Valuation of Synthetic CDOs and Related Portfolio Credit Derivatives

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Abstract

This thesis develops and analyzes synthetic collateralized debt obligation (CDO) pricing models within the conditional independence framework. The models that are developed in this thesis produce the correlation skew observed in the single tranche CDO market while still retaining analytical tractability and ease of implementation. Using the market standard one factor Gaussian copula model as a starting point, the model is extended to incorporate stochastic correlation similar in spirit to stochastic volatility models used in the equity options market. Incorporation of stochastic correlation results in the production of significant correlation skews and provides a much better fit to market prices than existing models. An explicit expression is derived for the copula corresponding to the dependency structure imposed by the stochastic correlation model. The large homogeneous portfolio loss distribution is also derived and closed form expressions are developed for the expected tranche losses in the large limit case. The second proposed model incorporates unpredictable external shocks to the one factor Gaussian copula model and is termed the shock-Gaussian (SG) model. It is shown that such a model is also capable of producing steep correlation skews and the corresponding large homogeneous loss distribution is also derived. Pricing algorithms for CDO Squared derivative products are also considered and a new methodology is proposed that overcomes the overlap problem associated with such transactions and allows one to incorporate the correlation skew effect in its valuation. Although the models developed in this thesis are primarily concerned with pricing synthetic CDOs, they are general portfolio credit models and can be used to price any portfolio credit derivative whose price depends only on the distribution of default times.
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Chapter 1

Portfolio Credit Derivatives

1.1 Introduction

This chapter provides a survey of credit derivative modelling techniques. Credit derivatives are financial products where the payoff depends on the occurrence of a credit event. By credit event we mean the failure to pay financial obligations or the declaration of bankruptcy. In most cases the word 'default' will be used to signify a credit event. Although there exists credit derivatives such as options on credit default swaps and options on synthetic collateralized debt obligation tranches where the payoff is a function of the credit spread or credit rating, in this thesis we only concentrate on credit derivatives where the payoff is a function of the default times and recovery values and we do not consider products with option type payoffs. Hence, we will be concerned only with models that capture the distribution of the default times and recovery rates. In fact, when we proceed to look at portfolio credit derivatives we will make the assumption that recovery rates are a fixed constant and only concentrate on the modelling of the joint distribution of default times.

Section 1.2 of this chapter starts off by reviewing three different credit derivative products: the credit default swap (CDS), Nth-to-default baskets, and collateralized debt obligations (CDO). We only give a qualitative review of Nth-to-default baskets and CDOs and postpone a full quantitative treatment until chapter 2. Section 1.3 provides a literature review and description of single issuer credit
models. The class of single issuer credit models can be decomposed into structural and reduced form models and we review both approaches. Intuitively, the structural approach aims to model the inner workings of the firm and explain the event of default in terms of the interaction of variables such as the asset value and outstanding debt issued by the firm. Reduced form models, on the other hand, model the probability of default directly using Poisson processes and does not aim to link the cause of default to any internal variables of the firm. Section 1.4 provides an overview of credit portfolio models and the concept of copulas. Section 1.5 outlines the structure of this thesis.

1.2 A Primer on Credit Derivatives Products

There are many types of credit derivatives ranging from simple credit default swaps to complex options on credit tranches. In this thesis we are only concerned with the valuation of portfolio credit derivatives where the payoff depends only on the times of default and the realized recovery values. By default we mean a missed payment on a scheduled cash flow, such as not paying the interest on a loan on the interest payment dates or not paying back the principal of the loan at the maturity of the loan. In this thesis we exclude derivatives whose payoff depends on the credit spread level (such as an option on a credit default swap) or on the issuers credit rating.

Although the models developed in this thesis can be used to price any portfolio credit derivative where the payoff is a function of the default times, we concentrate mainly on the pricing of synthetic collateralized debt obligations (CDOs). For completeness, we also review in this section credit default swaps and Nth-to-default baskets. We also briefly mention CDO squared derivatives but postpone a full mathematical treatment of these products until chapter 2.

1.2.1 Credit Default Swaps

Credit default swaps (CDS) are the basic building blocks of synthetic portfolio credit derivatives. CDS are simple insurance contracts, where the entity that is
insured is a corporate or sovereign bond. There are two parties to a CDS contract, the protection seller and the protection buyer, and the contract is characterized by a final maturity date $T$. The protection buyer makes periodic (usually quarterly in arrears) premium payments to the protection seller expressed as a fixed percentage of the notional amount of the insured bond. This fixed percentage payment is known as the CDS rate and the payments made by the protection buyer is known as the fixed or premium leg of the CDS contract. If the bond does not default during the term of the contract, then these premium payments are made until the maturity of the contract. If a default does happen by the reference entity before the final maturity date, then the protection buyer ceases to make any premium payments except for a one-off accrued coupon payment if the default happens in between premium payment dates. The protection seller on the other hand, makes a payment to the protection buyer to compensate him for the loss in value of the reference entity due to the credit event. This payment equals the difference between the par value and the price of the reference entity just after the default and is adjusted by the notional of the contract. There are two ways this payment can be made, either by physical settlement or cash settlement. In the physical settlement procedure, the protection buyer is required to deliver the notional amount of the reference entity to the protection seller in return for the notional value paid in cash. In the cash settlement procedure, only a cash payment is made from the protection seller to the protection buyer, the amount of which is equal to the reference entity par value minus the recovery value of the reference entity. The recovery value is calculated by referencing dealer quotes or observable market prices over some period after the default has happened. The payment made by the protection seller is known as the floating leg or protection leg of the CDS contract.

The fair CDS rate is the CDS rate that equates the present value of the fixed and floating leg of the CDS contract. Clearly, the two main factors that determine the present value of the floating and fixed leg is the timing of default and the recovery value given default. In order to find the fair CDS rate using a single issuer credit model, the model must be capable of modelling the distribution of the default time and the recovery value given default. It is usually the hazard
rate function that is modelled and used to value a CDS contract. The hazard rate is simply defined as the probability of defaulting in the next small time interval given that it has not already defaulted, or mathematically: 

$$h(t)dt = P[t \leq \tau < t + dt | \tau > t]$$

where \( \tau \) represents the default time and \( h(t) \) is the hazard rate function of the issuer. The cumulative survival probability of an obligor is related to the hazard rate function via the following expression:

$$Q(t) = P[\tau > t] = \exp\left[ - \int_0^t h(s)ds \right]$$

And the cumulative default probability is simply:

$$F(t) = P[\tau \leq t] = 1 - \exp\left[ - \int_0^t h(s)ds \right]$$

By differentiating the cumulative default probability, we obtain the default time probability density function:

$$f(t) = \frac{P[t < \tau \leq t]}{dt} = h(t)Q(t)$$

Assuming a constant recovery rate, \( R \), and contract maturity, \( T \), the present value of the floating leg can be computed via the integral:

$$PV_{\text{float}} = \int_0^T N(1-R)D(0,s)f(s)ds = N(1-R)\int_0^T D(0,s)h(s)Q(s)ds$$

where \( D(t, T) \) is the discount factor at time \( t \) for maturity \( T - t \), and \( N \) is the notional of the CDS contract. The expression for the present value of the floating leg essentially computes the discounted default payment at time \( t \), weighted by the probability that the default will occur in the small time interval \([t, t + dt]\), and we integrate this over all times from now to the contract maturity to reflect the fact that the default may occur at any time during this period.

Assuming that the fixed leg premium payments are paid at the times \( t_1, \ldots, t_n \), the present value of the fixed leg can be expressed as:
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\[
P V_{\text{fixed}} = CN \sum_{i=1}^{n} \alpha(t_{i-1}, t_i)Q(t_i)D(0, t_i)
+ CN \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \alpha(t_{i-1}, s)Q(s)h(s)D(0, s)ds
\]

where \( C \) is the CDS rate and \( \alpha(t_{i-1}, t_i) \) is the accrual factor for the time interval \([t_{i-1}, t_i]\). The first summation in the above expression for the present value of the fixed leg represents the PV of receiving the premium payments at the premium payment dates. The second summation reflects the present value of any accrual payment made if the obligor defaults in between premium payment dates.

In practice, CDS contracts have become liquid enough that practitioners use them to back out the default time distribution. This is done by making an assumption on the recovery rate and assuming a piecewise constant hazard rate function between the maturity dates of the liquid CDS contracts. A bootstrapping methodology is then employed to find the piecewise constant hazard rate function which reprices all of the different maturity CDS's. The bootstrapping procedure uses the above expressions for the present value of the fixed and floating leg to find the implied hazard rate function.

1.2.2 Nth-to-Default Baskets

N\(th\)-to-default contracts are very similar to standard credit default swaps except that now we have a basket of reference entities or bonds and a default event is triggered at the time of the N\(th\) default in the basket. Just like a CDS, an N\(th\)-to-default contract has two parties, a protection buyer and a protection seller. The payments made by the protection buyer are termed the fixed or premium leg of the contract and the payments made by the protection seller are termed the floating leg or protection leg of the contract. The contract is characterized by a final maturity date \(T\), and the protection buyer makes periodic payments (usually quarterly in arrears) to the protection seller, expressed as a fixed percentage of the notional of the contract (the fixed percentage is known as the N\(th\)-to-default swap rate). Premium payments are made until the final maturity date or the
date of the \( N^{th} \) default in the reference basket, whichever is first. An accrual payment is also made to the protection seller if the \( N^{th} \) default happens between premium payment dates. If there are at least \( N \) defaults in the basket before the final maturity date, then the protection seller makes a payment to the protection buyer to compensate him for the loss at the time of the \( N^{th} \) default. The payment is equal to the par value minus the recovery value of the \( N^{th} \) defaulted entity and is thus dependent on the identity of the \( N^{th} \) defaulted obligor. As for a CDS, this payment may be settled either physically or in cash.

In order to value a \( N^{th} \)-to-default basket we must have a model that captures the joint default time distribution of the reference entities and the recovery value given default of each reference entity. There are many different ways to model the joint default times and a review of the various approaches are made in section 1.4 of this chapter.

1.2.3 Collateralized Debt Obligations

Collateralized debt obligations (CDO) are one of the more complex derivatives in the family of portfolio credit derivative products. There are many different types of CDOs and the underlying entities that make up the portfolio range from corporate bonds and bank loans to mortgage and asset backed securities. CDOs that are comprised of bonds are known as collateralized bond obligations (CBO) and CDOs that are comprised of loans are called collateralized loan obligations (CLO). CDOs are mainly created to transfer the credit risk of a loan or bond portfolio to investors so as to free up regulatory capital requirements of the originating bank. In the creation of a CDO, the reference assets are transferred to a Special Purpose Vehicle (SPV) and the SPV issues notes to investors of varying credit rating. It is the proceeds from the issuance of the notes that are used to fund the purchase of the collateral for the SPV. The issued notes (usually termed A note, B note,, and D note, where note A has the highest credit rating and note D the lowest) offer the investors a fixed or floating interest payment where the highest rated note receives the lowest interest and the riskiest note offers the highest interest to compensate the investor for the extra credit risk that he is
taking on. These rated notes are said to reference a specific tranche of the portfolio. By buying the safest note, the investor is said to participate in the senior tranche. The most junior note is usually termed the equity note and the second most riskiest note is said to reference the mezzanine tranche. The interest payments on the notes are funded from the interest and prepayment proceeds from the loan collateral held in the SPV. The interest and prepayment income from the collateral is distributed to the CDO notes in a top-down manner where first the A note interest is paid, then the B note interest is paid and so on. If there is not enough income from the portfolio to pay all the interest on the notes, then the lowest rated note, note D, is the first to suffer missed interest payments. The next note to suffer missed interest payments if the interest proceeds are still insufficient, despite diverting cash flows from the D note, is the C note. This pattern continues and the A note is the last to suffer any missed interest payments. If the collateral starts experiencing defaults, then the notional of the lowest rated note, D, is reduced correspondingly by an amount proportional to the loss in collateral due to default. Once the lowest rated note has been completely eliminated due to default of the collateral pool, then it is the next lowest rated note (C) that starts absorbing losses. The exact mechanism by which defaults are absorbed by the notes sequentially is transaction or deal specific and each CDO transaction has its own prioritized payment schedule called the waterfall structure. The interest and principal payments made to the note holders usually have their own separate waterfall structure with over-collateralization (OC) and interest coverage (IC) tests to provide extra protection to the most senior notes.

CDO transactions that are conducted primarily with the objective of transferring credit risk and freeing regulatory capital requirements are termed balance sheet CDOs. Sometimes the CDO originating firm buys the equity notes with the objective of earning a large spread income as the equity notes trap excess interest proceeds and such CDOs are referred to as arbitrage CDOs. By participating in the equity tranche, the CDO issuer is indicating to the market that it believes the collateral will experience very few defaults and signals its trust in the transaction. From the investors perspective, CDO notes offer tailor made risk-return profiles and those with a high return high risk appetite can participate
in the equity tranche while those that are very risk averse may participate in the super senior tranche. Those that lie somewhere in the middle of the risk return spectrum may take part in the mezzanine tranche.

CDO transactions that involve the purchase of the collateral by an SPV as described above are known as cash CDOs. As already mentioned, the payoff profile for a cash CDO is deal specific with an intricate cash flow structure. Such deals are usually modelled using Monte Carlo simulations. In this thesis we will not consider the valuation of cash CDOs. Instead, the bulk of this thesis is devoted to the modelling and valuation of synthetic CDOs.

Synthetic CDO transactions are CDOs where the underlying portfolio consists of credit default swaps. These are unfunded transactions and the word 'synthetic' refers to the fact that exposure to credit risk is gained synthetically via credit default swaps without buying any defaultable assets. Unlike cash CDOs, synthetic CDOs have a well defined generic payoff that is far more accessible to mathematical analysis. Synthetic CDOs are characterized by the tranche attachment and detachment points, and a final maturity date $T$. For example, if the portfolio has a total notional of 100m, then the tranche attachment and detention points could be 3m and 6m corresponding respectively to the 3% attachment point and 6% detachment point. The tranche notional in this case is 3m (6m-3m). If the total portfolio loss, which is the loss incurred due to the default of the reference entities underlying the individual CDS's, is denoted by $L$, then the loss incurred on the tranche is $L_{\text{tranch}} = \max(L-A,0) - \max(L-D,0)$ which is a call spread on the portfolio loss with $D$ and $A$ corresponding to the detachment and attachment points respectively. The protection buyer in a synthetic CDO transaction pays a periodic (usually quarterly in arrears) premium payments to the protection seller, where the payments are a fixed percentage of the outstanding tranche notional. The outstanding tranche notional is simply the original tranche notional minus the tranche loss. This fixed premium percentage is known as the CDO coupon rate. The protection seller pays the protection buyer an amount equal to the incremental loss that occurs on the tranche at the times on which the losses occur. The maximum possible total amount that the protection seller can pay is limited to the tranche notional.
The payments made by the protection buyer is termed the fixed or premium leg of the CDO transaction. The payment made by the protection seller is termed the floating or protection leg of the CDO transaction. The fair CDO coupon or premium rate is the coupon that equates the present value of the fixed leg and floating leg. Just like in the case of an $N$th-to-default basket, the valuation of a synthetic CDO requires a framework that models the joint default times of all the underlying entities that make up the portfolio. In Chapter 2 we give a full mathematical treatment of the pricing of synthetic CDO tranches. The example in the subsection below shows the possible cash flows that may occur in a synthetic CDO transaction.

**Synthetic CDO Example**

Consider a synthetic CDO consisting of a pool of 100 credit default swaps, each with a notional of 1m. The originating bank issues three notes: Note A, note B, and note C. Note C references the equity tranche with attachment/detachment points of [0%-3%]. Note B is characterized by the attachment/detachment points [3%-7%], and note A references the points [7%-15%]. Notes A, B, and C have respective notional values of 3m, 4m, and 8m. Note A has an annual coupon of 10bps, note B has an annual coupon of 50bps, and note C has an annual coupon of 500bps. This CDO transaction is depicted in figure 1.1. Synthetic CDOs are unfunded, so there is no exchange of note principal during the life of the transaction.

Assuming that each note has a maturity of five years, consider the default scenario where at the end of the first year the pool of credit default swaps experience a loss of 2m, and at the end of year four there is a further portfolio loss of 5m. According to this scenario, each note will receive the cash flow stream shown in table 1.1.

From table 1.1 it is clear that note C is the riskiest since it is the first note to absorb any portfolio losses and have a reduction in coupon payments. Note A has the least risk since it is the last note to suffer any losses. This concludes the simple CDO transaction example.

Liquid synthetic CDO tranches that reference liquid indices of credit default
Figure 1.1: Typical synthetic CDO transaction. 100 underlying credit default swaps, each with a notional of 1m. Three notes, C, B, and A, referencing tranches with attachment/detachment points [0%-3%], [3%-7%], and [7%-15%] respectively. Note A is paid an annual coupon of 10bps, note B is paid an annual coupon of 50bps, and note C receives 500bps per annum.
swaps have now given practitioners a way of assessing the success of potential portfolio credit models. Two such liquid indices are the ITRAXX and CDX credit indices that both reference 125 liquid credit default swaps. In chapter 2 we will show the market convention for quoting tranche values using implied correlations.

A further credit portfolio product that is an extension of CDOs is the CDO squared (CDO\(^2\)) derivative. CDO\(^2\) are the same as standard CDOs, except that the underlying portfolio consists of tranches of other CDOs rather than loans, bonds, or credit default swaps. The different tranches that are underlying the parent CDO each reference a different portfolio. However, it is often the case that a given obligor belongs to more than one portfolio, so that there is an overlap of issuers amongst the underlying portfolios. This means that a default of a given obligor may have a much greater impact on the parent portfolio loss than would usually be the case with standard CDOs. CDO\(^2\) are usually popular in low credit spread environments, where standard credit default swaps cannot provide the necessary yield to produce high spread tranches. A full quantitative description and a new pricing methodology for CDO\(^2\) will be presented in chapter 2.

Having qualitatively reviewed the main credit derivative products, we proceed with an introduction and literature review of the various credit risk modelling techniques. We start off with the case of single issuer modelling methodologies and then proceed with the modelling of portfolio products.
1.3 Single Issuer Credit Risk Modelling

Single issuer credit risk modelling can be divided into two separate schools of thought: the 'structural' approach or the 'reduced form' approach. Both approaches have their associated set of advantages and disadvantages, and we proceed with an overview of them both.

1.3.1 Structural Models

Structural credit models aim to model the default risk of a firm by modelling the internal variables of the firm, most notably the asset value process. Merton (1974) was the first to apply the structural approach to corporate bond pricing using the option pricing framework of Black and Scholes (1973). In his paper, Merton considers the simple case of a zero coupon debt that matures at time $T$ and which has a face value of $D$. He postulates that the firm will default on its debt if the asset value of the firm is below the face value of debt at the maturity of the debt. The rationale behind the model is that if the asset value exceeds the face value of debt, then it is always possible for a firm to sell its assets and pay off the debt. In the Merton model, a firm can only default on the maturity of the debt and the equity value is simply the residual value of the assets left after the debt has been paid. Since the equity value can never go below zero (bond holders can at most take control of the assets of the firm in the case of bankruptcy and cannot demand extra payment from the equity holders if the asset value is insufficient to pay the outstanding debt), the value of the equity at the maturity of the debt is $E_T = \max(A_T - D, 0)$, where $E_T$ is the equity value at time $T$, $A_T$ is the asset value at time $T$ and $D$ is the face value of debt. This is simply a European call option on the assets of the firm with a strike level equal to the face value of debt. As a result, it is possible to price the equity given the value and volatility of the assets and the level of debt. Consequently, the payoff to the bondholders is $\min(A_T, D) = D - \max(D - A_T, 0)$, which is simply the face value of debt minus a put option on the assets of the firm. Again, the debt can be valued, as Merton showed, using the Black and Scholes option pricing formula. Using the Merton model, it is possible to obtain explicit formulas for the spread on the debt, and
the recovery rate is determined endogenously with in the model.

An implicit and often unrealistic assumption of the Merton model is that it assumes the asset value is a tradable commodity which can be used for replicating the payoff of the debt. It also assumes that we can observe the exact value of assets today and its instantaneous volatility, both of which are untrue given that the asset value is estimated from balance sheet data which is only periodically updated. Since the Merton model uses the Black and Scholes formula to price equity and debt as options on the assets of the firm, it assumes that the risk-neutral asset value process is modelled by a geometric Brownian motion or lognormal process:

\[ dA_t = rA_t dt + \sigma A_t dW_t \]

where \( r \) is the constant instantaneous risk free rate of interest, \( \sigma \) is the asset volatility and \( W_t \) is a Brownian motion process. If the face value of debt is \( D \), then the probability of default by time \( T \) is given by:

\[
P[A_T < D] = P[A_0 \exp((r - \frac{1}{2}\sigma^2)T + \sigma(W_T - W_0)) < D] \\
= P \left( W_T - W_0 \leq \frac{\ln(D/A_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma} \right) \\
= \Phi \left( \frac{\ln(D/A_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)
\]

where \( \Phi(\cdot) \) is the cumulative standard Gaussian distribution function. The point to note here, and which will help explain the origin of the CreditMetrics portfolio model, is that the probability of the firm defaulting is equivalent to the probability of a standard Gaussian random variable falling below some modified threshold \( K \) given by:

\[
K = \frac{\ln(D/A_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}
\]

As stated before, a major disadvantage of the Merton model is that defaults can only happen at the maturity of the debt, no matter what level the asset value was prior to that. Black and Cox (1976) go one step further than Merton
(1974) and assume that default happens at the first time the asset value crosses some given boundary or barrier. The time varying boundary, $K(t)$, is of the form $K(t) = ke^{-\gamma(T-t)}$, for some constants $k$ and $\gamma$. Longstaff and Schwartz (1995) introduce stochastic interest rates to the first passage model framework. They assume that the interest rates are governed by a Vasicek (1977) process which is correlated with the asset value and the default boundary is a constant value. Briys and de Varenne (1997) use the Black and Cox (1976) discounted barrier model and introduce stochastic interest rates using a generalized short rate Vasicek model. Leland (1994) and Leland and Toft (1996) consider the case of constant interest rates, a constant dividend payment and tax relief benefits arising from debt payments. They postulate that a firm defaults upon the first time the asset value crosses a constant barrier, and this barrier level is determined endogenously by the equity holders to maximize their wealth.

One major disadvantage with all the structural models reviewed so far is that in each case the asset value process is a continuous process. This means that if the asset value is above the default barrier, then the probability of it defaulting within the next time interval $[0, t]$ approaches zero as $t \to 0$. The implication of this is that short term credit spreads tend to zero, a feature which is not observed empirically. Short term credit spreads are usually greater than zero as there is always the possibility of the firm defaulting today due to, for example, fraudulent accounting activities. Zhou (1997) introduced a jump diffusion asset value model which produced positive credit spreads for short maturities as there is always the possibility that the asset value may jump below the default barrier immediately.

### 1.3.2 Reduced Form Models

Reduced form or intensity based credit models do not attempt to explain economically the default generating process but rather they directly model the default probability via a Poisson process. A right continuous integer valued stochastic counting process $N_t, t \geq 0$, is called a Poisson process if it has independent increments, and the increment $N_t - N_s, t \geq s$, has a Poisson distribution with parameter $\lambda(t - s)$. An obligor is assumed to default upon the first jump time.
of the Poisson process. The word 'intensity' usually refers to the parameter λ in
the Poisson process. One of the main distinctions between reduced form mod-
els and structural models is that for intensity based credit models the default
times are totally unpredictable stopping times. For a time dependent param-
eter, λ(t), we have what is termed an inhomogeneous Poisson process. In this
case the probability of not having a jump in the time interval [s, t], for t ≥ s, is
P[N_t - N_s = 0] = exp(- \int_s^t \lambda(u) du). If we wish to simulate the first jump time of
an inhomogeneous Poisson process, then for a standard uniform random variable,
U ∈ [0, 1], the first jump time is given by:

\[ \tau = \inf \{ t : \exp \left( - \int_0^t \lambda(u) du \right) \leq U \} \]  \hspace{1cm} (1.1)

If the intensity parameter is now a stochastic process λ(t, ω) defined on a suit-
able probability triple (Ω, F, P), where F is the filtration and P is a probability
measure on the filtration, then the counting process becomes what is known as a
Cox process (also called a doubly stochastic process). In this case the first jump
time is given by:

\[ \tau = \inf \{ t : \exp \left( - \int_0^t \lambda(u, \omega) du \right) \leq U \} \]  \hspace{1cm} (1.2)

where U is independent from the filtration generated from λ(t, ω). Lando (1998)
applied the Cox process to the pricing of defaultable securities using iterated
expectations. If we condition on a sample path of the stochastic intensity then the
Cox process framework reduces to the inhomogeneous Poisson process. Assuming
λ(t, ω) is adapted to some filtration Ft, then we have P[N_t - N_s = 0|F_t] =
exp(- \int_s^t \lambda(u) du). It then follows via iterated expectations that the probability
of no jumps (and hence no default) in the Cox process framework is given by:

\[
P[N_t - N_s = 0|F_0] = E[1_{\{N_t - N_s = 0\}}|F_0] \\
= E[E[1_{\{N_t - N_s = 0\}}|F_t]|F_0] \\
= E[\exp \left( - \int_s^t \lambda(u, \omega) du \right)]
\]

The expectation is taken with respect to all possible paths of the intensity process.
Practically all intensity models can be reduced to the Cox framework. The popularity of Cox models in the pricing of defaultable securities arises due to the expression for the probability of default. In particular, the probability that an obligor will survive to time $t$ is given by $P[T > t] = E \left[ e^{- \int_0^t \lambda(u) du} \right]$. Such expressions are frequently encountered in interest rate modelling where $\lambda(t, \omega)$ usually represents the short rate process and closed form formulae exist for the special case of affine short rate models. Furthermore, it is possible, just like in interest rate modelling, to calibrate intensity models to fit the entire credit spread curve. As an example of defaultable bond pricing in the reduced form framework consider an environment where the risky bondholder receives $R_T$ at maturity $T$ if the bond defaults at time $T < t$. If the bond does not default, the bond holder receives par at the maturity date. The price of the risky bond with unit notional can be expressed as:

$$P_R(0, T) = E \left[ e^{-\int_0^T h(u) + r(u) du} + e^{-\int_0^T r(u) du} \int_0^T R_V(t) e^{-\int_0^t h(u) du} dt \right]$$

where $h(t)$ is the intensity, $r(t)$ is the continuously compounded spot interest rate, $P[T \geq T] = E[ e^{-\int_0^T h(u) du} ]$ and $P[t < \tau \leq t + dt] = E[ h(t) e^{-\int_0^t h(u) du} dt ]$.

One aspect that is not so clear in reduced form models is the recovery rate given default. Usually structural models specify the recovery rate endogenously whereas the nature of the recovery rate must be specified exogenously for intensity based credit risk models. Reduced form modelling can be traced back to Jarrow and Turnbull (1995) where they assume a constant intensity and fixed recovery rate. Duffle and Singleton (1999) consider a model where the recovery rate is a fraction of the pre-default value of the bond. In this case a particularly simple expression arises for the price of a risky zero coupon bond. If the bond loses a fraction $1 - R_V$ of its pre-default value at the default time $T$, then the price of the risky zero coupon bond can be expressed as:

$$P_R(0, T) = E \left[ e^{-\int_0^T h(u)(1 - R_V) + r(u) du} \right]$$

$h(u)(1 - R_v)$ is effectively a new thinned intensity process.

Other reduced form models include Madan and Unal (1998) whom assume that the intensity is a function of an asset value process and that the recovery
rate is an unpredictable random variable.

1.4 Portfolio Credit Risk Modelling

One of the first attempts at modelling the default behavior of a portfolio of credit sensitive instruments was the CreditMetrics model presented in Gupton et al (1997). The original technical paper is geared towards the risk management of credit portfolios and as a result the primary input data consists of credit ratings and the associated historical default probabilities. However, the CreditMetrics model can easily be adapted to the pricing of portfolio credit derivatives by replacing historic default probabilities with risk neutral default probabilities.

The CreditMetrics model is essentially based on the structural default model of Merton (1974) where it is assumed that asset returns are lognormally distributed and a default occurs if on the maturity of a bond the assets of a firm are below its liabilities. The CreditMetrics model simplifies matters a bit by assuming the asset value to be standard normally distributed with the default threshold chosen so that the marginal default probability of each obligor matches the default probabilities given by the rating agencies or those implied in the market for each time horizon. For example, if the cumulative default probability for obligor $i$ at time horizon $t$ is $P[T_i \leq t] = P_i(t)$, then the default threshold for time horizon $t$ is $C_i(t) = \Phi^{-1}(P_i(t))$, where $\Phi^{-1}(\cdot)$ is the inverse Gaussian cumulative distribution function. It was shown in section 1.3 that the Merton model effectively reduces to a model where an obligor defaults if a standard normally distributed random variable falls below a certain threshold.

Default correlation is introduced into the model by correlating the asset values of the firms underlying the portfolio. Hence, it is assumed that the asset values of the obligors are jointly normally distributed with a given correlation matrix $\Sigma$. The algorithm for calculating the loss distribution of a portfolio consisting of $N$ obligors at a given time horizon using the CreditMetrics model is thus as follows:

- Calculate the default threshold for each obligor using the expression $C_i(t) = \Phi^{-1}(P_i(t))$, where $P_i(t)$ is the cumulative default probability for obligor $i$ for time horizon $t$. 

• Simulate $N$ correlated standard normal random variables $A_1, \ldots, A_N$ using the matrix multiplication $\bar{A} = \bar{X}^T \Lambda$, where $\bar{A} = [A_1, \ldots, A_N]^T$, $\bar{X} = [X_1, \ldots, X_N]^T$ and $\Sigma = \Lambda \Lambda^T$. $X_1, \ldots, X_N$ are all independent standard Gaussian variables. $\Lambda$ is obtained by performing the Cholesky decomposition on the correlation matrix $\Sigma$. The variable $A_i$ represents the asset value of obligor $i$.

• Determine which obligors have asset values that are below their respective default thresholds.

• For those defaulted obligors determine the loss amount using a possibly random recovery rate.

• Repeat the above procedure many times to build up a Monte Carlo generated loss distribution.

Possible extensions to the CreditMetrics model involve using a multivariate Student-t distribution (which is still elliptical and is thus completely characterized by its correlation matrix) in order to incorporate tail dependence into the model.

The CreditMetrics model is popular due to its simplicity and availability of equity time series data to compute correlations and to use them as a proxy for asset return correlations. It is straightforward to use the CreditMetrics model for CDO pricing, however it is less clear how to use it for $N^{th}$-to-default pricing which depends on the time and identity of the $N^{th}$ defaulted asset. One possible way of using the CreditMetrics model to identify the $N^{th}$ defaulted asset would be to sample a default time for all defaulted issuers using their conditional marginal default probability functions, conditional on the event of default, which can readily be determined from the original unconditional marginal default time distribution using Bayes’ Theorem.

Finger (1999) used a factor based approach to redefine the CreditMetrics model. Here, rather than using a full blown correlation matrix, default dependence is introduced via a few common market variables. For example, the normally distributed asset value for each obligor is of the form:

$$A_i = \rho_1 M_1 + \ldots + \rho_n M_n + \sqrt{1 - \rho_1^2 - \ldots - \rho_n^2} \varepsilon_i$$
where $M_1, \ldots, M_n$ and $\epsilon_i$ are all independent standard normally distributed random variables. $M_1, \ldots, M_n$ are commonly referred to as the market factors and typically no more than two factors are used. The factor coefficients, $\rho_i$, can be estimated by performing some sort of principal component factor decomposition on the correlation matrix. One such decomposition algorithm is given in Andersen et al (2003). The interesting aspect of factor based models is that conditional on a realization of the market factors, all obligors default independently with conditional default probabilities equal to:

$$P[A_i \leq C_i(t)|M_1 = m_1, \ldots, M_n = m_n] = \Phi\left(\frac{C_i(t) - \rho_1 m_1 - \cdots - \rho_n m_n}{\sqrt{1 - \rho_1^2 - \cdots - \rho_n^2}}\right)$$

This opens up a whole host of semi-analytical methods for computing the portfolio loss distribution, avoiding the need to perform lengthy and noisy Monte Carlo simulations. Merino and Nyfeler (2002) and Gregory and Laurent (2003b) use the Fourier Transform technique to compute the portfolio loss distribution. Andersen et al (2003) proposes a recursive algorithm to compute the portfolio loss distribution in the factor based setup. A full analysis of these two semi-analytical methods will be presented in chapter 2 of this thesis. The semi-analytical factor based approach to portfolio credit modelling has effectively become the market standard methodology for pricing and hedging portfolio credit derivatives.

The main disadvantage of the CreditMetrics model and similar factor based models is that they are static models that only give information on the portfolio loss distribution for certain time horizons. They do not tell us how the portfolio loss distribution may evolve over time.

Hull and White (2001) propose a structural first passage type portfolio model where an obligor defaults upon the first time its asset value crosses a time dependent barrier. The default barrier is piecewise constant between certain time intervals and the assets values are correlated Brownian motion processes. This is a dynamic model where we can observe the joint evolution of credit worthiness of each obligor over time. A further advantage of this model is that it is possible to incorporate a term structure of correlations. There are, however, a number of disadvantages of such a model. The first is that it is computationally intensive as the full asset value path must be simulated for each obligor.
calibration routine to calculate the default barriers is also a nontrivial task and is computationally intensive.

Davis and Violet (2000) present an alternative infectious default model, where the default probability of a given obligor jumps upwards at the event of a default of another obligor. This model was presented as a methodology to add fat tails to an otherwise binomial distribution. It is not clear how this model can be calibrated to the individual obligor default probabilities for a heterogeneous portfolio.

Reduced form based portfolio credit risk models provides another possible method of introducing a dynamic model where we can observe the evolution of the joint default probabilities in a continuous time framework. There are two different routes to introduce default dependence in the reduced form models. The first method is to simply correlate the intensity processes of the individual obligors. Conditional on the realization of the intensity paths, each obligor defaults independently. For example, we can introduce correlation via correlating the driving Brownian motion processes:

\[
\begin{align*}
    d\lambda_i(t) &= \alpha_i(\lambda_i(t), t)dt + \sigma_i(\lambda_i(t), t)dW_i(t) \\
    d\lambda_j(t) &= \alpha_j(\lambda_j(t), t)dt + \sigma_j(\lambda_j(t), t)dW_j(t) \\
    E[dW_i dW_j] &= \rho dt
\end{align*}
\]

where \( \lambda_i(t) \) represents the intensity process for obligor \( i \) and \( \lambda_j(t) \) represents the intensity process for obligor \( j \). Duffie and Garleanu (2001) provide an example of such a framework by assuming that the intensity of each obligor is governed by a general affine jump diffusion process. Correlation is introduced into this model by representing the intensity of each obligor as the summation of idiosyncratic and systemic affine jump diffusion processes, where the systemic component is common to all obligors. This results in correlation amongst the default intensity of the underlying obligors and it is also possible to have common jumps in the intensities. CreditRisk+ (1997) introduces default dependence by assuming that the intensity processes of the individual obligors are driven by common shared variables that have a gamma distribution. A given gamma distributed variable has an influence on a given obligor depending on which industry and country it
resides in. The problem with introducing default dependence via correlating the individual obligor intensities is that the level of default dependence produced by this approach is usually insufficient to reflect the high level of empirical default correlation observed in the market. Introducing default contagion in this framework, where the default of one obligor will trigger an upward jump in the intensity of the surviving obligors, will help achieve higher levels of default dependence.

The second method of introducing default dependence in the intensity based portfolio models is to have common Cox processes, where the event arrival time of a common process will result in the simultaneous defaults of all the obligors belonging to some subset of the portfolio. Duffie (1998) and Giesecke (2003) both used this approach to achieve sufficiently high levels of default dependence. The main disadvantage with this approach is that it is necessary to specify a Cox process for each possible simultaneous joint default event, and it is very easy for such a framework to become complicated and cumbersome. As intensity based portfolios models are dynamic models, it is unrealistic to assume that multiple obligors default exactly at the same time. It is also the case that such an approach does not include default infection, so the intensity of a surviving obligor is unaffected if a default is triggered in the portfolio. Default infection is a realistic and desirable feature of continuous time dynamic models.

Li (1999) proposed a general and powerful multivariate default model in his influential paper using the concept of copulas. A copula is simply a multivariate probability distribution function with uniform marginal distributions. We give a detailed mathematical overview of copulas in section 1.4.1. In his paper, Li also shows that the CreditMetrics model is a special case of the more general Gaussian copula model.

The Li model utilizes a simple default time generating procedure using copulas. For a portfolio of n obligors the algorithm can be summarized as follows:

- Generate a set, \( u_1, \ldots, u_n \), of n correlated uniformly distributed random variables from a given copula function. \( u_i \in [0,1] \).

- Let \( F_i(t) = P[\tau_i \leq t] \) denote the cumulative default time distribution for obligor \( i \). The default time of obligor \( i \) is determined by \( \tau_i = F_i^{-1}(u_i) \).
• Given a complete set of default times, one for each obligor, calculate the payoff of the portfolio derivative in question and average over many Monte Carlo trials.

A number of points must be emphasized for this model. The first is that if the copula is continuous, then there cannot be simultaneous defaults at a given time, hence the time and identity of the \( n \)th defaulted asset is always known. The second point is that under this model all obligors default at one time or another.

Schönbucher and Schubert (2001) present a framework for combining the intensity based approach with the copula model of Li (2000) so as to incorporate the advantages of both approaches in a unified framework. The link between intensity models and copulas can be seen by examining expression 1.2, where the time of default is the first time the exponential integrated intensity process hits a barrier that is generated from a standard uniform random variable. The uniform random variable is independent from the filtration generated by the intensity process. Schönbucher and Schubert (2001) pointed out that in order to introduce default correlation amongst the obligors, then all that is required is to introduce correlation amongst the uniform variables \( U_i \), where \( U_i \) is the default barrier for obligor \( i \). Copulas fit conveniently in this framework as by definition copulas are multivariate distribution functions with uniform margins. The default time generating algorithm for a portfolio of \( K \) obligors now becomes:

1. For each obligor \( i \), simulate the path of the intensity process \( h_i(t) \). Note that the intensity processes may be correlated themselves.

2. Simulate correlated uniform random variables \( u_1, \ldots, u_K \) from a copula function \( C(u) \). These uniform random variables should be independent from the filtration generated by the intensity process.

3. The default time of obligor \( i \) is \( \tau_i = \inf\{t : \gamma_i(t) \leq u_i\} \), where \( \gamma_i(t) = \exp(-\int_0^t h_i(u)du) \).

For the case of a single obligor the Schönbucher and Schubert (2001) approach reduces to the standard Cox process framework. This is also the case if the
uniform random variables are independent or equivalently are generated by the product copula. Within the Schönbucher and Schubert framework the filtration becomes very important as the intensity of obligor \(i\) will be different if we observe the default behavior of the entire portfolio or if we restrict our filtration to just the default information of obligor \(i\). Their paper also presents an explicit expression for the dynamics of the intensity process for a given obligor (which depends on the default history of the other obligors) and the jump in the intensity caused by the default of another obligor. Although the framework of Schönbucher and Schubert is theoretically very appealing, in practice it is too computationally intensive to be used for frequent valuation and hedging purposes. In this framework not only do we have the burden of having to simulate the path of the intensity process for each and every obligor in the portfolio, but we must also simulate (for each intensity path) random draws from a given copula function to determine the default barriers.

1.4.1 Copulas - An Overview

In summary, a copula function is a multidimensional distribution function defined on the unit cube \([0, 1]^n\) with uniform margins. For completeness, a more mathematically rigorous definition is provided which is derived from Embrechts et al (2001).

**Definition 1**

A \(n\)-copula is a function \(C\) from \([0, 1]^n\) to \([0, 1]\) with the following properties:

1. For every \(u\) in \([0, 1]^n\), \(C(u) = 0\) if at least one coordinate of \(u\) is 0 and \(C(u) = u_k\) if all coordinates of \(u\) equal 1 except \(u_k\).

2. Let \(B = [a, b] = \prod_{i=1}^{n} [a_i, b_i]\) be an \(n\)-box whose vertices are in \([0, 1]^n\), with \(a \leq b\). The volume of an \(n\)-box with corners \(a\) and \(b\) is positive, i.e.

\[
\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+i_2+\cdots+i_n} C(u_{i_1}, \ldots, u_{i_n}) \geq 0
\]

where \(u_{ij} = a_j\) for \(i_1 = 1\) and \(u_{ij} = b_j\) for \(i_1 = 2\), and \(j = 1, \ldots, n\).
**Theorem 1** Sklar’s Theorem. Let $H$ be an $n$-dimensional distribution function with margins $F_1, \ldots, F_n$. Then there exits an $n$-copula $C$ such that for all $x \in \mathbb{R}^n$

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

If $F_1, \ldots, F_n$ are all continuous then $C$ is unique. Conversely, if $C$ is an $n$-copula and $F_1, \ldots, F_n$ are distribution functions, then the function $H$ defined above is an $n$-dimensional distribution function with margins $F_1, \ldots, F_n$.

Sklar’s Theorem allows the full dependence structure of a multivariate distribution function to be specified by decomposing it into marginal distributions and a corresponding copula function.

**Corollary 1** Let $H$ be an $n$-dimensional distribution function with continuous margins $F_1, \ldots, F_n$ and copula $C$. Then for any $u \in [0, 1]^n$

$$C(u_1, \ldots, u_n) = H(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n))$$

This powerful result allows us to construct copula functions from known multivariate distributions and their respective marginal distributions. For example, the Gaussian copula is defined as:

$$C(u_1, \ldots, u_n) = \Phi^a_\Sigma(\Phi_1^{-1}(u_1), \ldots, \Phi_n^{-1}(u_n))$$

Where $\Phi^a_\Sigma$ is the multivariate standard Gaussian distribution with correlation matrix $\Sigma$. If it is assumed that the correlated normal random variables can be represented by a one factor structure, i.e. each normal random variable has the representation $Y_i = \rho_i X + \sqrt{1 - \rho_i^2} \varepsilon_i$ for $X$ and $\varepsilon_i$ both standard normal independent random variables, then we have the one factor Gaussian copula:

$$C(u_1, \ldots, u_n) = \int_{-\infty}^{\infty} \phi(x) \prod_{i=1}^n \Phi(\frac{\Phi^{-1}(u_i) - \rho_i x}{\sqrt{1 - \rho_i^2}}) dx$$

To simulate $N$ correlated uniform random variables from a Gaussian copula, we make use of the following simulation procedure:

- Simulate $N$ independent standard normal random variables $\overline{X} = [X_1, \ldots, X_N]^T$
• Perform the Cholesky decomposition on the correlation matrix $\Sigma$ to obtain the matrix $\Lambda$ such that $\Sigma = \Lambda\Lambda^T$.

• Obtain the set of correlated Gaussian variables using the matrix multiplication $\vec{Z} = \vec{X}^T \Lambda$, $\vec{Z} = [Z_1, \ldots, Z_N]^T$.

• Map the correlated standard Gaussian random variables using the mapping $U_i = \Phi(Z_i)$, where $\Phi(\cdot)$ is the cumulative standard Gaussian distribution function. The vector $\vec{U} = [U_1, \ldots, U_N]^T$ represents the correlated uniform random variables.

Note that for any n-copula $C$, $n \geq 3$, each k-dimensional margin of $C$ is a k-copula.

Dependence can also be expressed via the useful survival copula:

$$\hat{C}(\hat{F}_1(x_1), \ldots, \hat{F}_n(x_n)) = \hat{H}(x_1, \ldots, x_n) = P[X_1 \geq x_1, \ldots, X_n \geq x_n]$$

where $\hat{F}_i(x) = 1 - F_i(x)$. Note that copula functions are invariant under monotonically increasing transformations of the variables. If transformation is monotonically decreasing, the copula of the transformed variables is the survival copula.

One important topic that copulas raise is the measure of dependence between random variables. The most commonly used measure of dependence is the linear correlation coefficient defined for two random variables $X$ and $Y$ as

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

where $|\rho(X, Y)| = 1$ for perfect linear dependence $Y = aX + b$. The linear correlation coefficient is popular primarily due to the ease with which it can be computed and because it is the natural measure of dependence for commonly used distribution functions like the multivariate Gaussian or Student-t distribution. The major shortcomings of the linear correlation coefficient is that it only measures the degree of linear dependence, so using this measure on two totally dependent random variables defined by, say, $Y = X^2$ will not give a result of $|\rho(X, Y)| = 1$, which will lead the unsuspecting observer to conclude that $Y$ and $X$ are not totally dependent.

Two alternative measures of dependence are Kendall's tau and Spearman's rho.
Definition 2 Kendall’s tau for the random vector \((X,Y)^T\) is defined as

\[
\tau(X,Y) = P[(X - \bar{X})(Y - \bar{Y}) > 0] - P[(X - \bar{X})(Y - \bar{Y}) < 0]
\]

where \((\bar{X}, \bar{Y})^T\) is an independent copy of \((X,Y)^T\). If \((X,Y)^T\) have copula \(C\), Kendall’s tau for \((X,Y)^T\) is given by:

\[
\tau(X,Y) = 4 \int \int_{[0,1]^2} C(u,v)dC(u,v) - 1
\]

Definition 3 Spearman’s rho for the random vector \((X,Y)^T\) is defined as

\[
P_s(X,Y) = 3(P[(X - \bar{X})(Y - Y') > 0] - P[(X - \bar{X})(Y - Y') < 0])
\]

where \((X,Y)^T, \bar{X}, Y^T\), and \((X',Y')^T\) are independent copies. It can be shown that if \((X,Y)^T\) have copula \(C\) then:

\[
P_s(X,Y) = 12 \int \int_{[0,1]^2} uvdC(u,v) - 3
\]

Furthermore, if \(X \sim F\) and \(Y \sim G\), letting \(U = F(X)\) and \(V = G(Y)\), then:

\[
P_s(X,Y) = 12 \int \int_{[0,1]^2} uvdC(u,v) - 3 = 12E(UV) - 3
\]

\[
= \frac{E(UV) - \frac{1}{4}}{\frac{1}{12}} = \frac{Cov(U,V)}{\sqrt{Var(U)Var(V)}} = \rho(F(X), G(Y))
\]

Some important properties of these two measures of dependence for two continuous random variables \(X\) and \(Y\) with copula \(C\) are the following:

- \(-1 \leq \tau(X,Y) \leq 1\), \(-1 \leq P_s(X,Y) \leq 1\).
- \(\tau(X,X) = P_s(X,X) = 1\), \(\tau(X,-X) = P_s(X,-X) = -1\).
- If \(X\) and \(Y\) are independent \(\tau(X,Y) = P_s(X,Y) = 0\).
- \(\tau(-X,Y) = \tau(X,-Y) = -\tau(X,Y), P_s(-X,Y) = P_s(X,-Y) = -P_s(X,Y)\).
- If \(Y\) is an almost surely increasing function of \(X\), then \(\tau(X,Y) = P_s(X,Y) = 1\).
• If $Y$ is an almost surely decreasing function of $X$, then $\tau(X, Y) = P_s(X, Y) = -1$.

• If $\alpha$ and $\beta$ are almost surely increasing functions defined on the range of $X$ and $Y$ respectively, then $\tau(\alpha(X), \beta(Y)) = \tau(X, Y), P_s(\alpha(X), \beta(Y)) = P_s(X, Y)$.

Another further useful property is stated next. Let $X$ and $Y$ be continuous random variables with copula $C$, and let $\kappa$ denote Kendall’s tau or Spearman’s rho. The following are true:

1. $\kappa(X, Y) = 1 \iff C = M$.
2. $\kappa(X, Y) = -1 \iff C = W$.

where $M(u, v) = \min(u, v)$ and $W(u, v) = \min(u + v - 1, 0)$ are the Fréchet-Hoeffding Bounds for joint distribution functions.

1.5 Thesis Overview

1.5.1 Chapter 2

In chapter 2 we start by reviewing the conditional independence approach to portfolio credit risk modelling. This approach introduces default correlation via some common systematic random variables, which when conditioned on results in independent default time generation for the underlying obligors. We also show how to value $N^{th}$-to-default baskets and synthetic CDOs in the conditional independence framework. The main contribution we make in this chapter is the introduction of a new pricing algorithm for $CDO^2$ derivatives which overcomes the obligor overlap problem associated with the valuation of such transactions. Two concrete examples are given of conditionally independent models, namely the Archimedean copula and the one factor Gaussian copula. The concept of the large homogeneous portfolio is also reviewed at this stage. The chapter ends by reviewing the compound and base correlation methodologies to quote implied
correlations. Implied correlations are analogous to implied volatilities in the equity options market. The advantages and disadvantages of both approaches are listed and it is shown that the commonly used base correlation methodology for pricing nonstandard tranches is not an arbitrage free methodology.

1.5.2 Chapter 3

Using the market standard one factor Gaussian copula model to back-out implied base correlations results in a pronounced base correlation skew which shows that this model is insufficient to capture the full default dependency structure implied in the credit derivatives market. This chapter extends the one factor Gaussian copula by incorporating stochastic correlation and the extended model provides a good fit to market prices as opposed to existing portfolio default models. The corresponding stochastic correlation copula is derived and we also present the corresponding large homogeneous portfolio loss distribution. Closed form single tranche CDO prices are also derived under the large homogeneous portfolio approximation. All material presented in this chapter is original and new.

1.5.3 Chapter 4

Probability distributions implied in the financial markets, such as equity return distributions, are characterized by fat tails and the possibility of realizing extreme market scenarios. This effect is especially true in the credit derivatives market and this chapter introduces a new model that explicitly incorporates the possibility of ruin scenarios. Using an asset value based portfolio model as a starting point we incorporate external shocks to the portfolio in the form of Poisson jump processes. Upon the arrival of an external shock a given subset of the portfolio defaults simultaneously and different jumps imply a different degree of devastation inflicted on the portfolio. Such a model provides a possible explanation for the single tranche prices observed in the market, i.e. market participants explicitly price in the possibility of extreme credit default scenarios that cannot be generated from solely using a multivariate Gaussian distribution. The proposed model is termed the shock-Gaussian model and we provide a recursive algorithm
to compute the portfolio loss distribution for this type of model. The large homogenous portfolio loss distribution is derived for the shock Gaussian model. Closed form single tranche CDO prices are also derived under the large homogenous portfolio approximation. All material presented in this chapter is original and new.

1.5.4 Appendix A - A Primer On Gaussian Quadrature
The Gaussian quadrature numerical integration technique is widely used in the implementation of credit portfolio models. If the reader decides to implement any of the models presented in this thesis, then an efficient implementation will require the use of Gaussian quadrature. Appendix A presents an overview of the Gaussian Quadrature technique.

1.5.5 Appendix B - Useful Gaussian Integrals
Appendix B lists some integrals that are used in this thesis. The derivation of the integrals are also provided.
Chapter 2

Factor Models and Implied Correlations

In this chapter we present the conditional independence framework for pricing portfolio credit derivatives. The advantage of such an approach is that it allows semi-analytical pricing methodologies which are considerably more accurate and faster than Monte Carlo simulations. In section 2.1 of this chapter we show how to construct the portfolio loss distribution under the conditional independence framework using either the Fourier Transform or a recursive algorithm. Having establishing the concept of conditional independence, in section 2.2 we give two specific examples, the Archimedean copula and one factor Gaussian copula model. The one factor Gaussian copula model is generally regarded as the market standard pricing and hedging model for single tranche CDOs. In section 2.2 we also review the concept of the large homogeneous portfolio (LHP) as first demonstrated by Vasicek (1987). The LHP allows one to quickly obtain a closed form solution to the approximate portfolio loss distribution. In section 2.3 we specifically show how to price single tranche CDOs and $N^{th}$-to-default baskets using conditionally independent models. The main contribution we make in this chapter is the introduction of a new pricing methodology to price CDO squared derivatives using a combination of the conditional independence approach and Monte Carlo simulations and this novel approach is presented in section 2.3. The last part of this chapter, section 2.4, then proceeds to introduce the concepts of
implied compound and base correlation. Advantages and disadvantages of both approaches are presented and we show that the commonly used base correlation approach to pricing is not an arbitrage free methodology.

2.1 Conditional Independence Framework

The conditional independence approach is defined by the feature that given the realization of certain latent random variables, the default time of each obligor is independent from the default times of the other obligors in the portfolio. Assuming that the set of latent random variables can be represented by the vector $\mathbf{M} = [M_1, \ldots, M_k]^T$, we then have:

$$E[\tau_i \tau_j | \mathbf{M} = \mathbf{m}] = E[\tau_i | \mathbf{M} = \mathbf{m}] E[\tau_j | \mathbf{M} = \mathbf{m}]$$  \hspace{1cm} (2.1)

where $\tau_i$ is the default time of obligor $i$, $\tau_j$ is the default time of obligor $j$, and $E[\cdot | \mathbf{M} = \mathbf{m}]$ represents the conditional expectation. It is also assumed that the possibly random recovery rates are also independent conditional on the market factors:

$$E[R_i R_j | \mathbf{M} = \mathbf{m}] = E[R_i | \mathbf{M} = \mathbf{m}] E[R_j | \mathbf{M} = \mathbf{m}]$$  \hspace{1cm} (2.2)

where $R_i$ is the random recovery rate of obligor $i$ and $R_j$ is the random recovery rate of obligor $j$.

Intuitively the vector $\mathbf{M}$ could represent the state of the macro economy or of a specific sector or region. The total portfolio loss at the time horizon $t$, represented by the random variable $L(t)$, is simply equal to the sum of the losses on all the underlying obligors:

$$L(t) = \sum_{i=1}^n l_i(t)$$

where we have assumed that there are $n$ underlying obligors in the portfolio and $l_i(t)$ is the loss incurred on obligor $i$ due to a credit event that may have occurred within the fixed time horizon $t$. However, to obtain the portfolio loss
distribution we cannot simply sum the loss distributions of the individual obligors. Mathematically this can be expressed as: 

$$f_{L(t)}(l) \neq \sum_{i=1}^{n} f_{i(t)}(l)$$

where 

$$f_{i(t)}(l) = P[l \leq l_i(t) < l + dl]$$

is the loss density function of obligor $i$ for time horizon $t$ and $f_{L(t)}(l)$ is the portfolio loss density function. In the conditional independence approach, the default times and recovery rates are assumed to be independent for each obligor given a realization of the vector $\overline{M}$. It then follows that the conditional portfolio loss distribution is the convolution of the individual obligor conditional loss distributions:

$$f_{L(t)|\overline{M}}(l|\overline{m}) = f_{i_1(t)|\overline{M}}(l|\overline{m}) \otimes f_{i_2(t)|\overline{M}}(l|\overline{m}) \otimes \ldots \otimes f_{i_n(t)|\overline{M}}(l|\overline{m}) \quad (2.3)$$

where 

$$f_{i(t)|\overline{M}}(l|\overline{m}) = P[l \leq l_i(t) < l + dl|\overline{M} = \overline{m}]$$

and

$$f_{i(t)}(l) \otimes f_{i(t)}(l) = \int_{-\infty}^{\infty} f_{i(t)}(\gamma) f_{i(t)}(l - \gamma) d\gamma$$

The unconditional portfolio loss distribution is then obtained by integrating the conditional portfolio loss distribution with respect to the probability density function of the vector $\overline{M}$.

$$f_{L(t)}(l) = \int_{-\infty}^{\infty} f_{L(t)|\overline{M}}(l|\overline{m}) f_{\overline{M}}(\overline{m}) d\overline{m} \quad (2.4)$$

where $f_{\overline{M}}(\overline{m})$ is the multidimensional probability density function for the vector $\overline{M}$.

The conditional portfolio loss distribution $f_{L(t)|\overline{M}}(l|\overline{m}) = f_{i_1(t)|\overline{M}}(l|\overline{m}) \otimes f_{i_2(t)|\overline{M}}(l|\overline{m}) \otimes \ldots \otimes f_{i_n(t)|\overline{M}}(l|\overline{m})$ involves nested integrals which are impractical to compute for a credit portfolio that usually contains 100 obligors or more. Two methods appear in the credit literature that computes the conditional portfolio loss distribution in a fast manner. The first method presented in Merino and Nnyfeler (2002) and Gregory and Laurent (2003b) uses the Fast Fourier Transform and the second method presented in Andersen et al (2003) proposes a recursive algorithm. The two methods are now reviewed.
2.1.1 Using the Fourier Transform to Compute the Conditional Portfolio Loss Distribution

The Fourier Transform of a function \( f(t) \) is defined as:

\[
\Psi(f(t)) = F(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t) dt
\]

where \( j = \sqrt{-1} \) is the imaginary number and an important result from the theory of Fourier Transforms is that the convolution of two functions in the time domain is equivalent to multiplying the Fourier transforms of the two functions in the frequency domain. Expressed mathematically, this can be written as:

\[
f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau = \Psi^{-1}(F(j\omega)G(j\omega))
\]

where \( F(j\omega) \) is the Fourier Transform of \( f(t) \), \( G(j\omega) \) is the Fourier Transform of \( g(t) \) and \( \Psi^{-1} \) is the inverse Fourier Transform operator: \( \Psi^{-1}(\Psi(f(t))) = f(t) \).

In order to quickly compute the conditional portfolio loss distribution given by the convolution integral

\[
f_{L(t)M}(l|m) = \sum_{i=1}^{n} f_{L(t_{i})M}(l|m) \otimes f_{L(t_{i})M}(l|m) \otimes \ldots \otimes f_{L(t_{n})M}(l|m)
\]

we simply compute the Fourier Transform of each obligor's conditional loss distribution and multiply them together and finally compute the inverse Fourier Transform of the resultant product:

\[
f_{L(t)M}(l|m) = \Psi^{-1}\left( \prod_{i=1}^{n} \Psi(f_{L(t_{i})M}(l|m)) \right) \quad (2.5)
\]

In practice the conditional obligor loss distributions are assumed to be discrete and so the Fast Fourier Transform (FFT) algorithm is used compute the Fourier Transform. The FFT algorithm has a computational complexity of order \( O\left(\frac{N}{2} \log_2 N\right) \). See Press et al (1992) for more information on the FFT. An important point to note is that if the length of the discrete conditional loss distribution \( f_{L(t_{i})M}(l|m) \) is \( N \) and the length of \( f_{L(t_{i})M}(l|m) \) is \( K \), then the length of \( f_{L(t_{i})M}(l|m) \otimes f_{L(t_{i})M}(l|m) \) is \( (N + K - 1) \). So when multiplying \( \Psi(f_{L(t_{i})M}(l|m)) \) and \( \Psi(f_{L(t_{i})M}(l|m)) \) it is vital that both conditional loss distributions are zero padded so that they both have lengths equal to \( (N + K - 1) \). If there are \( n \) obligors in the portfolio and the length of each obligors conditional loss distribution is
then the conditional loss distribution of each obligor must be zero padded to be of length \((nk - n + 1)\).

In order to obtain the unconditional portfolio loss distribution, we integrate over the probability distribution of the market factors:

\[
I_l(l) = \int_{-\infty}^{\infty} \Psi^{-1}\left(\prod_{i=1}^{n} [\Psi(f_{i(0)lv})(l|m)]\right) f_{Ml}(m) dm
\]  

(2.6)

and due to the linearity of Fourier Transforms this can be rewritten as:

\[
I_l(l) = \Psi^{-1}\left(\int_{-\infty}^{\infty} \prod_{i=1}^{n} [\Psi(f_{i(0)lv})(l|m)] f_{Ml}(m) dm\right)
\]  

(2.7)

which means that we only need to perform the inverse Fourier Transform once.

Note that up to now we have been talking about the obligor conditional loss distribution \(f_{i(0)lv}(l|m)\), however, in practice it is usually the case that we specify the conditional default probability \(P_i(t|m) = E[1\{t<t}\] = \(P_l(t|m)\), where \(\tau_i\) is the default time of obligor \(i\), and from this conditional default probability we construct the conditional obligor loss distribution. If recovery rates are constant, then the conditional obligor loss distribution is:

\[
f_{i(0)lv}(l|m) = (1 - P_l(t|m)) \delta(l) + P_l(t|m) \delta(l - N_i(1 - R_i))
\]  

(2.8)

where \(N_i\) is the obligor notional, \(R_i\) is its constant recovery rate and \(\delta(x)\) is the unit delta function defined by \(\delta(0) = 1\) and \(\delta(x) = 0\) for \(x \neq 0\). If the recovery rate is assumed to have the conditional density \(f_{R_iM}(R|M)dR = P[R \leq R_i < R + dR|M = m]\), then the conditional obligor loss distribution now becomes:

\[
f_{i(0)lv}(l|m) = (1 - P_l(t|m)) \delta(l) + (1 - \delta(l)) P_l(t|m) f_{R_i|M} \left(1 - \frac{l}{N_i|m}\right)
\]  

(2.9)

That concludes our discussion regarding the use of Fourier Transforms to compute the loss distribution of a portfolio containing credit risky securities. We now proceed to review the recursive algorithm as first proposed in Andersen et al (2003) which is an alternative method to the Fourier Transform technique to compute the portfolio loss distribution.
2.1.2 The Recursive Algorithm

For the case of constant recovery rates there is a simple procedure to build the conditional loss distribution without the need to resort to Fourier Transforms. We call this methodology the recursive algorithm method for building the portfolio loss distribution and it was first applied to the pricing of single tranche CDOs in Andersen et al (2003). This methodology also makes use of the fact that conditional on the realization of the market factors, all obligors default independently. Notation wise we denote the conditional probability of obligor i defaulting by time t as \( P_i(t|m) \) where we explicitly state that it is a function of the vector of realized market factors \( m \). It is also assumed that each obligor has a loss given default value of \( \lambda = N_i(1 - R_i) \), where \( N_i \) is its notional amount, and \( R_i \) is its fixed recovery rate. Next let \( \chi \) represent the portfolio of obligors excluding the obligor i and denote the conditional loss distribution of the portfolio \( \chi \) by \( f_{L(\chi)|M}(l|m) \).

By adding obligor i to portfolio \( \chi \), the modified loss distribution \( f_{L(\chi+i)|M}(l|m) \) of the portfolio \( \chi \) plus obligor i may be obtained via expression (2.10).

\[
f_{L(\chi+i)|M}(l|m) = f_{L(\chi)|M}(l|m)(1 - P_i(t|m)) + f_{L(\chi)|M}(l - z_i|m)P_i(t|m) \quad (2.10)
\]

Provided \( l > z_i \), otherwise we simply have \( f_{L(\chi+i)|M}(l|m) = f_{L(\chi)|M}(l|m)(1 - P_i(t|m)) \). The rationale behind expression (2.10) is that in order to have a loss of \( l \) on the modified portfolio \( \chi + i \), it is possible to either have a loss of \( l \) on the original portfolio \( \chi \) and no default by obligor \( i \), or we can have a loss of \( l - z_i \) on the portfolio \( \chi \) and a default by obligor \( i \) which will contribute an additional loss of \( z_i \). In practice, to use expression (2.10) to build the conditional loss distribution we start with the empty basket where the portfolio \( \chi \) will be the null portfolio and add one obligor at a time to the portfolio. At each addition of an obligor we update the conditional loss distribution using expression (2.10). In order to feasibly implement (2.10) on a computer it is required to approximate each obligors exposure by a multiple of a base loss unit. For example, if the loss given default of obligor \( i \) is \( z_i \), then we approximate \( z_i \) by \( z_i \approx k_i u \) where \( u \) is the smallest lost unit and \( k_i \) is some integer value obtained via rounding the expression \( z_i/u \) to the nearest integer. The computational complexity of the recursive algorithm grows as the base loss unit \( u \) is made smaller. On the other hand, for large values
of $u$, the rounding error may become too significant to ignore. It is up to the implementer to choose a suitable value of $u$ in order to find the optimal trade-off between speed and accuracy.

### 2.2 Conditionally Independent Models - Examples

So far we have defined the notion of conditional independence as models where the default times of the obligors are independent conditional on the realization of certain random variables. We now present two specific examples, the Archimedean copula model, and the one factor Gaussian copula model. In both examples we also present the corresponding large homogeneous portfolio approximation which provides a quick back-of-the-envelope method for calculating the portfolio loss distribution.

#### 2.2.1 Archimedean Copula Model

This subsection presents a correlated default model using the Archimedean copula and which can be placed in the conditional independence framework. The material here closely follows Schönbucher (2002). The Archimedean copula is a copula function $C : [0, 1]^N \rightarrow [0, 1]$ which can be represented in the following form:

$$C(u_1, \ldots, u_n) = \varphi^{-1}\left(\sum_{i=1}^{n} \varphi(u_i)\right)$$

(2.11)

where $\varphi(x)$ is a strictly decreasing function $\varphi : [0, 1] \rightarrow \mathbb{R}_+$ with $\varphi(1) = 0$, $\varphi(0) = \infty$ and is called the generator function of the copula. $\varphi^{-1}(x)$ is its corresponding inverse function.

The task of constructing an Archimedean copula now translates to the problem of finding a suitable generator function. In order to do this we make use of the following fact: If $F(x)$ is a cumulative distribution function of a positive random variable with $F(x = 0) = 0$, $f(x) = \frac{dF(x)}{dx}$, and $\hat{F}(y) = \int_{0}^{\infty} e^{-yx} f(x)dx$ is its Laplace Transform, then $\varphi(t) = \hat{F}^{-1}(t)$ is the generator of an Archimedean
copula for any dimension, where \( \hat{F}^{-1}(t) \) is the inverse function of \( \hat{F}(t) \). So, in order to find a suitable generator function, we must first specify a positive random variable. Possible candidates include, as pointed out in Schönbucher (2002), the Gamma distribution, the alpha-stable distribution and the Logarithmic series distribution. The Archimedean copulas corresponding to these distributions are known as the Clayton, Gumbel, and Frank copula respectively.

To simulate from an Archimedean copula with generator \( \varphi(\cdot) \), we make use of the following procedure:

1. Draw \( U_1, \ldots, U_N \) independent uniformly distributed random variables on \([0,1]\).
2. Draw a mixing variable \( Y \) with the following properties:
   - \( Y \) is independent from \( U_1, \ldots, U_N \) and has a cumulative probability distribution function denoted by \( G(\cdot) \).
   - The Laplace transform of \( Y \) is \( \varphi^{-1}(s) = \int_0^\infty e^{-sy} dG(y) \).
3. Define \( X_i = \varphi^{-1}\left( -\frac{1}{\varphi} \ln(U_i) \right) \).
4. Then the joint distribution function of the \( X_i, 1 \leq i \leq N \) is: 
   \[
P[X_1 \leq x_1, \ldots, X_N \leq x_N] = \varphi^{-1}\left( \sum_{i=1}^N \varphi(x_i) \right)
\]

To adapt the above Monte Carlo procedure to be used in the conditional independence framework, we must first decide on the conditioning variable. It is clear from the above simulation procedure that the random variable \( Y \) introduces dependency amongst the variables \( X_i \), and as such if we condition on a realization of \( Y \) then all the \( X_i \)’s become independent. Since the \( X_i \)’s are uniform random variables, we assume obligor \( i \) defaults by time \( t \) if \( X_i \leq P_i(t) \), where \( P_i(t) = P[T_i < t] \) and \( T_i \) is the default time of obligor \( i \). The conditional probability that obligor \( i \) will default by time \( t \) can be expressed as:

\[
P[X_i \leq P_i(t) | Y = y] = P[\varphi^{-1}\left( -\frac{1}{y} \ln(U_i) \right) \leq P_i(t)]
\]
\[
\begin{align*}
&= P[-\frac{1}{y} \ln(U_i) \geq \varphi(P_i(t))] \\
&= P[\ln(U_i) \leq -y\varphi(P_i(t))] \\
&= P[U_i \leq \exp(-y\varphi(P_i(t)))] \\
&= \exp(-y\varphi(P_i(t))) (2.12)
\end{align*}
\]

Hence, to construct the portfolio loss distribution we first condition on a realization of \( Y = y \), and given this we calculate the conditional default probability for each obligor using expression (2.12). Next we use either the Fourier Transform method or the recursive method to compute the conditional portfolio loss distribution and finally integrate this with respect to the probability distribution of \( Y \) to obtain the unconditional loss distribution.

We now proceed to introduce the concept of the large homogeneous portfolio (LHP) loss distribution as first pioneered by Vasicek (1987). In the case of a homogenous portfolio, all obligors are assumed to have identical default probabilities, recovery rates, and notional value. By large, we mean that the number of obligors tend to infinity, while the total notional remains at a constant value \( N \). This means that the notional of each obligors tends to zero in the limit and the portfolio becomes increasingly granular. In the derivation of the LHP loss distribution, a further assumption is made which is that conditional on a set of latent random variables all the obligors default independently and have the same conditional default probabilities. Since all obligors default independently and have the same conditional default probability, the Law of Large Numbers can be applied to calculate the conditional loss fraction. There are different forms of the Law of Large numbers (LLN) depending on what convergence measure is used. For the case of convergence in probability, we have the following version:

**Theorem 2** The law of large numbers for a sequence of independent, identically distributed random variables \( X_1, X_2, \ldots, X_n \) with common finite mean \( \mu \). As \( n \) tends to \( \infty \), the sample mean \( \frac{X_1 + \ldots + X_n}{n} \) converges in probability to the mean \( \mu = E[X] \), in which \( X \) is a random variable obeying the common probability law of \( X_1, X_2, \ldots, X_n \).

Proof: See Parzen (1960), Chapter 10, Theorem 3B.
CHAPTER 2. FACTOR MODELS AND IMPLIED CORRELATIONS

To apply the LLN to the construction of the portfolio loss distribution we start by assuming that there are $n$ obligors in the portfolio, that the notional of each obligor is $N/n$, and the corresponding loss given default is $(1 - R)N/n$ were $R$ is the recovery rate. Denoting the conditional probability that each obligor will default by time $t$ by $P(t|m)$, and letting $h_i(m)$ be a random variable that takes the value 1 with probability $P(t|m)$ and the value 0 with probability $1 - P(t|m)$, then the total conditional portfolio loss is given by: $L(t|m) = \sum_{i=1}^{n} h_i(m)(1 - R)N/n = \sum_{i=1}^{n} X_i/n$, where $X_i = h_i(m)(1 - R)N$. If the number of obligors tend to infinity, then we have by the LLN that the conditional portfolio loss is equal to: $L(t|m) = \lim_{n \to \infty} \sum_{i=1}^{n} X_i/n = E[X_i] = P(t|m)(1 - R)N$. This means that once we condition on a realization of the vector of random variables $M = m$, then we know exactly what the portfolio loss will be and it is a function of the obligor conditional default probability. Intuitively, this means that if the conditional default probability is, say, 5% then if the portfolio is sufficiently large we would expect 5% of the portfolio to default.

Returning back to the Archimedean copula, we denote the conditional probability of default by time $t$ as $p_t(y) = P[X_i \leq P(t)|Y = y]$ where $P(t)$ is the obligor default probability and is the same for each obligor since the portfolio is homogeneous. Using the LLN, the conditional portfolio loss is $p_t(y)N(1 - R) = l$. The probability that the portfolio loss is less than $l$ is equivalent to the condition: $P[L(t) \leq l] = P[p_t(Y) \leq \frac{l}{N(1 - R)}]$. Evaluating this condition further we have:

$$P[L(t) \leq l] = P[p_t(Y) \leq \frac{l}{N(1 - R)}]$$

$$= P[exp(-Y\varphi(P(t))) \leq \frac{l}{N(1 - R)}]$$

$$= P[-\frac{\ln(l/N(1 - R))}{\varphi(P(t))}]$$

$$= P[Y \geq \frac{-\ln(l/N(1 - R))}{\varphi(P(t))}]$$

$$= 1 - G\left(\frac{-\ln(l/N(1 - R))}{\varphi(P(t))}\right) \tag{2.13}$$

Expression (2.13) provides a closed form formula for the portfolio loss distribution where the dependence structure is given by an Archimedean copula. It is
simply a function of the distribution of $Y$ and its Laplace Transform. Clearly this is far easier to implement than the Fourier Transform method or the recursive algorithm method for computing the portfolio loss distribution.

In the next section we present another conditionally independent portfolio credit model that is by far the most popular model used in industry.

### 2.2.2 The One Factor Gaussian Copula Model

The one factor Gaussian copula has become the market standard pricing model for synthetic CDOs and is the same as the factor CreditMetrics model presented in Finger (1999) but in this case there is only one market factor in the asset value of each obligor. Specifically, the asset value of each obligor is given by expression (2.14):

$$A_i = \rho_i M + \sqrt{1 - \rho_i^2} \varepsilon_i$$  \hspace{1cm} (2.14)

where $M$ and $\varepsilon_i$ are both standard Gaussian random variables and $\rho_i \in [0, 1]$ is the correlation parameter. Given this specification of the asset value, the linear correlation coefficient between two obligors is $\text{Corr}(A_i, A_j) = \rho_i \rho_j$. It is assumed that obligor $i$ defaults by time $t$ if its asset value falls below some time dependent threshold $C_t$. The probability of default by time $t$ is:

$$P[\tau_i \leq t] = P[A_i \leq C_t]$$  \hspace{1cm} (2.15)

The threshold, $C_t$, is calculated so that the marginal default probability of the obligor matches those implied in the credit default swap market:

$$C_t = \Phi^{-1}(P_i(t))$$  \hspace{1cm} (2.16)

where $P_i(t) = P[\tau_i \leq t]$. The conditioning variable in the one factor Gaussian model is $M$, commonly referred to as the market factor, and conditional on $M$ all obligors default independently. The conditional default probability given a realization of the market factor is given by (2.17):
\[ P[A_t \leq C_t | M = m] = \Phi\left( \frac{\Phi^{-1}(P_t(t)) - \rho t m}{\sqrt{1 - \rho^2}} \right) \]  

(2.17)

Hence, once we condition on the realization of the market factor we can use the Fast Fourier Transform or the recursive algorithm given by (2.10) to compute the conditional portfolio loss distribution. In order to obtain the unconditional loss distribution we integrate the conditional loss distribution with respect to the distribution of the market factor, which is a standard Gaussian random variable. This integration has to be performed numerically using some quadrature routine. For integrals of the form \( \int_{-\infty}^{\infty} g(x)\phi(x)dx \), for some function \( g(x) \) and \( \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \), there is a fast quadrature routine called Gaussian quadrature. Standard integration techniques can exactly integrate a \( N-1 \) degree polynomial function by evaluating the function at \( N \) points or abscissas. Gaussian quadrature allows one to exactly integrate a polynomial of degree \( 2N-1 \) using only \( N \) points and, conveniently, the range of integration is the whole real line. Appendix A gives a detailed discussion behind the mechanics of Gaussian Quadrature since it is widely used in the implementation of credit portfolio models.

The explicit copula defined by the one factor Gaussian dependency structure is given by (2.18):

\[ C(u_1, \ldots, u_n) = \int_{-\infty}^{\infty} \phi(x)\prod_{i=1}^{n} \Phi\left( \frac{\Phi^{-1}(u_i) - \rho t x}{\sqrt{1 - \rho^2}} \right) dx . \]  

(2.18)

and the default times are simulated using the mapping \( \tau_i = F^{-1}(\Phi(A_i)) \) where \( F(t) = P[\tau_i \leq t] \).

Vasicek (1987) developed the large homogeneous portfolio loss distribution formula for the one factor Gaussian copula model and we now review his work. Just as for the LHP derivation for the Archimedean copula, all obligors are assumed to have identical default probabilities \( P(t) \), recovery rate \( R \), and the total portfolio notional is \( N \). It is also assumed that all obligors have the same correlation coefficient \( \rho \). Conditional on a realization of the market the factor \( M = m \), the conditional default probability of each obligor is given by expression (2.17) and we denote this by the short hand notation \( P(t|m) \). Since the portfolio consists of an infinite number of obligors, then by using the Law of Large Numbers,
we deduce that the percentage number of obligors to default is precisely $P(t|m)$. Hence the total loss incurred on the portfolio is $N(1-R)P(t|m) = l$. By inverting this relationship, we find that the value of $m$ that gives us a total loss of $l$ is:

$$m = \frac{\Phi^{-1}(P(t)) - \Phi^{-1}(l/(1-R)N)\sqrt{1-\rho^2}}{\rho} = G$$

(2.19)

which is equal to some value $G$. It is straightforward to see that $P[L(t) \leq l] = P[M \geq G]$, where $L(t)$ is the total portfolio loss. This allows us to obtain the cumulative distribution function of $L(t)$:

$$P[L(t) \leq l] = \Phi\left(\frac{\Phi^{-1}(l/(1-R)N)\sqrt{1-\rho^2} - \Phi^{-1}(P(t))}{\rho}\right)$$

(2.20)

Having presented two concrete examples of conditionally independent models, we proceed to develop the general framework and demonstrate how the conditional independence methodology may be used to compute the prices of single tranche CDOs and $N^{th}$-to-default baskets using semi-analytical techniques.

### 2.3 Pricing Portfolio Credit Derivatives Under the Conditional Independence Approach

In this section we show how to price single tranche CDOs, $N^{th}$-to-default baskets, and CDO squared derivatives using the conditional independence approach presented in the previous section.

#### 2.3.1 Single Tranche CDOs

In this section we proceed to define the payoff of a synthetic CDO and show how semi-analytic methods may be used to price the various tranches.

Assume a reference portfolio with cumulative loss at time $T$ of $L(T)$. A tranche is defined by its lower attachment point $K^L$ and upper detachment point $K^U$. The cumulative loss incurred on a tranche by time $T$ can be expressed as:

$$L^{\text{trch}}(T) = \max(\min(K^U, L(T)) - K^L, 0)$$

(2.21)
Concentrating now on the premium leg (fixed leg), the protection buyer pays a periodic coupon (usually quarterly in arrears) expressed as a percentage of the remaining notional on the tranche. Assuming for now that coupon payments are continuous in time and that the tranche notional is \( N = K^U - K^L \), the coupon payment at time \( t + dt \) calculated at time \( t \) is:

\[
C_{t+dt} = C(N - L^{trch}(t))dt
\]

where \( C \) is the continuous coupon rate. Given that the portfolio loss is stochastic and that the portfolio loss at time \( t' > 0 \) will not be know at time \( t = 0 \), the present value of the fees paid by the protection buyer may be calculated as the discounted expected coupon payments, where the expectation is taken under the risk neutral measure. The expected coupon payment at time \( t + dt \) is given by:

\[
E[C_{t+dt}] = C(N - E[L^{trch}(t)])dt
\]

It then follows that the present value of the fixed leg is simply the summation of the discounted expected coupon payments where the summation is taken with respect to time:

\[
PV_{fixed} = C \int_0^T D(0, t)(N - E[L^{trch}(t)])dt 
\]

(2.22)

where \( D(t, T) \) is the discount factor at time \( t \) for maturity \( T - t \). The expected tranche loss at time \( t \) is calculated by the following expression:

\[
E[L^{trch}(t)] = \int_0^{L_{max}} \max(\min(K^U, l) - K^L, 0)f_{L(t)}(l)dl
\]

(2.23)

where \( f_{L(t)} \) is the portfolio loss distribution at time \( t \) and \( L_{max} \) is the maximum possible loss the portfolio may experience. Concentrating now on the loss paying leg (the floating leg), payments are made by the protection seller each time a loss is incurred on the tranche and the incremental loss payment \( ds_t \) made at time \( t \) is:

\[
ds_t = L^{trch}(t) - L^{trch}(t - dt)
\]

The sum of the discounted loss payment is then given by:
\[ \int_0^T D(0,t) dt = \int_0^T D(0,t) (L^{trch}(t) - L^{trch}(t - dt)) \]

The present value of the floating leg is then simply the expectation of this summation:

\[ PV_{float} = E \left[ \int_0^T D(0,t) dt \right] \tag{2.24} \]

The tranche par coupon rate is simply the value of \( C \) that equates the present value of the floating leg and fixed leg:

\[ C = \frac{E \left[ \int_0^T D(0,t) dt \right]}{\int_0^T D(0,t) (N - E[L^{trch}(t)]) dt} \tag{2.25} \]

In practice it is not possible to find closed form solutions for the continuous integrals and approximations are derived by dividing the time line into small discrete time intervals \( t_0 = 0, \ldots, t_n = T \). The discrete time approximation of the floating leg then becomes:

\[ PV_{float} \approx \sum_{i=0}^{n-1} \left( \frac{D(0,t_{i+1}) + D(0,t_i)}{2} \right) (E[L^{trch}(t_{i+1})] - E[L^{trch}(t_i)]) \tag{2.26} \]

The coupon payments made by the fixed leg are usually not continuous but are paid quarterly in arrears. Let us assume that coupons are paid on the dates \( \tau_1, \ldots, \tau_m \). The present value of the fixed leg is then calculated by the mid point approximation where the coupon payment \( c_i \) made at time \( \tau_i \) is calculated based on the average surviving tranche notional between the coupon dates \( \tau_{i-1} \) and \( \tau_i \). Hence, we have

\[ PV_{fixed} \approx C \Delta \sum_{i=1}^{m} D(0,\tau_i) \left( \frac{N - E[L^{trch}(\tau_i)] + N - E[L^{trch}(\tau_{i-1})]}{2} \right) \tag{2.27} \]

where \( \Delta \) is the day count fraction.
As can be seen from equations (2.26) and (2.27) the task of computing the price of a CDO tranche essentially involves calculating the expected tranche loss for a series of discrete time intervals. In the semi-analytical framework we can calculate the cumulative portfolio loss distribution at a set of time intervals using either the Fast Fourier Transform or the recursive algorithm given by (2.10) and from this obtain the expected tranche loss.

Having presented the semi-analytical method to price synthetic CDO tranches, we proceed to price N\textsuperscript{th}-to-default contracts.

### 2.3.2 Nth-to-Default Baskets

In a N\textsuperscript{th}-to-default contract the two legs of the contract are usually termed the premium leg and the default leg. The contract is characterized by a final maturity date \(T\) and a periodic (usually quarterly) coupon, \(C\), that the protection buyer pays the protection seller until the time of the N\textsuperscript{th} default or final maturity date, whichever is first. If the N\textsuperscript{th} default occurs before the maturity date, then at the time of the default date the protection buyer pays any accrued coupon payment and receives from the protection seller a payment of \(N_i(1-R_i)\), where \(N_i\) is the notional of the N\textsuperscript{th} defaulted obligor and \(R_i\) is its respective recovery rate. Hence, N\textsuperscript{th}-to-default contracts are identical to standard credit default swaps except that the default payment is contingent on the N\textsuperscript{th} defaulted obligor. Before we proceed to derive the pricing algorithm we define the notation that we will use. The probability that obligor \(i\) will default by time \(t\) is \(P[r_i \leq t] = P_i(t)\) and the survival probability is given by \(Q_i(t) = 1 - P_i(t)\). Let the probability that there will be \(N\) defaults by time \(t\) be denoted by \(P^N(t)\), and let \(Q^N(t)\) be the probability that there will be less than \(N\) defaults by time \(t\), \(Q^N(t) = \sum_{i=0}^{N-1} P^i(t)\). Assume coupon payments of \(C\) are received at equal time intervals of \(t_0, \ldots, t_n\) where \(t_{i+1} - t_i = \Delta\). The present value of a coupon payment at time \(t_i\) is simply the value of the discounted payment at time \(t_i\) weighted by the probability that the payment will be received: \(D(0, t_i)Q^N(t_i)C\), where \(D(0, t)\) is the discount factor. The present value of an accrual payment (AC) between the dates \([t_i, t_{i+1}]\) is given by:
\[ AC = \int_{t_i}^{t_{i+1}} D(0,t)C \frac{(t-t_i)}{\Delta} dP^N(t) \]

And the present value of the premium leg is the sum of the present value of the coupon payments and accrual payments over all time intervals:

\[ PV_{\text{premium}} = C \sum_{i=0}^{n-1} \left( D(0,t_{i+1})Q^N(t_{i+1}) + \int_{t_i}^{t_{i+1}} D(0,t) \frac{(t-t_i)}{\Delta} dP^N(t) \right) \quad (2.28) \]

Hence, to price the premium leg of a \( N^{th} \)-to-default contract, the problem boils down to a specification of \( P^N(t) \).

For each obligor we have the individual default and survival probabilities \( P_i(t) \), and \( Q_i(t) \), inferred from market spreads. We make the assumption that conditional on a vector of market factors, \( \bar{M} \), each obligor defaults independently. Assuming there are \( j \) obligors in the portfolio, let \( K(t) = \sum_{i=1}^{j} 1_{\{t_i \leq t\}} \) be a counting process that counts the number of defaults in the portfolio by time \( t \).

The probability generating function of \( K(t) \) is defined as \( \psi_{K(t)}(u) = E[u^{K(t)}] = \sum_{i=0}^{j} P(K(t) = i)u^i \). Let \( K_i(t) = 1_{\{t_i \leq t\}} \) be the default indicator of obligor \( i \), we then have:

\[ \psi_{K_i(t)}(u) = E[u^{K_i(t)}] = Q_i(t) + uP_i(t) \quad (2.29) \]

and

\[ \psi_{K_i(t)|\bar{M}}(u|\bar{m}) = \frac{E[u^{K_i(t)}|\bar{M} = \bar{m}]}{E[u^{K_i(t)}]} = Q_i(t|\bar{M} = \bar{m}) + uP_i(t|\bar{M} = \bar{m}) \]

where \( \psi_{K_i(t)|\bar{M}}(u|\bar{m}) \) is the conditional probability generating function. Since the probability generating function of a sum of independent random variables is the product of their individual probability generating functions, it follows:

\[ \psi_{K(t)|\bar{M}}(u|\bar{m}) = \prod_{i=1}^{j} \left( Q_i(t|\bar{M} = \bar{m}) + uP_i(t|\bar{M} = \bar{m}) \right) \quad (2.30) \]
and the unconditional probability generating function is simply obtained by taking the expectation with respect to the market factors $\overline{M}$:

$$
\psi_{K(t)}(u) = E[\psi_{K(t)|\overline{M}}(u|\overline{m})] = \int_{-\infty}^{\infty} \prod_{i=1}^{j} \left( Q_i(t|\overline{M} = \overline{m}) + uP_i(t|\overline{M} = \overline{m}) \right) f_{\overline{M}}(\overline{m}) d\overline{m} \quad (2.31)
$$

Finally the value of $P_k(t)$ is simply the coefficient of $u^k$ in the polynomial given by $\psi_{K(t)}(u)$. Hence, in summary, to obtain $P^k(t)$ we compute the conditional probability generating function of the default indicator function for each obligor and multiply the individual probability generating functions together to obtain the conditional probability generating function for the portfolio default counting process. We then integrate the product with respect to the probability distribution of the market factors to obtain the unconditional probability generating function. The resultant probability generating function is a polynomial in $u$ and the value of $P^k(t)$ is given by the coefficient of $u^k$. Note that this approach involves multiplying polynomials and the coefficients of a polynomial, $y(u)$, obtained by multiplying the polynomials $f(u)$ and $g(u)$ such that $y(u) = f(u)g(u)$, is given by performing the discrete convolution integral on the coefficients of $f(u)$ and $g(u)$. Stately mathematically, if the coefficient of $y(u)$ can be expressed as $C[y(u)]$, then we have $C[y(u)] = C[f(u)] \otimes C[g(u)]$, where $h(n) \otimes d(n) = \sum_{x} h(x)d(n - x)$ and $x$ and $n$ are discrete variables. As stated in section 2.1.1 the convolution integral can be efficiently computed using the Fast Fourier Transform.

We now proceed to derive an expression for the present value of the default leg. If the $n^{th}$ defaulted obligor is $i$, then at the time of the $n^{th}$ default the protection buyer receives a payment of $N_i(1 - R_i)$, where $N_i$ is the notional of obligor $i$ and $R_i$ is its recovery rate. Let $K^{-i}(t)$ be the total number of defaults by time $t$ in the portfolio where we have removed obligor $i$ from the portfolio. When we condition on the market factors $\overline{M}$, all obligors default independently and as a result the conditional probability that obligor $i$ will default in the time interval $[t, t+dt]$ and be the $n^{th}$ defaulted asset is $P[K^{-i}(t) = n-1|\overline{M} = \overline{m}]dP_i(t|\overline{M} = \overline{m})$. 
Note that we have implicitly assumed that two or more obligors cannot default simultaneously. Hence, the present value of a default payment occurring in the time interval \([t, t+dt]\) due to the \(n^{th}\) default of obligor \(i\) conditional on a realization of the market factor is given by:

\[
N_i(1 - R_i)D(0,t)P[K^{-i}(t) = n - 1|\bar{M} = \bar{m}]dP_i(t|\bar{M} = \bar{m})
\]

and the present value of a default payment occurring in the time interval \([t, t+dt]\) conditional on the market factor is simply the sum of the above expression across all obligors in the portfolio:

\[
\sum_{i=1}^{j} N_i(1 - R_i)D(0,t)P[K^{-i}(t) = n - 1|\bar{M} = \bar{m}]dP_i(t|\bar{M} = \bar{m})
\]

The reason as to why we can simply sum the probabilities across the different obligors is because the different default scenarios are disjoint events. To get the total PV of the default leg conditional on the market factor requires us to integrate the above expression with respect to time to reflect the fact that the \(n^{th}\) default can occur at any time:

\[
P_{\text{default}|\bar{M}} = \int_0^T \sum_{i=1}^{j} N_i(1 - R_i)D(0,t)P[K^{-i}(t) = n - 1|\bar{M} = \bar{m}]dP_i(t|\bar{M} = \bar{m})
\]

and finally the unconditional case is given by taking the expectation with respect to the market factor:

\[
P_{\text{default}} = E\left[\int_0^T \sum_{i=1}^{j} N_i(1 - R_i)D(0,t)P[K^{-i}(t) = n - 1|\bar{M} = \bar{m}]dP_i(t|\bar{M} = \bar{m})\right]
\]

(2.32)

where \(P[K^{-i}(t) = n - 1|\bar{M} = \bar{m}]\) is given by the coefficient of \(u^{n-1}\) in the polynomial \(\prod_{h=1,h\neq i} Q_h(t|\bar{M} = \bar{m}) + uP_h(t|\bar{M} = \bar{m})\).

Having now obtained expressions for the present value of the premium leg (2.28), and the default leg (2.32), the fair value of the \(n^{th}\)-to-default coupon is simply the value of \(C\) in (2.28) that equates the present value of the default leg and premium leg. This concludes the section on \(N^{th}\)-to-default baskets and we proceed with the analysis of CDO squared derivatives in the next section.
2.3.3 CDO Squared Valuation

Synthetic CDO squared portfolio derivatives (CDO$^2$) are CDOs where the underlying entities are tranches of other synthetic CDOs. CDO$^2$ allows one to have underlying entities with high spreads in periods of low credit spreads by referencing the risky equity tranches of other CDOs. These derivatives are becoming increasingly popular and it is important to be able to efficiently value these products. The main issue that complicates the valuation of CDO$^2$ is that the underlying portfolios may have common or shared obligors. For example, a given CDO$^2$ transaction may have ten underlying portfolios each consisting of 100 obligors, but there are in total only 400 distinct obligors in the entire parent portfolio. This point complicates the application of the conditional independence approach to CDO$^2$ pricing, since even after conditioning on the market factors the underlying portfolios are not independent. Note that the overlap phenomenon is also beneficial to the credit derivatives structurer as by varying the degree of overlap, it is possible to vary the degree of correlation underlying the parent portfolio whereas in standard synthetic CDOs, the market determines the level of correlation.

In this section we first define the payoff for a CDO$^2$ transaction and review how the recursive portfolio loss building algorithm can be extended to price CDO$^2$ products as first presented in Baheti et al (2004). We then present a new methodology for pricing CDO$^2$ which treats the portfolio overlap problem as an equivalent increase in portfolio correlation. The accuracy of this method is compared to a complete Monte Carlo pricing methodology.

We now proceed to define the exact payoff for a CDO$^2$ transaction.

Pricing CDO$^2$ - Definitions and Concepts

Synthetic CDO$^2$ are CDOs where the underlying entities are single tranches of other synthetic CDOs. We call the CDO that has other CDO tranches as underlying as the 'parent' CDO. Assume the parent CDO references $N$ underlying tranches and that each of these tranches references a different portfolio which we call the 'child' portfolio. Each obligor in the parent portfolio belongs to one or more of the $N$ child portfolios, and a given obligor belonging to more than one
child portfolio may have a different notional in each child portfolio that it belongs to.

Before we proceed to understand how losses from the underlying child portfolios are distributed to a given tranche of the parent portfolio, the following notation will be defined.

- \( n_{i,j} \): Notional amount of obligor \( i \) in child portfolio \( j \).
- \( K_j^L \): Lower tranche attachment point referencing child portfolio \( j \).
- \( K_j^U \): Upper tranche detachment point referencing child portfolio \( j \).
- \( L_j(t) \): Total loss at time \( t \) for child portfolio \( j \).
- \( K_P^{PL} \): Lower attachment point of tranche referencing the parent portfolio.
- \( K_P^{PU} \): Upper detachment point of tranche referencing the parent portfolio.
- \( L_P(t) \): Total loss at time \( t \) for parent portfolio.

Given the above notation, the loss incurred by time \( t \) to the tranche referencing child portfolio \( j \) is given by

\[
L_j^{trch}(t) = \max(\min(K_j^U, L_j(t)) - K_j^L, 0)
\]

The total loss to the parent portfolio is simply the sum of the losses on the tranches referencing the child portfolios:

\[
L_P(t) = \sum_{i=1}^{N} L_i^{trch}(t)
\]

and the loss on the parent tranche is:

\[
L_{Ptrch}(t) = \max(\min(K_P^{PU}, L_P(t)) - K_P^{PL}, 0)
\]

If we can compute the expected parent tranche loss for a series of time intervals, then it is possible to price the synthetic CDO\(^2\) using the expressions (2.36) and (2.37) for the present values of the floating and fixed legs:

\[
PV_{float} = \sum_{i=0}^{n-1} \left( \frac{D(0, t_{i+1}) + D(0, t_i)}{2} \right) (E[L_{Ptrch}(t_{i+1})] - E[L_{Ptrch}(t_i)])
\]
\[ PV_{\text{fixed}} = C \Delta \sum_{i=1}^{m} D(0, \tau_i) \left( \frac{2(K^{PU} - K^{PL}) - E[LP_{\text{trch}}(\tau_i)] - E[LP_{\text{trch}}(\tau_i-1)]}{2} \right) \]

where \( \Delta \) is the day count fraction, \( D(t, T) \) is the discount factor at time \( t \) for maturity \( T - t \), and \( C \) is the tranche coupon. The par coupon of the CDO\(^2 \) tranche is the value of \( C \) which equates the present value of the fixed leg and floating leg. This is exactly the same pricing formula used to price standard synthetic CDOs presented in section 2.3.1. The task now consists of extending the conditional independence framework to calculate the loss distribution and hence the expected tranche loss at a series of time horizons for the parent portfolio. The difficulty presented here is that even if we condition on the common market factors, the loss on the child portfolios are not necessarily independent due to overlapping obligors in the multiple child portfolios.

Baheti et al (2004) extended the recursive algorithm given by expression (2.10) to price CDO\(^2 \) under the conditional independence framework. In this case rather than building a single conditional loss distribution a multivariate conditional loss distribution is constructed where each marginal represents the conditional loss for a specific child portfolio. Let \( f_{L_1(t), \ldots, L_N(t) | \bar{M}}(l_1, \ldots, l_N | \overline{m}) \) denote the multivariate conditional loss distribution for some multidimensional portfolio \( \chi \) that excludes obligor \( i \). The marginal random variable \( L_j(t) \) represents the loss on child portfolio \( j \) for time horizon \( t \) and \( \bar{M} \) is the vector of market factors. Let us now include obligor \( i \) in the portfolio by adding it to some of the child portfolios where the notional contribution of obligor \( i \) to child portfolio \( j \) is denoted by \( n_{ij} \).

The multivariate recursive algorithm is now given by:

\[
f_{L_1(t), \ldots, L_N(t) | \bar{M}}(l_1, \ldots, l_N | \overline{m}) = f_{L_1(t), \ldots, L_N(t) | \bar{M}}(l_1, \ldots, l_N | \overline{m})(1 - P_i(t | \overline{m}))
+ f_{L_1(t), \ldots, L_N(t) | \bar{M}}(l_1 - n_{i1}, \ldots, l_N - n_{iN} | \overline{m})P_i(t | \overline{m})
\]

(2.38)

provided \( \bigcap_{j=1}^{N}(l_j > n_{ij}) \), otherwise we have:

\[
f_{L_1(t), \ldots, L_N(t) | \bar{M}}(l_1, \ldots, l_N | \overline{m}) = f_{L_1(t), \ldots, L_N(t) | \bar{M}}(l_1, \ldots, l_N | \overline{m})(1 - P_i(t | \overline{m}))
\]

\[ P_i(t | \overline{m}) \] is the conditional default probability of obligor \( i \).
given the realization of the market factors $\overline{M} = \overline{m}$. Similar to the implementation of expression (2.10), we start with the null portfolio and add an obligor to the child portfolios one at a time. Given this multivariate conditional loss distribution, it is possible to obtain the conditional expected loss on the parent tranche. This would require to perform an $N$ dimensional integral over the distribution $\int_{\mathcal{L}(t), \ldots, L_N(t) \mid \overline{M}} (l_1, \ldots, l_N \mid \overline{m})$. This is clearly a very computationally intensive methodology were the computational complexity grows exponentially with the number of child portfolios. This method is not really feasible for parent tranches that reference more than three child portfolios. To obtain the unconditional parent tranche expected loss we have to further integrate the conditional expected loss over the probability distribution of the market factors.

**Alternative CDO Squared Pricing Methodology - Overcoming the Overlap Problem**

This section introduces a fast method for pricing CDO$^2$ using a combination of the conditional independence framework and Monte Carlo simulations. We begin by assuming that conditional on a set of market factors, each obligor defaults independently and denote the conditional default probability of obligor $i$ by time $t$ as $P[\tau_i \leq t \mid \overline{M} = \overline{m}] = P_i(t \mid \overline{m})$, where $\overline{m}$ is the vector of realized market factors. The first step is to compute the conditional loss distribution of each child portfolio using either the Fourier Transform method (see Gregory and Laurent, 2003) or the recursive algorithm given in Andersen et al (2003) for a given realization of the market factors. If there was no overlap of the obligors in the child portfolios, then the conditional portfolio loss distributions would be independent. However, there is usually considerable overlap between the child portfolios and as a result there is a degree of correlation between the conditional child portfolio losses. So the second step is to calculate the linear correlation coefficient between the conditional child portfolio losses. It is shown in the next section how to calculate this. Denote the calculated correlation matrix of the conditional losses up to time $t$ on the child portfolios by $\sum_{\overline{m}}(t)$, where the subscript denotes that this matrix is dependent on the conditioning variables. The dimension of $\sum_{\overline{m}}(t)$ is the same as the number of underlying child portfolios and is typically less than a 10x10 matrix. The third
step involves Monte Carlo simulation: using a Gaussian copula with correlation matrix $\Sigma_m(t)$, we simulate $N$ correlated uniform random variables $U_1, \ldots, U_N$, where we have assumed that we have $N$ underlying child portfolios. Conditional child portfolio loss values are generated using the mapping $Y_i(t) = F_i^{-1}(U_i|M)$, where $F_i^{-1}(\cdot|M)$ is the conditional inverse cumulative distribution loss function for the $i^{th}$ child portfolio, $F_i(U_i|M) = P[L_i(t) \leq u|M = m]$. Clearly the loss variables $Y_i(t), i = 1, \ldots, N$ have the correct marginal loss distributions. From this set of loss variables $Y_i(t), i = 1, \ldots, n$, we compute the loss to the parent CDO tranche. We repeat this many times to get a conditional expected loss value for the parent CDO tranche. Finally, the conditional expected loss on the parent tranche is integrated over the probability distribution of the market factors to get the unconditional expected parent tranche loss.

There is one important point to note about this method for pricing CDO$^2$. $\Sigma_m(t)$ is the correlation matrix of the conditional losses of the child portfolios. The generated random variables $Y_i(t), i = 1, \ldots, N$, represent the conditional losses on the child portfolios for time horizon $t$ and we would like the correlation between $Y_i(t)$ and $Y_j(t)$, for $i, j \in [1, \ldots, N]$ to correspond to the elements of the matrix $\Sigma_m(t)(i, j)$. However, we use this correlation matrix $\Sigma_m(t)$ in the Gaussian copula which produces uniform random variables $U_1 = \Phi(X_1), \ldots, U_n = \Phi(X_n)$, where the $X_i$ are obtained from the matrix product $\bar{X} = \varepsilon \Lambda$, where $\Sigma_m(t) = \Lambda \Lambda^T$ and $\bar{X} = [X_1, \ldots, X_N]^T$, $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_N]^T$, for $\varepsilon_i \sim \mathcal{N}(0, 1)$ and $\varepsilon_i$ and $\varepsilon_j$ are independent for all $i, j$. From the above construction, the correlation between $X_i$ and $X_j$ corresponds to the element $\Sigma_m(t)(i, j)$ of the correlation matrix. The random variables $Y_i(t)$ representing the conditional loss values of the child portfolios are computed via $Y_i(t) = F_i^{-1}(U_i|M)$, and the question now remains of whether the correlation between $Y_i(t)$ and $Y_j(t)$ is the same as the correlation between $X_i$ and $X_j$. To answer this we first note that linear correlations are unaltered under linear transformations. However, the $X_i$'s undergo nonlinear transformations to produce the $Y_i(t)$'s which would suggest that the linear correlation between the $Y_i(t)$'s are no longer given by the matrix $\Sigma_m(t)$. A simple argument can show that the correlations between the random variables $Y_i(t)$'s are in fact very close to those given by the matrix $\Sigma_m(t)$. Recall that the
random variables $Y_i(t)$ represent samples from the conditional loss distributions of the child portfolios. The conditional loss distributions are generated by summing many independent random variables, and by the Central Limit Theorem the conditional loss distributions should be approximately normal with a certain mean and variance. Hence, $Y_i(t)$ is generated approximately by a linear transformation of $X_i$ and as a result the correlations should be preserved.

The Monte Carlo routine outlined above has very fast convergence speed when compared to other Monte Carlo credit loss models. Usually when simulating a credit loss model the Monte Carlo generated variables are binary default indicators taking the value one (for default) or zero (for survival of the obligor), and due to the low default probabilities in the corporate bond market, most of the outcomes of the simulations are zero and hence make negligible contribution to the convergence of the Monte Carlo simulation. In the routine outlined above, however, each copula generated uniform variable is mapped to a loss value corresponding to the conditional loss distribution of a child portfolio. Hence, each generated variable makes a positive contribution to the convergence of the simulation and as a result much fewer simulations are required. Figure 2.1 displays the speed of convergence of the methodology for pricing CDO$^2$. The figure displays the amount of dispersion or variance of the expected loss on the parent tranche for various number of simulations. With no variance reduction techniques employed, the proposed Monte Carlo procedure quickly converges for about 5000 simulations. Note also that in usual credit portfolio loss simulations, a loss variable is generated for each obligor in the portfolio which is usually in excess of a hundred. In the case of our method we only generate as many random variables as there are child portfolios which generally does not exceed ten. This fact also speeds up the Monte Carlo routine significantly.

Calculating the Linear Dependence Caused by the Overlap Problem

We now turn to the task of calculating the linear correlation coefficient between the conditional losses of any two child portfolios that arise due to the overlap of common obligors in the portfolios. Let the cumulative loss by time $t$ of child portfolio $A$ be denoted by $L^A(t)$ and the loss for child portfolio $B$ be $L^B(t)$. 
Figure 2.1: Convergence of Monte Carlo Routine. Spreads 100bps, recovery 40%, asset correlation is 30%, obligor notional is 100,000. 5 child portfolios each referencing the tranche [3%-12%]. Parent tranche defined by attachment points [3%-20%]. Each child portfolio has 100 obligors and there is an overlap of 30% i.e. 30 obligors are common to each child portfolio. There are 380 distinct obligors in the parent portfolio.
Suppose there are $N$ obligors in child portfolio $A$ and $M$ obligors in child portfolio $B$. Also assume that the first $K$ obligors are common to both portfolios, but not necessarily with equal notional values. Denote the notional of obligor $i$ belonging to child portfolio $A$ by $l_i^A$ and the notional of obligor $j$ belonging to child portfolio $B$ by $l_j^B$. The loss on child portfolio $A$ by time $t$ can be expressed as:

$$L^A(t) = \sum_{i=1}^{N} 1_{(\tau_i < t)} l_i^A$$

(2.39)

and similarly for portfolio $B$:

$$L^B(t) = \sum_{j=1}^{M} 1_{(\tau_j < t)} l_j^B$$

(2.40)

Since we are considering the conditional loss amounts, the default times are independent for each obligor. The conditional linear correlation coefficient between $L^A(t)$ and $L^B(t)$ is defined as:

$$\text{Corr}(L^A(t), L^B(t) | \bar{M} = \bar{m}) = \frac{E[L^A(t)L^B(t) | \bar{M} = \bar{m}] - E[L^A(t) | \bar{M} = \bar{m}]E[L^B(t) | \bar{M} = \bar{m}]}{\sqrt{\text{Var}[L^A(t) | \bar{M} = \bar{m}]\text{Var}[L^B(t) | \bar{M} = \bar{m}]}$$

(2.41)

In order to compute the correlation coefficient, we must find tractable expressions for each term in expression (2.41). To do this we first denote the conditional default probability of obligor $i$ by time $t$ as $P[\tau_i \leq t | \bar{M} = \bar{m}] = P_i(t | \bar{m})$. In the case of the one factor Gaussian dependency structure we have $P_i(t | \bar{m}) = \Phi\left(\frac{\Phi^{-1}(P_i(t)) - \beta_i \bar{m}}{\sqrt{1-\beta_i^2}}\right)$ where $P_i(t)$ is the unconditional default probability, $\beta_i$ is the correlation parameter and $\bar{m}$ is the realized market factor. It then follows that the conditional expected cumulative loss on child portfolio $A$ by time $t$ is:

$$E[L^A(t) | \bar{M} = \bar{m}] = \sum_{i=1}^{N} P_i(t | \bar{m}) l_i^A$$

(2.42)

and similarly for portfolio $B$

$$E[L^B(t) | \bar{M} = \bar{m}] = \sum_{j=1}^{M} P_j(t | \bar{m}) l_j^B$$

(2.43)
We have assumed that recovery rates are zero for simplicity. If recovery rates where not zero the loss would be simply adjusted by replacing $l_i^A$ with $l_i^A(1 - R_i)$, where $R_i$ is the recovery rate of obligor $i$. Since all obligors default independently, it is straightforward to write down the conditional variance of child portfolios $A$ and $B$:

$$
\text{Var}[L^A(t)|\overline{M} = \overline{m}] = \sum_{i=1}^{N} P_i(t|\overline{m})(1 - P_i(t|\overline{m}))l_i^A
$$

$$
\text{Var}[L^B(t)|\overline{M} = \overline{m}] = \sum_{j=1}^{M} P_j(t|\overline{m})(1 - P_j(t|\overline{m}))l_j^B
$$

In order to obtain an expression for the linear correlation coefficient, all that remains now is to compute $E[L^A(t)L^B(t)|\overline{M} = \overline{m}]$.

$$
E[L^A(t)L^B(t)|\overline{M} = \overline{m}] = E\left[\sum_{i=1}^{N} 1_{\{\tau_i < t\}}l_i^A \sum_{j=1}^{M} 1_{\{\tau_j < t\}}l_j^B|\overline{M} = \overline{m}\right]
$$

The first $K$ obligors are common to both portfolios, i.e. $\tau_i = \tau_j$ for $i = j$, $i \in [0, \ldots, K]$, $j \in [0, \ldots, K]$. Splitting each portfolio into their common and uncommon parts, we have:

$$
E[L^A(t)L^B(t)|\overline{M} = \overline{m}] = E\left[\left(\sum_{i=1}^{K} 1_{\{\tau_i < t\}}l_i^A + \sum_{i=K+1}^{N} 1_{\{\tau_i < t\}}l_i^A\right)\left(\sum_{j=1}^{K} 1_{\{\tau_j < t\}}l_j^B + \sum_{j=K+1}^{M} 1_{\{\tau_j < t\}}l_j^B\right)|\overline{M} = \overline{m}\right]
$$

Expanding the above expression term by term and noting that $E[1_{\{\tau_i < t\}}|\overline{M} = \overline{m}] = P_i(t|\overline{m})$, we have:

$$
E[L^A(t)L^B(t)|\overline{M} = \overline{m}] = \sum_{i=1}^{K} P_i(t|\overline{m})l_i^A l_i^B + \sum_{i=K+1}^{N} P_i(t|\overline{m})l_i^A l_i^B
$$

$$
+ \sum_{i=1}^{K} \sum_{j=K+1}^{M} P_i(t|\overline{m})P_j(t|\overline{m})l_i^A l_j^B
$$

$$
+ \sum_{i=K+1}^{N} \sum_{j=1}^{K} P_i(t|\overline{m})P_j(t|\overline{m})l_i^A l_j^B
$$
Hence, given expressions (2.41)-(2.46), we may compute the conditional linear correlation coefficient.

Summarizing the CDO Squared Pricing Algorithm

In this section we provide the reader with a summary of the CDO\(^2\) pricing algorithm presented in the section above. It is assumed that the CDO\(^2\) references \(N\) underlying child tranches. Note that the order of the list below is intended to maximize algorithm efficiency.

1. Fix a time horizon \(t = t_1\).

2. Let the market factors be represented by the vector \(\bar{M}\). Obtain the discrete set of conditioning values \(\bar{M}_1 = \bar{m}_1, \bar{M}_2 = \bar{m}_2, \ldots, \bar{M}_K = \bar{m}_K\) corresponding to the nodes of integration.

3. For each conditional value of the market factors, \(\bar{M}_j = \bar{m}_j \ j = 1, \ldots, K\), compute the conditional default probability for each obligor. This will result in a vector of conditional default probabilities for each obligor: \(P_i(t_1|\bar{m}_j), \ j = 1, \ldots, K\).

4. Using either the Fourier transform method or recursive algorithm, compute the conditional loss distribution of each child portfolio for all values of the conditional market factors. Hence, each child portfolio will have a different conditional loss distribution for each realization of the market factors. Denote the conditional loss distribution for child portfolio \(v\) at time \(t\) as \(F_{v,i}(t|\bar{m}_j), \ j = 1, \ldots, K\).

5. Using expressions (2.41)-(2.46) calculate the pairwise linear correlation coefficient between each pair of conditional child portfolio losses. Do this for each realization of the market factors. This will result in a correlation matrix for each conditional value of the market factors: \(\sum_{\bar{m}_j}(t_1), \ j = 1, \ldots, K\).
6. Perform Cholesky decomposition on each of the correlation matrices to obtain the set of matrices $\Lambda_{m_j}$ such that $\sum_{m_j}(t_1) = \Lambda_{m_j} \Lambda_{m_j}^T$ for $j = 1, \ldots, K$.

7. Generate $N$ independent Gaussian random variables $g_1, \ldots, g_N$. Produce correlated Gaussian random variables using the linear transformation $\Sigma_j = \Lambda_{m_j} \bar{g}$, $j = 1, \ldots, K$, where $\bar{g} = [g_1, \ldots, g_N]^T$ and $\Sigma_j = [x_{1j}, \ldots, x_{Nj}]^T$.

8. Map the correlated random variables to conditional child portfolio loss values using the mapping $Y_{ij}(t_1) = F_{x_{1j}}^{-1}(\Phi(x_{1j} | \bar{m}_j)), \ldots, Y_{Nj}(t_1) = F_{x_{Nj}}^{-1}(\Phi(x_{Nj} | \bar{m}_j))$, $j = 1, \ldots, K$, where $F_{x_{v}^{-1}}(\bar{m}_j)$ is the inverse cumulative conditional loss distribution for child portfolio $v$ at time horizon $t_1$ and $\Phi$ is the cumulative Gaussian distribution function.

9. Use expressions (2.33) to (2.35) to calculate the conditional loss on the parent tranche.

10. Repeat steps 7-9 many times to obtain the expected conditional parent tranche loss $E[L^{P\text{trch}}(t_1) | \bar{M} = \bar{m}_j]$, $j = 1, \ldots, K$.

11. Integrate the conditional expected parent tranche loss with respect to the distribution of market factors to obtain the unconditional expected tranche loss $E[L^{P\text{trch}}(t_1)] = \sum_{j=1}^{K} E[L^{P\text{trch}}(t_1) | \bar{M} = \bar{m}_j] W(\bar{m}_j)$ where $W(\bar{m}_j)$ represents the weight of integration corresponding to the node $\bar{m}_j$ and is dependent on the type of numerical integration used.

12. Repeat steps 1-11 for time horizons $t = t_2, \ldots, t_m$ such that we obtain the parent tranche expected loss for a series of time horizons $E[L^{P\text{trch}}(t)]$, $t = t_1, \ldots, t_m$.

13. Use expressions (2.36) and (2.37) to compute the parent tranche premium.

**Numerical Results**

In this section we provide numerical results for the new CDO$^2$ pricing algorithm proposed above which, for the purposes of reporting the results, we call the semi-Monte Carlo (semi-MC) method. Since CDO$^2$ spreads are simply a function
of the expected loss on the master tranche, the output we compare is the expected tranche loss on the master tranche. For the computation of the master tranche expected loss, the following assumptions where made: The underlying factor model was taken to be the one factor Gaussian copula and that the master tranche referenced five different underlying CDO tranches. Each underlying child portfolio consisted of 100 obligors each with fixed recovery rates of 40%. It was also assumed that all obligors had the same spread equal to 100bps, and same notional amounts equal to 100,000. The first $K$ obligors for each child portfolio are assumed to be the same which means that the percentage overlap of any two child portfolios is $K\%$. The attachment and detachment points for the underlying tranches are the same for each child tranche. Given this setup, the semi-MC method was implemented to compute the expected master tranche loss for a time horizon of one year and for varying levels of obligor overlap and various attachment and detachment points for the underlying tranches. The results are compared to a full Monte Carlo (MC) implementation which simulated the default times of each obligor using the one factor Gaussian copula model. Antithetic variate reduction technique was implemented and 100,000 simulations were conducted to get the expected tranche loss using the full Monte Carlo approach. Tables 2.1 to 2.6 display the results. It is useful to note that since all obligors have the same notional, spread and recovery rates and that the degree of overlap is the same across all child portfolios, then the linear correlation coefficient between the conditional losses of the child portfolios (as given by the expressions 2.41-2.46) is equal to the percentage overlap. For example if there are 20 obligors common to all child portfolios (each child portfolio has 100 obligors) then the percentage overlap is 20% which means that the linear correlation as given by expressions 2.41-2.46 is also 20%.

### 2.4 Quoting Implied Correlations

Standardized liquid single tranche CDOs have given practitioners the opportunity to calibrate their credit portfolio models to quoted prices. In particular, quoted prices are being used to imply out the asset correlation parameter that is a key
CHAPTER 2. FACTOR MODELS AND IMPLIED CORRELATIONS

<table>
<thead>
<tr>
<th>Master Tranche</th>
<th>Expected Loss (MC)</th>
<th>Expected Loss (semi-MC)</th>
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</thead>
<tbody>
<tr>
<td>Equity 0%-3%</td>
<td>94.13%</td>
<td>92.49%</td>
</tr>
<tr>
<td>Junior Mezzanine 3%-6%</td>
<td>87.80%</td>
<td>87.15%</td>
</tr>
<tr>
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<tr>
<td>Senior 9%-12%</td>
<td>75.46%</td>
<td>76.18%</td>
</tr>
<tr>
<td>Super Senior 12%-22%</td>
<td>61.22%</td>
<td>62.36%</td>
</tr>
</tbody>
</table>

Table 2.1: Expected Loss on Master Tranche. Underlying tranche attachment/detachment points [0%-3%]. Percentage overlap=30%.

<table>
<thead>
<tr>
<th>Master Tranche</th>
<th>Expected Loss (MC)</th>
<th>Expected Loss (semi-MC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity 0%-3%</td>
<td>9.58%</td>
<td>9.63%</td>
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<tr>
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<tr>
<td>Senior 9%-12%</td>
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<td>2.96%</td>
</tr>
<tr>
<td>Super Senior 12%-22%</td>
<td>1.72%</td>
<td>1.75%</td>
</tr>
</tbody>
</table>

Table 2.2: Expected Loss on Master Tranche. Underlying tranche attachment/detachment points [3%-9%]. Percentage overlap=30%.

<table>
<thead>
<tr>
<th>Master Tranche</th>
<th>Expected Loss (MC)</th>
<th>Expected Loss (semi-MC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity 0%-3%</td>
<td>0.04%</td>
<td>0.05%</td>
</tr>
<tr>
<td>Junior Mezzanine 3%-6%</td>
<td>0.02%</td>
<td>0.02%</td>
</tr>
<tr>
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<td>0.01%</td>
<td>0.01%</td>
</tr>
<tr>
<td>Senior 9%-12%</td>
<td>0.01%</td>
<td>0.01%</td>
</tr>
<tr>
<td>Super Senior 12%-22%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Table 2.3: Expected Loss on Master Tranche. Underlying tranche attachment/detachment points [9%-22%]. Percentage overlap=30%.
Master Tranche | Expected Loss (MC) | Expected Loss (semi-MC) 
---|---|---
Equity 0%-3% | 86.26% | 84.87% 
Junior Mezzanine 3%-6% | 78.41% | 79.78% 
Senior Mezzanine 6%-9% | 72.03% | 75.10% 
Senior 9%-12% | 67.13% | 70.83% 
Super Senior 12%-22% | 60.47% | 59.40% 

Table 2.4: Expected Loss on Master Tranche. Underlying tranche attachment/detachment points [0%-3%]. Percentage overlap=70%.

Master Tranche | Expected Loss (MC) | Expected Loss (semi-MC) 
---|---|---
Equity 0%-3% | 7.38% | 7.58% 
Junior Mezzanine 3%-6% | 5.03% | 5.13% 
Senior Mezzanine 6%-9% | 3.67% | 3.69% 
Senior 9%-12% | 3.01% | 2.99% 
Super Senior 12%-22% | 2.05% | 2.00% 

Table 2.5: Expected Loss on Master Tranche. Underlying tranche attachment/detachment points [3%-9%]. Percentage overlap=70%.

Master Tranche | Expected Loss (MC) | Expected Loss (semi-MC) 
---|---|---
Equity 0%-3% | 0.05% | 0.04% 
Junior Mezzanine 3%-6% | 0.02% | 0.02% 
Senior Mezzanine 6%-9% | 0.01% | 0.01% 
Senior 9%-12% | 0.01% | 0.01% 
Super Senior 12%-22% | 0.00% | 0.00% 

Table 2.6: Expected Loss on Master Tranche. Underlying tranche attachment/detachment points [9%-22%]. Percentage overlap=70%.
component of structural portfolio models. Although not yet fully established, it appears that the standard agreed upon model used for backing-out implied asset correlations is the Gaussian one factor copula model. Due to its simplicity and ease of implementation the Gaussian one factor copula model has become analogous to the Black-Scholes model used for quoting implied volatilities in the equity market.

Two different approaches have been developed to imply out asset correlations using the Gaussian one factor copula model. The two different methodologies lead to different values of implied correlation known as 'Compound Correlation' and 'Base Correlation'. Note that since we are trying to imply a unique correlation value both compound and base correlation assume a flat correlation structure.

2.4.1 Compound Correlation

Compound correlation is defined as the single flat correlation value that gives a certain single CDO tranche a market value of zero using the one factor Gaussian copula model to compute the market value. For example, suppose we have a generic credit portfolio model, where each obligor in the portfolio defaults once its asset returns falls below a certain threshold. In order to calculate the present value of a single tranche CDO we need the following information:

- $c_1$: Tranche lower attachment point.
- $c_2$: Tranche upper detachment point.
- $Spread$: Vector of credit spreads, where each element $i$ represents the credit spread of obligor $i$.
- $g$: Linear correlation coefficient of asset returns.
- $D(t, T)$: Discount factor for a starting date $t$ and maturity $(T - t)$.
- $Recovery$: Vector of recovery rates, where element $i$ represents the constant or stochastic recovery rate of obligor $i$.  

• \textit{coupon}[^c1, c2]: The coupon paid to the protection seller for the tranche \([c1, c2]\).

Given the above information the PV of a single tranche CDO can be compactly expressed by the functional form:

\[
\text{TranchePV}(c1, c2, \text{Spread}, \text{Recovery}, g, D(t, T), \text{coupon}[^c1, c2])
\]

The compound correlation is simply the correlation value, \(g\), that produces a tranche PV of zero.

2.4.2 Base Correlation

Base correlation computes the compound correlation for a series of equity tranches where the lower attachment point is zero, hence the name 'base correlation'. In order to calculate base correlation for a series of equity tranches a bootstrapping methodology is used. For example, providing protection on a tranche defined by the strikes \(c1\) and \(c2\), where \(c2 > c1\), is equivalent to providing protection on an equity tranche with upper detachment point \(c2\) and buying protection on an equity tranche with upper detachment point \(c1\). Computation of the base correlation then proceeds as follows:

1. Find the value of \(g_{c1}\) such that the base tranche \(c1\) has zero PV:

\[
\text{TranchePV}(0, c1, \text{Spread}, \text{Recovery}, g_{c1}, D(t, T), \text{coupon}[0, c1]) = 0
\]

2. Find the value of \(g_{c2}\) such that the tranche \([c1, c2]\) has zero PV by decomposing the tranche into two equity tranches:

\[
\text{TranchePV}(0, c2, \text{Spread}, \text{Recovery}, g_{c2}, D(t, T), \text{coupon}[c1, c2])
\]

\[
-\text{TranchePV}(0, c1, \text{Spread}, \text{Recovery}, g_{c1}, D(t, T), \text{coupon}[c1, c2]) = 0
\]

where \(g_{c1}\) was found in step 1. \(g_{c1}\) and \(g_{c2}\) are the base correlations corresponding to the equity tranches \([0, c1]\) and \([0, c2]\) respectively.
2.4.3 Advantages and Disadvantages of Base vs Compound Correlation

Using the one factor Gaussian copula model to calculate compound correlation results in an implied smile where the correlation for equity and senior tranches are higher than the compound correlation for mezzanine tranches. Using the one factor Gaussian copula model to calculate base correlations generally results in a correlation skew where implied correlations increase monotonically as the upper detachment point is increased. The cause of the implied correlation smile/skew is due to the inadequacy of the Gaussian flat correlation dependency structure to model the actual dependence structure of the portfolio. Figure 2.2 shows the compound correlation smile and base correlation skew typically observed in the market. Note that for the equity tranche which has zero attachment point, both compound and base correlation are equivalent.

At first glance compound correlation seems more appealing because it is intuitively easy to understand whereas base correlation is relatively more complex.
due to its bootstrapping methodology. Despite this, the popularity of base correlation is growing due to a number of distinct advantages over the compound correlation method. The most striking disadvantages of compound correlation are:

1. There are often two solutions for mezzanine tranches.

2. Not possible to extend compound correlation for the pricing of non-standard tranches on standard portfolios of CDS's.

Base correlation does not suffer from either of these two disadvantages. Firstly because base correlation refers to equity tranches, and the PV of an equity tranche is in general a monotonic function with respect to correlation, base correlation has only one solution in most cases. The reason for the uniqueness (in most cases) of base correlations can be seen by analyzing the pricing formula for the premium and default leg of synthetic tranche given by equations (2.26) and (2.27). Both expressions consist of the expected loss on a tranche for a series of time intervals and the expected loss on a base tranche is guaranteed to be a monotonic function with respect to correlation. This ensures that the premium leg given by (2.27) is also a strictly monotonic function with respect to correlation. However, the default leg, given by (2.26), which is composed of the difference between the expected loss on a tranche for consecutive time intervals is not guaranteed to be strictly monotonic as the difference between the expected loss on a tranche for two consecutive time intervals is not guaranteed to be strictly monotonic with respect to correlation. While this means that the value or par spread of a base tranche is not strictly monotonic with respect to correlation, in almost all practical cases base correlations are unique. Note that in some cases there may be no solution. The second advantage of base correlations is that they are functions of the upper detachment point (the lower attachment point is zero) and so non-standard tranches can be priced by interpolating or extrapolating the base correlations calculated from standard tranches.

A further advantage of the base correlation framework is that the sum of the expected losses on the individual tranches that span the whole capital struc-
ture is equal to the expected loss on the entire portfolio. For example consider the tranches with attachment and detachment points \([0\% - X_1\%], [X_1\% - X_2\%], [X_2\% - X_3\%], \) and \([X_3\% - 100\%]\). Using the base correlation framework, the expected loss on the tranche \([X_1\% - X_2\%]\) is calculated via the expression

\[
E_{\{X_1\% - X_2\%\}}[L] = E_{\{X_2\%\}}[\min(L, X_2)] - E_{\{X_1\%\}}[\min(L, X_1)],
\]

where \(E_{\{X_i\%\}}\) indicates that the expectation is calculated using the correlation \(\varrho(X_i)\) corresponding to the base tranche \([0\% - X_i\%]\). We then have for \(X_0\% = 0\%\) and \(X_4\% = 100\%\),

\[
\sum_{i=0}^{3} E_{\{X_i\% - X_{i+1}\%\}}[L] = \sum_{i=0}^{3} E_{\{X_{i+1}\%\}}[\min(L, X_{i+1})] - E_{\{X_i\%\}}[\min(L, X_i)]
\]

\[
= E_{\{X_{100}\%\}}[\min(L, X_{100})]
\]

\[
= E[L]
\]

since the expected loss on the entire portfolio is independent of correlation. This property of base correlation is not valid for the compound correlation framework, i.e. \(\sum_{i=0}^{3} E_{\{X_i\% - X_{i+1}\%\}}[L] \neq E[L]\).

Despite the apparent attractiveness of base correlation it too has some serious drawbacks as outlined in a recent paper by Willemann (2004). Willemann finds some fundamental flaws in the use of base correlation as a method to quote implied correlations. The first inconsistency to note is that increasing the degree of correlation between the obligors does not necessarily translate to an increase in the base correlation of tranches with high detachment points and may even result in a decrease in base correlation. The reason for this is due to the bootstrapping methodology of base correlation. While increasing the underlying obligor correlation will result in an increase in base correlation for tranches with low detachment points and may even result in a decrease in base correlation, base correlation for higher detachment points (which are bootstrapped from lower detachment point base correlations) may actually decrease as the increase in the underlying obligor correlation has already been captured by the base correlation of tranches with low detachment points.

Another danger with base correlation arises when base correlations are interpolated to price nonstandard tranches. It is possible that the tranche priced by interpolation has significant error relative to the true model tranche spread.
and in some cases for extremely steep skews introduce arbitrage by imposing a negative expected loss on the interpolated tranche. In order to demonstrate this we consider tranches with a width of only 1% called tranchlets. For example the small tranche defined by the attachment and detachment points [3%-4%] is classified as a tranchlet. In order to avoid arbitrage in a CDO model one requirement is that tranchlet break even spreads are strictly decreasing with respect to the attachment point. So, the break even spread on the tranchlet [x%-x+1%] should be strictly greater than the break even spread on the tranchlet [z%-z+1%] for z% strictly greater than x%. However this condition is usually violated when interpolating base correlations to price tranches. In figure 2.3 we plot the break even spread for a series of tranchlets using the base correlation methodology. It should first be noted that the spreads are not strictly decreasing. Secondly the spreads exhibit sudden downward drops in value rather than following a smooth curve. In order to remedy this situation we tried different interpolation schemes other than linear interpolation. Both cubic and quadratic splines do give smoother curves but do not solve the arbitrage issue relating to strictly decreasing spreads. The base correlation skew used in the construction of the tranchlet prices in figure 2.3 was a rather moderate skew were the following correlations; [19%, 28%, 34%, 39%, 48%], corresponded to the following detachment points; [3%, 6%, 9%, 12%, 22%]. If we used more extreme base correlation skews such as [47%, 68%, 80%, 88%, 99%] corresponding to the same detachment points, then the situation is far more severe as quite a few of the tranchlet spreads become negative as demonstrated in figure 2.4. Hence the base correlation methodology is far from a coherent framework for pricing CDO tranches.

Although base correlations are, unlike compound correlations, unique, the actual base correlation figure will depend on the set of detachment points used for bootstrapping. For example, suppose there exits the two sets of tranche prices for the following percentage attachment and detachment points [0% - 3%, 3% - 10%, 10% - 15%], [0% - 5%, 5% - 7%, 7% - 15%] on a given reference portfolio. Applying the base correlation bootstrapping method will result in different implied correlations for the 15% detachment point which is common to both sets. This means that base correlations depend on the detachment points.
Figure 2.3: Tranchlet spreads using moderate base correlation skew

Figure 2.4: Tranchlet spreads using steep base correlation skew
2.4.4 The Relation between Base Correlation and Implied Loss Distribution

The relation between base correlations and the implied loss distribution was first noted by Turc et al (2005). If there exists prices for 'European' base tranche CDOs for all detachment points, then it would be possible to compute the implied loss distribution from the base correlation skew. By 'European' CDO, we mean a CDO where the protection buyer pays a one off upfront premium and makes no further periodic coupon payments. The protection seller, on the other hand, pays the total losses suffered on the tranche at the maturity of the deal to the protection buyer. There are no intermediate cash flows during the life of the trade.

Assuming a detachment point of $D$ and independence between interest rates and portfolio losses, the fair upfront premium of a 'European' base tranche can be expressed as:

$$C = D(0, T)\mathbb{E}^Q[\min(L_T, D)]$$

where $C$ is the upfront premium, $D(t, T)$ is the discount factor at time $t$ for maturity $T - t$, and we have assumed a deal maturity of $T$. $L_T$ represents the total cumulative portfolio loss up to time $T$. The expectation is taken under the risk neutral measure, $Q$. Explicitly writing the expectation in terms of integrals, we have:

$$\mathbb{E}^Q[\min(L_T, D)] = \int_0^D l f_{LT}(l)dl + D(1 - F_{LT}(D))$$

(2.47)

where

$$\frac{dF_{LT}(l)}{dl} = f_{LT}(l)$$

and

$$f_{LT}(l) = \frac{P[l < L_T \leq l + dl]}{dl}$$

Using integration by parts, equation (2.47) can be rewritten as:
\[ E^Q[\min(L_T, D)] = D - \int_0^D F_{L_T}(l)dl \quad (2.48) \]

Differentiating (2.48) with respect to the detachment point \( D \), we have:

\[ \frac{\partial E^Q[\min(L_T, D)]}{\partial D} = 1 - F_{L_T}(D) \quad (2.49) \]

Alternatively, consider computing the expected tranche loss using the Gaussian copula with a base correlation parameter of \( \rho(D) \). \( \rho(D) \) represents the base correlation skew function where the argument \( D \) represents the detachment point. It is assumed that this function is chosen so that the one factor Gaussian copula model reprices all 'European' base tranches quoted in the market. Mathematically, this can be expressed as:

\[ C = D(0, T)E^{GG(\rho(D))}[\min(L_T, D)] \quad (2.50) \]

where \( E^{GG(\rho(\cdot))}[\cdot] \) indicates that the expectation is calculated using the Gaussian copula with correlation parameter \( \rho(D) \). Since we have calibrated to the observed premium, \( C \), the expectation computed using the one factor Gaussian copula is the same as the expectation computed using the risk neutral loss distribution:

\[ E^Q[\min(L_T, D)] = E^{GG(\rho(D))}[\min(L_T, D)] \quad (2.51) \]

Differentiating the Gaussian copula expectation term with respect to the detachment point, we have:

\[ \frac{\partial E^{GG(\rho(\cdot))}[\min(L_T, D)]}{\partial D} = \frac{\partial}{\partial D} \left( \int_0^D f_{L_T}^{GG(\rho(D))}(l)dl + D(1 - F_{L_T}^{GG(\rho(D))}(D)) \right) \quad (2.52) \]

where \( f_{L_T}^{GG(\rho(D))}(l) \) and \( F_{L_T}^{GG(\rho(D))}(l) \) represents respectively the marginal and cumulative loss density produced by the one factor Gaussian copula model. Again, using integration by parts, expression (2.52) can be decomposed to produce the following:

\[ \frac{\partial E^{GG(\rho(\cdot))}[\min(L_T, D)]}{\partial D} = \frac{\partial}{\partial D} \left( D - \int_0^D F_{L_T}^{GG(\rho(D))}(l)dl \right) \]
\[
F_{LT}^{GC(\rho(D))} (D) = 1 - \int_0^D \frac{\partial F_{LT}^{GC(\rho(D))} (l)}{\partial D} dl
\]

\[
= 1 - F_{LT}^{GC(\rho(D))} (D) - \frac{\partial \rho(D)}{\partial D} \int_0^D \frac{\partial F_{LT}^{GC(\rho(D))} (l)}{\partial \rho} dl
\]

(2.53)

It is also the case that we have:

\[
\frac{\partial E^{GC(\rho(D))}[\min(L_T, D)]}{\partial \rho} = - \int_0^D \frac{\partial F_{LT}^{GC(\rho(D))} (l)}{\partial \rho} dl
\]

(2.54)

Combining expressions (2.49), (2.51), (2.53), and (2.54), we conclude that the relation between the market implied loss distribution and the base correlation skew is given by expression (2.55):

\[
F_{LT}(D) = F_{LT}^{GC(\rho(D))} (D) - \frac{\partial \rho(D)}{\partial D} \frac{\partial E^{GC(\rho(D))}[\min(L_T, D)]}{\partial \rho}
\]

(2.55)

2.5 Conclusion

In this chapter we presented the conditional independence approach to pricing portfolio credit derivatives, namely single tranche CDOs, Nth-to-default baskets and CDO squared transactions. Such an approach allows semi-analytical pricing which involves only numerical integration and avoids noisy Monte Carlo simulation. A new methodology was introduced to price CDO\(^2\) which is a mixture of the conditional independence approach and Monte Carlo simulation and is significantly faster than existing approaches to pricing CDO\(^2\). The new methodology approximates the dependence caused by obligor overlaps across the child portfolios via an equivalent linear dependence structure using the Gaussian copula. Finally the concept of implied compound and base correlation skew was introduced and a comparative analysis made of the two methodologies. It was also shown that the widely used base correlation framework is not arbitrage free and it is vital to develop skew producing models to price portfolio credit derivatives in an arbitrage free manner. The remainder of this thesis is devoted to the development of correlation skew producing models within the conditional independence
framework.
Chapter 3

Stochastic Correlation Model

This chapter introduces a novel stochastic correlation model that is capable of producing the base correlation skew observed in the single tranche CDO market. It is the first structural portfolio credit model of its kind that incorporates stochastic correlation. Section 3.1 of this chapter lists some Gaussian integrals that are used throughout this thesis. Section 3.2 introduces the general framework of the stochastic correlation model and we further divide this model class into two subclasses called Type 1 and Type 2, where in the Type 1 case the default times are independent conditional on the realization of two random variables whereas in the Type 2 case the default times of the obligors are independent conditional on the realization of just one random variable. Explicit copulae are derived for the Type 1 and Type 2 cases and it is shown how to simulate the default times from these copulae. Section 3.3 presents a specific and tractable example of the general framework and the behavior of the tranche prices are examined with respect to the model parameters. A number of useful analytic results are also presented that provides a measure of the degree of dependence between the asset value of two obligors. In section 3.4 and 3.5 we derive the formula for the large homogeneous portfolio limit loss distribution and expected tranche loss for the Type 1 model which permits very fast pricing of CDO tranches using a skew producing model. Finally, section 3.6 provides a conclusion to the chapter.
3.1 Useful Gaussian Integrals

Proofs of the integral values can be found in Appendix B. The following notation will be used:

- $\phi(y)$ - Standard normal density function.
- $\Phi(y)$ - Standard normal cumulative distribution function.
- $\Phi(y, m; \beta)$ - Bivariate standard normal cumulative distribution function with correlation coefficient $\beta$.
- $a, b, c, d$ are real constants.
- $E(\cdot)$ denotes the expectation of a random variable.
- $Y$ is a standard normal random variable.

Integral 1

$$ \int_{-\infty}^{\infty} \Phi(ay + b)\phi(y)dy = \Phi\left(\frac{b}{\sqrt{1 + a^2}}\right) $$

Integral 2

$$ \int_{-\infty}^{\infty} \Phi(ay + b)\Phi(cy + d)\phi(y)dy = \Phi\left(\frac{b}{\sqrt{1 + a^2}}, \frac{d}{\sqrt{1 + c^2}}, \frac{ac}{\sqrt{1 + a^2\sqrt{1 + c^2}}}\right) $$

Integral 3 Define $\alpha = \frac{a}{\sqrt{1 + a^2}}$. Then

$$ E[Y\Phi(ay + b)] = \alpha \phi(b\sqrt{1 - \alpha^2}) $$

Integral 4 Define $\alpha = \frac{a}{\sqrt{1 + a^2}}$. Then

$$ E[Y^2\Phi(ay + b)] = \frac{-b\alpha}{\sqrt{1 + a^2}} E[Y\Phi(ay + b)] + \Phi\left(\frac{b}{\sqrt{1 + a^2}}\right) $$

Integral 5

$$ E[Y^2\Phi(ay + b)\Phi(cy + d)] = \Phi\left(\frac{b}{\sqrt{1 + a^2}}, \frac{d}{\sqrt{1 + c^2}}, \frac{ac}{\sqrt{1 + a^2\sqrt{1 + c^2}}}\right) $$

$$ + \frac{a}{\sqrt{2\pi}} \frac{e^{-\frac{\eta^2(1-\alpha^2)}{2}}}{1 + a^2} \left[\phi(\omega\sqrt{1 - \eta^2}) - b\alpha \Phi\left(\frac{\omega}{\sqrt{1 + \eta^2}}\right)\right] $$
CHAPTER 3. STOCHASTIC CORRELATION MODEL

\[ c - \frac{e^{-\frac{\psi}{2}(1-\gamma^2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\psi^2}{2}} \left[ \psi \phi\left(\frac{\kappa}{\sqrt{1 + \psi^2}} - d\phi\left(\frac{\kappa}{\sqrt{1 + \psi^2}}\right)\right) \right] \]

Where we have \( \alpha = \frac{a}{\sqrt{1+\psi^2}}, \eta = \frac{\psi}{\sqrt{1+\psi^2}}, \vartheta = \frac{\psi}{\sqrt{1+\psi^2}}, \omega = \frac{-\vartheta c}{\sqrt{1+\psi^2}} + d, \) and \( \gamma = \frac{c}{\sqrt{1+\psi^2}}. \)

\[ \zeta = \frac{a}{\sqrt{1+\psi^2}}, \psi = \frac{c}{\sqrt{1+\psi^2}}, \kappa = \frac{-\vartheta c}{\sqrt{1+\psi^2}} + b. \]

Integral 6

\[ \int_{-\infty}^{\infty} \Phi(ay + b)\phi(y)dy = \Phi\left(\frac{b}{\sqrt{1 + a^2}}, c; \frac{-a}{\sqrt{1 + a^2}}\right) \]

3.2 The General Framework

Recall the one factor Gaussian copula model introduced in chapter 2 where the asset value of each obligor takes the form \( A_i = \rho M + \sqrt{1 - \rho^2} \varepsilon_i, \) and where \( M \) is the market factor, \( \varepsilon_i \) is the idiosyncratic noise term, and \( \rho \) is the correlation coefficient. Both \( M \) and \( \varepsilon_i \) are assumed to be independent standard Gaussian random variables and an obligor defaults if its asset value falls below some threshold level. This model is popular because only one parameter, \( \rho, \) needs to be calibrated other than the default thresholds. The correlation parameter \( \rho \) can now be implied from the market quotes of single tranche CDOs and the one factor Gaussian model has become a market standard model just like the Black-Scholes model used in equity derivatives. Implementation of the model also avoids Monte-Carlo simulation, since by conditioning on the market factor all obligors default independently with conditional default probabilities equal to \( P_i(t|M = m) = \Phi\left(\frac{C_i - \rho m}{\sqrt{1 - \rho^2}}\right), \) where \( C_i = \Phi^{-1}(P_i(t)) \) is the default threshold of obligor \( i \) and \( m \) is the conditioning variable. As a result of this, either the Fourier Transform method or the recursive algorithm presented in chapter 2 may be used to compute the portfolio loss distribution. The former approach involves calculating the single obligor conditional loss distribution given a certain realization of the market factor. The conditional portfolio loss distribution is calculated by taking the convolution of the individual loss distributions, or equivalently by multiplying their Fourier transforms. Denoting the conditional portfolio loss distribution by \( f_{L(t|M)}(l|m), \) the unconditional loss distribution is obtained by
integrating the conditional loss distribution with respect to the probability distribution of the market factor: 
\[ f_{L(t)}(l) = \int f_{L(t)|M}(l|m)f_M(m)dm, \]
where \( f_M \) is the distribution of the market factor. The Fourier Transform method should be used if recovery rates are assumed to be stochastic. The recursive algorithm, on the other hand, is simple to implement for the case of constant recovery rates such that the loss given default of obligor \( i \) is equal to \( z_i = N_i(1 - R_i) \), where \( N_i \) is the obligor notional and \( R_i \) is its recovery rate. Assuming that we have calculated the conditional loss distribution, \( f^X_{L(t)|M}(l|m) \), for some portfolio denoted by \( \chi \), then adding a new obligor, denoted by \( i \), to the portfolio will result in a modified loss distribution \( f^{X+i}_{L(t)|M}(l|m) \) such that the following relationship holds:
\[ f^{X+i}_{L(t)|M}(l|m) = f^X_{L(t)|M}(l|z_i = m)P_i(t|M = m), \]
for \( l \geq z_i \), otherwise \( f^{X+i}_{L(t)|M}(l|m) = f^X_{L(t)|M}(l|m)(1 - P_i(t|M = m)) \), where \( P_i(t|M = m) \) is the conditional probability that obligor \( i \) will default by time \( t \). Using this relationship, the conditional portfolio loss distribution may be calculated starting with the base case of the empty portfolio and adding one obligor at a time to the portfolio.

However, despite its popularity, the one factor Gaussian model fails to price all tranches simultaneously. The aim of this chapter is to extend the one factor Gaussian model to better fit all tranche prices simultaneously while still retaining its analytical tractability. This is accomplished by assuming the correlation parameter \( \rho \) to be stochastic. The inspiration for making the correlation stochastic came from the stochastic volatility models that extend the Black-Scholes model to better fit the range of equity option prices. The stochastic correlation model is again a factor model, where conditional on the market factor and the correlation parameter, all obligors default independently. We now proceed to develop the model of which we specify two different forms, called Type 1 and Type 2.

### 3.2.1 Stochastic Correlation Model: Type 1

The asset value of each obligor in the portfolio can be represented as:
\[ A_i = \frac{1}{\sigma_i}(g_i(Y)M + \sqrt{1 - g_i(Y)^2}\varepsilon_i - \mu_i) \]  
(3.1)
where $M$ is the common market factor with distribution $F_M$, and $\varepsilon_i$ is the idiosyncratic risk with distribution $F_{\varepsilon_i}$. $M$ and $\varepsilon_i$ are independent of one and other. $g_i(Y)$ is the stochastic factor coefficient. $Y$ is a random variable with distribution $F_Y$ that is also linked to the common factor $M$ via the joint distribution $F_{MY}$. $g_i(\cdot)$ is a monotone mapping function $g_i : \mathbb{R} \mapsto [0, 1]$. The constants $\mu_i$ and $\sigma_i$ are introduced so that the asset value has zero mean and unit variance. The rationale behind the model is that at times of market depression the obligors joint default probability increases while at times of market prosperity the default risk of each firm is largely driven by each firm's idiosyncratic risk. The model presented in (3.1) is similar in spirit to the model presented in Anderson and Sidenius (2004) but the model presented there sets the factor coefficients as deterministic functions of the market factors. It is also the case that in their model the factor coefficients are discontinuous step functions such as $g(x) = \alpha_11_{\{x \leq k\}} + \alpha_21_{\{x > k\}}$. In this paper we assume that the factor coefficients are linked to the market factor via a joint distribution function which is continuous and smooth. This introduces an extra parameter which facilitates calibration of the model and increases its flexibility.

Returning back to the model where the asset value is given by (3.1), it is assumed that obligor $i$ defaults once its asset value falls below a threshold $C_i$. The conditional default probability given $M$ and $Y$ can be expressed by:

$$P(A_i \leq C_i|M = m, Y = y) = P\left(\frac{1}{\sigma_i}\left(g_i(y)m + \sqrt{1 - g_i(y)^2}\varepsilon_i - \mu_i\right) \leq C_i\right) = P\left(\varepsilon_i \leq \frac{\sigma_iC_i + \mu_i - g_i(y)m}{\sqrt{1 - g_i(y)^2}}\right) = \mathcal{F}_\varepsilon\left(\frac{\sigma_iC_i + \mu_i - g_i(y)m}{\sqrt{1 - g_i(y)^2}}\right) \quad (3.2)$$

The unconditional default probability is given by the following expression:

$$P[A_i \leq C_i] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{F}_\varepsilon\left(\frac{\sigma_iC_i + \mu_i - g_i(y)m}{\sqrt{1 - g_i(y)^2}}\right) f_{MY}(m, y) \, dm \, dy \quad (3.3)$$

where $f_{MY}$ is the joint density function of $M$ and $Y$.

Given expression (3.3) one may find the threshold, $C_i$, associated with a given default probability using a simple root searching algorithm. So once a suitable
joint density function, \( f_{MY}(m, y) \), and a mapping function \( g(y) \), is decided upon, the thresholds \( C_i \) may be computed and the conditional independence framework invoked to calculate the portfolio loss distribution. The conditioning variables in this case are the random variables \( Y \) and \( M \). Once the cumulative portfolio loss distribution is calculated for a series of time horizons, the price of any CDO tranche may be computed. Chapter 2 gives a detailed description of how to price a CDO tranche given the cumulative portfolio loss distribution for a series of time intervals. It effectively involves calculating the cumulative tranche expected loss for time intervals defined on an appropriate grid.

The copula corresponding to this setup is given by the following proposition:

**Proposition 1** The copula corresponding to the dependence structure defined by the stochastic correlation model Type 1 is:

\[
C_1(u_1, \ldots, u_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} F_{\xi^i} \left( \sigma_i \gamma_i^{-1}(u_i) + \mu_i - g_i(y) m \right) f_{MY}(m, y) dm dy
\]

where \( \gamma_i(x) = P[A_i \leq x] \) and is given by expression (3.3) and \( \gamma_i^{-1}(\cdot) \) is the inverse function of \( \gamma_i(\cdot) \).

**Proof:**

\[
P[A_1 \leq x_1, \ldots, A_n \leq x_n] = E[P[A_1 \leq x_1, \ldots, A_n \leq x_n|M = m, Y = y]]
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} F_{\xi^i} \left( \sigma_i x_i + \mu_i - g_i(y) m \right) f_{MY}(m, y) dm dy
\]

and the result follows from an application of Sklar’s Theorem.

The default time of obligor \( i \) is obtained by sampling the asset value \( A_i \) and using the mapping given by expression (3.4).

\[
\tau_i = F_i^{-1}(\gamma_i(A_i))
\]

where \( \gamma_i(x) = P[A_i \leq x] \) is given by expression (3.3) and \( F_i^{-1}(\cdot) \) is the inverse of the function \( F_i(t) = P[\tau_i \leq t] \).
3.2.2 Stochastic Correlation Model: Type 2

In this case the asset value of each obligor is given by:

\[ A_i = \frac{1}{\sigma_i} \left( g_i(Y_i)M + \sqrt{1 - g_i(Y_i)^2} \varepsilon_i - \mu_i \right) \]  

(3.5)

Again we have that \( M \) has the distribution function \( F_M \), \( \varepsilon_i \) is the idiosyncratic risk with distribution \( F_\varepsilon \), and \( Y_i \) is linked to the common factor via the joint distribution \( F_{MY_i} \). Although this looks very similar to the Type 1 model asset value specified by expression (3.1), there is a subtle difference which is that the stochastic correlation driver, \( Y_i \), is now specific to each obligor. Hence, \( Y_i \) and \( Y_j \) have respective distributions \( F_{Y_i} \) and \( F_{Y_j} \). In this model the correlation for each obligor is linked to the market factor but it is ‘noisy’ in that the correlation for each obligor fluctuates randomly and independently around some common level. All obligors default independently if we condition on the market factor \( M \) (previously in the Type 1 model we had to condition on both \( M \) and \( Y \) for the default times to be independent). The conditional default probability is given by the following expression:

\[ P(A_i \leq C_i|M = m) = \int_{-\infty}^{\infty} F_\varepsilon \left( \frac{\sigma_i C_i + \mu_i - g_i(y_i)m}{\sqrt{1 - g_i(y_i)^2}} \right) f_{Y_i|M}(y_i|m)dy_i \]  

(3.6)

And the unconditional default probability is given by:

\[ P(A_i \leq C_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_\varepsilon \left( \frac{\sigma_i C_i + \mu_i - g_i(y_i)m}{\sqrt{1 - g_i(y_i)^2}} \right) f_{Y_i|M}(y_i|m)dy_i f_M(m)dm \]  

(3.7)

The copula corresponding to the Type 2 stochastic correlation model is given by the following proposition:

**Proposition 2** The copula corresponding to the dependence structure defined by the stochastic correlation model Type 2 is:

\[ C_2(u_1, \ldots, u_n) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \int_{-\infty}^{\infty} F_\varepsilon \left( \frac{\sigma_i \kappa_i^{-1}(u_i) + \mu_i - g_i(y_i)m}{\sqrt{1 - g_i(y_i)^2}} \right) f_{Y_i|M}(y_i|m)dy_i f_M(m)dm \]

where \( \kappa_i(x) = P[A_i \leq x] \) and is given by expression (3.7) and \( \kappa_i^{-1}(\cdot) \) is the inverse function of \( \kappa_i(\cdot) \).
Proof:

\[
P[A_1 \leq x_1, \ldots, A_n \leq x_n] = E[P[A_i \leq x_1, \ldots, A_n \leq x_n|M = m]]
\]

\[
= \int_{-\infty}^{\infty} \prod_{i=1}^{n} \int_{-\infty}^{\infty} f_i(x_i) \left( \frac{\sigma_i x_i + \mu_i - g_i(y_i)m}{\sqrt{1 - g_i^2(y_i)}} \right) f_{Y|M}(y_i|m) dy_i f_M(m) dm
\]

and the result follows from an application of Sklar’s Theorem.

The default time of obligor \(i\) is obtained by sampling the asset value \(A_i\) and using the mapping given by expression (3.8).

\[
\tau_i = F_i^{-1}(\kappa_i(A_i))
\]

where \(\kappa_i(x) = P[A_i \leq x]\) is given by expression (3.7) and \(F_i^{-1}(\cdot)\) is the inverse of the function \(F_i(t) = P[\tau_i \leq t]\).

For the case where the correlation driver \(Y\) is equal to the market factor \(M\), then the Type 1 and Type 2 models become equivalent. For the case where \(Y\) is independent from \(M\), and \(M\) and \(\varepsilon_i\) are both standard Gaussian random variables, then the asset value distribution for both the Type 1 model (expression 3.1) and Type 2 model (expression 3.5) are both standard Gaussian. In order to see this consider the asset value process \(A = \rho M + \sqrt{1 - \rho^2} \varepsilon\), where \(M\) and \(\varepsilon\) are both standard Gaussian random variables and \(\rho\) is the stochastic correlation with distribution \(f_\rho \in [0, 1]\). The characteristic function for a standard Gaussian random variable, \(X\), is \(E[e^{iuX}] = e^{-\frac{1}{2}u^2}\), and a sufficient condition to see whether a random variable is Gaussian is to check whether its characteristic function is equal to \(e^{-\frac{1}{2}u^2}\). Now, \(E[e^{iuA}] = E[E[e^{iuA}|\rho = \rho]] = e^{-\frac{1}{2}u^2}\), since conditional on a realization of \(\rho = \rho\), \(A\) is a standard Gaussian random variable.

In the next section we give an example of the model where we specify the joint density function, \(f_{MY}(m,y)\), and a mapping function \(g(y)\).

### 3.3 A Tractable Example

We now present a specific example of the models given in (3.1) and (3.5) which leads to some useful analytic results. The following assumptions are made: \(M\) and
\( \varepsilon_i \) are standard normal random variables and are independent of one and other; \( M \sim N(0, 1), \varepsilon_i \sim N(0, 1) \). In the case of the Type 1 model \( Y \) is related to \( M \) via the expression \( Y = \beta M + \sqrt{1 - \beta^2} \zeta \) where \( \zeta \) is an independent standard normal random variable. This implies that \( Y \) is also a standard normal random variable but is correlated with \( M \) with a linear correlation coefficient of \( \beta^2 \). For the Type 2 case we have \( Y = \beta M + \sqrt{1 - \beta^2} \zeta \) where the noise term in the correlation driver is issuer specific. Finally the monotone mapping function is chosen to be \( g_i(\cdot) = \Phi(a_i Y + b_i) \) where \( \Phi(\cdot) \) is the cumulative Gaussian distribution function and \( a_i \) and \( b_i \) are real constants. Hence, under this setup, the asset value for the Type 1 case is given by (3.9).

\[
A_i = \frac{1}{\sigma_i} \left( \Phi(a_i Y + b_i) M + \sqrt{1 - \Phi^2(a_i Y + b_i)} \varepsilon_i - \mu_i \right) \quad (3.9)
\]

And the asset value for the Type 2 case is given by expression (3.10).

\[
A_i = \frac{1}{\sigma_i} \left( \Phi(a_i Y + b_i) M + \sqrt{1 - \Phi^2(a_i Y + b_i)} \varepsilon_i - \mu_i \right) \quad (3.10)
\]

Given this setup, propositions 3 and 4 both apply for the Type 1 and Type 2 case.

**Proposition 3** \( \mu_i \) is given by the expression

\[
\mu_i = \beta \alpha \phi(b_i \sqrt{1 - \alpha^2})
\]

where \( \alpha = \frac{a_i}{1 + a_i^2} \).

**Proof:** \( \mu_i = E[\Phi(a_i Y + b_i)M + \sqrt{1 - \Phi(a_i Y + b_i)^2} \varepsilon_i] = E[\Phi(a_i Y + b_i)M] \) by independence of \( \varepsilon_i \) and the fact that \( E[\varepsilon_i] = 0 \). \( M \) can be expressed as \( M = \beta Y + \sqrt{1 - \beta^2} \zeta \), so

\[
\mu_i = E[\Phi(a_i Y + b_i)(\beta Y + \sqrt{1 - \beta^2} \zeta)]
\]

\[
= \beta E[Y \Phi(a_i Y + b_i)] + E[\Phi(a_i Y + b_i) \sqrt{1 - \beta^2} \zeta]
\]

\[
= \beta \alpha \phi(b \sqrt{1 - \alpha^2})
\]

\(^1\)A more general mapping function which also leads to tractable results is \( g(\cdot) = \sum_i \Phi(a_i Y + b_i) \). By increasing the number of terms, and hence parameters, it is possible to reach an arbitrary degree of accuracy in the calibration to market prices.
by independence of $\zeta$ and $E[\zeta] = 0$ and application of integral 3.

**Proposition 4** $\sigma_i^2$ is given by the expression

$$\sigma_i^2 = 1 + \beta^2 \left( E[Y^2 \Phi^2(a_i Y + b_i)] - \Phi \left( \frac{b_i}{\sqrt{1 + a_i^2}}, \frac{b_i}{\sqrt{1 + a_i^2}}; \frac{a_i^2}{1 + a_i^2} \right) \right) - \mu_i^2$$

Where $E[Y^2 \Phi^2(aY + b)]$ is evaluated using integral 5.

**Proof:**

$$\sigma_i^2 = \text{Var} \left[ \Phi(a_i Y + b_i)M + \sqrt{1 - \Phi^2(a_i Y + b_i)\varepsilon_i} \right] = E \left[ \left( \Phi(a_i Y + b_i)M + \sqrt{1 - \Phi^2(a_i Y + b_i)\varepsilon_i} \right)^2 \right] - \mu_i^2 = E[M^2 \Phi^2(a_i Y + b_i)] + 1 - E[\Phi^2(a_i Y + b_i)] - \mu_i^2$$

Since $\varepsilon_i$ is independent, $E[\varepsilon_i] = 0$ and $E[\varepsilon_i^2] = 1$. Now,

$$M^2 = (\beta Y + \sqrt{1 - \beta^2} \zeta)^2 = \beta^2 Y^2 + 2 \beta Y \sqrt{1 - \beta^2} \zeta + (1 - \beta^2) \zeta^2$$

So,

$$E[M^2 \Phi^2(a_i Y + b_i)] = \beta^2 E[Y^2 \Phi^2(a_i Y + b_i)] + (1 - \beta^2)E[\Phi^2(a_i Y + b_i)]$$

And the result follows by an application of Integral 2 and Integral 5.

Before we proceed to analyze some numerical results, the concept of compound and base correlation is reviewed. Compound correlation is simply the correlation number that when plugged into the one factor Gaussian copula model reprices a specific tranche exactly. A given compound correlation is defined by the attachment and detachment points of a tranche. Base correlation is the correlation number that reprices a tranche with an attachment point of zero (hence base) using the one factor Gaussian model. In order to use the base correlation framework on a tranche that does not have a zero attachment point, the tranche in question is decomposed into two base tranches. For example, providing protection on a tranche defined by the attachment and detachment points $[x\%-y\%]$ is equivalent to providing protection on the tranche $[0\%-y\%]$ and selling protection on the tranche $[0\%-x\%]$. As long as we have the correlation number such
that the tranche \([0\%-x\%]\) is priced correctly, it is possible to find the correlation corresponding to the base tranche \([0\%-y\%]\) such that the tranche \([x\%-y\%]\) is priced correctly. Hence, given a host of tranche prices, it is possible to decompose these tranches into base tranches and to use a bootstrapping methodology to find the implied correlation for these base tranches such that the original tranches are priced correctly. Base correlation is only defined by the tranche detachment point.

For a more detailed discussion of compound and base correlation see chapter two.

In order to see the improvement that this model produces over the standard one factor Gaussian model, figures 3.1 and 3.2 display the compound and base correlation skews produced by the Type 1 and Type 2 models and the skews quoted in the market for the iTraxx Europe Series 2 indices for 13 October 2004.\(^2\) The average spread is 37bps and 5 years to maturity. For the calibration of the Type 1 model the correlation parameters where set to \(a = -0.716\), \(b = -0.750\), and \(\beta = 55\%\) for each obligor. For the Type 2 model the parameters where set to \(a = -0.384\), \(b = -0.549\), and \(\beta = 90\%\). It was also assumed that recovery rates where 40\%, the continuously compounded interest rate was 2\%, and that the portfolio consisted of 100 obligors each with a notional of 100,000. As can be seen from figures 3.1 and 3.2, the skews produced by the model are very close to the skews produced in the market.

Table 3.1 presents these results in tabular format by displaying the tranche prices produced by the stochastic correlation model and the prices quoted for the iTraxx Europe Series 2 indices for 13 October 2004. These market prices are the ones used in the construction of the compound and base correlations shown in figures 3.1 and 3.2. As can be seen from table 3.1, the model can produce prices that are very close to those quoted in the market. It should be noted that the equity tranche has an upfront protection payment expressed as a percentage of the tranche notional plus a running spread expressed in basis points.

Returning back to the analysis of the stochastic correlation model, we now concentrate on the nature and distribution of the stochastic correlation function
\[
g(Y) = \Phi(aY + b).
\]

The correlation function determines the probability density function of the random asset correlation and it is important to be able to obtain

\(^2\)This data was taken from O'Kane and Livesey, 2004
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Figure 3.1: Compound Correlation. Market iTraxx vs Stochastic Correlation Model

Figure 3.2: Base Correlation. Market iTraxx vs Stochastic Correlation Model
Table 3.1: Comparison between the stochastic correlation model prices and market prices.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Upfront</th>
<th>Market Prices</th>
<th>Type 1 Model</th>
<th>Type 2 Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%-3%</td>
<td>24.25%</td>
<td>500bps</td>
<td>500bps</td>
<td>500bps</td>
</tr>
<tr>
<td>3%-6%</td>
<td>0%</td>
<td>137.5bps</td>
<td>135.5bps</td>
<td>160.3bps</td>
</tr>
<tr>
<td>6%-9%</td>
<td>0%</td>
<td>47.5bps</td>
<td>58.5bps</td>
<td>58.1bps</td>
</tr>
<tr>
<td>9%-12%</td>
<td>0%</td>
<td>34.5bps</td>
<td>30.1bps</td>
<td>33.1bps</td>
</tr>
<tr>
<td>12%-22%</td>
<td>0%</td>
<td>15.5bps</td>
<td>15.5bps</td>
<td>15.5bps</td>
</tr>
</tbody>
</table>

a closed form formula for this distribution. Once the model is calibrated to market prices, such expressions allow one to obtain the ‘implied asset correlation distribution’. The following propositions give the moments and distribution of \( g(Y) = \Phi(aY + b) \).

**Proposition 5** The mean of the stochastic correlation function \( g(Y) = \Phi(aY + b) \) is equal to:

\[
E[g(Y)] = \Phi\left( \frac{b_i}{\sqrt{1 + a_i^2}} \right)
\]

**Proof:** A direct application of integral 1.

**Proposition 6** The variance of the stochastic correlation function is equal to:

\[
\text{Var}[g(Y)] = \Phi\left( \frac{b_i}{\sqrt{1 + a_i^2}}, \frac{b_i}{\sqrt{1 + a_i^2}}; \frac{a_i^2}{1 + a_i^2} \right) - \Phi^2\left( \frac{b_i}{\sqrt{1 + a_i^2}} \right)
\]

**Proof:** \( \text{Var}[\Phi(aY + b)] = E[\Phi^2(aY + b)] - E^2[\Phi(aY + b)] \), and the result follows from an application of integral 1 and 2.

**Proposition 7** The distribution of the stochastic correlation function is given by:

\[
P[\Phi(aY + b) \leq l] = \Phi\left( \frac{\Phi^{-1}(l) - b_i}{|a_i|} \right)
\]
and its probability density is:

\[
P[l \leq \Phi(a_i Y + b_i) \leq l + dl]/dl = \frac{1}{|a_i|} \phi \left( \frac{\Phi^{-1}(l) - b_i}{|a_i|} \right) \frac{1}{\phi(\Phi^{-1}(l))}
\]

Proof: \( P[\Phi(a_i Y + b_i) \leq l] = P[a_i Y + b_i \leq \Phi^{-1}(l)] = \Phi \left( \frac{\Phi^{-1}(l) - b_i}{|a_i|} \right) \), since \( Y \) is a standard normal random variable. To get the density simply differentiate the cumulative distribution function.

Given the Gaussian specification for the density of \( Y \) and the form of the mapping function \( g(Y) = \Phi(aY + b) \), a whole host of correlation distributions may be generated. This is demonstrated in figure 3.3 for various parameterizations of \( a \) and \( b \).

It was stated earlier that the correlation function \( g(Y) = \Phi(aY + b) \) is linked to the market factor, \( M \), via the relation \( Y = \beta M + \sqrt{1 - \beta^2} \zeta \) for \( \zeta \sim N(0, 1) \). We next provide expressions that measure the degree of dependence between the stochastic correlation term, \( g(Y) = \Phi(aY + b) \), and \( M \), the market factor.
Such expressions allow one to see how correlated the asset correlation is with the market index. We provide two different measures of dependence, the linear correlation coefficient and Kendall’s tau.

**Proposition 8** The linear correlation coefficient between \( g(Y) = \Phi(aY + b) \) and \( M \) is given by

\[
\text{Corr}(\Phi(aY + b), M) = \frac{\beta a \phi(b\sqrt{1 - \alpha^2})}{\sqrt{\Phi\left(\frac{b}{\sqrt{1+\alpha^2}}, \frac{b}{\sqrt{1+\alpha^2}}; \frac{\alpha^2}{1+\alpha^2}\right) - \Phi^2\left(\frac{b}{\sqrt{1+\alpha^2}}\right)}}
\]

Where \( \alpha = \frac{a}{\sqrt{1+a^2}} \).

**Proof:**

\[
\text{Corr}(\Phi(aY + b), M) = \frac{E[\Phi(aY + b)M]}{\sqrt{\text{Var}(\Phi(aY + b))\text{Var}(M)}}
\]

Now,

\[
E[\Phi(aY + b)M] = E[\Phi(aY + b)(\beta Y + \sqrt{1 - \beta^2} + \zeta)]
\]

\[
= \beta E[\Phi(aY + b)Y]
\]

and the result follows from integral 3 and proposition 6.

**Proposition 9** Kendall’s tau for the market variable \( M \) and \( g(Y) = \Phi(aY + b) \) is \( \tau(M, \Phi(aY + b)) = \frac{2}{\pi} \arcsin(\text{sign}(a)\beta^2) \)

**Proof:** For any two Gaussian random variables \( X_1 \) and \( X_2 \), Kendall’s tau is equal to \( \tau(X_1, X_2) = \frac{2}{\pi} \arcsin(\rho) \) where \( \rho \) is the linear correlation coefficient between \( X_1 \) and \( X_2 \). Now, \( \text{Corr}(M, aY + b) = \text{sign}(a)\beta^2 \) since \( Y = \beta M + \sqrt{1 - \beta^2} + \zeta \). Finally, the result follows from the fact that Kendall’s tau remains the same under strictly increasing functions of the variables \( \alpha(X_1) \), and \( \gamma(X_2) \), i.e. \( \tau(X_1, X_2) = \tau(\alpha(X_1), \gamma(X_2)) \).

Although in this model we assume that the correlation is stochastic, we can still compute the linear correlation coefficient between two asset processes. Such an expression may be useful in comparing the asset correlations implied by the
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CDO market and those calculated from historical time series equity data. Proposition 10 provides an expression for the degree of linear dependence between the asset value and the market factor and is applicable to both Type 1 and Type 2 models. Proposition 11 provides an expression for the linear correlation coefficient between two assets and is valid only for the Type 1 model.

**Proposition 10** The linear correlation coefficient between asset $i$ and the common market factor $M$ is given by the expression:

$$
\text{Corr}(A_i, M) = \frac{1}{\sigma_i} \left( \beta \mathbb{E}[\Phi(a_i Y + b_i) Y^2] + (1 - \beta^2) \Phi \left( \frac{b_i}{\sqrt{1 + \alpha_i^2}} \right) \right)
$$

where the expectation term can be evaluated using integral 4.

**Proof:**

$$
\text{Corr}(A_i, M) = \mathbb{E}[A_i M] = \mathbb{E} \left[ \frac{1}{\sigma_i} \left( \Phi(a_i Y + b_i) M + \sqrt{1 - \Phi^2(a_i Y + b_i)} \varepsilon_i - \mu_i \right) M \right]
$$

which, after simplifying due to the independence of $M, \varepsilon_i,$ and $\mathbb{E}[M] = \mathbb{E}[\varepsilon_i] = 0$ leaves

$$
= \frac{1}{\sigma_i} \mathbb{E}[\Phi(a_i Y + b_i) M^2]
$$

and substituting $M = \beta Y + \sqrt{1 - \beta^2} \zeta$ gives

$$
\text{Corr}(A_i, M) = \frac{1}{\sigma_i} \left( \beta \mathbb{E}[\Phi(a_i Y + b_i) Y^2] + (1 - \beta^2) \mathbb{E}[\Phi(a_i Y + b_i)] \right)
$$

And the result follows by an application of integral 1 and 4.

**Proposition 11** The correlation between two assets $A_i$ and $A_j$ is given by the expression:

$$
\text{Corr}(A_i, A_j) = \frac{1}{\sigma_i \sigma_j} \left( \beta \mathbb{E}[\Phi(a_i Y + b_i) \Phi(a_j Y + b_j) Y^2] \right)
$$

$$
+ (1 - \beta^2) \phi \left( \frac{b_i}{\sqrt{1 + \alpha_i^2}}, \frac{b_j}{\sqrt{1 + \alpha_j^2}}; \frac{a_i a_j}{\sqrt{1 + \alpha_i^2} \sqrt{1 + \alpha_j^2}} \right)
$$

$$
- \mu_i \beta \alpha_i \phi(b_i \sqrt{1 - \alpha_i^2}) - \mu_j \beta \alpha_j \phi(b_j \sqrt{1 - \alpha_j^2}) + \mu_i \mu_j
$$
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Where \( \alpha_i = \frac{a_i}{\sqrt{1+a_i^2}} \) and the first term involving the expectation can be calculated using integral 5.

\[
\text{Proof: } \text{Corr}(A_i, A_j) = E[A_iA_j] \text{ where }
\]
\[
A_i = \frac{1}{\sigma_i}(\Phi(a_i Y + b_i) M + \sqrt{1 - \Phi^2(a_i Y + b_i)} \varepsilon_i - \mu_i)
\]
\[
A_j = \frac{1}{\sigma_j}(\Phi(a_j Y + b_j) M + \sqrt{1 - \Phi^2(a_j Y + b_j)} \varepsilon_j - \mu_j)
\]

Since \( \varepsilon_i, \varepsilon_j \) and \( M \) are independent with zero mean, the expectation of the product of \( A_i \) and \( A_j \) simplifies to:

\[
E[A_iA_j] = \frac{1}{\sigma_i\sigma_j} \left( E[\Phi(a_i Y + b_i)\Phi(a_j Y + b_j)M^2] - \mu_iE[\Phi(a_j Y + b_j)M] - \mu_jE[\Phi(a_i Y + b_i)M] + \mu_i\mu_j \right)
\]

Substituting \( M = \beta Y + \sqrt{1 - \beta^2}\zeta \) in the above expression and again cancelling terms involving products of independent random variables yields

\[
E[A_iA_j] = \frac{1}{\sigma_i\sigma_j} \left( \beta^2 E[\Phi(a_i Y + b_i)\Phi(a_j Y + b_j)Y^2] + (1 - \beta^2)E[\Phi(a_i Y + b_i)\Phi(a_j Y + b_j)] 
- \mu_i \beta E[\Phi(a_j Y + b_j)Y] - \mu_j \beta E[\Phi(a_i Y + b_i)Y] + \mu_i\mu_j \right)
\]

And the result follows from an application of integral 2, 3 and 6.

Continuing with the analysis of the model we now turn our attention to finding a quick method to compute the default threshold for a given default probability. Recall that the default thresholds are computed using expression (3.3) which involves a double integral and a root searching algorithm. Expression (3.3) is valid for both Type 1 and Type 2 models. This double integral can in fact be reduced to a one dimensional integral using the following proposition.

**Proposition 12** The default threshold for obligor \( i \) can be calculated with the following one dimensional quadrature and a suitable root search algorithm:

\[
P_i = \int_{-\infty}^{\infty} \Phi \left( \frac{\omega}{\sqrt{1+\nu^2}} \right) \phi(y) dy
\]
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Where $P_i$ is the required default probability, $\omega = \frac{\sigma_i C_i + \mu_i - \Phi(a_i y + b_i)\beta y}{\sqrt{1 - \Phi^2(a_i y + b_i)}}$ and $\nu = \frac{-\Phi(a_i y + b_i)\sqrt{1 - \beta^2}}{\sqrt{1 - \Phi^2(a_i y + b_i)}}$

Proof:

$$P_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi \left( \frac{\sigma_i C_i + \mu_i - \Phi(a_i y + b_i)m}{\sqrt{1 - \Phi^2(a_i y + b_i)}} \right) \phi_{MY}(m, y; \beta^2) dm dy$$

where $\phi_{MY}(m, y; \beta^2)$ is the standard bivariate normal distribution function with a correlation coefficient equal to $\beta^2$. This can be written as:

$$P_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi \left( \frac{\sigma_i C_i + \mu_i - \Phi(a_i y + b_i)m}{\sqrt{1 - \Phi^2(a_i y + b_i)}} \right) \phi_{MY}(m|y) dm \phi_Y(y) dy$$

where $\phi_{MY}(m|y)$ is the conditional marginal distribution of $M$ given $Y$. In particular, the marginal distribution is Gaussian with mean $\beta y$ and variance $(1 - \beta^2)$. This means that the integral can be rewritten as:

$$P_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi \left( \frac{\sigma_i C_i + \mu_i - \Phi(a_i y + b_i)(\sqrt{1 - \beta^2}m + \beta y)}{\sqrt{1 - \Phi^2(a_i y + b_i)}} \right) \phi_M(m) dm \phi_Y(y) dy$$

And the result follows by making the substitutions $\omega = \frac{\sigma_i C_i + \mu_i - \Phi(a_i y + b_i)\beta y}{\sqrt{1 - \Phi^2(a_i y + b_i)}}$ and $\nu = \frac{-\Phi(a_i y + b_i)\sqrt{1 - \beta^2}}{\sqrt{1 - \Phi^2(a_i y + b_i)}}$ and applying integral 1.

3.3.1 Impact of Correlation Parameters on Tranche Prices

Having established the stochastic correlation model, it is of practical interest to observe what effect the three parameters $a$, $b$, and $\beta$ have on tranche prices. $a$ and $b$ are the parameters in the stochastic correlation function $g(Y) = \Phi(aY + b)$ and $\beta^2$ is the correlation between $Y$ and $M$, where $M$ is the market factor. The impact of asset correlation on tranche prices is well documented (see Turc and Very 2004). It is known that an increase in asset correlation causes the equity tranche par spread to decrease monotonically. Conversely, the par spread of the super senior tranche increases monotonically for an increase in asset correlation. The mezzanine tranche behaves in a more complex manner where for low values of correlation it behaves like a super senior tranche but for high values of correlation
its response is similar to an equity tranche. For the stochastic correlation model we can no longer talk about the effect of correlation on tranche par spreads but rather we consider the effect of the stochastic correlation parameters on the tranche par spreads. Looking first at the effect of varying $\beta$ we calculate for the Type 1 model the par spread for various tranches and values of $\beta$. During the calculation of tranche par spreads it was assumed that the portfolio consisted of 100 obligors each with a credit spread of 37bps, recovery rates of 40% and notional equal to 100,000. The equity tranche had an upfront payment of 24.25%, $a=-0.43$, $b=-0.38$ and the continuously compounded interest was set to 2%. The results are displayed in figure 3.4 which shows that the equity tranche clearly has the greatest sensitivity to $\beta$. The spread of senior tranches increase as $\beta$ is increased while the spread of the equity tranche decreases as $\beta$ increases. This has the same effect as simply increasing the correlation in the one factor Gaussian model.

Next we consider for the Type 1 model the response of the tranche par spreads for changes in the parameters $a$ and $b$ which completely specify the distribution of the random asset correlation. During the calculation of the tranche par spreads the same portfolio was used and the parameter $\beta$ was set to $\beta=0.6$ for all cases.
Figures 3.5 to 3.9 graph the tranche par spreads for varying values of the parameters $a$ and $b$. Only negative values of $a$ and $b$ are considered as this seems the most plausible range given the tranche prices in the market. From figures 3.5 and 3.9 it can be seen that both the equity and super senior tranche preserve their monotonicity. Increasing the magnitude of parameter $a$ reduces the par spread of the equity tranche while causing the spread of the super senior tranche to increase. Conversely increasing the magnitude of $b$ causes the par spread of the equity tranche to increase while the spread of the super senior tranche decreases.

The reason for this monotonic behavior can be seen by examining the asset correlation between two asset as given by proposition 11. An increase in the magnitude of parameter $a$ for a given value of $b$ increases the linear correlation coefficient between two assets, and an increase in the magnitude of parameter $b$ for a given value of $a$ causes the linear correlation coefficient to decrease in value. Given our knowledge of the response of the equity and super senior tranche to changes in asset correlation, it is now clear why the equity and super senior tranche respond monotonically to the parameters $a$ and $b$. The behavior of the mezzanine tranches is relatively more complex and is no longer monotone.

All tranches do respond in a continuous and smooth manner which will aid any optimization procedure to calibrate the model. One such optimization procedure that exploits the monotonic response of the equity and senior tranche and should in most cases fit three tranches prices exactly would be the following:

1. For a fixed value of $\beta$, perform a two dimensional root search algorithm to find the values of $a$ and $b$ such that the equity and super senior tranches are priced to match the market prices.

2. Repeat step 1 for different values of $\beta$ to find the 'best fit' skew, or vary $\beta$ until a mezzanine tranche is calibrated.

Clearly, the above procedure is by no means an efficient algorithm.

The response of the tranche prices with respect to the parameters $a$, $b$, and $\beta$ for the Type 2 model is very similar to the results for the Type 1 model. However, for the Type 2 model the senior tranche does not seem to respond monotonically.
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Figure 3.5: Equity (0%-3%) Model Type 1 Tranche Prices vs Parameter a and b

Figure 3.6: Junior Mezzanine (3%-6%) Model Type 1 Tranche Prices vs Parameter a and b
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Figure 3.7: Senior Mezzanine (6%-9%) Model Type 1 Tranche Prices vs Parameter a and b

Figure 3.8: Senior (9%-12%) Model Type 1 Tranche Prices vs Parameter a and b
to the parameters $a$ and $b$. This is not a major obstacle however as base tranches (those with zero attachment point) do appear to respond monotonically and any tranche can be decomposed into two base tranches.

### 3.4 Large Homogeneous Portfolio Loss Distribution for the Type 1 Model

In this section we derive the large homogeneous portfolio (LHP) loss distribution for the Type 1 model (It was not possible to derive a LHP distribution for the Type 2 case). In this setup it is assumed that the asset value is governed by expression 3.9 and that the portfolio consists of an infinite number of obligors with a total notional equal to $N$. Each obligor has the same correlation mapping function $g(Y) = \Phi(aY + b)$ and parameter $\beta$. It is also assumed that each obligor has the same probability of default and as a result the same default threshold $C$. We proceed now to derive the portfolio loss distribution under these conditions.
much in the same way as Vasicek (1987).

The default probability of an obligor given the market factor \( M \) and correlation driver \( Y \) is:

\[
P(A \leq C|M = m, Y = y) = \Phi \left( \frac{\sigma C + \mu - g(y)m}{\sqrt{1 - g(y)^2}} \right) = q
\]

which we assume is equal to some value \( q \). Since the portfolio is assumed to consist of an infinite number of obligors, each with the same default threshold and mapping function \( g(\cdot) \), then by the Law of Large Numbers exactly this proportion of the portfolio will default for the given realization of the variables \( M \) and \( Y \). If we assume a constant recovery rate of \( R \) and a total portfolio notional of \( N \) then the total portfolio loss conditional on \( M \) and \( Y \) is \( qN(1 - R) = l \). Solving for the market variable \( M \) which gives a total loss of \( l \), we have:

\[
m = \frac{\sigma C + \mu - \Phi^{-1}(l/N(1 - R))\sqrt{1 - g^2(y)}}{g(y)} = G
\]  

(3.11)

The probability that the portfolio loss will be less than \( l \) given \( Y = y \) can be expressed as:

\[
P[L \leq l|Y = y] = P[M \geq G]
\]

where \( G \) is given in equation (3.11). But we have \( M = \beta y + \sqrt{1 - \beta^2} \zeta \). Hence,

\[
P[L \leq l|Y = y] = P[M - \beta y \geq G - \beta y] = P \left[ \frac{M - \beta y}{\sqrt{1 - \beta^2}} \geq \frac{G - \beta y}{\sqrt{1 - \beta^2}} \right]
\]

And so we have,

\[
P[L \leq l|Y = y] = \Phi \left( \frac{\beta y - G}{\sqrt{1 - \beta^2}} \right)
\]

The unconditional loss distribution is simply obtained by integrating over the distribution of \( Y \):

\[
P[L \leq l] = \int_{-\infty}^{\infty} \Phi \left( \frac{\beta y - G}{\sqrt{1 - \beta^2}} \right) \phi(y) dy
\]  

(3.12)
3.5 Single Tranche CDO Pricing with the Large Homogeneous Portfolio applied to the Type 1 Model

The pricing algorithm presented in chapter 2 for single tranche CDOs effectively consists of evaluating the expected loss on a tranche for various time horizons. The expected loss on a tranche can be computed as the difference of the expected loss on two base tranches. We now proceed to derive under the large homogeneous portfolio (LHP) limit, closed form formulas for the Type 2 model for the expected loss on a tranche which has a zero attachment point and a detachment level of $D$ (a base tranche with detachment level $D$) and where the asset value is given by expression 3.9. Under the LHP assumption all obligors have the same correlation mapping function, $g(Y) = \Phi(\alpha Y + b)$, and the same notional, default probability, and recovery rate $R$. It is also assumed that there are an infinite number of obligors in the portfolio and that the total notional of the portfolio is $N$. As for the tractable example, $M$ and $Y$ are standard normal random variables with a correlation coefficient equal to $\beta^2$. The expected loss on a base tranche with detachment level $D$ can be expressed as:

$$E[min(L(t), D)] = E[D1_{L(t)\geq D} + L(L(t)<D)]$$  \hspace{1cm} (3.13)

Consider

$$E[1_{L(t)\geq D}] = P[L(t) \geq D] = E[E[1_{L(t)\geq D}|Y = y]]$$

Define $G$ by

$$G = \frac{\sigma C + \mu - \Phi^{-1}(D/N(1-R))\sqrt{1-g^2(y)}}{g(y)} \hspace{1cm} (3.14)$$

This is the value of $M$ that gives a portfolio loss equal to $D$ given $Y = y$. Then we have:

$$E[1_{L(t)\geq D}|Y = y] = P[M \leq G]$$
Since $M = \beta y + \sqrt{1 - \beta^2} \zeta$, it follows that,

$$E[1_{\{L(t) \geq D\}}|Y = y] = P\left[\frac{M - \beta y}{\sqrt{1 - \beta^2}} \leq \frac{G - \beta y}{\sqrt{1 - \beta^2}}\right] = \Phi\left(\frac{G - \beta y}{\sqrt{1 - \beta^2}}\right)$$

And by integrating over the distribution of $Y$ we obtain:

$$E[1_{\{L(t) \geq D\}}] = \int_{-\infty}^{\infty} \Phi\left(\frac{G - \beta y}{\sqrt{1 - \beta^2}}\right) \phi(y)dy \quad (3.15)$$

This integral can be very efficiently implemented using a quadrature scheme such as Gaussian quadrature. We now wish to evaluate the second term $E[L1_{\{L(t) < D\}}]$.

$$E[L1_{\{L(t) < D\}}] = E[E[L1_{\{L(t) < D\}}|Y, M]] = E\left[\Phi\left(\frac{\sigma C + \mu - g(Y)M}{\sqrt{1 - g^2(Y)}}\right) N(1 - R)1_{\{M > G\}}\right]$$

Where the expectation is taken with respect to the joint distribution of $M$ and $Y$. In order to understand how this expression arises, we note that conditional on $M = m$ and $Y = y$, the portfolio loss is $pN(1 - R)$ where $p = \Phi\left(\frac{\sigma C + \mu - g(y)m}{\sqrt{1 - g^2(y)}}\right)$ is the conditional default probability. Also, when conditioned on $Y$, the event $(L(t) < D)$ is equivalent to the event $(M > G)$ where $G$ is given by expression 3.14, which explains the change in indicator functions. Conditioning first on $Y$, this expectation can be broken down into:

$$= E\left[\Phi\left(\frac{\sigma C + \mu - g(Y)M}{\sqrt{1 - g^2(Y)}}\right) N(1 - R)1_{\{M > G\}}|Y\right]$$

But $M = \beta Y + \sqrt{1 - \beta^2} \zeta$, so:

$$= E\left[\Phi\left(\frac{\sigma C + \mu - g(Y)(\beta Y + \sqrt{1 - \beta^2} \zeta)}{\sqrt{1 - g^2(Y)}}\right) N(1 - R)1_{\{\beta Y + \sqrt{1 - \beta^2} \zeta > G\}}|Y\right]$$

$$= E\left[\Phi\left(\frac{\sigma C + \mu - g(Y)(\beta Y + \sqrt{1 - \beta^2} \zeta)}{\sqrt{1 - g^2(Y)}}\right) N(1 - R)1_{\{\zeta > \frac{G - \beta Y}{\sqrt{1 - \beta^2}}\}}|Y\right]$$

$$= E\left[\int_{\frac{G - \beta Y}{\sqrt{1 - \beta^2}}}^{\infty} \Phi\left(\frac{\sigma C + \mu - g(Y)(\beta Y + \sqrt{1 - \beta^2} \zeta)}{\sqrt{1 - g^2(Y)}}\right) N(1 - R)\phi(\zeta) d\zeta\right]$$
CHAPTER 3. STOCHASTIC CORRELATION MODEL

Where the expectation is with respect to the random variable $Y$ and $\phi(\zeta)$ is the density function of $\zeta$ and is a standard normal density function. Making the substitution $u = -\zeta$, the integral becomes:

$$
E \left[ \int_{-\infty}^{\beta Y - G} \Phi \left( \frac{\sigma Y + \mu - g(Y) (\beta Y - \sqrt{1 - \beta^2} u)}{\sqrt{1 - g^2(Y)}} \right) N(1 - R) \phi(u) du \right]
$$

Using integral 6,

$$
\int_{-\infty}^{c} \Phi(a y + b) \phi(y) dy = \Phi \left( \frac{b}{\sqrt{1 + a^2}}, \frac{-a}{\sqrt{1 + a^2}} \right)
$$

we have:

$$
E[L1_{L(t)<D}] = \int_{-\infty}^{\infty} \Phi \left( \frac{b}{\sqrt{1 + a^2}}, \frac{-a}{\sqrt{1 + a^2}} \right) \phi(y) dy
$$

where

$$
a = g(y) \sqrt{1 - \beta^2} / \sqrt{1 - g^2(y)}
$$

$$
b = \sigma Y + \mu - g(Y) \beta y / \sqrt{1 - g^2(y)}
$$

$$
c = \beta y - G / \sqrt{1 - \beta^2}
$$

Using expressions 3.13, 3.14, 3.15, and 3.16 for the expected tranche loss, rapid computation of CDO tranche prices are possible. To demonstrate the LHP approximation we price a series of CDOs consisting of 500 obligors each for various parameterizations of the model. In each case it was assumed that the recovery rates were 40% and that each obligor had a notional of 100,000. The maturity is five years with a continuously compounded interest rate of 2%. The result are reported in tables 3.2-3.5 and it can be seen that the LHP approximates well the true tranche prices for sufficiently large portfolios.
### Table 3.2: LHP CDO pricing. Credit spread=37bps, \( a=-0.4, b=-0.35, \beta=60\% \).

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Upfront</th>
<th>Exact Prices</th>
<th>LHP Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity 0%-3%</td>
<td>24.25%</td>
<td>493bps</td>
<td>498bps</td>
</tr>
<tr>
<td>Junior Mezzanine 3%-6%</td>
<td>0%</td>
<td>152bps</td>
<td>151bps</td>
</tr>
<tr>
<td>Senior Mezzanine 6%-9%</td>
<td>0%</td>
<td>69bps</td>
<td>64bps</td>
</tr>
<tr>
<td>Senior 9%-12%</td>
<td>0%</td>
<td>31bps</td>
<td>35bps</td>
</tr>
<tr>
<td>Super Senior 12%-22%</td>
<td>0%</td>
<td>16bps</td>
<td>15bps</td>
</tr>
</tbody>
</table>

### Table 3.3: LHP CDO pricing. Credit spread=70bps, \( a=-0.1, b=-0.2, \beta=40\% \).

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Upfront</th>
<th>Exact Prices</th>
<th>LHP Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity 0%-3%</td>
<td>24.25%</td>
<td>1460bps</td>
<td>1491bps</td>
</tr>
<tr>
<td>Junior Mezzanine 3%-6%</td>
<td>0%</td>
<td>564bps</td>
<td>553bps</td>
</tr>
<tr>
<td>Senior Mezzanine 6%-9%</td>
<td>0%</td>
<td>198bps</td>
<td>207bps</td>
</tr>
<tr>
<td>Senior 9%-12%</td>
<td>0%</td>
<td>102bps</td>
<td>90bps</td>
</tr>
<tr>
<td>Super Senior 12%-22%</td>
<td>0%</td>
<td>22bps</td>
<td>22bps</td>
</tr>
</tbody>
</table>

### Table 3.4: LHP CDO pricing. Credit spread=70bps, \( a=-0.6, b=-0.5, \beta=75\% \).

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Upfront</th>
<th>Exact Prices</th>
<th>LHP Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity 0%-3%</td>
<td>24.25%</td>
<td>1429bps</td>
<td>1465bps</td>
</tr>
<tr>
<td>Junior Mezzanine 3%-6%</td>
<td>0%</td>
<td>338bps</td>
<td>326bps</td>
</tr>
<tr>
<td>Senior Mezzanine 6%-9%</td>
<td>0%</td>
<td>147bps</td>
<td>151bps</td>
</tr>
<tr>
<td>Senior 9%-12%</td>
<td>0%</td>
<td>110bps</td>
<td>94bps</td>
</tr>
<tr>
<td>Super Senior 12%-22%</td>
<td>0%</td>
<td>46bps</td>
<td>49bps</td>
</tr>
</tbody>
</table>
Table 3.5: LHP CDO pricing. Credit spread=50bps, \( a=-0.2 \), \( b=-0.2 \), \( \beta=45\% \).

### 3.6 Conclusion

In this chapter we presented a new structural portfolio default model that is capable of producing the correlation skews observed in the single tranche CDO market. The model is a simple extension of the one factor Gaussian copula model where the factor coefficient is made to be stochastic. Such a model can also be interpreted as a stochastic correlation model similar to the stochastic volatility models used in the equity options markets. A specific tractable example of the model was presented that allows a diverse range of correlation probability distribution functions to be specified by simply calibrating two parameters. The model still preserves the monotonicity of the equity tranche par spread with respect to the correlation parameters, enabling a direct calibration routine that uses a two dimensional root search algorithm. Under the large homogenous portfolio limit, the portfolio loss distribution was derived and closed form expressions where obtained for the expected tranche loss. Such expressions provide good approximations for portfolios consisting of 500 obligors or more. A complete skew producing model like the stochastic correlation model presented in this chapter allows one to price nonstandard tranches in an arbitrage free manner and can be used as an underlying model to incorporate skews in the valuation of more complex portfolio credit derivatives such as CDO squared transactions.
Chapter 4

Shock-Gaussian Model

We present in this chapter an extension of the Gaussian one-factor copula model that is capable of producing the correlation skew observed in the single tranche CDO market. The model is a simple mixture of structural and intensity based default models. The asset values of the obligors underlying the portfolio are assumed to be jointly normally distributed, however, it is also assumed that the portfolio is subject to random shock events that can cause multiple simultaneous defaults. The model, termed the 'shock-Gaussian' model, is effectively a mixture of the one factor Gaussian copula and the Marshall-Olkin copula. Chapter 4 proceeds as follows: Section 4.1 introduces shock models and section 4.2 introduces the general framework of the shock-Gaussian model. Section 4.3 derives the portfolio loss distribution and section 4.4 analyzes the simplifying case when the portfolio is assumed to be homogeneous. Section 4.4 also derives the large homogeneous portfolio (LHP) limit loss distribution and presents closed form results for the expected tranche loss.

4.1 Introduction to Shock Models

Shock models, such as the one introduced by Giesecke (2003), are a subset of the reduced form models in that default occurs upon the first jump time of a Poisson process. The main differentiating feature of shock models is in the way default correlations are handled. In standard intensity models, default correla-
tion is introduced by allowing the intensity processes of the respective obligors to be correlated. Conditional on the realization of a particular path for the intensity processes, each obligor defaults independently. It is also possible to combine the intensity framework with a copula as described in Schönbucher and Schubert (2001). In the case of shock models, the default of a given obligor is determined not only by its individual idiosyncratic intensity process, but also on one or more intensity processes that represent the credit risk for a particular industry or cohort. Upon the first jump time associated with the intensity for a particular industry or cohort, all obligors belonging to that industry or cohort will default simultaneously. By allowing a given subset of the portfolio to default simultaneously, shock models incorporate default correlation from a top-down approach in contrast to standard structural and intensity models of default.

The following subsection will present a mathematical treatment of shock models and closely follows the work of Giesecke (2003).

4.1.1 Shock Model - Two Obligor Case

Consider Poisson counting processes $N_1, N_2,$ and $N$ with respective constant intensities $\lambda_1$, $\lambda_2$, and $\lambda$. $\lambda_1$ can be considered as the idiosyncratic intensity process associated with obligor $i$. $\lambda$ can be considered as a systematic intensity process which influences both obligors. The default time of obligor $i$ can be expressed as:

$$\tau_i = \inf \{ t \geq 0 : N_i(t) + N(t) > 0 \}$$

(4.1)

which means that obligor $i$ may default either by the first Poisson jump of its idiosyncratic counting process $N_i(t)$, or by the macro process $N(t)$. The survival probability is:

$$S_i(t) = P[\tau_i > t] = P[N_i(t) + N(t) = 0] = e^{-(\lambda_i + \lambda)t}$$

(4.2)

The joint survival probability of two obligors is:

$$S(t, u) = P[\tau_1 > t, \tau_2 > u] = P[N_1(t) = 0, N_2(u) = 0, N(t \vee u) = 0]$$
\[ e^{-\lambda_1 t - \lambda_2 u - \lambda (u+v)} \]
\[ = e^{-(\lambda_1 + \lambda) t - (\lambda_2 + \lambda) u + \lambda (u+v)} \]
\[ = S_1(t)S_2(u) \min(e^{\lambda_1 t}, e^{\lambda_2 u}) \quad (4.3) \]

Letting \( \hat{C} : [0,1]^2 \rightarrow [0,1] \) denote the survival copula we have:

\[ \hat{C} = S(S_1^{-1}(u), S_2^{-1}(v)) \]
\[ = \min(uv^{1-\theta_1}, uv^{1-\theta_2}) \quad (4.4) \]

where \( \theta_i = \frac{\lambda_i}{\lambda_i + \lambda} \).

\( \hat{C}_\theta = \min(uv^{1-\theta_1}, uv^{1-\theta_2}) \) is known as the Marshall Olkin copula function.

The Marshall Olkin copula is bounded below by the product copula and bounded above by the Fréchet upper bound copula, corresponding to the cases \( (\lambda = 0 \text{ or } \lambda_1, \lambda_2 \rightarrow \infty) \) and \( (\lambda \rightarrow \infty \text{ or } \lambda_1 = \lambda_2 = 0) \) respectively:

\[ uv \leq \hat{C}_\theta \leq \min(u,v) \]

for \( \theta \in [0,1]^2 \).

This implies that only positive default dependence is possible. The standard copula function defined by \( C(u,v) = F(F_1^{-1}(u), F_2^{-1}(v)) \) where \( F(t,u) = P[\tau_1 \leq t, \tau_2 \leq u] \) and \( F_i(t) = 1 - S_i(t) \) is related to the survival copula via the relation:

\[ C(u,v) = \hat{C}(1-u,1-v) + u + v - 1 \]
\[ = \min([1-v][1-u]^{1-\theta_1}, [1-u][1-v]^{1-\theta_2}) + u + v - 1 \quad (4.5) \]

### 4.1.2 Shock Model - Multi-Obligor Case

Assume there exists a sequence of Poisson counting processes \( N_j, j \in [1, \ldots, m] \) and assume that obligor \( i \) will default upon the first arrival time of a counting process \( N_j \) if \( a_{ij} = 1 \), where \( a_{ij} = (0,1) \). The matrix \( \{a_{ij}\} \) indicates whether shock \( j \) will effect obligor \( i \) or not. And so the default time can be specified as:

\[ \tau_i = \inf \{ t \geq 0 : \sum_{k=1}^{m} a_{ik} N_k(t) > 0 \} \]
and the survival function is given by:

$$S_i(t) = \exp\left(-\sum_{k=1}^{m} a_{ik} \lambda_k t\right)$$

The joint survival function is expressed as:

$$S(t_1, \ldots, t_n) = \exp\left(-\sum_{k=1}^{m} \lambda_k \max(a_{1k} t_1, \ldots, a_{nk} t_n)\right)$$

(4.6)

The survival copula can be constructed via

$$\hat{C}(u_1, \ldots, u_n) = S(S_1^{-1}(u_1), \ldots, S_n^{-1}(u_n)).$$

The two dimensional marginal survival copula is given by:

$$\hat{C}(u_i, u_j) = \min(u_i^{1-\theta_i}, u_j^{1-\theta_j})$$

where

$$\theta_i = \frac{\sum_{k=1}^{m} a_{ik} a_{jk} \lambda_k}{\sum_{k=1}^{m} a_{ik} \lambda_k}$$

$$\theta_j = \frac{\sum_{k=1}^{m} a_{ik} a_{jk} \lambda_k}{\sum_{k=1}^{m} a_{jk} \lambda_k}$$

Spearman’s rank correlation corresponding to this copula is (see Giesecke, 2003):

$$\rho_{ij}^S = \frac{3\theta_i \theta_j}{2\theta_i + 2\theta_j - \theta_i \theta_j}$$

(4.7)

4.2 Shock Gaussian Model - The General Framework

Let the portfolio consist of $N$ obligors where obligor $i$ has recovery rate $R_i$ and notional $l_i$. An obligor may default if either its asset value falls below some threshold or if an external shock event hits the obligor. Let $N_i(t)$ denote the Poisson process that models the external shocks that may hit obligor $i$. The default time of obligor $i$ can then be expressed as

$$\tau_i = \inf\{t \geq 0 : (A_i(t) \leq C(t)) \cup (N_i(t) \geq 1)\}$$
where \( A_i(t) \) is the asset value process of the obligor and \( C(t) \) is the time varying default threshold. Let \( P_i(t) = P[\tau_i \leq t] \) represent the cumulative default probability. There are two ways that obligor \( i \) can default before time \( t \); either if the asset value crosses the default barrier or if the obligor is hit by a shock event. In order to simplify the problem we make the assumption used in Merton (1974) and in the CreditMetrics model (Gupton et al, 1997), namely that the obligor can default due to diffusion if the asset value is below a threshold on a certain date and do not consider a first passage type framework. The asset value is normalized and of the form \( A_i = \sqrt{\beta} M + \sqrt{1-\beta} \varepsilon_i \) where \( M \) and \( \varepsilon_i \) are independent standard normal random variables. Now, it is assumed that the shock event is independent from the asset value process. Using the identity \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \), we have

\[
P_i(t) = P[A_i \leq C(t)] + P[N_i(t) \geq 1] - P[A_i \leq C(t)]P[N_i(t) \geq 1]
\]

therefore,

\[
P[A_i \leq C(t)] = \frac{P_i(t) - P[N_i(t) \geq 1]}{1 - P[N_i(t) \geq 1]}
\]

and the default threshold may be obtained from expression 4.10

\[
C(t) = \Phi^{-1} \left( \frac{P_i(t) - P[N_i(t) \geq 1]}{1 - P[N_i(t) \geq 1]} \right)
\]

Clearly we must have \( P_i(t) > P[N_i(t) \geq 1] \). Hence in order to calculate the default thresholds \( C(t) \), we must first make an assumption regarding the nature of the shock process \( N_i(t) \).

It is assumed that the shock indicator process is made up of idiosyncratic and macro shock parts:

\[
N_i(t) = N^I_i(t) + N^M_i(t)
\]

where \( N^I_i(t) \) represents idiosyncratic default shock due to, for example, fraudulent accounting practices. \( N^M_i(t) \) represents a systemic shock event in which the occurrence of such a shock will cause a subset of the portfolio to default. This effect is supposed to mimic how an external macro shock could have a devastating
effect on a given industry or cohort. Given this setup we have that the probability of a shock event hitting obligor $i$ by time $t$ is:

$$P[N_i(t) \geq 1] = P[(N_i^I(t) \geq 1) \cup (N_i^M(t) \geq 1)]$$

Assuming $N_i^I(t)$ and $N_i^M(t)$ are independent, this simplifies to

$$P[N_i(t) \geq 1] = P[(N_i^I(t) \geq 1)] + P[(N_i^M(t) \geq 1)] - P[(N_i^I(t) \geq 1)]P[(N_i^M(t) \geq 1)]$$

If the intensity process of $N_i^I(t)$ is stochastic, then $P[(N_i^I(t) \geq 1)]$ can be readily computed as

$$P[(N_i^I(t) \geq 1)] = 1 - E[e^{-\int_0^t h_i^I(u)du}]$$

Where $h_i^I(t)$ is the intensity driving the idiosyncratic counting process $N_i^I(t)$. Closed form results exist for the expectation term for certain classes of the intensity process such as the Ornstein-Uhlenbeck or CIR processes. It now remains to specify the nature of $N_i^M(t)$. Assume the existence of $k$ Poisson processes $N_i^1(t), \ldots, N_k(t)$. $N_i^M(t)$ is equal to the sum of some subset of these counting processes:

$$N_i^M(t) = \sum_{s=1}^k a_isN_s(t)$$

where $a_is$ takes the value either 0 or 1 and indicates whether obligor $i$ is affected by the shock process $N_s(t)$. For example, $N_j(t)$ could represent a shock to a given industry and $N_k(t)$ a shock to a certain region. We call the shock processes $N_j(t), j = 1, \ldots, k$, ‘macro shock’ processes. We can again associate with $N_j(t), j = 1, \ldots, k$, a given intensity such that $P[N_j(t) \geq 1] = 1 - E[e^{-\int_0^t h_j(u)du}]$. It then follows naturally that the probability that obligor $i$ is hit by a macro shock is given by:

$$P[N_i^M \geq 1] = 1 - E[e^{-\int_0^1 \sum_{j=1}^k a_{ij}h_j(u)du}]$$

This basically tells us that the macro shock intensity seen by obligor $i$ is the sum of the intensities of the counting processes $a_{ij}N_j(t), j = 1, \ldots, k$. It is up to the user to specify which counting process effect obligor $i$ by specifying $a_{ij} = (0, 1)$.

We have now fully specified the building blocks of the model. The steps below give a summary of how the model should be parameterized.
1. Obtain for each obligor the marginal default probabilities \( P(\tau \leq t) \).

2. Specify for each obligor an idiosyncratic shock intensity such that the probability of an idiosyncratic shock hitting obligor \( i \) is given by \( P[(N_i^i(t) \geq 1)] = 1 - E[e^{-\int_0^t k_i^i(u)du}] \).

3. Specify a set of \( k \) intensities for the \( k \) macro shock indicator processes \( \bar{N}_j(t) \), \( j = 1, \ldots, k \).

4. Choose which macro shocks will affect obligor \( i \) by specifying the vector \( [a_{i1}, \ldots, a_{ik}]^T \), where \( a_{ij} = (0, 1) \). The probability that a macro shock will hit obligor \( i \) is then given by \( P[\bar{N}_i^M \geq 1] = 1 - E[e^{-\int_0^t \sum_{j=1}^k a_{ij} \bar{N}_j(u)du}] \).

5. The probability that obligor \( i \) will be hit by a shock, whether idiosyncratic or not is simply \( P[N_i(t) \geq 1] = P[(N_i^i(t) \geq 1)] + P[(\bar{N}_i^M(t) \geq 1)] - P[(N_i^i(t) \geq 1)]P[(\bar{N}_i^M(t) \geq 1)] \).

6. Using the following expression, calculate the asset value thresholds, \( C(t) = \Phi^{-1}\left( \frac{P(i \notin \Omega) \cdot P[N_i(t) \geq 1]}{1 - P[N_i(t) \geq 1]} \right) \).

We now proceed to derive the expressions for the portfolio loss distribution using the recursive method of Anderson et al (2003).

### 4.3 Deriving the Portfolio Loss Distribution

This section derives the cumulative portfolio loss distribution for time horizon \( t \). Recall the existence of \( k \) macro shock processes \( \bar{N}_j(t), j = 1, \ldots, k \). Assume that some subset \( \pi \subset \{1, \ldots, k\} \), of the shock processes has experienced jumps by time \( t \), expressed mathematically as \( \bar{N}_j(t) \geq 1 \) for \( j \in \pi \) and \( \bar{N}_n(t) = 0 \) for \( n \notin \pi \). Now, we know by construction exactly which obligors have defaulted due to the jumps in the processes \( \{\bar{N}_j(t), j \in \pi\} \) and denote the set of corresponding defaulted obligors by \( \Theta \). Conversely by conditioning on the macro shock processes, we also know which obligors did not default and let us denote the set of surviving obligors by \( \Upsilon = \Omega - \Theta \), where \( \Omega \) is the set of all obligors in the portfolio. Letting
A denote the event $\Lambda = (\mathcal{N}_j(t) \geq 1, j \in \pi; \mathcal{N}_n(t) = 0, n \notin \pi)$, the conditional default probability of a surviving obligor $i$ is given by:

$$P[\tau_i \leq t|\Lambda] = P[(A_i \leq C(t)) \cup (N^I_i(t) \geq 1)]$$

In other words, obligor $i$ may default either if the asset value falls below its default threshold or if an idiosyncratic shock hits it. Since both these events are independent, we have:

$$P[\tau_i \leq t|\Lambda] = P[A_i \leq C(t)] + P[N^I_i(t) \geq 1] - P[A_i \leq C(t)].P[N^I_i(t) \geq 1]$$

If we now also condition on the market variable $M = m$, the conditional default probabilities become:

$$P[\tau_i \leq t|\Lambda, M = m] = \Phi\left(\frac{C(t) - \beta m}{\sqrt{1 - \beta^2}}\right)(1 - P[N^I_i(t) \geq 1]) + P[N^I_i(t) \geq 1]$$

(4.11)

Since we have conditioned on the market factor $M = m$, all obligors now default independently and we can thus invoke the recursive algorithm for computing the conditional portfolio loss distribution.

We stated before that $\Theta$ is the set of all obligors that have defaulted due to the occurrence of macro shock events. The loss as a result of this is denoted by $L_\Theta = \sum_{k \in \Theta} R_k(1 - R_k)$. Let us also denote the cumulative portfolio loss at time $t$ by $L(t)$. It is clear that $P[L(t) < L_\Theta|\Lambda] = 0$ since the conditional loss is already at least $L_\Theta$. For $l \geq L_\Theta$ we have the following expression:

$$P[L(t) \leq l|\Lambda] = P[L^{-\Theta}(t) \leq l - L_\Theta|\Lambda]$$

where $L^{-\Theta}(t)$ represents the loss to the portfolio where the defaulted assets belonging to set $\Theta$ are removed. Conditioning now also on the market variable $M = m$, the conditional portfolio loss density becomes:

$$P[L(t) = l|\Lambda, M = m] = P[L^{-\Theta}(t) = l - L_\Theta|\Lambda, M = m]$$
Denoting by $L^{-\Theta-k}(t)$ the loss on the portfolio where the $k^{th}$ obligor and the obligors belonging to set $\Theta$ have been removed, the following recursive algorithm holds:

$$P[L^{-\Theta}(t) = l - L_\Theta | \Lambda, M = m] =$$

$$P[L^{-\Theta-k}(t) = l - L_\Theta - l_k(1 - R_k) | \Lambda, M = m].P[\tau_k \leq t | \Lambda, M = m]$$

$$+ P[L^{-\Theta-k}(t) = l - L_\Theta | \Lambda, M = m].(1 - P[\tau_k \leq t | \Lambda, M = m])$$

Note that $P[\tau_k \leq t | \Lambda, M = m]$ is given by expression (4.11). Obligor $k$ is assumed to be a member of $\Upsilon$, i.e. it is not hit by any macro shocks. This recursive algorithm can now be used to build the conditional portfolio loss distribution. The only difference here with the recursive algorithm of Andersen et al (2003) is that we exclude all obligors that defaulted due to the occurrence of macro shocks. It follows that the portfolio loss distribution conditional only on the event $\Lambda$ is given by:

$$P[L(t) = l | \Lambda] = \int_{-\infty}^{\infty} P[L(t) = l | \Lambda, M = m]1_{\{l \geq L_\Theta\}} f(m) dm$$

where $f(m)$ is the density function of the market factor. Finally, the unconditional portfolio loss distribution is obtained by integrating over all possible combinations of macro shock events:

$$P[L(t) = l] = \sum_{\nu \subseteq \{1, \ldots, k\}} P[\Lambda_\nu].P[L(t) = l | \Lambda_\nu]$$

where $\forall \nu \subseteq \{1, \ldots, k\}$ represents all possible subsets of $\{1, \ldots, k\}$ and $\Lambda_\nu$ denotes the event $\Lambda_\nu = (N_j(t) \geq 1, j \in \nu; N_n(t) = 0, n \notin \nu)$.

We have presented the general framework of the Shock-Gaussian model.

### 4.4 Homogeneous Portfolio Case

#### 4.4.1 Specifying Shock Events

In this section we concentrate on a homogeneous portfolio where every obligor has the same notional amount, credit spread, and recovery rate. Focusing on a homogeneous portfolio allows us to observe the key features of the shock-Gaussian
model. It is assumed that for a given time horizon, a number of mutually exclusive shock events exist each with varying degrees of devastation inflicted on the portfolio if it occurs. For example, a very severe shock event may cause 50% of the portfolio to default, whereas a minor shock event may cause just 5% of the portfolio to default. Let shock event \( S_k \) lead to \( n_k \) defaults out of the \( N \) obligors in the portfolio. The identity of the defaulted obligors is not of concern since each obligor has equal recovery rates and credit spreads. Given that shock event \( S_k \) occurs, the probability that obligor \( i \) will default due to it is \( v_t^{i,k} \).

In order to see the validity of this we first note that the number of ways \( n_k \) obligors can be chosen out of a total of \( N \) is \( \binom{N}{n_k} \) and the total number of ways that \( n_k \) obligors may be chosen such that one of them is always obligor \( i \) is \( \frac{(N-1)!}{(N-1-(n_k-1))!(n_k-1)!} \). Let there be \( m \) different shock events each with varying degree of severity and assume that they are mutually exclusive, so we have \( P(\bigcap_{k=0}^{m} S_k) = 0 \). Why do we impose the mutually exclusive condition? Because a medium shock event, say, may be equivalent to two small shock events. By making the shock events mutually exclusive, we are not ruling out any severity of shocks, but we are avoiding the need to deal with combinations of shock events when constructing the portfolio default distribution. Given that there are a total of \( m \) mutually exclusive macro shock events, the probability that obligor \( i \) will default due to a macro shock event by time \( t \) is:

\[
P[N_i^M(t) \geq 1] = \sum_{j=1}^{m} \frac{n_j}{N} P_l(S_j)
\]

where \( P_l(S_j) \) is the probability that shock event \( S_j \) occurs by time \( t \). Note it is assumed that \( \sum_{j=0}^{m} P_l(S_j) = 1 \), where \( S_0 \) is the event that no macro shocks happen. These shock probabilities may be expressed in terms of an intensity rate, where the intensity rate may be stochastic so that \( P_l(S_j) = 1 - E[e^{-\int_0^t \lambda_j(u)du}] \), and \( \lambda_j(u) \) is the intensity of the macro shock event \( S_j \). Hence, we have:

\[
P[N_i^M(t) \geq 1] = \sum_{j=1}^{m} \frac{n_j}{N} (1 - E[e^{-\int_0^t \lambda_j(u)du}])
\]

A given obligor may also default due to an idiosyncratic shock event, which
is specific only to that obligor. Letting \( P_t(S_t^i) \) denote the probability that the idiosyncratic shock will hit obligor \( i \) by time \( t \), we have:

\[
P_t(S_t^i) = P[N_t^i(t) \geq 1] = 1 - E \left[ e^{-\int_0^t \lambda'(u) du} \right]
\]

where \( \lambda'(u) \) is the intensity governing the counting process \( N_t^i(t) \). Since the portfolio is homogeneous, all obligors have the same idiosyncratic shock probability \( P_t(S_t^i) \) and hence identical shock intensities. We do not assume the macro shocks and the idiosyncratic shock is mutually exclusive. The probability that obligor \( i \) will default due to a shock event is then given by:

\[
P[N_t(t) \geq 1] = P[(N_t^M(t) \geq 1) \cup (N_t^I(t) \geq 1)]
\]

where \( N_t^M(t) \) and \( N_t^I(t) \) are both independent. We can then use equations (4.8) to (4.10) to compute the default threshold \( C(t) \).

### 4.4.2 Calculating the Portfolio Loss Distribution

Conditional on the realization of the market factor \( M \), the conditional default probability of each obligor when no macro shocks happen is given by:

\[
q_t(m) = P[\tau \leq t | M = m] = \Phi \left( \frac{C(t) - \sqrt{\beta m}}{\sqrt{1 - \beta}} \right) + P^I - \Phi \left( \frac{C(t) - \sqrt{\beta m}}{\sqrt{1 - \beta}} \right) \cdot P^I
\]

where \( P^I = P_t(S_t^i) \) is the probability of an idiosyncratic shock event happening by time \( t \). Since each obligor has equal default probability and defaults independently given a realization of the market factor and no macro shocks, the cumulative distribution of the number of defaults \( \bar{N} \) by time \( t \) is given by:

\[
P[\bar{N} \leq n | (S = S_0)] = \int_0^\infty \sum_{i=0}^n b(N, i, q_t(m)) \phi(m) dm
\]

where \( b(N, j, p) = \frac{\binom{N}{j} \cdot p^j (1 - p)^{N-j}}{\binom{N_j}{j}} \) is the binomial distribution and \( \phi(.) \) is the standard normal density function. \((S = S_0)\) is the event of no macro shocks. Now assume that by time \( t \) shock event \( S_k \) has occurred which has resulted in \( n_k \) defaults. This means that there are \( N - n_k \) obligors remaining in the portfolio.
Letting $\bar{N}$ denote the number of defaults in the portfolio, the portfolio default distribution conditional on the shock event $S_k$ is given by:

$$P[\bar{N} = j | S = S_k] = \int_{-\infty}^{\infty} b(N - n_k, j - n_k, q_k(m))\phi(m)dm$$

for $j \geq n_k$, otherwise $P[\bar{N} = j | S = S_k] = 0$. The unconditional default distribution is given by:

$$P[\bar{N} = j] = \sum_{k=0}^{m} P_k(S_k)P[\bar{N} = j | S = S_k]$$

$S_0$ is the event that no macro shocks happen, i.e. $n_0 = 0$.

### 4.4.3 Numerical Results

Having presented the shock Gaussian model using a homogeneous portfolio, we proceed by showing some numerical results regarding the fit to market prices and the behavior of tranche prices with respect to certain model parameters. In the analysis it was assumed that each obligor in the reference portfolio had a credit spread of 37bps, and a recovery rate of 40%. The portfolio consisted of 100 obligors with a total portfolio notional of 100m and the continuously compounded interest rate was 2%. When calibrating to market prices we assumed the existence of a single macro shock event that caused 25 obligors in the portfolio to default. The idiosyncratic shock intensity was set to 10bps, the macro shock to 38.2bps, and linear asset correlation to 11%.

Figures 4.1 and 4.2 display the compound and base correlations respectively for the shock-Gaussian model. The market data used to produce the market implied compound correlation smile and base correlation skew was the same as the market data used to compare the results of the stochastic correlation model in chapter 3. The market data refers to the ITRAXX Europe Series 2 indices for 13th October 2004 where the index credit spread was 37bps. Table 4.1 presents the results in tabular format. Note that the prices produced by the shock-Gaussian model are by no means the 'best fit' to market data. Introducing more macro shocks and using a sophisticated optimization routine will produce better results.
Figure 4.1: Compound Correlation. Market iTraxx vs Gaussian-Shock Model

Figure 4.2: Base Correlation. Market iTraxx vs Gaussian-Shock Model
Next we look at the response of tranche par spreads to movements in the intensities of the idiosyncratic and macro shocks. In this analysis it was assumed that we had a reference portfolio of 100 obligors each with a credit spread of 37bps and a recovery rate of 40%. There is only one macro shock that causes 60 obligors to default. The correlation was set at 10.25% and the continuously compounded interest rate was 2%.

Figures 4.4 to 4.8 show the variations in tranche prices for movements in the idiosyncratic shock intensities. The par spread of the equity tranche increases for an increase in the idiosyncratic shock intensity, whereas all other tranches have a reduction in par spreads. The reason why the equity tranche spread increases and the senior tranche spread decreases for an increase in the idiosyncratic shock intensity can be seen by analyzing the effect of the idiosyncratic shock on default correlation. The default correlation between two obligors $A$ and $B$ is defined as:

$$
\varrho = \frac{P_{AB} - P_A P_B}{\sqrt{P_A(1 - P_A)P_B(1 - P_B)}}
$$

where $P_A$ is the default probability of obligor $A$, $P_B$ is the default probability of obligor $B$, and $P_{AB}$ is the joint default probability of obligor $A$ and $B$. All default probabilities, $P_A$, $P_B$, and $P_{AB}$, are defined with respect to some time horizon. Note that the default correlation is different from the asset correlation used in the Gaussian copula. In the case of the shock-Gaussian model when we exclude any macro shocks, the default correlation is given by:
where $\Phi(\cdot, \cdot; \rho)$ is the bivariate Gaussian cumulative distribution function with correlation parameter $\rho$. $C_A$ and $C_B$ are given by:

$$C_A = \Phi^{-1}\left(\frac{P_A - P_I}{1 - P_I}\right)$$

$$C_B = \Phi^{-1}\left(\frac{P_B - P_I}{1 - P_I}\right)$$

where $P_I$ is the idiosyncratic shock probability. Figure 4.3 shows the response of the default correlation between two obligors for varying degrees of idiosyncratic shock probability. It can be seen that increasing the idiosyncratic shock probability reduces the default correlation and as a result the equity tranche par spread increases and the senior tranche par spread decreases. All tranche par spreads appear to vary linearly for changes in the idiosyncratic shock intensity.

Figures 4.9 to 4.13 show the response of the tranche par spreads for changes in the macro shock intensity. The responses are opposite to those observed for changes in the idiosyncratic shock intensity. The par spread of the equity and junior mezzanine tranche decrease for an increase in the macro shock intensity while all other tranche par spreads increase. This is what we would expect as an increase in the macro shock intensity means that the chances of a catastrophic scenario where the senior tranches will suffer losses increases. Again, all tranche spreads appear to respond linearly to variations in the macro shock intensity.

### 4.4.4 Large Homogeneous Portfolio Distribution

In this section we derive a closed form expression for the portfolio loss distribution assuming a homogeneous and infinitely granular portfolio. The following assumptions are made:

- The total portfolio notional is $N$. 

$$\rho = \frac{\Phi(C_A, C_B; \rho) - P_A P_B}{\sqrt{P_A (1 - P_A) P_B (1 - P_B)}}$$
Figure 4.3: Default Correlation vs Idiosyncratic Shock Probability. $P_A=10\%$, $P_B=10\%$, $\rho=30\%$.

Figure 4.4: [0%-3%] Equity Tranche Idiosyncratic Jump Response (bps)
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Figure 4.5: [3%-6%] Junior Mezzanine Tranche Idiosyncratic Jump Response (bps)

Figure 4.6: [6%-9%] Mezzanine Tranche Idiosyncratic Jump Response (bps)
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Figure 4.7: [9%-12%] Senior Tranche Idiosyncratic Jump Response (bps)

Figure 4.8: [12%-22%] Super Senior Tranche Idiosyncratic Jump Response (bps)

Figure 4.9: [0%-3%] Equity Tranche Macro Jump Response (bps)
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Figure 4.10: [3%-6%] Junior Mezzanine Macro Jump Response (bps)

Figure 4.11: [6%-9%] Mezzanine Tranche Macro Jump Response (bps)

Figure 4.12: [9%-12%] Senior Tranche Macro Jump Response (bps)
• All assets have the same default probability $P$, recovery rate $R$, and asset correlation coefficient $\beta^2$.

• Each asset has the same idiosyncratic shock probability $P^I$.

• The macro shock event $S_k$ causes $n_k\%$ of the portfolio to default.

• Shock event $S_k$ occurs with probability $P_k$.

• $j$ different and mutually exclusive macro shock events exits.

• There is an infinite number of obligors in the portfolio.

Let us start by conditioning on the realization of a shock event $S_k$ and market factor $M = m$. In this case the total loss incurred on the portfolio is:

$$n_k(1 - R)N + N(1 - n_k)q(m)(1 - R) = l$$ (4.12)

which we assume is equal to some value $l$. $q(m)$ is the conditional probability that a obligor will default either due to its asset value falling below the default threshold or if it is hit by an idiosyncratic shock event. $q(m)$ is given by expression 4.13:

$$q(m) = \Phi\left(\frac{C - \beta m}{\sqrt{1 - \beta^2}}\right)(1 - P^I) + P^I$$ (4.13)
\[ C = \Phi \left( \frac{P - P^S}{1 - P^S} \right) \]  
\[ P^S = P^M + P^I - P^M P^I \]  
\[ P^M = \sum_{k=1}^{j} n_k P_k \]  

The first term in equation (4.12) arises due to the realization of the macro shock event. The second term represents the obligors that survived the macro shock but defaulted either due to the idiosyncratic shock or due to the asset value falling below the threshold barrier. We know that exactly \( q(m) \) of the remaining portfolio defaulted by applying the Law of Large Numbers.

The value of \( m \) that gives a portfolio loss of \( l \) conditional on the macro shock \( S_k \) is:

\[ m = \frac{1}{\beta} \left( C - \sqrt{1 - \beta^2} \Phi\left( \frac{1}{1 - P^I} \left( \frac{l - n_k(1 - R)N}{N(1 - n_k)(1 - R)} - P^I \right) \right) - G \right) \]  

if \( l > n_k(1 - R)N + P^I N(1 - n_k)(1 - R) \). And so we have:

\[ P[L \leq l | S_k] = P[M > G|S_k] \mathbb{1}_{\{l > n_k(1 - R)N + P^I N(1 - n_k)(1 - R)\}} \]  

where,

\[ P[M > G|S_k] = \Phi \left( \frac{1}{\beta} \left( \sqrt{1 - \beta^2} \Phi^{-1} \left( \frac{1}{1 - P^I} \left( \frac{l - n_k(1 - R)N}{N(1 - n_k)(1 - R)} - P^I \right) \right) - C \right) \right) \]

and finally integrating with respect to the probability of shock events, we have:

\[ P[L \leq l] = \sum_{k=1}^{j} P_k P[L \leq l | S_k] \]

Combining equations (4.18), (4.19) and (4.20) gives us an expression for the portfolio loss distribution.
4.4.5 Single Tranche CDO Pricing with the Large Homogeneous Portfolio Approximation

The pricing algorithm presented in chapter 2, section 2.3.1, for single tranche CDOs effectively consists of evaluating the expected loss on a tranche for various time horizons. The expected loss on a tranche can be computed as the difference of the expected loss on two base tranches. Recall that a base tranche is a tranche with zero attachment point. The expected loss on a base tranche with detachment level \( D \) can be expressed as:

\[
E[\min(L, D)] = E[D1_{(L>D)} + L1_{(L\leq D)}] \tag{4.21}
\]

The first term in expression (4.21) can be evaluated as:

\[
E[D1_{(L>D)}] = DP[L > D] = D(1 - P[L \leq D]) \tag{4.22}
\]

and an expression for \( P[L \leq D] \) was already found in equations (4.18)-(4.20). Hence, all that remains is to evaluate the term \( E[L1_{(L \leq D)}] \). Using iterated expectations, we have:

\[
E[L1_{(L \leq D)}] = E[E[L1_{(L \leq D)}|M = m, S_k]] \tag{4.23}
\]

where we have conditioned on a realization of the market factor and macro shock event \( S_k \). In the case of an infinitely large portfolio, the portfolio loss conditional on \( M = m \) and \( S = S_k \) is given by expression 4.12. Hence, equation 4.23 can be rewritten as:

\[
E[L1_{(L \leq D)}] = E[E[(n_k(1 - R)N + N(1 - n_k)q(m)(1 - R))1_{(L \leq D)}|M = m, S_k]] \tag{4.24}
\]

\[
= (1 - R)NE[n_k1_{(L \leq D)}|M = m, S_k] + (1 - R)NE[(1 - n_k)q(m)1_{(L \leq D)}|M = m, S_k] \tag{4.25}
\]
Now, given the realization of the macro shock $S_k$ and provided $N(1 - R)n_k + P^t N(1 - n_k)(1 - R) < D$, the value of $m$ that gives a portfolio loss equal to $D$ is:

$$m = \frac{1}{\beta} \left( C - \sqrt{1 - \beta^2} \Phi^{-1} \left( \frac{1}{1 - P^t} \left( \frac{D - n_k(1 - R)N}{N(1 - n_k)(1 - R) - P^t} \right) \right) \right) = G \quad (4.26)$$

which we say is equal to some value $G$. Hence, we can replace the indicator function $1_{\{L < D\}}$ with $1_{\{M > G\}}$ in equation 4.25:

$$E[L1_{\{L \leq D\}}] = (1 - R)N E[n_k 1_{\{M > G\}} | M = m, S_k]$$

$$+ (1 - R)N E[(1 - n_k)q(m)1_{\{M > G\}} | M = m, S_k] \quad (4.27)$$

Evaluating the expectation with respect to the market factor $M$, expressions 4.27 becomes:

$$E[L1_{\{L \leq D\}}] = (1 - R)N E[n_k \Phi(-G)]$$

$$+ (1 - R)N \int_G q(m)\phi(m)dm \quad (4.28)$$

where $\phi(m)$ is the standard Gaussian probability density function and $q(m)$ is given by expression 4.13. Substituting expression 4.13 in 4.28 we get:

$$E[L1_{\{L \leq D\}}] = (1 - R)N E[n_k \Phi(-G)]$$

$$+ (1 - R)N \int_0^\infty \Phi \left( \frac{C - \beta m}{\sqrt{1 - \beta^2}} \right) \phi(m)dm$$

$$+ (1 - R)N E[(1 - n_k)P^t \int_G \phi(m)dm] \quad (4.29)$$

Making the substitution $z = -m$ in 4.29 we have:

$$E[L1_{\{L \leq D\}}] = (1 - R)N E[n_k \Phi(-G)]$$
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\[ + (1 - R)N E[(1 - n_k)(1 - P^I) \int_{-\infty}^{-G} \Phi\left(\frac{C + \beta z}{\sqrt{1 - \beta^2}}\right) \phi(z) dz] \]

\[ + (1 - R)N E[(1 - n_k)P^I(1 - \Phi(G))] \]

(4.30)

Using the integral \( \int_{-\infty}^{b} \Phi(ay + b)\phi(y)dy = \Phi\left(\frac{b}{\sqrt{1 + \alpha^2}}, k; \frac{-\alpha}{\sqrt{1 + \alpha^2}}\right) \), expression 4.30 simplifies to:

\[ E[L_1^{\{L \leq D\}}] = (1 - R)N E[n_k \Phi(-G)] \]

\[ + (1 - R)N E[(1 - n_k)(1 - P^I)\Phi(C, -G; -\beta)] \]

\[ + (1 - R)N E[(1 - n_k)P^I(1 - \Phi(G))] \]

(4.31)

All that remains now is to compute the expectation with respect to the distribution of the macro shocks.

\[ E[L_1^{\{L \leq D\}}] = (1 - R)N \sum_{k=0}^{j} P_k n_k \Phi(-G)1_{\{H < D\}} \]

\[ + (1 - R)N(1 - P^I) \sum_{k=0}^{j} P_k(1 - n_k)\Phi(C, -G; -\beta)1_{\{H < D\}} \]

\[ + (1 - R)NP^I \sum_{k=0}^{j} P_k(1 - n_k)(1 - \Phi(G))1_{\{H < D\}} \]

(4.32)

where \( H = n_k(1 - R)N + P^I N(1 - n_k)(1 - R) \). Combining expressions 4.26 and 4.32 gives us a closed form result for \( E[L_1^{\{L \leq D\}}] \) in the large portfolio limit. Hence, we have obtained expressions for \( E[D_1^{\{L \leq D\}}] \) and \( E[L_1^{\{L \leq D\}}] \) and it is now possible to rapidly compute the expected loss on a tranche and as a result rapidly compute tranche prices.

We now proceed to compare the tranche prices computed using the large homogeneous portfolio approximation method with the exact tranche prices of a sufficiently large portfolio. By sufficiently large we mean a portfolio consisting of
<table>
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<th>Tranche</th>
<th>Upfront</th>
<th>Exact Prices</th>
<th>LHP Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity 0%-3%</td>
<td>24.25%</td>
<td>2529bps</td>
<td>2397bps</td>
</tr>
<tr>
<td>Junior Mezzanine 3%-6%</td>
<td>0%</td>
<td>1017bps</td>
<td>1001bps</td>
</tr>
<tr>
<td>Senior Mezzanine 6%-9%</td>
<td>0%</td>
<td>410bps</td>
<td>401bps</td>
</tr>
<tr>
<td>Senior 9%-12%</td>
<td>0%</td>
<td>161bps</td>
<td>168bps</td>
</tr>
<tr>
<td>Super Senior 12%-22%</td>
<td>0%</td>
<td>33bps</td>
<td>34bps</td>
</tr>
</tbody>
</table>

Table 4.2: LHP CDO pricing. Credit spread=100bps, idiosyncratic shock intensity=3bps, macro intensity=0bps, macro shock magnitude =0 obligors, correlation=15%, recovery=40%.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Upfront</th>
<th>Exact Prices</th>
<th>LHP Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity 0%-3%</td>
<td>0%</td>
<td>709bps</td>
<td>721bps</td>
</tr>
<tr>
<td>Junior Mezzanine 3%-6%</td>
<td>0%</td>
<td>384bps</td>
<td>373bps</td>
</tr>
<tr>
<td>Senior Mezzanine 6%-9%</td>
<td>0%</td>
<td>191bps</td>
<td>173bps</td>
</tr>
<tr>
<td>Senior 9%-12%</td>
<td>0%</td>
<td>98bps</td>
<td>100bps</td>
</tr>
<tr>
<td>Super Senior 12%-22%</td>
<td>0%</td>
<td>64bps</td>
<td>56bps</td>
</tr>
</tbody>
</table>

Table 4.3: LHP CDO pricing. Credit spread=77bps, idiosyncratic shock intensity=0bps, macro intensity=50bps, macro shock magnitude =495 obligors, correlation=25%, recovery=40%.

500 obligors. The result are reported in tables 4.2-4.5 and it can be seen that the LHP approximates well the true tranche prices for sufficiently large portfolios.

### 4.5 Conclusion

In this chapter we presented a new extension of the one factor Gaussian copula model that is capable of producing the correlation skew observed in the single tranche CDO market. The new model incorporates external Poisson shocks that have a devastating effect on the portfolio. This feature models extreme macro shock events such as an unforeseen global or regional disaster. Idiosyncratic shock
### Table 4.4: LHP CDO pricing. Credit spread=77bps, idiosyncratic shock intensity=0bps, macro intensity=14bps, macro shock magnitude =100 obligors, second macro intensity=20bps, second macro shock magnitude=275 obligors, correlation=15%, recovery=40%.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Upfront</th>
<th>Exact Prices</th>
<th>LHP Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity 0%-3%</td>
<td>24.25%</td>
<td>1619bps</td>
<td>1606bps</td>
</tr>
<tr>
<td>Junior Mezzanine 3%-6%</td>
<td>0%</td>
<td>605bps</td>
<td>600bps</td>
</tr>
<tr>
<td>Senior Mezzanine 6%-9%</td>
<td>0%</td>
<td>213bps</td>
<td>215bps</td>
</tr>
<tr>
<td>Senior 9%-12%</td>
<td>0%</td>
<td>95bps</td>
<td>94bps</td>
</tr>
<tr>
<td>Super Senior 12%-22%</td>
<td>0%</td>
<td>34bps</td>
<td>34bps</td>
</tr>
</tbody>
</table>

### Table 4.5: LHP CDO pricing. Credit spread=67bps, idiosyncratic shock intensity=1bps, macro intensity=1bps, macro shock magnitude =20 obligors, correlation=15%, recovery=0%.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Upfront</th>
<th>Exact Prices</th>
<th>LHP Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity 0%-3%</td>
<td>24.25%</td>
<td>1413bps</td>
<td>1393bps</td>
</tr>
<tr>
<td>Junior Mezzanine 3%-6%</td>
<td>0%</td>
<td>550bps</td>
<td>538bps</td>
</tr>
<tr>
<td>Senior Mezzanine 6%-9%</td>
<td>0%</td>
<td>194bps</td>
<td>198bps</td>
</tr>
<tr>
<td>Senior 9%-12%</td>
<td>0%</td>
<td>84bps</td>
<td>81bps</td>
</tr>
<tr>
<td>Super Senior 12%-22%</td>
<td>0%</td>
<td>17bps</td>
<td>18bps</td>
</tr>
</tbody>
</table>
events were also included that capture unpredictable defaults due to, say, fraudulent accounting practices. The new model, termed the 'shock-Gaussian' model, was presented in the conditional independence framework and a modified recursive algorithm was presented to compute the portfolio loss distribution. Finally, the large homogeneous portfolio loss distribution was derived along with closed form expressions for the expected tranche loss. The pricing accuracy of the large homogeneous portfolio approximation was compared to the exact tranche prices and was found to provide good accuracy for a sufficiently large portfolio that consisted of 500 obligors or more.
Chapter 5

Conclusion and Future Work

In this thesis we developed two new credit portfolio models that are capable of producing the base correlation skew observed in the single tranche CDO market. It is crucial to develop a credit portfolio model that produces the base correlation skew since the current market standard of pricing CDO tranches using an interpolation scheme is not arbitrage free. Both models were developed in the conditional independence framework which allows semi-analytic computations of the portfolio loss distribution. The first such model incorporated stochastic correlation in the one factor Gaussian copula model. Stochastic correlation allows the default correlation to increase in times of market depression while allowing idiosyncratic risk to determine the health of a firm in times of market prosperity. The stochastic correlation model produced single tranche CDO spreads that were very close to those observed in the credit derivatives market. An explicit expression for the stochastic correlation copula was found and a closed form solution was derived for the portfolio loss distribution in the large homogeneous portfolio limit. Closed form expressions were also found for the expected loss on a tranche in the large portfolio limit that allows rapid pricing of CDO tranches. The large portfolio limit provided a good approximation to the spread of CDO tranches that referenced portfolios of 500 obligors or more. The second proposed model was termed the 'shock-Gaussian' model and incorporated external shocks to the portfolio that is capable of causing many simultaneous defaults. In the shock-Gaussian model an obligor may default either if its asset value falls below
a certain threshold or if it is hit by an external Poisson shock. Incorporation of shock effects in the one factor Gaussian copula allows one to model unpredictable defaults such as those due to fraud or an unforeseen macro level disaster. The shock-Gaussian model also produced base correlation skews similar to those observed in the credit derivatives market. The large homogeneous loss distribution was derived for the shock-Gaussian model and closed form expressions were found for the expected loss on a tranche. The large portfolio approximation method worked well for portfolios consisting of 500 obligors or more. It is possible to mix the stochastic correlation model with the shock-Gaussian model. A new algorithm was also presented that prices CDO\(^2\) derivatives. The new algorithm is a mixture of the conditional independence approach and Monte Carlo simulation. It effectively consists of approximating the overlap amongst the child portfolios via a linear dependence structure and simulates the losses on the child portfolios. The Monte Carlo routine is fast since we only simulate the loss on the child portfolios (no more than 10 usually) rather than simulate the default times of all the underlying obligors. Any portfolio default model may be used with the new CDO\(^2\) pricing algorithm.

There are a number of issues not addressed in this thesis that are becoming increasingly important in the credit derivatives market. The first such issue is the need to develop a CDO model that not only fits the correlation skew at a single maturity but also simultaneously fits the correlation skew at other maturities. For example at the time the research for this thesis was conducted only the 5 year maturity index single tranche CDOs were traded. However, the market is evolving and liquidly traded tranches are now available for the 3 year, 5 year, 7 year, and 10 year maturities. Hence there exists a growing need to develop models that fit the entire correlation surface, not just a skew. One possible way of fitting the term structure of tranche spreads is by extending the base correlation framework to incorporate time dependent correlation. Pricing a single tranche CDO effectively consists of computing the cumulative expected loss on a tranche for a series of time intervals. Letting the set of time intervals be denoted by \( t = t_1, \ldots, t_n \), it is possible to compute the cumulative expected tranche losses \( E_{t_1}^{\text{trch}}(t_1), \ldots, E_{t_n}^{\text{trch}}(t_n) \), where \( E_{t_i}^{\text{trch}}(t_i) \) denotes the cumulative expected loss at
time $t_i$ on a tranche that was computed using the one factor Gaussian copula model with correlation coefficient equal to $\rho_{t_i}$. It is possible to bootstrap a set of time dependent correlations $\{\rho_{t_i}\}$ to fit the term structure of single tranche CDO spreads. However, there are two major disadvantages to this approach. The first problem is that $E^{trch}_{\rho_{t_i}}(t_i)$ represents the cumulative expected loss, so there is an inconsistency in that we are using different correlations for overlapping time periods. The second and far more serious problem is that this method is not arbitrage free. The reason for this is that by using different correlations for different time periods, we no longer ensure that the cumulative tranche expected loss is a strictly increasing function of time. Mathematically the arbitrage violation can be expressed as:

$$E^{trch}_{\rho_{t_i}}(t_i) \neq E^{trch}_{\rho_{t_j}}(t_j)$$

for $t_i > t_j$.

An alternative approach to term structure modelling that is more intuitive and arbitrage free is to use a first passage framework with forward correlations. To demonstrate this approach we assume that the asset value of an obligor at time $t$ is of the form:

$$A(t) = \rho_t M + \sqrt{1-\rho_t^2} \varepsilon$$

where $M$ and $\varepsilon$ are both independent standard Gaussian random variables. Assuming a discrete set of default barriers $C_{t_1}, \ldots, C_{t_n}$, the probability that the obligor will default within the time interval $[t_i, t_{i+1}]$ can be expressed as:

$$P[t_i < \tau \leq t_{i+1}] = P[A(t_1) > C_{t_1} \cap A(t_2) > C_{t_2} \cap \ldots \cap A(t_i) > C_{t_i} \cap A(t_{i+1}) \leq C_{t_{i+1}}]$$

where $\tau$ represents the default time of the obligor. Given this default generating mechanism, the following recurrencs relation holds:

$$P[\tau \leq t_{i+1} \mid M = m] - P[\tau \leq t_i \mid M = m] =
\left[ \Phi \left( \frac{C_{t_{i+1}} - \rho_{t_{i+1}} m}{\sqrt{1-\rho_{t_{i+1}}^2}} \right) - \Phi \left( \max_{n=1}^{i} \left( \frac{C_{t_n} - \rho_{t_n} m}{\sqrt{1-\rho_{t_n}^2}} \right) \right) \right]^+$$
where the operator $[x]^+$ is defined as $[x]^+ = \max(x, 0)$. Integrating over the
distribution of $M$ we obtain the following expression for the default probability:

$$P[\tau \leq t_{i+1}] =$$

$$\int_{-\infty}^{\infty} \left[ \Phi \left( \frac{C_{t_{i+1}} - \rho_{t_{i+1}} m}{\sqrt{1 - \rho_{t_{i+1}}^2}} \right) - \Phi \left( \max_{n=1}^{i} \left( \frac{C_{t_n} - \rho_{t_n} m}{\sqrt{1 - \rho_{t_n}^2}} \right) \right) \right]^+ \phi(m) dm + P[\tau \leq t_i]$$

We can use this reoccurrence relation to recursively compute the default
thresholds to match the marginal default probabilities. Once all the thresholds
have been calibrated, the single obligor conditional default probability can be
expressed as:

$$P[\tau \leq t_i | M = m] = \Phi \left( \max_{n=1}^{i} \left( \frac{C_{t_n} - \rho_{t_n} m}{\sqrt{1 - \rho_{t_n}^2}} \right) \right)$$

This expression can be used in the conditional independence framework to price
single tranche CDOs semi-analytically. The advantage of this approach is that
the correlations, $\rho_{t_i}$, can be interpreted as actual forward correlations and for
the case of constant correlations, the model reduces to the standard one factor
Gaussian copula model. It is possible to have a slightly different definition of the
asset value and assume that the idiosyncratic noise term is also time dependent:

$$A(t) = \rho_t M + \sqrt{1 - \rho_t^2} \varepsilon(t)$$

where $\varepsilon(t)$ is independent for each $t$. Given this definition of the asset value
process the following reoccurrence relation holds:

$$P[\tau \leq t_{i+1} | M = m] - P[\tau \leq t_i | M = m] =$$

$$\prod_{n=1}^{i} \left( 1 - \Phi \left( \frac{C_{t_n} - \rho_{t_n} m}{\sqrt{1 - \rho_{t_n}^2}} \right) \right) \Phi \left( \frac{C_{t_{i+1}} - \rho_{t_{i+1}} m}{\sqrt{1 - \rho_{t_{i+1}}^2}} \right)$$

Again, we can recursively compute the default thresholds, $C_{t_i}$, and use the con-
dditional independence approach to price synthetic single tranche CDOs. Exper-
imenting with these forward correlation models and assessing if they can easily
be calibrated to the single tranche CDO market is clearly an important area for
future research.
The models presented in this thesis are capable of pricing any portfolio credit derivative where the payoff is a function of the default times and recovery rates only. New credit portfolio products are being developed where the payoff depends not only on the times of default but also on the credit spread levels. One such product is a European option on a single tranche CDO. Clearly a totally different approach must be taken to model such complex products which takes into account the time evolution of tranche spreads. Due to the large dimensionality involved in modelling CDOs, it is probably best to take the portfolio loss distribution as the basic building block of the model and find suitable no arbitrage constraints on the loss dynamics.
Bibliography


Appendix A

A Primer On Gaussian Quadrature

We are often interested in numerically computing the integral:

\[ V(f) = \int_a^b f(x)dx \] (A.1)

The usual practice is to use a polynomial \( q_k(x) \), where the subscript \( k \) denotes its degree, to interpolate \( f(x) \) between \( k + 1 \) points. We then have,

\[ V(f) \approx \int_a^b q_k(x)dx \] (A.2)

If we use the Lagrange form of \( q_k(x) \), then \( q_k(x) \) is expressed as the sum of \( k + 1 \) functionals, i.e. \( q_k(x) = \sum_{i=1}^{k+1} l_i(x) f(x_i) \) where \( x_i, i = 1 : k + 1 \), represent the points of interpolation. For example suppose we want the Lagrange representation of a polynomial of degree one that interpolates \( f(x) \) at the points \( a \) and \( b \), then \( P_1(x) = \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b) \), where \( P_k(x) \) is shorthand for ‘polynomial of degree \( k \)’. As a result of this, the approximate integral in (A.2) is expressed as:

\[ V(f) \approx \int_a^b \sum_{i=1}^{k+1} l_i(x) f(x_i)dx \]

\[ V(f) \approx \sum_{i=1}^{k+1} f(x_i) \left( \int_a^b l_i(x)dx \right) \]
Letting \( w_i = \left( \int_a^b \ell_i(x) \, dx \right) \) we have:

\[
V(f) \simeq \sum_{i=1}^{k+1} f(x_i)w_i
\]

Using polynomials of different degrees results in different approximation integrals. For degree zero we have the midpoint rule, degree one produces the trapezoidal rule and using a second degree polynomial gives Simpson’s rule. Clearly, if the original function is a polynomial of order \( k \), then using an interpolating polynomial of order \( k \) gives exact results. Stated differently, to exactly evaluate \( \int_a^b f(x) \, dx \) where \( f(x) = P_{k-1}(x) \), then we need to evaluate \( f(x) \) at \( k \) points. Although using higher degree polynomials usually results in more accurate integrals, the usual practice is to use a relatively low order (one or two) polynomial and to use it in what is called a composite framework. In the composite version, the domain of integration is usually divided into \( N \) subintervals and for each subinterval we perform the polynomial technique presented above. Hence, we are essentially integrating a piecewise polynomial function. Since we are simply replicating an integration formula over many subintervals, the nodes of integration are fixed and are defined by the type of integration and number of subintervals. To keep the composite integration formulae compact, the nodes of integration are usually placed equidistant from one and other.

Having viewed the most common numerical integration technique, we proceed to develop the Gauss Quadrature methodology.

The first distinction in Gaussian Quadrature is the choice of interpolating nodes. Previously the nodes were fixed at equidistance and the corresponding weights, \( w_i \), used to compute the integral. In Gauss Quadrature, the nodes are not fixed at equidistance from one another but chosen so that for a given number of nodes, the quadrature exactly computes the integral of a polynomial of as high a degree as possible. By careful placing of the nodes Gaussian Quadrature can exactly integrate all polynomials up to degree \( 2k - 1 \) using only \( k \) points. Previously using \( k \) nodes would only exactly integrate a function up to order \( k - 1 \). Another distinction with Gaussian Quadrature is that we consider the evaluation of weighted integrals of the form \( V_H(f) = \int_a^b f(x)H(x) \, dx \) where \( H(x) \)
is a positive function.

Approximating \( f(x) \) by a polynomial \( P_{k-1}(x) \) we have:

\[
V_H(f) \approx \int_a^b P_{k-1}(x)H(x)dx = \sum_{i=1}^k w_i f(x_i) \quad \text{(A.3)}
\]

Where

\[
w_i = \int_a^b l_i(x)H(x)dx
\]

The following theorem is a key ingredient in Gaussian Quadrature.

**Theorem 3** If \( f \) is a function of degree \( 2k - 1 \), \( \int_a^b P_{k-1}(x)H(x)dx \) is equal to \( V_H(f) \) if and only if

\[
\int_a^b h_k(x)P_{k-1}(x)H(x)dx = 0 \quad \text{(A.4)}
\]

where \( P_{k-1} \) is some arbitrary polynomial of degree \( k - 1 \), \( h_k(x) = (x - x_1)(x - x_2)\ldots(x - x_k) \) and the points \( x_1, x_2, \ldots, x_k \) are the nodes of interpolation.

**Proof:** Let \( f(x) \) be an arbitrary polynomial of degree \( 2k - 1 \), and let \( P_{k-1}(x) \) be the polynomial that interpolates \( f(x) \) at the points \( x_1, \ldots, x_k \). The nodes are the roots of the polynomial \( f(x) - P_{k-1}(x) \), and so:

\[
f(x) - P_{k-1}(x) = (x - x_1)(x - x_2)\ldots(x - x_k)g_{k-1}(x) = h_k(x)g_{k-1}(x)
\]

Where \( h_k(x) = (x - x_1)(x - x_2)\ldots(x - x_k) \) and for some \( (k - 1) \) degree polynomial \( g_{k-1} \). Rearranging,

\[
f(x) = h_k(x)g_{k-1}(x) + P_{k-1}(x)
\]

So,

\[
V_H(f) = \int_a^b f(x)H(x)dx = \int_a^b h_k(x)g_{k-1}(x)H(x)dx + \int_a^b P_{k-1}(x)H(x)dx
\]

And from (A.4) it follows that

\[
V_H(f) = \int_a^b P_{k-1}(x)H(x)dx
\]
Hence, (A.4) is a sufficient condition to conclude that for a suitable choice of integrating nodes \( \int_a^b P_{k-1}(x)H(x)\,dx \) is equivalent to integrating \( \int_a^b f(x)H(x)\,dx \) where \( f(x) \) is of degree \( 2k - 1 \). To show that it is a necessary condition express \( f(x) \) as \( f(x) = h_k(x)P_{k-1}(x) \), where \( P_{k-1}(x) \) is some polynomial of degree \( k - 1 \) and assume \( \int_a^b f(x)H(x)\,dx \neq 0 \). But \( \sum_{i=1}^k w_i h_k(x_i)P_{k-1}(x_i) = 0 \) since the \( x_i \)'s \( \) are the roots of \( h_k(x) \), which shows that (A.4) is a necessary condition.

To show that the theorem does not work for degree greater than \( 2k - 1 \), consider \( f(x) = h_k^2(x) \) which has degree \( 2k \). Then,

\[
V_H(f) = \int_a^b h_k^2(x)H(x)\,dx = \sum_{i=1}^k w_i h_k^2(x_i) = 0
\]

which is a contradiction.

The next question we aim to answer is how to find the function \( h_k(x) \) such that \( \int_a^b P_{k-1}(x)h_k(x)H(x)\,dx = 0 \) so that the abscissas is just the roots of \( h_k(x) \).

Start by defining the inner product with respect to \( H(x) \) as \( \langle f, y \rangle_H = \int_a^b f(x)y(x)H(x)\,dx \). \( f \) and \( y \) are orthogonal w.r.t \( H(x) \) if \( \langle f, y \rangle_H = 0 \). Hence, for (A.4) to hold we require:

\[
\langle h_k, P_{k-1} \rangle_H = 0 \tag{A.5}
\]

For any \( k - 1 \) degree polynomial \( P_{k-1}(x) \). We look for a sequence of polynomials \( h_1, h_2, \ldots, h_k \) for which \( \langle h_k, P_{k-1} \rangle_H = 0 \) hold. Construction of the orthogonal polynomials relies on the following theorem.

**Theorem 4** For each weight function \( H(x) \) there is a unique system of orthogonal polynomials \( h_k \) with leading coefficient one for which \( \langle h_k, P_{k-1} \rangle_H = 0 \). The sequence starts with \( h_{-1}(x) = 0, \ h_0(x) = 1 \). For \( k > 0 \) the polynomials are given by:

\[
h_{k+1}(x) = (x - \gamma_k)h_k(x) - \phi_k h_{k-1}(x) \tag{A.6}
\]

where

\[
\gamma_k = \frac{\langle xh_k, h_k \rangle_H}{\langle h_k, h_k \rangle_H}, \quad \phi_k = \frac{\langle h_{k+1}, h_{k+1} \rangle_H}{\langle h_k, h_k \rangle_H}
\]
Proof: The theorem is proved via induction. By construction in (A.6) all polynomials have leading coefficient of one. For $k = 1$, we have,

$$< h_1, P_0 >_H = < h_1, C >_H = C < h_1, h_0 >_H$$

where $C$ is a constant.

$$C < h_1, h_0 >_H = C < (x - \gamma_0) h_0, h_0 >_H$$

$$= C[< x h_0, h_0 >_H - \gamma_0 < h_0, h_0 >_H] = 0$$

Which proves the case for $k = 1$. Now suppose $< h_k, P_{k-1} >_H = 0$ holds for all $k = 1, \ldots, n$. Since $P_{k-1}$ represents a general polynomial of degree $k - 1$ it can be replaced by the polynomial $h_{k-1}$, and it follows that

$$< h_{n+1}, h_n >_H = < x h_n, h_n >_H - \gamma_n < h_n, h_n >_H - 0 = 0 \quad (A.7)$$

Moreover,

$$< h_{n+1}, h_{n-1} >_H = < x h_n, h_{n-1} >_H - 0 - \phi_n < h_{n-1}, h_{n-1} >_H$$

$$= < x h_n, h_{n-1} >_H - < h_n, h_n >_H \quad (A.8)$$

$h_n$ and $h_{n-1}$ have leading coefficient one and as a result $h_{n-1}$ can be expressed as $h_{n-1} = x^{n-1} + d_{n-2}$, where $d_{n-2}$ is some polynomial of degree $n - 2$. Using this together with the fact that $h_n$ is orthogonal to any polynomial of degree $k - 1$, (A.8) simplifies to:

$$= < h_n, x^n >_H - < h_n, h_n >_H$$

$$= < h_n, x^n >_H - < h_n, x^n >_H - < h_n, b_{n-1} >_H = 0$$

Where we have decomposed $h_n$ into $h_n = x^n + b_{n-1}$, for some $n - 1$ degree polynomial $b_{n-1}$. It is also clear that for any $n - 2$ degree polynomial $q_{n-2},$
\[ <h_{n+1}, q_{n-2}>_H = 0, \text{ since } h_{n+1} \text{ is orthogonal to } h_{n-1}, \text{ and } h_{n-1} \text{ can be decomposed into the sum } h_{n-1} = q_{n-2} + v_{n-1}, \text{ for some polynomial } v_{n-1}. \text{ Now, the interpolating polynomial } P_n \text{ can be written as } P_n(x) = ah_n(x) + bh_{n-1}(x) + q_{n-2}(x) \text{ for some constants } a \text{ and } b \text{ and polynomial } q_{n-2}, \text{ since } h_n \text{ and } h_{n-1} \text{ have leading coefficient equal to one. As a result of this, we can state that:}
\]
\[ <h_{n+1}, P_n>_H = a <h_{n+1}, h_n>_H + b <h_{n+1}, h_{n-1}>_H + <P_{n+1}, q_{n-2}> = 0 \]

Which concludes the proof by induction.

So, in summary, the Gaussian Quadrature technique requires the following steps to be evaluated:

1. Decide on the number of integration nodes, say \( n \).
2. Find the set of orthogonal polynomials, \( h_n \), with respect to a certain weight function \( H(x) \).
3. Find the roots of the polynomial, \( h_n \), by using some root finding algorithm. The roots of the polynomial are the nodes of integration.
4. Find the corresponding weights, \( w_i \). The weight may be computed by the formula \( w_i = \frac{<h_{n-1}, h_n>_H}{h'_n(x_i)h'_n(x_j)} \), where \( h'_n(x_j) \) is the derivative of \( h_n \) evaluated at the root \( x_j \). The proof of this is not given in this thesis.

Press et al (1992) gives code for generating the nodes and weights for various weight functions \( H(x) \), including \( H(x) = e^{-\frac{x^2}{2}} \), which is of particular importance to integrals involving the Gaussian Density.
Appendix B

Useful Gaussian Integrals

The following notation will be used:

$\phi(y)$ - Standard normal density function.

$\Phi(y)$ - Standard normal cumulative distribution function.

$\Phi(y, m; \beta)$ - Bivariate standard normal cumulative distribution function with correlation coefficient $\beta$.

$a, b, c, d$ are real constants.

$E(\cdot)$ denotes the expectation of a random variable.

$Y$ is a standard normal random variable.

Proofs of integrals 1, 3, 4, and 6 are taken from Andersen and Sidenius (2004).

Integral 1

$$\int_{-\infty}^{\infty} \Phi(ay + b)\phi(y)dy = \Phi \left( \frac{b}{\sqrt{1 + a^2}} \right)$$
APPENDIX B. USEFUL GAUSSIAN INTEGRALS

Proof: Let $W = -aY + Z$ where $Y \sim N(0,1)$, $Z \sim N(0,1)$ and $Y$ and $Z$ are independent. From this construction we have $W \sim N(0, \sqrt{1 + a^2})$ and $\text{Prob}[W \leq b] = \Phi\left(\frac{b}{\sqrt{1 + a^2}}\right)$. Using the law of iterated expectations we also have that:

$$\text{Prob}[W \leq b] = E[1_{\{W \leq b\}}] = E[E[1_{\{W \leq b\}}|Y]]$$

$$= E[\text{Prob}[Z \leq b + ay]] = \int_{-\infty}^{\infty} \Phi(ay + b)\phi(y)dy$$

Integral 2

$$\int_{-\infty}^{\infty} \Phi(ay + b)\Phi(cy + d)\phi(y)dy = \Phi\left(\frac{b}{\sqrt{1 + a^2}}, \frac{d}{\sqrt{1 + c^2}}; \frac{ac}{\sqrt{1 + a^2\sqrt{1 + c^2}}}\right)$$

Proof: Let

$$Y_1 = -aY + W_1$$
$$Y_2 = -cY + W_2$$

Where $Y \sim N(0,1)$, $W_1 \sim N(0,1)$, $W_2 \sim N(0,1)$ and are each independent from one another. From this construction we have that the linear correlation coefficient between $Y_1$ and $Y_2$ is:

$$\text{Corr}(Y_1, Y_2) = \frac{E[Y_1Y_2]}{\sigma_{Y_1}\sigma_{Y_2}} = \frac{ac}{\sqrt{1 + a^2\sqrt{1 + c^2}}}$$

And it follows that:

$$\text{Prob}[Y_1 \leq b, Y_2 \leq d] = \Phi\left(\frac{b}{\sqrt{1 + a^2}}, \frac{d}{\sqrt{1 + c^2}}; \frac{ac}{\sqrt{1 + a^2\sqrt{1 + c^2}}}\right)$$

Now, using the law of iterated expectations:

$$\text{Prob}[Y_1 \leq b, Y_2 \leq d] = E[1_{\{Y_1 \leq b, Y_2 \leq d\}}]$$
\[ E[E[1(Y_1 \leq b, Y_2 \leq d)|Y = y]] = E[Prob[W_1 \leq b + ay, W_2 \leq d + cy]] \]

But since \( W_1 \) and \( W_2 \) are independent, we have:

\[ Prob[Y_1 \leq b, Y_2 \leq d] = E[\Phi(ay + b)\Phi(cy + d)] \]

**Integral 3**

Define \( \alpha = \frac{a}{\sqrt{1 + \alpha^2}} \). Then

\[ E[Y\Phi(ay + b)] = \alpha\phi(b\sqrt{1 - \alpha^2}) \]

Proof:

\[ E[Y\Phi(ay + b)] = \int_{-\infty}^{\infty} y\phi(y)\Phi(ay + b)dy \]

Using \( \frac{d\phi(y)}{dy} = -y\phi(y) \)

\[ = -\int_{-\infty}^{\infty} \frac{d\phi(y)}{dy}\Phi(ay + b)dy \]

And integrating by parts gives:

\[ = -[\phi(y)\Phi(ay + b)]_{-\infty}^{\infty} + a\int_{-\infty}^{\infty} \phi(y)\phi(ay + b)dy \]

\[ = \frac{a}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2 + a^2y^2 + 2aby + b^2)}dy \]

\[ = ae^{-\frac{1}{2}b^2(1-\alpha^2)} \frac{2\pi}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y\sqrt{1+\alpha^2} + b\alpha)^2}dy \]

Making the substitution \( z = y\sqrt{1 + a^2} + b\alpha \) we have that:

\[ = \frac{ae^{-\frac{1}{2}b^2(1-\alpha^2)}}{\sqrt{2\pi}} \frac{1}{\sqrt{1 + a^2}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}dz \]
Integral 4

Define $\alpha = \frac{a}{\sqrt{1+a^2}}$. Then

$$E[Y^2\Phi(aY+b)] = \frac{-b\alpha}{\sqrt{1+a^2}} E[Y\Phi(aY+b)] + \phi\left(\frac{b}{\sqrt{1+a^2}}\right)$$

Proof:

$$E[Y^2\Phi(aY+b)] = \int_{-\infty}^{\infty} y^2 \phi(y) \Phi(ay+b) dy$$

Using $\frac{d^2\phi(y)}{dy^2} = -\phi(y) + y^2\phi(y)$, the integral can be written as:

$$= \int_{-\infty}^{\infty} \phi(y) \Phi(ay+b)dy + \int_{-\infty}^{\infty} \frac{d^2\phi(y)}{dy^2} \Phi(ay+b)dy$$

The first term is simply integral 1 in our list of Gaussian integrals. Using integration by parts on the second integral, we have:

$$\int_{-\infty}^{\infty} \frac{d^2\phi(y)}{dy^2} \Phi(ay+b)dy = \left[\frac{d\phi(y)}{dy} \Phi(ay+b)\right]_{-\infty}^{\infty} - a \int_{-\infty}^{\infty} \frac{d\phi(y)}{dy} \phi(ay+b)dy$$

$$= a \int_{-\infty}^{\infty} y\phi(y)\phi(ay+b)dy$$

$$= \frac{a}{2\pi} e^{-\frac{1}{2}b^2(1-a^2)} \int_{-\infty}^{\infty} ye^{-\frac{1}{2}(y\sqrt{1+a^2}+b\alpha)^2} dy$$

Making the substitution $z = y\sqrt{1+a^2} + b\alpha$:

$$= \frac{a}{\sqrt{2\pi}} e^{-\frac{1}{2}b^2(1-a^2)} \int_{-\infty}^{\infty} \left(\frac{z - b\alpha}{\sqrt{1+a^2}}\right) \frac{1}{\sqrt{2\pi}\sqrt{1+a^2}} e^{-\frac{1}{2}z^2} dz$$
\begin{align*}
&= \frac{a}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}b^2(1-a^2)}}{(1+a^2)} \left[ \int_{-\infty}^{\infty} ze^{-\frac{1}{2}z^2} dz + \frac{-b\alpha}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz \right] \\
&= \frac{a}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}b^2(1-a^2)}}{(1+a^2)} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ze^{-\frac{1}{2}z^2} dz + \frac{-b\alpha}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz \right] \\
&= \frac{-ab\alpha e^{-\frac{1}{2}b^2(1-a^2)}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}b^2(1-a^2)}}{(1+a^2)} = \frac{-b\alpha^2 \phi(b\sqrt{1-a^2})}{\sqrt{1+a^2}}
\end{align*}

And using integral (3) this can be expressed as:

\[ \frac{-b\alpha}{\sqrt{1+a^2}} E[Y\Phi(aY+b)] \]

Integral 5

\[ E[Y^2\Phi(aY+b)\Phi(cY+d)] = \Phi\left(\frac{b}{\sqrt{1+a^2}}, \frac{d}{\sqrt{1+c^2}}, \frac{ac}{\sqrt{1+a^2}\sqrt{1+c^2}}\right) \]

\[ + \frac{a}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}b^2(1-a^2)}}{1+a^2} \left[ \psi\phi(\psi^2 - \varphi^2) - b\alpha \Phi\left(\frac{\varphi}{\sqrt{1+\psi^2}}\right) \right] \]

\[ + \frac{c}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}c^2(1-c^2)}}{1+c^2} \left[ \psi\phi(\psi^2 - \varphi^2) - d\gamma \Phi\left(\frac{\kappa}{\sqrt{1+\psi^2}}\right) \right] \]

Where we have \( \alpha = \frac{a}{\sqrt{1+a^2}}, \eta = \frac{c}{\sqrt{1+c^2}}, \varphi = \frac{\eta}{\sqrt{1+\varphi^2}}, \varphi = \frac{\varphi}{\sqrt{1+\psi^2}} + d, \) and \( \gamma = \frac{c}{\sqrt{1+c^2}}, \) \( \gamma = \frac{\gamma}{\sqrt{1+\psi^2}} \) \( + b. \)

Proof:

\[ E[Y^2\Phi(aY+b)\Phi(cY+d)] = \int_{-\infty}^{\infty} y^2\Phi(aY+b)\Phi(cY+d)\phi(y) dy \]

Using \( \frac{d^2\phi(y)}{dy^2} = y^2\phi(y) - \phi(y) \) this can be expressed as:

\[ \int_{-\infty}^{\infty} \left( \frac{d^2\phi(y)}{dy^2} + \phi(y) \right) \Phi(aY+b)\Phi(cY+d)dy \]
APPENDIX B. USEFUL GAUSSIAN INTEGRALS

Let us consider for now the term:
\[ \int_{-\infty}^{\infty} \phi(y) \Phi(ay + b) \Phi(cy + d) dy \]
from integral (2) this is equal to:
\[ \int_{-\infty}^{\infty} \Phi(ay + b) \Phi(cy + d) \phi(y) dy = \Phi \left( \frac{b}{\sqrt{1 + a^2}}, \frac{d}{\sqrt{1 + c^2}}; \frac{ac}{\sqrt{1 + a^2 \sqrt{1 + c^2}}} \right) \]
All that remains now is to evaluate the term:
\[ \int_{-\infty}^{\infty} \frac{d^2 \phi(y)}{dy^2} \Phi(ay + b) \Phi(cy + d) dy \]
Using integration by parts, the above term evaluates to:
\[ = \left[ \frac{d\phi(y)}{dy} \Phi(ay + b) \Phi(cy + d) \right]_{-\infty}^{\infty} - a \int_{-\infty}^{\infty} \frac{d\phi(y)}{dy} \phi(ay + b) \Phi(cy + d) dy \]
\[ - c \int_{-\infty}^{\infty} \frac{d\phi(y)}{dy} \phi(cy + d) \Phi(ay + b) dy \]
\[ = a \int_{-\infty}^{\infty} y \phi(y) \phi(ay + b) \Phi(cy + d) dy + c \int_{-\infty}^{\infty} y \phi(y) \phi(cy + d) \Phi(ay + b) dy \]
We only evaluate the first term in this integral as the second term is an identical integral but with different variables. Hence, the first term can be written as:
\[ a \int_{-\infty}^{\infty} y \phi(y) \phi(ay + b) \Phi(cy + d) dy = \frac{ace^{-\frac{\alpha^2}{2}(1-\alpha^2)}}{2\pi} \int_{-\infty}^{\infty} y \Phi(cy + d) e^{-\frac{1}{2}(y\sqrt{1+\alpha^2}+b\alpha)^2} dy \]
where \( \alpha = \frac{a}{\sqrt{1+\sigma^2}} \). Making the substitution \( z = y\sqrt{1+\alpha^2}+b\alpha \), this expression becomes:
\[ = \frac{ace^{-\frac{\alpha^2}{2}(1-\alpha^2)}}{2\pi(1+\alpha^2)} \int_{-\infty}^{\infty} (z-b\alpha) \Phi \left( c \left( \frac{z-b\alpha}{\sqrt{1+\alpha^2}} + d \right) \right) e^{-\frac{1}{2}z^2} dz \]
By making the further substitutions \( \omega = \frac{z-b\alpha}{\sqrt{1+\alpha^2}} + d \) and \( \eta = \frac{c}{\sqrt{1+\alpha^2}} \) the integral reduces to:
\[ ae^{-\frac{a^2}{2}} \frac{1}{2\pi(1+a^2)} \int_{-\infty}^{\infty} (z - b\alpha) \Phi(\eta z + \omega)e^{-\frac{1}{2}z^2} \, dz \]

And the result follows from an application of integral (1) and (3).

\section*{Integral 6}

\[ \int_{-\infty}^{\infty} \Phi(ay + b)\phi(y) \, dy = \Phi(\frac{b}{\sqrt{1+a^2}}, c; \frac{-a}{\sqrt{1+a^2}}) \]

Proof: Let \( X = -aY + W \) where \( W \sim N(0,1) \) and so \( X \sim N(0, \sqrt{1+a^2}) \). Thus,

\[ \text{Prob}[X \leq b, Y \leq c] = \int_{-\infty}^{c} \text{Prob}(X \leq b | Y = y)\phi(y) \, dy \]

\[ = \int_{-\infty}^{c} \Phi(ay + b)\phi(y) \, dy \]

But since the correlation between \( X \) and \( Y \) is \( \frac{-a}{\sqrt{1+a^2}} \), we have \( \text{Prob}[X \leq b, Y \leq c] = \Phi(\frac{b}{\sqrt{1+a^2}}, c; \frac{-a}{\sqrt{1+a^2}}) \).