PRICING CONVERTIBLE BONDS WITH EQUITY, INTEREST AND CREDIT RISK

by

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ABSTRACT

This thesis presents a model to price convertible bonds. It is the first model to my knowledge that combines a stock price tree calibrated to the implied volatility surface with an interest rate model of the users choice and a probability of default as a proxy for credit risk.

The aim was to develop a pricing model which enables security pricing for hybrid derivatives with equity, interest and default risk, using observable market inputs from the equity and bond markets.

The model gives the user the flexibility to choose any interest rate model they desire. Normally convertible bond models implemented on a finite difference grid or a 2 factor 3-D tree are restricted to Markovian interest rate models which can be implemented via a recombining lattice. The latest advances in interest rate modelling in the form of multi-factor HJM and Libor Market Models, that are now becoming increasing popular by practitioners, however tend to be non Markovian. The implementation of these models is restricted to inefficient non-recombining lattices/trees or Monte Carlo simulations.

By designing the model so the stochastic interest rate factor is integrated through a Monte-Carlo simulation the convertible bond pricing model is open to the entire spectrum of Markovian and non Markovian interest rate models. This feature now allows convertible bond practitioners to compare how the convertible bond pricing model differs under different interest rate models. This is important as usually no single model can satisfactorily price and risk manage all exotic trades, hence traders like to keep a selection of different models available. Risk managers also benefit by having a spread of model evaluations to keep a check on model error.

Credit risk has been integrated using the CreditGrades models to ascertain the probability of default. This completely removes the ambiguity of trying to determine which discount rates to use on different portions of the bond. The use of a static credit premium above the risk free rate to capture credit risk is replaced by a dynamic probability of default. All discounting in this scenario is done via the risk free rate.

Results prove promising with the model delivering accurate prices with fast computation times.

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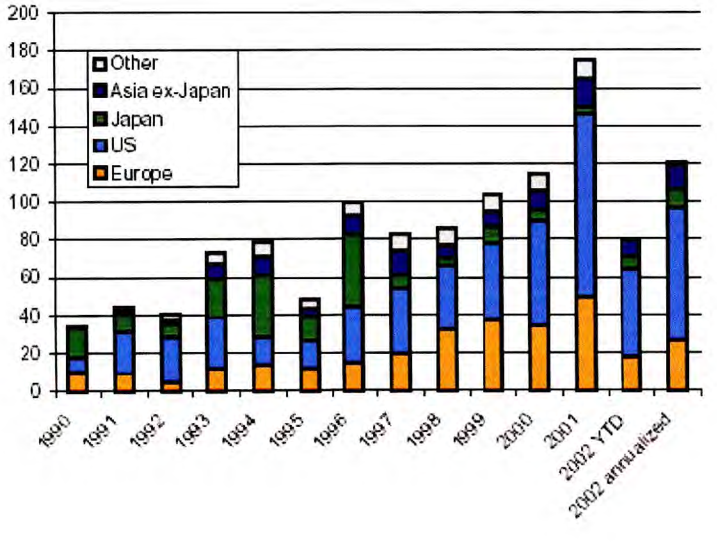
Chapter 1

INTRODUCTION

Convertible bonds have existed for over a century as an investment instrument. The very first convertible was issued around 150 years ago in the USA in order to finance a railroad company. The number of issues of convertible bonds has been rising noticeably over for the last couple of decades, especially accelerating in recent times. The figure below highlights the pace of growth in the yearly issuance of European and global convertible bonds since 1990.

European and total market new convertible issuance (in millions of EUR), 1990 to September 10, 2002[1].



Convertible bonds are a combination of equities and bonds and possess various highly attractive characteristics. A fundamental feature of convertibles is that they offer high, equity-like upside potential while strongly limiting downside risk. A more detailed introduction to convertible bonds is discussed in chapter 2.

Although this investment instrument offers numerous advantages, even professional portfolio managers seldom make use of convertible bonds on a regular basis. Perhaps this is largely due to the relatively low level of recognition convertible bonds receive and the complications in valuing them.

These complications arise due to the hybrid nature of convertible bonds, which expose them to many sources of uncertainty. These multiple sources of risk in relation to convertible bonds have received considerable attention from academia and convertible bond market practitioners as they attempt to combine them into a viable model to price convertible bonds. Chapter 2 reviews the literature to date from academia and the private sector, which attempts to value convertible bonds using a variety of approaches and techniques.

Despite the extensive research into convertible bonds there still doesn't seem to be a comprehensive and coherent model for pricing convertible bonds, which is largely accepted by the market. This is evident by the growth over the last 5 years in convertible bond arbitrage, where traders attempt to exploit discrepancies in the prices

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[1] Source: Bloomberg, Goldman Sachs.

of traded convertible bonds to generate profit. Hedge funds and proprietary trading arms of the investment banks have been particularly active in stripping under/over valued convertible bonds and trading the subsequent components of the convertible bond in their respective markets for the correct valuations.

This thesis attempts to develop existing methodologies and contribute additional ideas to the current convertible bond pricing literature. I ultimately aim to develop a model to price convertible bonds that incorporates the three most important sources of risk that any convertible bond would have to incorporate accurately to successfully price convertible bonds. These sources of risk are equity, interest and credit risk[2].

A vital characteristic of any model is that it yields meaningful and consistent prices. The single most important feature of my pricing model is that it delivers arbitrage free prices for convertible bonds. The model I develop therefore with a combined equity, interest and credit process to price convertible bonds must be consistent with observed market parameters and vanilla products in all the underlying markets.

This can only be achieved if we can firstly be assured the individual processes are correctly modelled before we attempt to combine them. Chapters 3, 4 and 5 therefore take each risk (equity, interest and credit) independently and attempt to model the single processes to be consistent and calibrated with observed market characteristics and data. Once we have no arbitrage functions for the three sources of risk, chapter 6 combines the elements of the 3 previous chapters to create the convertible bond model. Chapter 7 then goes on to test the model using current actively traded convertible bonds. Chapter 8 finally concludes the thesis and gives thoughts on further research.

[2] Some convertible bonds are denominated in a foreign currency, which subsequently creates another source of uncertainty in the form of FX risk. I however will be focusing my attention to convertible bonds denominated in the domestic currency.

Chapter 2

INTRODUCTION TO CONVERTIBLE BONDS

2.1 A basic description of convertible bonds

Convertible Bonds (CBs) are fixed income instruments that can be converted into a fixed number of shares of the issuer at the option of the investor. Bonds that are convertible into shares other than the issuer's are called exchangeable bonds.

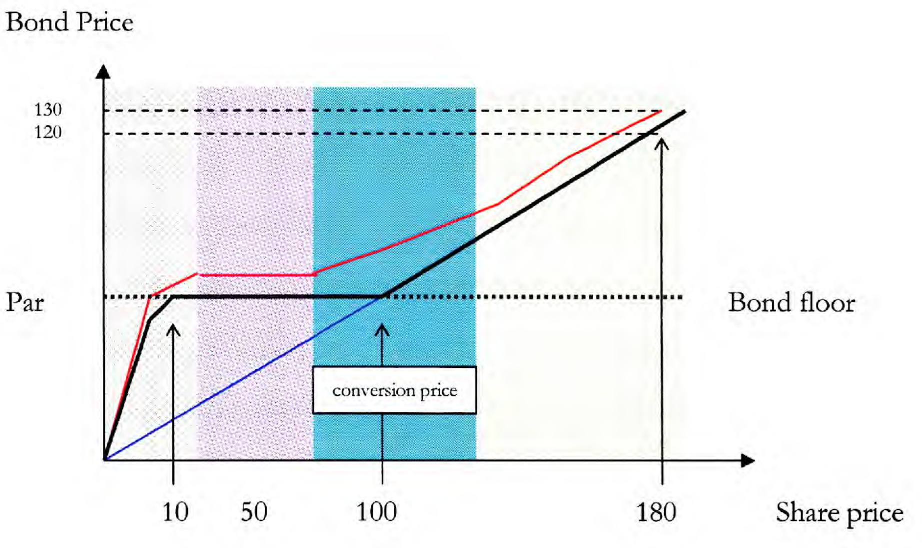
Convertibles are fascinating hybrid securities. On the one hand, they have the benefits of debt instruments that pay fixed coupons and will be redeemed at maturity at a pre-specified price. On the other hand, the embedded conversion option provides the investor with a participation in the upside potential of the underlying equity.

The conversion right provides the bondholder with a better-of-two-choices option. At maturity, the convertible bonds are worth the higher of;

(a) The redemption value (the price at which the issuer had agreed to buy the bonds back) or;

(b) The market value of the underlying shares.

In other words, a convertible bond is a straight bond with an embedded equity call option. Due to this call option, the convertible will participate in any increase of the underlying equity, while the fixed income portion provides capital protection, should the share price fall. The pay off profile is illustrated below



The x-axis displays the underlying share price while the y-axis represents the price of the convertible bond. The blue diagonal line expresses the intrinsic value called parity. Parity represents the value that the investor would receive upon conversion of the bond. Parity is a lower boundary for the price of the convertible.

The value of the bond on the maturity date is represented by the black bold line. The value at maturity is simply the higher of the bonds redemption value and the market value of the shares if the bond was converted. The bold line kinks at the critical point where conversion into shares is more profitable than redeeming the bond at par value, in our diagram this occurs when the share price is 100. So at maturity there is an obvious choice to make for the bondholder.

During its lifetime, there are generally deviations from these intrinsic values. Since the bond and underlying equity are traded daily on the stock exchange, it is exposed to certain influences (e.g. time value and volatility). The thin red line shows the theoretical value of the convertible prior to maturity. For example, at a share price of 180, the value of the bond is no longer at the price it would be at maturity (120) but at a higher level of approximately 130.

The red line outlines the convertible's fair value. If the share price increases, the fair value of the convertible bond rises as well. As the share price increases, the relationship between the share price and the convertible bond becomes more direct until the bond price behaviour and risk profile resemble characteristics of the underlying equity.

If the share price falls, the bond's sensitivity to its underlying share price will decrease and the bond will not decline to the same extent as the equity. The level, which will prevent the convertible from falling further down, is shown in the graph above as the bond floor (dashed line), which is also a lower boundary for the price of the convertible. This bond floor will cease to exist at very low share prices as the risk of the company going bankrupt and failing to honour their debt obligations is considerably higher. The bond price will then approach zero as the stock price falls further.

The partitions in the convertible price in the diagram above are explained below

1) Grey shaded area : Junk or busted convertibles

Similarly to straight debt, a convertible contains the risk of the issuer not being able to repay the principal at maturity. This credit risk is expressed in the graph as the steep fall of the bond floor as well as the bond price on the left-hand side as the share price reaches zero (indicating poor performance and possibly bankruptcy).

2) Purple shaded area : Out-of-the-money

Convertible bonds where the underlying share price trades significantly below the conversion price have low equity sensitivity and behave like fixed income securities. The main factors effecting the value of the convertible bond in this scenario are the interest rate level and the issuer's credit spread.

3) Blue shaded area : At-the-money

Convertible bonds where the underlying share price trades close to the conversion price are considered balanced convertibles because of their asymmetric payoff profile. They have a medium sensitivity to changes in the underlying equity.

These bonds are affected by the share price performance and volatility movements as well as changes in interest rates and the issuer's credit profile. The majority of new issues are launched as balanced convertibles.

4) Green shaded area : In-the-money

Convertible bonds where the underlying share price trades significantly above the conversion price are highly sensitive to changes in the equity, whereas their sensitivity to changes in interest rates and/or credit spreads is low. These bonds trade at an insignificant premium or even a small discount to parity. Deep-in-the-money convertibles will almost certainly be converted into the underlying shares at maturity and will be subject to the same value drivers as the underlying equity.

2.2 Standard features of convertible bonds

Convertible bonds can include an array of complicated embedded options, which provide insurances for both the issuer and holder of convertible bonds.

Convertible bonds are usually callable, this feature gives the issuer the option to call back the instrument prior to maturity at a price specified in the prospectus. As this feature could possibly limit the holder profits, it inclusion into an issuance will cause the price of a convertible bond to trade at a lower price than a non- callable convertible bond.

The holder of a convertible bond however usually has a period of time where the issuer is not permitted to call the bond back from them. This period is usually at the start of the issuance and is referred to as the call protection period. There are two main types of protection, which may be used exclusively or together;

• Hard Call Protection: The issuer cannot call back the convertible for the time stated in the prospectus under any circumstances.

• Soft Call Protection: This protection implies that the bond cannot be called unless the stock trades above a pre-defined level for a certain period of time.

The call protection period is an attractive feature for the holder of the CB and consequently a issue offering call protection will trade higher than a issue offering no call protection. Similarly as the hard call protection is more desirable to a buyer than the soft call protection, CBs with hard call protection will trade at a higher price than identical CBs with soft call protection.

Convertible bonds can also be puttable, this feature gives the holder the option to sell back the instrument prior to maturity at prices and dates specified ex ante. As this feature could possibly limit the holders losses, it inclusion into an issuance will cause the price of a convertible bond to trade at a higher price than a non- puttable convertible bond. The embedded puts tend to be European, so the holder can only exercise on maturity. The number of puts embedded into a CB differs in every issue, with the greater the number the more attractive for the buyer and consequently causing the price of the CB to be more expensive.

2.3 Sources of risk inherent in convertible bonds

As already stated, a convertible bond is a hybrid security that exhibits characteristics of both fixed income and equity securities. As a consequence of this, holders of CB's are subject to many sources of risk. A CB is exposed to the same or even more risks than its constituents.

The main risks convertible bond holders face are:

• Equity market risk:

At high share prices the CB price approaches the parity line and it behaves like pure-equity and thus shares the benefits of a rising market. At low share prices the CB value falls to a lower rate and flattens out to a constant level and at maturity it is likely the redemption would be invoked rather than conversion. The relationship between equity volatility and CB's is that a share with a higher volatility has a higher chance of ending up with a value significantly greater than the conversion price and thus has the potential to be worth more.

Equity risk can be hedged by shorting the underlying stock against the long convertible position. Such hedging produces a very small beta risk and thus a market neutral position.

• Interest rate risk:

As for every bond, the fixed income component of a CB moves inversely to interest rate changes and its sensitivity to these changes depends on how closely the CB is trading in relation to its bond floor. Conversely the embedded option values move in line with interest rate changes. A short position is the underlying stock to neutralize equity risk also serves as convenient hedge here as stock prices and interest rates are inversely related- however this may not cover the entire exposure and commonly, interest rate risk is hedged with treasury futures or interest rate swaps.

• Credit risk:

The exposure comes from the long convertible bond position. Like a conventional bond the holder is subject to the issuer defaulting on coupon payments and the final redemption value at maturity. Whilst the market compensates the holder for this risk by offering it a yield premium over the risk less rate, a holder must balance off this added premium against the probability of default. A short position in the underlying stock to neutralize equity risk also serves as convenient hedge here as stock prices and credit risk are inversely related- however this may not cover the entire exposure and typically credit risk is hedged with credit default swaps (CDS) or by shorting a plain bond or another not identical CB from the same issuer.

• Liquidity risk:

A CB investor is subject to liquidity risk if the long position is not as liquid as expected. Liquidity risk can also occur due to the size of an issue or because of the low credit quality of the issuer. There is no hedging possibility for such risk.

• Currency risk:

Some CB issuances are denominated in a foreign currency. This introduces an element of currency uncertainty for the investor. To hedge the currency risk the investor usually utilises currency options or forward contracts.

2.4 Literature review on the pricing of convertible bonds

The academic literature on the valuation of convertible bonds started with one school of thought before we experienced a structural change in the thinking of academics and practitioners.

In its infancy, convertible bond research was based on the "structural" approach for valuing risky non-convertible debt (e.g. Merton, 1974; Black and Cox, 1976; Longstaff and Schwartz, 1995). In this approach, the basic underlying state variable is the value of the issuing firm. The firm's debt and equity are claims contingent on the firm's value, and options on its debt and equity are compound options on this variable. In general terms, default occurs when the firm's value becomes sufficiently low that it is unable to meet its financial obligations.

While in principle this is an attractive framework, it is subject to the same criticisms that have been applied to the valuation of risky debt by Jarrow and Turnbull (1995). In particular, because the value of the firm is not a traded asset, parameter estimation is difficult. Also, any other liabilities which are more senior than the convertible must be simultaneously valued.

Due to these shortfalls we then saw researchers adopting a new approach where they propose models of convertible bonds where the basic underlying factor is the issuing firm's stock price (augmented in some cases with additional random variables such as an interest rate). As this is a traded asset, parameter estimation is simplified (compared to the structural approach). Moreover, there is no need to estimate the values of all other more senior claims.

In the following section I provide a review of the academic literature in chronological order, beginning with the papers which used the underlying firms value as its state variable and then progressing to the more recent literature which is based on the firms stock price.

2.5 Firm value models

The valuation of convertible bonds based on the modern Black-Scholes-Merton contingent claim pricing literature starts with Ingersoll [1977] and Brennan and Schwartz [1977].

In his paper Ingersoll develops arbitrage arguments to derive several results concerning the optimal conversion strategy (for the holder) and call strategy (for the issuer) as well as analytical solutions for convertible bonds in a variety of special cases. For example, an important result is that he decomposes the value of non-callable convertible bond CB into a discount bond (with the same principal as the convertible bond) and a warrant with an exercise price equal to the face value of the bond. His assumption of no dividends on the equity leads to the result that it is never optimal to convert prior to maturity.

Ingersoll then generalises his result to price convertible bonds with calls. In this case the convertible bond is decomposed into a discount bond, a warrant and an additional term representing the cost of the call which reduces the value of the callable convertible bond relative to the non-callable convertible bond. Ingersoll is able to solve analytically for the price of the convertible bond because of his assumption of no dividends and no coupons.

Brennan and Schwartz [1977] use finite difference methods to solve the partial differential equation for the price of a convertible bond with call provisions, coupons and dividends. Later Brennan and Schwartz [1980] numerically solved a two-factor partial differential equation for the value of the convertible bond. This modelled both the value of the firm and also the interest rate stochastically. Brennan and Schwartz found that often the additional factor representing stochastic interest rates had little impact on the convertible bond price.

Nyborg [1996] extends this model to include a put provision and floating coupons. He introduces coupons into the convertible valuation by assuming that they are financed by selling the risk-free asset. In his simple but worthwhile extension he uses Rubinstein's[1983] diffusion model to value the risky and risk-less assets of the firm separately and gets an analytical solution for the value of the convertible bond. Dividends can also be handled in this model if they are assumed to be a constant fraction of the risky assets. He also analyses the impact of other debt in the capital structure of the firm (senior debt, junior debt and debt with a different maturity to the convertible bond).

When the coupons are financed through the sale of risky assets an analytical solution is no longer possible. For pricing derivative securities such as convertible bonds subject to credit risk the above structural models view derivatives as contingent claims not on the financial securities themselves, but as compound options on the assets underlying the financial securities.

In the Merton [1974] model increasing the volatility of the assets of the firm increases the credit spread with respect to the risk free rate. Varying the volatility of the assets of the firm stochastically has the result of varying the credit spread of the compound option stochastically. Geske's [1979] compound option pricing model has the volatility of the equity being negatively correlated to the value of the firm. As the value of the firm decreases, the leverage increases and the volatility of the equity increases and vice versa. Thus the firm value models easily capture some appealing properties.

The papers of Ingersoll, Nyborg and Brennan and Schwartz assume that the value of the firm as a whole is composed of equity and convertible bonds and they model the value of the firm as a geometric Brownian motion. The advantage of firm value models is that it is relatively easy to model the value of the convertible bond when the firm is in financial distress. Furthermore, firm value models such as the compound option model reproduce the empirical observation that as the value of the firm decreases, leverage increases and the volatility of the equity increases and vice versa.

2.6 Equity value models

The more recent literature considers the convertible bond to be a security contingent on the equity and (for more complicated models) the interest rate rather than the value of the firm. The equity is then modelled as a geometric Brownian motion. The advantage of modelling equity rather than firm value is that firm value is not directly observable and has to be inferred. Additionally, the true complex nature of the capital structure of the firm can make it difficult to model whereas the price of equity is explicitly observable in the market.

In their Quantitative Strategies Research Notes, Goldman Sachs [1994] consider the issue of which discount rate to use when valuing a convertible bond. They consider two extreme situations:

Firstly where the stock price is far above the conversion price and the conversion option is deep in-the-money and is certain to be exercised. Here they use the risk-free rate as they argue that the investor is certain to obtain stock with no default risk.

Second they consider the situation where the stock price is far below the conversion price and the conversion option is deep out-of-the-money. Here the investor owns a risky corporate bond and will continue to receive coupons and principal in the absence of default. The appropriate rate to use here is the risky rate which they obtain by adding the issuer's credit spread to the risk-less rate.

They use a simple one-factor model with a binomial tree for the underlying stock price. However, at each node they consider the probability of conversion and use a discount factor that is an appropriately weighted arithmetic average of the risk-less and risky rate.

At maturity T the probability of conversion is either 1 or 0 depending on whether the convertible is converted or not. Backward induction is then used to determine the probability at earlier nodes, i.e. the conversion probability is the arithmetic average of the two future nodes. If at a node the bond is put then the probability is set to zero and if the bond is converted the probability is set to one.

The methodology seems somewhat incoherent i.e., the investor is assumed to receive stock through conversion even in the event of default but the stock is not explicitly modelled as having zero value in this eventuality. Moreover, prior to default there is no compensating rate for the risk of default (this intensity rate will be formally defined later) entering into the drift of the stock as one would expect. Finally the model makes no mention of any recovery in the event of default on the debt.

The approach used by Goldman Sachs is formalized by Tsiveriotis and Fernandes [1998]. In their paper they decompose the value of the convertible bond into a cash account and an equity account. They then write down two coupled partial differential equations:

The first equation for a holder who is entitled to all cash flows and no equity flows, that an optimally behaving holder of the corresponding convertible bond would receive, this is therefore, discounted at the risky rate (as defined above).

The second equation represents the value of the payments to the convertible bond related to payments in equity and is therefore, discounted at the risk-free rate.

The equations are coupled because any free boundaries associated with the call, put and conversion options are located using the PDE related to the equity payments and these are the boundary conditions used for the PDE related to the cash payments.

The model outlined by Tsiveriotis and Fernandes is again a one factor model in the underlying equity. It is better than the Goldman Sachs model in the sense that the correct weighting (for example taking into account coupons) rather than a probability weighting is used for discounting the risky and risk-less components of the convertible bond price.

Although, the Tsiveriotis and Fernandes model is more careful about modelling the cash and equity cash flows it suffers from the same theoretical inconsistencies as Goldman Sachs. For example the intensity rate does not enter the drift on the equity process, the equity price is not explicitly modelled as jumping to zero in the event of default and any recovery from the bond is omitted.

Ho and Pfeffer [1996] describe a two-factor convertible bond pricing model. Unlike the two factor model of Brennan and Schwartz the Ho and Pfeffer model can be calibrated to the initial term structure.

The interest rate factor is modelled using the Ho and Lee [1986] model. Ho and Pfeffer use a two dimensional binomial tree as their pricing algorithm. The authors appear to discount all cash flows at the risky (i.e., risk free plus credit spread) rate which implies the equity price goes to zero in the event of bond default and therefore, the intensity rate enters into the drift on the equity. However, this is implicit in their model and is not actually stated in the paper. Furthermore, any recovery on the bond in the event of default is omitted from the model.

Moreover, from an empirical point of view, they use a constant spread over the risk free rate at all points to capture the credit risk. Goldman Sachs and Tsiveriotis and Fernandes are likewise guilty of this and it means that the credit spread is assumed fixed irrespective of whether the equity price is very high or very low. Empirically, the credit spread grows as equity prices deteriorate.

Recently, an alternative approach has emerged. This is known as the "reduced-form" approach. It is based on developments in the literature on the pricing of risky debt (see, e.g. Jarrow and Turnbull, 1995; Duffie and Singleton, 1999; Madan and Unal, 2000). In this setting default is exogenous, the "consequence of a single jump loss event that drives the equity value to zero and requires cash outlays that cannot be externally financed" (Madan and Unal, 2000, p. 44). The probability of default over the next short time interval is determined by a specified hazard rate.

When default occurs, some portion of the bond (either its market value immediately prior to default, or its par value, or the market value of a default-free bond with the same terms) is assumed to be recovered. Authors who have used this approach in the convertible bond context include Davis and Lischka (1999), Takahashi et al. (2001), Hung and Wang (2002), and Andersen and Buffum (2003).

As in models such as that of Tsiveriotis and Fernandes (1998), the basic underlying state variable is the firm's stock price (though some of the authors of these papers also consider additional factors such as stochastic interest rates or hazard rates).

Davis and Lischka [1999] use a Jarrow and Turnbull [1995] style stochastic hazard rate to capture credit risk and a extended Vasicek or Hull and White [1994] and [1996] interest rate model in their convertible bond pricing model.

The Jarrow and Turnbull model can be calibrated so that the hazard rate reproduces the survival probabilities observed in the market. Davis and Lischka describe three possible models:

1) The first has a stochastic equity process (including the intensity rate in the drift), an extended Vasicek interest rate process and a deterministic intensity rate;

2) The second model has a stochastic equity process (including the intensity rate in the drift), an extended Vasicek intensity rate process and a deterministic interest rate; and

3) The third model has a stochastic equity process (including the intensity rate in the drift), an extended Vasicek interest rate process and an intensity rate following a perfectly negatively correlated arithmetic Brownian motion process with respect to the equity process.

The first and second models have considerable symmetry the only difference comes through the impact of the recovery rate. The third model is described as a 2 ½ factor model. It is intuitively appealing and certainly preferable to modelling the intensity rate as an ad-hoc function of the equity level. However, the arithmetic Brownian motion of the intensity process implies that the intensity rate can become negative.

The inclusion of the intensity rate in the drift of the equity (in the event of no-default), a zero equity price in the event of default and the inclusion of a recovery rate makes these models more coherent with theory. The ability to correlate the intensity rate with the equity price is also appealing from an empirical point of view.

Quinlan [2000] highlights the difficulty of parameter estimation once a model has been selected. Long-term equity implied volatilities do not exist, dividend forecasts must be estimated, determining the credit spread for subordinated debt can be difficult if the firm is not rated and correlations between the interest rate process and the equity process are difficult to measure and are non-stationary. Moreover, assumptions must be made about when the issuer will call a convertible, if it can be called. North American issuers will usually do this when parity rises 15-30% above the call price. But there is no rule that applies in all cases[3].

[3] This literature review is sourced from: The Valuation of Convertible Bonds: A study of alternative pricing models – Grimwood and Hodges, 2002.

Chapter 3

MODELLING THE EQUITY PROCESS

3.1 Introduction

Modelling the stock price process is perhaps the most fundamental aspect of any model which is subject to equity uncertainty. Since the pioneering work of the Black-Scholes option pricing model and its consequent popularity equity modelling has been based on the Black-Scholes framework.

As Derman and Kani[4] state there are two important but independent features of the Black-Scholes theory. The primary feature of the theory is that it is preference free – the values of contingent claims do not depend upon investors' risk preferences. Therefore, you can value an option as though the underlying stock's expected return is riskless. This risk neutral valuation is allowed because you can hedge an option with stock to create an instantaneously riskless portfolio.

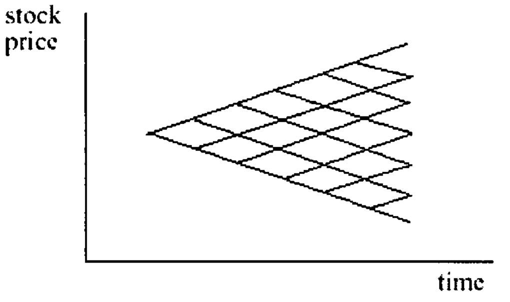
4 Derman & Kani 1994 – The volatility smile and its Implied Tree

A secondary feature of the BS theory is its assumption that stock prices evolve lognormally with a constant local volatility σ at any time and market level. This stock price evolution over an infinitesimal time dt is described by the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dW \qquad \textbf{[Eq 3.1]}$$

Where S is the stock price, μ is its expected return and W a standard brownian motion. The Black-Scholes formula for a call with strike K and time to expiration t, when the riskless rate is r, $C_{BS}(S,\sigma, r, t, K)$ follows from applying the general method of risk-neutral valuation to a stock whose evolution is specifically assumed to follow equation 3.1.

In the Cox-Ross-Rubinstein (CRR) binomial implementation of the process above, the stock evolves along a risk-neutral binomial tree with constant logarithmic stock price spacing, corresponding to constant volatility, as illustrated schematically in the figure below



The binomial tree corresponding to the risk-neutral stock evolution is the same for all options on that stock, irrespective of their strike level or time to expiration. The stock tree cannot "know" about which option we are valuing on it.

Market options prices are not exactly consistent with theoretical prices derived from the BS formula. Nevertheless, the success of the BS framework has led traders to quote a option's market price in terms of whatever constant local volatility σ_{imp} makes the BS formula value equal to the market price. We call σ_{imp} the Black-Scholes *equivalent* or *implied volatility*, to distinguish it from the theoretically constant local volatility σ assumed by the BS theory. In essence, σ_{imp} is a means of quoting prices.

Generally if we observe option prices from a cross sectional view (identical options with only the strike price differing) we see changes in implied volatility as we move through the spectrum of options from deep out of the money options to deep in the money options. This asymmetry is commonly called the volatility "skew." Secondly if observe options over a term structure (identical options with only maturity differing) we also see a non constant implied volatility as we move through the spectrum of options from shortest to maturity to longest to maturity. This variation is generally called the volatility "term structure." The volatility skew and term structure are collectively known as the volatility "smile."

So from observed option prices the implied volatility as backed out from the BS equation differs over different strikes and maturities. This suggests a discrepancy between theory and the market. The situation then arises where is it probably incorrect to calculate options prices using a constant volatility in the BS formula.

There have been various attempts to extend the BS theory to account for the volatility smile. One approach incorporates a stochastic volatility factor, another allows for discontinuous jumps in the stock price. These extensions cause several practical difficulties. First, since there are no securities with which to directly hedge the volatility or the jump risk, options valuation is in general no longer preference free. Second, in these multifactor models, options values depend upon several additional parameters whose values must be estimated. This often makes confident option pricing difficult.

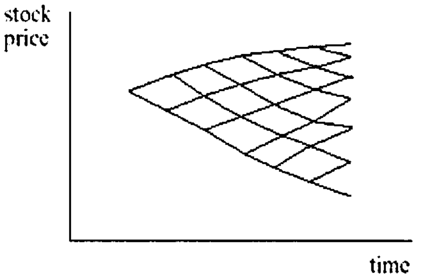
I want to use an equity process, which allows me to develop an arbitrage-free CB model that fits the smile, is preference-free, avoids additional factors and can be used to value options from easily observable data.

The most natural and minimal way to extend the BS model to accommodate the smile is to replace the original SDE with a new equation

$$\frac{dS}{S} = \mu dt + \sigma(S,t)dW \qquad \textbf{[Eq 3.2]}$$

where $\sigma(S,t)$ is the *local volatility function* that is dependent on both stock price and time .

If we were to relate equation 3.2 to a new binomial framework , a distorted or implied tree, drawn schematically below, will prevail to replace the regular CRR binomial tree shown earlier. Options prices for all strikes and expirations, obtained by interpolation from known options prices, will determine the position and the probability of reaching each node in the implied tree.



Whilst the theory of accommodating the volatility smile will be developed in continuous time I have highlighted how the smile effects the traditional CRR tree as I will be implementing the smile in a discrete time setting to ultimately find a analytical solution for the price of a convertible bond. This approach is required due to the multifactor model required to price convertible bonds making it near impossible to find a closed form solution.

I will be modelling the equity process using an implied binomial tree (IBT) as opposed to a regular CRR tree which has been used in all the literature to date. An implied binomial tree is a generalization of the Cox, Ross, and Rubinstein binomial tree (CRR) for option pricing (CRR [1979]).

Implied binomial tree techniques, like the CRR technique, build a binomial tree to describe the evolution of the values of an underlying asset. An IBT differs from CRR because the probabilities attached to outcomes in the tree are inferred from a collection of actual option prices, rather than simply deduced from the behavior of the underlying asset.

These option implied risk-neutral probabilities (or alternatively, the closely related risk-neutral state-contingent claim prices) are then available to be used to price other options. Jackwerth (1999) reviews two inter-related strands of the literature: how to infer probability distributions from option prices, and how to build implied binomial trees.

The best known practical methods for implementing IBT include Rubinstein (1994), Derman and Kani (1994), and Jackwerth (1997). We compare and contrast these three in Table 1.

Table 1

Properties of Competing Implied Binomial Tree Models	Rubinstein 1994	Derman/Kani 1994	Jackwerth 1997
IBT constructed backwards from ending nodes?	Yes	No	Yes
Ability to use intermediate-maturity options in IBT construction?	No	Yes	Yes
Ability to use other than European-style options in IBT construction?	No	No	Yes
Requires extrapolation and interpolation in IBT construction?	No	Yes	No
Assumes all paths leading to a given node are equally likely?	Yes	No	No
Approximately lognormal distribution of ending nodal probability?	Yes	No	No

I will be following the Derman and Kani method of implementing a binomial tree as it fits the entire smile unlike the Rubinstein tree and allows for interpolation in its construction, which is not useable in the Jackwerth tree. The following section describes the continuous time theory behind the Derman and Kani (1994) paper in detail. For specific details on its implementation please refer to their paper as I will implement their model for my equity process which will be seen in chapter 7 and 8 without highlighting a step by step guide.

3.2 The Continuous- Time theory of accommodating the volatility smile[5]

In this section we will investigate the continuous time theory associated with the stock price diffusion process

$$\frac{dS}{S} = r(t)dt + \sigma(S,t)dW \qquad \textbf{[Eq 3.3]}$$

where *r(t)* is the expected instantaneous stock price return, which is assumed to be a deterministic function of time, and $\sigma(S,t)$ is the local volatility function which is assumed to be a (path-independent) function of stock price and time. Here *W(t)* denotes the standard Brownian motion.

Let $\Phi(S, S', t)$ denote the transition probability function associated with the diffusion equation 3.3. It is defined as the probability that the stock price reaches the value S' at time t given its starting value S at time 0. It is well known that this function satisfies both the backward and forward Kolmogorov equations together with the boundary condition $\Phi(S, S', 0) = \delta(S'-S)$, where $\delta(x)$ is the Dirac delta function. The backward equation reads,

$$\frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2\Phi}{\partial S^2} + r(t)\frac{\partial\Phi}{\partial S} - \frac{\partial\Phi}{\partial t} = 0 \qquad \textbf{[Eq 3.4]}$$

while the forward equation is the formal adjoint

$$\frac{1}{2}\frac{\partial^2}{\partial S'^2}(\sigma^2(S',t)S'^2\Phi) - r(t)\frac{\partial}{\partial S'}(S'\Phi) - \frac{\partial \Phi}{\partial t} = 0 \qquad \textbf{[Eq 3.5]}$$

Let *D(t)* denote the standard discount function

$$D(t) = \exp\left(-\int_0^t r(t')d(t')\right) \qquad \textbf{[Eq 3.6]}$$

Then the value of a standard European call option with spot price *S*, strike price *K*, and time to expiration *t* is given by

$$C(S,K,t) = D(t)\int_K^\infty \Phi(S,S',t)(S'-K)dS' \qquad \textbf{[Eq 3.7]}$$

Differentiating equation 3.7 once with respect to strike price *K* leads to the following relationship between a strike spread and the integrated distribution function:

$$D(t)\int_K^\infty \Phi(S,S',t)dS' = \frac{\partial}{\partial K}C(S,K,t) \qquad \textbf{[Eq 3.8]}$$

[5] from Derman & Kani 1994 – The volatility smile and its Implied Tree

Differentiating equation 3.7 twice with respect to strike price K leads to the following relationship between a butterfly spread and the distribution function:

$$D(t)\Phi(S,K,t) = \frac{\partial^2}{\partial K^2} C(S,K,t) \qquad \textbf{[Eq 3.9]}$$

The left side of equation 3.9 is the familiar Arrow-Debreu price in this theory. It is the price of a security whose payoff function is given by $\delta\ (S'-K)$. If, for a given stock level, the prices (and therefore, all partial derivatives with respect to the strike) of call options of all strikes and all maturities were to be available, Equation 3.9 would entirely specify the distribution functions of this theory.

However, the stock distribution function is not necessarily sufficient to determine the diffusion process completely. Different diffusion processes can have the same distribution functions. Remarkably, though, all the parameters of the diffusion process in Equation 3.3 are uniquely specified by the stock price distribution.

To show this, Derman & Kani establish that the standard European call option prices $C\,(S\,, K\,, t)$ in this theory satisfy the following "forward" equation:

$$\frac{1}{2}\sigma^2(K,t)K^2\frac{\partial^2}{\partial K^2} - r(t)K\frac{\partial C}{\partial K} - \frac{\partial C}{\partial t} = 0 \qquad \textbf{[Eq 3.10]}$$

Derman & Kani's proof is a variation of the original proof by Dupire[6]. They begin by multiplying both sides of equation 3.5 by (S' - K) and integrating with respect to S'. This leads to:

$$\frac{1}{2}D(t)\int_{K}^{\infty}\frac{\partial^2}{\partial S'^2}\left[\sigma^2(S',t)S'^2\Phi(S,S',t)\right](S'-K)dS'$$
$$-r(t)D(t)\int_{K}^{\infty}\frac{\partial}{\partial S'}\left[S'\Phi(S,S',t)\right](S'-K)dS' \qquad \textbf{[Eq 3.11]}$$
$$-D(t)\int_{K}^{\infty}\frac{\partial}{\partial t}\Phi(S,S',t)(S'-K)dS'=0$$

Integrating the first term on the left side of equation 3.11 by parts and then substituting from equation 3.8 leads to

$$\frac{1}{2}D(t)\int_{K}^{\infty}\frac{\partial^2}{\partial S'^2}\left[\sigma^2(S',t)S'^2\Phi(S,S',t)\right](S'-K)dS'=$$
$$\frac{1}{2}\sigma^2(K,t)K^2\frac{\partial^2}{\partial K^2}C(S,K,t)+\text{boundary terms at infinity} \qquad \textbf{[Eq 3.12]}$$

[6] Bruno Dupire 1994 – Pricing with a Smile, Risk Magazine

Integrating the second term on the left hand side of equation 3.11 by parts and then substituting from equation 3.9 leads to

$$r(t)D(t)\int_K^\infty \frac{\partial}{\partial S'}[S'\Phi(S,S',t)](S'-K)dS' =$$

$$-r(t)D(t)\int_K^\infty S'\Phi(S,S',t)dS' + \text{boundary terms to infinity} \qquad \textbf{[Eq 3.13]}$$

$$-r(t)\left[C(S,K,t) - k\frac{\partial}{\partial K}C(S,K,t)\right] + \text{boundary terms to infinity}$$

Finally using equation 3.7, the last term on the left hand side of equation 3.11 can be written in the form

$$D(t)\int_K^\infty \frac{\partial}{\partial t}\Phi(S,S',t)(S'-K)dS' = r(t)C(S,K,t) + \frac{\partial}{\partial t}C(S,K,t) \qquad \textbf{[Eq 3.14]}$$

Let us assume that $\Phi(S, S', t)$ approaches zero sufficiently fast for large values of S' so that all the boundary terms above vanish. Then equations 3.12 through to 3.14 can be combined to yield equation 3.10.

Equation 3.7 shows that, in the theory defined by the diffusion of equation 3.3, the distribution function $\Phi(S, K, t)$ completely determines call option prices $C(S, K, t)$ for all values of strike and time. Conversely, from equation 3.9, call prices determine the distribution. Furthermore equation 3.11 can be used in this theory to derive the local volatility function σ (S,t), from the known call option prices (and their known derivatives).

Combining these facts we can see that the stock price diffusion process of equation 3.3 is entirely determined from the knowledge of the stock price distribution function, as asserted earlier.

Derman & Kani explain the analysis above also in a more general theory. They reiterate that knowledge of the stock price distributions do not necessarily allow the unique deduction of the diffusion process. This is the case, for example, where the drift in the diffusion process depends on the path the stock price takes as well as on time, and therefore call option prices cannot be described in terms of a distribution function alone. If the drift function is an *a priori* known (path-independent) function of spot price and time, they show that the knowledge of call option prices is in fact sufficient to derive the underlying diffusion.

Consider a diffusion process whose drift is any known function $r(S, t)$ of the spot price and time, satisfying the following diffusion equation:

$$\frac{dS}{S} = r(S,t)dt + \sigma(S,t)dW \quad \textbf{[Eq 3.15]}$$

The Arrow-Debreu price $\Lambda(S, S', t)$ is the price of a security which pays one unit if the stock price $S(t)$ at any time t attains value S', and zero otherwise. $\Lambda(\ldots)$ can be computed as the expected discounted value of its payoff as follows:

$$\Lambda(S,S',t) = E_{(S,0)}\left[\exp\left(-\int_0^t r(S(t'),t')dt'\right)\partial(S(t)-S')\right] \qquad \textbf{[Eq 3.16]}$$

where $E_{(S,0)}[\ldots]$ is the expectation conditional on the initial stock price S at $t = 0$. The price of a standard European call option with spot price S, strike price K, and time to expiration t is defined by:

$$C(S,K,t) = E_{(S,0)}\left[\exp\left(-\int_0^t r(S(t'),t')dt'\right)\partial(S(t)-K)^+\right] \qquad \textbf{[Eq 3.17]}$$

Using equation 3.16 it is possible to rewrite this in terms of Arrow-Debreu prices as

$$C(S,K,t) = \int_K^\infty \Lambda(S,S',t)(S-K)dS' \qquad \textbf{[Eq 3.18]}$$

Differentiating this equation once with respect to K leads to the following more general form of equation 3.8:

$$\int_K^{\infty} \Lambda(S,S',t)dS' = \frac{\partial}{\partial K}C(S,K,t) \qquad \textbf{[Eq 3.19]}$$

and differentiating twice leads to a more general form of equation 3.9:

$$\Lambda(S,K,t) = \frac{\partial^2}{\partial K^2}C(S,K,t) \qquad \textbf{[Eq 3.20]}$$

It is known that $\Lambda(S, S', t)$ satisfies the following forward Kolmogorov differential equation:

$$\frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2(S',t)S'\Lambda\right) - r(t)\frac{\partial}{\partial S'}(S'\Lambda) - \frac{\partial \Lambda}{\partial t} = r(S',t)\Lambda \qquad \textbf{[Eq 3.21]}$$

This equation is analogous to equation 3.5 satisfied by the transition probability function, and can be used in the same manner to derive a forward equation for European call option prices similar to equation 3.10. So, multiplying both sides of equation 3.21 by $(S'-K)$ and integrating with respect to S', and then assuming similar boundary conditions at infinity, leads to the following equation:

$$\frac{1}{2}\sigma^2(K,t)K^2\frac{\partial^2 C}{\partial K^2} - r(K,t)K\frac{\partial C}{\partial K} + \int_K^\infty \frac{\partial}{\partial S}C(S,S',t)\frac{\partial}{\partial S}r(S',t) - \frac{\partial C}{\partial t} = 0$$

[Eq 3.22]

For a given spot price S, if the local drift function $r(S, t)$ and European call option prices corresponding to all strikes and expirations are known, then we can use equation 3.22 to find the local volatility $\sigma(S, t)$ for all values S of and t . This completes the specification of the diffusion process associated with equation 3.15.

In a discrete time framework the implied binomial tree will ensure the stock price process will follow the process defined above which is determined by the exogenous local drift function and the prices of European puts and calls defining the local volatility.

When implementing the tree we must ensure it remains arbitrage free. The transition probabilities P_i at any node in the implied tree must lie between 0 and 1. If $P_i > 1$, the stock price S_{i+1} at the up-node at the next level will fall below the forward price F_i Similarly, if $P_i < 0$, the stock price S_i at the down-node at the next level will fall above the forward price F_i .

Either of these conditions allows riskless arbitrage. Therefore, as we move through the tree node by node, we demand that each newly determined node's stock price must lie between the neighboring forwards from the previous level that is $F_i < S_{i+1} < F_{i+1}$.

If the stock price at a node violates the above inequality, we override the option price that produced it. Instead we choose a stock price that keeps the logarithmic spacing between this node and its adjacent node the same as that between corresponding nodes at the previous level. This procedure removes arbitrage violations (in this one-factor model) from input option prices, while keeping the implied local volatility function smooth.

Chapter 4

MODELLING THE INTEREST RATE PROCESS

4.1 Introduction

The convertible bond pricing literature to date has not been conclusive on the impact of incorporating interest rate uncertainty into a convertible bond model. Intuitively a convertible bond may be deemed a traditional fixed income instrument with potential to convert it into an equity instrument. With this school of thought, it would be considered imperative that we accommodate interest rate uncertainty in any convertible bond pricing model. However some studies have shown it has little impact on the pricing results of models whilst others studies shows its inclusion does add accuracy to pricing models[7].

I believe it is essential to incorporate interest rate uncertainty to any convertible pricing model as the underlying instrument is fundamentally a fixed income security and it is inherently related to the other sources of risk. Excluding it will not allow us to model the inter-play between equity prices, credit premiums and interest rates.

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7 See chapter 2

After deciding to incorporate interest rate uncertainty into my model the question then is firstly how do we model interest rates? And secondly how do we integrate a chosen interest rate model into the convertible model framework of this thesis. The next section will answer the first question followed by a subsequent section addressing the second question.

4.2 Interest rate models[8]

The modelling of the term structure of interest rates has produced a variety of approaches since the advent of arbitrage-free pricing theory and it continues to occupy the efforts of both academics and practitioners.

Unlike for other asset classes (equities, foreign exchange), where the lognormal Black-Scholes framework is universally accepted, no such agreement exists with regard to interest rate modelling. One reason for this is that the phenomenon we are attempting to model – the random fluctuation of the whole yield curve – is much more complex than the movements of a single stock or index price. One can intuitively relate this to the difference in the dynamics of a scalar variable (in the case of an index) and a vector (representing the yield curve).

A second reason, that is perhaps more fundamental from a market perspective, relates to the nature of the vanilla market in interest rate derivatives. This consists of caps/floors and swaptions, which the market prices using the Black framework where the respective forward Libor and swap rate underlyings are lognormal but the discount factors are non-stochastic. Thus the market standard for the purposes of hedging must regard these vanilla instruments to be independent, where the volatility matrix for swaption prices has for the most part no bearing on the volatility curve associated with

[8] from Han Lee 2000 – Interest Rate Risk – models similarities and differences, Risk Magazine

the cap/floor market. Moreover, the assumption of simultaneous lognormal behaviour in the Libor and swap rates is not mathematically easy to reconcile. Nevertheless, the goal of interest rate modelling is to provide a framework under which a large class of interest rate sensitive securities can be priced in a consistent manner.

The term structure of interest rates or the yield curve can be described in a variety of different ways, which are equivalent: Zero Coupon (or discount) bond prices, yields, spot rates, instantaneous short rates, instantaneous forward rates and discrete Libor rates.

Due to the numerous variables used to describe interest rates we find there are consequently a range of interest rate models with differing features and characteristics. The models developed to date can be categorised into three families: spot/short rate, forward rate and market models. Although all three of these prescriptions are mathematically consistent (by definition of a term structure model), each approach leads to distinct development, implementation and calibration issues.

Spot/short rate models (pioneered by Vasicek 1997) attempt to describe the bond dynamics through directly modelling the short-term interest rate. Heath Jarrow and Morton (HJM) established the general framework where these principles are satisfied and formulated the interest rate dynamics explicitly in terms of the continuously compounded forward rate. Market models are a class of models within the HJM framework that describe variables directly observed in the market, such as the discretely compounding Libor and swap rates.

Models have been formulated using two approaches: (1) a general equilibrium framework, where interest rate changes are derived from economic agents who maximize expected utility; and (2) the no-arbitrage approach, which assumes that financial markets have no arbitrage opportunities. Examples of the general equilibrium approach include the early short rate models of Vasicek (1977), Dothan (1978), Cox, Ingersoll, and Ross (CIR) (1985), Brennan and Schwartz (1979) and Longstaff and Schwartz (1992). Models based on arbitrage arguments are some more advanced short rate models by Black, Derman & Toy (1990), Hull & White (1990), Black-Karasinski (1991) and the entire family of models based on forward rates and Libor rates pioneered by Ho and Lee (1986), Heath, Jarrow, and Morton (HJM) (1992) and Brace, Gatarek& Musiela (1997).

In all the models the yield curve is described through stochastic differential equations driven by a diffusion term and a drift term. Based on the arbitrage-free principle, the market price of risk is removed by the choice of the drift (this occurs in the portfolio replication argument for stock options in Black-Scholes, where the drift is equal to the risk free rate). This is performed in different ways. Spot rate models have to match the initial yield curve that implicitly holds information on investor choice and hence market price of risk, through the drift function. Models formulated with instantaneous forward rates explicitly relate the choice of volatility function to the form of the drift (imposed through the HJM condition), in order for the no-arbitrage principle to hold. Similarly, for market models the drift is adjusted to ensure that the model remains arbitrage-free.

Spot/Short rate models

The first generation of models developed were generally short rate-based. This choice was due to a combination of mathematical convenience and tractability, or numerical ease of implementation. Furthermore, the most widely used of these models are one-factor models, in which the entire yield curve is specified by a single stochastic state variable, in this case the spot or short-term rate. Examples of these include the models of Vasicek, Cox, Ingersoll & Ross, Dothan, Hull & White, Black Derman & Toy (BDT), and Black-Karasinski.

These models are distinguished by the exact specification of the short rate dynamics through time, in particular the form of the diffusion process, and hence the underlying distribution of the short rate.

The earliest short rate models by Vasicek, Cox Ingersoll & Ross, and Dothan are known as equilibrium models, their general form for describing the changes in the short rate, is as follows:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\, r_t^{\gamma}\, dW_t \qquad \textbf{[Eq 4.1]}$$

r_t = current level of the instantaneous rate

κ = speed of the mean reversion

θ = rate to which the short rate reverts

σ = volatility of the short rate

γ = proportional conditional volatility exponent

W_t = standard Brownian motion

The first important feature of this type of model is mean reversion of the short-term rate. This feature is appealing since it presumes that when rates become very high or very low, they will tend to revert to "normal" levels. The speed of reversion is determined by the parameter κ. This parameter ultimately affects the shape of the yield curve. If κ is high, the yield curve quickly trends toward the long-run yield rate θ. If κ is low, the yield curve slowly trends toward θ.

The difference between the Vasicek, CIR, and Dothan models primarily revolves around the parameter γ (the exponent). Vasicek assumes it to be 0, CIR assumes it to be 0.5, and Dothan assumes it to be 1.0. The basic question distinguishing the models is whether the conditional volatility of changes in interest rates is proportional to the level of the rate. This subsequently determines the parameter γ.

These models are criticized because they do not fit the existing term structure. Although parameters can be chosen to minimize errors from today's yield curve, the fit will not be perfect. Such a discrepancy has led to these models to be unacceptable in practice as its usage could lead to arbitrage. They are consequently unsuitable to use in the model I am building in this thesis. It is apparent to find a suitable interest rate model for pricing convertible bonds we will have to focus on arbitrage models which are consistent with the existing term structure and observed volatilities.

In the short rate world this leaves us with models by Hull & White (1990), Black Derman & Toy (BDT)(1990), and Black-Karasinski (1991).

Forward rate models

An alternative approach to modelling the term structure was offered by the Heath, Jarrow & Morton (HJM) structure. In contrast to the spot rate approach, they model the entire yield curve as a state variable, providing conditions in a general framework that incorporates all the principles of arbitrage-free pricing and discount bond dynamics. The HJM methodology uses as the driving stochastic variable the instantaneous forward rates, the evolution of which is dependent on a specific (usually deterministic) volatility function.

Because of the relationship between the spot rate and the forward rate, any spot rate model is also an HJM model. In fact, any interest rate model that satisfies the principles of arbitrage-free bond dynamics must be within the HJM framework.

Heath, Jarrow, and Morton (1992) use the no-arbitrage argument to develop the process for the *forward* rate implied by the relationship of bond prices. Assuming the risk neutral process for the bond price $P(t,T)$ has the form:

$$dP(t,T) = r(t,T)P(t,T)dt + \sigma(t,T)P(t,T)dW_t \qquad \textbf{[Eq 4.2]}$$

$P(t,T)$ = instantaneous forward rate at time t with maturity T

$r(t,T,)$ = risk neutral drift of the forward rate process

$\sigma(t,T,)$ = volatility of the forward rate process

W_t = standard Brownian motion

This equation reflects the fact since a discount bond is a traded security providing no income, its expected return at time t in a risk neutral world must be $r(t)$. Regardless how the volatility function is defined it has to incorporate the boundary condition that at maturity we are certain of a default free discount bonds value and so we must have:

$$\sigma(t,t)=0 \qquad \textbf{[Eq 4.3]}$$

Hence from the standard forward price equation derived from discount bond prices and using equation 4.2 we get the following relationship:

$$f(t,T_1,T_2)=\frac{\ln[P(t,T_1)]-\ln[P(t,T_2)]}{T_2-T_1} \qquad \textbf{[Eq 4.4]}$$

Applying Ito's lemma to equation 4.2 we can determine the diffusion process followed by the log of the discount bond prices:

$$d\ln P(t,T_1)=\left[r(t)-\frac{1}{2}\sigma(t,T_1)^2\right]dt+\sigma(t,T_1)dW_t \qquad \textbf{[Eq 4.5]}$$

$$d\ln P(t,T_2)=\left[r(t)-\frac{1}{2}\sigma(t,T_2)^2\right]dt+\sigma(t,T_2)dW_t \qquad \textbf{[Eq 4.6]}$$

Using these results in equation 4.4 gives us the following process for forward rates:

$$df(t,T_1,T_2)=\left[\frac{\sigma(t,T_2)^2-\sigma(t,T_1)^2}{2(T_2-T_1)}\right]dt+\left[\frac{\sigma(t,T_1)-\sigma(t,T_2)}{(T_2-T_1)}\right]dW_t \qquad \textbf{[Eq 4.7]}$$

Taking this process to the limit gives us the process for the instantaneous forward rates:

$$df(t,T) = [\sigma(t,T)\sigma_T(t,T_2)]dt - [\sigma_T(t,T_2)]dW_t \qquad \textbf{[Eq 4.8]}$$

HJM find that by imposing the no-arbitrage argument to term structure movements, the drift of the forward rate process can be stated in terms of volatilities. Thus, the structure of the volatility becomes the most important element of the HJM model. Different functional forms of the volatility reveal an entire family of HJM models. The family is extended by the ability of the model to incorporate several factors of uncertainty rather than the one factor discussed above. This is done to improve calibration but at a cost of tractability, implementation difficulties and slower execution times.

Market models

The motivation for the development of market models arose from the fact that, although the HJM framework is appealing theoretically, its standard formulation is based on continuously compounded rates and is therefore fundamentally different from actual forward Libor and swap rates as traded in the market. The lognormal HJM model was also well known to exhibit unbounded behaviour (producing infinite values) in contrast to the use of lognormal Libor distribution in Black's formula for caplets. The construction of a mathematically consistent theory of a term structure with discrete Libor rates being lognormal was achieved by Miltersen, Sandmann & Sondermann, and developed by Brace, Gatarek & Musiela (BGM). Jamshidian developed an equivalent market model based on lognormal swap rates.

4.3 Choosing an Interest rate model for convertible bonds

To use any interest rate model for pricing of contingent claims, it must be calibrated to the market. Besides matching the initial yield curve, the prices of caps/floors and swaptions are required. Ideally the model is capable of providing an analytical formula for these vanilla instruments, but otherwise a very efficient numerical algorithm is necessary. With this criterion in mind we can eliminate the use of equilibrium models in this thesis.

Spot and forward arbitrage models must derive the appropriate quantities from the underlying state variables to construct the equivalent of the option pricing formulae. By construction, market models are based on observable rates in the market and hence (in some measure) readily price-standard instruments. The process of calibrating any model must start with making the choice of distribution or volatility function.

Spot rate models require a specification of the dynamics, examples of which include a normal or Gaussian distribution (Hull-White), lognormal (Black-Karasinski) or something in between (eg, the 'square root' type model equivalent of the Cox-Ingersoll-Ross model). Variables derived from the spot rate, such as the zero-coupon and Libor or swap rates, will have a distribution dependent on that of the short rate; for example the discount bond is lognormal for Gaussian spot rate models such as Hull-White. For forward rate models, the critical factor in determining the behaviour of a model is the form of the (HJM) volatility function.

For reasons of analytic tractability, the most common models in this category are the Gaussian forward rate models, so called when the volatility function is independent of the forward rate itself. In market models there is a choice in both the distribution of the underlying market variable, or perhaps a function of that variable, and in the functional form of the volatility.

For use in exotic derivatives, models are required that price and hedge tradeable products. These should capture all risks associated with the product. Usually no single model can satisfactorily price and risk manage all exotic trades, hence traders like to keep a selection of different models available. Risk managers also benefit by having a spread of model evaluations to keep a check on model error.

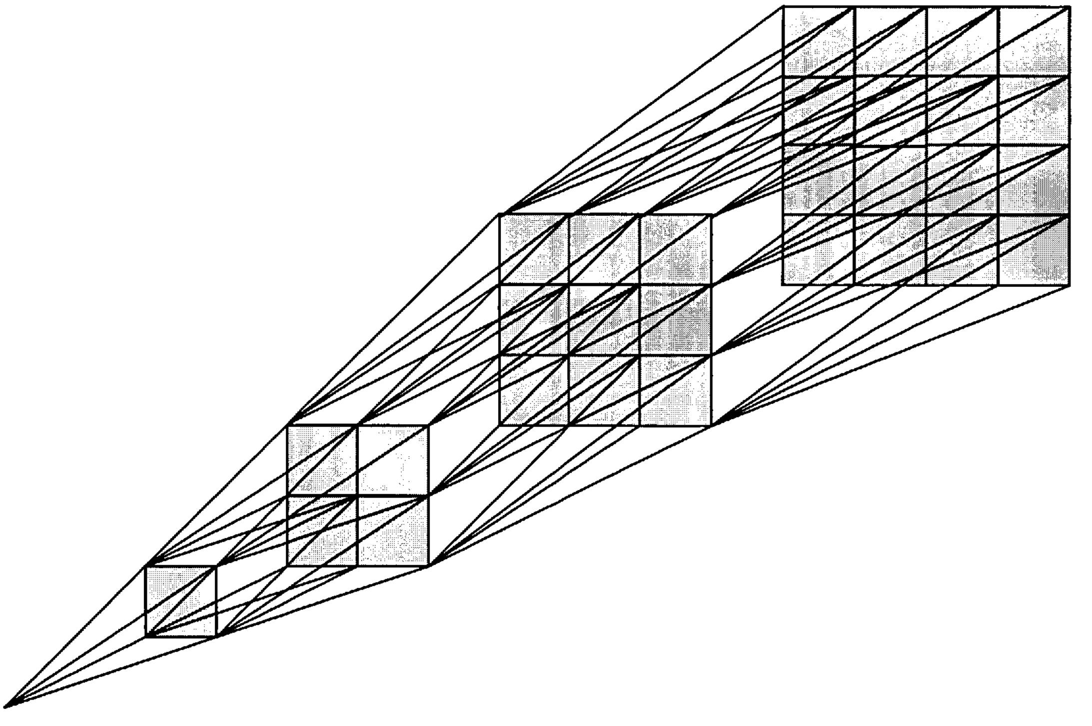
Convertible bond models to date have largely ignored this risk management technique. The literature usually specifies a specific interest rate model it will incorporate into its convertible pricing framework and is usually limited to its choice by the nature of some interest rate models and the implementation technique they are using.

The interest rate models presented so far have been introduced in a continuous time framework. Although some continuous time models may lead to closed form solutions for simple cash flows such as non-callable bonds, convertible bonds are more complicated. To use the model's dynamics in our framework we have to use discrete time intervals for the interest rate process. This is done through either a Monte-Carlo simulation of the interest rate process or by describing the evolution of interest rates via trees/lattices.

A majority of convertible bond models are implemented via trees or finite difference grids. In such approaches the equity process and interest rate process can be combined on a 3-D Quad tree where every node branches out to 4 possible future nodes rather than the customary 2 or 3 in binomial or trinomial trees. The four possible future states cover all eventualities in a default free 2 factor model:

- Stock price rises and interest rates rise
- Stock price rises and interest rates fall
- Stock price falls and interest rates rise
- Stock price falls and interest rates fall

The figure below depicts a 4 stage 3-D tree



This problem with this methodology is any convertible bond model, which incorporates a two factor model like this is usually limited to its choice of interest rate models it can use. This is due to the fact that discrete time implementation of interest rate models using trees falls into two categories. There are those interest rate models which are Markovian and can be implemented using a regular recombining tree and there are those interest rate models which are non-Markovian and can only be implemented using non recombining or bushy trees.

The recombining interest rate tree combined with a recombining equity tree is depicted above. This combination is highly practical and efficient as the possible nodes is limited to the number of time steps, $(n+1)^2$, where n is the number of time steps. If we were however to combine a non recombining interest rate tree with a recombining equity tree we would find this computationally unfeasible with unacceptable execution times. This is due to the number of time steps increasing in the interest rate tree alone by 2^n, where n is the number of time steps.

Consequently most of the convertible pricing literature uses Markovian models of the interest rate – usually one factor no arbitrage models of the short rate. Unfortunately the more advanced forward rate and market models are largely non- Markovian. They have to be implemented using Monte-Carlo simulations and have largely been ignored in the convertible bond pricing environment.

I want to develop a convertible pricing model which will allow the use of Monte Carlo simulations in its modelling of the interest rate. Such a feature allows the use of all the previously used interest rate models in convertible bond pricing and opens up the use of non Markoivian models also. The use of Monte-Carlo simulations in modelling interest rates gives the convertible bond model total flexibility as Markovian and non Markovain interest rate models can be simulated whilst both cannot be discretiszed into recombining trees. This methodology also addresses the problem that no single model can satisfactorily price and risk manage all exotic trades, hence convertible bond traders will be able like to keep a selection of different models available.

Whichever model is used, there will be freedom to adjust the parameters governing the 'indeterminate' parts of the effective volatility for the pricing of particular products. In practice this adjustment is made depending on the needs of the user (eg, whether buying or selling the option and how aggressive or conservative the trader is), and by indirect means such as by calibrating to other quoted prices in the market when possible.

The interaction between the implied binomial tree described in the previous chapter and the Monte Carlo simulation of interest rates discussed in this chapter is detailed in chapter 6. In chapter 7 when the model is tested I will use a simulation of a BDT interest rate model, however any model the reader desires can be used to compare how prices vary according to interest rate model and observed market prices. A more detailed analysis of the BDT model is provided in Appendix B.

Chapter 5

MODELLING THE CREDIT RISK PROCESS

5.1 Introduction[9]

Theories concerning credit risk modelling have evolved reasonably quickly over time. They began with the "balance-sheet" approach, typified by the classic work of Altman [1968]. In this approach, historical data on defaults is used along with firm characteristics to fit logit-type models for credit risk. Altman developed a proxy for the default probabilities, now well-known as the "Z-score".

This was followed by a theory that takes the modelling basis to be the value of the firm. Also known as the so-called "structural" models, which were initially suggested in the seminal paper of Black and Scholes [1973], and developed in substantial detail in Merton [1974]. Structural models assume that the value of a firm is continuous in time and, given the dynamics of firm value through time and appropriate terminal and boundary conditions, derive the value of the firm's debt.

[9] Refer to Carayannopoulos(2001) and Das(2004)

Merton (1974) developed one of the first models, which assumes that default is allowed only at the maturity of the debt. Subsequent structural models relax some of the unrealistic assumptions of his model. Default can instead occur anytime during the life of the bond and default is triggered when the value of the firm reaches a certain threshold level.

Structural models for convertible bonds were initially developed by Ingersoll (1977a, 1977b) and Brennan and Schwartz(1977). They follow the same principles as the structural models for the valuation of regular bonds, and allow for the possibility of equity conversion through a set of appropriate terminal and boundary conditions. Brennan and Schwartz (1980) extend their previous work and allow for the uncertainty inherent in interest rates by introducing the short-term risk-free interest rate as an additional stochastic variable.

Empirical investigations of structural convertible bond valuation models are limited. King (1986) examines a sample of 103 American convertible bonds and concludes that when market prices are compared with model valuations, the means are not significantly different. Carayannopoulos (1996b), using a structural model that allows for the stochastic nature of interest rates, in a study of monthly data for 30 US convertible bonds finds that market prices are significantly lower than model prices when the conversion option is deep-out-of- the-money, i.e., when the conversion value of the convertible bond is low relative to the straight bond value of the security.

This class of models uses publicly traded equity and option market prices to determine default probabilities, based on a measure known as the "distance-to-default", embodied in the approach taken by KMV (see Crosbie [1999]). The KMV model relies almost exclusively on equity market information. It is also common to combine the balance-sheet approach with the structural one, leading to so-called "hybrid" models (as in the approach adopted by Moody's Risk Management Services before KMV and Moody's merged).

Most of the problems associated with the practical application of structural models are circumvented with the use of reduced-form models. Unlike structural ones, reduced-form models do not condition default exclusively on firm value, and unobservable parameters associated with firm value need not be estimated for model implementation. These models also view risky debt as paying off a fraction of each promised dollar if bankruptcy occurs. However, the time of bankruptcy is treated as an exogenous process and does not depend explicitly on firm value.

A typical reduced-form model assumes that an exogenous variable drives default, and the conditional probability of default (also called hazard or intensity rate) during any time interval is nonzero. Furthermore, it is assumed that, upon default, bondholders receive a fraction of the bond's face value, known as the recovery rate that is known a priori.

In general, the value of a corporate bond is equal to the present value of its future cash flows discounted at a risky rate. The risky rate has two components: the risk-free short-term rate and a credit risk premium while one or both components may very through time. The credit risk premium is assumed to be a function of the (risk-neutral) probability of default and the recovery rate, if default occurs. One set of reduced models employs a credit-rating based approach in which default is depicted through a gradual change in ratings driven by a Markov transition matrix. Others depict the default process through the evolution of default spreads or equivalently, the joint evolution of the conditional probability of default and recovery rate.

Classic models in this genre are those of Jarrow and Turnbull (1995), Madan and Unal(1995),(2000), and Duffie and Singleton (1999). Mamaysky (2002) extends the Duffie-Singleton approach to linkages with equity risk, through the dividend process, an idea presented initially in Jarrow (2001). Default times may also be simulated or computed directly off the rating transition matrix. Such an approach may be applied directly to the transition matrix, (see Jarrow, Lando and Turnbull (1997), Das and Tufano [1996]), or it may be based on changes in firm asset values (the approach adopted by RiskMetrics).

All these "pure" approaches have been hybridized by mixing information from other markets into those models. Within the class of structural models, the KMV approach has been modified by enhancing the information set beyond the distance to default measure (see Sobehart, Stein, Mikityanskaya and Li [2000]). For example, the approach used by Moody's combines distance to default with balance-sheet information to determine default probabilities.

Another variant of the structural model has been developed by RiskMetrics and is called CreditGrades (see Finkelstein, Lardy, Pan, Ta and Tierney [2002]). Their approach is a variation on the standard Merton model with additional constraints to ensure that the default probabilities are consistent with observed spreads in the debt market.

Each of these approaches, but for the reduced-form models, requires some element of data that is not market observable. Structural models are based on the value of the firm, which needs to be extracted from an inversion over stock and option prices. The balance sheet models require the use of accounting information, which is not validated by a trading process. While the reduced-form models do not suffer from this deficiency, they extract default probabilities from debt prices and utilize no information from the equity markets.

5.2 Convertible bonds and credit risk[10]

While it has long been realized that a framework for pricing convertible bonds should ideally incorporate elements of both equity and debt modelling, practical efforts in this direction have long been somewhat lacking. In particular, there seems to have been considerable confusion and disagreement about how to appropriately and consistently apply a default-adjusted discount operator to cashflows generated by convertible bonds.

Early papers with an *ad hoc* approach to discounting include McConell and Schwarz (1986), Cheung and Nelken (1994), and Ho and Pfeffer (1996). Many of these models do not explicitly model bankruptcy, and as compensation uniformly apply a somewhat arbitrary risky spread to the risk-free discount rate.

More recent papers recognize that equity and debt components of convertible bonds are subject to different default risk and attempt more sophisticated schemes. An often-quoted example is Tsivioritis and Fernandes (TF) (1998) (later extended by Yigitbasioglu (2001) to multiple factors), which effectively splits the convertible bond into cash and equity components, with only the former being subject to credit risk.

[10] See Andersen & Buffman(2002)

A related approach was promoted by Goldman Sachs (1994) and involves careful weighting of risky and risk-free discounting in a binomial lattice. The TF splitting scheme is analyzed in detail in Ayache *et al* (2002) who conclude that it is inherently unsatisfactory due to its unrealistic assumption of stock prices being unaffected by bankruptcy.

With the advances of credit derivatives theory, in particular the reduced-form approach of Jarrow and Turnbull (1995), the foundation for convertible bond models has recently improved significantly. A key development has been the inclusion of stock price dynamics that explicitly incorporate default events, as well as the explicit modelling of stock and bond recoveries in default. Most commonly, default is modeled as a Poisson event that drives stock prices into some low value and coupon bond prices (and convertible bonds) into a certain, fixed percentage of their notional values. Representative, and quite similar, papers include Davis and Lischka (1999) and Takahashi *et al* (2001).

The credit risk of a convertible bond is modelled by assuming that any risky cash flows, which include coupons and redemption payments on the bond, are discounted at the risk free rate plus a credit premium. That premium in many of the previous literature applies to all bond cash flows regardless of under what circumstances they occur. In particular it does not depend on the on the prevailing equity levels. Hence the models are indifferent and apply the same credit premium when equity prices are high and the underlying company is doing well and when then company is not performing well and share prices fall as a consequence. Intuitively we know this is not realistic and is a downfall of models, which use a uniform credit premium

The feature of previous models that credit premiums do not change as the equity price changes is overly simplistic and is for many companies grossly untrue. Negative correlations usually exist between credit premiums and equity prices.

An improved model might include a stochastic credit spread that incorporates the ability to build an equity credit correlation into the model. But this will cause additional complexity and computational inefficiencies to a model which already has interest rates and equity process modelled stochastically and attributes extra value to a convertible bond for its credit option.

I consider a approach which makes modelling the stochastic process followed by credit spreads easier. The approach extracts a probability of default (PD) as a function of equity prices and interest rates, and hence, once the stochastic processes for equity and interest rates are set in the model, the stochastic process for PDs is automatically derived. It is important to note that this is just as feasible in the Duffie-Singleton [1999] model, however, in a setting in which correlated default is to be analyzed.

Accounting for credit risk with this in mind is achieved by adding the process for default probability [$\lambda(t)$] to the lattice. Rather than add an extra dimension to the lattice model by embedding a separate $\lambda(t)$ process, we define one-period default probability functions at each node on the lattice, by making default a function of equity prices and interest rates at each node. There are two reasons for this. First, equity prices already reflect credit risk, and hence there is a connection between $\lambda(t)$ and equity prices. Second, default probabilities are empirically known to be connected to the term structure, and hence, may be modeled as such. Therefore, it seems appropriate modelling the default risk at each node as a function of the level of equity and the term structure at each node.

Specifying a conditional $\lambda(t)$ at each node, i.e. rather than add a separate default probability process, we simply make $\lambda(t)$'s a function of the state variables of equity and interest rates. This can be referred to as an endogenous default approach. If in fact, default probabilities were added as a separate stochastic process (which we denote the exogenous approach, as in David and Lischka [1999] or Andersen and Buffum [2002]), the question of consistency conditions between $\lambda(t)$, equity and interest rates would create a complex situation to resolve.

By positing a functional relationship of $\lambda(t)$ to the other variables, we are able to obtain a consistent lattice as well as a more parsimonious one. We impose the condition that is required of default intensities to conform to the behaviour required. This is not a new approach. A similar endogenous default intensity extraction has been implemented in Das and Sundaram [2000], Carayannopoulos and Kalimipalli [2001], and Acharya, Das and Sundaram [2002].

Various possible parameterizations of the default intensity function may be used. For example, the following model (subsuming the parameterization of Carayannopoulos and Kalimipalli [2001]) prescribes the relationship of the default intensity ξ(t) to the stock price *S(t)*, short rate *r(t)*, and time on the tree $(t - t_0)$.

$$\xi(t) = h(y)e^{[a_0 + a_1 r(t) - a_2 \ln S(t) + a_3 (t - t_0)]}$$

$$\xi(t) = h(y)\left[\frac{e^{[a_0 + a_1 r(t) + a_3 (t - t_0)]}}{S(t)^{a_2}}\right]$$

For $a_2 \geq 0$, we get that as S(t) ⟶ 0, ξ(t) ⟶ 1, and as S(t) ⟶ 1, ξ(t) ⟶ 0. Further, we may also specify the function h(y), based on a state variable y (such as the debt-equity ratio) through which other influences on the default intensity function may be imposed. This function must satisfy consistency conditions depending on its choice of state variable.

However as an objective of this thesis it was imperative to use widely available and acceptable data in its calibration. Fortunately probability of defaults for companies, which differ according to stock price levels, are already publicly available from the widely regarded credit grades platform discussed earlier. I will be using the probability of defaults as generated by the Credit Grades model as the credit risk factor in my tree as discussed above. A full description of the Credit Grades model is highlighted in Appendix A.

5.3 Recovery rates

In addition to the probability of default of the issuer, a recovery rate is required. In the state in which default occurs, this recovery rate is applied. The recovery rates may be treated as constant, or as a function of the state variables. It may also be pragmatic to express recovery as a function of the default intensity, supported by the empirical analysis of Altman, Brooks, Resti and Sironi [2002].

A critical aspect of a corporate bond default is the severity of the loss incurred. Eventually, most bond default resolutions provide bondholders with some amount of recovery, which may take the form of cash, other securities, or even physical assets. The recovery rate, defined here as the percentage of par value returned to the bondholder, is a function of several variables.

These variables include the seniority of the issue within the issuer's capital structure, the quality of collateral (if any), the overall state of the economy, and the thickness of the market for corporate assets.

In July 2001 Moody's KMV published research, which collected, from several sources, prices for many of the US convertible bonds that, defaulted between 1970 and 2000. For each defaulted issue, they considered the convertibility, seniority, date of default, and the price approximately one month after default.

The data revealed considerable volatility in average defaulted bond prices year-over-year, as well as some degree of correlation with macroeconomic variables and the risk of default. The low recovery rates of 1990 correspond to a peak in the corporate default rate and an economic recession in the US. Interpretation of the 1981 and 1979 lows for the average defaulted convertible bond prices should be tempered by the fact that sample sizes for these years are critically low.

Overall, the average defaulted convertible bond price series tracks that of the non-convertible bonds closely suggesting that these instruments react similarly to prevailing business conditions.

The average range for recovery values of defaulted convertible bonds, is between \$28.00 and \$34.07 in the Moody's KMV study . For the purpose of this study I will treat the recovery rate as a constant and use a value of 30% of par value. This can be extended if the reader desires to incorporate a recovery value which is a function of many variables such as stock price, economic conditions etc etc.

Chapter 6

THE CONVERTIBLE BOND MODEL

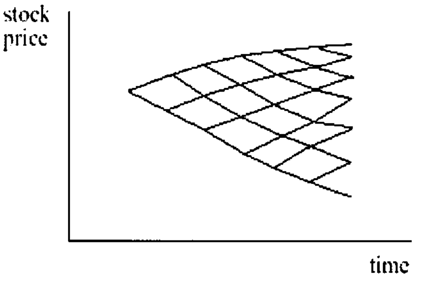
6.1 Introduction

The three previous chapters laid the foundations for adopting the three sources of risk I am addressing in this thesis. The purpose of this chapter is to combine the elements in a coherent manner to price convertible bonds. Chapter 3 detailed the continuous time theory behind the Derman & Kani implied binomial tree. The analysis assumed a deterministic interest rate through all stages of the tree and did not consider the possibility of default.

The use of a stochastic interest rates in pricing convertible bonds and allowing the possibility of a convertible bond issuer defaulting are pivotal to my analysis and were discussed in considerable detail in chapters 4 and 5. Incorporating these features into the original work of Derman & Kani will be the initial focus of this chapter. Once this has been shown I will progress to explain how I aim to price convertible bonds. Here it will shown that the ambiguity of discounting at risk free and risky rates which has plagued many previous models is completely removed and replaced by the probability of default. Now discounting is categorically only through the risk free rate and the credit risk is reflected through the probability of default.

6.2 The basic implied binomial tree revisited [11]

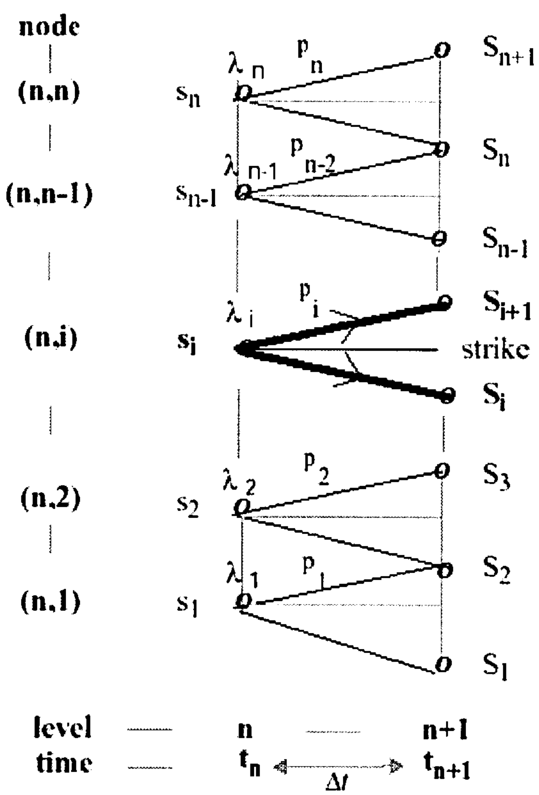
In the new binomial framework , a distorted or implied tree, drawn schematically below, will replace the regular binomial tree shown earlier. Options prices for all strikes and expirations, obtained by interpolation from known options prices, will determine the position and the probability of reaching each node in the implied tree.



Derman and Kani and use induction to build an implied tree with uniformly spaced levels, Δt apart. Assume they have already constructed the first n levels that match the implied volatilities of all options with all strikes out to that time period. The figure below shows the n^{th} level of the tree at time t_n, with n implied tree nodes and their already known stock prices s_i.

[11] from Derman & Kani 1994 – The volatility smile and its Implied Tree

Constructing the $(n+1)^{th}$ Level of the Implied Tree



NOTATION

r: known forward riskless interest rate at this level

s_i: known stock price at node (n,i) at level n: node i: also the strike for options expiring at level $n+1$

F_i: known forward price at level $n+1$ of the known price s_i at level n

S_i: unknown stock price at node $(n+1,i)$

λ_i: known Arrow-Debreu price at node (n,i)

p_i: unknown risk-neutral transition probability from node (n,i) to node $(n+1,i+1)$

They call the continuously compounded forward riskless interest rate at the *nth* level *r*. In general this rate is time-dependent and can vary from level to level; for notational simplicity they avoid attaching an explicit level index to this and other variables used here.

The aim is to determine the nodes of the $(n+1)th$ level at time t_{n+1}. There are $n+1$ nodes to fix, with $n+1$ corresponding unknown stock prices S_i. The figure above shows the ith node at level n, denoted *by* **(n,i)** in boldface. It has a known stock price s_i and evolves into an "up" node with price S_{i+1} and a "down" node with price S_i at level $n+1$, where the forward price corresponding to s_i is $F_i = e^{r\Delta t} s_i$. p_i is the probability of making a transition into the up node. We call λ_i the Arrow-Debreu price at node (n,i);

The Arrow-Debreu price is computed by forward induction as the sum over all paths, from the root of the tree to node (n,i), of the product of the risklessly discounted transition probabilities at each node in each path leading to node (n,i). All λ_i at level n are known because earlier tree nodes and their transition probabilities have already been implied out to level n.

There are $2n+1$ parameters that define the transition from the nth to the $(n+1)th$ level of the tree, namely the $n+1$ stock prices S_i and the n transition probabilities p_i. These must be determined to be consistent with the observed smile.

The nodes at the $(n+1)th$ level can be implied by using the tree to calculate the theoretical values of $2n$ known quantities – the values of n forwards and n options, all expiring at time t_{n+1} – and requiring that these theoretical values match the interpolated market values. This provides $2n$ equations for these $2n+1$ parameters. The one remaining degree of freedom is used to make the center of the tree coincide with the center of the standard CRR tree that has constant local volatility.

If the number of nodes at a given level is odd, choose the central node's stock price to be equal to spot today; if the number is even, make the average of the natural logarithms of the two central nodes' stock prices equal to the logarithm of today's spot price. We now derive the *2n* equations for the theoretical values of the forwards and the options.

The implied tree is risk-neutral. Consequently, the expected value, one period later, of the stock at any node *(n,i)* must be its known forward price.

$$F_i = p_i S_{i+1} + (1 - p) S_i \qquad \textbf{[Eq 6.1]}$$

where F_i is known. There are n of these forward equations, one for each i.

The second set of equations expresses the values of the n independent options, one for each strike s_i equal to the known stock prices at the *nth* level, that expire at the *(n+1)th* level. The strike level s_i splits the up and down nodes, S_{i+1} and S_i, at the next level, as shown in the figure above. This ensures that only the up (down) node and all nodes above (below) it contribute to a call (put) struck at s_i. These n equations for options, derived below, together with forward price equation and our choice in centering the tree, will determine both the transition probabilities p_i that lead to the *(n+1)th* level and the stock prices S_i at the nodes at that level.

Let $C(s_i, t_{n+1})$ and $P(s_i, t_{n+1})$, respectively, be the known market values for a call and put struck today at s_i and expiring at t_{n+1}. We know the values of each of these calls and puts from interpolating the smile curve at time t_{n+1}. The theoretical binomial value of a call struck at K and expiring at t_{n+1} is given by the sum over all nodes j at the $(n+1)th$ level of the discounted probability of reaching each node $(n+1, j)$ multiplied by the call payoff there;

$$C(K,t_{n+1}) = e^{-r\Delta t}\sum_{j=1}^{n}\{\lambda_j p_j + \lambda_{j+1}(1-p_{j+1})\}\max\left(S_{j+1}-K,0\right) \qquad \textbf{[Eq 6.2]}$$

When the strike K equals s_i, the contribution from the transition to the first in-the-money up node can be separated from the other contributions, which, using the forward pricing equation, can be rewritten in terms of the known Arrow-Debreu prices, the known stock prices s_i and the known forwards $F_i = e^{r\Delta t}\ s_i$,

$$e^{r\Delta t}C(s_i,t_{n+1}) = \lambda_i p_i (S_{i+1}-s_i) + \sum_{j=1+1}^{n}\lambda_j\left(F_j - s_i\right) \qquad \textbf{[Eq 6.3]}$$

The first term depends upon the unknown p_i and the up node with unknown price S_{i+1}. The second term is a sum of already known quantities. Since we know both F_i and $C(s_i, t_{n+1})$ from the smile, we can simultaneously solve equation 6.1 and equation 6.3 for S_{i+1} and the transition probability p_i in terms of S_i; which gives rise to;

$$S_{i+1} = \frac{S_i\left[e^{r\Delta t}C(s_i, t_{n+1}) - \Sigma\right] - \lambda_i s_i (F_i - S_i)}{\left[e^{r\Delta t}C(s_i, t_{n+1}) - \Sigma\right] - \lambda_i (F_i - S_i)}$$ **[Eq 6.4]**

and;

$$p_i = \frac{(F_i - S_i)}{(S_{i+1} - S_i)}$$ **[Eq 6.5]**

where Σ denotes the summation term in equation 6.3.

We can use these equations to find iteratively the S_{i+1} and p_i for all nodes above the center of the tree if we know S_i at one initial node. If the number of nodes at the *(n+1)th* level is odd (that is, *n* is even), we can identify the initial S_i, for $i = n/2 + 1$, with the central node whose stock price we choose to be today's spot value, as in the CRR tree. Then we can calculate the stock price S_{i+1} at the node above from equation 6.4, and then use equation 6.5 to find the p_i. We can now repeat this process moving up one node at a time until we reach the highest node at this level. In this way we imply the upper half of each level.

If the number of nodes at the *(n+1)th* level is even (that is, *n* is odd), we start instead by identifying the initial S_i and S_{i+1}, for $i = (n+1)/2$, with the nodes just below and above the center of the level. The logarithmic CRR centering condition we chose is equivalent to choosing these two central stock prices to satisfy $S_i = S^2 / S_{i+1}$, where $S = s_i$ is today's spot price corresponding to the CRR-style central node at the previous level. Substituting this relation into equation 6.4 gives the formula for the upper of the two central nodes for even levels:

$$S_{i+1} = \frac{S\left[e^{r\Delta t}C(S,t_{n+1}) + \lambda_i S - \Sigma\right]}{\lambda_i F_i - \left[e^{r\Delta t}C(S,t_{n+1})\right] + \Sigma} \quad \textit{for } i = n/2 \qquad \textbf{[Eq 6.6]}$$

Once we have this initial node's stock price, we can continue to fix higher nodes as shown above.

In a similar way we can fix all the nodes below the central node at this level by using known put prices. The analogous formula that determines a lower node's stock price from a known upper one is shown here;

$$S_i = \frac{S_{i+1}\left[e^{r\Delta t}P(s_i,t_{n+1}) - \Sigma\right] + \lambda_i s_i (F_i - S_{i+1})}{\left[e^{r\Delta t}P(s_i,t_{n+1}) - \Sigma\right] + \lambda_i (F_i - S_{i+1})} \qquad \textbf{[Eq 6.7]}$$

where here Σ denotes the sum

$$\sum_{j=1}^{i+1} \lambda_j (s_j - F_j) \qquad \textbf{[Eq 6.8]}$$

This summation applies to all nodes below the one with price s_i at which the put is struck. If you know the value of the stock price at the central node, you can use equation 6.7 and equation 6.5 to find, node by node, the values of the stock prices and transition probabilities at all the lower nodes.

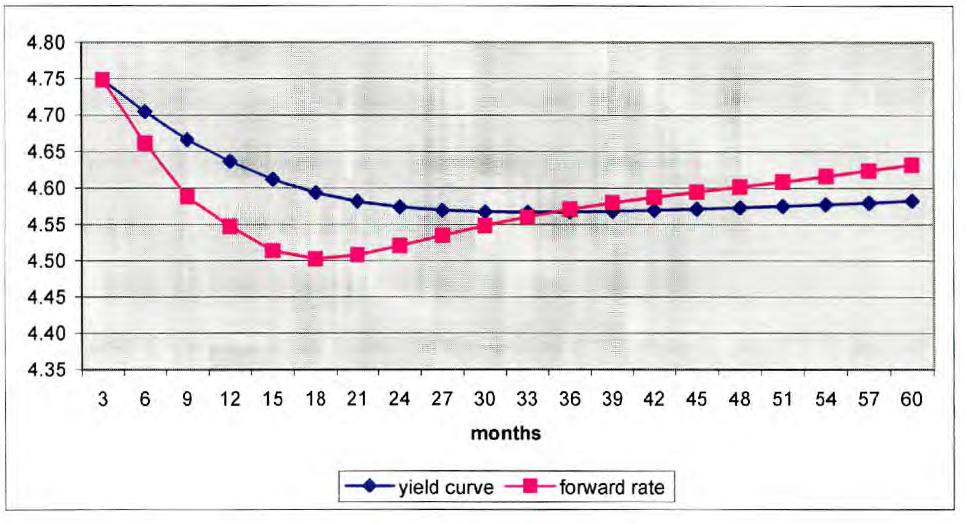
By repeating this process at each level, we can use the smile to find the transition probabilities and node values for the entire tree. If we do this for small enough time steps between successive levels of the tree, using interpolated call and put values from the smile curve, we obtain a good discrete approximation to the implied risk-neutral stock evolution process.

Derman and Kani have shown that you can use the volatility smile of liquid options, as observed at any instant in the market, to construct an entire implied tree. This tree will correctly value all standard calls and puts that define the smile. In the continuous time limit, the risk neutral stochastic evolution of the stock price in their model has been completely determined by market prices for European-style standard options.

From the analysis above it is clear that the binomial implied tree as devised by Derman & Kani derives stock prices which reflect the volatility surface which almost surely means uneven spacing between the nodes at any level of the timeframe in question. Due to this uneven spacing the methodology is implemented via a flexible binomial tree rather than a static finite difference grid which constrains future possible nodes to predefined outcomes.

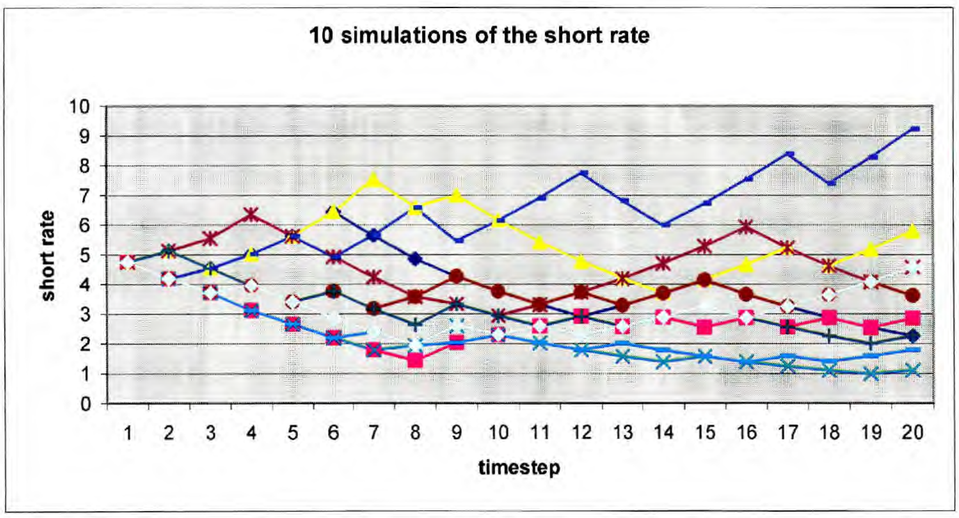
6.3 The implied binomial tree and stochastic interest rates

The volatility surface and the term structure of interest rates are the key inputs into the implied binomial tree. In a world where we didn't consider interest rates as a variable in our model the input into the implied binomial tree is the current observed yield curve. From the yield curve we can imply the forward rates for the timesteps we decide to use when building the tree. The current 5 year yield curve and the implied quarterly forward rates are shown below

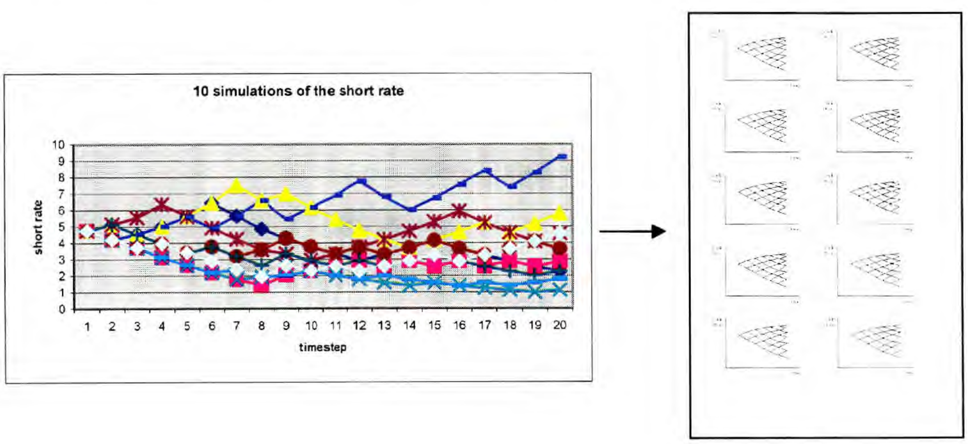


With this data and the spectrum of option prices across strikes and maturities the model creates a unique binomial tree. However as chapter 5 indicates this model to price convertible bonds incorporates stochastic interest rates via a Monte-Carlo simulation. Each simulation will give rise to a new path for the short rate which when consequently used in the Derman & Kani framework will create a unique binomial tree specific to that simulation.

The figure below shows 10 possible simulations of the short rate



Each of these simulations will create a new implied binomial tree



This will be the case of for no matter how many simulations we decide to run in the interest rate Monte Carlo model. I have restricted the figure above to 10 simulations from the BDT model for illustrative purposes only, in practice the simulation is run until there is convergence in the distribution of its results. By re-evaluating the tree for every simulation I have now incorporated a stochastic interest rate factor into the Derman & Kani model.

6.4 The implied binomial tree and credit risk

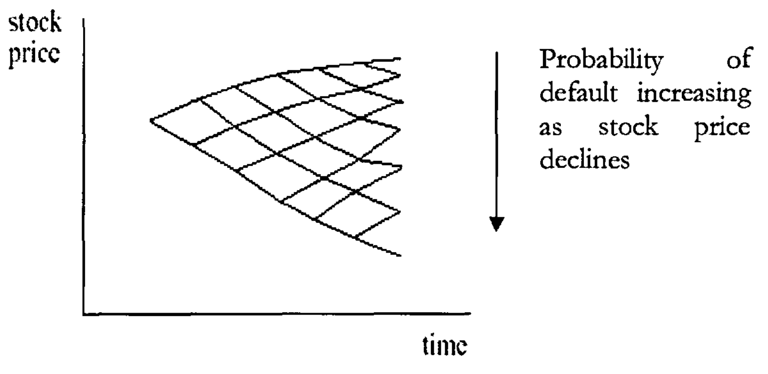
As mentioned in chapter 5, I will be including a probability of default at each node of the tree to accommodate credit risk rather than using any variant of the traditional approach where risky and risk free discount rates are used to recursively value the equity and debt portions of the convertible bond respectively.

CreditGrades developed by the RiskMetrics Group provide industry-standard, company-specific risk measures that provide a robust and transparent source for default probabilities and credit spreads. . Their model derives a probability of default as a function of numerous variables including the stock price and interest rates.

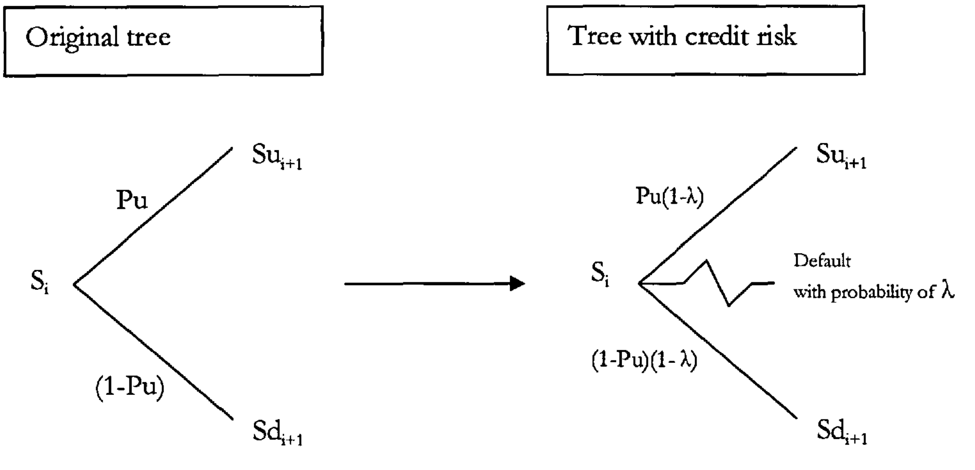
The previous section already highlighted in the implied binomial tree that stock prices were a function of interest rates and the option price matrix. Hence as we vary the stock price parameter in the Credit Grades model we are also accounting for interest rate variability too.

We now move away from a model where there is a uniform credit risk regardless of the stock price of a company to a model that is dynamically derived from it.

The figure below depicts how the probability of default will related to a implied binomial tree:



This will be incorporated to every node of every tree in every simulation of the binomial tree. The figure below shows how incorporating default changes the original implied tree:



In the original tree where S_i is known we have already shown how the future possible stock prices and the probabilities of reaching them are derived. In the tree that incorporates credit risk whilst the possible future states remain the same the probability of reaching them are changed. The original probabilities of Pu and (1-Pu) are now multiplied by the probability of survival (1- λ) to give the true probability of reaching those nodes.

The new third scenario is the underlying company defaults with a probability of λ. If this occurs a common assumption is to treat the stock price as if it has jumped to zero. At default however convertible bond values do not jump to zero and as discussed in chapter 6 bond holders receive approximately 30% of par value. This factor will be used in pricing convertible bonds

It is important to note once we implement credit risk in this manner we move away from a complete market analysis to a incomplete market analysis. We are no longer able to define prices of instruments by creating a portfolio of other instruments, which match its payoff. Whilst this is unfortunate it is wholly realistic, as convertible bonds with all their embedded features cannot be matched by using vanilla instruments.

6.5 The combined pricing model

In the previous sections I have explained in detail how the implied binomial tree has been modified to incorporate a stochastic interest rate and accommodate credit risk.

If a modified implied tree (which is a function of the option price matrix, a particular simulation of the short rate and the credit grades given default probabilities) is given I will now explain how convertible bonds are priced from it.

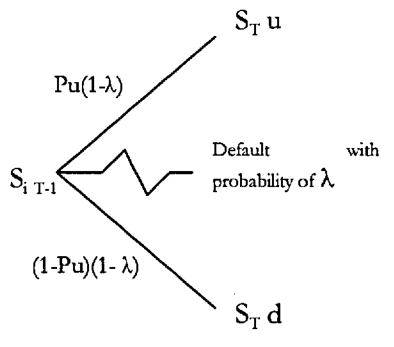
The method involves stepping backwards in time through the nodes of the tree and solving recursively for the value of the convertible bond at time 0.

At maturity time T, the value of the convertible bond at each of the nodes is determined by the following boundary condition;

CB = Max(Conversion ratio X stock price at that node, redemption price of bond)

[Eq 6.9]

Prior to maturity, the value of the convertible bond is the maximum of the discounted expected value of future cash flows plus the cash flows that are paid in that time period and the conversion and option features of the convertible bond. So for example at the timestep just before maturity i.e T-1 the value of a convertible bond at any particular node is the higher of the expected value of the bond in the next time step(T) discounted back one time step plus the coupons the bond pays at time T-1 and the conversion and other embedded options in the bond . The expected value at time T is illustrated and calculated as follows:



If $S_{i\ T-1}$ is a node one step before maturity and with boundary condition stated above the expected value at T for the convertible bond is:

$E[CB_T]$ = Pu(1-λ)*max(S_T u*CR, redemption price) +

(1-P)(1-λ)*max(S_T d*CR, redemption price)+

λ* recovery value of 30% par **[Eq 6.10]**

This is discounted back at the short rate simulated for the time period in question from time T-1- T to give:

$$e^{(\eta_{[T-1,T]})[T-(T-1)]}\mathrm{E}[CB_T] \qquad \textbf{[Eq 6.11]}$$

We the include the coupons or any other payments due at time T-1:

$$e^{(\eta_{[T-1,T]})[T-(T-1)]}\mathrm{E}[CB_T] + coupons \qquad \textbf{[Eq 6.12]}$$

In more general terms

$$e^{(\eta_{[t-1,t]})[t-(t-1)]}\mathrm{E}[CB_t] + coupons_{t-1} \qquad \textbf{[Eq 6.13]}$$

Equation 6.13 is how we calculate the discounted expected value of the convertible bond to the timestep and node we are working at, however the convertible bond is a complicated instrument and the price as calculated by equation 6.13 may not be the convertible bond price we allocate to that node. This is due to the conversion feature and embedded options convertible bonds have.

Convertible bonds are convertible to the underlying equity normally throughout the life of the bond. Hence at any node if the value of conversion is greater than what equation 6.13 derives we value the convertible bond at that node as the conversion value. If the convertible bond is also callable and we are at a node in the tree that triggers the conditions for it to become active the value of the convertible is no longer a simple comparison between conversion and the value as derived by equation 6.13. This is because if the call is lower than equation 6.13 then this is what is compared to the conversion value. Additionally any puts, which have a strike price higher than all of the above factors will cause us to value the convertible at these nodes as the put value.

In summary at any time and node before maturity the value of the convertible bond is:

CB value = max[min(equation 6.13, issuer call value), conversion value, put value]

[Eq 6.14]

This is continued until we find the price at timestep 0 of the bond.

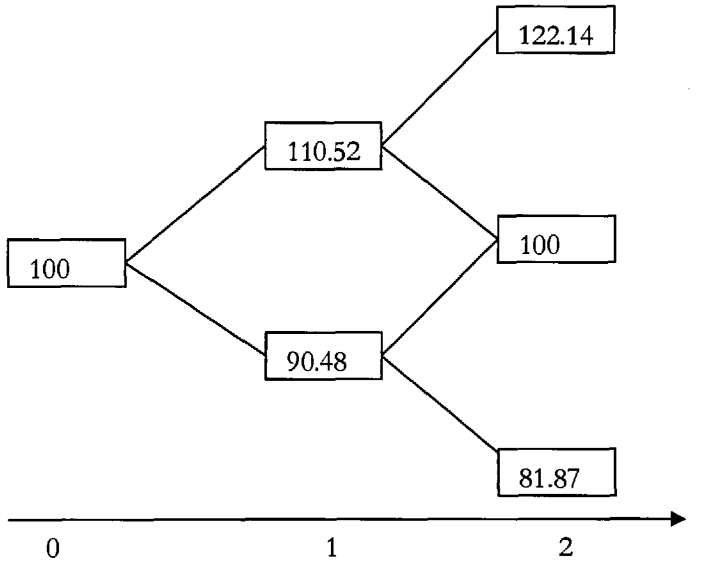
This process is repeated for every simulation of the interest rate model the reader chooses to use to ultimately gain a distribution of convertible bond prices which once has converged gives us a average price and a indication how variable this price could become.

To illustrate the mechanics of the model I will give a brief example of the construction of the tree and the recursive procedure used to derive a price for a convertible bond.

Derman and Kani show an example which takes the current value of a security as 100, with a dividend yield is zero, and annually compounded riskless interest rate of

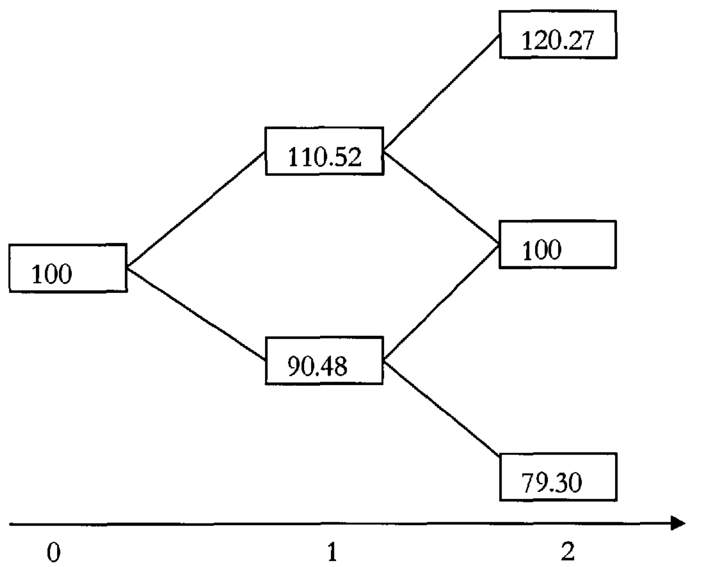
3% per year for all maturities. They assume that the annual implied volatility of an at-the-money European call is 10% for all expirations, and that implied volatility increases (decreases) linearly by 0.5 percentage points with every 10 point drop (rise) in the strike. This defines the hypothetical smile.

With the variables above a standard CRR binomial stock tree will evolve as follows over one year time step for two periods.

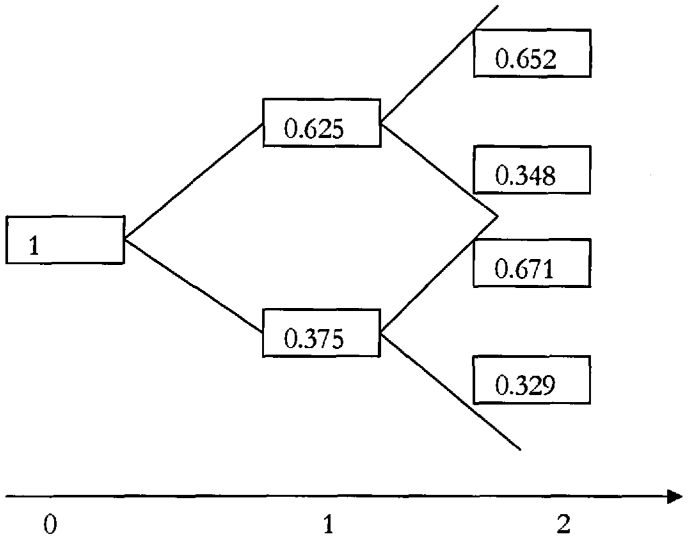


Derman and Kani highlight that the CRR binomial stock tree has a local volatility of 10% everywhere. This tree produces no smile with a transition probability at every node of 0.625 and is the discrete binomial analog of the continuous-time BS equation.

They then go on to derive their implied binomial tree (which has been explained in chapter 3). The tree above now changes to:



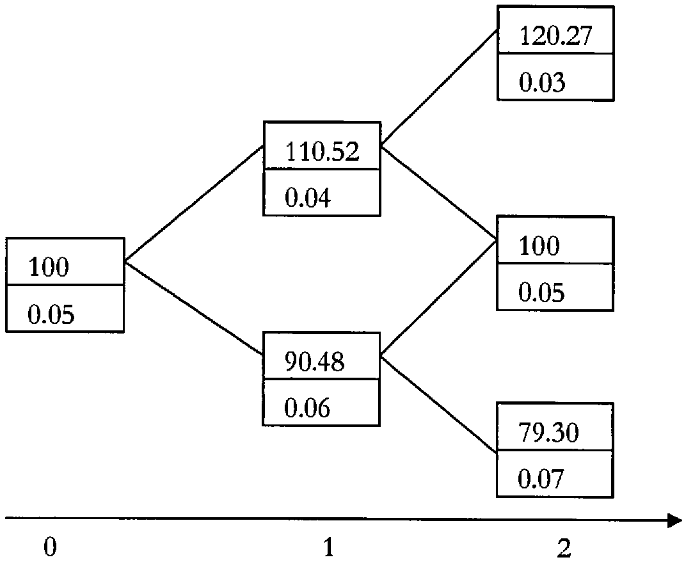
With a probability at each node of ;



In the convertible bond model we build upon this basic model. This convertible bond model will derive the implied binomial tree above for every simulation of the short rate conducted in the interest rate model. The case above keeps interest rates constant and I will continue with this to keep the example simple

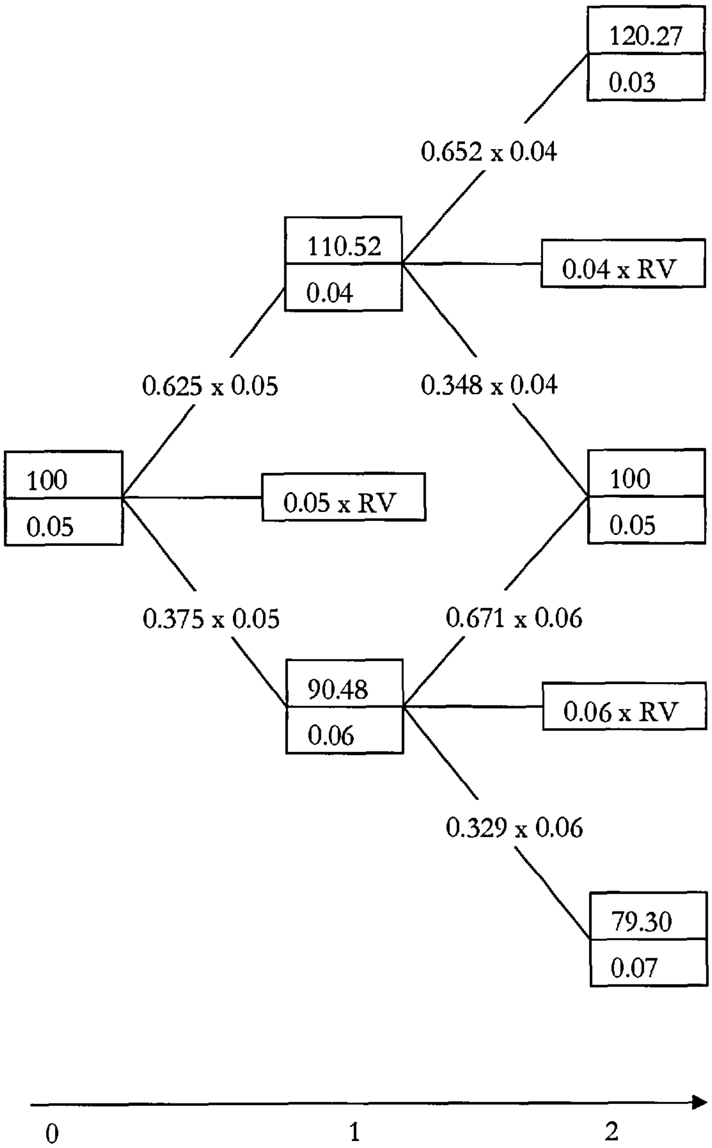
Now from Credit Grades we derive the probability of default for each node on the tree. We have a range of default probabilities from Credit Grades which are a function of the stock price. We find the probability of default for each node by interpolating between these Credit Grades inputs

For example



The tree above shows the implied binomial tree with the implied stock price in the top box at each node and the probability of default for that stock price in the bottom box at each node.

We now have a final tree which combines the implied binomial tree stock prices with the unique probabilities and the probability of default


120.27
0.03
0.652 x 0.04
110.52
0.04
0.04 x RV
0.625 x 0.05
0.348 x 0.04
100
0.05
0.05 x RV
100
0.05
0.375 x 0.05
0.671 x 0.06
90.48
0.06
0.06 x RV
0.329 x 0.06
79.30
0.07
0
1
2

We use this final tree to work back recursively to price the convertible bond subject to the boundary conditions as specified by equations 6.9 to 6.14.

This procedure is repeated for as many simulations we conduct on the interest rate model. This eventually gives a average CB bond price which is the models final output.

Chapter 7

TESTING THE MODEL & RESULTS

7.1 Introduction

Chapter 6 discussed in detail how the model theoretically works. This chapter is devoted to applying this theory to price convertible bonds currently trading actively in the market. I will test my model on 5 current convertible bonds with varying levels of embedded options to ensure it can accommodate varying levels of complexity that are possible in convertible bonds.

In selecting the convertible bonds to price, I chose companies listed within the FTSE 100 due to the fact that their underlying equity is the most liquid and the vanilla option markets for these companies are likely to be widely traded in considerable volume. This feature is particularly important in my model, as the most important input into the construction of the implied binomial tree is the option price matrix. Additionally the model does not focus on FX risk and hence convertible bonds were chosen that are denominated in sterling and are convertible into equity denominated in sterling.

With this criterion above in mind I have chosen the five following bonds

Issuer	Coupon & Maturity
BAA PLC	2.94% 04/04/2008
Friends Provident PLC	4.25% 11/12/2007
Legal and General PLC	2.75% 18/12/2006
Scottish & Southern Energy PLC	3.73% 29/10/2009
WPP Group PLC	2.00% 11/04/2007

These bonds will all be dealt with in more detail in the following sections, however before we discuss them on a individual basis the Monte Carlo simulation of the stochastic interest rate factor which is the same for all the bonds will be developed.

7.2 Interest Rate Monte Carlo simulation

As discussed in earlier chapters this model is adaptable to any interest rate model the reader chooses to use. For purposes of testing I will use the arbitrage free model developed by Black Derman & Toy (1990). Its principle inputs are the current yield curve and the volatilities of these rates throughout the curve.

The yield curve and the respective volatilities as of April 3 2005 is a shown below[12]

months	rate	volatility
3	4.75%	
6	4.70%	10.00%
9	4.67%	10.00%
12	4.64%	10.60%
15	4.61%	11.06%
18	4.59%	11.52%
21	4.58%	11.98%
23	4.57%	12.33%
27	4.57%	12.32%
30	4.57%	12.31%
33	4.57%	12.39%
36	4.57%	12.37%
39	4.57%	12.35%
32	4.57%	12.33%
35	4.57%	12.32%
38	4.57%	12.30%
51	4.57%	12.27%
53	4.58%	12.25%
57	4.58%	12.22%
60	4.58%	12.19%

12 Data from Bloomberg and volatilities are historic as calculated from Bloomberg

With these inputs the BDT tree can be implemented which can consequently be used to generate simulations through the tree. Simulations of a 1000 iterations seemed to find paths which converged. The prices of zero coupon bonds priced by the tree were very close to what the yield curve presumes they are as illustrated below.

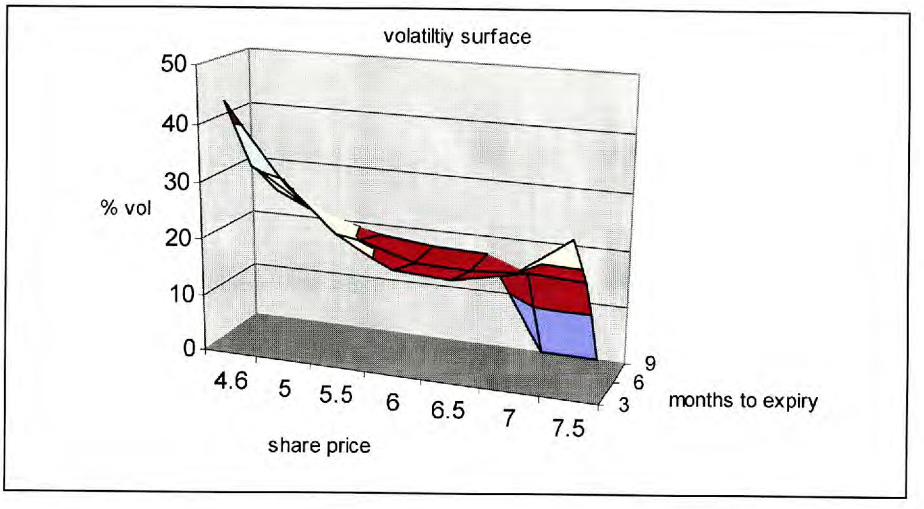
Time	ZCB YC	ZCB BDT	Error
3	0.95	0.95	0.00%
6	0.91	0.91	0.00%
9	0.87	0.87	-0.02%
12	0.83	0.83	-0.02%
15	0.80	0.80	-0.03%
18	0.76	0.76	-0.06%
21	0.73	0.73	-0.11%
23	0.70	0.70	-0.18%
27	0.67	0.67	-0.23%
30	0.63	0.63	-0.31%
33	0.61	0.62	-0.30%
36	0.59	0.59	-0.38%
39	0.56	0.57	-0.59%
32	0.53	0.53	-0.70%
35	0.51	0.52	-0.82%
38	0.39	0.50	-0.95%
51	0.37	0.38	-1.07%
53	0.35	0.36	-1.22%
57	0.33	0.33	-1.35%
60	0.31	0.32	-1.50%

7.3 BAA PLC

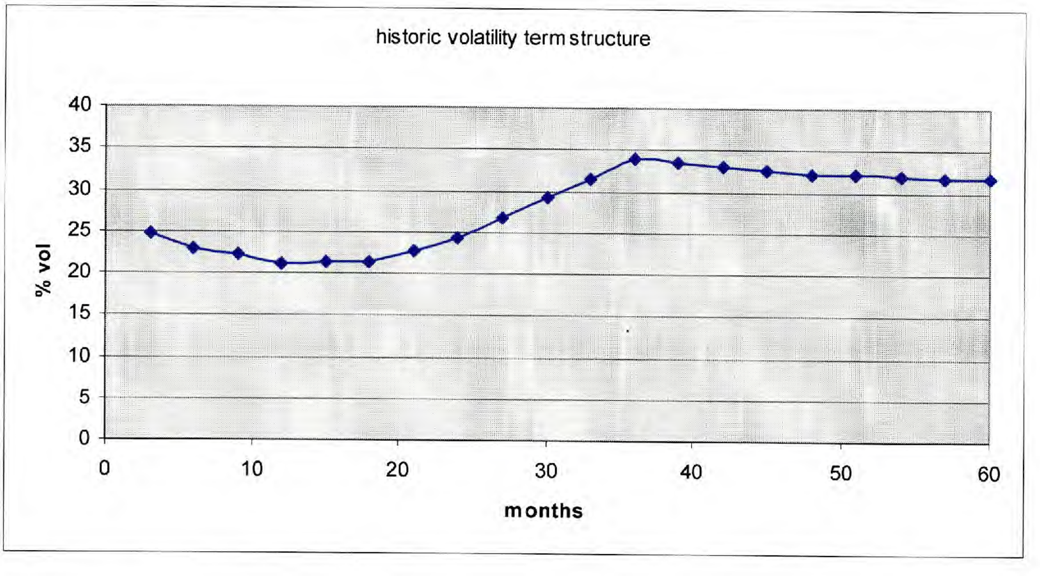
The key inputs are

Share Price	£6.01
Number of quarterly timesteps	12
Option Price matrix timesteps	3
Volatility term structure timesteps	9
Conversion ratio	12.5
Hard Call	none
Soft Call	Call applicable from 18/04/2006 to Maturity. Stock must exceed £10.40 for 20 business days out of 25 days to be triggered at a strike of par
Put	None
Probability of default	See Credit grades
Observed Market price of Bond(bid-ask)	£96.3370-£96.8370

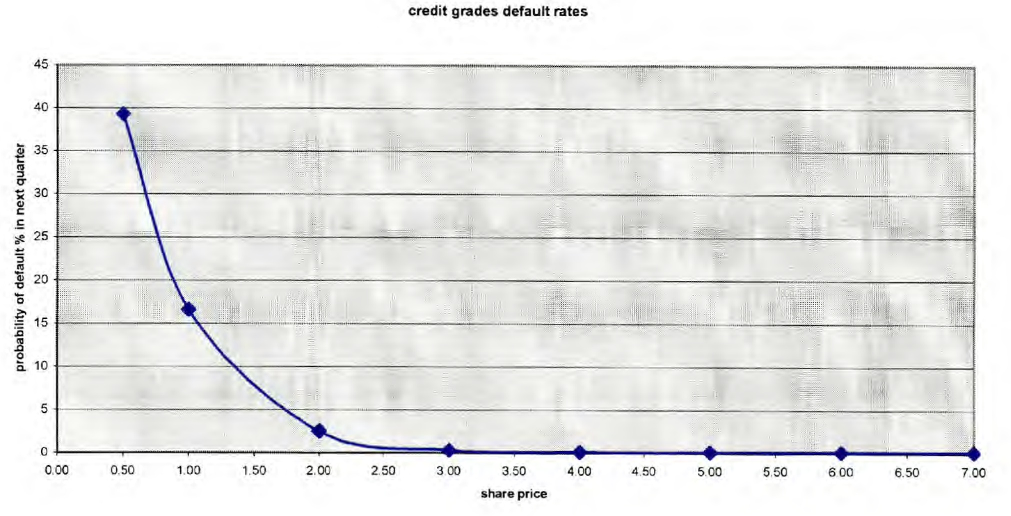
The implied volatility surface



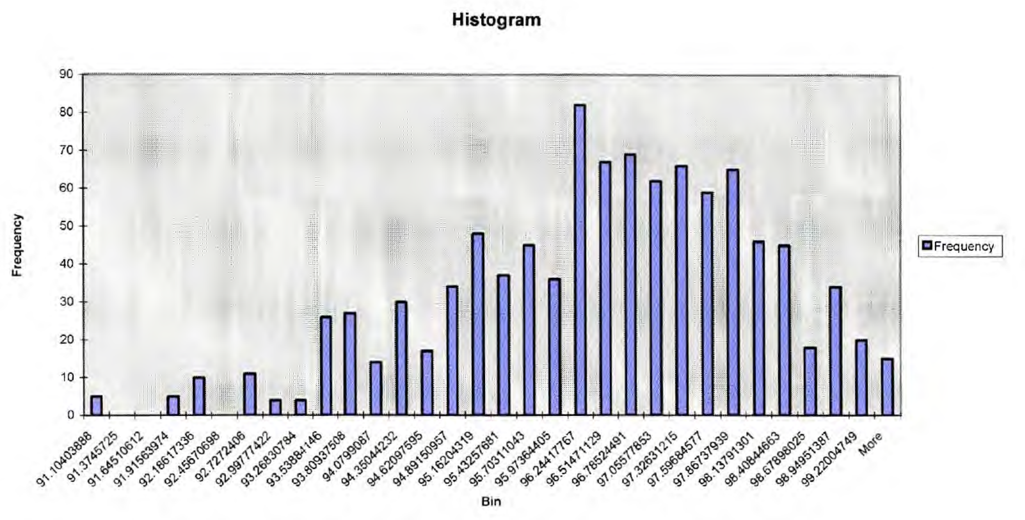
The volatility term structure



and credit grades defaults of



The model derived an average price £96.90978. The distribution around the price after 1000 simulations is shown below

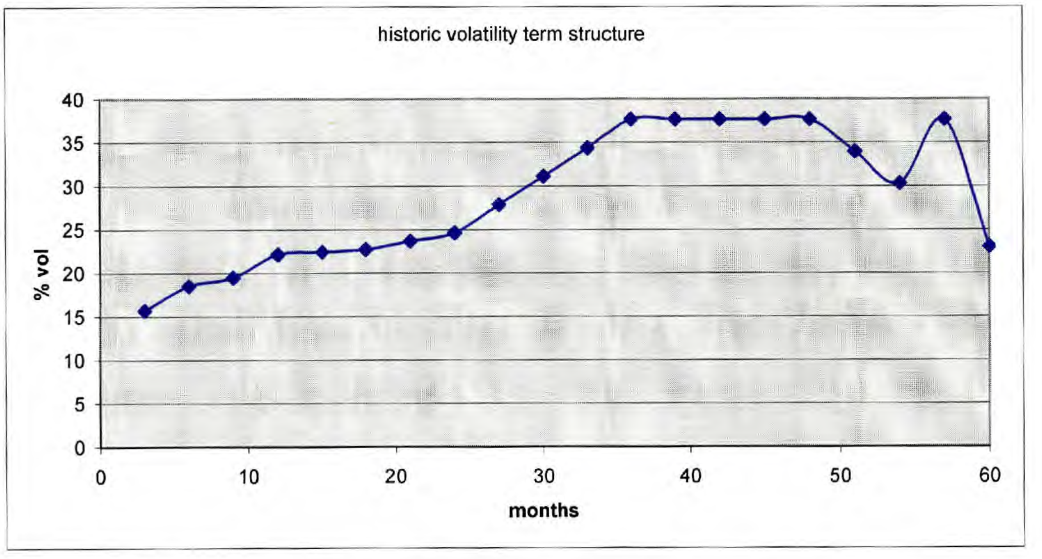


7.4 Friends Provident PLC

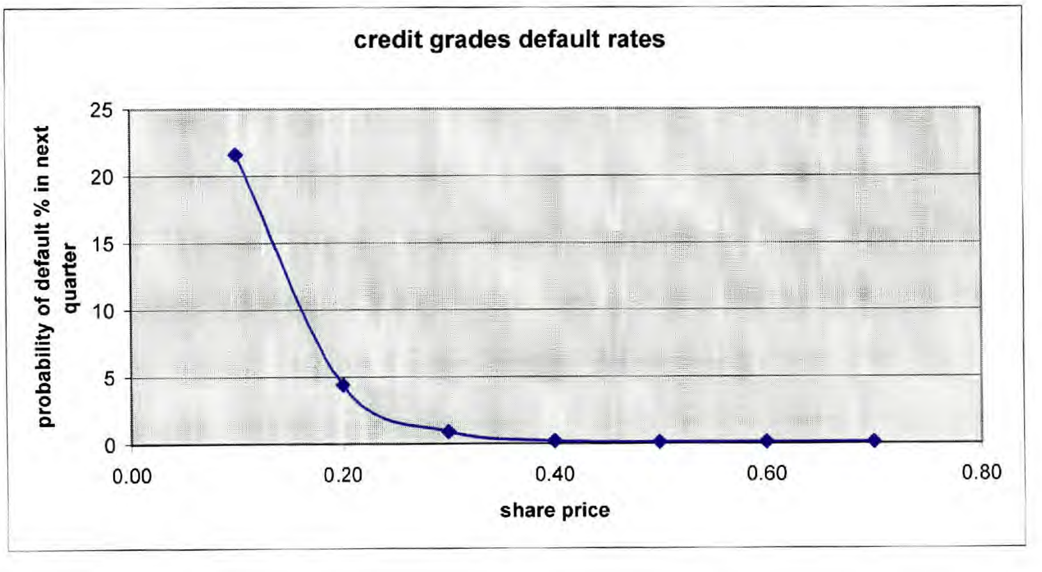
The key inputs are

Share Price	£1.7175
Number of quarterly timesteps	11
Option Price matrix timesteps	0
volatility term structure timesteps	11
Conversion ratio	58.37953
Hard Call	None
Soft Call	Call applicable from 27/12/2005 to Maturity. Stock must equal or exceed £2.223 to be triggered at a strike of par
Put	None
Probability of default	See Credit grades
Observed Market price of Bond(bid-ask)	£111.61-£112.11

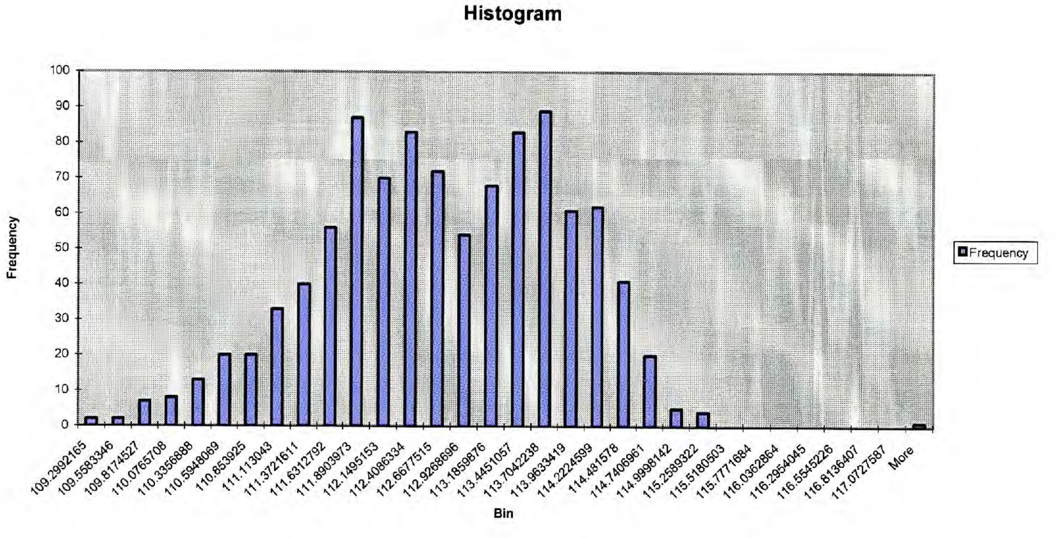
The volatility term structure



and credit grades defaults of



The model derived an average price £112.61. The distribution around the price after 1000 simulations is shown below

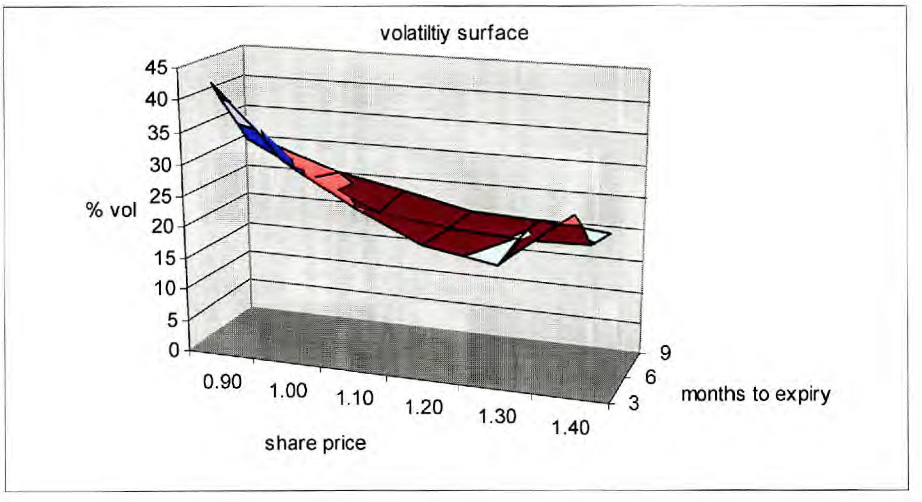


7.5 Legal & General

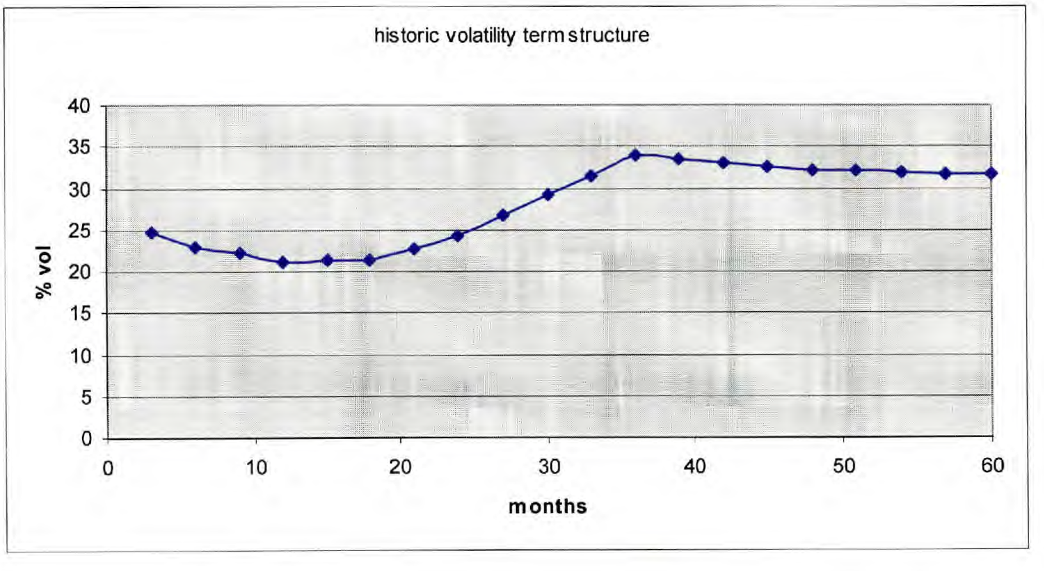
The key inputs are

Share Price	£1.1.45
Number of quarterly timesteps	7
Option Price matrix timesteps	3
volatility term structure timesteps	4
Conversion ratio	54.347827
Hard Call	None
Soft Call	Call applicable from 03/01/2005 to Maturity. Stock must equal or exceed £2.448 for 20 consecutive days out of 30 days to be triggered at a strike of par
Put	None
Probability of default	See Credit grades
Observed Market price of Bond(bid-ask)	£96.8442-£96.3442

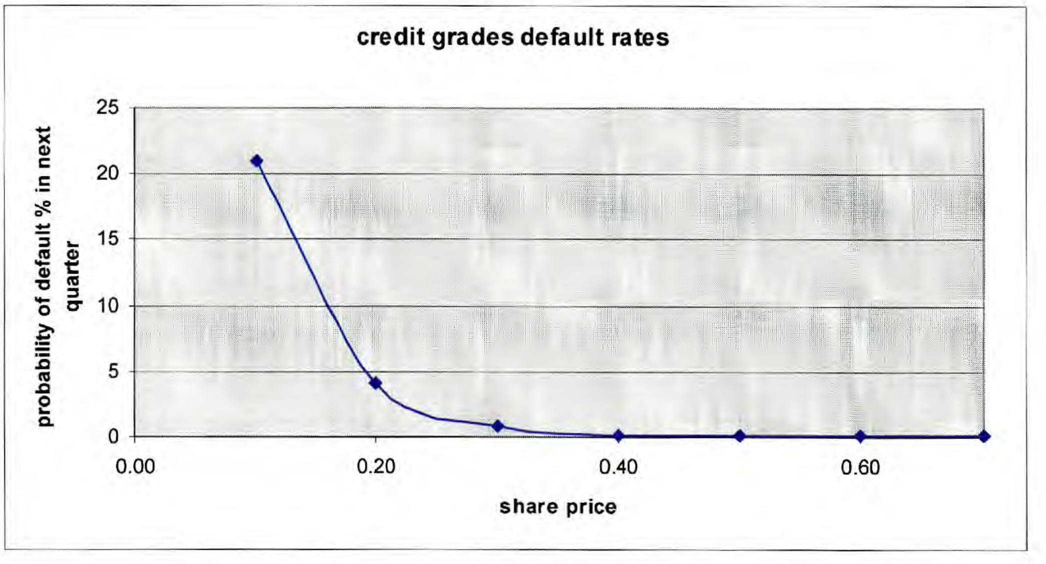
The implied volatility surface



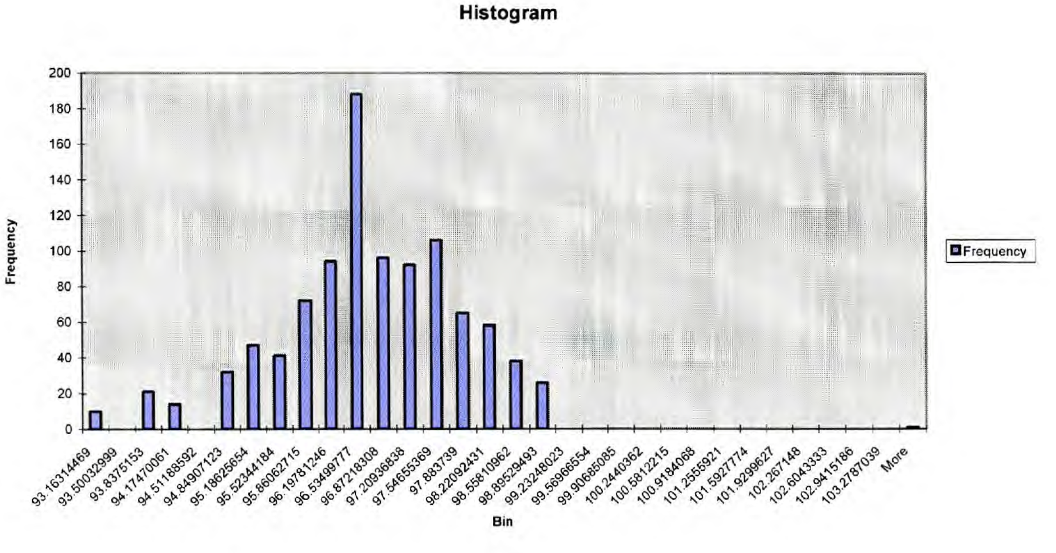
The volatility term structure



and credit grades defaults of



The model derived an average price £97.88045. The distribution around the price after 1000 simulations is shown below

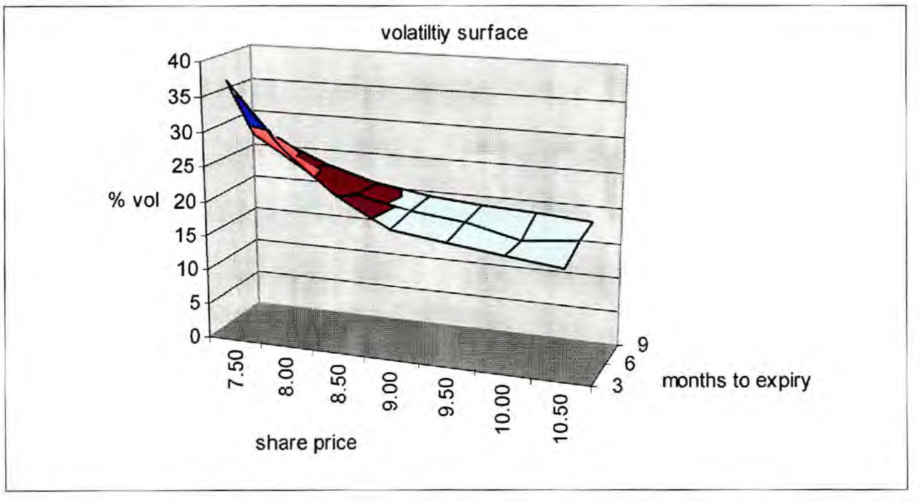


7.6 SSE PLC

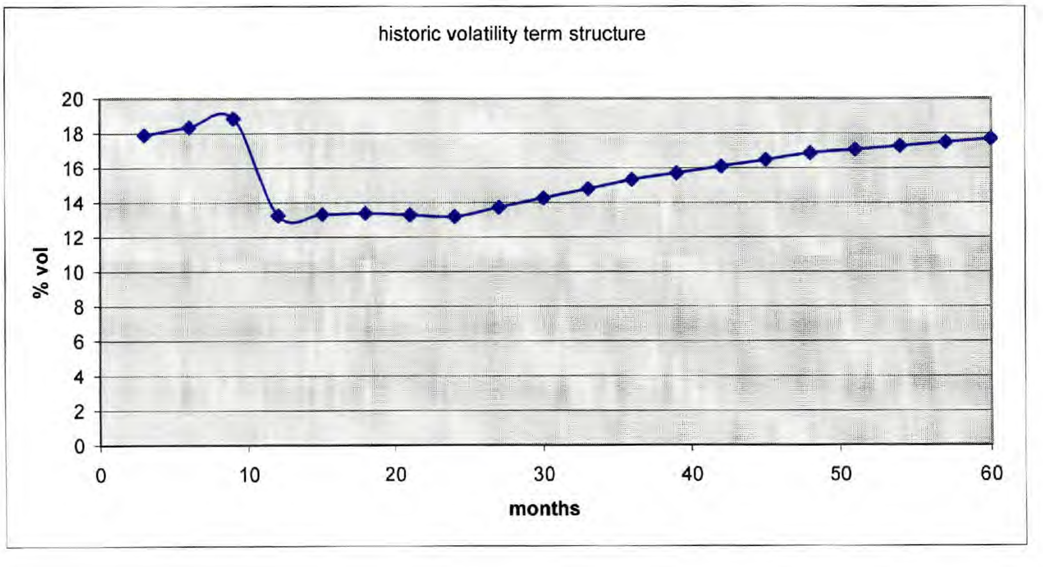
The key inputs are

Share Price	£9.315
Number of quarterly timesteps	18
Option Price matrix timesteps	3
volatility term structure timesteps	15
Conversion ratio	11.111
Hard Call	none
Soft Call	None
Put	Puttable at par on 29/10/2007
Probability of default	See Credit grades
Observed Market price of Bond(bid-ask)	£109.0324-£109.5324

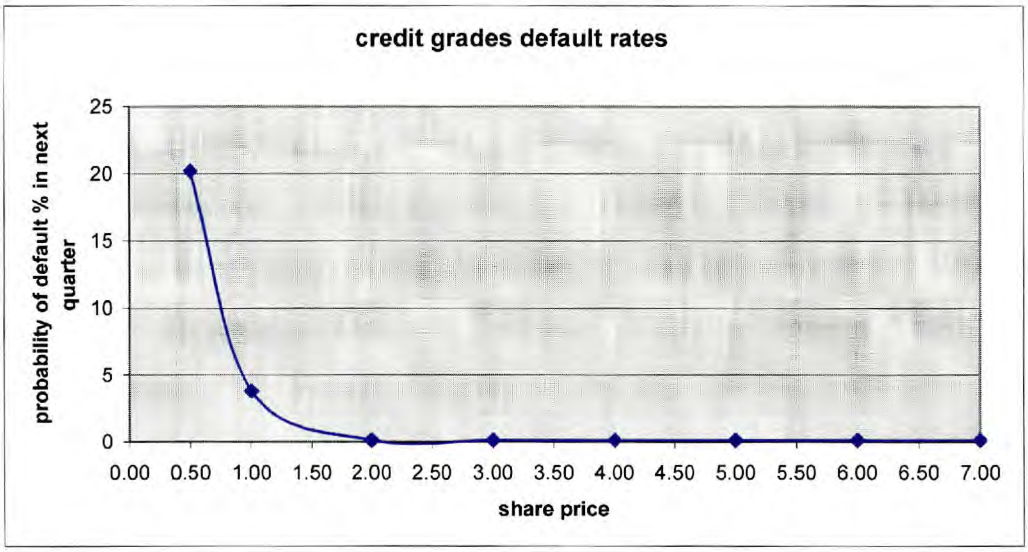
The implied volatility surface



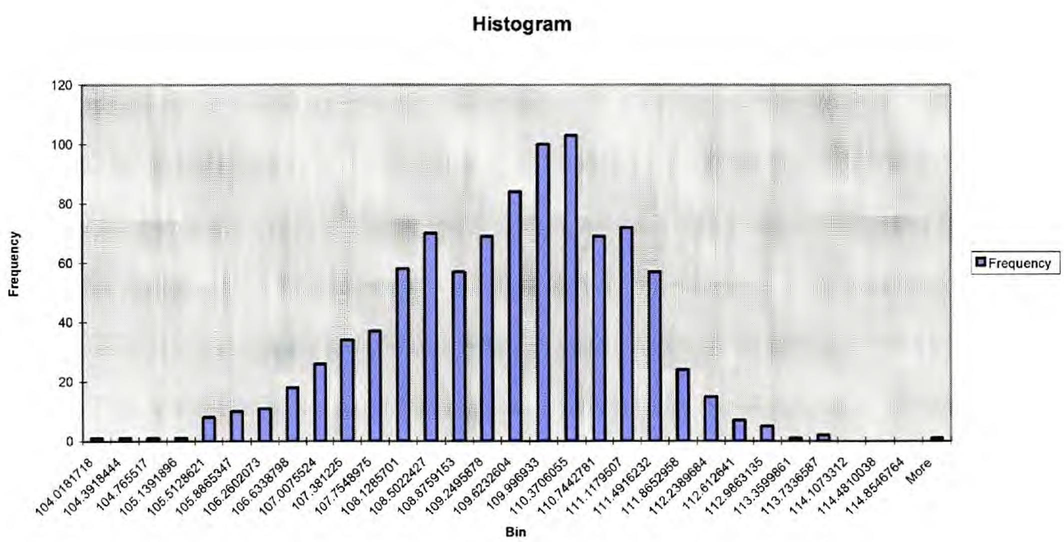
The volatility term structure



and credit grades defaults of



The model derived an average price £109.3623. The distribution around the price after 1000 simulations is shown below

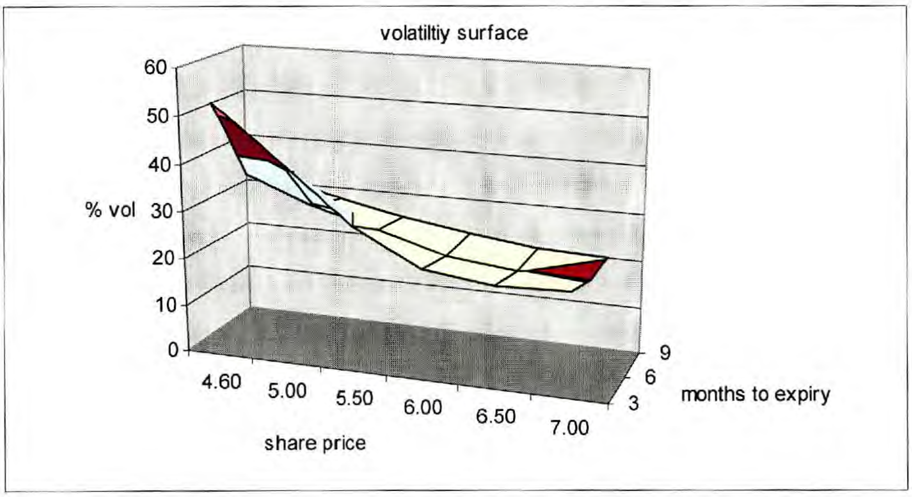


7.7 WPP Group PLC

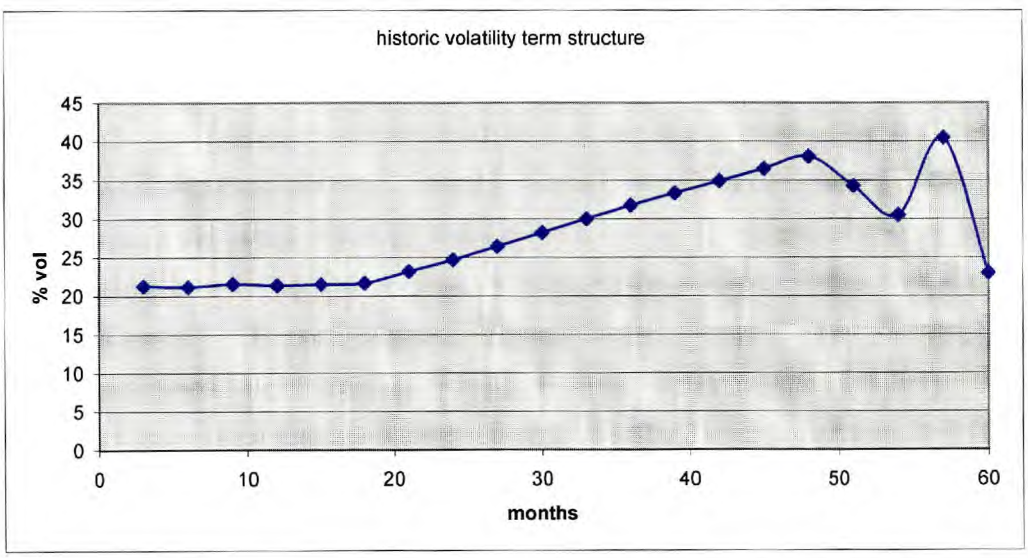
The key inputs are

Share Price	£6.16
Number of quarterly timesteps	8
Option Price matrix timesteps	3
volatility term structure timesteps	5
Conversion ratio	9.30233
Hard Call	None
Soft Call	None
Put	None
Probability of default	See Credit grades
Observed Market price of Bond(bid-ask)	£98.9891-£99.4891

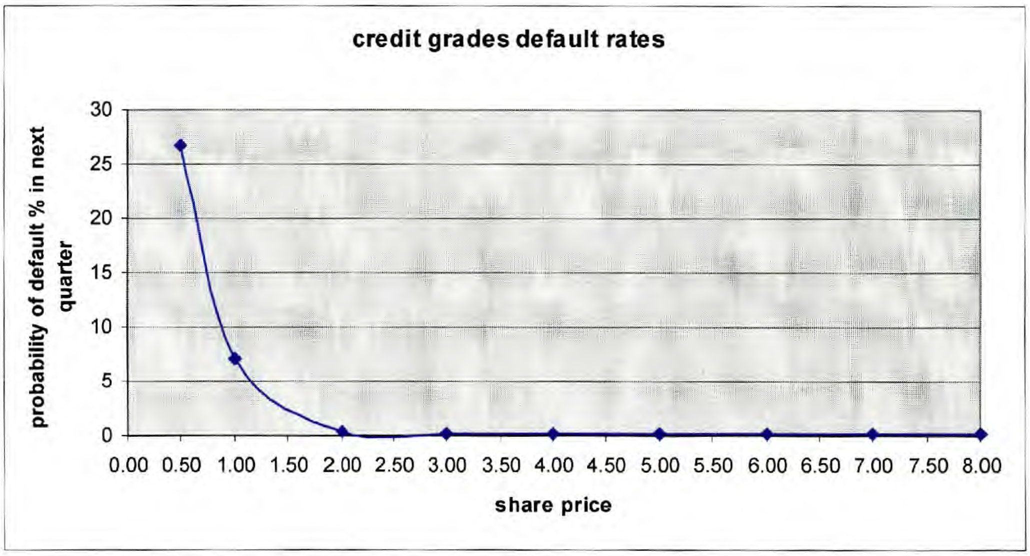
The implied volatility surface



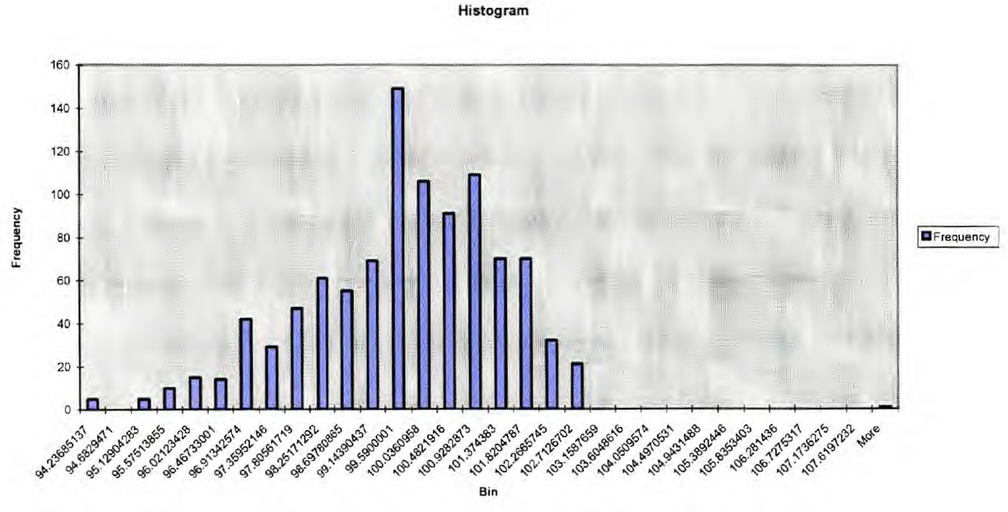
The volatility term structure



and credit grades defaults of



The model derived an average price £99.4712. The distribution around the price after 1000 simulations is shown below



Chapter 9

CONCLUSION

This thesis presents a model that embeds major forms of security risk, enabling the pricing of convertible bonds. It combines the derivation and calibration of the implied tree from Derman & Kani with an interest rate model of the users choice and a probability of default derived from credit grades in regards to interest rate and credit risk.

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The aim was to develop a pricing model which accommodated multiple sources of risks to price convertible bonds using observable market inputs from the equity and bond markets.

The model is the first I am aware of where the user has the flexibility to choose any interest rate model they desire. Normally convertible bond models implemented on a finite difference grid or 2 factor 3-D tree are restricted to Markovian interest rate models which can be implemented via a recombining lattice. The latest advances in interest rate modelling in the form of multi-factor HJM and Libor Market Models, that are now becoming increasing popular by practitioners, however tend to be non

Markovian. The implementation of these models is restricted to Monte Carlo simulations.

By designing the model so the stochastic interest rate factor is integrated through a Monte-Carlo simulation I have opened the convertible bond pricing model to the entire spectrum of Markovian and non Markovian interest rate models. This feature now allows convertible bond practitioners to compare how the convertible bond pricing model differs under different interest rate models. This is important as usually no single model can satisfactorily price and risk manage all exotic trades, hence traders like to keep a selection of different models available. Risk managers also benefit by having a spread of model evaluations to keep a check on model error.

Credit risk has been integrated using the CreditGrades models to ascertain the probability of default at each node of the tree. This completely removes the ambiguity of trying to determine which discount rates to use on different portions of the bond. The use of a static credit premium above the risk free rate to capture credit risk is replaced by a dynamic probability of default. All discounting in this scenario is done via the risk free rate.

The calibration and testing of the models on five of the most liquid convertible bonds in the UK proves promising. The model derives results which seem to be fractionally higher than the market observed prices as quoted by Bloomberg on April 13th 2005. The table below summarises

Issuer	Market Price (bid-ask)	Model Price
BAA	£96.3370-£96.8370	£96.90978
Friends Provident	£111.61-£112.11	£112.61
Legal & General	£96.8442-£97.3442	£97.88045
SSE	£109.0324-£109.5324	£109.3623
WPP	£98.9891-£99.4891	£99.4712

The model slightly over prices the convertible bonds in relation to the observed market prices. This is largely expected as traders also consider non quantifiable risks such as liquidity and prospectus risk before quoting prices. There is no exact science in incorporating this type of risk and how it affects prices largely depends on the traders intuition and judgement. They normally take the price of a convertible bond given via a model and shave off some value to accommodate non quantifiable risk. With this taken into consideration the prices achieved by the model is increasingly more accurate than making a direct comparison in the table above.

The model seems promising and robust and could be improved further by reducing the timesteps in the model at the expense of computation time. At quarterly timesteps computation time was very quick and the model can be implemented easily in many environments from spreadsheets to dedicated programmes.

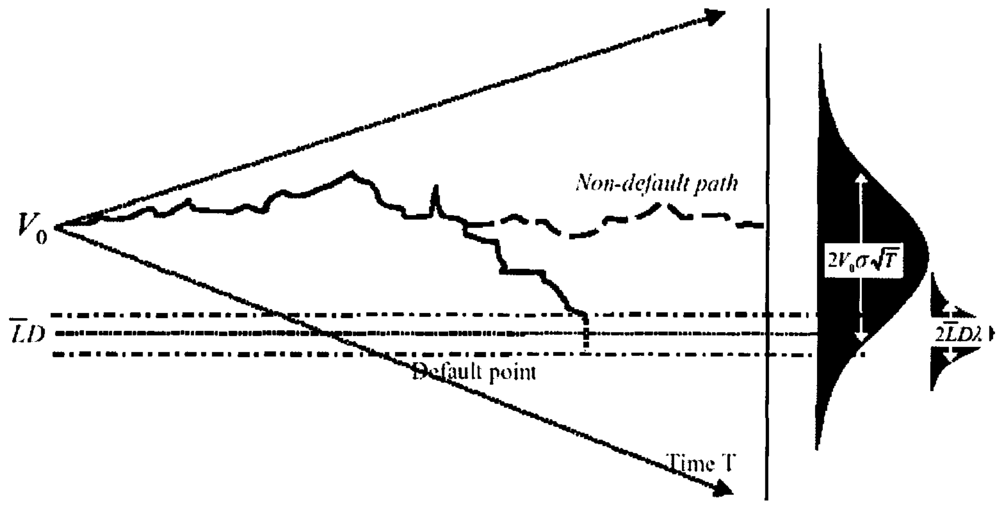
APPENDIX A

The purpose of the CreditGrades™ model is to establish a robust but simple framework linking the credit and equity markets. The relationship between corporate debt and equity was first formally proposed by Black and Scholes (1973) and Merton (1973). These authors observed that equity may be modelled as an option on a firm's assets, and that the value of a firm's debt is simply the value of its assets in excess of the equity value. The approach was further developed by Black and Cox (1976) and later by Leland (1993). According to their approach (which is commonly referred to as the structural model), an event of default occurs when the asset value of a firm crosses a predetermined default barrier or threshold.

CreditGrades uses the structural model framework to develop a link between credit and equity derivatives. This section includes the details from chapter 2 of the Credit Grades technical paper which highlights the technical details of the model. It is included here for completeness as it is used to derive the probability of default at each timestep in the model.

For the most part, the CreditGrades model can be viewed as a practical implementation of the standard structural model. It employs approximations for the asset value, volatility and drift terms which relate all of these quantities to market observables. In this framework, credit is valued as an exotic equity derivative whose pricing formula can be expressed in closed form. The resulting formula is appealingly simple and yet can approximate any sophisticated model relying on similar fundamental assumptions. See Finkelstein (2001), Finkelstein and Lardy (2001), Lardy (2001a), Lardy (2001b), Lardy and Pradier (2001) and Pan (2001) for further detail. One departure from the standard structural model is made to address its artificially low short-term spreads. These low spreads occur because assets that begin above the barrier cannot reach the barrier immediately by diffusion only. Hull and White (2001) confront this issue using a time-dependent default barrier which is calibrated to market spreads. An alternative approach is to incorporate jumps into the asset value process. In the CreditGrades approach, the uncertainty is modelled in the default barrier, motivated by the fact that one cannot expect to know the exact level of leverage of a firm except at the time the firm actually defaults. The uncertainty in the barrier admits the possibility that the firm's asset value may be closer to the default point than we might otherwise believe. This leads to higher short-term spreads than are produced without the barrier uncertainty. Thus the standard deviation of recovery value takes on an important role in the calculation of the probability of default and its term structure.

The basic assumptions of the model are illustrated in the figure below.



The model begins with a stochastic process V *and* defines default as the first time V crosses the default barrier. V may be thought of intuitively as the asset value (on a per share basis) process for the firm, although as will be discussed below, the model will not identify V exactly with the firm's asset value. The model defines the default barrier as the amount of the firm's assets that remain in the case of default. This quantity is simply the recovery value that the debt holders receive, $L \cdot D$, where L is the average recovery on the debt and D is the firm's debt-per-share.

CreditGrades assumes that the asset value evolves as a geometric Brownian motion;

$$\frac{dV_t}{V_t} = \sigma dW_t + \mu_D dt \qquad \text{[A1.1]}$$

where W is a standard Brownian motion, σ is the asset volatility, and μ_D is the asset drift. The model assumes for now that $\mu_D = 0$; this is discussed further on in the analysis.

Because the standard structural model, with the asset value evolving by pure diffusion and the default barrier fixed, produces unrealistic short-term credit spreads, CreditGrades introduces randomness to the average recovery value L. The introduction of uncertain recovery value is based on empirical studies of recovery rates. One prevalent finding of these studies is an extreme variance of the distribution of recoveries. In addition to some industrial sector dependence, the recovery rate can be greatly affected by factors such as whether default is triggered by financial or operational difficulties and whether the company will be restructured or liquidated. The model assumes that the recovery rate L follows a lognormal distribution with mean L^* and percentage standard deviation λ. Specifically,

$$L^* = \mathbf{E}\, L \qquad [A1.2]$$

$$\lambda^2 = Var \quad \ln(L) \qquad [A1.3]$$

$$LD = L^* D e^{\left(\lambda Z - \frac{\lambda^2}{2}\right)} \qquad [A1.4]$$

where Z is a standard normal random variable. The random variable Z is independent of the Brownian motion W. Z is unknown at $t = 0$ and is only revealed at the time of default. Intuitively, by letting Z be random, the model captures the uncertainty in the actual level of a firm's debt-per-share. Thus, there is some true level of L that does not evolve through time, but that we are unable to observe with certainty. With the uncertain recovery rate, the default barrier can be hit unexpectedly, resulting in a jump-like default event.

For an initial asset value V_0, default does not occur as long as

$$V_0 e^{\left(\sigma W_t - \frac{\sigma^2 t}{2}\right)} > L^* D e^{\left(\lambda Z - \frac{\lambda^2}{2}\right)} \qquad [A1.5]$$

The survival probability of the company at time t is then given by the probability that the asset value (A1.1) does not reach the barrier (A1.3) before time t.

Introducing a process

$$X_t = \sigma W_t - \lambda Z - \frac{\sigma^2 t}{2} - \frac{\lambda^2}{t} \qquad \text{[A1.6]}$$

Then equation A1.5 can be rewritten as

$$X_t > \ln\left(\frac{L^* D}{V_0}\right) - \lambda^2 \qquad \text{[A1.7]}$$

Notice that for $t = 0$, Xt is normally distributed with

$$\mathbf{E}X_t = -\frac{\sigma^2}{2}\left(t + \frac{\lambda^2}{\sigma^2}\right) \qquad \text{[A1.8]}$$

$$\mathbf{Var}X_t = \sigma^2\left(t + \frac{\lambda^2}{\sigma^2}\right) \qquad \text{[A1.9]}$$

Note that if λ does not equal 0, X_0 has positive variance. The model approximates the process X with a Brownian motion X^* with drift $-\sigma^2/2$ and variance rate σ^2. It then stipulates that X^* starts in the past at $-\Delta t = -\lambda^2/\sigma^2$ with $X^*_{-\Delta t} = 0$. It can be seen that for $t \geq 0$, the moments of X^*_t agree with the moments of X_t above. Intuitively this approximation replaces the uncertainty in the default barrier with an uncertainty in the level of the asset value at time 0; since it is the distance between the asset value and the default barrier that ultimately drives the model, this approximation has little impact.

The model then makes use of the distributions for first hitting time of Brownian motion. In particular, for the process $Y_t = a_t + bW_t$ with constant a and b, we have (see, for example, Musiela and Rutkowski (1998))

$$\mathbf{P}\{Y_s > y, \forall s < t\} = \Phi\left(\frac{at-y}{b\sqrt{t}}\right) - e^{\frac{2ay}{b^2}}\Phi\left(\frac{at+y}{b\sqrt{t}}\right) \qquad \text{[A1.10]}$$

To apply this result to X^*, we set $a = -\sigma^2/2$, $b = \sigma$ and $y = \ln(L^*D/V_0) - \lambda^2$, and substitute t with $t + \lambda^2/\sigma^2$, we obtain a closed form formula for the survival probability up to time t,

$$\mathbf{P}(t) = \Phi\left(-\frac{A_t}{2} + \frac{\ln(d)}{A_t}\right) - d \cdot \Phi\left(-\frac{A_t}{2} - \frac{\ln(d)}{A_t}\right) \qquad \text{[A1.11]}$$

where

$$d = V_0 e^{\lambda^2} \quad \text{[A1.12]}$$

$$A_t^2 = \sigma^2 t + \lambda^2 \quad \text{[A1.13]}$$

Note that the survival probability given by (A1.11) implicitly includes the possibility of default in the period (- *t*, 0], producing counterintuitive result that there is a non-zero probability of default at *t* = 0. This particular fact may be considered a technical artifact of the modelling assumptions, specifically the lognormality of the default barrier. At the same time, though, this feature aids in obtaining a simple formula for survival probability and in producing reasonable spreads for short (6-month to 2-year) maturity instruments.

An alternative to the approximation with X^* is to integrate over the barrier distribution. This approach yields an expression for the survival probability that contains the cumulative bivariate normal distribution:

$$\mathbf{P}(t) = \Phi_2\left(-\frac{\lambda}{2}+\frac{\ln(d)}{\lambda}, -\frac{A_t}{2}+\frac{\ln(d)}{\lambda}; \frac{\lambda}{A_t}\right) - d\cdot\Phi_2\left(\frac{\lambda}{2}+\frac{\ln(d)}{\lambda}, -\frac{A_t}{2}-\frac{\ln(d)}{\lambda}; \frac{\lambda}{A_t}\right) \quad \text{[A1.13]}$$

For practical purposes, the numerical differences between the survival probabilities given by the two approaches are marginal.

To convert the CreditGrades survival probability to a credit price, The model must specify two additional parameters: the riskfree interest rate r and the recovery rate R on the underlying credit. Note that R differs from L^* in that R is the expected recovery on a specific class of a firm's debt, while L^* is the expected recovery averaged over all debt classes. The asset specific recovery R for an unsecured debt is usually lower than L^* since the secured debt will have a higher recovery.

To price a Credit Default Swap (CDS), we solve for the continuously compounded spread c^* such that the expected premium payments on the CDS equate to the expected loss payouts. For a constant risk-free interest rate r and the survival probability function given by the CreditGrades model, the par spread for a CDS with maturity t may be expressed as

$$c^* = r(1-R)\frac{1-P(0)+e^{r\xi}\left(G(t+\xi)-G(\xi)\right)}{P(0)-P(t)e^{-rt}-e^{r\xi}\left(G(t+\xi)-G(\xi)\right)} \qquad \text{[A1.15]}$$

where $\xi = \lambda^2/\sigma^2$, and the function G is given by Rubinstein and Reiner (1991):

$$\mathbf{G}(u) = d^{z+\frac{1}{2}}\Phi\left(-\frac{\ln(d)}{\sigma\sqrt{u}} - z\sigma\sqrt{u}\right) - d^{-z+\frac{1}{2}}\cdot\Phi_2\left(-\frac{\ln(d)}{\sigma\sqrt{u}} - z\sigma\sqrt{u}\right) \qquad \text{[A1.16]}$$

with

$$z = \sqrt{\left(\frac{1}{4} + \frac{2r}{\sigma^2}\right)}$$

In practice, we see little difference between spreads calculated by assuming continuous fee payments and those calculated using the market standard of quarterly payments. For simplicity, we calculate the CreditGrades spread as above and adjust for the market's Act/360 pricing convention.

In order to implement the survival probability formula (A1.11), it is necessary to link the initial asset value V_0 and the asset volatility σ to market observables. This is accomplished by examining the boundary conditions. We focus on long-term tenors $(t > \lambda^2/\sigma^2)$, since the short-term default probability is mainly driven by the level of λ.

Let S denote the firm's equity price and σ_S the equity volatility. In general, the equity and asset volatilities are related through;

$$\sigma_S = \sigma\frac{V}{S}\frac{\partial S}{\partial V} \qquad \text{[A1.17]}$$

Define the distance to default measure η as the number of annualised standard deviations separating the firm's current equity value from the default threshold:

$$\eta = \frac{1}{\sigma}\ln\left(\frac{V}{LD}\right) = \frac{V}{\sigma_S S}\frac{\partial S}{\partial V}\ln\left(\frac{V}{LD}\right) \quad \text{[A1.18]}$$

Clearly, η plays an important role in determining the survival probability through (A1.11), and so the model focuses on the behavior of η in the boundary cases.

The first boundary condition is the behaviour of V near the default threshold $L \cdot D$. The model assume that as default approaches (that is, $S/(LD) << 1$), the value of the equity (which we denote by S) approaches zero. Thus,

$$V \big|_{S=0} = LD \quad \text{[A1.19]}$$

At the boundary and;

$$V \approx L \cdot D + \frac{\partial V}{\partial S} S \quad \text{[A1.20]}$$

near the default threshold. Substituting into (A1.18), we see that;

$$\eta \approx \frac{1}{\sigma_S} \quad \text{[A1.21]}$$

near the boundary.

The second boundary condition is far from the default barrier (that is, $S >> LD$). Here, the assumption is;

$$\frac{S}{V} \rightarrow 1 \quad [A1.22]$$

that is that the asset and equity values increase at the same rate. This leads to an approximation for η:

$$\eta \cong \frac{1}{\sigma_S} \ln\left(\frac{S}{LD}\right) \quad [A1.23]$$

The simplest expressions for V and η that simultaneously satisfies the near default boundary conditions ((A1.19) and (A1.21)) and the far from default conditions ((A1.22) and (A1.23)) are $V = S + LD$ and

$$\eta = \frac{S + LD}{\sigma_S S} \ln\left(\frac{S + LD}{LD}\right) \quad [A1.23]$$

Thus for the initial asset value V_0 at time $t = 0$, we have

$$V_0 = S_0 + L^* D \quad [A1.25]$$

where S_0 is the current stock price. This also gives

$$\sigma = \sigma_S \frac{S}{S + LD} \qquad \text{[A1.26]}$$

relating the asset volatility to the observable equity volatility.

Equation (A1.26) shows that for a stable asset volatility, the equity volatility increases with declining stock price, and eventually reaches very high levels for a company at the brink of default. This dependence of equity volatility on the stock price is evident in a pronounced volatility skew in equity option markets, especially for high yield names. It often makes sense to use a reference share price S^* and equity volatility s^* (either historical or implied) to determine an asset volatility and keep it stable for some period of time. In this case, the asset volatility will be given by

$$\sigma = \sigma_S^* \frac{S^*}{S^* + LD} \qquad \text{[A1.27]}$$

In deriving (A1.11), another assumption has been that the asset value has zero drift (μ_V = 0). It is important to note that for pricing credit, it is not the asset drift itself, but rather the drift of the asset relative to the default boundary that is relevant. The model assumes that on average over time a firm issues more debt to maintain a steady level of leverage, or else pays dividends so that the debt has the same drift as the stock price.

Given (A1.25), to avoid arbitrage the same drift should be assigned to the asset value V, implying that the drift of the assets relative to the default barrier is indeed zero. For given debt-per-share and estimation of recovery value, using (2.25) and (2.26), we obtain a closed form formula that involves only market observable parameters.

Survival probability (Lardy, Finkelstein, Khuong-Huu and Yang (2000))

$$\mathbf{P}(t) = \Phi\left(-\frac{A_t}{2} + \frac{\ln(d)}{A_t}\right) - d \cdot \Phi\left(-\frac{A_t}{2} - \frac{\ln(d)}{A_t}\right) \quad [A1.28]$$

is expressed as a function of market observable parameters

$$d = \frac{S_0 + L^* D}{LD} e^{\lambda^2} \quad [A1.29]$$

and

$$A_t^2 = \left(\sigma_S^* \frac{S^*}{S^* + LD}\right)^2 t + \lambda^2 \quad [A1.30]$$

- S_0: initial stock price,
- S^*: reference stock price,
- $\sigma_S{}^*$: reference stock volatility,
- D: debt-per-share,
- L^*: global debt recovery,
- λ: percentage standard deviation of the default barrier.

The debt-per-share D is based on financial data from consolidated statements. The model first calculates all liabilities that participate in the financial leverage of the firm. These include the principal value of all financial debts, short-term and long-term borrowings and convertible bonds. Additionally, quasi-financial debts are included such as capital leases, under-funded pension liabilities or preferred shares. Non-financial liabilities such as accounts payable, deferred taxes and reserves are not included. Debt-per-share is then the ratio of the value of the liabilities to the equivalent number of shares. The equivalent number of shares includes the common shares outstanding, as well as any shares necessary to account for other classes of shares and other contributors to the firm's equity capital. In practice, the financial data used in the debt-per-share calculation should be adjusted for recent events that are already priced in by the market. The details of the CreditGrades debt-per-share calculation are provided in the full paper (Appendix B).

The mean (L^*) and the percentage standard deviation (λ) of the global recovery L are estimated using the Portfolio Management Data and Standard & Poor's database (Hu and Lawrence (2000)). The database contains actual recovery data for approximately 300 non-financial U.S. firms that defaulted from 1987 to 1997. Defaulted instruments include bonds and bank loans. Based on the study of these historical data, L^* and λ are estimated to be 0.5 and 0.3, respectively. A lower λ is expected for the financial sector due to the sector specific government regulations.

APPENDIX B[13]

The term structure model developed in 1990 by Fischer Black, Emanuel Derman and William Toy is a yield-based model which has proved popular with practitioners for valuing interest rate derivatives such as caps and swaptions etc. The Black, Derman and Toy model (BDT model) is a one-factor short-rate (no-arbitrage) model – all security prices and rates depend only on a single factor, the short rate – the annualized one-period interest rate.

The current structure of long rates (yields on zero-coupon Treasury bonds) for various maturities and their estimated volatilities are used to construct a tree of possible future short rates. This tree can then be used to value interest-rate-sensitive securities. Several assumptions are made for the model to hold:

- Changes in all bond yields are perfectly correlated.
- Expected returns on all securities over one period areequal.
- The short rates are log-normally distributed
- There exists no taxes or transaction costs.

As with the original *Ho and Lee* model, the model is developed algorithmically, describing the evolution of the term structure in a discrete-time binomial lattice framework.

[13] from Implementation of the BDT model. Summer 2003, Klose & Yuan

Although the algorithmic construction is rather opaque with regard to its assumptions about the evolution of the short rate, several authors have shown that the implied continuous time limit of the BDT model. As we take the limit of the size of the time step to zero, the limit is given by the following stochastic differential equation:

$$d \ln r(t) = \left[\theta(t) - \frac{\partial \sigma(t) / \partial t}{\sigma(t)} \ln r(t) \right] + \sigma(t) dz$$

This representation of the model allows us to understand the assumption implicit in the model. The BDT model incorporates two independent functions of time, θ(t) and σ(t), chosen so that the model fits the term structure of spot interest rates and the term structure of spot rate volatilities.

In contrast to the Ho and Lee and Hull and White model, in the BDT representation the short rates are log-normally distributed; with the resulting advantage that interest rates cannot become negative. An unfortunate consequence of the model is that for certain specifications of the volatility function σ(t) the short rate can be mean-fleeing rather than mean-reverting.

It is popular among practitioners, partly for the simplicity of its calibration and partly because of its straightforward analytic results. The model furthermore has the advantage that the volatility unit is a percentage, confirming with the market conventions.

The BDT model offers in comparison to the Ho-Lee model more flexibility. In the case of constant volatility the expected yield of the Ho-Lee model moves exactly parallel, but the BDT model allows more complex changes in the yield-curve shape.

It must be stressed that using BDT, which is a one-factor model, does not mean that the yield curve is forced to move parallel. The crucial point is that only one source of uncertainty is allowed to affect the different rates. In contrast to linearly independent rates, a one factor model implies that all rates are perfectly correlated. Of course, rates with different maturity are not perfectly correlated.

Mainly three advantages using one factor models rather than two or three factor models can be mentioned:

1. It is easier to implement
2. It takes much less computer time
3. It is much easier to calibrate

The ease of calibration to caps is one of the advantages in the case of the BDT model. It is considered by many practitioners to outperform all other one-factor models.

The BDT model suffers from two important disadvantages:

• Substantial inability to handle conditions where the impact of a second factor could be of relevance because of the one-factor model

• Inability to specify the volatility of yields of different maturities independently of future volatility of the short rate

An exact match of the volatilities of yields of different maturities should not be expected and, even if actually observed, should be regarded as a little more than fortuitous.

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