Bu_i \ , \ v_i \in \mathcal{V}_K^*, \ \bar{r}_i \in \bar{\mathcal{R}}_{a,K}$$

for some set of linearly independent vectors $\{d_i\}$, $i \in \underline{s}$, such that span $\{d_i\} = \mathcal{D}$.

Define $F_3 : \mathcal{D} \to \mathcal{U}$ by

$$F_3 d_i = -u_i \ , \ i \in \underline{s}.$$

Then

$$(BF_3 + G)d_i = v_i + A\bar{r}_i$$

whence

$$\mathrm{Im}(BF_3+G) \subset \mathcal{V}_K^* \oplus \bar{\mathcal{R}}_{a,K} .$$

The rest of the "only if" part is identical to the proof of Theorem 4.6.

<=) By hypothesis there exists a law $u = F_1x + F_2\dot{x} + F_3d$ such that the closed loop response with $\underline{x}(0-) = 0$ satisfies $\underline{z}(t) = D\underline{x}(t) = 0$, $t \geq 0$.

Since the pencil $s(I-BF_2) - (A+BF_1)$ is regular, there exists by Theorem 4.3, subspaces $\mathcal{V}^{\square}$ and $\mathcal{W}_b^*$ such that

$$\mathcal{V}^{\square} \oplus \mathcal{W}_b^* = \mathcal{X} \quad ; \qquad E\mathcal{V}^{\square} \oplus A\mathcal{W}_b^* = \mathcal{X}$$

$$(A+BF_1)\mathcal{V}^{\square} \subset (I-BF_2)\mathcal{V}^{\square}; \quad (I-BF_2)\mathcal{W}_b^* \subset (A+BF_1)\mathcal{W}_b^*$$

Let $M : \mathcal{X} \to \mathcal{X}$ be the usual map defined by

$$M^{-1}x = \begin{cases} (I-BF_2)x & , \quad x \in \mathcal{V}^{\square} \\ (A+BF_1)x & , \quad x \in \mathcal{W}_b^* \end{cases} .$$

Let $Q_v : \mathcal{X} \to \mathcal{V}^{\square}$ be the projection on $\mathcal{V}^{\square}$ along $\mathcal{W}_b^*$ and $Q_w : \mathcal{X} \to \mathcal{W}_b^*$ be the projection on $\mathcal{W}_b^*$ along $\mathcal{V}^{\square}$. Then pre-multiplying the equation

$$(I-BF_2)\dot{x} = (A+BF)x + (BF_3+G)d$$

respectively, by $Q_vM$ and $Q_wM$ we obtain the following decomposition.

$$\dot{x}_1 = Lx_1 + G_1d \quad , \quad x_1 = Q_vx$$

$$J\dot{x}_2 = x_2 + G_2d \quad , \quad x_2 = Q_wx$$

where

$$J = M(I-BF_2)|\mathcal{W}_b^* ; \quad L = M(A+BF_1)|\mathcal{V}^{\square}$$

$$G_1 = Q_vM(BF_3+G) \; ; \quad G_2 = Q_wM(BF_3+G) .$$

Since $\underline{z}(t) = 0$, $t \geq 0$, we must have

$$\mathcal{V} := \langle L|\mathcal{G}_1\rangle \subset \mathcal{K} \text{ and } \mathcal{W} := \langle J|\mathcal{G}_2\rangle \subset \mathcal{K}$$

and note that $\mathcal{V} \subset \mathcal{V}^{\square}$ and $\mathcal{W} \subset \mathcal{W}_b^*$.

Thus

$$Q_{\mathcal{V}} M \ \mathrm{Im}(BF_3+G) \subset \mathcal{V} \iff M \ \mathrm{Im}(BF_3+G) \subset \mathcal{V} \oplus \mathcal{W}_b^* . \tag{4.103}$$

and

$$Q_{\mathcal{W}} M \ \mathrm{Im}(BF_3+G) \subset \mathcal{W} \iff M \ \mathrm{Im}(BF_3+G) \subset \mathcal{V}^{\square} \oplus \mathcal{W} . \tag{4.104}$$

From (4.103-4) it follows that

$$\begin{aligned} M \ \mathrm{Im}(BF_3+G) &\subset (\mathcal{V} \oplus \mathcal{W}_b^*) \cap (\mathcal{V}^{\square} \oplus \mathcal{W}) \\ &= \mathcal{W} + (\mathcal{V} \oplus \mathcal{W}_b^*) \cap \mathcal{V}^{\square} = \mathcal{W} \oplus \mathcal{V} \end{aligned} \tag{4.105}$$

since $\mathcal{W}_b^* \cap \mathcal{V}^{\square} = 0$ and $\mathcal{V} \subset \mathcal{V}^{\square}$ .

From (4.105) and the definition of $M$ it now follows that

$$\mathrm{Im}(BF_3+G) \subset (I-BF_2)\mathcal{V} + (A+BF_1)\mathcal{W}.$$

By Proposition (4.4), $\mathcal{V}$ is controlled invariant and thus $\mathcal{V} \subset \mathcal{V}_K^*$. By Proposition 4.5, $\mathcal{W} \subset \mathcal{R}_{a,K}^*$ .

Therefore

$$\mathrm{Im}(BF_3+G) \subset (I-BF_2)\mathcal{V}_K^* + (A+BF_1)\mathcal{R}_{a,K}^*$$

whence

$$\mathcal{G} \subset \mathcal{V}_K^* + A\mathcal{R}_{a,K}^* + \mathcal{B} = \mathcal{V}_{b,K}^* . \qquad \square$$

Remarks 4. :

1) The reason for the controllability hypothesis in Theorems 4.6 and 4.7 is the same as that mentioned in Remark 4.3..

2) It should be noted that the proof of the "if" part of Theorem 4.7 is applicable (set $F_3 = 0$) to the situation of Theorem 4.6. In other words, if we achieve exact disturbance decoupling by a regular P.D. law, $u = F_1x + F_2\dot{x}$, then $\mathcal{G} \subset \mathcal{V}^*_{b,K}$. However, as pointed out before, we cannot ensure the converse statement. It should be noted that the role of the feedback map $F_3$ is to eliminate the component of $\mathcal{G}$ on $\mathcal{B}(\text{mod } \mathcal{V}^*_K)$.

3) It is interesting to remark that the condition $\mathcal{G} \subset \mathcal{V}^*_{b,K}$ corresponds to an other type of control law which achieves exact disturbance decoupling. Willems [4.19] has shown that the law

$$u = \sum_{i=1}^{n} F_i d^{(i)} + Fx \tag{4.106}$$

attains the same objective, i.e. *exact* disturbance decoupling from d to z if and only if $\mathcal{G} \subset \mathcal{V}^*_{b,K}$.

The law suggested in Theorem 4.6 is somewhat easier than (4.106) since it avoids the measurement of derivatives of the disturbance. But, of course, there are practical situations where the disturbance is unmeasurable and neither of such laws is applicable.

4) If we consider the law $u = F_1x + F_3d$ then the disturbance decoupling problem is solvable if and only if $\mathrm{Im}G \subset V_K^* + B$ (see exercise 4.10 in [4.20]). Since $V_{b,K}^* \supset V_K^* + B$ it follows that the law introduced in Theorem 4.7 can solve DDP in a situation where the above law cannot do it.

## IV.4 NONSINGULAR P.D. LAWS AND ALMOST CONTROLLED INVARIANT SUBSPACES

Thus far we have dealt with P.D. laws for which the map $(I-BF_2)$ is singular and the pencil $s(I-BF_2) - (A+BF_1)$ is regular. By using this type of law we have been able to obtain *exact* results in problems involving almost controlled invariant subspaces.

Henceforth we consider the sequence of controlled invariant subspaces $V_\epsilon$ introduced in [4.14] such that $V_\epsilon \xrightarrow[\epsilon\to 0]{} V_{b,K}^*$. It is shown that there exists a sequence of P.D. laws, $u = F_1x + F_{2\epsilon}\dot{x}$, such that

$$(I-BF_{2\epsilon})^{-1}(A+BF_1)V_\epsilon \subset V_\epsilon \text{ where } (I-BF_{2\epsilon}) \text{ is nonsingular.}$$

We shall see that $F_{2\epsilon} \xrightarrow[\epsilon\to 0]{} F_2$ where $F_2$ is as in Theorem 4.4. In this sense the regular P.D. law defined in Theorem 4.4 may be regarded as a limit case of the above sequence of P.D. laws.

The following definition simply establishes a terminology to be followed in the text.

Definition 4.2: A P.D. law, $u = F_1x + F_2\dot{x}$, is termed nonsingular if $(I-BF_2)$ is a nonsingular map for some $F_2$.

We first show some simple properties of invariance under a non-singular P.D. law.

Let $G^*$ be a transformation induced by a nonsingular P.D. law, i.e.

$G^* : (A,B) \to ((I-BF_2)^{-1}(A+BF_1),\ (I-BF_2)^{-1}B)$ and denote
$\hat{A} := (I-BF_2)^{-1}(A+BF_1)$.

Proposition 4.6: Let $\phi^u = \mathcal{B} + A\mathcal{B} + \ldots A^u\mathcal{B}$, $u \in \underline{n}$. Then $\phi^u$ is invariant under $G^*$.

Proof: First note that

$$(I-BF_2)^{-1}(\mathcal{B} + \mathcal{T}) = \mathcal{B} + \mathcal{T} \tag{4.107}$$

for any subspace $\mathcal{T}$.

Hence

$$(I-BF_2)^{-1}\mathcal{B} + \hat{A}(I-BF_2)^{-1}\mathcal{B} + \ldots + \hat{A}^u(I-BF_2)^{-1}\mathcal{B}$$

$$= (I-BF_2)^{-1}[\mathcal{B} + \hat{A}(\ldots(\mathcal{B} + \hat{A}\mathcal{B}))\ldots]$$

$$= \mathcal{B} + (A+BF_1)\mathcal{B} + \ldots + (A+BF_1)^u\ \mathcal{B}$$

$$\mathcal{B} + A\mathcal{B} + \ldots + A^u\ \mathcal{B} = \phi^u$$ □

Corollary 4.3: The reachable subspace $\langle A|\mathcal{B}\rangle$ and the controllability indices of the pair $(A,B)$ are invariant under $G^*$.

Proof: The statement about $\langle A|B\rangle$ follows from Proposition 4.6 and the statement on the controllability indices follows from the fact that such indices are determined from dim $\phi^u$ [4.20]. □

The above invariance property can also be extended to the sequences of (almost) controlled invariant subspaces $\mathcal{V}^u$, $\mathcal{R}_a^u$, $\mathcal{S}^u$, $\mathcal{R}^u$,

$u \in \underline{n}$, defined in (1.3-4) and (1.9-1).

Also consider the *fixed* map $A_t := ((A+BF)|\mathcal{V}_K^*) (\text{mod } \mathcal{R}_K^*)$, $\forall F \in F(\mathcal{V}_K^*)$ (see Section I.2.5 for an interpretation of such map) where $K$ is a given subspace.

<u>Proposition 4.7:</u> The sequences $\mathcal{V}^u$, $\mathcal{R}_a^u$, $\mathcal{S}^u$, $\mathcal{R}^u$, $u \in \underline{n}$ and the map $A_t$ are invariant under $G^*$.

<u>Proof:</u>

a) Invariance of $\mathcal{V}^u$

Consider the sequence

$$\hat{\mathcal{V}}^u = K \cap \hat{A}^{-1}(\mathcal{V}^{u-1} + (I-BF_2)^{-1}\mathcal{B}); \quad \hat{\mathcal{V}}^0 = X.$$

By (4.107) it follows that

$$\hat{\mathcal{V}}^u = K \cap \hat{A}^{-1}(\hat{\mathcal{V}}^{u-1} + \mathcal{B}) \quad . \tag{4.108}$$

Then from (1.3) and (4.108) we obtain $\mathcal{V}^1 = \hat{\mathcal{V}}^1 = K$. Suppose that $\mathcal{V}^{u-1} = \hat{\mathcal{V}}^{u-1}$. Hence

$$\hat{\mathcal{V}}^u = \{x \in K \mid (I-BF_2)^{-1}(A+BF_1)x = v+b, \; v \in \hat{\mathcal{V}}^{u-1}, \; b \in \mathcal{B}\}$$

which implies $Ax \in \mathcal{V}^{u-1} + \mathcal{B}$, so that $\hat{\mathcal{V}}^u \subset \mathcal{V}^u$.

Now let $x \in \mathcal{V}^u$. Then, $Ax = v+b$, for some $v \in \mathcal{V}^{u-1}$ and some $b \in \mathcal{B}$. Hence

$$(A+BF_1)x = v + b' \quad , \quad b' := BF_1 + b, \ \forall F_1$$

and

$$(A+BF_1)x = (I-BF_2)(v+\tilde{b})$$

for any $F_2$ such that $(I-BF_2)$ is nonsingular and where $\tilde{b} \in \mathcal{B}$ is given by $\tilde{b} = (I-BF_2)^{-1}(b'+BF_2 v)$.

Thus $(I-BF_2)^{-1}(A+BF_1)x \in \hat{\mathcal{V}}^{u-1} + \mathcal{B}$, i.e., $\mathcal{V}^u \subset \hat{\mathcal{V}}^u$ and the result follows.

The invariance of the sequences $\mathcal{R}_a^u$, $\mathcal{S}^u$, $\mathcal{R}^u$, under $G^*$ is proved in a similar way.

b) Invariance of $A_t$.

Let $F_a$ be a map such that $(I-BF_2)^{-1}(A+BF_1+BF_a)\mathcal{V}_K^* \subset \mathcal{V}_K^*$ and denote $A_{F_a} := (I-BF_2)^{-1}(A+BF_1+BF_a)$. Analogously, let $F_b$ be a map such that $(A+BF_b)\mathcal{V}_K^* \subset \mathcal{V}_K^*$ and denote $A_{F_b} := A+BF_b$.

Also consider the maps $\bar{A}_{F_a}$ and $\bar{A}_{F_b}$ induced in $\mathcal{V}_K^*/\mathcal{R}_K^*$ and let $P : \mathcal{V}_K^* \rightarrow \mathcal{V}_K^*/\mathcal{R}_K^*$ be the canonical projection. Then

$$\bar{A}_{F_a}\bar{x} - \bar{A}_{F_b}\bar{x} = PA_{F_a}x - PA_{F_b}x = P(A_{F_a}x - A_{F_b}x).$$

Let $(I-BF_2)^{-1}Ax := \omega$, which implies $\omega = Ax + BF_2\omega$.

Hence

$P(A_{F_a}x-A_{F_b}x) = P[B(F_2\omega-F_b x) + (I-BF_2)^{-1}(BF_1 x+BF_a x)] \in P(\mathcal{B}\cap\mathcal{V}_K^*)$ by using (4.107) and by noting that $(A_{F_a}-A_{F_b})x \in \mathcal{V}_K^*$.

But $P(\mathcal{B}\cap\mathcal{V}_K^*) \subset P\mathcal{R}_K^* = 0$ and the claim follows. □

Corollary 4.4: Let $K$ := ker C. Then the set of transmission polynomials of the triple (C,A,B) and the set of infinite-zeros of $C(sI-A)^{-1}B$ are invariant under $G^*$.

Proof: By definition (see II.1.4), the set of transmission polynomials of (C,A,B) is the set of invariant polynomials of the map $A_t$.

The set of infinite-zeros of $C(sI-A)^{-1}B$ is invariant under $G^*$ since there are $\dim\left(\frac{B+V_K^*}{V_K^*}\right)$ infinite-zeros (see Definition 2.3) with orders determined from $\dim\left(\frac{S^u+V_K^*}{V_K^*}\right)$ [4.5]. □

The above results have shown that a nonsingular P.D. law preserves all important structural features of a pair (A,B) relative to a given subspace $K$.

## IV.4.1 Approximation of Almost Controlled Invariant Subspaces

Let $K$ be a given subspace and consider the supremal $L_p$-almost controlled invariant subspace "contained" in $K$ and denoted, as usual, by $V_{b,K}^*$.

Trentelman [4.14] has constructed a sequence of controlled invariant subspaces $V_\epsilon$ which approach $V_{b,K}^*$ as $\epsilon \to 0$. In the approximation process the state feedback maps $F_\epsilon$ for which $(A+BF_\epsilon)V_\epsilon \subset V_\epsilon$ are such that $F_\epsilon \to \infty$. This is an intrinsic property of the almost controlled invariant subspace $V_{b,K}^*$, namely, one requires high gain state feedback to approach it.

In this section we show that the subspaces $V_\epsilon$ can be made invariant under a nonsingular P.D. law of the type $(I-BF_{2\epsilon})^{-1}(A+BF_1)V_\epsilon \subset V_\epsilon$. The main difference with respect to using state feedback only is that the sequence of derivative feedback maps $F_{2\epsilon}$ converge to a finite map $F_2$. This can be useful in application in which one wishes to avoid high gain state feedback.

To facilitate the notation we now denote the state feedback map by $F$ and not by $F_1$ as we had been doing previously.

Let $K$ := ker D. From Section II.1.4 we have that

$$V^*_{b,K} = V^*_K + R^*_{b,K} = V^*_K \oplus \bar{R}_{b,K}. \tag{4.109}$$

From (2.23) and Definition 2.3 it follows that

$$\bar{R}_{b,K} = M_1 \oplus \dots \oplus M_p \oplus M_{p+1} \oplus \dots \oplus M_q \tag{4.110a}$$

where $q = \dim \dfrac{V^*_K + B}{V^*_K}$ is the number of infinite-zeros of $D(sI-A)^{-1}B$.

From (2.22-4) we also have that

$$M_i = b_i + A_F b_i + \dots + A_F^{n_i} b_i \qquad , i \in \underline{p} \tag{4.110b}$$

$$M_i = b_i \ , \quad i \in \{p+1,\dots,q\} \tag{4.110c}$$

where $p = \dim(\bar{B} \cap K)$ and $\bar{B}$ is a subspace such that

$$B = \bar{B} \oplus B \cap V^*_K \ .$$

In (4.110c) span $\{b_i\} = B^*$ where

$$\bar{\mathcal{B}} = \bar{\mathcal{B}} \cap \mathcal{K} \oplus \mathcal{B}^*$$

and in (4.110b) $F$ is defined on $\bar{R}_{a,K}$ (see Theorem 1.4) with

$$\bar{R}_{a,K} = L_1 \oplus L_2 \oplus \ldots \oplus L_p \tag{4.111}$$

$$L_i = b_i + A_F b_i + \ldots + A_F^{n_i - 1} b_i, \quad i \in \underline{p}.$$

Since $V_K^* \cap \bar{R}_{a,K} = 0$ we can define $F$ on $V_K^*$ so that $A_F V_K^* \subset V_K^*$ and

$$\sigma[A_F | V_K^*] = \Lambda_r \cup \Lambda_z \tag{4.112}$$

where $\Lambda_r := \sigma[A_F | R_K^*]$ is a pre-specified symmetric set of dim $R_K^*$ complex numbers and $\Lambda_z := \sigma[(A_F | V_K^*) (\text{mod } R_K^*)]$ is fixed for all $F \in F(V_K^*)$.

Let $\Lambda_{i,\epsilon} := \{\lambda_{j,i}(\epsilon)\}$, $i \in \underline{q}$ be a set defined by

$$\Lambda_{i,\epsilon} := \{\lambda_{j,i}(\epsilon)\}, \; i \in \underline{p}, \; j \in \{1,2,\ldots,n_i+1\}$$

$$\Lambda_{i,\epsilon} := \{\lambda_{1,i}(\epsilon)\}, \; i \in \{p+1,\ldots,q\}$$

such that $\lambda_{j,i}(\epsilon)$ is a real number with $|\lambda_{j,i}(\epsilon)| \xrightarrow[\epsilon \to 0]{} \infty$ .

Also let $\Lambda_c$ be a symmetric set of $n - \dim V_{b,K}^*$ complex numbers such that $\Lambda_c \cap \Lambda_z = \phi$.

We then obtain :

Theorem 4.8; Let the pair (A,B) be controllable and consider the subspaces $\mathcal{V}^*_{b,K}$, $\bar{\mathcal{R}}_{b,K}$ and the set $\Lambda_{i,\epsilon}$, $i \in \underline{q}$, $\Lambda_c$, $\Lambda_r$ and $\Lambda_z$ above defined. Then there exist

a) a sequence of controlled invariant subspaces $\mathcal{V}_\epsilon$ such that $\mathcal{V}_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \bar{\mathcal{R}}_{b,K}$ (which implies $\mathcal{V}_\epsilon \oplus \mathcal{V}^*_K \underset{\epsilon\to 0}{\longrightarrow} \mathcal{V}^*_{b,K}$)

b) maps F and $F_{2\epsilon} \underset{\epsilon\to 0}{\longrightarrow} F_2$ such that $(I-BF_{2\epsilon})^{-1}A_F\mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon$,

$$\sigma[(I-BF_{2\epsilon})^{-1}A_F|\mathcal{V}_\epsilon] = \bigcup_{i=1}^{q} \Lambda_{i,\epsilon} \text{ and } (I-BF_2)\bar{\mathcal{R}}_{b,K} \subset A_F\bar{\mathcal{R}}_{b,K}.$$

Morever

$$\sigma[(I-BF_{2\epsilon})^{-1}A_F] = \bigcup_{i=1}^{q} \Lambda_{i,\epsilon} \cup \Lambda_c \cup \Lambda_r \cup \Lambda_z.$$

Proof: This involves two steps:

i) Definition of the map F.

Let F be defined on $\bar{\mathcal{R}}_{a,K}$ and also on $\mathcal{V}^*_K$ so that (4.112), holds. As in (4.68) we can write

$$\bar{\mathcal{R}}_{b,K} = \bar{\mathcal{R}}_{a,K} \oplus \tilde{\mathcal{R}}_{b,K} \tag{4.113}$$

where by (4.110-111)

$$\tilde{\mathcal{R}}_{b,K} = \mathcal{B}^* \oplus \text{span}\{A_F^{n_i} b_i\},\ i \in \underline{p}.$$

Since the pair (A,B) is controllable, then by Theorem 1.8 there exist a controlled invariant subspace $C$ and a map $F_0$ such that

$$X = V^*_{b,K} \oplus C$$

and

$$\sigma[(A+BF_0)|C] = \Lambda_c.$$

Let $F|C = F_0|C$ and note from (4.109) and (4.113) that we obtain the following decomposition

$$X = V^{\square} \oplus \bar{R}_{a,K} \oplus \tilde{R}_{b,K}$$

with

$$V^{\square} := V^*_K \oplus C .$$

Note that we have not defined yet F on $\tilde{R}_{b,K}$. Thus, as in Step 2 of Theorem 4.4, it follows that F can be defined on $\tilde{R}_{b,K}$ so that $P_{\bar{B}} A_F|\tilde{R}_{b,K}$ is nonsingular, where $P_{\bar{B}}$ is the projection on $\bar{B}$ along $A\bar{R}_{a,K} \oplus V^{\square}$. This implies (see (4.73)) that

$$X = V^{\square} \oplus A_F\bar{R}_{b,K}. \tag{4.114}$$

The reason for the above definition will become clear in the next step.

<u>Step 2</u>: Construction of $V_\epsilon$ and definition of $F_{2\epsilon}$.

The sequence of subspace $\mathcal{V}_\epsilon$ used here is similar to that constructed in [4.14].

Let $\delta_{j,i}(\epsilon) := \frac{1}{\lambda_{j,i}(\epsilon)}$ and consider vectors $v_{j,i}(\epsilon)$ defined

by

$$v_{j,i}(\epsilon) := (I-\delta_{j,i}(\epsilon)A_F)^{-1}A_F v_{j-1,i}(\epsilon); v_{1,i}(\epsilon) := (I-\delta_{1,i}(\epsilon)A_F)^{-1}b_i \quad \ldots \quad (4.115a)$$

for $i\in\underline{p}$, $j\in\{1,2,\ldots,n_i+1\}$ and

$$v_{1,i}(\epsilon) := (I-\delta_{1,i}(\epsilon)A_F)^{-1}b_i, \ i\in\{p+1,\ldots,q\} . \qquad (4.115b)$$

Note that $(I-\delta_{j,i}(\epsilon)A_F)$ is invertible for $|\delta_{j,i}(\epsilon)| \to 0$ *and*

$$v_{j,i}(\epsilon) \xrightarrow[\epsilon\to 0]{} A_F^{j-1}b_i \ .$$

Furthermore, for $\epsilon$ sufficiently small, the vectors $v_{j,i}(\epsilon)$ introduced in (4.115), are linearly independent (See [4.8] on convergence of subspaces).

Thus, let

$$\mathcal{R}_{i,\epsilon} := \text{span } \{v_{j,i}(\epsilon)\} \ , \quad i \in \underline{q}$$

and

$$\mathcal{V}_\epsilon := \mathcal{R}_{1,\epsilon} \oplus\ldots\oplus \mathcal{R}_q(\epsilon).$$

Note that $\mathcal{V}_\epsilon$ is a subspace spanned by real vectors, since

$\lambda_{j,i}(\epsilon)$ is real, and that $\mathcal{V}_\epsilon \xrightarrow[\epsilon\to 0]{} \bar{\mathcal{R}}_{b,K} = \mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_q$ (see (4.110)). It has been shown in [4.14] that $\mathcal{V}_\epsilon$ is a controlled invariant subspace.

For $\epsilon$ sufficiently small we can write

$$\mathcal{X} = \mathcal{V}^{\square} \oplus \mathcal{V}_\epsilon \quad .$$

Define $F_{2\epsilon} : \mathcal{X} \to \mathcal{U}$ by

$$\begin{aligned} \theta F_{2\epsilon} v_{1,i}(\epsilon) &= b_i \ , \quad i \in \underline{q} \\ F_{2\epsilon} v_{j,i}(\epsilon) &= 0, \qquad i \in \underline{p},\ j \in \{2,\ldots,n_i+1\} \\ F_{2\epsilon} | \mathcal{V}^{\square} &= 0. \end{aligned} \tag{4.116}$$

From (4.115 - 116) we now obtain

$$(I-BF_{2\epsilon})v_{1,i}(\epsilon) = \delta_{1,i}(\epsilon)A_F\, v_{1,i}(\epsilon),\ i \in \underline{q} \tag{4.117}$$

and

$$(I-BF_{2\epsilon})v_{j,i}(\epsilon) = \delta_{j,i}(\epsilon)A_F\, v_{j,i}(\epsilon) + A_F\, v_{j-1,i},$$

$$i \in \underline{p},\ j \in \{1,\ldots,n_i+1)\}.$$

Assume for the moment that $(I-BF_{2\epsilon})$ is nonsingular. Then from (4.116-117) it follows that in the basis $\{v_{j,i}(\epsilon)\}$ , $i \in \underline{q}$

$$\mathrm{Mat}[(I-BF_{2\epsilon})^{-1}A_F | \mathcal{R}_{i,\epsilon}] = M_{i,\epsilon}$$

where

$$M_{i,\epsilon} := \begin{bmatrix} u^i_{1,1}(\epsilon) & u^i_{2,1}(\epsilon) & \cdot & \cdot & u^i_{n_i+1,1}(\epsilon) \\ 0 & u^i_{2,2}(\epsilon) & \cdot & \cdot & u^i_{n_i+1,2}(\epsilon) \\ 0 & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \cdot & \cdot & \vdots \\ 0 & 0 & \cdot & \cdot & u^i_{n_i+1,n_i+1}(\epsilon) \end{bmatrix}$$

with

$$u^i_{j,k}(\epsilon) = (-1)^{j-k}\lambda_{j,i}(\epsilon) \ldots \lambda_{k,i}(\epsilon) \text{ for } k < j$$

$$u^i_{j,j} = \lambda_{j,i}(\epsilon).$$

Since the subspaces $\mathcal{R}_{i,\epsilon}$, $i \in \underline{q}$, are independent it follows that

$$(I-BF_{2\epsilon})^{-1}A_F \mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon$$

with

$$\sigma[(I-BF_{2\epsilon})^{-1}A_F|\mathcal{V}_\epsilon] = \bigcup_{i=1}^{q} \Lambda_{i,\epsilon} .$$

Also, from the definition of $F$ and $F_{2\epsilon}$, it follows that

$$(I-BF_{2\epsilon})^{-1}A_F \mathcal{V}^{\square} \subset \mathcal{V}^{\square}$$

with

$$\sigma[(I-BF_{2\epsilon})^{-1}A_F|\mathcal{V}^{\square}] = \Lambda_c \cup \Lambda_r \cup \Lambda_z$$

which establishes the claim on the configuration of eigenvalues.

Since $v_{1,i}(\epsilon) \underset{\epsilon\to 0}{\to} b_i$, it follows that $F_{2\epsilon} \underset{\epsilon\to 0}{\to} F_2$ such that

$$(I-BF_2)b_i = 0 \qquad i \in \underline{q}$$

and

$$F_2 A_F^{j-1} b_i = 0, \quad i \in \underline{p} \quad j \in \{1,\ldots,n_i+1\}$$

so that

$$(I-BF_2)\bar{R}_{b,K} \subset A_F \bar{R}_{a,K} \subset A_F \bar{R}_{b,K}$$

It remains to show that $(I-BF_{2\epsilon})$ is indeed nonsingular. From (4.116) it follows that

$$\operatorname{Im}(I-BF_{2\epsilon}) = V_1 + (V_2 \oplus V^{\square}) \tag{4.118}$$

where

$$V_1 := \operatorname{span}\{v_{1,i}(\epsilon) - b_i\} \quad , \quad i \in \underline{q}$$

$$V_2 := \operatorname{span}\{v_{j,i}(\epsilon)\}, \ i \in \underline{p}, \ j \in \{2,\ldots,n_i+1\}.$$

From (4.115) we obtain that

$$v_{1,i}(\epsilon) - b_i = \delta_{1,i}(\epsilon) A_F \, v_{1,i}(\epsilon), \quad i \in \underline{q}$$

and

$$v_{j,i}(\epsilon) = A_F v_{j-1,i}(\epsilon) + \delta_{j,i}(\epsilon) A_F \, v_{j,i}(\epsilon), \ i \in \underline{p}, \ j \in \{2,\ldots,n_i+1\} \ . \tag{4.119}$$

Since $\operatorname{span}\{A_F v_{j,i}(\epsilon)\} \underset{\epsilon \to 0}{\to} \operatorname{span}\{A_F^j b_i\} = A_F \bar{R}_{b,K}$, it follows that for $\epsilon$ sufficiency small the vectors $\{A_F v_{j,i}(\epsilon)\}$ are linearly independent and also independent from $V^{\square}$ (see (4.114)).

Therefore the sum in (4.118) is a direct one, i.e.

$$\mathrm{Im}(I-BF_{2\epsilon}) = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}^{\square} = \mathcal{X}$$

so that $(I-BF_{2\epsilon})$ is nonsingular

□

Comments:

1) The theorem also holds if the sets $\Lambda_{i,\epsilon}$, $i \in \underline{p}$ are taken to be sets of $n_i+1$ symmetric complex numbers. In this case the vectors $v_{j,i}(\epsilon)$ shown in (4.115) are in general complex. In order to avoid the definition of complex feedback maps, Trentelman [4.14] has suggested a nice procedure to compute real vectors $\hat{v}_{j,i}(\epsilon)$ from the vectors $v_{j,i}(\epsilon)$. The only modification in the proof of the theorem is that the maps $F_{2\epsilon}$ are now defined on the real vectors $\hat{v}_{j,i}(\epsilon)$.

2) Theorem 4.4 may be considered the limit case of Theorem 4.8 in the sense of convergence of the maps $F_{2\epsilon} \xrightarrow[\epsilon\to 0]{} F_2$ and $\mathcal{V}_\epsilon \xrightarrow[\epsilon\to 0]{}$ $\bar{\mathcal{R}}_{b,K}$. The action of the map F defined in Theorem 4.8 is entirely analogous to the action of the map $F_1$ of Theorem 4.4. Note that if we set $\delta_{i,j}(\epsilon) = 0$ in (4.119) then $F_{2\epsilon}$ is replaced by $F_2$ with $\ker(I-BF_2) = \bar{\mathcal{B}}$ and $\mathrm{Im}(I-BF_2) = A_F\bar{\mathcal{R}}_{a,K} \oplus \mathcal{V}^{\square}$, as in Theorem 4.4.

It is not difficult to see from Theorem 4.4. and 4.8, Corollary 4.4. and Lemma 2.2. that the strictly proper transfer matrix

$$G_p(s) = D(sI-(I-BF_{2\epsilon})^{-1}A_F)^{-1}(I-BF_{2\epsilon})^{-1}B$$

has p infinite-zeros of respective orders $n_i+1$, $i \in \underline{p}$, and q-p infinite zeros of order one (see (4.110)).

3) We have not considered the distributional convergence of the system under the nonsingular P.D. law of Theorem 4.8 to the system under regular P.D. law of Theorem 4.4. This remains to be worked out and we suggest the reference [4.4] for a good analysis on a related subject.

A similar result holds for the subspace $V^*_{a,K}$, the supremal almost controlled invariant subspace contained in $K$ .

Let $\tilde{\Lambda}_{i,\epsilon} := \{\lambda_{j,i}(\epsilon)\}$, $i \in \underline{p}$, $j \in \underline{n}_i$, be a set of real numbers such that $|\lambda_{j,i}(\epsilon)| \underset{\epsilon\to\infty}{\to} \infty$ and let $\tilde{\Lambda}_c$ be a symmetric set of n-dim $V^*_{a,K}$ complex numbers such that $\tilde{\Lambda}_c \cap \Lambda_z = \phi$ . Hence we have :

<u>Theorem 4.9</u>: Let the pair (A,B) be controllable and consider the subspaces $V^*_{a,K}$, $\bar{R}_{a,K}$ and the sets $\tilde{\Lambda}_{i,\epsilon}$, $i \in \underline{p}$, $\tilde{\Lambda}_c$, $\Lambda_r$ and $\Lambda_z$. Then there exist :

a) a sequence of controlled invariant subspaces $V_\epsilon$ such that $V_\epsilon \underset{\epsilon\to 0}{\to} \bar{R}_{a,K}$ (which implies $V_\epsilon \oplus V^*_K \underset{\epsilon\to 0}{\to} V^*_{a,K}$) .

b) maps F and $F_{2\epsilon} \underset{\epsilon\to 0}{\to} F_2$ such that $(I-BF_{2\epsilon})^{-1}A_F V_\epsilon \subset V_\epsilon$,
$\sigma[(I-BF_{2\epsilon})^{-1}A_F|V_\epsilon] = \bigcup_{i=1}^{p} \tilde{\Lambda}_{i,\epsilon}$ and $(I-BF_2)\bar{R}_{a,K} \subset A_F\bar{R}_{a,K}$ .

Moreover

$$\sigma[(I-BF_{2\epsilon})^{-1}A_F] = \bigcup_{i=1}^{p} \tilde{\Lambda}_{i,\epsilon} \cup \tilde{\Lambda}_c \cup \Lambda_r \cap \Lambda_z .$$

Proof: The subspaces $V_\epsilon$ and the maps $F_{2\epsilon}$ are constructed identically to those of Theorem 4.8.

We shall just point out a fact about the definition of F. By Theorem 1.8 there exist a controlled invariant subspace $C$ and a map $F_c$ such that

$$X = V^*_{a,K} \oplus C$$

and

$$\sigma[(A+BF_c)|C] = \tilde{\Lambda}_c.$$

From Theorem 1.4 we have that

$$\bar{R}_{a,K} = \bar{R}^u_a = B_i \oplus A_F B_2 \oplus \dots \oplus A_F^{u-1} B_u, \text{ for some } u \in \underline{n}$$

where $\{B_i\}$, $i \in \underline{u}$ is a chain in $\bar{B}$ and from the proof of such a theorem $B_1 = \bar{B} \cap K$.

By Remark 1.1b we have that F need not be defined on $A_F^{u-1}B_u$. Thus, analogously to step 2 of Theorem 4.4 consider the following decompositions

$$X = V^{\square} \oplus \bar{R}^{u-1}_a \oplus A_F^{u-1}B_u$$

$$X = V^{\square} \oplus A_F R^{u-2}_a \oplus B_1$$

where

$$V^{\square} := V^*_K \oplus C.$$

By using identical arguments we have that F can be defined on $A_F^{u-1}\mathcal{B}_u$ so that $P_{\mathcal{B}_1}(A+BF)|A_F^{u-1}\mathcal{B}_u$ is invertible, where $P_{\mathcal{B}_1}$ is the projection on $\mathcal{B}_1$ along $\mathcal{V}^{\square} \oplus A_F R_a^{u-2}$. This implies that $P_{\bar{\mathcal{R}}_{a,K}} A_F|\bar{\mathcal{R}}_{a,K}$ is nonsingular and then as in (4.73b) we can write

$$\mathcal{X} = \mathcal{V}^{\square} \oplus A_F \bar{\mathcal{R}}_{a,K} \quad .$$

Let F be defined on $\bar{\mathcal{R}}_{a,K}$ as shown above. Further, define F on $\mathcal{V}_K^*$ so that (4.112) holds and also $F|\mathcal{C} = F_c|\mathcal{C}$.

The rest of the proof is identical to that of Theorem 4.8.

□

### IV.4.2 Almost Disturbance Decoupling by a Nonsingular P.D. Law

Consider again the linear system

$$\dot{x} = Ax + Bu + Gd$$
$$z = Dx \tag{4.120}$$

where

$$x \in \mathcal{X} := \mathbb{R}^n; \; u \in \mathcal{U} := \mathbb{R}^m; \; d \in \mathcal{D} := \mathbb{R}^s; \; z \in \mathcal{Z} := \mathbb{R}^{\ell} \; .$$

The almost disturbance decoupling (ADDP) by state feedback has already been introduced in Section IV.3.4.(ADDP) requires a state feedback map $F_\epsilon$ such that in the closed loop system with $\underline{x}(0) = 0$ there holds $\|\underline{z}\|_{L_p} \leq \epsilon \|\underline{d}\|_{L_p}$.

Let $K$ := ker D. If Im G $\subset V^*_{b,K}$ then (ADDP) is solvable and the state feedback map $F_\epsilon$ involved in the solution is such that $F_\epsilon \underset{\epsilon\to 0}{\to} \infty$ .

The main objective of this section is to show that if Im G $\subset V^*_{b,K}$ then we can also use a nonsingular P.D. law to achieve the same goal, i.e. $\|\underline{z}\|_{L_p} \leq \epsilon \|\underline{d}\|_{L_p}$. The main difference in using a nonsingular P.D. law, $u = Fx + F_{2\epsilon}\dot{x}$, is that the maps involved in the solution are such that F is finite and $F_{2\epsilon} \underset{\epsilon\to 0}{\to} F_2$, where $F_2$ is also a finite map. Such a result is expected in view of Theorem 4.7 on *exact* disturbance decoupling by a regular P.D. law and, in view of Theorem 4.8 on the approximation of $V^*_{b,K}$, by making use of a nonsingular P.D. law.

We first recall a lemma stated in [4.14].

<u>Lemma 4.2</u>: Fix $1 \leq p \leq \infty$. Suppose there exists a sequence of state feedback maps $F_\epsilon$ such that $\| De^{A_{F_\epsilon} t} G\|_{L_1} \underset{\epsilon\to 0}{\to} 0$. Then (ADDP) is solvable.

<u>Proof</u>: Since the $L_p$-induced norm of a convolution operator is bounded by the $L_1$ - norm of its kernel

$$\|\underline{z}\|_{L_p} \leq \| De^{A_{F_\epsilon} t} G\|_{L_1} \|\underline{d}\|_{L_p} .$$

□

The above lemma shows that (ADDP) requires the $L_1$-norm of the closed loop impulse response to be arbitrarily small.

We now introduce a modified version of (ADDP) which makes use of a nonsingular P.D. law and which we shall denote by $(ADDP)^o$.

<u>Definition 4.3</u>: For fixed $1 \leq p \leq \infty$ the almost disturbance decoupling in the $L_p$-sense, $(ADDP)^o$, is said to be solvabe if $\forall \epsilon > 0$ there exists a nonsingular P.D. law, represented by the pair $(F, F_{2\epsilon})$, such that

in the closed loop system with $\underline{x}(0) = 0$

$$\| \underline{z} \|_{L_p} \leq \epsilon \, \| d \|_{L_p} .$$

Let $A_\epsilon := (I-BF_{2\epsilon})^{-1}(A+BF)$. We then have the following result analogous to that described in Lemma 4.2.

Lemma 4.3: Fix $1 \leq p \leq \infty$. Suppose there exists a sequence of nonsingular P.D. laws represented by the pair $(F_{2\epsilon}, F_2)$ such that $\| De^{A_\epsilon t}(I-BF_{2\epsilon})^{-1} G\|_{L_p} \xrightarrow[\epsilon \to 0]{} 0$. Then $(ADDP)^o$ is solvable.

Proof: Identical to that of Lemma 4.2.

The following definition is also needed:

Definition 4.4: For $\epsilon > 0$, let $\Lambda_\epsilon^m := \{\lambda_1(\epsilon), \ldots, \lambda_k(\epsilon)\}$ be a multiplicity set such that $\lambda_i(\epsilon)$ is real, $\lambda_i(\epsilon) = \lambda_j(\epsilon) := \lambda(\epsilon)$ for $i \neq j \in \underline{k}$. We shall say that $\Lambda_\epsilon^m$ is a set of infinite root loci with common growth $\alpha$ and asymptotic direction $\bar{\lambda}$ in $\mathbb{R}^-$, if there is a real $\alpha > 0$ such that $\epsilon^\alpha \lambda(\epsilon) \xrightarrow[\epsilon \to 0]{} \bar{\lambda}$, whenever $\lambda(\varepsilon) \in \Lambda_\epsilon^m$.

Let $\mathcal{M}$ be an almost controllability subspace described by

$$\mathcal{M} = \mathcal{b} \oplus A_F \mathcal{b} + \ldots + A_F^{k-1} \mathcal{b}$$

and define the following vectors

$$v_i(\epsilon) := (I - \frac{1}{\lambda(\epsilon)} A_F)^{-1} A_F \, v_{i-1}, \quad v_1(\epsilon) := (I - \frac{1}{\lambda(\epsilon)} A_F)^{-1} b$$

where $\lambda(\epsilon) \in \Lambda_\epsilon^m$.

Let $V_\epsilon$ := span $\{v_i(\epsilon)\}$, $i \in \underset{\cdot}{k}$. Then as in the proof of Theorem 4.8 we have that $V_\epsilon \underset{\epsilon\to 0}{\longrightarrow} M$ and that there exists a sequence of derivative feedback maps $F_{2\epsilon}$ such that

$$(I-BF_{2\epsilon})^{-1}A_F\, V_\epsilon \subset V_\epsilon$$

with

$$\text{Mat}[(I-BF_{2\epsilon})^{-1}\, A_F|V_\epsilon] = M_\epsilon$$

where

$$M_\epsilon := \begin{pmatrix} u_{1,1}(\epsilon) & u_{2,1}(\epsilon) & \cdot\ \cdot & u_{k,1}(\epsilon) \\ 0 & u_{2,2}(\epsilon) & \cdot\ \cdot & u_{k,2}(\epsilon) \\ 0 & 0 & \cdot\ \cdot & \cdot \\ \vdots & \vdots & \cdot\ \cdot & \cdot \\ 0 & 0 & \cdot\ \cdot & u_{k,k}(\epsilon) \end{pmatrix}$$

with

$$u_{i,j}(\epsilon) = (-1)^{i-j}\,\lambda^{i-j+1}(\epsilon) \quad \text{for } j < i$$

$$u_{i,i}(\epsilon) = \lambda(\epsilon)\ .$$

Let $I_\epsilon$ := diag$(u_{1,1}(\epsilon),\ldots,u_{k,k}(\epsilon))$ and let $N_\epsilon$ be the nilpotent matrix defined by $N_\epsilon := M_\epsilon - I_\epsilon$. Note that $N_\epsilon^k = 0$. We then have the following lemma:

Assume that $b \oplus A_F b \oplus \ldots \oplus A_F^{k-2} b \subset \ker D$. Then

<u>Lemma 4.4</u>: $\| De^{I_\epsilon t} t^j N_\epsilon^j v_i(\epsilon) \|_{L_p} \underset{\epsilon \to 0}{\longrightarrow} 0$ , for $i \in \underline{k}$ $j \in \{0,\ldots,k-1\}$ and $1 \leq p < \infty$:

<u>Proof</u>: This result is proven in [4.14 , Lemma 6.2, 6.3].

The next lemma establishes that the $L_p$-norm of the closed loop impulse response in the direction of the vectors $v_i(\epsilon)$, $i \in \underline{k}$, can be made arbitrarily small.

Let $A_\epsilon := (I-BF_{2\epsilon})^{-1} A_F$. Hence

<u>Lemma 4.5</u>: Assume that $b \oplus A_F b \oplus \ldots \oplus A_F^{k-2} b \subset \ker D$. Let $\Lambda_\epsilon^m$ be a set as given in Definition 4.4. Let $F_{2\epsilon}$ and $v_i(\epsilon)$ as described above. Then for $i \in \underline{k}$ and all $1 \leq p < \infty$

$$\| De^{A_\epsilon t} v_i(\epsilon) \|_{L_p} \underset{\epsilon \to 0}{\longrightarrow} 0.$$

<u>Proof</u>: For $I_\epsilon$ and $N_\epsilon$ as previously defined there holds

$$I_\epsilon \; N_\epsilon = N_\epsilon \; I_\epsilon \tag{4.121}$$

which implies [4.2]

$$e^{(I_\epsilon + N_\epsilon)t} = e^{I_\epsilon t} e^{N_\epsilon t} .$$

Thus

$$e^{A_\epsilon t} v_i(\epsilon) = e^{M_\epsilon t} v_i(\epsilon)$$

$$= e^{I_\epsilon t} (I + tN_\epsilon + \ldots \frac{t^{k-1}}{(k-1)!} N_\epsilon^{k-1}) v_i(\epsilon).$$

By Lemma 4.4, the result follows. □

Comments:

1) Note that the commutativity in (4.121) plays an important rôle. It is for this reason that we have chosen the rather poor multiplicity set $\Lambda_\epsilon^m$. In [4.14] (see Lemmas 6.2 and 6.3) the set $\Lambda_\epsilon^m$ in Definition 4.4 is replaced by the more general set $\Lambda_\epsilon := \{\lambda_1(\epsilon), \ldots, \lambda_k \epsilon\}$ where now $\Lambda_\epsilon$ is a symmetric set of $k$ complex numbers, $\epsilon^\alpha \lambda_i(\epsilon) \xrightarrow[\epsilon\to 0]{} \bar{\lambda}_i$, $i \in \underline{k}$, and, apart from the symmetry, we can choose $\lambda_i(\epsilon) \neq \lambda_j(\epsilon)$ for $i \neq j$.

In this case the elements of the matrix $M_\epsilon$ are given by

$$u_{i,j}(\epsilon) = (-1)^{i-j}\, \lambda_i(\epsilon) \ldots \lambda_j(\epsilon), \quad \text{for} \quad j < i$$

$$u_{i,i}(\epsilon) = \lambda_i(\epsilon)$$

and it is easy to see that if $I_\epsilon := \operatorname{diag}(\lambda_1(\epsilon), \ldots, \lambda_k(\epsilon))$ with $\lambda_i(\epsilon) \neq \lambda_j(\epsilon)$, $i \neq j$, and $N_\epsilon := M_\epsilon - I_\epsilon$ then $N_\epsilon I_\epsilon$ and $I_\epsilon N_\epsilon$ do not commute.

This seems to show that the asymptotic configuration of eigenvalues (as for example, that given by Definition 4.4) in the closed loop system is important to achieve the result of Lemma 4.5.

2) Lemma 6.3 in 4.14 states that, relative to $\Lambda_\epsilon^m$, there exists a sequence of state feedback maps $F_\epsilon$ such that

$$\| D e^{A_{F_\epsilon} t}\, v_{\underline{i}}(\epsilon) \|_{L_p} \xrightarrow[\epsilon\to 0]{} 0 \qquad 1 \leq p < \infty$$

where as usual $A_{F_\epsilon} = A + BF_\epsilon$ and $F_\epsilon$ is a high gain map such that $A_{F_\epsilon} \mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon$ with $\sigma[A_{F_\epsilon} | \mathcal{V}_\epsilon] = \Lambda_\epsilon^m$ .

Lemma 4.5 has shown that we can attain the same objective, in the sense that $\| De^{(\cdot)t} v_i(\epsilon) \|_{L_p} \xrightarrow[\epsilon\to 0]{} 0$, by either using the high gain operator $A+BF_\epsilon (F_\epsilon \xrightarrow[\epsilon\to 0]{} \infty)$ or the operator $(I-BF_{2\epsilon})^{-1} A_F$ where $F$ is finite and $F_{2\epsilon} \xrightarrow[\epsilon\to 0]{} F_2$, where $F_2$ is also finite. In any case we can choose $\lambda(\epsilon) \in \Lambda_\epsilon^m$ to go to $-\infty$ along any real asymptote $\bar{\lambda}$.

The next theorem is a combination of the results obtained in Theorem 4.8 and Lemma 4.5 and it is a key one for the solution of $(ADDP)^o$.

<u>Theorem 4.10</u>: Let $(A,B)$ be controllable and consider the subspaces $\mathcal{V}^*_{b,K}, \bar{\mathcal{R}}_{b,K}$ . Let $\Lambda_c$, $\Lambda_r$ and $\Lambda_z$ be as in Theorem 4.8 with $\Lambda_c \subset \mathbb{C}^-$.

For $i \in \underline{q}$, let $\Lambda^m_{i,\epsilon}$ be a multiplicity set of $n_i+1$ infinite root loci with common growth $\alpha_i$ and asymptotic direction $\bar{\lambda}_i$ in $\mathbb{R}^-$. Then there exist

a) a sequence of controlled invariant subspaces $\mathcal{V}_\epsilon \xrightarrow[\epsilon\to 0]{} \mathcal{V}^*_{b,K}$

b) maps $F$ and $F_{2\epsilon} \xrightarrow[\epsilon\to 0]{} F_2$ such that $(I-BF_{2\epsilon})^{-1} A_F \mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon$ ,

$\sigma[I-BF_{2\epsilon})^{-1} A_F | \mathcal{V}_\epsilon] = \bigcup_{i=1}^{q} \Lambda^m_{i,\epsilon}$ and $(I-BF_2)\bar{\mathcal{R}}_{b,K} \subset A_F \bar{\mathcal{R}}_{a,K}$.

Let $A_\epsilon := (I-BF_{2\epsilon})^{-1} A_F$. Then for such maps $F$ and $F_{2\epsilon}$ :

c) $\| De^{A_\epsilon t} \mathcal{V}^*_{b,K} \|_{L_p} \xrightarrow[\epsilon\to 0]{} 0 \qquad 1 \le p < \infty$

d) $\sigma[A_\epsilon] = \bigcup_{i=1}^{q} \Lambda^m_{i,\epsilon} \cup \Lambda_c \cup \Lambda_r \cup \Lambda_z$ .

<u>Proof</u>; Items a, b and d have been proved in Theorem 4.8.

It is also clear from the proof of Theorem 4.8 that $A_\epsilon V_K^* \subset V_K^*$. Thus by using (4.109) it follows that we have to show that

$$\| De^{A_\epsilon t} \bar{R}_{b,K} \|_{L_p} \xrightarrow[\epsilon\to 0]{} 0 \qquad 1 \leq p < \infty. \qquad (\mathscr{L})$$

As in Theorem 4.8 let

$$V_\epsilon := R_{1,\epsilon} \oplus \dots \oplus R_{q,\epsilon}$$

where

$$R_{i,\epsilon} := \text{span}\{v_{j,i}(\epsilon)\} \ , \ i \in \underline{q}, \ j \in \{1,2,\dots,n_i+1\}$$

with $n_i = 0$ for $i \in \{p+1,\dots,q\}$ (see (4.110c) and also (2.21)) and $v_{j,i}(\epsilon) \xrightarrow[\epsilon\to 0]{} A_F^{j-1} b_i$.

Let $A_{i,\epsilon} := A_\epsilon | R_{i,\epsilon}$ and note that from (4.110-111)

$$b_i + A_F b_i + \dots + A_F^{n_i - 1} b_i \subset K \ , \qquad i \in \underline{p} \ .$$

Hence by Lemma 4.5 we have that

$$\| De^{A_{i,\epsilon} t} v_{j,i}(\epsilon) \|_{L_p} \xrightarrow[\epsilon\to 0]{} 0 \ , \ i \leq p < \infty, \ j \in \{1,2,\dots,n_i+1\}.$$

From the proof of Theorem 4.8 we also have

$$A_\epsilon | C = A_{F_C} | C, \ A_{F_C} := A+BF_C$$

where $\mathcal{C}$ is a controlled invariant subspace such that $\mathcal{V}^*_{b,K} \oplus \mathcal{C} = \mathcal{X}$ and $\sigma[A_{F_c}|\mathcal{C}] = \Lambda_c$.

To prove item c, it follows from $(\mathcal{L})$ that it suffices if

$$\| De^{A_\epsilon t} A_F^{\ell-1} b_s \|_{L_p} \xrightarrow[\epsilon\to 0]{} 0 \qquad s \in \underline{q},\ \ell \in \{1,2,\ldots,n_i+1\} .$$

Now, for $\epsilon$ sufficiently small we can write

$$\mathcal{X} = \mathcal{V}^*_K \oplus \mathcal{C} \oplus \mathcal{V}_\epsilon \tag{4.122}$$

which implies that the vector $A_F^{\ell-1} b_s$ can be written as

$$A_F^{\ell-1} b_s = \sum_{i=1}^{q} \sum_{j=1}^{n_i+1} \alpha_{j,i}(\epsilon) v_{j,i}(\epsilon) + w(\epsilon) + c(\epsilon)$$

where $\alpha_{j,i}(\epsilon)$ are real scalars, $w(\epsilon) \in \mathcal{V}^*_K$, $c(\epsilon) \in \mathcal{C}$.

Since $v_{\ell,s}(\epsilon) \xrightarrow[\epsilon\to 0]{} A_F^{\ell-1} b_s$, we must have

$$\alpha_{j,i}(\epsilon) \xrightarrow[\epsilon\to 0]{} 0 \quad \text{for } (j,i) \neq (\ell,s),\ \alpha_{\ell,s}(\epsilon) \xrightarrow[\epsilon\to 0]{} 1,\ w(\epsilon) \xrightarrow[\epsilon\to 0]{} 0 \text{ and } c(\epsilon) \xrightarrow[\epsilon\to 0]{} 0 .$$

Also note that since $c(\epsilon) \to 0$ and $\sigma[A_{F_c}|\mathcal{C}] \subset \mathbb{C}^-$ then

$$\| De^{A_{F_c} t} c(\epsilon) \|_{L_p} \xrightarrow[\epsilon\to 0]{} 0 .$$

Hence

$$\| De^{A_\epsilon t} A_F^{\ell-1} b_s \|_{L_p} \leq \sum_{i=1}^{q} \sum_{j=1}^{n_i+1} |\alpha_{j,i}(\epsilon)| \; \| De^{A_{i,\epsilon} t} v_{j,i}(\epsilon) \|_{L_p} +$$

$$+ \; \| De^{A_\epsilon t} c(\epsilon) \|_{L_p} \xrightarrow[\epsilon\to 0]{} 0$$

and c) is proved.

The arguments used in the proof of the above theorem are similar to those used in Theorem 6.1 [4.14].

We finally obtain :

<u>Corollary 4.5:</u> Suppose that $\operatorname{Im} G \subset V^*_{b,K}$. Then $(ADDP)^o$ is solvable.

<u>Proof</u>: Note that $\operatorname{Im} G \subset V^*_{b,K}$ implies $\operatorname{Im} G + B \subset V^*_{b,K}$ and that

$$\operatorname{Im}(I-BF_{2\epsilon})^{-1}G \subset \operatorname{Im} G + B \subset V^*_{b,K} \quad .$$

The result now folllows from Theorem 4.10c and Lemma 4.3. □

The final conclusion is that given that $\operatorname{Im} G \subset V^*_{b,K}$, then we can almost decouple the disturbance $(\| \underline{z}\|_{L_p} \leq \epsilon \| \underline{d}\|_{L_p})$ by either using high gain state feedback or a finite gain nonsingular P.D. law .

## IV.5 P.I.D. OBSERVERS

Consider the system

$$\Sigma : \dot{x} = Ax \; ; \quad y = Cx$$

where

$$x \in X := \mathbb{R}^n \quad ; \quad y \in Y := \mathbb{R}^r .$$

Let $L$ be a given subspace and let $S^*_L$ be the infimal conditionally invariant subspace that contains $L$ . We have seen in Chapter I that there exists an observer $\Sigma^{obs}$ described by

$$\sum^{obs} : \dot{w} = Kw + Ly \tag{4.123}$$

such that $\underline{w}(0) = \underline{x}(0) \pmod{S_L^*}$ yields $\underline{w}(t) = \underline{x}(t) \pmod{S_L^*}$, $\forall t \in \mathbb{R}$.

Note that $\sum^{obs}$ is an integral (I) type observer in the sense that $w = f(\int y)$.

Now consider $S_{b,L}^*$, the infimal $L_p$ - almost conditionaly invariant subspace "containing" $L$, $1 \leq p < \infty$ (see Definition 1.13 and Theorem 1.3 (dual)). It is clear from the comments in [4.18] that in order to arbitrarily accurately estimate $x(\text{mod } S_{b,L}^*)$, i.e. $\| \underline{w} - \underline{x}(\text{mod } S_{b,L}^*) \|_{L_p} \leq \epsilon$, the integral observer $\sum^{obs}$ has to be a high gain one ($K \to \infty$, $L \to \infty$).

In this section we show that there exists a P.I.D. observer which *exactly* estimates $\underline{x}(\text{mod } S_{b,L}^*)$. P.I.D. observers are those which in addition to integral action also admit a proportional (P) and differentiating (D) action, i.e. if $w$ is the state variable of the observer then $w = f(\int y, y, \dot{y}, \ldots, y^{(n)})$.

It is clear that if $w(s) = T(s)\, y(s)$ is the equation of a P.I.D. observer in the complex variable $s$, then $T(s)$ is a rational matrix. (For integral observers, $T(s)$ is is strictly proper).

Since a rational matrix can be realized by a generalized linear system [4.16] we shall say that an observer is a P.I.D type observer if it is described by

$$\sum_{P.I.D.} : \begin{cases} \dot{w}_1 = Kw_1 + Ly & (4.124a) \\ w_2 = M_0 y + M_1 \dot{y} + \ldots + M_n y^{(n)} & (4.124b) \end{cases}$$

where $w^T = (w_1^T, w_2^T)^T$ is the state variable of the P.I.D. observer. Note that (4.124b) corresponds to the forced response of a subsystem of the type $J\dot{w}_2 = w_2 + My$ where $J$ is nilpotent map (see (3.35)).

The main objective of this section is to give a dual interpretation to theorems 4.4 and 4.8 which leads to :

a) the construction of a P.I.D. observer which *exactly* estimates $x(\text{mod } S^*_{b,L})$ and

b) the construction of an observer which *exactly* estimates $x(\text{mod } S^*_\epsilon)$ where $S^*_\epsilon \xrightarrow[\epsilon\to 0]{} S^*_{b,K}$.

The following lemma shows a decomposition of $X/S^*_{b,L}$.

<u>Lemma 4.6</u>: Let $\bar{N}_{b,L}$ be a subspace as in (1.78). Then

$$\frac{X}{S^*_{b,L}} = \frac{X}{S^*_L} \oplus \frac{X}{\bar{N}_{b,L}} .$$

<u>Proof</u>: The above direct sum is to be considered as an external direct sum of two vector spaces, which in this case are quotient spaces.

By Lemma 1.2'-a we obtain

$$N^*_{b,L} = \bar{N}_{b,L} \cap N^*_L . \tag{4.125}$$

Now, from (1.70) and (4.125)

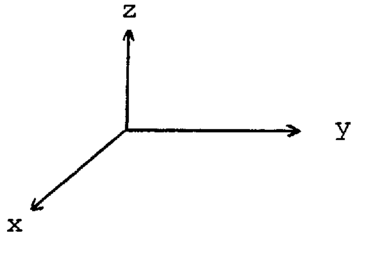
$$S^*_{b,L} = N^*_{b,L} \cap S^*_L = \bar{N}_{b,L} \cap N^*_L \cap S^*_L$$

and by using (1.67) and Lemma 1.2'-b, it follows that

$$S^*_{b,L} = \bar{N}_{b,L} \cap S^*_L \quad \text{with } \bar{N}_{b,L} + S^*_L = X$$

and the result follows. □

For a geometric interpretation of Lemma 4.6., recall that for $x \in X$ and for a given subspace $K$, the element $\bar{x} \in X/K$ is the coset of $X(\text{mod } K)$ and $\bar{x} = x + K$. Geometrically, $\bar{x}$ is the hyperplane passing through $x$ obtained by parallel translation of $K$.




$y \oplus z := \bar{N}_{b,L}$

$x \oplus z := S^*_L$

$z := S^*_{b,L}$

In the above figure, the cosets of $X/S^*_L$ are all the two dimensional planes which are parallel to $S^*_L$ and similarly for the cosets of $X/\bar{N}_{b,L}$.

When we perform the externaldirect sum of the two quotient spaces we obtain all the lines which are parallel to $S^*_{b,L}$, which are exactly the cosets of $X/S^*_{b,L}$.

We know how to construct an integral observer for $X/S_L^*$. The question that arises now is whether we may define an observer on a subspace $Z$ such that

$$\bar{N}_{b,L} \oplus Z = X.$$

The answer is yes and we shall see in a moment that the observer defined on $Z$ is in fact a P.D. observer. Note that if $\{z_i\}$ is a basis for $Z$ then $\bar{z}_i = z_i + \bar{N}_{b,L}$ is a basis for $X/\bar{N}_{b,L}$.

Thus by adding the output of such an observer with the output of observer for $X/S_L^*$ we then obtain by Lemma 4.6 an estimate of $x(\text{mod } S_{b,L}^*)$.

In order to facilitate the presentation of the ensuing theorem we state a trivial fact from linear algebra.

Fact (*): Consider the dual space $X'$ and let $W^\perp$ and $V^\perp$ be subspace of $X'$ such that

$$X' = W^\perp \oplus V^\perp$$

Then

$$X = V \oplus W.$$

<u>Proof</u>: First note that $(X')' = X$ and let $\{x_1',\ldots,x_k',x_{k+1}',\ldots,x_n'\}$ be a basis for $X'$ such that

$$W^\perp = \text{span}\{x_1',\ldots,x_k'\} \text{ and } V^\perp = \text{span}\,\{x_{k+1}',\ldots,x_n'\}.$$

Let $\{x_j\}$ , $j \in \underline{n}$, be the unique set such that $x_i' x_j = \delta_{ij}$ . Then $\{x_1,\ldots,x_k,x_{k+1},\ldots,x_n\}$ is a basis for $X$ with

$$\text{span}\{x_1,\ldots,x_k\} = V \text{ and } \text{span}\{x_{k+1},\ldots,x_n\} = W.$$

□

The next theorem is a mere dualization of Theorem 4.4.

Theorem 4.11: Let the pair (C,A) be observable. Then there exists a P.I.D. observer

$$\sum_{\text{P.I.D.}} \begin{cases} \dot{w}_1 = Kw_1 + Ly \\ \\ w_2 = M_0 y + M_1 \dot{y} + \ldots + M_n y^{(n)} \end{cases}$$

such that $\underline{w}_1(0) = 0$, $\underline{w}_2(0-) = 0$ yields for all $\underline{x}(0) \in S^*_{b,L}$, $\underline{w}_1(t) + \underline{w}_2(t) = \underline{x}(t) \pmod{S^*_{b,L}}$ , $t \in \mathbb{R}^+$.

Proof: Our aim is to define a system on a subspace $Z$ which complements $\bar{N}_{b,L}$ . For this write

$$(I-L_2C)\dot{x} = (A+L_1C)x - L_1 y - L_2 \dot{y} \tag{4.126}$$

where $L_1$ and $L_2$ are maps to be defined.

From the duality principle established by Willems [4.18] we have that the subspace $\bar{N}^{\perp}_{b,L}$ is the "$\bar{R}_{b,L}$" of the sequence (1.12) with respect to the pair $(A^T,C^T)$.

Since the pair $(A^T,C^T)$ is controllable, then as in the proof of Theorem 4.4 (see (4.59)) it follows that the dual space $X'$

admits the following decomposition

$$X' = S^{\square\perp} \oplus \bar{N}_{b,L}^{\perp} \tag{4.127}$$

where

$$S^{\square\perp} = S_L^{*\perp} \oplus L_i^{\perp} \tag{4.128}$$

and $L_i^{\perp}$ is a controlled invariant subspace relative to $(A^T, C^T)$.

Moreover, by the proof of Theorem 4.4 there are maps $L_1^T$ and $L_2^T$ such that

$$X' = (I-L_2C)^T S^{\square\perp} \oplus (A+L_1C)^T \bar{N}_{b,L}^{\perp} \tag{4.129}$$

and such that the pencil $s(I-L_2C)^T - (A+L_1C)^T$ is regular. Hence by Theorem 4.3 there are maps

$$L_v^T := (I-L_2C)^T S^{\square\perp} | (A+L_1C)^T | S^{\square\perp} \tag{4.130a}$$

$$J^T := (A+L_1C)^T \bar{N}_{b,L}^{\perp} | (I-L_2C)^T | \bar{N}_{b,L}^{\perp} \tag{4.130b}$$

such that the eigenvalues of $L_v^T$ and the Jordan structure of the nilpotent map $J^T$ determine, respectively, the finite-zeros and the infinite-zeros of the pencil $s(I-L_2C)^T - (A+L_1C)^T$.

Let $Z := S^{\square}$. Then by using fact(*) we obtain from (4.127-8) the following decomposition for $X$

$$X = \bar{N}_{b,L} \oplus Z \tag{4.131}$$

where

$$Z = S_L^* \cap L_i \text{ , with } S_L^* + L_i = X.$$

The decomposition of (4.129) also yields

$$\mathcal{X} = (A+L_1C)^{-1}\bar{N}_{b,L} \oplus \mathcal{Z} \quad . \tag{4.132}$$

Let $\mathcal{X}_1'$ and $\mathcal{X}_2'$ denote, respectively, the decompositions (4.127) and (4.129) and, similarly, let $\mathcal{X}_1$ and $\mathcal{X}_2$ denote the decompositions (4.131-2). Hence, by using (4.130) it follows that if

$$(A+L_1C)^T : \mathcal{X}_1' \to \mathcal{X}_2' \ , \ (I-L_2C)^T : \mathcal{X}_1' \to \mathcal{X}_2'$$

then

$$\mathrm{Mat}(A+L_1C)^T = \begin{pmatrix} L_v^T & 0 \\ 0 & I \end{pmatrix} \qquad \mathrm{Mat}(I-L_2C)^T = \begin{pmatrix} I & 0 \\ 0 & J^T \end{pmatrix} .$$

By considering the dual bases $\mathcal{X}_1$ and $\mathcal{X}_2$ we then obtain

$$(A+L_1C) : \mathcal{X}_2 \to \mathcal{X}_1, \quad (I-L_2C) : \mathcal{X}_2 \to \mathcal{X}_1$$

with

$$\mathrm{Mat}(A+L_1C) = \begin{pmatrix} L_v & 0 \\ 0 & I \end{pmatrix} \qquad \mathrm{Mat}(I-L_2C) = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} . \tag{4.133}$$

Let $P_z$ be the projection on $\mathcal{Z}$ along $\bar{N}_{b,L}$. Then from (4.126) and (4.133) we obtain the following generalized linear system on $\mathcal{Z}$

$$J\dot{z} = z - \hat{L}_1 y - \hat{L}_2 \dot{y} \tag{4.134}$$

where

$$\hat{L}_1 := P_Z L_1 \; ; \quad \hat{L}_2 := P_Z L_2 \; .$$

The solution of (4.134) has the form shown in ( 3.37) and is given by

$$\underline{z}(t) = \sum_{i=0}^{q-1} M_i \, \underline{y}^{(i)}(t), \quad t \geq 0$$

where

$$M_0 := \hat{L}_1 \; ; \; M_1 := (J\hat{L}_1 + \hat{L}_2); \ldots ; M_{q-2} := (J^{q-1}\hat{L}_1 + J^{q-2}\hat{L}_2); M_{q-1} := J^{q-1}\hat{L}_2$$

and $q$ is the nilpotency index of $J$.

Define an integral observer for $X/S_L^*$ (see (1.60)) by

$$\dot{w}_1 = Kw_1 + Ly$$

where

$$K := (A+L'C)(\text{mod } S_L^*) \; ; \; L := L'(\text{mod} S_L^*)$$

and $L' \in L(S_L^*)$.

Also define a P.D. observer for $X/\bar{N}_{b,L}$ by

$$w_2 = \sum_{i=0}^{q-1} M_i \, y^{(i)} \quad .$$

Let $e := (w_1+w_2) - (x(\text{mod} S_L^*)+z)$. Then by using Lemma 4.6 it follows that $\underline{w}_1(0) = 0$, $\underline{w}_2(0-) = 0$ yields for all $\underline{x}(0) \in S_{b,L}^*$, $\underline{e}(t) = 0$, $t \geq 0$, i.e.

$$\underline{w}_1(t) + \underline{w}_2(t) = \underline{x}(t)(\text{mod } S_{b,L}^*) .$$

□

Comments:

1) Note that the error dynamics are dictated by the eigenvalues of K, i.e., $\dot{e} = Ke$.

2) It can also be shown that there exists a P.I.D. observer which estimates $X/\overset{*}{S}_{a,L}$, where $\overset{*}{S}_{a,L}$ is the infimal almost conditionally invariant subspace which contains $L$.

The next theorem contains dual interpretations of some results shown in Theorem 4.8. By dualing result a) we obtain that there exists a sequence of conditionally invariant subspaces $S_\epsilon$ such that $S_\epsilon \xrightarrow[\epsilon\to 0]{} \bar{N}_{b,L}$. This implies by Lemma 1.2'-b that, for $\epsilon$ sufficiently small, $S_\epsilon + \overset{*}{S}_L = X$. Let $\overset{*}{S}_\epsilon := S_\epsilon \cap \overset{*}{S}_L$. Hence as in Lemma 4.6 we define the following external direct sum

$$\frac{X}{\overset{*}{S}_\epsilon} = \frac{X}{S_\epsilon} \oplus \frac{X}{\overset{*}{S}_L} \quad .$$

Note that $\overset{*}{S}_\epsilon \xrightarrow[\epsilon\to 0]{} \overset{*}{S}_{b,L}$ . Hence $x + \overset{*}{S}_\epsilon \xrightarrow[\epsilon\to 0]{} x + \overset{*}{S}_{b,L}$ so that $X/\overset{*}{S}_\epsilon \xrightarrow[\epsilon\to 0]{} X/\overset{*}{S}_{b,L}$. We then obtain

Theorem 4.12: Let $\overset{*}{S}_\epsilon$ be as defined above. Then there exists an observer described by

$$\begin{cases} \dot{w}_1 = K_1 w_1 + L_1 y \\ \dot{w}_2 = K_{2\epsilon} w_2 + M_0 y + M_{1\epsilon}\dot{y} \; ; \; z = w_2 + L_{2\epsilon} y \end{cases}$$

such that $\underline{w}_1\,(0) = 0$, $\underline{z}\,(0) = 0$ yields for all $\underline{x}(0) \in S_\epsilon^*$,
$\underline{w}_1(t) + \underline{z}\,(t) = \underline{x}(t) \,(\mathrm{mod}\, S_\epsilon^*)$.

Proof: By dualizing result b of Theorem 4.8 we obtain maps L and $L_{2\epsilon} \xrightarrow[\epsilon\to]{} L_2$ such that $A^L(I-L_{2\epsilon}C)^{-1}S_\epsilon \subset S_\epsilon$ and $A^{L^{-1}}\bar{N}_{b,L} \subset (I-L_2C)^{-1}\bar{N}_{b,L}$.

We recall the notation $A^L := A+LC$ and we remark that $(I-L_{2\epsilon}C)$ is nonsingular.

Now write

$$(I-L_{2\epsilon}C)\dot{x} = A^L x - Ly - L_{2\epsilon}\dot{y}$$

and let $\hat{x} := (I-L_{2\epsilon}C)x$. Hence

$$\dot{\hat{x}} = A^L(I-L_{2\epsilon}C)^{-1}\hat{x} - Ly - L_{2\epsilon}\dot{y} \quad . \tag{4.133}$$

Since $A^L(I-L_{2\epsilon}C)^{-1}S_\epsilon \subset S_\epsilon$ then there exists a map $K_{2\epsilon}$ (which is unique) such that $K_{2\epsilon}P = PA^L(I-L_{2\epsilon}C)^{-1}$ where $P : X \to X/S_\epsilon$ is the canonical projection. Let $M_0 := -PL$ and $M_{1\epsilon} := -PL_{2\epsilon}$. Then from (4.133) it follows that

$$\dot{\hat{x}}(\mathrm{mod}\ S_\epsilon) = K_{2\epsilon}\hat{x}(\mathrm{mod} S_\epsilon) + M_0 y + M_{1\epsilon}\dot{y}.$$

Since

$$\hat{x}(\mathrm{mod} S_\epsilon) = (I-L_{2\epsilon}C)x + S_\epsilon = x(\mathrm{mod} S_\epsilon) - L_{2\epsilon}y$$

then

$$\begin{cases} \dot{w}_2 = K_{2\epsilon} w_2 + M_0 y + M_{1\epsilon}\dot{y} \\ z = w_2 + L_{2\epsilon} y \end{cases}$$

represents an observer for $\underline{x}(t)\,(\mathrm{mod}\,S_{\epsilon})$.

Let

$$\dot{w}_1 = K_1 w_1 + L_1 y$$

be an observer for $\underline{x}(t)\,(\mathrm{mod}\,S_L^*)$ (see (1.60)) with

$$K_1 := (A+L'C)\,(\mathrm{mod}\,S_L^*); \quad L_1 := -L'\,(\mathrm{mod}\,S_L^*)$$

and $L' \in L(S_L^*)$.

Let $e_1 := w_1 - x(\mathrm{mod}\,S_L^*)$ and $e_2 := z - x(\mathrm{mod}\,S_{\epsilon})$. Hence

$$\dot{e}_1 = K_1 e_1 \quad : \quad \dot{e}_2 = K_{2\epsilon} e_2 \quad .$$

Since $S_{\epsilon}^* = S_{\epsilon} \cap S_L^*$, then $\underline{x}(0)\,(\mathrm{mod}\,S_{\epsilon}^*) = 0$ implies $\underline{x}(0)\,(\mathrm{mod}\,S_L^*) = \underline{x}(0)\,(\mathrm{mod}\,S_{\epsilon}) = 0$. Since $(\underline{w}_1(0), \underline{z}(0)) = 0$ we then obtain $(\underline{e}_1(0), \underline{e}_2(0) = 0)$ so that $\underline{e}(t) := \underline{e}_1(t) + \underline{e}_2(t) = 0,\ t \geq 0$.

Comments:

1) It is easy to see that the transfer function matrix $X_2(s)$ from $y$ to $z$ is given by

$$X_2(s) = (sI-K_{2\epsilon})^{-1}(M_0 + sM_{1\epsilon}) + L_{2\epsilon}$$

and note that $X_2(s)$ is proper. Thus we can exactly estimate $\underline{x}(t)(\mathrm{mod}\,S_\epsilon)$, $S_\epsilon \xrightarrow[\epsilon\to 0]{} \bar{N}_{b,L}$, by using an observer with a proper transfer matrix whereas Theorem 4.11 has shown that to exactly estimate $\underline{x}(t)\,\mathrm{mod}(\bar{N}_{b,L})$ $X_2(s)$ degenerates into a polynomial transfer matrix.

2) Let $\Lambda_\epsilon := \sigma[K_{2\epsilon}] = \sigma[A^L(I-L_{2\epsilon}C)^{-1}(\mathrm{mod}S_\epsilon]$. Then by using again a duality argument (see the proof of Theorem 4.8) we have that $|\Lambda| \xrightarrow[\epsilon\to 0]{} \infty$ .

3) It should be noted that the controllability hypothesis of Theorem 4.8 (relative to the pair $(A^T,C^T)$) has been used only to find the controlled invariant subspace $L_i^\perp$ (see (4.128)). Since the observer constructed in Theorem 4.12 is not concerned with the estimation of $\underline{x}(t)(\mathrm{mod}\ L_i)$, it follows that the observability of the pair (C,A) (equivalent to the controllability of $(A^T,C^T)$) is not needed as hypothesis in such a theorem.

4) It can also be shown that there exists an observer of the same type shown in Theorem 4.12 which estimates $\underline{x}(t)(\mathrm{mod}S_\epsilon \cap S_L^*)$ where $S_\epsilon \cap S_L^* \xrightarrow[\epsilon\to 0]{} S_{a,L}^*$ .

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# CONCLUSIONS

The intention of this final section is to give some general comments about the thesis and also to point out some directions of future research which are connected with the work developed here.

By using geometric properties of sliding subspaces described in Chapter I and a suitable state space decomposition for an invertible system, we have constructed in Chapter II an output feedback map R such that the asymptotes of the closed loop map (A+g BRC), $g \to \infty$, take on pre-assigned values. We have also seen that the i eigenvectors associated with an asymptote $\alpha\left(\lambda^{i} \xrightarrow[g\to\infty]{} g\alpha\right)$ converge to a direction which can be chosen to belong to any subspace $\mathcal{B}_i' \subset \mathcal{B}$ such that $\mathcal{B}_i' \oplus \mathcal{B}_i = \mathcal{B}_{i-1}$ (see the comments after Proposition 2.2).

It would be interesting in this context to investigate in more detail the limiting process of the sequence of (A+g BRC)-invariant subspaces associated with eigenvalues that go to infinity in order to find out the reason for the collapsing of the i eigenvectors into only one direction.

In the simple case with rank CB = m, such a collapse does not occur : we have m first order asymptotes and the m-dimensional (A+g BRC)-invariant subspace associated with such asymptotes converge to the m-dimensional subspace $\mathcal{B}$. It is worthwhile to note that an (A+g BRC)-invariant subspace is an (A,B,C)-invariant subspace [4.12, 1.16 ], i.e., it is simultaneously a controlled and a conditionally invariant subspace. Thus in the case rank CB = m we have a sequence of m-dimensional (A,B,C) invariant subspaces (with arbitrary spectrum) converging to the subspace $\mathcal{B}$ which is simultaneously an almost controlled invariant subspace and a conditionally invariant subspace (see the consequences of rank CB = m in II.2.3). It would be valuable to have similar interpretatious for the case rank CB < m.

The work carried out here on the topics of generalized linear systems (Chapter III) and regular pencils (Chapter IV) was partly guided by the suggestions given by Verghese et al [3.13]. They have pointed out the need for a geometric language to deal with such subjects and the usefulness of such an approach is clearly manifested at various points in the text. For example, we have obtained a clearer picture of the controllable and unobservable infinite-zeros due to the association of such zeros with controllable and unobservable subspaces. This yields an alternative way of computing a controllable and observable generalized linear system. As another example, we have obtained a constructive regularity condition for a pencil in the sense that one can establish computational algorithms based on the sequences (4.34) and (4.36) to check for the regularity and if the pencil turns out to be regular one can easily obtain the maps L and J which yield the modal decomposition of the regular pencil.

The primary objective of Chapter III has been that of understanding more about geometric structural properties of a plant which is naturally modelled by a generalized linear system. The results on zero placement by state feedback and output feedback have illustrated the possibilities of altering the dynamical structure and have also demonstrated the similarities with ordinary linear systems: for example, the infinite-zeros can be assigned if and only if they are controllable. Thus a generalized linear system with controllable infinite-zeros (and of course, a rational transfer matrix $C(sE-A)^{-1}B$) can be converted into a linear system with proper transfer matrix. It is also worthwhile to note that a generalized linear system can be transformed into a linear system with strictly proper transfer matrix by state derivative feedback and for this to occur it is necessary and sufficient that $\text{Im } E + \mathcal{B} = \mathcal{X}$. This follows immediately from the equivalence : $\exists F$ such that $(E+BF)$ is nonsingular $\iff$ zero eigenvalues of $E$ are controllable $\iff$ $\text{Im } E + \mathcal{B} = \mathcal{X}$.

The other important reason for studying generalized linear systems and regular pencils is that an ordinary linear system $(\dot{x} = Ax + Bu)$ may become a generalized linear system under a PD law. Indeed, several results in Chapter IV involve a regular PD law which yields a generalized linear system.

The sections concerning PD laws and PID observers are perhaps the most interesting from the synthesis point of view because they point to an area of research, namely the use of PID compensators in multivariable control problems, which apparently has not been exploited yet.

It would be very interesting to have general constructive principles for a PID compensator in the same spirit as those described by Schumacher [4.12] with respect to PI compensators. By a PID compensator we mean a time invariant linear system with the form

$$\dot{w} = Kw + L_1 y + L_2 \dot{y} \tag{C.1}$$

$$u = Mw + F_1 y + F_2 \dot{y} \; . \tag{C.2}$$

We believe that the properties described in Chapter IV concerning the relationship between a linear system under a (regular and non-singular) PD law and (almost) invariant subspaces will be of utility in obtaining those principles.

Regarding the disturbance decoupling problem we also believe that the results of Theorems 4.7 and 4.10 could be useful in the search of solution for the following extensions :

1. Consider the linear system

$$\dot{x} = Ax + Bu + Gd$$

$$z = Dx$$

where, as usual, $d$ is a vector of disturbances and $z$ denotes the to-be-controlled outputs.

We can then formulate the following problem which we term the disturbance decoupling problem by a PID compensator and disturbance feedforward : does there exist a PID compensator given by (C.1) and

$$u = Mw + F_1 y + F_2 \dot{y} + F_3 d$$

such that in the closed loop system the transfer matrix from $d$ to $z$ is zero? If so, give existence conditions and a procedure for the computation of such a compensator.

2. Willems [4.18] has formulated and solved the almost disturbance decoupling problem by measurement feedback (ADDPM) which is as follows : does there exist a PI compensator $\dot{w} = Kw + Ly$; $u = Mw + Fy$ such that in the closed loop system with $(\underline{x}(0), \underline{\omega}(0)) = 0$ there holds $\| z \|_{L_p} \leq \epsilon \| d \|_{L_p}$ ?

He has shown that (ADDPM) is solvable if and only if

$$\text{Im } G \subset \mathcal{V}^*_{b,\ker D} \text{ and } \mathcal{S}^*_{b,\text{Im}G} \subset \ker D. \tag{C.3}$$

By keeping symmetry with $(\text{ADDP})^{\circ}$ (see Definition 4.3) we are then led to the following problem : does there exist a PID compensator given by (C.1) and (C.2) and which involves finite maps such that in the closed loop system with $(\underline{x}(0), \underline{\omega}(0)) = 0$ there holds $\| z \|_{L_p} \leq \epsilon \| d \|_{L_p}$ ? If so, find existence conditions and give a procedure for the construction of such a P.I.D. compensator.

It is reasonable to conjecture (by analogy between (ADDP) and $(\text{ADDP})^{\circ}$) that under the same conditions (C.3) there exists a P.I.D. compensator with the required properties that solves the last problem.

It is our opinion that in order to construct the above PID compensator we shall have first to understand more about the state space synthesis of a high gain compensator which solves (ADDPM). Such a synthesis procedure does not exist at the present time. In [4.18] the PI compensator which solves (ADDPM) is obtained through a realization of a transfer matrix which is computed as the solution of an equation.

We then conclude this thesis in the hope that the above questions are significant and interesting, and that they point to valuable future research.