# ABSOLUTELY CONTINUOUS SPECTRUM OF A TYPICAL SCHRÖDINGER OPERATOR WITH AN OPERATOR-VALUED POTENTIAL

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*In memory of Sergei Naboko, our friend and colleague*

## 1. Main results

Let $\mathfrak{H}$ be a separable Hilbert space and let $V$ be a measurable function from $\mathbb{R}_+$ to the set of bounded self-adjoint operators on $\mathfrak{H}$. Measurability of $V$ means that the function $x \mapsto \langle V(x)h, h\rangle$ is measurable for each $h \in \mathfrak{H}$. We study the absolutely continuous spectrum of the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + \alpha V, \qquad V^* = V, \tag{1}$$

acting in the space $L^2(\mathbb{R}_+, \mathfrak{H})$. Here, $\alpha$ is a real parameter. We impose the condition

$$\int_{\mathbb{R}_+} \|V(x)\|^2 dx < \infty. \tag{2}$$

The domain of $H$ consists of $W_0^2(\mathbb{R}_+, \mathfrak{H})$-functions. This class of functions can be viewed as the countably infinite orthogonal sum of Sobolev spaces $W_0^2(\mathbb{R}_+)$. The generalized second derivatives of $W_0^2(\mathbb{R}_+)$-functions are square integrable and the functions themselves vanish at $x = 0$.

*Definition.* We say that an essential support of the absolutely continuous spectrum of the operator $H$ contains $[0,\infty)$, if the spectral projection $E_\alpha(\Omega)$ of $H$ corresponding to any Borel set $\Omega \subset [0,\infty)$ is different from zero $E_\alpha(\Omega) \neq 0$ as soon as the Lebesgue measure of $\Omega$ is positive.

Operators with square integrable potentials were studied by P. Deift and R. Killip [1] in the case where $\mathfrak{H} = \mathbb{C}$. The main result of [1] states that the absolutely continuous spectrum of the operator $-d^2/dx^2 + V$ covers the positive half-line $[0,\infty)$, if $V \in L^2(\mathbb{R}_+)$.

We consider the case where the space $\mathfrak{H}$ is infinitely dimensional and give a different proof of the following theorem by S. Denisov [4].

**Theorem 1.1.** *Let $V$ satisfy the condition* (2). *Then an essential support of the absolutely continuous spectrum of the operator* (1) *contains* $[0,\infty)$ *for almost every* $\alpha \in \mathbb{R}$.

Besides the article [4], one can also find a close discussion of similar operator families in the papers [7] and [8]. In all mentioned publications, the properties of the absolutely continuous spectrum are established for almost every value of the real parameter $\alpha$. However, if $\|V(x)\| \le C(1+|x|)^{-2/3-\delta}$ with $\delta > 0$, then the absolutely continuous spectrum fills the positive half-line $\mathbb{R}_+$ for all $\alpha$ (see [5]). Instead of using hyperbolic pencils considered in [4], we obtain Theorem 1.1 by an application of Lemma 2.1.

## 2. Auxiliary lemma

*Notations.* Throughout the text, $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of a complex number $z$. For a self-adjoint operator $B = B^*$ and a vector $g$ of a Hilbert space the expression $((B-k-i0)^{-1}g,g)$ is always understood as the limit

$$\Big((B-k-i0)^{-1}g,g\Big) = \lim_{\varepsilon\to 0}\Big((B-k-i\varepsilon)^{-1}g,g\Big), \qquad \varepsilon>0,\ k\in\mathbb{R}.$$

The following simple statement plays a very important role in our proof.

**Lemma 2.1.** *Let $B$ be a self-adjoint operator in a separable Hilbert space $\mathfrak{H}$ and let $g\in\mathfrak{H}$. Then the function*

$$\eta(k) := \operatorname{Im}\Big((B-k-i0)^{-1}g,g\Big) \geq 0$$

*is integrable over $\mathbb{R}$. Moreover,*

$$\int_{-\infty}^{\infty}\eta(k)dk \leq \pi||g||^2.$$

*and*

$$\int_{-\infty}^{\infty}\frac{\eta(k)}{k^2+1}dk \leq \pi||(B^2+I)^{-1/2}g||^2.$$

*Proof.* Let $E_B(\cdot)$ be the spectral measure of the operator $B$. Then

$$\Big((B-z)^{-1}g,g\Big) = \int_{\mathbb{R}}(t-z)^{-1}d\big(E_B(-\infty,t)g,g\big), \quad z\in\mathbb{C}\setminus\mathbb{R}.$$

Therefore, according to the Stieltjes-Perron inversion formula,

$$\pi^{-1}\eta(k) = \frac{d}{dk}\big(E_B(-\infty,k)g,g\big), \qquad \text{for almost every } k\in\mathbb{R}.$$

Consequently, for any nonnegative measurable function $f$ on $\mathbb{R}$,

$$\int_{\mathbb{R}} f(k)\eta(k)dk \leq \pi\int_{\mathbb{R}} f(k)d\big(E_B(-\infty,k)g,g\big) = \pi\big(f(B)g,g\big).$$

□

## 3. Entropy

Let $\mu$ be a nonegative finite Borel measure on the real line $\mathbb{R}$. As any other measure it is decomposed uniquely into a sum of three terms

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sc},$$

where the first term is pure point, the second term is absolutely continuous and the last term is a continuous but singular measure on $\mathbb{R}$. Obviously, $\mu(-\infty,\lambda)$ is a monotone function of $\lambda$, therefore, it is differentiable almost everywhere. In particular, the limit

$$\mu'(\lambda) = \lim_{\epsilon\to 0}\frac{\mu(\lambda-\epsilon,\lambda+\epsilon)}{2\epsilon}$$

exists for almost every $\lambda\in\mathbb{R}$. It is also clear that

$$\mu_{ac}(\Omega) = \int_{\Omega}\mu'(\lambda)\,d\lambda, \qquad \forall\Omega\subset\mathbb{R},$$

which means $\mu' = \mu'_{ac}$.

Let $\Omega_0 = \{\lambda : \ \mu'(\lambda) > 0\}$ A measurable set $\Omega \subset \mathbb{R}$ is called an essential support of $\mu_{ac}$, if the Lebesgue measure of the symmetric difference

$$\Omega_0 \triangle \Omega := \Big(\Omega_0 \setminus \Omega\Big) \cup \Big(\Omega \setminus \Omega_0\Big)$$

is zero. So, an essential support of $\mu_{ac}$ coincides with the set where $\mu' > 0$ up to a set of measure zero. As we see, the study of the essential support of the a.c. part of the measure $\mu$ is reduced to the study of the set $\Omega_0 = \{\lambda : \ \mu'(\lambda) > 0\}$. Let $\Omega$ be a measurable set. One of the ways to show that $\mu'(\lambda) > 0$ for almost every $\lambda \in \Omega$ relies on the study of the quantity

$$S_\Omega(\mu) := \int_\Omega \log \mu'(\lambda)\, d\lambda.$$

Due to Jenssen's inequality, $S_\Omega < \infty$ for sets of finite Lebesgue measure $|\Omega| < \infty$. So, the entropy in this case can diverge only to the negative infinity.

But if

$$S_\Omega(\mu) > -\infty, \qquad \text{while} \quad |\Omega| < \infty,$$

then

$$\mu'(\lambda) > 0 \qquad \text{a.e. on } \Omega.$$

Very often one can obtain an estimate for $\mu'$ by an analytic function from below. In this case we will use the following statement

**Proposition 3.1.** *Let a function $F(\lambda) \neq 0$ be analytic in the neighborhood of an interval $[a, b] \subset \mathbb{R}$. Suppose that*

$$\mu'(\lambda) > |F(\lambda)|^2, \qquad \text{for all } \lambda \in \Omega \subset [a, b]. \tag{3}$$

*Then*

$$S_\Omega(\mu) := \int_\Omega \log \mu'(\lambda)\, d\lambda \geq C > -\infty,$$

*where the constant $C = C([a, b], F)$ depends on the interval $[a, b]$ and the function $F$.*

*Proof.* This proposition follows from the fact that zeros of analytic functions are isolated and have finite multiplicities. □

In applications to Schrödinger operators, one often has an estimate of the form (3) for a sequence of measures $\mu_n$ that converges to $\mu$ weakly

$$\mu_n \to \mu \qquad \text{weakly}.$$

In this situation, one can still derive a certain information about the limit measure $\mu$ from the information about $\mu_n$.

*Definition.* Let $\rho$, $\nu$ be finite Borel measures on a compact Hausdorff space, $X$. We define the entropy of $\rho$ relative to $\nu$ by

$$S(\rho|\nu) = \begin{cases} -\infty, & \text{if } \rho \text{ is not } \nu-\text{ac} \\ -\int_X \log(\frac{d\rho}{d\nu}) d\rho, & \text{if } \rho \text{ is } \nu-\text{ac}. \end{cases} \tag{4}$$

**Theorem 3.1.** (cf.[6]) *The entropy $S(\rho|\nu)$ is jointly upper semi-continuous in $\rho$ and $\nu$ with respect to the weak topology. That is, if $\rho_n \to \rho$ and $\nu_n \to \nu$ as $n \to \infty$ weakly, then*

$$S(\rho|\nu) \geq \limsup_{n\to\infty} S(\rho_n|\nu_n).$$

Now, we will use this theorem in order to prove the following statement.

**Proposition 3.2.** *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic in the neighborhood of $[a, b]$. Let $\mu_n$ be a sequence of finite nonnegative Borel measures on the real line $\mathbb{R}$ converging to $\mu$ weakly. Suppose that*

$$\mu_n'(\lambda) > |F(\lambda)|^2, \qquad \text{for all } \lambda \in \Omega_n \subset [a, b],$$

*where the measurable sets $\Omega_n$ satisfy*

$$\Big|[a, b] \setminus \Omega_n\Big| < b - a - \varepsilon.$$

*Then $\mu'(\lambda) > 0$ on a subset of $[a, b]$ whose measure is not smaller than $b - a - \varepsilon$*

*Proof.* Let us denote the characteristic function of the set $\Omega_n$ by $\chi_n$. Since $L^2$-norms of $\chi_n$ are uniformly bounded, this sequence of functions has a weakly convergent subsequence. Therefore without loss of generality, one can assume that

$$\chi_n \to \chi, \qquad \text{weakly in} \quad L^2(\mathbb{R}).$$

This, of course, implies that the corresponding measures $\chi_n d\lambda$ also converge weakly to $\chi d\lambda$. Even though, $\mathbb{R}$ is not compact, we can still use Theorem 3.1 and show (see [7]) that

$$\int_{\mathbb{R}} \log\Big(\frac{\mu'(\lambda)}{\chi(\lambda)}\Big)\, \chi(\lambda)\, d\lambda \geq \limsup_{n\to\infty} \int_{\mathbb{R}} \log\Big(\frac{\mu_n'(\lambda)}{\chi_n(\lambda)}\Big)\, \chi_n(\lambda)\, d\lambda > -\infty$$

Thus, we see that $\mu' > 0$ on the support of the function $\chi$. However, we still need to know how big this set is. For that purpose, we first observe that

$$\int_a^b \chi(\lambda)\, d\lambda = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_n(\lambda)\, d\lambda \geq b - a - \varepsilon.$$

On the other hand, it is easy to show that $0 \leq \chi \leq 1$. Therefore, the Lebesgue measure of the support of the function $\chi$ is not smaller than $b - a - \varepsilon$. $\square$

Since we deal with a family of operators depending on a parameter $\alpha$, we also need a modification of the previous statement, suitable in the case when measures depend on the parameter $\alpha$ as well. Let $\mathcal{M}$ be the topological space whose elements are nonnegative Borel measures $\mu$ on $\mathbb{R}$ having the property $\mu(\mathbb{R}) = 1$. We define the topology on $\mathcal{M}$ to be the one that is induced by the weak-$*$ topology. Finally, let $\mathfrak{M}(\mathbb{R})$ be the class of continuous functions from $\mathbb{R}$ to $\mathcal{M}$. We are ready to state the following result.

**Proposition 3.3.** *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic in the neighborhood of $[a, b]$. Let $\mu_n(\cdot, \alpha)$ be a sequence of $\alpha$-dependent families of finite nonegative Borel measures on $\mathbb{R}$ converging to $\mu(\cdot, \alpha)$ weakly for every $\alpha \in \mathbb{R}$. Suppose the function $\alpha \mapsto \mu_n(\cdot, \alpha)$ belongs to $\mathfrak{M}(\mathbb{R})$ for each $n \in \mathbb{N}$. Finally, assume that the derivative of $\mu_n$ with respect to $d\lambda$ satisfies*

$$\mu_n'(\lambda, \alpha) > |F(\lambda)|^2, \qquad \text{for all } (\lambda, \alpha) \in \Omega_n \subset [a, b] \times [\alpha_1, \alpha_2],$$

*where the measurable sets $\Omega_n$ obey*

$$\Big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega_n\Big| < (b - a)(\alpha_2 - \alpha_1) - \varepsilon.$$

*Then $\mu'(\lambda, \alpha) > 0$ on a subset of $[a, b] \times [\alpha_1, \alpha_2]$ whose measure is not smaller than $(b-a)(\alpha_2-\alpha_1)-\varepsilon$.*

The proof of this statement is a counterpart of the proof of the preceding proposition and it is left to the reader as an exercise. A similar statement is proven in [7].

We conclude this section by a discussion of the following simple claim.

**Proposition 3.4.** *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic on a neighborhood of the interval $[a, b]$. Let $\mu(\cdot, \alpha)$ be an $\alpha$-dependent family of finite nonegative measures on $\mathbb{R}$. Suppose that the derivatives of $\mu$ with respect to the Lebesgue measure $d\lambda$ satisfy the estimate*

$$\mu'(\lambda, \alpha) \geq |F(\lambda)|^2(1 - \Psi(\lambda, \alpha)), \qquad \text{where} \quad \int_{\alpha_1}^{\alpha_2} \int_a^b |\Psi(\lambda, \alpha)| d\lambda d\alpha < \varepsilon/2.$$

*Then*

$$\mu'(\lambda, \alpha) \geq \frac{1}{2}|F(\lambda)|^2, \qquad \text{for all} \quad (\lambda, \alpha) \in \Omega,$$

*where the measureable set $\Omega$ obeys*

$$\Big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega\Big| \leq \varepsilon. \tag{5}$$

*Proof.* According to Chebyshev's inequality,

$$\Psi(\lambda, \alpha) \leq 1/2$$

on a set satisfying the condition (5). □

## 4. The case of a compactly supported $V$

In this section, we assume that $V$ belongs to the class $\mathfrak{V}$ described below.

**Definition.** We say that a bounded measurable function $V$ from $\mathbb{R}_+$ to the set of bounded self-adjoint operators on $\mathfrak{H}$ belongs to the class $\mathfrak{V}$ if

1) there is a bounded interval $[0, R]$ containing the support of $V$ and such that $V(x + R/2)$ is an odd function of $x$:

$$V(x + R/2) = -V(-x + R/2), \qquad \forall x \in [0, R/2]. \tag{6}$$

2) the range of the operator $V(x)$ is a finite dimensional subspace $\mathfrak{H}_0 \subset \mathfrak{H}$ which stays the same when one changes $x$.

Our proof of Theorem 1.1 is based on the relation between the derivative of the spectral measure and the so called scattering amplitude. Both objects should be introduced properly. While the spectral measure can be defined for any self-adjoint operator, the scattering coefficient will be introduced only for a Schrödinger operator. Let $f$ be a square integrable function from $\mathbb{R}_+$ to $\mathfrak{H}$. It is very well known that the quadratic form of the resolvent of $H$ can be written as a Cauchy integral

$$((H - z)^{-1} f, f) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{t - z}, \qquad \operatorname{Im} z \neq 0.$$

The measure $\mu$ in this representation is called the spectral measure of $H$ corresponding to the element $f$.

Let us introduce the scattering amplitude. Since the support of the potential $V$ is compact, there exists an $R$, such that $V(x) = 0$ for $x > R$. Take any bounded compactly supported function $f$ that also vanishes for $x > R$. Then

$$\big[(H - z)^{-1} f\big](x) = e^{ik|x|} A_f(k), \qquad \text{for } x > R,\ k^2 = z,\ \operatorname{Im} k \geq 0,\ A_f(k) \in \mathfrak{H}. \tag{7}$$

Clearly, the relation

$$\mu'(\lambda) = \pi^{-1} \lim_{z \to \lambda + i0} \operatorname{Im}((H - z)^{-1} f, f) = \pi^{-1} \lim_{z \to \lambda + i0} \operatorname{Im} z ||(H - z)^{-1} f||^2 \tag{8}$$

implies the formula

$$\pi \mu'(\lambda) = \sqrt{\lambda} \|A_f(k)\|^2, \qquad k^2 = \lambda > 0. \tag{9}$$

To prove (9), define $\chi_X$ to be the characteristic function of a set $X \subset \mathbb{R}_+$. Since the limit

$$\lim_{z\to\lambda+i0} ||\chi_{[0,b]}(H-z)^{-1}f||^2$$

(along the vertical directions) exists and is finite for each $b>0$, we infer from (8) that

$$\mu'(\lambda) = \pi^{-1} \lim_{z\to\lambda+i0} \operatorname{Im} z ||\chi_{[R,\infty)}(H-z)^{-1}f||^2.$$

Now (9) follows by (7), since

$$||\chi_{[R,\infty)}(H-z)^{-1}f||^2 = \frac{e^{-2\operatorname{Im} kR}}{2\operatorname{Im} k}\|A_f(k)\|^2, \qquad \text{for } \operatorname{Im} k > 0.$$

The remaining arguments in this paper will be devoted to a lower estimate of $\|A_f(k)\|$.

For our purposes, it is sufficient to assume that $f$ is the product of the characteristic function of the unit interval $[0,1]$ times a unit vector $\tau \in \mathfrak{H}$. Traditionally, $H$ is viewed as an operator obtained by a perturbation of

$$H_0 = -\frac{d^2}{dx^2}.$$

In its turn, $(H-z)^{-1}$ can be viewed as an operator obtained by a perturbation of $(H_0-z)^{-1}$. The theory of such perturbations is often based on the second resolvent identity

$$(H-z)^{-1} = (H_0-z)^{-1} - (H-z)^{-1}\,\alpha V(H_0-z)^{-1}, \tag{10}$$

which turns out to be useful for our reasoning. As a consequence of (10), we obtain that

$$A_f(k) = F_0(k)\tau - A_g(k), \qquad z = k^2+i0,\ k>0, \tag{11}$$

where $g(x) = \alpha V(H_0-z)^{-1}f$ and the number $F_0(k) \in \mathbb{C}$ is defined by

$$(H_0-z)^{-1}f = e^{ik|x|}F_0(k)\tau, \qquad \text{for } x>1. \tag{12}$$

We will shortly show that, without loss of generality, one can assume that $V(x)\tau = 0$ inside the unit interval $[0,1]$. In this case,

$$g = F_0(k)h_k, \quad \text{where} \quad h_k(x) = \alpha e^{ik|x|}V\tau. \tag{13}$$

According to (11),

$$2\|A_f(k)\|^2 \geq |F_0(k)|^2 - 2\|A_g(k)\|^2,$$

which can be written in the form

$$2\pi\mu'(\lambda) \geq |F_0(k)|^2\Big(\sqrt{\lambda} - 2\operatorname{Im}\Big((H-z)^{-1}h_k,\, h_k\Big)\Big), \qquad z = \lambda+i0, \tag{14}$$

due to (9) and (13). Therefore, in order to establish the presence of the absolutely continuous spectrum, we need to show that the quantity $\operatorname{Im}\Big((H-z)^{-1}h_k,\, h_k\Big)$ is small.

Let us define $\eta$ setting

$$\alpha^2 k^{-2}\eta(k,\alpha) := \frac{1}{k}\operatorname{Im}\Big((H-z)^{-1}h_k,\, h_k\Big) \geq 0, \qquad z = k^2+i0.$$

Obviously, $\eta$ is positive for all real $k \neq 0$, because we agreed that $z = k^2 \pm i0$ if $\pm k > 0$. This is very convenient. Since $\eta \geq 0$, we can conclude that $\eta$ is small on a rather large set if the integral of this function is small. That is why we will estimate

$$J(V) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta(k,\alpha)}{(\alpha^2+k^2)}\frac{|k|\,dk d\alpha}{(k^2+1)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta(k,tk)}{(k^2+1)(t^2+1)}\,dkdt. \tag{15}$$

We will employ a couple of tricks, one of which is related to the involvment of an additional parameter $\varepsilon$. Instead of dealing with the operator $H$, we will deal with $H+\varepsilon I$ where $\varepsilon>0$ is small. We will first obtain an integral estimate for the quantity

$$\eta_\varepsilon(k,\alpha)=\frac{k}{\alpha^2}\mathrm{Im}\Big(\big(H+\varepsilon-z\big)^{-1}h_k,\,h_k\Big),\qquad z=k^2+i0.$$

Then, since

$$\eta(k,\alpha)=\lim_{\varepsilon\to0}\eta_\varepsilon(k,\alpha)\qquad \text{a.e. on } \mathbb{R}\times\mathbb{R},$$

we conclude by Fatou's Lemma that

$$J(V)\le\liminf_{\varepsilon\to0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\eta_\varepsilon(k,\alpha)}{(\alpha^2+k^2)}\frac{|k|\,dkd\alpha}{(k^2+1)}.$$

The second trick is to set $\alpha=kt$ and represent $\eta_\varepsilon$ in the form

$$\eta_\varepsilon(k,kt)=\mathrm{Im}\Big((B+1/k-i0)^{-1}H_\varepsilon^{-1/2}v,\,H_\varepsilon^{-1/2}v\Big)\tag{16}$$

where $v=V\tau$, $H_\varepsilon=-d^2/dx^2+\varepsilon I$ and $B$ is the bounded selfadjoint operator defined by

$$B=H_\varepsilon^{-1/2}\Big(-2i\frac{d}{dx}+tV\Big)H_\varepsilon^{-1/2}.$$

This operator is bounded, because $H_\varepsilon^{-1/2}$ is a continuous mapping from $L^2(\mathbb{R}_+,\mathfrak{H})$ to $W_0^1(\mathbb{R}_+,\mathfrak{H})$, while the middle factor $(-2i\frac{d}{dx}+tV)$ is a continuous mapping from $W_0^1(\mathbb{R}_+,\mathfrak{H})$ to $L^2(\mathbb{R}_+,\mathfrak{H})$. Since the quadratic form of the operator $B$ is real, this operator is symmetric, and hence it is self-adjoint.

In order to justify (16) at least formally, one has to introduce the operator $U$ of multiplication by the function $\exp(ikx)$. Using this notation, we can represent $\eta_\varepsilon$ in the following form

$$\eta_\varepsilon(k,tk)=k\mathrm{Im}\Big(U^{-1}(H+\varepsilon-z)^{-1}Uv,\,v\Big),\qquad z=k^2+i0.$$

Since we deal with a unitary equivalence of operators, we can employ the formula

$$\Big(U^{-1}(H+\varepsilon-z)^{-1}Uv,v\Big)=\Big((U^{-1}HU+\varepsilon-z)^{-1}v,v\Big),\qquad z=k^2+i0.$$

On the other hand, since $H$ is a differential operator and $U$ is an operator of multiplication, the commutator $[H,U]:=HU-UH$ can be easily found:

$$\Big[H,U\Big]=kU\Big(-2i\frac{d}{dx}+k\Big)\Big|_{D(H)}\qquad\text{on } D(H).\tag{17}$$

Using the formula $U^{-1}HU=H+U^{-1}[H,U]$, we infer from (17) that

$$U^{-1}HU+\varepsilon-z=H_\varepsilon+k\Big(-2i\frac{d}{dx}+tV\Big)=H_\varepsilon^{1/2}(I+kB)H_\varepsilon^{1/2}.$$

If $\tilde k$ belongs to the upper half plane then so does $-1/\tilde k$. Consequently,

$$k\Big(U^{-1}(H+\varepsilon-z)^{-1}Uv,v\Big)=\Big(H_\varepsilon^{-1/2}(B+1/k-i0)^{-1}H_\varepsilon^{-1/2}v,v\Big),\qquad z=k^2+i0.\tag{18}$$

In fact, (18) holds for $\mathrm{Im}\,k>0$ when $U$ is not a unitary operator, but we only need it for $k\in\mathbb{R}$.

Since $B$ is a self-adjoint operator, $\pi^{-1}\eta_\varepsilon(k,kt)$ coincides with the derivative of the spectral measure of the operator $B$ corresponding to the element $H_\varepsilon^{-1/2}v$. According to Lemma 2.1, the latter observation implies that

$$\int_{-\infty}^{\infty}\frac{\eta_\varepsilon(k,kt)}{(1+k^2)}dk\le\pi\Big((B^2+I)^{-1}H_\varepsilon^{-1/2}v,H_\varepsilon^{-1/2}v\Big),$$

which leads to

$$\int_{-\infty}^{\infty}\frac{\eta_\varepsilon(k,kt)}{(1+k^2)}dk\le\pi\Big(B^{-1}H_\varepsilon^{-1/2}v,B^{-1}H_\varepsilon^{-1/2}v\Big)=\pi||B^{-1}H_\varepsilon^{-1/2}v||^2,\tag{19}$$

provided $B$ is invertible. Our further arguments will be related to the estimate of the quantity in the right hand side of (19). We will show now that $B$ has an unbounded inverse having the property

$$\lim_{\varepsilon\to 0}||B^{-1}H_\varepsilon^{-1/2}v||^2\le\int_{\mathbb{R}_+}\|V(x)\|^2dx,\qquad D(B^{-1})=\ \mathrm{Ran}(B)\subset W_0^1(\mathbb{R}_+,\mathfrak{H}).\tag{20}$$

Our proof of (20) is based on the representation

$$B^{-1}H_\varepsilon^{-1/2}v=H_\varepsilon^{1/2}T^{-1}v,\tag{21}$$

where $T\subset T^*$ is the first order differential (symmetric) operator defined by

$$T=-2i\frac{d}{dx}+tV,\qquad D(T)=D(H_\varepsilon^{1/2})=W_0^1(\mathbb{R}_+,\mathfrak{H}).$$

As we will see, $H_\varepsilon^{-1/2}$ is a one-to-one mapping of $D(T^{-1})$ onto $D(B^{-1})$, and (21) holds for all $v\in D(T^{-1})$. To establish (21), observe that the equality $B=H_\varepsilon^{-1/2}TH_\varepsilon^{-1/2}$ leads to the relations $\mathrm{Ran}(B)\subset D(H_\varepsilon^{1/2})$ and $H_\varepsilon^{1/2}B=TH_\varepsilon^{-1/2}$. The latter of the two relations clearly implies (21) provided $T$ is invertible and $v\in D(T^{-1})$.

On the other hand, one can establish invertibility of $T$ by deriving an explicit formula for $T^{-1}$ (which is also an unbounded operator). For that purpose we define $U_0$ to be the unitary operator of multiplication by the solution of the differential equation

$$\frac{d}{dx}U_0(x)=\frac{it}{2}U_0(x)V(x),\qquad U_0(0)=I.$$

The object on the right hand side is the composition of two operators in $\mathfrak{H}$. The solution of this differential equation exists on all of $\mathbb{R}_+$ since the equation is linear and $V\in\mathfrak{V}$. Now we see that

$$T=-2iU_0^{-1}\Big[\frac{d}{dx}\Big]U_0,\qquad\text{and}\qquad T^{-1}=\frac{i}{2}U_0^{-1}\Big[\frac{d}{dx}\Big]^{-1}U_0.$$

Since $[\frac{d}{dx}]^{-1}$ is just the simple integration with respect to $x$ and $\frac{d}{dx}U_0\tau=\frac{i}{2}tU_0V\tau$,

$$\begin{aligned}\big[T^{-1}v\big](x)=\frac{i}{2}U_0^{-1}(x)\int_0^x U_0(y)V(y)\tau dy=\\ \frac{1}{t}U_0^{-1}(x)(U_0(x)-I)\tau=\frac{1}{t}(I-U_0^{-1}(x))\tau.\end{aligned}\tag{22}$$

Note that due to the condition (6), the function $T^{-1}v$ is compactly supported, which leaves no doubt about the relation $v\in D(T^{-1})$. Combining (21) with (22) and using the fact that $\|H_\varepsilon^{1/2}u\|^2=\|\frac{d}{dx}u\|^2+\varepsilon\|u\|^2$ for all $u\in D(H_\varepsilon^{1/2})$, we conclude that

$$\lim_{\varepsilon\to 0}||B^{-1}H_\varepsilon^{-1/2}v||^2=\lim_{\varepsilon\to 0}||H_\varepsilon^{1/2}T^{-1}v||^2=\int_{\mathbb{R}_+}\|V(x)U_0^{-1}(x)\tau\|^2dx.\tag{23}$$

Thus, (20) is established. The relations (19), (20) lead to the inequality

$$J(V)\le\pi^2\int_{\mathbb{R}_+}\|V(x)\|^2dx,$$

where the quantity $J(V)$ from (15). However, we can say more:

**Lemma 4.1.** *Let $T > 0$. Let $V$ be a potential of the class $\mathfrak{V}$ such that*

$$V(x)\tau = 0, \qquad \text{for all} \quad x < T. \tag{24}$$

*Then*

$$J(V) \le \pi^2 \int_T^\infty \|V(x)\|^2 dx. \tag{25}$$

*Proof.* If (24) holds, then $U_0(x)\tau = \tau$ for all $x < T$. Therefore, the right hand side of (23) can be estimated as follows

$$\int_{\mathbb{R}_+} \|V(x)U_0^{-1}(x)\tau\|^2 dx \le \int_T^\infty \|V(x)\|^2 dx.$$

□

## 5. Approximations of potentials and spectral measures

**Proposition 5.1.** *Let $T > 0$. Let $\tilde V$ be the potential*

$$\tilde V(x) = V(x) - \langle\cdot,\tau\rangle V(x)\tau - \langle\cdot, V(x)\tau\rangle\tau + \langle V(x)\tau,\tau\rangle\langle\cdot,\tau\rangle\tau, \qquad \text{for all} \quad x < T, \tag{26}$$

*and let*

$$\tilde V(x) = V(x), \qquad \text{for all} \quad x > T. \tag{27}$$

*Then*

$$\Big(H - z\Big)^{-1} - \Big(-\frac{d^2}{dx^2} + \alpha\tilde V - z\Big)^{-1} \in \mathfrak{S}_1 \tag{28}$$

*is a trace class operator for any $z$ with* $\operatorname{Im} z > 0$.

*Proof.* Using Hilbert's identity, we obtain

$$\Big(H - z\Big)^{-1} - \Big(-\frac{d^2}{dx^2} + \alpha\tilde V - z\Big)^{-1} = \alpha\Big(H - z\Big)^{-1}(\tilde V - V)\Big(-\frac{d^2}{dx^2} + \alpha\tilde V - z\Big)^{-1}.$$

Consequently, it is sufficient to prove that

$$\Gamma := \Big(-\frac{d^2}{dx^2} - z\Big)^{-1}(\tilde V - V)\Big(-\frac{d^2}{dx^2} - z\Big)^{-1} \in \mathfrak{S}_1.$$

Observe now that $\tilde V(x) - V(x)$ is a finite rank operator of the form

$$\tilde V(x) - V(x) = w_1(x)\langle\cdot, e_1(x)\rangle e_1(x) + w_2(x)\langle\cdot, e_2(x)\rangle e_2(x), ,$$

where $w_j \in L^1(\mathbb{R}_+)$ are real valued compactly supported functions and $e_j(x)$ are unit vectors in $\mathfrak{H}$. Since $(-\frac{d^2}{dx^2} - z)^{-1}$ is an integral operator whose integral kernel $r(x,y)$ satisfies

$$\sup_x \int_0^\infty |r(x,y)|^2 dy + \sup_y \int_0^\infty |r(x,y)|^2 dx < \infty,$$

the operators $G_j(z)$ defined by

$$\big[G_j(z)u\big](x) = \int_0^\infty |w_j(x)|^{1/2}\langle r(x,y)u(y), e_j(x)\rangle e_j dy$$

are Hilbert-Schmidt operators. It remains to note that

$$\Gamma = G_1^*(\bar z)\Omega_1 G_1(z) + G_2^*(\bar z)\Omega_2 G_2(z)$$

where $\Omega_j$ are bounded. □

According to Birman's theorem (see [2],[3]), we can now state the following result.

**Proposition 5.2.** *Let $\tilde V$ be defined as in (26). Then the absolutely continuous parts of the operators $H$ and $-\frac{d^2}{dx^2}+\alpha\tilde V$ are unitary equivalent.*

Let $\delta>0$. The latter proposition allows one to assume that there is a $T>0$ having the following properties:
1) $V(x)\tau=0$ for all $x<T$.
2) the value of the integral $\int_T^\infty\|V(x)\|^2dx$ is smaller than $\delta$.
If that is not true, we replace $V$ by $\tilde V$ defined by (26) for a sufficiently large $T>0$.

Now we use the inequality (14) and employ Proposition 3.4 with

$$F(\lambda)=(2\pi)^{-1/2}F_0(\sqrt{\lambda})\lambda^{1/4}\qquad\text{and}\qquad \Psi(\lambda)=\frac{2\mathrm{Im}((H-z)^{-1}h_k,h_k)}{\sqrt{\lambda}}.$$

According to Lemma 4.1, we obtain the following result.

**Theorem 5.1.** *Let $0<a<b<\infty$, let $0<\alpha_1<\alpha_2<\infty$ and let $T>1$. For any $\varepsilon>0$ there is a number $\delta>0$ such that for any potential $V$ of the class $\mathfrak{V}$ having the properties*

$$1)\quad V(x)\tau=0\qquad\text{for all}\quad x<T,\qquad\text{and}\quad 2)\quad \int_T^\infty\|V(x)\|^2dx<\delta,$$

*the derivative $\mu'(\lambda)=\mu'(\lambda,\alpha)$ of the spectral measure satisfies the inequality*

$$\mu'(\lambda,\alpha)\geq(4\pi)^{-1}|F_0(\sqrt{\lambda})|^2\lambda^{1/2},\qquad\text{for all}\quad(\lambda,\alpha)\in\Omega,$$

*where the measurable set $\Omega$ obeys*

$$\big|[a,b]\times[\alpha_1,\alpha_2]\setminus\Omega\big|\leq\varepsilon.$$

The proof of the next statement is left to the reader as an exercise.

**Proposition 5.3.** *Let $V$ be a measurable operator-valued function obeying*

$$\int_{\mathbb{R}_+}\|V(x)\|^2dx<\infty.$$

*Assume that*

$$V(x)\tau=0,\qquad\text{for all}\quad x<T,\tag{29}$$

*where $T>0$ is a fixed number. Then there is a sequence of compactly supported operator-valued functions $V_n\in\mathfrak{V}$ having the following three properties:*

*1)*

$$V_n(x)\tau=0,\qquad\text{for all}\quad x<T,$$

*2)*

$$\int_T^\infty\|V_n(x)\|^2dx\leq 2\int_T^\infty\|V(x)\|^2dx,\tag{30}$$

*and*

*3)*

$$\int_0^K\|\big(V_n(x)-V(x)\big)u(x)\|^2dx\to0,\qquad\text{as}\quad n\to\infty,\qquad\text{for any } u\in L^\infty(\mathbb{R}_+,\mathfrak{H})\quad\text{and any } K>0.$$

Another statement, that we are going to use, deals with the spectral measures of operators whose potentials $V_n$ approximate the function $V$.

**Proposition 5.4.** *Let $V \in L^2(\mathbb{R}_+,\mathfrak{H})$ and $V_n \in L^2(\mathbb{R}_+,\mathfrak{H})$ obey (30) for some $T>0$. Let $\mu_n$ and $\mu$ be the spectral measures of the operators $H_n$ and $H$ with potentials $\alpha V_n$ and $\alpha V$, correspondingly. Assume that*

$$\int_0^K \|\big(V_n(x)-V(x)\big)u(x)\|^2dx \to 0, \qquad \text{as} \quad n\to\infty, \qquad \text{for any } u \in L^\infty(\mathbb{R}_+,\mathfrak{H}) \quad \text{and any } K>0.$$

*Then*

$$\mu_n \to \mu \qquad \text{weakly}, \quad \text{as} \quad n\to\infty, \quad \text{for all} \quad \alpha\in\mathbb{R}.$$

The *proof* of this proposition is rather standard. First observe that the set of finite linear combinations of functions of the form $\phi_z(t)=\mathrm{Im}\Big(1/(t-z)\Big)$ with $\mathrm{Im} z>0$ is dense in the space of functions that are continuous on $\mathbb{R}$ and decay at infinity. Consequently, it suffices to show that

$$\int_{\mathbb{R}} \phi_z(t)d\mu_n(t) \to \int_{\mathbb{R}} \phi_z(t)d\mu(t), \qquad \text{as} \quad n\to\infty$$

for each $z\in\mathbb{C}_+$. According to the definition of the measures $\mu_n$ and $\mu$, that is the same as showing that

$$\mathrm{Im}\Big((H_n-z)^{-1}f,f\Big) \to \mathrm{Im}\Big((H-z)^{-1}f,f\Big), \qquad \text{as} \quad n\to\infty.$$

The latter follows from the identity

$$\Big((H_n-z)^{-1}f,f\Big)-\Big((H-z)^{-1}f,f\Big)=\Big((H_n-z)^{-1}(V-V_n)(H-z)^{-1}f,f\Big),$$

since the condition $(H-z)^{-1}f\in W_0^1(\mathbb{R}_+,\mathfrak{H})$ implies that $\|(V-V_n)(H-z)^{-1}f\|\to 0$ as $n\to\infty$.
□

According to Proposition 3.2, the assertion below follows from Theorem 5.1 combined with Propositions 5.3 and 5.4.

**Theorem 5.2.** *Let $0<a<b<\infty$, let $0<\alpha_1<\alpha_2<\infty$ and let $T>1$. For any $\varepsilon>0$ there is a number $\delta>0$ such that for any potential $V\in L^2(\mathbb{R}_+,\mathfrak{H})$ having the properties*

$$1)\quad V(x)\tau=0 \qquad \text{for all} \quad x<T, \qquad \text{and} \quad 2) \quad \int_T^\infty \|V(x)\|^2dx<\delta,$$

*the derivative $\mu'(\lambda)=\mu'(\lambda,\alpha)$ of the spectral measure is positive*

$$\mu'(\lambda,\alpha)>0, \qquad \text{for all} \quad (\lambda,\alpha)\in\Omega, \tag{31}$$

*where the measurable set $\Omega$ obeys*

$$\Big|[a,b]\times[\alpha_1,\alpha_2]\setminus\Omega\Big|\le\varepsilon.$$

Let $E_\alpha(\cdot)$ be the operator-valued spectral measure of $H$. Let also

$$\Omega_\alpha=\{\lambda\in[a,b]: \quad (\lambda,\alpha)\in\Omega\}$$

be the cross-section of $\Omega$. One can conclude from the inequality (31) that, for any measurable subset $X\subset[a,b]$, the condition $E_\alpha(X)=0$ implies the relation

$$\Big|\Omega_\alpha\cap X\Big|=0.$$

Using the unitary equivalence claimed by Proposition 5.2, we obtain

**Theorem 5.3.** *Let $0<a<b<\infty$, let $0<\alpha_1<\alpha_2<\infty$. Assume that $V\in L^2(\mathbb{R}_+,\mathfrak{H})$. Then for any $\varepsilon>0$, there is a measurable set $\Omega(\varepsilon)\subset[a,b]\times[\alpha_1,\alpha_2]$ obeying*

$$\Big|[a,b]\times[\alpha_1,\alpha_2]\setminus\Omega(\varepsilon)\Big|\le\varepsilon,$$

*such that, for any Borel set $X \subset [a, b]$ and the cross-section $\Omega_\alpha(\varepsilon)$ defined by*

$$\Omega_\alpha(\varepsilon) = \{\lambda \in [a, b] : \quad (\lambda, \alpha) \in \Omega(\varepsilon)\},$$

*the condition $E_\alpha(X) = 0$ implies the equality*

$$\big|\Omega_\alpha(\varepsilon) \cap X\big| = 0.$$

Take now a monotonically decreasing sequence $\varepsilon_n$ converging to 0, as $n \to \infty$, and set

$$\tilde{\Omega} = \bigcup_{n=1}^{\infty} \Omega(\varepsilon_n).$$

Obviously, $\tilde{\Omega}$ is a subset of full measure in $[a, b] \times [\alpha_1, \alpha_2]$. Consequently,

$$\tilde{\Omega}_\alpha = \{\lambda \in [a, b] : \quad (\lambda, \alpha) \in \tilde{\Omega}\}$$

is a subset of full measure in $[a, b]$ for almost every $\alpha \in [\alpha_1, \alpha_2]$.

Take now an arbitrary Borel subset $X \subset [a, b]$. If $|X \cap \tilde{\Omega}_\alpha| > 0$ then there is an integer number $n$ for which

$$\big|\Omega_\alpha(\varepsilon_n) \cap X\big| > 0.$$

The latter condition implies that $E_\alpha(X) \neq 0$. Thus, the essential support of the absolutely continuous spectrum of $H$ contains the interval $[a, b]$ for all $\alpha$ such that

$$|\tilde{\Omega}_\alpha| = b - a. \tag{32}$$

It remains to note that (32) holds for almost every $\alpha \in [\alpha_1, \alpha_2]$.

This completes the proof of Theorem 1.1

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