Elastic properties and thermodynamic anomalies of supersolids

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We study a supersolid in the context of a Gross-Pitaevskii theory with a nonlocal effective potential. We employ a homogenization technique which allows us to calculate the elastic moduli, supersolid fraction, and other state variables of the system. Our methodology is verified against numerical simulations of elastic deformations. We can also verify that the long-wavelength Goldstone modes that emerge from this technique agree with Bogoliubov theory. We find a thermodynamic anomaly that the supersolid does not obey the thermodynamic relation $\partial P/\partial V|_N = -n\,(\partial P/\partial N|_V)$, which we claim is a feature unique to supersolids.

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I. INTRODUCTION

A supersolid is a phase of matter that displays both crystalline order and superfluidity in the form of nonclassical rotational inertia (NCRI). A key requirement is that both the continuous translational symmetry and the $U(1)$ global gauge symmetry of the system is spontaneously broken in the ground state.

There have been several attempts to understand this quantum phase of matter experimentally and to understand its properties theoretically. While it has not been observed in bulk ^{4}He [1,2], there are hints that such a phase may exist in the second monolayer of ^{4}He on graphite [3] where NCRI has been measured in a density regime near layer completion with an anomalous temperature dependence of the specific heat capacity. More recent experiments in ultracold dysprosium and rubidium atoms [4–6] have observed the spontaneous breaking of continuous translational symmetry together with long-range phase correlation.

The first considerations of a supersolid phase were made by Andreev and Lifshitz [7] and Chester [8] who considered the possibility of a supersolid phase in ^{4}He. They argued that the superfluid fraction of a supersolid would be reduced from 100% due to the coupling of the phonons of the crystalline structure to the $U(1)$ phase of the condensate wave function. This was further developed by Leggett [9].

Later attempts at theoretical work have taken a phenomenological symmetry-based approach starting with the work of Nozières [10] and Dorsey *et al.* [11–13]. Son [14] has described how Galilean invariance puts constraints on the form of the Lagrangian that describes the low-energy dynamics of supersolids.

There have also been approaches starting from microscopic Hamiltonians based on Gross-Pitaevskii theory [15] and Bogoliubov theory [16]. It is natural to ask whether these approaches give rise to the same predictions for the low-energy properties of the supersolid system, such as elasticity and superfluidity.

Moreover, there has been intensive study on the superfluid fraction and excitation spectrum of the supersolid phase. However, there has been comparatively little study on the elastic properties. In this paper we will build on the homogenization technique of Rica and coworkers [17–19] to obtain an effective low-energy theory that agrees well with Bogoliubov theory and numerical calculations. Importantly, the original formulation of the homogenization technique obtained a value for the bulk modulus that was not in agreement with the expected bulk modulus in the superfluid phase. Furthermore, a triangular supersolid would be expected to possess certain symmetries in the Cauchy elastic tensor, and these were not satisfied in the original formulation. Finally, the velocities of the long-wavelength excitations in the previous work also did not agree with the results based on Bogoliubov theory [16], in the sense that the velocities obtained on a particular supersolid phase using the Bogoliubov technique were not consistent with those obtained via the original formulation of homogenization.

With some important corrections that will be derived in this paper, we will show that the homogenization approach can be reconciled with other techniques, producing the predictions for all the elastic and superfluid properties of the supersolid phase. More specifically, we provide a method with the Gross-Pitaevskii approximation that calculates the elastic constants and superfluid density (phase stiffness) which is valid for any particular ground-state solution of a Gross-Pitaevskii Lagrangian. It should be noted that, in the limit of zero temperature, where a homogeneous superfluid has a superfluid fraction of 100%, a supersolid has a reduced superfluid fraction [9,20]. It is important to clarify that this is not an enhancement of the "normal fraction" which is found in superfluids at nonzero temperatures and carries

entropy. Rather, it is indicative of a separate phase that spontaneously breaks translational symmetry. In fact, it is a consequence of the fact that the "phonons" of the crystalline structure couples to the $U(1)$ phase of the condensate wave function. [21–25].

The outline of the paper is as follows. In Sec. II, we review the Gross-Pitaevskii theory for Bose condensates and how a finite-range interaction can give rise to a supersolid phase. In Sec. III, we lay the framework for the homogenization theory, specifying the deformation procedure and rigorously defining all necessary steps to calculate a long-wavelength theory of the low-lying excitations. Section IV uses results from Sec. III to derive elastic constants analytically and finds that they agree with elastic constants that we obtain numerically. In Sec. V, we will consider the additional $U(1)$ gauge and derive the supersolid fraction, as well as a coarse-grained Lagrangian which is the analytic expansion around the ground state of the elastic strain and $U(1)$ fields. Section VI uses the effective Lagrangian and considers the additional energetic contributions of flow across the system, and of work done by the strain on the surroundings. In doing so, we derive a long-wavelength effective Lagrangian which describes the Goldstone modes of the system. Section VII solves for the excitation velocities and discusses some of the key features of the theory. In Sec. VIII, we verify the excitation velocities through use of Bogoliubov fluctuations in a Bloch theorem context, and find excellent agreement between the two seemingly disparate theories. In Sec. IX, we will discuss a thermodynamic anomaly in the compressibility that we believe is unique to a supersolid.

We subsequently make the claim that since we have satisfied the symmetry requirements for an effective theory imposed by Son [14], have verified our elastic constants numerically, and have verified the velocities of the low-lying excitations through two independent and seemingly unconnected techniques, then we must have the correct effective Lagrangian for a supersolid. Furthermore, the technique outlined in this paper can now be used and applied directly to other more complicated and realistic systems.

II. MEAN-FIELD SUPERSOLIDS

In this paper, we study a Bose-Einstein condensate (BEC) of particles of mass m with a finite-range interaction $U(\mathbf{r})$ in a Gross-Pitaevskii theory. The condensate wave function $\psi(\mathbf{r},t)$ is a complex-valued function that can be written in number-phase representation (or Madelung form) as $\psi(\mathbf{r},t)=\sqrt{\rho(\mathbf{r},t)}e^{i\phi(\mathbf{r},t)}$. The Lagrangian of the system is

$$\mathcal{L}=-\int_\Omega\left[\hbar\rho\frac{\partial\phi}{\partial t}+\frac{\hbar^2}{2m}\left(\rho(\nabla\phi)^2+\frac{1}{4\rho}(\nabla\rho)^2\right)+\frac{1}{2}\rho(\mathbf{r})\int_\Omega U(|\mathbf{r}-\mathbf{r}'|)\rho(\mathbf{r}')\,\mathbf{dr}'\right]d\mathbf{r}, \tag{1}$$

where Ω is the spatial domain of the system. In the absence of any current or twisted boundary conditions, the phase ϕ is spatially uniform in the ground state and can subsequently be set to zero. The ground-state density can be found by variational calculus and is given by

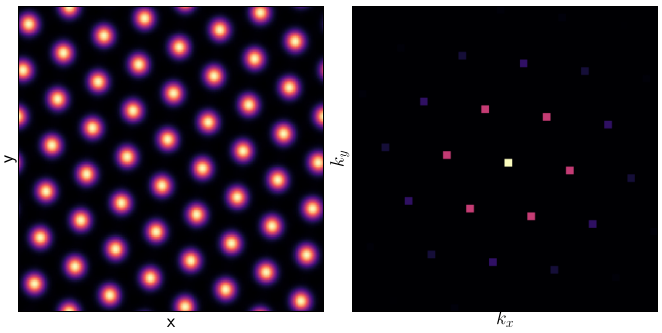

FIG. 1. A spatially modulated condensate. Left: real space density distribution. Right: Fourier transform of density distribution implying nonzero momenta in the condensate.

$$\frac{\hbar^2}{4m}\left(\frac{(\nabla\rho)^2}{2\rho^2}-\frac{\nabla^2\rho}{\rho}\right)+\int_\Omega U(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\,\mathbf{dr}'=\mu, \tag{2}$$

where μ is the chemical potential to enforce the constraint $\int_\Omega\rho(\mathbf{r})\,d\mathbf{r}=N$, the total number of particles in the system Eq. (2) is a nonlinear equation which we solve numerically via a Crank-Nicholson scheme. By evolving the Lagrangian in imaginary time numerically, we obtain the (real) ground-state wave function as the steady-state solution, using a Runge-Kutta algorithm for the temporal evolution. The numerics are performed on a 2D triangular grid in real space with periodic boundary conditions to respect the hexagonal symmetry of the spatially modulated condensate that we expect.

A necessary feature (as shown by Heinonen *et al.* [26]) for the spontaneous development of a density wave in the condensate is the presence of an interaction potential which at least has nonzero Fourier components in $\mathbf{k}^4$ and upwards [26]. This rules out the use of the typical contact potential for supersolid formation.

To illustrate this, consider a toy model of a soft-core repulsive interaction with a finite-range of a in two dimensions:

$$U(\mathbf{r})=\mathbf{U_0}\boldsymbol{\Theta}(|\mathbf{r}|-\mathbf{a}), \tag{3}$$

where Θ is the Heaviside step function. For this simple interaction, the system is controlled by a single dimensionless parameter,

$$\Lambda=\frac{\pi U_0 m a^2}{\hbar^2}na^2, \tag{4}$$

for a system of average number density n.

The phase diagrams for the soft-core system are described in detail in various works [27–29]. We find that our results are in agreement, with small Λ corresponding to a superfluid phase and that for $\Lambda>37$ the system spatially orders into the supersolid phase. This is in agreement with a theoretical subcritical limit calculated by During *et al.* [19]. This supersolid phase is characterized by a triangular lattice in 2D, illustrated in Fig. 1.

The bulk of this paper is concerned with an effective long-wavelength theory for this supersolid phase where the Bose condensate is spatially modulated. We will use these numerics

to verify our formalism by calculating the elastic constants which are dependent on the ground state.

III. HOMOGENIZATION FOR ELASTICITY

The spontaneously broken translational symmetry in a supersolid means that the system must possess Goldstone modes which are the elastic modes. In this section, we will build on the work of Refs. [17,19] to derive the elastic response of the system. The goal here is to consider an elastic deformation on the ground state, e.g., stretching/shearing as described by some strain tensor, and then to expand the Lagrangian to second order in said strain tensor and recover the elastic moduli through the generalized Hooke's law.

We follow the metholodology of Josserand *et al.* [17], but our results have some key differences. Namely, we believe some key terms were missed which skews the elastic constants to ones that do not agree with simple calculations in the superfluid limit and lead to physically inconsistent elastic moduli. More specifically, the Cauchy elastic tensor does not obey internal symmetries consistent with a triangular lattice, and possesses negative elastic moduli which suggest an unstable state. Moreover, we conduct a different treatment of the system with regard to the canonical ensemble and the degrees of expansion in the strain tensor.

The analysis will take the following structure: we will first carefully define the notion of an elastic deformation on the system and provide the basis for the calculation. We then expand the Lagrangian to second-order in the elastic strain tensor. We demand that the deformed system also be a ground state, and therefore solve the Euler-Lagrange equations for the new system, which will allow us to determine elastic constants.

We begin by describing a physical deformation by considering the displacement, $u(r)$, of points in the material. The displacement can be written as

$$\mathbf{r}' = \mathbf{r} - \mathbf{u}(\mathbf{r}), \tag{5}$$

moving points $\mathbf{r}$ (over some domain Ω) in the undeformed ground state to points $\mathbf{r}'$ (with some domain Ω'). This basis describes the deformed material in the laboratory frame of an external observer. To clarify, this is considered an *active* transformation which physically acts on the material and changes the domain of the system in the laboratory frame. The basis $\mathbf{r}'$ is a Cartesian coordinate system in the laboratory frame of the external observer, but is a non-Cartesian coordinate system in the frame of the material.

We then define an inverse *passive* transformation which maps the coordinate system $\mathbf{r}'$ to the coordinate system $\mathbf{r}''$, which is a non-Cartesian basis in the material frame that has the same domain as the inital $\mathbf{r}$ (domain Ω). This passive transformation is defined by the displacement $\mathbf{u}'(\mathbf{r}')$, where we note that this does not change the energy of the system in any way. It is simply a redefinition of coordinates in order for us to express the actual change in the energy that occured during the **active** transformation defined in Eq. (5).

This passive transformation is given by the form

$$\mathbf{r}'' = \mathbf{r}' + \mathbf{u}'(\mathbf{r}'), \tag{6}$$

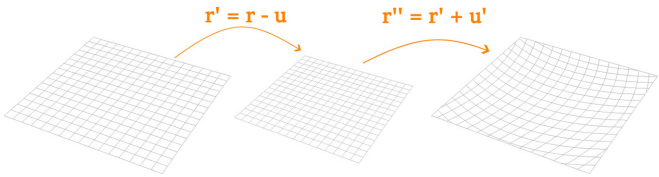

FIG. 2. A schematic of the deformation procedure. The leftmost image depicts an undeformed system with a basis $\mathbf{r}$, which then undergoes an *active* transformation given by some strain to become the middle system. The rightmost image depicts the system after the *passive* transformation, such that the final coordinate system $\mathbf{r}''$ coincides exactly with $\mathbf{r}$, but now there are additional curvatures in the space.

where the basis $\mathbf{r}''$ coincides identically with the basis $\mathbf{r}$ in the material frame, i.e., $\mathbf{r}'' = \mathbf{r}$, but now the space has additional components in its gradients, as well as a Jacobian associated with its volume element. Note that the passive transformation $\mathbf{u}'(\mathbf{r}')$ is in the deformed laboratory frame given by the coordinates $\mathbf{r}'$ and importantly is a different transformation to $\mathbf{u}(\mathbf{r})$.

These extra curvatures are necessary to describe the full effect of the displacement procedure on the system. We now relabel the basis $\mathbf{r}''$ as $\mathbf{r}$, for convenience, where again we must stress that even though the coordinate systems $\mathbf{r}$ and $\mathbf{r}''$ coincide, the space is no longer Cartesian and has an associated Jacobian and curvature. This procedure is illustrated schematically in Fig. 2.

As of now, the above transformation is completely general, but to proceed we need to specify a particular class of transformation which allows us to use linear response theory and elasticity theory. We make the distinction that $\mathbf{u}(\mathbf{r})$ is such that all gradients in space are small constants and that gradients in time are considered small velocities.

More specifically, we choose the tensor $\partial_i u_k$, which is a dimensionless number, s.t. $\partial_i u_k \sim \delta$, where $\delta \ll 1$. The time derivative of $\mathbf{u}$, i.e., $\partial_t u_k$, is treated as small in the same sense as the phase perturbation, which we will later cover in Sec. V. This is referred to as a uniform deformation, in that the strain tensor is constant and small everywhere, which allows us to treat it as a perturbative parameter which we will expand up to harmonic terms.

We now refer to $u_{ik} \equiv \partial_i u_k$ as the *strain tensor*, and endeavour to expand $\mathcal{L}$ in powers of u_{ik}. To employ linear elastic theory we have to keep all terms up to $\mathcal{O}(\delta^2)$ in the Lagrangian. We also need to relate the gradient of the transformation $\mathbf{u}'$ in the primed deformed basis in the laboratory frame to the gradient of the transformation $\mathbf{u}$ in the material basis. It is relatively easy to show that

$$\partial'_i u'_k = \partial_i u_k + \partial_i u_l \partial_l u_k \equiv u_{ik} + u_{il} u_{lk} \tag{7}$$

up to $\mathcal{O}(\delta^2)$ in the strain tensor (note that we are using the Einstein summation convention). This allows us now to express derivatives in the primed (deformed laboratory frame) basis in terms of derivatives in the unprimed (deformed material frame) basis to find

$$\partial'_i = \partial_i + \partial_k (u_{ik} + u_{il} u_{lk}). \tag{8}$$

To simplify the expressions that we will derive in this paper, we introduce notation for a set of key tensors that appear

regularly:

$$\epsilon_{ik} = -\tfrac{1}{2}(u_{ik} + u_{ki}), \quad \Delta_{ik} = \tfrac{1}{2}u_{il}u_{kl},$$
$$\omega_{ik} = \tfrac{1}{2}u_{li}u_{lk}, \quad \chi_{ik} = \tfrac{1}{2}(u_{li}u_{kl} + u_{lk}u_{il}). \tag{9}$$

Since the deformation actively changes the domain of the system, there is an associated change of volume. We refer to the deformations described in Eqs. (5) and (6) to define the differential volume elements as

$$\int_{\Omega'} d\mathbf{r}' = \int_{\mathbf{\Omega}} \mathcal{J}_{\mathbf{r}'\to\mathbf{r}}\, \mathbf{dr}, \tag{10}$$

with the Jacobian

$$\mathcal{J}_{\mathbf{r}'\to\mathbf{r}} = \det\left(\frac{\partial r_i'}{\partial r_k}\right) = 1 + \epsilon_{ll} + M_{iklm}\, u_{ik}u_{lm}, \tag{11}$$

where $M_{iklm} \equiv (\delta_{ik}\delta_{lm} - \delta_{im}\delta_{kl})/2$. A negative ϵ_{ll} corresponds to a reduction in volume. In two dimensions, $M_{iklm}u_{ik}u_{lm} = u_{xx}u_{yy} - u_{xy}u_{yx}$. The quantities ϵ_{ll} and $M_{iklm}u_{ik}u_{lm}$ are sometimes called the first and second strain invariants as they are independent of the basis in which the strain tensor is written. We stress now that the coordinate system $\mathbf{r}$ describes the deformed system in the frame of the material: it has the same domain as the undeformed material but is now a non-Cartesian space.

To understand how this deformation affects the Lagrangian, we need to understand how the density changes as a function of the deformation. After the deformation occurs, the particles in the system will reorganise themselves in such a way as to minimise the total energy of the new configuration, leading to a new density described by $\rho(\mathbf{r}')$. The reorganization of particles due to a deformation can be expressed as the predeformation density plus a component, $\tilde{\rho}$, which completely accounts for all changes. The component $\tilde{\rho}$ covers both local density changes at length scales within a unit cell of the supersolid and also includes the change in average density due to total number conservation and a change in volume.

We consider that this new density is written in the $\mathbf{r}'$ basis. We stress that this is not the density in the material frame, but rather the density in the laboratory frame of the external observer. We can then use the passive transformation to write the density in the $\mathbf{r}''$ basis, which is the material frame that now has some curvature. Without loss of generality we can express this new density at some point $\mathbf{r}'$ as

$$\rho(\mathbf{r}') = \rho_0(\mathbf{r}'') + \tilde{\rho}(\mathbf{r}''), \int_{\Omega'}\rho(\mathbf{r}')\, d\mathbf{r}' = \int_{\Omega}\rho_0(\mathbf{r}'')\, d\mathbf{r}'', \tag{12}$$

where ρ_0 is the predeformation ground-state density profile. As of now this is simply a mapping to a scalar function that has been subject to an active transformation.

We stress that from this point we drop the $\mathbf{r}''$ notation and refer to the coordinate as $\mathbf{r}$ where now the coordinate system is non-Cartesian and has a Jacobian associated with it.

The general form of $\tilde{\rho}$ is determined by minimizing the energy of the deformed system, subject to particle conservation. For linear elastic theory, we only need to consider only a perturbative strain on the system and expect that, since the strain is perturbative, the density change $\tilde{\rho}$ must be analytic in the strain tensor. We truncate the Taylor expansion to $\mathcal{O}(\delta^2)$ in the strain tensor and use the ansatz

$$\tilde{\rho}(\mathbf{r}) = \rho_1^{\mathbf{ik}}(\mathbf{r})\,\epsilon_{\mathbf{ik}} + \rho_2^{\mathbf{iklm}}(\mathbf{r})\,\mathbf{u_{ik}u_{lm}}, \tag{13}$$

which we will justify later. This allows us to expand the new normalization condition for particle conservation,

$$\int_\Omega (\rho_0 + \tilde{\rho})(1 + \epsilon_{ll} + M_{iklm}\, u_{ik}u_{lm})\, d\mathbf{r} = \int_{\mathbf{\Omega}} \rho_{\mathbf{0}}\, \mathbf{dr}, \tag{14}$$

and after collecting powers of the strain tensor obtain normalization requirements for the different terms in the expansion

$$\int_\Omega \left(\rho_1^{ik} + \rho_0\delta^{ik}\right) d\mathbf{r} = 0,$$
$$\int_\Omega \left[\rho_2^{iklm} + \rho_0(M_{iklm} - \delta_{ik}\delta_{lm})\right] d\mathbf{r} = 0. \tag{15}$$

Due to the changing energy of the system (and indeed the change in average density), we can expect to see a change in the chemical potential μ too. In a similar spirit as that of Eq. (13), we can analytically expand the new chemical potential in the strain tensor as

$$\mu = \mu_0 + \mu_1^{ik}\,\epsilon_{ik} + \mu_2^{iklm}\,u_{ik}u_{lm}, \tag{16}$$

where the first term is the chemical potential of the undeformed system, and the constants μ_1^{ik} and μ_2^{iklm} encodes how the chemical potential changes due to the deformation and are to be determined by the least-action principle.

We additionally need to consider how the displacement field u_{ik} changes the interaction potential U. We note that the interaction is a function of separation between particles, but importantly is originally written in terms of the separation $\Delta r'$ as measured in the *real laboratory frame coordinates*: $U = U(\Delta r')$. For a uniform strain, we find immediately that the laboratory-frame separation $\Delta\mathbf{r}'$ can be written in terms of the material-frame separation $\Delta\mathbf{r}$ such that

$$(\Delta r')^2 = (\delta_{ik} + 2\epsilon_{ik} + 2\Delta_{ik})(\Delta r)_i(\Delta r)_k, \tag{17}$$

which is exactly equivalent to that of Landau and Lifshitz [30]. The tensor in the above equation is sometimes referred to as the finite strain tensor. A Taylor expansion of the interaction for small strain gives

$$U(\Delta r') = U(\Delta r) + (\epsilon_{ik} + \Delta_{ik})f_{ik}(\Delta r) + \epsilon_{ik}\epsilon_{lm}W_{iklm}(\Delta r), \tag{18}$$

with

$$f_{ik}(\mathbf{r}) \equiv \frac{r_i r_k}{|\mathbf{r}|}\frac{\partial U(|\mathbf{r}|)}{\partial|\mathbf{r}|},$$
$$W_{iklm}(\mathbf{r}) \equiv \frac{r_i r_k r_l r_m}{2|\mathbf{r}|^{\mathbf{2}}}\left(\frac{\partial^2 U}{\partial|\mathbf{r^2}|} - \frac{1}{|\mathbf{r}|}\frac{\partial U}{\partial|\mathbf{r}|}\right). \tag{19}$$

To summarize, we have now defined a perturbative deformation which changes the geometry of our system and have obtained the volume changes and curvature changes due to said deformation. We have also derived a new density and chemical potential, both of which we will solve for using the least-action principle. We now have all the ingredients to formulate a theory of linear response of the system to elastic deformations.

IV. ELASTICITY THEORY

We can now develop a linear elastic theory by expanding the Lagrangian to second order in the strain tensor and finding the response, $\tilde{\rho}$, of the density to a small applied strain using the principle of least action. The full details are given in Appendix A where we calculated and collected all the first- and second-order components of u_{ik} in the Lagrangian. This should be of the form:

$$L(\Omega') \simeq L_0(\Omega) + |\Omega|\pi_{ik}u_{ik} - \frac{|\Omega|}{2}A_{iklm}u_{ik}u_{lm}, \tag{20}$$

where the first term is the ground-state Lagrangian, the second term contains the coupling to the stress tensor π_{ik}, and the third term is related to the Cauchy elastic tensor B_{iklm}, as carefully discussed in Bavaud *et al.* [31]. ($|\Omega|$ is the total volume of the predeformed system.)

Let us consider the term linear in the strain tensor. The pressure tensor is related to the stress tensor π_{ik} by $P_{ik} = -\pi_{ik}$. We will be studying a two-dimensional supersolid whose spatial modulation forms a triangular lattice. The C_3 symmetry of the ground state dictates that $P_{ik} = P\delta_{ik}$, i.e., the pressure is isotropic. Collecting the $\mathcal{O}(u_{ik})$ terms in the expansion of the Lagrangian [see Eqs. (A12) and (A12)], we can identify the pressure as

$$P_{ik} = \int_\Omega \left[\frac{\hbar^2}{4m}\left(\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0} - \nabla^2\rho_0\,\delta_{ik}\right) - \frac{1}{2}\rho_0(f_{ik} * \rho_0)\right]\frac{d\mathbf{r}}{|\Omega|}, \tag{21}$$

which will be used in subsequent calculations. The above expression can be directly calculated for any ground-state density ρ_0. In the case of a spatially homogeneous superfluid, it leads to what we expect: $P_{ik} = \frac{1}{2}\mu_0 n\delta_{ik}$. (See Appendix A.)

Let us now turn to the the second-order terms in the expansion (20). It is important to note that the tensor A_{iklm} is *not* the Cauchy elastic tensor B_{iklm} which provides the tensor in the Hooke's law for the stress/strain relationship between a deformed state stress and the strain that induced the deformation, i.e., $\pi_{ik}(\Omega') = \pi_{ik}(\Omega) - B_{iklm}(\Omega)u_{lm}$. It can be shown [31] that

$$B_{iklm} = A_{iklm} + P_{ik}\delta_{lm} - P_{im}\delta_{kl}, \tag{22}$$

which we will derive independently later in Eq. (44) by considering work done by the expansion on the surroundings.

For a supersolid with C_3 symmetry, we also expect the elastic tensor to obey the indicial symmetries $B_{iklm} = B_{klim} = B_{lkmi} = B_{mlki}$ so that it only contains two independent quantities—the bulk modulus $K = c_{44}$ and the shear modulus $G = c_{66}$:

$$B_{iklm} = K\delta_{ik}\delta_{lm} + G(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl}). \tag{23}$$

It should be noted that A_{iklm} is not indicially symmetric in the same way as B_{iklm}. Elastic theory [30] in 2D posits that the bulk modulus K is given by B_{xxyy}, and the shear modulus G is given by B_{xyxy} (or all equivalent indicially symmetric elements). For a triangular lattice or a homogeneous system, the tensor also possesses the symmetry: $B_{xxxx} = B_{xxyy} + 2B_{xyxy}$. Note also that we expect the changes to the chemical potential to have the same symmetry as the pressure, i.e., $\mu_1^{ik} = \mu_1\delta_{ik}$. We will later make use of these properties to simplify the expressions for the elastic constants.

We can deduce the elastic tensor by identifying the $\mathcal{O}(u_{ik}^2)$ terms in the expansion of the Lagrangian with A_{iklm}, and then using (22) to obtain B_{iklm}. [See Appendix (A24) and the derivative rules in Appendix C.] We find that $B_{iklm} = |\Omega|^{-1}\int_\Omega b_{iklm}(\mathbf{r})\,\mathbf{dr}$, where

$$\begin{aligned} b_{iklm}(\mathbf{r}) = &+ \frac{\hbar^2}{4m}\left(\frac{\partial_i\rho_0\partial_l\rho_0}{\rho_0}\delta_{km} + \frac{\partial_k\rho_0\partial_l\rho_0}{\rho_0}\delta_{im} - \frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0}\delta_{lm}\right) + \frac{1}{2}(f_{il} * \rho_0)\rho_0\delta_{km} - \frac{1}{2}(f_{ik} * \rho_0)\rho_0\delta_{lm} + \frac{1}{2}(f_{im} * \rho_0)\rho_0\delta_{kl} \\ &+ \{[(U\delta_{ik} + 2f_{ik})\delta_{lm} + W_{iklm}] * \rho_0\}\rho_0 + \rho_1^{lm}\left[\frac{\hbar^2}{4m}\left(2\frac{\partial_{ik}\rho_0}{\rho_0} - \frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\right) + (f_{ik} + \delta_{ik}U) * \rho_0 + \mu_1^{ik}\right], \end{aligned} \tag{24}$$

with $(g * \rho)_\mathbf{r} \equiv \int d\mathbf{r}'\mathbf{g}(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')$ for any function g and ρ and where f_{ik} and W_{iklm} are defined in Eq. (19).

We note that this result for the elastic constants depends on the ground-state density ρ_0 as well as the shift in the density ρ_1 and shift in the chemical potential μ_1 as defined in Eqs. (13) and (16). In other words, we need to deduce how the density profile has changed in response to the applied strain. This is achieved by solving the Euler-Lagrange equations for the variable $\tilde{\rho}$ as defined in Eq. (12). This involves solving the following equation:

$$\begin{aligned} &\frac{\hbar^2}{4m}\nabla\cdot\left(\frac{\nabla\tilde{\rho}}{\rho_0}\right) - U * \tilde{\rho} + \frac{\hbar^2}{4m}\left(\frac{(\nabla\rho_0)^2}{\rho_0^3} - \frac{\nabla^2\rho_0}{\rho_0^2}\right)\tilde{\rho} \\ &\quad = -\left(\mu_2^{iklm} + \mu_1^{ik}\delta_{lm} + \mu_0 M_{iklm}\right)u_{ik}u_{lm} + \epsilon_{ik}\left[-\mu_1^{ik} + \frac{\hbar^2}{4m}\left(2\frac{\partial_{ik}\rho_0}{\rho_0} - \frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\right)\right. \\ &\qquad \left. + \int_\Omega \left(f_{ik}(\mathbf{r} - \mathbf{r}') + \delta_{\mathbf{ik}}\mathbf{U}(\mathbf{r} - \mathbf{r}')\right)\rho_0(\mathbf{r}')\,\mathbf{dr}'\right], \end{aligned} \tag{25}$$

which is an integrodifferential equation that can be solved either by using Bloch's theorem (only applicable in periodic boundary conditions) or solving directly via writing the linear operator as a matrix and using standard iterative matrix solvers. More details on solving the equation and finding the chemical potentials are provided in Appendix (A15).

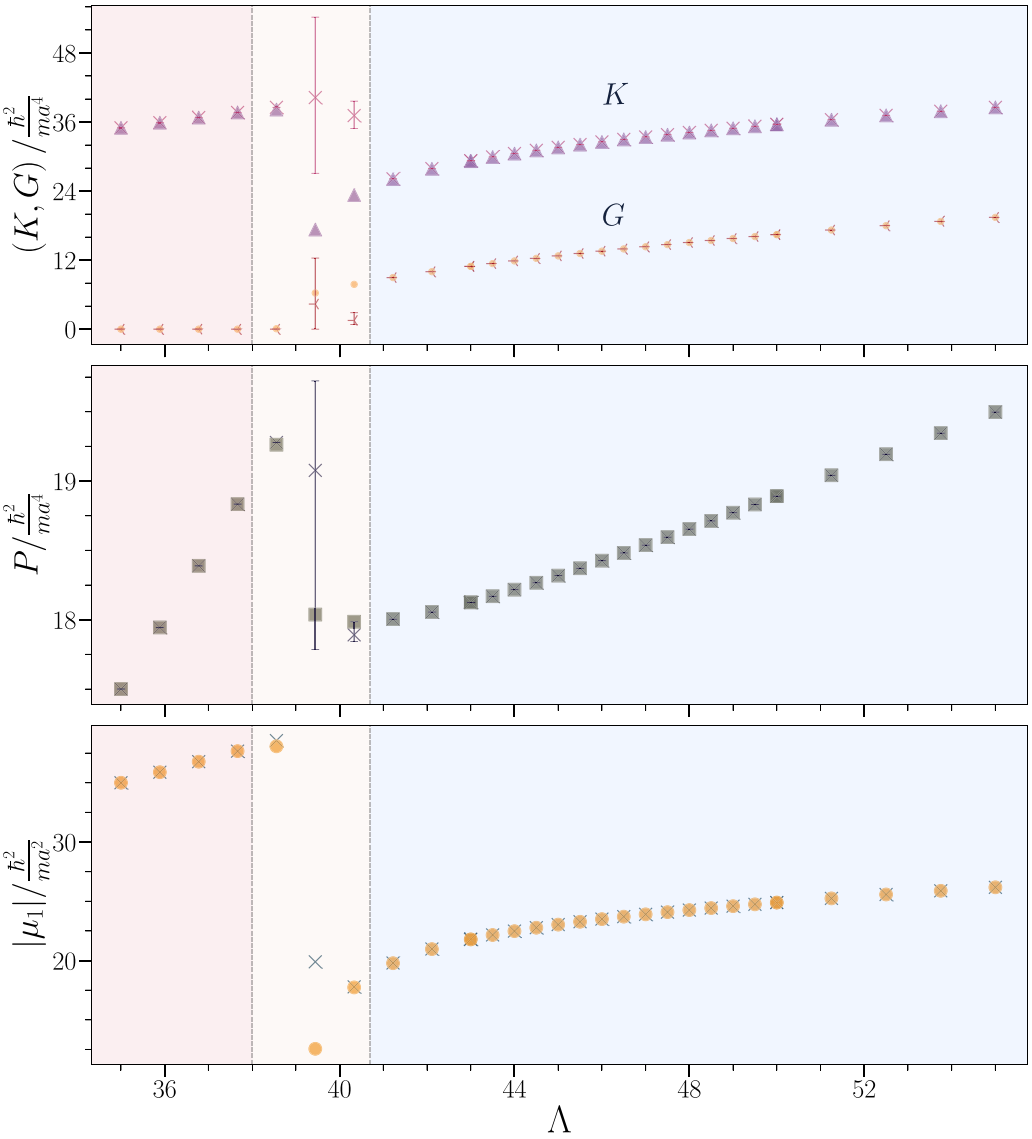

FIG. 3. A comparison of the various elastic constants obtained via homogenization to those from numerics (homogenization constants are shown in solid markers, numerics in crosses). The top figure shows the bulk modulus and shear modulus, the middle figure shows the pressure, and the bottom figure shows the absolute value of μ_1. The agreement between homogenization and numerics is very strong on the whole, but falls apart in the phase transition region. This is likely due to numerical instabilities close to the phase transition which affect the numerically obtained constants but not the homogenization constants.

It must be noted that we actually have 8 equations to solve, as for each pair of indices (i, k) there is a different right-hand side (RHS) source function, and for each of these we need to solve the appropriate equation for both ρ_1^{ik} and μ_1^{ik}. This generates 8 equations for a 2D system and 18 equations for a 3D system. Once we have the solutions for both ρ_1^{ik} and μ_1^{ik}, we can calculate the other functions in the expression for Eq. (24) and obtain the elastic tensor B_{iklm}.

Let us check these results for the case of a homogeneous system, where the modulation vanishes and we can check our results against the known results for elastic moduli of a superfluid. We can see that the elastic tensor is greatly simplified by the disappearance of the derivative terms $\partial_i \rho_0$, and the chemical potential is analytically known: $\mu = U * \rho_0$. We can find the convolutions $f_{ik} * \rho_0$ and $W_{iklm} * \rho_0$, solve Eq. (A15), and use this to find that

$$B_{iklm} = \mu n \delta_{ik} \delta_{lm}. \tag{26}$$

Using the definitions of K and G from the Cauchy elastic tensors (23), we see here that we obtain the expected results of $K = \mu n$ and no shear modulus: $G = 0$.

We will now present the numerical calculation of these homogenization results for the elastic constants. The results are shown in Fig. 3 (solid markers). In our numerics, we see some small asymmetry, e.g., $B_{iklm} \approxeq B_{lmik}$, in the results for the elastic tensor. The discrepancy with the ideal form of Eq. (23) decreases as we reduce the discretization and increase the system size in the numerics. With that in mind, we choose to define the bulk modulus K as the average of the values of B_{xxyy} and B_{yyxx}, the pressure P as the average of P_{xx} and P_{yy},

and the shear modulus G as the average of B_{xyxy}, B_{xyyx}, B_{yxyx}, and B_{yxxy}. We then use the minimal and maximal values that the elastic moduli *could have taken* to generate "error bars" for the elastic constants. It should be stressed that these error bars are not measures of some statistical error, but rather a quantification of how much the elastic constants deviate due to the crystal symmetry being only approximately triangular (as opposed to exactly triangular) as a result of discretization error in our numerics.

We now turn to comparing these numerical results from homogenization with a direct calculation of the elastic constants by finding how the energy of the ground state of the Gross-Pitaevskii equation changes when we deform the simulation cell at fixed particle number (with periodic boundary conditions). We begin with an undeformed ground state obtained via numerically solving the Gross-Pitaevskii with some given average density, area and interparticle interaction. We then apply a small strain to the system by changing the geometry of the simulation cell by a strain tensor u_{ik}, while keeping the total number of particles fixed, and solving the Gross-Pitaevskii equation for the new geometry to find the deformed ground state. For sufficiently small strain u_{ik}, we expect that such a procedure will yield a ground state with a Lagrangian of the form of Eq. (20), where the tensor u_{ik} is now a control parameter in our computational experiment. This strain is small in the same sense as defined in the expansion of the Lagrangian. More precisely, we consider $u_{ik} \ll 1$, and then try to find the harmonic terms in the energy perturbation.

By numerically extracting the Lagrangian for a system which has been deformed by some strain u_{ik}, we are able to find the pressure and elastic constants by judicious selection of strains and subsequent fitting of the coefficients. The full procedure is described in Appendix D, where we show the extraction of elastic constants specifically for a 2D case in a triangular geometry.

Our results are presented in Fig. 3, where we have plotted the bulk and shear modulus K and G, the pressure P and the shift in the chemical potential for unit strain: $\mu_1^{ik} = \mu_1 \delta_{ik}$. These elastic constants should ideally be the same as those obtained via homogenization theory. In reality, they suffer from numerical errors due to being unable to perform an arbitrarily small strain, as well as convergence errors in the new ground state when close to the first-order phase transition between the homogeneous superfluid and the supersolid. Scanning across the whole range of the interaction parameter Λ, we find that, outside of a small region where the phase transition occurs, the elastic constants obtained numerically are in excellent agreement with those obtained analytically, with a typical discrepancy of $\sim$0.1%.

We found that an applied numerical strain which was smaller than $\sim$5% (by this we mean $u_{ik} \sim 0.05$) would generate this excellent agreement, but if the applied strain was $\sim$10%, the numerical value would diverge from that of homogenization. We suspect that this is due to the introduction of anharmonic elastic moduli which are now no longer negligible.

In summary, we have developed an elastic theory for a supersolid using homogeneration theory and calculated its elastic properties using the solution of the Gross-Pitaevskii equation. Our homogenization results agree well with results from the numerical simulation of a strained supersolid.

V. $U(1)$ PHASE AND FRACTIONAL INERTIA

In this section, we consider the application of a phase perturbation to the mean-field supersolid, and analyze the coupling of the phase to the strain field. We follow the metholodology of Josserand *et al.* [17], but our results have some key differences that we will point out.

We begin by specifying the nature of the phase perturbation. We consider imposing a small superflow to the system in the form of a phase gradient. This is applied to the supersolid *in the deformed state*, i.e., the laboratory frame of the deformed material. The system will respond to this long-wavelength perturbation by generating phase fluctuations, $\tilde{\phi}$, at small wavelengths within a unit cell of the supersolid. As with $\tilde{\rho}$ in the previous section, $\tilde{\phi}$ is a correcting function to the respective perturbation applied to the ground state. Namely, $\tilde{\rho}$ is the function which allows the density to settle into the energetically optimal configuration for a given strain, and $\tilde{\phi}$ is the function which will allow the phase field to settle into the energetically optimal configuration for a given applied phase gradient. We mathematically represent this as

$$\phi(\mathbf{r}') = \phi_0(\mathbf{r}') + \tilde{\phi}(\mathbf{r}), \tag{27}$$

where $\phi(\mathbf{r}')$ is the total phase in the laboratory frame $\mathbf{r}'$, $\phi_0(\mathbf{r}')$ is the applied perturbative phase in the laboratory frame, and $\tilde{\phi}(\mathbf{r})$ is the phase-correction provided by the material in the material frame of reference. We express $\tilde{\phi}$ in terms of the material frame deformed coordinate system $\mathbf{r}$ in keeping with linear response theory, similar to how we expressed $\tilde{\rho}$ in terms of the material frame coordinate system $\mathbf{r}$.

To apply perturbation theory to the phase field, we specify that $\partial_i \phi_0(\mathbf{r}')$ is small and constant within the scope of the unit cell of the supersolid, in a similar way to the strain tensor u_{ik}. This is equivalent to stating that the phase change across the unit cell is much smaller than 2π, i.e., $\Delta\phi \ll 2\pi$. Note also that we are not considering the generation of vortices.

We note that since the phase is applied in the deformed frame, the total phase in the material frame can be expanded to obtain

$$\phi(\mathbf{r}') = \phi_0(\mathbf{r}) - \mathbf{u} \cdot \nabla\phi_0(\mathbf{r}) + \tilde{\phi}(\mathbf{r}), \tag{28}$$

where $\mathbf{u}$ is the displacement as defined in Eq. (5). This is the final expression for the phase field in the material frame of reference. We can now proceed to expand the Lagrangian in terms of the phase field, using the least action principle to find $\tilde{\phi}$ and the associated energy.

Using the transformation rules for the Lagrangian (Jacobian, derivatives, etc.) and the ϕ expansion given by Eq. (28), the phase-dependent part of the Lagrangian (1) can be expanded to $\mathcal{O}(\tilde{\phi}^2)$ as

$$\mathcal{L}_\phi = -\int_\Omega \Big\{ \hbar\rho_0 \partial_t \phi_0 (1 + \nabla \cdot \mathbf{u}) + \frac{\hbar^2 \rho_0}{2m} [(\nabla\phi_0)^2 + 2\mathbf{A} \cdot \nabla\tilde{\phi} + (\nabla\tilde{\phi})^2] \Big\} \mathbf{dr}, \tag{29}$$

where

$$\mathbf{A} = \nabla\phi_{\mathbf{0}} + (\nabla\phi_{\mathbf{0}} \cdot \nabla)\mathbf{u} + \frac{\mathbf{m}}{\hbar}\partial_{\mathbf{t}}\mathbf{u}. \tag{30}$$

The vector field $\mathbf{A}$ is a kind of convective flow. First of all, recall that the superfluid velocity is given by $\mathbf{v} = (\hbar/\mathbf{m})\nabla\phi_{\mathbf{0}}$. Also, we can define the material derivative of a field as

$$\frac{D}{Dt} = \partial_t + \mathbf{v} \cdot \nabla \tag{31}$$

to describe the time derivative of a function which is comoving with the superfluid velocity field.

Note here the appearance of the term $m/\hbar\partial_t\mathbf{u}$ which we previously stated as 'small' without further elaboration. Here we define this term to be small on the same order as $\nabla\phi_{\mathbf{0}}$, i.e., that the velocity induced by the elastic deformation is comparable to the velocity induced by the phase change.

We can rewrite $\mathbf{A}$ in term of these variables:

$$\mathbf{A} = \frac{\mathbf{m}}{\hbar}(\mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{u} + \partial_{\mathbf{t}}\mathbf{u}) = \frac{\mathbf{m}}{\hbar}\left(\mathbf{v} + \frac{\mathbf{Du}}{\mathbf{Dt}}\right). \tag{32}$$

A point of subtlety is that the actual deformation we have applied is $\mathbf{r}' = \mathbf{r} - \mathbf{u}$, and so $(\hbar/m)\mathbf{A} = \mathbf{v} - \mathbf{D}(-\mathbf{u})/\mathbf{Dt}$ is a relative velocity between the superflow and the motion of the lattice deformations. This is exactly the form predicted by Son [14] from symmetry based arguments. In fact, all terms in $\mathcal{L}_\phi$ satisfy the Galilean symmetry requirements imposed by Son. In this way we can show that we have microscopically satisfied the necessary Galilean symmetry requirements imposed on any supersolid system macroscopically.

Our phase Lagrangian $\mathcal{L}_\phi$ in Eq. (29) differs from the analogous form in Josserand *et al.* [17] by the existence of the term $\hbar\rho_0\partial_t\phi_0\nabla \cdot \mathbf{u}$. This term is in fact necessary to ensure that $\mathcal{L}_\phi$ satisfies Galilean symmetry constraints as pointed out by Son [14]. We also note that this affects the velocity of the excitations of the system, specifically in reference to (47) where this term is directly responsible for the n component of $\hbar(n - \varrho + \mu_1/\mathcal{E}'')\dot{\phi}\nabla \cdot \mathbf{u}$. If one were to calculate the velocities without this critical term, then one would see that it differs significantly from those predicted by Bogoliubov theory (shown in Sec. VIII).

Let us now solve for the short-distance phase response $\tilde{\phi}$ within a unit cell due to superflow and strain. We follow a similar procedure to Ref. [17]. By recognizing that the externally imposed phase twist and strains $\nabla\phi_0$ and $\partial_i u_k$ are defined as long-wavelength relative to the unit cell, we can treat $\mathbf{A}$ as constant within a unit cell. However, a uniform velocity with nonuniform density does not necessarily obey local mass conservation in equilibrium. So we need a correction to the phase field, $\tilde{\phi}$, to ensure that our system obeys the continuity equation in the ground state: $\nabla \cdot (\rho\nabla\phi) = 0$. The Euler-Lagrange equation for $\tilde{\phi}$ gives

$$\tilde{\phi} = K_i A_i \quad \text{with} \quad \partial_i\rho + \nabla \cdot (\rho\nabla\mathbf{K_i}) = \mathbf{0}. \tag{33}$$

This can be solved numerically in much the same way as Eq. (A16). Physically, this result expresses that the fact the phase twist (and therefore superflow) is not uniform within a unit cell because it costs less kinetic energy to introduce phase shifts in regions of low density compared to regions of high density. Using this solution for $\tilde{\phi}$, it can be shown that the phase-dependent Lagrangian can be written as

$$L_\phi = |\Omega|\frac{\hbar^2}{2m}\varrho_{ij}A_iA_j - \int_\Omega \left[\hbar\rho_0\partial_t\phi_0(1 + \nabla \cdot \mathbf{u}) + \frac{\hbar^2\rho_0}{2m}(\nabla\phi_0)^2\right] d\mathbf{r}, \tag{34}$$

with

$$\varrho_{ij} = -\frac{1}{|\Omega|}\int_\Omega \rho_0\partial_i K_j \, d\mathbf{r}, \tag{35}$$

which can be interpreted as a correction to the superfluid phase stiffness [see Eq. (37)]. Figure 4 shows the evolution of this quantity as we increase the interaction strength Λ showing a discontinuous transition at the superfluid-supersolid transition.

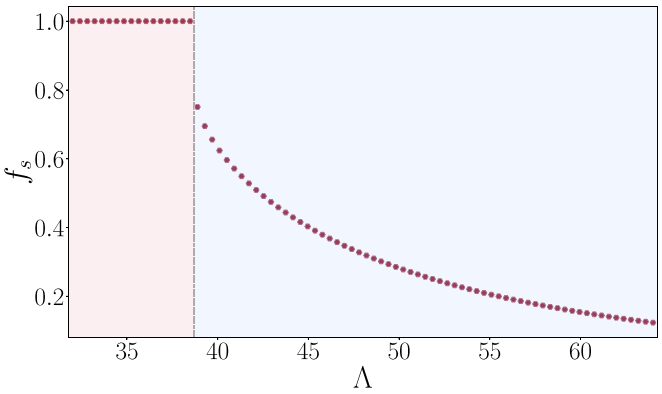

FIG. 4. The superfluid fraction $f_s \equiv 1 - \varrho/n$ as a function of Λ. We appear to observe a discontinuous transition at $\Lambda \sim 38$, which remains even after checking the transition region with a much finer Λ discretization. This appears to be numerical verification of Ref. [19], which predicted a first-order transition close to this value.

To restate the above, by carefully considering the dynamics of both an applied/induced phase and a strain deformation (both of which are long-wavelength relative to the unit cell), we calculate how the phase-dependent dynamics of our system are dependent on the parameters n and ϱ_{ij} given by the form of Eq. (34). n is simply the density of particles in the system, and corresponds to the phase-coherent BEC with components at both $\mathbf{k} = \mathbf{0}$ and $\mathbf{k} \neq \mathbf{0}$. We will later show that ϱ_{ij} is the part of the system that corresponds to the pattern formation, i.e., it only exists when $\psi_{\mathbf{k}\neq\mathbf{0}} \neq 0$. Consequently, we refer to ϱ_{ij} as being the "supersolid fraction" of the system. In this way we have separated out the pattern response of the system to the coupled phase-strain dynamical field from the standard superfluid response.

Finally, due to the C_3 symmetry, second-rank tensors representing bulk properties should be isotropic. We therefore make the assumption that $\varrho_{ij} = \varrho\delta_{ij}$, though in principle this is not necessary. We also note that the dependency in Eq. (34) on ϕ_0 and $\mathbf{u}$ can be taken out of the integral (due to the same argument that $\mathbf{A}$ is a constant in the unit cell). Combining

these two simplifications, we find that we can write

$$\frac{L_\phi}{|\Omega|} = -\hbar n\dot{\phi}_0(1+\nabla\cdot\mathbf{u}) - \frac{\hbar^2}{2m}\left[n(\nabla\phi_0)^2 - \varrho\left(\nabla\phi_0 + \frac{m}{\hbar}\frac{D\mathbf{u}}{Dt}\right)^2\right], \tag{36}$$

where $n = \int_\Omega \rho_0\, d\mathbf{r}/|\mathbf{\Omega}|$ is the average particle density. We use this result, along with the elastic expansion (24), to write out the full Lagrangian after the homogenization process as

$$\frac{L_{\rm hom}}{|\Omega|} = L_0 - \hbar n\dot{\phi}_0(1+\nabla\cdot\mathbf{u}) - \mathbf{P_{ik}u_{ik}} - \frac{1}{2}\mathbf{A_{iklm}u_{ik}u_{lm}} - \frac{\hbar^2}{2m}(n-\varrho)(\nabla\phi_0)^2 + \hbar\varrho\nabla\phi_0\cdot\frac{D\mathbf{u}}{Dt} + \frac{1}{2}m\varrho\left(\frac{D\mathbf{u}}{Dt}\right)^2. \tag{37}$$

This effective Lagrangian contains all the linear response of the system due to both strain and phase gradients. We can see that the phase stiffness is now given by $n-\varrho$ and ϱ is the inertia associated with the motion of the modulated pattern of the density. In summary, we have followed Ref. [17] and reproduced the same results for the superfluid phase stiffness of the supersolid. Our Lagrangian differs from that reference in the dynamical term. In the next section, we complete the development of an effective low-energy theory for the supersolid by considering the free energy of the system in the grand canonical ensemble. Then, we can finally proceed to discuss the macroscopic dynamics of the system.

VI. FREE ENERGY AND GRAND CANONICAL ENSEMBLE

In Secs. III–V, we developed a homogenized theory of the supersolid in the canonical ensemble at a fixed average particle density of n. We have integrated out the intra-unit-cell dynamics of the supersolid and have now obtained an effective Lagrangian $L_{\rm hom}(\phi_0, \mathbf{u})$ which describes the system under a given phase gradient and displacements $\mathbf{u}$. We have assumed that all the unit cells responded in the same way, i.e., the response has the same periodicity as the supersolid. This effective Lagrangian obeys all the necessary requirements of Galilean invariance, rotational invariance and $U(1)$ invariance as specified by Son [14] and Andreev and Lifshitz [7], so we would expect the dynamics to be phenomenologically correct.

In this section, we will consider the inter-cell dynamics and allow for particle flow between unit cells leading to variations in the density δn across the system at length scales much larger than the size of the unit cells of the supersolid.

It is important to note that the additional energy response due to flow between cells is crucial to obtaining the correct dynamics of the system. If we were to neglect this energy response, then we would find that the dynamics of the system would not match up with a Bogoliubov treatment. A similar phenomena was observed in Ref. [32], where the authors found that the bulk modulus computed via a homogeneous dilation technique was not in agreement with long-wavelength phonon predictions unless charge redistribution was taken into account.

We assume that the inter-cell number fluctuation is small, i.e., that $\delta n/n \ll 1$, and so we can expand the Lagrangian analytically in δn up to $\mathcal{O}(\delta n^2)$. We now consider the ground-state Lagrangian density $\mathcal{E}(n) = -L_0/|\Omega|$ which is a function of the average density n of the undeformed system. Under a change of the local density, this can be expanded as

$$\mathcal{E}(n+\delta n) \simeq \mathcal{E}(n) + \mathcal{E}'\delta n + \tfrac{1}{2}\mathcal{E}''(\delta n)^2 \tag{38}$$

and similarly the local pressure can be expanded as

$$P(n+\delta n) \simeq P(n) + P'\delta n. \tag{39}$$

These are the only terms we need to track in a theory that is an expansion up to second order in the total power of all perturbative fields. The coefficients, P' and $\mathcal{E}''$, can be calculated numerically by solving Gross-Pitaevskii equation at several densities. We can also use $\mathcal{E}' = \mu_0$ and $\mathcal{E}'' = \partial\mu/\partial n$ if we have the density dependence of the chemical potential. In fact, we have already calculated the pressure derivative in terms of the chemical potential shift per unit strain, μ_1, as defined in Eq. (16) with $\mu_1^{ik} = \mu_1\delta_{ik}$. To see this, we note that the pressure, $P = -(\partial E/\partial V)_N$, and the chemical potential, $\mu = (\partial E/\partial N)_V$, are related by a Maxwell relation so that

$$P' = \frac{\partial P}{\partial n} = V\left(\frac{\partial P}{\partial N}\right)_V = -V\left(\frac{\partial \mu}{\partial V}\right)_N = -\mu_1, \tag{40}$$

since $\delta V/V = u_{kk}$ is the strain. Similarly, we can use $\mathcal{E}'' = \partial\mu/\partial n$ if we have already obtained the density dependence of the chemical potential.

In general, the above analysis is valid for any tensorial quantity, but we have elected to use the symmetry properties of tensors that belong to a triangular lattice (i.e., $P_{ik} = P\delta_{ik}$ and $\mu_1^{ik} = \mu_1\delta_{ik}$) to simplify our calculations. The following analysis also uses this symmetry, but can be generalized to any tensorial quantity.

Finally, we need to include the work done by the change in local density on the local environment. Namely, if a part of the system expands due to an increase in particle number, then it does work on the surroundings which must compress. The work done caused by a total volume change of δV is

$$\delta W = -P\delta V = -P(\epsilon_{ll} + M_{iklm}u_{ik}u_{lm})|\Omega|. \tag{41}$$

To describe the dynamics of these density functions, we need to add this work done to the Lagrangian to obtain

$$\frac{L_{\rm dyn}}{|\Omega|} = \frac{L_{\rm hom}}{|\Omega|} - P(\epsilon_{ll} + M_{iklm}u_{ik}u_{lm}). \tag{42}$$

Comparing with our form of Eq. (37) for $L_{\rm hom}$ and the relation (22) between the Cauchy tensor B_{iklm} and A_{iklm}, we find that

$$-Pu_{ll} - \frac{A_{iklm}}{2}u_{ik}u_{lm} - P(\epsilon_{ll} + M_{iklm}u_{ik}u_{lm}) \simeq -\frac{1}{2}B_{iklm}u_{ik}u_{lm}. \tag{43}$$

Using this, we arrive at an effective quadratic Lagrangian that can describe the dynamics of the low-energy excitations of the

supersolid. This is the principal result of our paper:

$$\frac{L_{\rm dyn}}{|\Omega|} = -\mathcal{E} - \mu_0\delta n - \frac{1}{2}\mathcal{E}''(\delta n)^2 - \hbar n\dot{\phi}_0(1+\nabla\cdot\mathbf{u}) - \hbar\delta\mathbf{n}\dot{\boldsymbol{\phi}}_\mathbf{0} + \mu_\mathbf{1}\delta\mathbf{n}\nabla\cdot\mathbf{u} - \frac{1}{2}B_{iklm}u_{ik}u_{lm} - \frac{\hbar^2}{2m}(n-\varrho)(\nabla\phi_0)^2 + \hbar\varrho\nabla\phi_0\cdot\dot{\mathbf{u}} + \frac{1}{2}m\varrho\dot{\mathbf{u}}^2. \tag{44}$$

In this process we note that we have directly identified the Cauchy elastic tensor in two different ways, first by using the result from Bavaud [31], and second by directly considering the work the excitations would need to do on their environment.

To summarize, we have a coarse-grained Lagrangian to describe the low-energy excitations of the supersolid. In the next section, we will investigate these dynamics.

VII. EFFECTIVE SUPERSOLID LAGRANGIAN

In the previous sections, we have derived an effective Lagrangian (45) by integrating out the short-wavelength modes in the unit cell and coarse-graining the system such that a unit cell is now considered a point in the continuous field theory described by the fields $\{\mathbf{u}, \mathbf{n}, \phi\}$. This Lagrangian formally describes a low-energy effective field theory wherein the coupled Goldstone modes of the system are the low-lying excitations. We now set out to calculate the normal modes by integrating out the field δN and solving the resulting set of coupled equations.

By minimizing the Lagrangian for δn, we find

$$\delta n = \frac{\mu_1}{\mathcal{E}''}\nabla\cdot\mathbf{u} - \frac{\hbar}{\mathcal{E}''}\dot{\phi}, \tag{45}$$

which we substitute back into the Lagrangian to straightforwardly obtain

$$L = -\mathcal{E} - \hbar n\dot{\phi} + \frac{\hbar^2}{2}\left[\frac{\dot{\phi}^2}{\mathcal{E}''} - \frac{n-\varrho}{m}(\nabla\phi)^2\right] + \frac{1}{2}\left[m\varrho\dot{\mathbf{u}}^2 - \left(B_{iklm} - \frac{\mu_1}{\mathcal{E}''}\mu_1\delta_{ik}\delta_{lm}\right)u_{ik}u_{lm}\right] - \hbar\left(n-\varrho+\frac{\mu_1}{\mathcal{E}''}\right)\dot{\phi}\nabla\cdot\mathbf{u}. \tag{46}$$

This Lagrangian should be able to describe the Goldstone modes for the system. This effective Lagrangian is of the form predicted by Ye [33].

By counting the number of spontaneously broken continuous symmetries, we are able to predict the number of Goldstone modes that emerge. Since we have broken the $U(1)$ gauge symmetry, we expect to find one Goldstone mode. We also have a triangular lattice, which means we have broken two continuous translational and rotational symmetries, and so we expect to find two Goldstone modes. Therefore, we should find three Goldstone modes in total.

One can immediately find the equations of motion for both ϕ and $\mathbf{u}$, but it serves as a useful check to ensure we recover the Lagrangian for a Bogoliubov mode in the superfluid phase. In a superfluid phase, one can show that $B_{iklm} = U_{k=0}n^2\delta_{ik}\delta_{lm}$, $\mu_1 = -U_{k=0}n$, $\mathcal{E}'' = U_{k=0}$, and $\varrho = 0$. Consequently, the resultant Lagrangian is now

$$L = -\mathcal{E} - \hbar n\dot{\phi} + \frac{\hbar^2}{2}\left[\frac{\dot{\phi}^2}{U_{k=0}} - \frac{n}{m}(\nabla\phi)^2\right], \tag{47}$$

which corresponds exactly with the superfluid Bogoliubov mode.

Note that for the following derivation of the supersolid Goldstone velocities, it is assumed that the ground state is a supersolid and hence $\varrho \neq 0$. In a superfluid state ($\varrho = 0$), the following analysis does not apply, and instead one needs to refer to Eq. (48).

For the general supersolid case, we derive the equations of motion directly by assuming the fields ϕ and $\mathbf{u}$ are of a plane wave form $\sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\mathbf{t})}$. The vector equation for the variable $\mathbf{u}$ can be solved to provide both the shear and longitudinal mode. The Euler-Lagrange equation for the variable $\mathbf{u}$ is immediately found to be

$$m\varrho\ddot{\mathbf{u}} - \left(K + G - \frac{\mu_1}{\mathcal{E}''}\mu_1\right)\nabla(\nabla\cdot\mathbf{u}) - G\nabla^2\mathbf{u} - \hbar\left(n-\varrho+\frac{\mu_1}{\mathcal{E}''}\right)\nabla\dot{\phi} = 0, \tag{48}$$

where we have used Eq. (23) as we are analyzing a triangular lattice.

We can now solve this equation by noting that the vector field $\mathbf{u}$ can be split into longitudinal and transverse elements $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$. Taking the curl of Eq. (48) isolates the transverse motion ($\mathbf{u_t}$), giving

$$\nabla\times(m\varrho\ddot{\mathbf{u}}_t - G\nabla^2\mathbf{u_t}) = \mathbf{0}. \tag{49}$$

This is a wave equation for $\nabla\times\mathbf{u}$ from which we obtain the shear velocity

$$c_{\rm shear} = \sqrt{\frac{G}{m\varrho}}, \tag{50}$$

which is exactly equivalent to the shear velocity predicted in standard solid elasticity.

We examine longitudinal dynamics ($\mathbf{u_l}$) by taking the divergence of Eq. (49) we are examining the dynamics of $\mathbf{u}_l$. We find the two coupled equations

$$\hbar\left(\frac{\omega^2}{\mathcal{E}''} - \frac{n-\varrho}{m}\mathbf{k^2}\right)\Phi - i\omega\left(n-\varrho+\frac{\mu_1}{\mathcal{E}''}\right)\nabla\cdot\mathbf{u_l} = \mathbf{0}, \tag{51}$$

$$\left[m\varrho\omega^2 - \left(K - \frac{\mu_1}{\mathcal{E}''}\mu_1 + 2G\right)\mathbf{k^2}\right]\nabla\cdot\mathbf{u_l} + i\omega\hbar\mathbf{k^2}\left(\mathbf{n}-\varrho+\frac{\mu_\mathbf{1}}{\mathcal{E}''}\right)\mathbf{\Phi} = \mathbf{0} \tag{52}$$

in the variables $\mathbf{u}_l$ and ϕ. It can be shown easily that if we are in a superfluid regime the second equation is trivially satisfied, and the first equation finds the Bogoliubov frequency. These

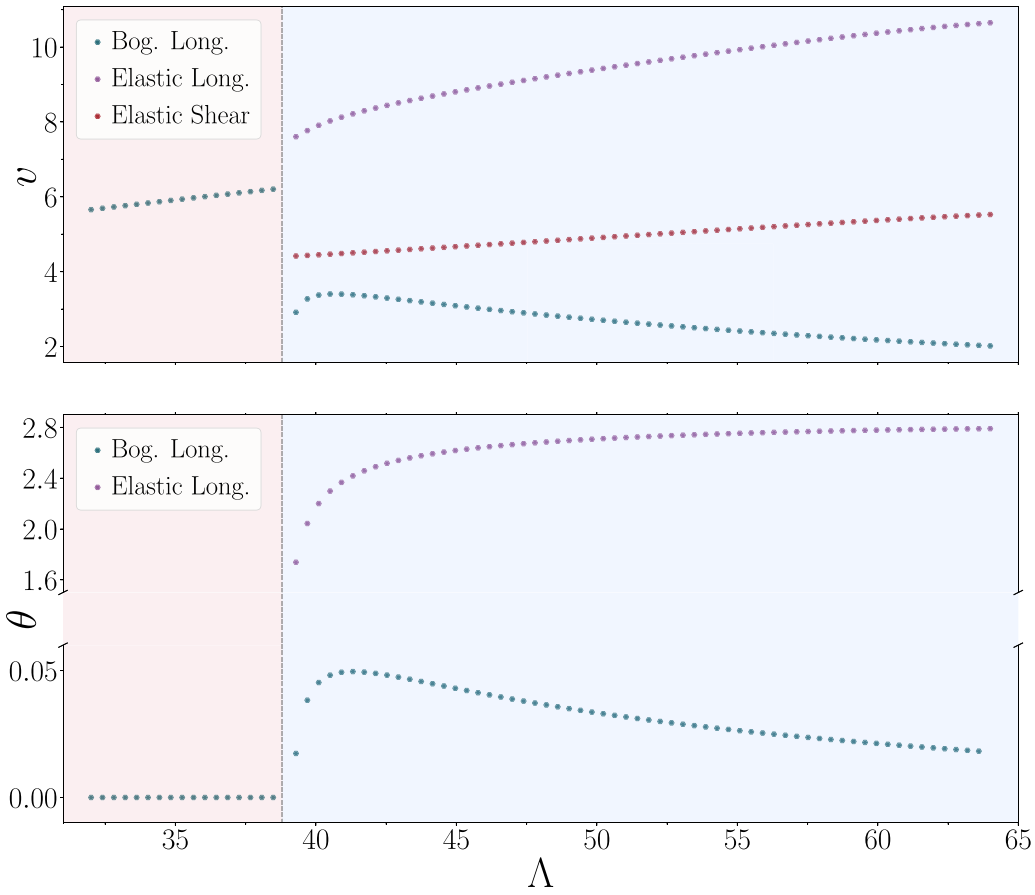

FIG. 5. Top: excitation velocities (in units of $\hbar/\sqrt{n}ma^2$) for all available sound modes of a system governed by a Heaviside interaction of strength Λ with a phase transition at $\Lambda \sim 38$, as calculated via homogenization. Note that in the superfluid phase the bulk mode is the Bogoliubov mode, hence the sudden appearance of two extra modes at the superfluid-supersolid transtion. Bottom: the eigenvector angle θ for the available longitudinal modes.

equations can be solved for ω as a function of $\mathbf{k}$ and leads to the following dispersion relation

$$\omega^2 = \frac{A \pm D}{2m\varrho}\mathbf{k}^2, \tag{53}$$

with

$$A = K + 2G + (n\mathcal{E}'' + 2\mu_1)(n-\varrho),$$
$$D^2 = A^2 - 4\varrho(n-\varrho)\left[\mathcal{E}''(K+2G) - \mu_1^2\right]. \tag{54}$$

We note that, in the case of the homogeneous superfluid ($\varrho = 0$), we obtain the superfluid velocity directly from (47)

$$\omega^2 = \frac{n\,U_{k=0}}{m}\mathbf{k}^2, \tag{55}$$

which is exactly equal to the Bogoliubov velocity. In the opposite limit of $\varrho = n$, the two modes have frequencies

$$\omega^2 = \frac{K+2G}{mn}\mathbf{k}^2 \quad \text{and zero.} \tag{56}$$

The zero mode corresponds to the collapse of the Bogoliubov mode and the linear dispersing mode has a velocity as expected for a normal solid with elastic moduli K and G.

It is interesting to note that, although we expect the Gross-Pitaevskii Lagrangian to struggle with describing a normal solid, the effective phenomenological Lagrangian that emerges actually returns to that of a normal solid in the limit $\varrho \to n$. Moreover, although all of the modes have a factor of $1/\varrho$ in their velocities which seems to indicate that the velocities should blow up as $\varrho \to 0$, the velocities appear to be well behaved as ϱ becomes small. This suggests that there is some implicit dependence on ϱ hidden in the values of B_{iklm} etc., which stops the velocities from blowing up. However, we should note that as we appear to see a first-order transition, the value of ϱ never really approaches zero in our numerics, but only jumps directly to zero at which point these equations are no longer valid.

We can also now examine the nature of the normal modes of the system. By writing Eqs. (51) and (52) in terms of a matrix equation, we can find the eigenvectors of the system for a given ω. These eigenvectors are the normal modes, and are some composite of the fields ϕ and $\mathbf{u}_l$. We can then subsequently define an eigenvector angle θ which describes the relative contribution of the fields ϕ and $\mathbf{u}_l$ to the normal mode. For example, in the superfluid the eigenvector angle is $\theta = 0$ as the normal mode is purely ϕ, i.e., $\mathbf{e}_n \equiv \mathbf{e}_\phi$. A supersolid with normal mode $\mathbf{e}_n = (0.5, 0.5)$ would have $\theta = \pi/4$, etc. The normal mode composition is illustrated in Fig. 5.

An interesting point of note is that the slow longitudinal mode (which is expected to die off in the limit of a solid) is a composite of both ϕ and $\mathbf{u}_l$, but is mostly ϕ at all points in the

phase diagram (i.e., θ is small). More importantly, it appears there is a critical point at which the slow longitudinal mode is "saturated," that is to say that there is a global maximum in the mixing of the slow longitudinal mode (but note that the mode is still mostly ϕ). This feature appears to be absent in the fast longitudinal mode. However, the fast longitudinal mode also exhibits interesting behaviour as the mixing angle does not tend to $\pi/2$ or $3\pi/2$ as one may initially expect (as we would expect the mode to tend to a purely elastic phonon mode in the limit of a solid), but instead tends to approximately $7\pi/8$. This is a curious feature of the fast longitudinal mode, and we are currently investigating the origin of this behavior.

The above holds for any system described by a Gross-Pitaevskii equation; we can now use these results in the specific case of a soft-core interaction. We find a ground state for the system numerically and by applying the procedure detailed above we can determine the velocities of all modes that exist. Importantly, we note that our Lagrangian is tuned *across the transition*, i.e., a single theory describes both the superfluid and supersolid regime and continuously varies across a first-order phase transition. Importantly, symmetry arguments by Son [14] dictate the functional form of any Lagrangian belonging to a supersolid and since our Lagrangian is a subset of those allowed by symmetry, we argue that we completely capture the phenomenological behavior of the supersolid as described by a Hartree-Fock approximation.

VIII. BOGOLIUBOV DISPERSION

An alternative technique to finding excitations of the system is to directly apply linearized oscillations in the number density and the phase. Namely, since $\psi = \sqrt{\rho}e^{i\phi}$, we can apply the Bogoliubov fluctuations $\rho \to \rho_0 + \delta\rho$ and ϕ. We expect to recover all the Goldstone modes accessible to the system, and expect the quantitative properties to be the same as those predicted in the elastic theory. More precisely, due to the broken $U(1)$ phase and the broken continuous translational symmetry, we expect to find three Goldstone modes, two of which are longitudinal and one of which is a shear mode. Since this is a completely independent technique to recovering the Goldstone modes, it serves as an important check of our results.

While both techniques access the speeds of sound, the elastic technique also provides elastic moduli and fractional inertia, whereas the bogoliubov technique provides a full band structure. In this way, the two techniques complement each other, as they verify common ground but also provide information that is inaccesible to the other.

By following a similar procedure to that outlined in Ref. [16], we are able to obtain eigenvalue equations which give the excitation frequencies ω of the Bloch waves in the system. These Bloch waves have a band structure, and the lowest band structure at the Γ point corresponds to the Goldstone modes of the system whose propagation velocities were independently derived in Ref. [16]. The band structure is fully described by the equations

$$\begin{aligned}[-\omega^2 + (\hat{T}^{\mathbf{k}} + \hat{U}^{\mathbf{k}})\hat{T}^{\mathbf{k}}](\Phi)^{\mathbf{k}} = 0,\\ [-\omega^2 + \hat{T}^{\mathbf{k}}(\hat{T}^{\mathbf{k}} + \hat{U}^{\mathbf{k}})](\delta\psi)^{\mathbf{k}} = 0,\end{aligned} \tag{57}$$

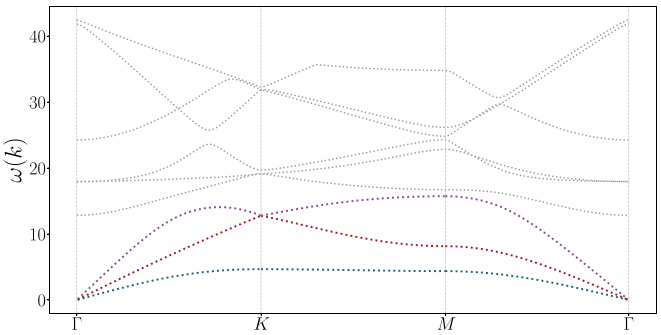

FIG. 6. Band structure for a supersolid at $\Lambda = 50$. The three Goldstone modes are highlighted as a fast longitudinal, slow longitudinal, and shear mode, colored in purple, green and red respectively.

with $\hat{T}$ and $\hat{U}$ defined as

$$\begin{aligned}\hat{T}^{\mathbf{k}}_{\mathbf{G},\mathbf{G}'} &= \left[\frac{1}{2}(\mathbf{G}+\mathbf{k})^2 - \mu\right]\delta_{\mathbf{G},\mathbf{G}'} + U_{\mathbf{G}-\mathbf{G}'}\rho_{\mathbf{G}-\mathbf{G}'},\\ \hat{U}^{\mathbf{k}}_{\mathbf{G},\mathbf{G}'} &= 2\sum_{\mathbf{G}''}\psi_{\mathbf{G}-\mathbf{G}''}U_{\mathbf{k}+\mathbf{G}''}\psi_{\mathbf{G}''-\mathbf{G}'},\end{aligned} \tag{58}$$

with $U_{\mathbf{q}}$, $\psi_{\mathbf{q}}$, and $\rho_{\mathbf{q}}$ the fourier transforms of the interaction potential, the superfluid order parameter and the density, respectively. We expect that the band structure obtained from solving these equations should find all the requisite Goldstone modes and that they should have the same velocities as predicted via homogenization.

Upon recovery of the band structure, illustrated in Fig. 6, one finds exactly $(D+1)$ Goldstone modes as expected. There is a band crossing at the K point of the fast longitudinal and shear modes, characteristic of a triangular lattice (note that this is not actually a crossing but a band touching, as shown by the $M \to \Gamma$ band structure).

For any particular given Λ, both the Bogoliubov and elastic techniques predict all three speeds of sound (if supersolid, only one speed if superfluid), and agree on the speeds with excellent precision. Subsequently, by deriving the excitations of the system in two independent ways and obtaining the same results to excellent numerical precision, we can reliably say we have obtained the correct dynamics.

IX. RESULTS AND DISCUSSION

We can see that we have now verified the technique of homogenization as a means of recovering the low-lying excitations of the system. This was achieved via means of two completely independent techniques, the results of which both agree to excellent accuracy. An advantage of homogenization over the Bogoliubov technique is that it also provides the elastic constants of the system, which cannot be obtained via Bogoliubov. Moreover, it provides the fractional inertia of the system, which is a quantity that is of great physical interest. A clear disadvantage however, is that homogenization does not provide the full spectrum of excitations, only the long-wavelength limit. We stress that this is not an issue, as the Bogoliubov technique is completely separate from homogenization, and so by utilising both techniques one can recover the full band structure from Bogoliubov and supplement it

with additional information in the long-wavelength limit from homogenization.

The effective Lagrangian that is the end result of homogenization elucidates interesting dynamics. By considering that the flow excitations of the system can be coupled to the strain excitations, we can see that the system is capable of strain-induced flow. Furthermore, by considering the application of an instantaneous strain to the system, we can see that flow will be induced and contribute to the elastic response of the system. Alternatively, if we apply a strain and keep it fixed for a long time so that the flow dissipates, then there will be a different elastic response. In this way we can see that the elastic modulus has a time dependence, and that the system is capable of dissipating energy via the flow of the system. This energy dissipation is likely to be via the two longitudinal modes which are coupled crystal and Bogoliubov phonons.

There is an interesting question of what the compressibility of the system is. When one considers the elastic strain, it is natural to define the compressibility of the system as $\kappa = -1/V\,(\partial V/\partial P)$. Typically in a solid, one would simultaneously be able to say $\kappa = 1/n\,(\partial n/\partial P)$, where n is the number density and μ the chemical potential. However, in our analysis, we can show that these two definitions do not agree, and are off by a considerable margin. These two values are defined as $\kappa = 1/B_{xxyy}$ and $\kappa = 1/(n\mu_1)$ in our notation, respectively.

The resolution of this seeming dilemma is to revisit the definition of the two compressibilities. The compressibility $\kappa = 1/B$ where $B = -V\,(\partial P/\partial V|_N)$ is the elastic bulk modulus of the system, taken at constant particle number. This is the compressibility of the system as defined via elastic theory, and is the one that we will take as the "true compressibility," and consequently B as the "true bulk modulus." Importantly, the derivative is taken under constant particle number. As such, to elucidate this a little further, we write

$$B_e = -V\left.\frac{\partial P}{\partial V}\right|_N, \tag{59}$$

where the subscript e denotes that this is the elastic bulk modulus. We can then write the compressibility as $\kappa_e = 1/B_e$. We can subsequently define the thermodynamic bulk modulus (using the typical definition with n but now removing the factor of V which is kept constant) as

$$B_t = N\left.\frac{\partial P}{\partial N}\right|_V \tag{60}$$

and the thermodynamic compressibility accordingly.

The key consideration is that the $B_e \neq B_t$ due to the fact that we are keeping different variables constant upon taking derivatives. If we want to connect these two definitions, then we need to use the cyclic relations of thermodynamics:

$$\left.\frac{\partial x}{\partial y}\right|_z = -\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial y}\right|_x. \tag{61}$$

We can then write

$$B_e = -V\left.\frac{\partial P}{\partial V}\right|_N = V\left.\frac{\partial P}{\partial N}\right|_V \left.\frac{\partial N}{\partial V}\right|_P \equiv B_t\,\frac{1}{n}\left.\frac{\partial N}{\partial V}\right|_P, \tag{62}$$

from which we can consider the following. In a regular material (solid, gas, liquid) we would expect that the derivative of the particle number with respect to the volume under constant pressure is simply the density of the system, and we would recover the thermodynamic bulk modulus $B_t = n\,(\partial P/\partial n|_V)$ from Eq. (62). We can then examine whether this is the case for our system, i.e., is $1/n\,(\partial N/\partial V|_P) = 1$?

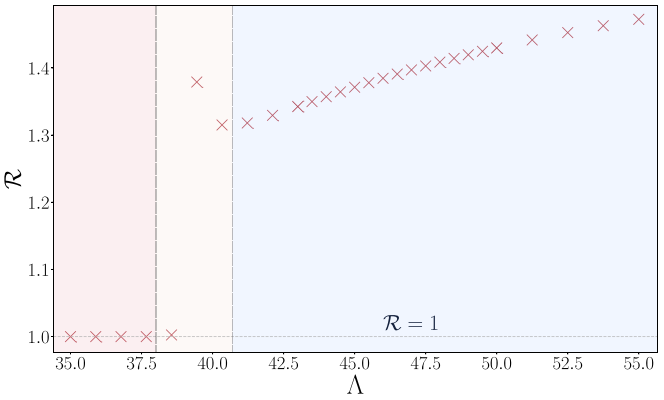

FIG. 7. Compressibility ratio $\mathcal{R}$ for a system governed by a soft-core interaction of strength Λ. $\mathcal{R}$ is obtained using the relation (63), where the values are calculated using homogenization. The expected value of $\mathcal{R} = 1$ is obtained in the superfluid phase, but is starkly different in the supersolid regime.

We can examine this quantity, henceforth denoted as $\mathcal{R}$ and called the "compressibility ratio," by choosing an initial volume and particle number V_0 and N_0 which will generate a P_0, adjusting the volume around V_0 with some δV, and then varying the particle number $N_0 + \delta N$ until the system reaches P_0. We can then numerically calculate $\mathcal{R}$ by finite differencing. If we expect the difference between the two compressibilities to be due to the fact that $\frac{1}{n}\frac{\partial N}{\partial V}|_P \neq 1$, then we would expect $\mathcal{R} = \frac{\kappa_t}{\kappa_e}$, or equivalently

$$\mathcal{R} = \frac{B_e}{B_t} \equiv \frac{K}{n\mu_1}. \tag{63}$$

Upon performing the numerical determination of $\mathcal{R}$, we find that it is within good agreement of the ratio of the two compressibilities as determined via homogenization, illustrated in Fig. 7. The percentage error between the homogenization and numerical $\mathcal{R}$ is typically between 3% and 5%. We believe this is an artifact of the numerical determination of $\mathcal{R}$, and that as the algorithm and precision of simulations for determining $\mathcal{R}$ is improved, the agreement will improve accordingly. While we have shown that the compressibility ratio obtained via homogenization is in good agreement with the numerical determination, we do not yet fully understand the nature of this anomalous result. In our search across the literature we have not found another example of this phenomenon, and we are currently investigating the origin of this effect. A possible explanation for the result is considering a pressure which can be written like $P(n, G(n, V))$, where G is the ordering wave vector. One can then subsequently show that the difference between bulk moduli can be written as

$$B_e = B_t - V\left.\frac{\partial P}{\partial G}\right|_n \left.\frac{\partial G}{\partial V}\right|_n, \tag{64}$$

where it is clear that the last term is the anomalous term.

This term vanishes in ordinary materials in various ways. For gases and liquids, there is no ordering vector, so this term never existed to begin with. A regular solid has well-defined atomic positions. So, if the density is specified, then the ordering wave vector is also specified. Upon changing the volume, there is no additional change in the ordering wave vector beyond that caused by the change in density. Supersolids are unique in that the ordering vector G experiences a commensuration effect. For instance, in a one-dimensional system, it needs to be an integer multiple of $2\pi/L$ with L being the length of the system. If we stretch the system by δL much less than the lattice spacing, then the fractional change in the ordering wave vector is $\delta G/G = -\delta L/L$ so that $L\partial G/\partial L = -G$. We believe this is a feature unique to supersolids, due to their ability to exist at a wide range of densities, each of which has their own ordering vector, whereas regular solids only have one density at which they can exist (at some given pressure) and therefore only have one ordering vector. This is (to the best of our knowledge) a new phenomena which could be of great interest to the characterization of supersolids.

Some currently open questions are as follows. Is this effect unique to the supersolid phase, or is it a more general phenomena? What kind of thermal transport is occurring in the phonon modes of a supersolid? Does the notion of the Landau criterion for superfluidity still hold in the supersolid phase, and if so, how does it change now that we have a band structure? These questions may potentially have avenues to solution via the techniques developed in this paper.

The successful development of homogenization techniques for the study of supersolids is a significant step forward in the study of these systems. We have shown that the homogenization technique is able to accurately capture the behaviour of the supersolid phase through comparison with well established Bogoliubov techniques. We were able to obtain previously unknown elastic properties of these materials as well as an effective Lagrangian which can be used to study the dynamics of the system. In contrast with typical phases of matter which have $\mathcal{R} = 1$, we found that the compressibility ratio of the supersolid phase is $\mathcal{R} > 1$. This is a new phenomena which we predict will arise in supersolids, and of which the origin is currently unknown.

ACKNOWLEDGMENT

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APPENDIX A: EXPANSION CALCULATION

We start by expanding the noninteracting part of $\mathcal{L}$ using the tools developed. We can express without expansion

$$(\nabla'\rho(\mathbf{r}'))^{\mathbf{2}} = [(\partial_{\mathbf{i}} + (\mathbf{u_{ik}} + \mathbf{u_{il}u_{lk}})\partial_{\mathbf{k}})(\rho_{\mathbf{0}}(\mathbf{r}) + \tilde{\rho}(\mathbf{r}))]^{\mathbf{2}}, \tag{A1}$$

where we subsequently drop the coordinates as it is clear everything is in the unprimed basis $\mathbf{r}$. We can re-express the prefactor

$$\frac{1}{\rho(\mathbf{r}')} = \frac{1}{\rho_0 + \tilde{\rho}} = \frac{1}{\rho_0} - \frac{\tilde{\rho}}{\rho_0^2} + \frac{\tilde{\rho}^2}{\rho_0^3} + \mathcal{O}(\tilde{\rho}^3) \tag{A2}$$

to make it more algebraically pliable. This allows us to write

$$\begin{aligned}\int_{\Omega'} \frac{\hbar^2}{2m}\frac{(\nabla'\rho(\mathbf{r}'))^{\mathbf{2}}}{4\rho(\mathbf{r}')}\,d\mathbf{r}' = \int_{\Omega}\Big\{\frac{\hbar^2}{8m}\Big(\frac{1}{\rho_0} - \frac{\tilde{\rho}}{\rho_0^2} + \frac{\tilde{\rho}^2}{\rho_0^3}\Big)(1 + \epsilon_{ll} + M_{iklm}u_{ik}u_{lm})[(\nabla\rho_0)^2 + 2(\boldsymbol{\nabla}\rho_0\cdot\boldsymbol{\nabla}\tilde{\rho}) - 2\epsilon_{ik}\partial_i\rho_0\partial_k\rho_0 \\ + 2\chi_{ik}\partial_i\rho_0\partial_k\rho_0 - 4\epsilon_{ik}\partial_i\rho_0\partial_k\tilde{\rho} + (\nabla\tilde{\rho})^2 + 2\Delta_{ik}\partial_i\rho_0\partial_k\rho_0]\Big\}\,d\mathbf{r},\end{aligned} \tag{A3}$$

which we can manipulate further. By isolating first- and second-order (in terms of the strain tensor) parts of the noninteracting Lagrangian, and using the relation (obtained via integration by parts)

$$\int_{\Omega}\frac{\boldsymbol{\nabla}\tilde{\rho}\cdot\boldsymbol{\nabla}\rho_0}{\rho_0}\,d\mathbf{r} = \int_{\boldsymbol{\Omega}}\tilde{\rho}\frac{(\nabla\rho_{\mathbf{0}})^{\mathbf{2}}}{\rho_{\mathbf{0}}^{\mathbf{2}}} - \tilde{\rho}\frac{\boldsymbol{\nabla}^{\mathbf{2}}\rho_{\mathbf{0}}}{\rho_{\mathbf{0}}}\,\mathbf{dr}, \tag{A4}$$

we are able to find the first-order component

$$\frac{\hbar^2}{4m}\Big[\Big(\frac{(\nabla\rho_0)^2}{2\rho_0^2} - \frac{\nabla^2\rho_0}{\rho_0}\Big)\tilde{\rho} + \epsilon_{ik}\Big(\frac{(\nabla\rho_0)^2}{2\rho_0}\delta_{ik} - \frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0}\Big)\Big] \tag{A5}$$

and the second-order component

$$\begin{aligned}&\frac{\hbar^2}{4m}\Big[(\Delta_{ik} + \chi_{ik})\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0}\Big] + \frac{\hbar^2}{4m}\frac{1}{2}\Big[-4\epsilon_{ik}\frac{\partial_i\rho_0\partial_k\tilde{\rho}}{\rho_0} + \frac{(\nabla\tilde{\rho})^2}{\rho_0} + 2\frac{\boldsymbol{\nabla}\tilde{\rho}\cdot\boldsymbol{\nabla}\rho_0}{\rho_0}\epsilon_{ll} - 2\epsilon_{ik}\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0}\epsilon_{ll} + \frac{(\nabla\rho_0)^2}{\rho_0}M_{iklm}u_{ik}u_{lm} \\ &-2\frac{\boldsymbol{\nabla}\tilde{\rho}\boldsymbol{\nabla}\rho_0}{\rho_0^2}\tilde{\rho} + 2\epsilon_{ik}\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\tilde{\rho} - \frac{(\nabla\rho_0)^2}{\rho_0^2}\tilde{\rho}\epsilon_{ll} + \frac{(\nabla\rho_0)^2}{\rho_0^3}\tilde{\rho}^2\Big],\end{aligned} \tag{A6}$$

respectively. We now turn to considering the interaction section of the Lagrangian.

By first noting that the interaction potential is centrosymmetric (and indeed we expect most effective potentials to be isotropic), we can write the interaction as a function of radial distance. We can express the change in the metric length of

distance by simply applying the deformation definition to $|\Delta\mathbf{r}'|^2$ and obtain the result

$$(\Delta r')^2 = (\delta_{ik} + 2\epsilon_{ik} + 2\Delta_{ik})(\Delta r)_i(\Delta r)_k, \tag{A7}$$

which is exactly equivalent to that of Landau and Lifshitz [30]. The tensor in the above equation is sometimes referred to as the finite strain tensor. Recalling the derivation of Eq. (19) we define the functions

$$U(\Delta r') = U(\Delta r) + (\epsilon_{ik} + \Delta_{ik}) f_{ik}(\Delta r) + \epsilon_{ik}\epsilon_{lm} W_{iklm}(\Delta r), \tag{A8}$$

with

$$f_{ik}(r) \equiv \frac{r_i r_k}{|r|}\frac{\partial U(|r|)}{\partial |r|}, \quad W_{iklm}(r) \equiv \frac{r_i r_k r_l r_m}{2|r|^2}\left(\frac{\partial^2 U}{\partial |r^2|} - \frac{1}{|r|}\frac{\partial U}{\partial |r|}\right). \tag{A9}$$

This can be used to express the interaction component of $\mathcal{L}$ by combining it with both the density mapping and the Jacobian to obtain

$$\begin{aligned} U = \frac{1}{2}\int_\Omega (\rho_0 + \tilde{\rho})_{r_1}[U(r_{12}) + (\epsilon_{ik} + \Delta_{ik}) f_{ik}(r_{12}) + \epsilon_{ik}\epsilon_{lm} W_{iklm}(r_{12})] \times (\rho_0 + \tilde{\rho})_{r_2}(1 + 2\epsilon_{ll} + \epsilon_{ll}\epsilon_{kk} + 2M_{iklm}u_{ik}u_{lm}) \\ \times dr_1\, d\mathbf{r_2}, \end{aligned} \tag{A10}$$

where $\mathbf{r_{12}} = \mathbf{r_1} - \mathbf{r_2}$. We note that, since the convolution is integrated, there is a term arising from $(\mathcal{J}_{\mathbf{r}'\to\mathbf{r}})^2$. Once again, this can be expanded and combined with the noninteracting component of $\mathcal{L}$, and we can fully express the first and second order components of the expansion. We make use of the Gross-Pitaevskii ground-state equation (2), as well as the normalization condition (14) to simplify our expressions, and we can show that

$$\mathcal{L}_1 = \int_\Omega \left[\rho_0\epsilon_{ll}(U * \rho_0 - \mu_0) + \frac{1}{2}\epsilon_{ik}(f_{ik} * \rho_0)\rho_0 + \frac{\hbar^2}{4m}\epsilon_{ik}\left(\frac{(\nabla\rho_0)^2}{2\rho_0}\delta_{ik} - \frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0}\right)\right] d\mathbf{r}. \tag{A11}$$

We can now make a direct connection between the linear component of the expansion and the pressure tensor P_{ik}. Recalling that the first-order component of the expansion is the stress tensor, and that the stress tensor is the negative of the pressure, we can write

$$\begin{aligned} P_{ik} &= \frac{1}{|\Omega|}\int_\Omega \left[(\mu_0 - U * \rho_0)\delta_{ik} - \frac{1}{2}f_{ik} * \rho_0 - \frac{\hbar^2}{4m}\left(\frac{(\nabla\rho_0)^2}{2\rho_0^2}\delta_{ik} - \frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\right)\right]\rho_0\, d\mathbf{r} \\ &= \frac{1}{|\Omega|}\int_\Omega \left[\frac{\hbar^2}{4m}\left(\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0} - \nabla^2\rho_0\,\delta_{ik}\right) - \frac{1}{2}\rho_0(f_{ik} * \rho_0)\right] d\mathbf{r}. \end{aligned} \tag{A12}$$

The second line consists of the ground-state density profile only This is our first key result which is obtaining the pressure of any ground-state system given a specified interaction. We can trivially calculate the expected pressure for a homogeneous superfluid by noting that $\rho_0 \equiv n$ and that $\mu_0 = U_{\mathbf{k}=\mathbf{0}}n$ where $U_{\mathbf{k}=\mathbf{0}} = \int_\Omega U(\mathbf{r})\mathbf{dr}$. The pressure simply reduces to $-\frac{1}{2}n^2\int_\Omega f_{ik}(\mathbf{r})\,\mathbf{dr}$. Integrating by parts gives the integral over f_{ik} as $-U_{\mathbf{k}=\mathbf{0}}\delta_{ik}$. So, we find that $P_{ik} = \frac{1}{2}\mu_0 n\delta_{ik}$, which is the expected result. We can now move on to the more difficult task of solving for the second-order components.

It is important to note that since we have allowed $\tilde{\rho}$ to have both first and second-order components in u_{ik}, then even single powers of $\tilde{\rho}$ must be taken into consideration for the $\mathcal{O}(u^2)$ components. The above analysis only collected the components of $\tilde{\rho}$ which were of $\mathcal{O}(u)$, but does not remove the singular powers of $\tilde{\rho}$ from the Lagrangian. It is necessary to calculate the functional form of $\tilde{\rho}$ as there will be terms like $\int_\Omega \tilde{\rho}^2\, d\mathbf{r}$ that will contribute to the perturbative Lagrangian, and as such we therefore need to solve the equations of motion for $\tilde{\rho}$. It is not good enough to simply know the normalization condition as we did for the $\mathcal{O}(u)$ component.

The first step in solving for $\tilde{\rho}$ is to collect all the terms in which it appears, regardless of first or second-order splitting. We take all the terms that contain $\tilde{\rho}$ and call them $\mathcal{L}_{\tilde{\rho}}$, finding

$$\begin{aligned} -\mathcal{L}_{\tilde{\rho}} = \frac{1}{2}\frac{\hbar^2}{4m}\Bigg[-4\epsilon_{ik}\frac{\partial_i\rho_0\partial_k\tilde{\rho}}{\rho_0} + \frac{(\nabla\tilde{\rho})^2}{\rho_0} + 2\frac{\nabla\tilde{\rho}\cdot\nabla\rho_0}{\rho_0}\epsilon_{ll} - 2\frac{\nabla\tilde{\rho}\cdot\nabla\rho_0}{\rho_0^2}\tilde{\rho} + 2\epsilon_{ik}\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\tilde{\rho} - \frac{(\nabla\rho_0)^2}{\rho_0^2}\tilde{\rho}\,\epsilon_{ll} + \frac{(\nabla\rho_0)^2}{\rho_0^3}\tilde{\rho}^2\Bigg] \\ + \mu_0\tilde{\rho} + \frac{1}{2}((U * \tilde{\rho})\tilde{\rho} + 4\epsilon_{ll}(U * \rho_0)\tilde{\rho} + 2\epsilon_{ik}(f_{ik} * \rho_0)\tilde{\rho}), \end{aligned} \tag{A13}$$

which we can solve by the standard Euler-Lagrange technique. A key detail now is that the new normalization condition changes the constraints we apply. Specifically, the equation we need to solve is

$$\frac{\partial\mathcal{L}_{\tilde{\rho}}}{\partial\tilde{\rho}} - \partial_i\frac{\partial\mathcal{L}_{\tilde{\rho}}}{\partial(\partial_i\tilde{\rho})} + \left(\mu_0 + \mu_1\epsilon_{ll} + \mu_2^{iklm}u_{ik}u_{lm}\right)(1 + \epsilon_{ll} + M_{iklm}u_{ik}u_{lm}) = 0, \tag{A14}$$

which can be shown via the strained normalization condition (14).

A straightforward calculation follows of differentiating the $\tilde{\rho}$ dependent Lagrangian and collating all terms. We make use of the zeroth-order equation of motion to simplify our expression and we find

$$\begin{aligned}&\frac{\hbar^2}{4m}\nabla\cdot\left(\frac{\nabla\tilde{\rho}}{\rho_0}\right)-U*\tilde{\rho}+\frac{\hbar^2}{4m}\left(\frac{(\nabla\rho_0)^2}{\rho_0^3}-\frac{\nabla^2\rho_0}{\rho_0^2}\right)\tilde{\rho}\\&\quad=-\big(\mu_2^{iklm}+\mu_1^{ik}\delta_{lm}+\mu_0 M_{iklm}\big)u_{ik}u_{lm}+\epsilon_{ik}\Bigg[-\mu_1^{ik}+\frac{\hbar^2}{4m}\left(2\frac{\partial_{ik}\rho_0}{\rho_0}-\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\right)\\&\qquad+\int_\Omega\big(f_{ik}(\mathbf{r}-\mathbf{r}')+\delta_{\mathbf{ik}}\mathbf{U}(\mathbf{r}-\mathbf{r}')\big)\rho_0(\mathbf{r}')\,\mathbf{dr}'\Bigg],\end{aligned}\tag{A15}$$

which contains both $\mathcal{O}(u^2)$ and $\mathcal{O}(u)$ terms. We can now begin to separate $\tilde{\rho}$ into first- and second-order components via Eq. (13), and consider solving the equation component by component. That is to say that we solve the lowest-order component first, and then use that solution to solve the next order component, and so on.

The first-order equation to solve is given by

$$\begin{aligned}&\frac{\hbar^2}{4m}\nabla\cdot\left(\frac{\nabla\big(\rho_1^{ik}\epsilon_{ik}\big)}{\rho_0}\right)-U*\big(\rho_1^{ik}\epsilon_{ik}\big)+\frac{\hbar^2}{4m}\left(\frac{(\nabla\rho_0)^2}{\rho_0^3}-\frac{\nabla^2\rho_0}{\rho_0^2}\right)\big(\rho_1^{ik}\epsilon_{ik}\big)\\&\quad=\epsilon_{ik}\Bigg[-\mu_1^{ik}+\frac{\hbar^2}{4m}\left(2\frac{\partial_{ik}\rho_0}{\rho_0}-\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\right)+\int_\Omega\big(f_{ik}(\mathbf{r}-\mathbf{r}')+\delta_{\mathbf{ik}}\mathbf{U}(\mathbf{r}-\mathbf{r}')\big)\rho_0(\mathbf{r}')\,\mathbf{dr}'\Bigg],\end{aligned}\tag{A16}$$

which we can solve numerically. We factor out ϵ_{ik} and solve the set of equations where we have one for each index (in 2D we have four equations to solve, in 3D there are nine). As this is an integrodifferential equation we can solve the equation by writing the left-hand side as a matrix operator on an unraveled vector and applying an iterative matrix solver algorithm which converges on some prespecified tolerance. We can introduce notation and write the first and second-order equations of motion as

$$L\big(\rho_1^{ik}\big)=F_1\big(\rho_0,\mu_1^{ik}\big),\qquad L\big(\rho_2^{iklm}\big)=F_2\big(\mu_0,\mu_1^{ik},\mu_2^{iklm}\big)\tag{A17}$$

for brevity. It is important to note that we do not actually need to solve the equation for ρ_2^{iklm} as it only appears in the Lagrangian when multiplied by a constant, thus we can directly use the normalization condition (14) and short circuit the task of solving it. We also do not need to find μ_2^{iklm} for a slightly different reason, which is that it appears in the Lagrangian to too high order and so is always discarded anyway. Thus, our only task now is to determine the first-order correction to the chemical potential μ_1^{ik} and the first-order solution to $\tilde{\rho}$ which is ρ_1^{ik}.

We first state $\rho_1^{ik}\equiv\rho_1^{ik}(\mathbf{r},\mu_{\mathbf{1}}^{\mathbf{ik}})$, i.e., it is now a function of the chemical potential. We specifically choose the functional form $\rho_1^{ik}=\hat{\rho}_1^{ik}+\mu_1^{ik}\rho_1'$, in a kind of "homogeneous + inhomogeneous" fashion. Namely, we demand that

$$L\big(\hat{\rho}_1^{ik}\big)=\frac{\hbar^2}{4m}\left(2\frac{\partial_{ik}\rho_0}{\rho_0}-\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\right)+\int_\Omega[f_{ik}(\mathbf{r}-\mathbf{r}')+\delta_{\mathbf{ik}}\mathbf{U}(\mathbf{r}-\mathbf{r}')]\rho_{\mathbf{0}}(\mathbf{r}')\,\mathbf{dr}',\tag{A18}$$

i.e., $\hat{\rho}_1^{ik}$ satisfies the equation of motion without any chemical potential considerations. Then we solve the equation

$$L\big(\mu_1^{ik}\rho_1'\big)=-\mu_1^{ik}\;\rightarrow\;L(\rho_1')=-1\tag{A19}$$

simultaneously, which we can do numerically as it is the same operator but just a different RHS. We can state this in the notation of Eq. (A17),

$$L\big(\hat{\rho}_1^{ik}\big)=F_1\big(\rho_0,\mu_1^{ik}=0\big),\quad L(\rho_1')=F_1\big(\rho_0=0,\mu_1^{ik}=1\big),\tag{A20}$$

which we can readily substitute into the "homogeneous" and "inhomogeneous" solutions to see that

$$L\big(\hat{\rho}_1^{ik}+\mu_1^{ik}\rho_1'\big)=F_1\big(\rho_0,\mu_1^{ik}\big),\tag{A21}$$

which is the "general" solution we were looking for. The final task is to now actually calculate μ_1^{ik} which we can do by using the normalization condition (14). Since we know what the first-order condition is, we can rearrange to find that

$$\mu_1^{ik}=-\int_\Omega\big(\rho_0\delta_{ik}+\hat{\rho}_1^{ik}\big)\,d\mathbf{r}\Bigg/\int_{\mathbf{\Omega}}\rho_{\mathbf{1}}'\,\mathbf{dr},\tag{A22}$$

which is the final ingredient.

We have now fully solved for $\tilde{\rho}$ (and discarded the parts that we don't need), so we can re-express $\mathcal{L}_{\tilde{\rho}}$ using integration by parts. The calculation is relatively straightforward and makes use of the base EoM (2), allowing one to find

$$-\mathcal{L}_{\tilde{\rho}}=\frac{1}{2}\left(2\mu_0\rho_2^{iklm}u_{ik}u_{lm}+\rho_1^{ik}\left[\frac{\hbar^2}{4m}\left(2\frac{\partial_{ik}\rho_0}{\rho_0}-\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\right)+(f_{ik}+\delta_{ik}U)*\rho_0+2\mu_0\delta_{ik}+\mu_1^{ik}\right]\epsilon_{ik}\epsilon_{lm}\right)+\mathcal{O}(\epsilon^3),\tag{A23}$$

which contains all the second-order contributions from $\tilde{\rho}$.

We can combine this with all the second-order contributions that *do not contain* $\tilde{\rho}$, and obtain a long expression

$$
\begin{aligned}
-\mathcal{L}_2 = \frac{1}{2}\Bigg[& 2(U*\rho_0)\rho_0 M_{iklm}u_{ik}u_{lm} + ((U*\rho_0)\rho_0\delta_{ik}\delta_{lm} + 2(f_{ik}*\rho_0)\rho_0\delta_{lm} + (W_{iklm}*\rho_0)\rho_0)\,\epsilon_{ik}\epsilon_{lm} \\
& + (f_{ik}*\rho_0)\rho_0\Delta_{ik} + 2\frac{\hbar^2}{4m}\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0}(\Delta_{ik}+\chi_{ik}) + \frac{\hbar^2}{4m}\left(-2\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0}\right)\delta_{lm}\epsilon_{ik}\epsilon_{lm} \\
& + \frac{\hbar^2}{4m}\frac{(\nabla\rho_0)^2}{\rho_0}M_{iklm}u_{ik}u_{lm} + 2\mu_0\rho_0(\delta_{ik}\delta_{lm} - M_{iklm})u_{ik}u_{lm} \\
& + \rho_1^{lm}\left[\frac{\hbar^2}{4m}\left(2\frac{\partial_{ik}\rho_0}{\rho_0} - \frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\right) + (f_{ik}+\delta_{ik}U)*\rho_0 + 2\mu_0\delta_{ik} + \mu_1^{ik}\right]\epsilon_{ik}\epsilon_{lm}\Bigg],
\end{aligned}
\tag{A24}
$$

which will be used in the calculation of the elastic constants. To summarize, we have expanded our Lagrangian up to second order in the strain tensor and collected all the terms. We have solved for $\tilde{\rho}$ such that the new state is a ground state with the strained normalization condition, and subsequently re-expressed the second-order Lagrangian with that solution in hand.

APPENDIX B: CONNECTING CAUCHY ELASTIC TENSOR TO EXPANSION

This section closely follows the calculation done by Bavaud *et al.* [31]. To begin with, we define an analytic expansion in the free energy of our system around a deformation $x'_\alpha = (\delta_{\alpha\beta} - u_{\alpha\beta})x_\beta$ with a mapping $\Omega \mapsto \Omega'$,

$$
F(\Omega') = F(\Omega) - |\Omega|\pi_{\alpha\beta}(\Omega)u_{\alpha\beta} + |\Omega|\tfrac{1}{2}A_{\alpha\beta\gamma\delta}(\Omega)u_{\alpha\beta}u_{\gamma\delta},
\tag{B1}
$$

where

$$
\pi_{\alpha\beta}(\Omega) = \frac{1}{|\Omega|}\frac{\partial F(\Omega')}{\partial(-u_{\alpha\beta})}\bigg|_{u=0} = -\frac{1}{|\Omega|}\frac{\partial F(\Omega')}{\partial u_{\alpha\beta}}\bigg|_{u=0},
\tag{B2}
$$

and similarly

$$
A_{\alpha\beta\gamma\delta}(\Omega) = \frac{1}{|\Omega|}\frac{\partial^2 F(\Omega')}{\partial u_{\alpha\beta}u_{\gamma\delta}}\bigg|_{u=0}.
\tag{B3}
$$

To clarify, we map between spaces $\Omega \mapsto \Omega'$ through the transformation $\mathbf{x} \mapsto \mathbf{x}'$.

It is important to note that the tensor $A_{\alpha\beta\gamma\delta}$ is *not* the elastic tensor. It is the expansion of the free energy due to the deformation, but the elastic tensor strictly connects the stress tensor to the strain. We are searching for Hooke's Law which maps strain in a deformed point to strain in an undeformed point but now with the notation

$$
\pi_{ik}(\Omega') = \pi_{ik}(\Omega) - B_{iklm}(\Omega)u_{lm}.
\tag{B4}
$$

An important additional feature of the elastic tensor B_{iklm} is that, for a sufficiently symmetric system, it has certain symmetry properties as described by Landau and Lifshitz [30], whereas the expansion tensor A_{iklm} *does not* necessarily have these symmetries.

We first define an additional two deformations, u' and u'' such that

$$
x''_\alpha = (\delta_{\alpha\beta} - u'_{\alpha\beta})x'_\beta, \quad x''_\alpha = (\delta_{\alpha\beta} - u''_{\alpha\beta})x_\beta.
\tag{B5}
$$

To clarify, we have an initial space Ω which we can deform with the transformation $\mathbf{u}$ into the space Ω'. We can then deform from $\Omega' \mapsto \Omega''$ with the deformation $\mathbf{u}'$, and we can *also* deform to the Ω'' space from the *original space* like $\Omega \mapsto \Omega''$ with the deformation $\mathbf{u}''$. We can combine the deformations to find the relation

$$
u''_{\alpha\beta} = u_{\alpha\beta} + u'_{\alpha\beta} - u'_{\alpha\sigma}u_{\sigma\beta}.
\tag{B6}
$$

We then use the above relation to expand the stress tensor that connects the Ω' and Ω'' space. Examining the expansion of the free energy from the Ω' space to the Ω'' space, we find

$$
F(\Omega'') = F(\Omega') - |\Omega'|\pi_{\alpha\beta}(\Omega')u'_{\alpha\beta} + \frac{|\Omega'|}{2}A_{\alpha\beta\gamma\delta}(\Omega')u'_{\alpha\beta}u'_{\gamma\delta}.
\tag{B7}
$$

Expanding $\pi_{\alpha\beta}(\Omega')$ and using Eqs. (B2) and (B6) we find

$$
\begin{aligned}
\pi_{\alpha\beta}(\Omega') &= -\frac{1}{|\Omega'|}\frac{\partial F(\Omega'')}{\partial u'_{\alpha\beta}}\bigg|_{u'=0} \\
&= -\frac{1}{|\Omega'|}\frac{\partial F(\Omega'')}{\partial u''_{\gamma\epsilon}}\frac{\partial u''_{\gamma\epsilon}}{\partial u'_{\alpha\beta}}\bigg|_{u'=0} \\
&= -\frac{1}{|\Omega'|}\frac{\partial F(\Omega'')}{\partial u''_{\gamma\epsilon}}(\delta_{\gamma\alpha}\delta_{\beta\epsilon} - \delta_{\gamma\alpha}\delta_{\beta\sigma}u_{\sigma\epsilon})\bigg|_{u'=0} \\
&= -\frac{1}{|\Omega'|}(\delta_{\beta\epsilon} - u_{\beta\epsilon})\frac{\partial F(\Omega'')}{\partial u''_{\alpha\epsilon}}\bigg|_{u'=0}.
\end{aligned}
\tag{B8}
$$

We can now turn our attention to the derivative in the above expression. Namely, since the derivative actually does not depend on u' explicitly (recall that we can map from $\Omega \mapsto \Omega''$ through u'' without needing u' at all), we can remove the evaluation at $u' = 0$. As such, the derivative is now not a number, but a function instead! This function describes the energy response of the mapping $\Omega \mapsto \Omega''$ through u'' for any arbitrary Ω'' and u''. That means it *must be equivalent* to the function which maps from $\Omega \mapsto \Omega'$ through u. Thus, we can write

$$
\frac{\partial F(\Omega'')}{\partial u''_{\alpha\epsilon}}\bigg|_{u'=0} = \frac{\partial F(\Omega')}{\partial u_{\alpha\epsilon}},
\tag{B9}
$$

and the final result from Eq. (B8) becomes

$$\pi_{\alpha\beta}(\Omega') = -\frac{1}{|\Omega'|}(\delta_{\beta\epsilon} - u_{\beta\epsilon})\frac{\partial F(\Omega')}{\partial u_{\alpha\epsilon}}. \tag{B10}$$

Since the derivative is no longer evaluated at $u = 0$, we now substitute in Eq. (B1) to find

$$\frac{\partial F(\Omega')}{\partial u_{\alpha\epsilon}} = -|\Omega|\pi_{\alpha\epsilon}(\Omega) + |\Omega|A_{\alpha\epsilon\gamma\delta}(\Omega)u_{\gamma\delta}. \tag{B11}$$

We now also expand the $1/|\Omega'|$ term to find

$$\frac{1}{|\Omega'|} = \frac{1}{|\Omega|}\frac{1}{\det(\mathbb{1} - \mathbf{U})} = \frac{1}{|\Omega|}[1 + \delta_{xy}u_{xy} + \mathcal{O}(2)], \tag{B12}$$

where the indices x and y are tensor indices of the same nature as α, β, etc. It is important to note that we are discarding terms of $\mathcal{O}(u^2)$ or higher as we are trying to derive Hooke's law, i.e., $F = kx$, and we do not want anharmonic terms.

We can now combine the results of Eqs. (B11) and (B12) into Eq. (B10) (discarding $\mathcal{O}(u_{ik}^2)$ terms) to find

$$\pi_{\alpha\beta}(\Omega') = \pi_{\alpha\beta} - (A_{\alpha\beta\gamma\delta} + \delta_{\gamma\beta}\pi_{\alpha\delta} - \delta_{\gamma\delta}\pi_{\alpha\beta})u_{\gamma\delta}, \tag{B13}$$

where everything on the RHS is a function of Ω. This is the expression of Hooke's Law as described in Eq. (B4) We can therefore find that

$$B_{iklm} = A_{iklm} + \pi_{im}\delta_{kl} - \pi_{ik}\delta_{lm}, \tag{B14}$$

with now the final task being to relate π_{ik} to the pressure tensor P_{ik}. Recalling that the stress tensor is just the negative of the pressure, we can write the final expression as

$$B_{iklm} = A_{iklm} + P_{ik}\delta_{lm} - P_{im}\delta_{kl}, \tag{B15}$$

which is the Cauchy elastic tensor. For the case of C_3 symmetry, we have the relation [30]

$$B_{xxxx} = B_{xxyy} + 2B_{xyxy}. \tag{B16}$$

APPENDIX C: TENSOR DIFFERENTIATION

We have shown in previous sections how to expand a Lagrangian under a strain tensor. Once we have that expansion, how do we recover the homogenized tensors as first and second derivatives: $\pi_{ik} = \partial F/\partial u_{ik}$ and $A_{iklm} = \partial^2 F/\partial u_{ik}\partial u_{lm}$?

We will now write out explicitly how expressions involving the tensors listed in Eq. (9) differentiate. Consider first $F = F_{ik}u_{ik}$ with $F_{ik} = F_{ki}$, its first derivative is $\partial F/\partial uik = F_{ik}$. Similarly, the first derivative of $F = F_{ik}\epsilon_{ik}$ is $\partial F/\partial u_{ik} = -F_{ik}$.

Let us now consider second derivatives. Suppose there is a term in the Lagrangian of the form $F = F_{iklm}u_{ik}u_{lm}$. We have the result $\partial^2 F/\partial u_{ik}\partial u_{lm} = 2F_{iklm}$. Consider now $F = F_{ik}\chi_{ik}$ where $\chi_{ik} = (u_{li}u_{kl} + u_{lk}u_{il})/2$. Since χ_{ik} is symmetric under $i \leftrightarrow k$, we can impose that F_{ik} is also symmetric under the same exchange.

$$\begin{aligned}\frac{\partial^2\chi_{ik}}{\partial u_{cd}\partial u_{ab}} &= \frac{1}{2}\frac{\partial}{\partial u_{cd}}(\delta_{la}\delta_{ib}u_{kl} + \delta_{la}\delta_{kb}u_{il} + \delta_{ka}\delta_{lb}u_{li} \\ &\quad + \delta_{ia}\delta_{lb}u_{lk}) \\ &= \frac{1}{2}(\delta_{ib}\delta_{kc}\delta_{ad} + \delta_{kb}\delta_{ci}\delta_{ad} + \delta_{ka}\delta_{bc}\delta_{id} + \delta_{ia}\delta_{bc}\delta_{kd}).\end{aligned} \tag{C1}$$

TABLE I. Top row lists terms that may appear in the expansion, and the bottom row shows their corresponding contributions to the elastic tensor after differentiation.

F	$F_{iklm}\epsilon_{ik}\epsilon_{lm}$	$F_{ik}\Delta_{ik}$	$F_{ik}\omega_{ik}$	$F_{ik}\chi_{ik}$
$\frac{\partial^2 F}{\partial u_{ik}\partial u_{lm}}$	$2F_{iklm}$	$F_{il}\delta_{km}$	$F_{km}\delta_{il}$	$F_{kl}\delta_{im} + F_{im}\delta_{kl}$

This allows us to obtain

$$\begin{aligned}\frac{\partial^2 F}{\partial u_{ik}\partial u_{lm}} &= \frac{1}{2}(F_{bc}\delta_{ad} + F_{cb}\delta_{ad} + F_{da}\delta_{bc} + F_{ad}\delta_{bc}) \\ &= F_{kl}\delta_{im} + F_{im}\delta_{kl},\end{aligned} \tag{C2}$$

using the symmetry of F_{ik} under exchange of indices. A similar calculation can be done for other tensor contractions involving the tensors in Eq. (9). This is listed in Table I.

It is important to briefly discuss the symmetry of each of these expressions. For terms that look like $F_{iklm}\epsilon_{ik}\epsilon_{lm}$, these already have all the symmetry properties we would want from our elastic tensors. However, we can immediately see that the contribution from $F_{ik}\Delta_{ik}$, $F_{ik}\omega_{ik}$ and $F_{ik}\chi_{ik}$ *do not* have the requisite symmetries. We instead need to *create* these symmetries from combinations of these tensors. This cannot be achieved *a priori* by adding in whatever we like. However, if we expect elastic theory to hold, then we would expect any expansion in any strain tensor (including ours) to lead us eventually back to symmetric expressions. This is a good test to check whether the expansion calculation has been done correctly.

APPENDIX D: NUMERICAL DEFORMATIONS

In our numerics, we use a spatial discretization consisting of a triangular lattice as a grid so that the numerical scheme is consistent with the symmetry of the supersolid phase under rotations. Therefore, each "pixel" is a parallelogram and the simulation cell is also a parallelogram with basis vectors $\mathbf{a}$ and $\mathbf{b}$ parallel to $(\sqrt{3}/2, 1/2)$ and (0,1) in Cartesian coordinates. This Appendix provides some algebraic details on how deformations $\mathbf{V}$ of the simulation cell can be used to extract the elastic tensors numerically using the energy change $F_2 \equiv Pu_{ll} + \frac{1}{2}A_{iklm}u_{ik}u_{lm}$ up to second order in the strain.

Let us write all positions and displacements in the basis of the triangular grid: $\mathbf{r} = \mathbf{r_a}\mathbf{a} + \mathbf{r_b}\mathbf{b}$ and $\mathbf{u} = \mathbf{u_a}\mathbf{a} + \mathbf{u_b}\mathbf{b}$. The strain tensor in this basis,

$$\mathbf{V} = \begin{pmatrix} \partial u_a/\partial r_a & \partial u_a/\partial r_b \\ \partial u_b/\partial r_a & \partial u_b/\partial r_b \end{pmatrix},$$

is related to the strain tensor in the Cartesian basis: $(\mathbf{U})_{\mathbf{ij}} \equiv \mathbf{u_{ij}}$ by

$$\mathbf{U} = \mathbf{T}^{-1}\mathbf{V}\mathbf{T}, \text{ with } \mathbf{T} = \begin{pmatrix} \frac{2}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & 1 \end{pmatrix}. \tag{D1}$$

In our numerics, we apply strains diagonal in the triangular-grid basis by deforming the simulation cell in the directions of $\mathbf{a}$ and $\mathbf{b}$. Using the above, we can calculate the energy changes we should expect to see. There are three

deformations that give linearly independent results.

$$\mathbf{V}=\begin{pmatrix}\epsilon & 0\\ 0 & 0\end{pmatrix}:\mathbf{F_2}=P\epsilon+\left(A_{xxxx}+\frac{A_{xyxy}}{3}\right)\frac{\epsilon^2}{2},$$

$$\mathbf{V}=\begin{pmatrix}0 & 0\\ 0 & \epsilon\end{pmatrix}:\mathbf{F_2}=P\epsilon+\left(A_{yyyy}+\frac{A_{xyxy}}{3}\right)\frac{\epsilon^2}{2},$$

$$\mathbf{V}=\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix}:\mathbf{F_2}=2P\epsilon+\left(\frac{A_{xxxx}+A_{yyyy}}{2}+A_{xxyy}\right)\epsilon^2. \tag{D2}$$

For our system with C_3 symmetry, there are only two independent elastic constants. Using Eqs. (B15) and (B16), we see that $A_{xxxx}=A_{xxyy}+P+2A_{xyxy}$. Therefore, we have three equations to solve for the elastic constants and the pressure for a given $\epsilon \ll 1$.

We have calculated these quantities at several values of ϵ to check for the self-consistency of this analysis. This gives a measure of the numerical errors of this technique.

A good question to ask is whether or not we are allowed to *a priori* assume the system is symmetric, and whether we are biasing towards a symmetric result. This is a valid question, but really what we are doing is seeking to compare the results we obtain with the analytical theory derived. Importantly, the theory assumes nothing about symmetry, so if the numerics agree with the theory, then this assumption is justified.

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