Efficient Quantum Algorithms for Stabilizer Entropies

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Stabilizer entropies (SEs) are measures of nonstabilizerness or "magic" that quantify the degree to which a state is described by stabilizers. SEs are especially interesting due to their connections to scrambling, localization and property testing. However, applications have been limited so far as previously known measurement protocols for SEs scale exponentially with the number of qubits. Here, we efficiently measure SEs for integer Rényi index $n > 1$ via Bell measurements. The SE of N-qubit quantum states can be measured with $O(n)$ copies and $O(nN)$ classical computational time, where for even n we additionally require the complex conjugate of the state. We provide efficient bounds of various nonstabilizerness monotones that are intractable to compute beyond a few qubits. Using the IonQ quantum computer, we measure SEs of random Clifford circuits doped with non-Clifford gates and give bounds for the stabilizer fidelity, stabilizer extent, and robustness of magic. We provide efficient algorithms to measure Clifford-averaged $4n$-point out-of-time-order correlators and multifractal flatness. With these measures we study the scrambling time of doped Clifford circuits and random Hamiltonian evolution depending on nonstabilizerness. Counterintuitively, random Hamiltonian evolution becomes less scrambled at long times, which we reveal with the multifractal flatness. Our results open up the exploration of nonstabilizerness with quantum computers.

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Stabilizer states and Clifford operations are essential to quantum information and quantum computing [1–3]. They are the cornerstone to run quantum algorithms on most fault-tolerant quantum computers, where Clifford operations are intertwined with non-Clifford gates [4,5]. To characterize the amount of non-Clifford resources needed to realize quantum states and operations the resource theory of nonstabilizerness has been put forward [6–14]. Stabilizer entropies (SEs) [15] are measures of nonstabilizerness with efficient algorithms for matrix product states [16–18] that have enabled the study of nonstabilizerness in many-body systems [16–24].

Recently, SEs have also been related to various important properties of quantum systems. SEs probe error-correction [25] and measurement-induced phase transitions [26,27], as well as relate to the entanglement spectrum [28] and property testing [29,30]. SEs are also connected to the participation entropy [31], which is helpful to understand Anderson [32] and many-body localization [33]. Further, recent works established a fruitful connection between out-of-time-order correlators (OTOCs) and nonstabilizerness [30,34,35]. OTOCs describe scrambling in quantum systems [36,37]. However, OTOCs are challenging to measure directly and often require an inverse of the time evolution [38]. Higher-order OTOCs and nonstabilizerness have been related to quantum chaos [34] and state certification [30].

The aforementioned properties make SEs highly interesting for experimental studies of quantum computers and simulators. However, the progress has so far been limited as all previously known measurement protocols for SEs scale exponentially with the number of qubits [14,39].

Here, we efficiently measure SEs with integer index $n > 1$ on quantum computers and simulators via Bell measurements on two copies of N-qubit quantum states. Our algorithms are practical to implement with $O(n)$ copies and $O(nN)$ classical postprocessing time, where even n requires also access to the complex conjugate of the state. We devise an efficient protocol to measure Clifford-averaged multifractal flatness and $4n$-point OTOCs where for odd n we do not require an inverse time evolution. We study the interplay of nonstabilizerness and scrambling and show that the number of Clifford gates needed for OTOCs to converge depends on the number of T gates. Further, we use the multifractal flatness to show that random Hamiltonian evolution stops being random for long evolution times. We also provide efficiently computable bounds to other nonstabilizerness monotones, which are otherwise intractable beyond a few qubits. Finally, we measure the Tsallis SE on the IonQ quantum computer and demonstrate SEs as efficient bounds for the robustness of magic, stabilizer extent, and stabilizer fidelity. Our work introduces methods to uncover the key features that characterize the power of quantum computers and simulators.

SE.—For an N-qubit state $|\psi\rangle$, the Rényi-n SE is given by [15]

$$M_n(|\psi\rangle) = (1-n)^{-1}\ln\left(\sum_{\sigma\in\mathcal{P}} 2^{-N}\langle\psi|\sigma|\psi\rangle^{2n}\right), \quad (1)$$

where n is the index of the SE and $\mathcal{P}$ is the set of 4^N Pauli strings. The Pauli strings are N-qubit tensor products $\sigma_{\boldsymbol{r}} = \bigotimes_{j=1}^N \sigma_{\boldsymbol{r}_{2j-1}\boldsymbol{r}_{2j}}$ with $\boldsymbol{r}\in\{0,1\}^{2N}$, where $\sigma_{00}=I_1$, $\sigma_{01}=\sigma^x$, $\sigma_{10}=\sigma^z$, and $\sigma_{11}=\sigma^y$ with ℓ-qubit identity matrix I_ℓ and Pauli matrices σ^k, $k\in\{x,y,z\}$. M_n is a faithful measure of nonstabilizerness for pure states, i.e., $M_n(|\psi_{\rm STAB}\rangle)=0$ only for pure stabilizer states $|\psi_{\rm STAB}\rangle$, and greater zero else [15]. Further, SEs are invariant under Clifford unitaries $U_{\rm C}$ with $M_n(U_{\rm C}|\psi\rangle) = M_n(|\psi\rangle)$, where Clifford unitaries map any Pauli string σ to another Pauli string σ' via $U_{\rm C}\sigma U_{\rm C}^\dagger = \sigma'$. Further, M_n is additive with $M_n(|\psi\rangle\otimes|\phi\rangle) = M_n(|\psi\rangle)+M_n(|\phi\rangle)$. M_n for $n<2$ is not a monotone under channels that can map a pure state to another pure state, while the case $n\geq 2$ remains an open problem.

Evaluating Eq. (1) requires an efficient measurement protocol for the nth moment of the Pauli spectrum

$$A_n(|\psi\rangle) = 2^{-N}\sum_{\sigma\in\mathcal{P}}\langle\psi|\sigma|\psi\rangle^{2n}, \quad (2)$$

which on first glance appears challenging due to the summation over exponentially many Pauli strings.

Algorithms.—We now provide two algorithms to efficiently measure $A_n(|\psi\rangle)$. First, we introduce Algorithm 1, which is efficient for odd $n>1$. We write A_n as the expectation value of an observable $\Gamma_n^{\otimes N}$ acting on $2n$ copies of $|\psi\rangle$ via the replica trick [16]

$$A_n = 2^{-N}\sum_{\sigma\in\mathcal{P}}\langle\psi|\sigma|\psi\rangle^{2n} = \langle\psi|^{\otimes 2n}\Gamma_n^{\otimes N}|\psi\rangle^{\otimes 2n}, \quad (3)$$

where $\Gamma_n = \frac{1}{2}\sum_{k=0}^3(\sigma^k)^{\otimes 2n}$. For even $n>1$, $2^{-N}\Gamma_n^{\otimes N}$ is a projector with two possible eigenvalues $\omega\in\{0,2^N\}$ as shown in the Supplemental Material (SM) A [40]. In contrast, for odd $n>1$ it is unitary and hermitian with eigenvalues $\omega\in\{-1,1\}$. This fact was previously pointed out in Ref. [49] for stabilizer testing.

To measure the operator $\Gamma_n^{\otimes N}$ we transform the operator into a diagonal eigenbasis. We first recall the Bell transformation acting on two qubits $U_{\rm Bell} = (H\otimes I_1)$CNOT, where $H = (1/\sqrt{2})(\sigma^x+\sigma^z)$ is the Hadamard gate, and CNOT $= \exp[i(\pi/4)(I_1-\sigma^z)\otimes(I_1-\sigma^x)]$. It turns out Γ_n is diagonalized by $U_{\rm Bell}$

ALGORITHM 1. SE without complex conjugate.

Input: Integer $n>1$; L repetitions;
Output: $A_n(|\psi\rangle)$

$A_n = 0$
for $k=1,\ldots,L$ **do**
  for $j=1,\ldots,n$ **do**
    Prepare $|\eta\rangle = U_{\rm Bell}^{\otimes N}|\psi\rangle\otimes|\psi\rangle$
    Sample in computational basis $\boldsymbol{r}^{(j)}\sim|\langle\boldsymbol{r}|\eta\rangle|^2$
  end
  $b=1$
  for $\ell=1,\ldots,N$ **do**
    $\nu_1 = \bigoplus_{j=1}^n r_{2\ell-1}^{(j)}$; $\nu_2 = \bigoplus_{j=1}^n r_{2\ell}^{(j)}$
    if n odd **then**
      $b = b\cdot(-2\nu_1\cdot\nu_2+1)$
    else
      $b = b\cdot 2(\nu_1-1)\cdot(\nu_2-1)$
    end
  end
  $A_n = A_n + b/L$
end

$$A_n = \langle\psi|^{\otimes 2n}\left\{U_{\rm Bell}^{\otimes n\,\dagger}\frac{1}{2}\left[(I_1\otimes I_1)^{\otimes n} + (\sigma^z\otimes I_1)^{\otimes n} + (I_1\otimes\sigma^z)^{\otimes n} + (-1)^n(\sigma^z\otimes\sigma^z)^{\otimes n}\right]U_{\rm Bell}^{\otimes n}\right\}^{\otimes N}|\psi\rangle^{\otimes 2n}. \quad (4)$$

Algorithm 1 utilizes this fact to provide an unbiased estimator for A_n, where $\oplus$ denotes binary addition. While Eq. (4) involves $2n$ copies of $|\psi\rangle$, the Bell transformation Eq. (4) can be written as tensor products. Thus, A_n is evaluated using only Bell measurements on two copies of the quantum state, which requires only a $2N$-qubit quantum computer. Then, via postprocessing A_n is computed as the parity of odd and even qubit index Bell measurement outcomes as derived in SM B [40].

We apply Hoeffding's inequality to bound the number of copies as $C = O(n\Delta\omega_n\epsilon^{-2})$, where ϵ is the error and $\Delta\omega_n$ the range of eigenvalues of $\Gamma_n^{\otimes N}$. For odd $n>1$, we have $\Delta\omega_n = 2$ and $C = O(n\epsilon^{-2})$. For even n, the eigenvalue spectrum of $\Gamma_n^{\otimes N}$ diverges and we require an exponential number of measurements. In SM C [40] we extend our algorithm to get gradients $\partial_k A_n$ via the shift rule for variational quantum algorithms.

Now, we provide Algorithm 2, which is efficient for any integer $n>1$ but requires access to the complex conjugate $|\psi^*\rangle$. We rewrite the SE as a sampling problem,

$$A_n = \mathop{\mathbb{E}}_{\sigma\sim\Xi(\sigma)}[\langle\psi|\sigma|\psi\rangle^{2n-2}], \quad (5)$$

where $\Xi(\sigma) = 2^{-N}\langle\psi|\sigma|\psi\rangle^2$ is the probability distribution of Pauli strings σ. The circuit for the algorithm is shown in Fig. 1. First, we prepare $|\psi^*\rangle\otimes|\psi\rangle$ on the quantum

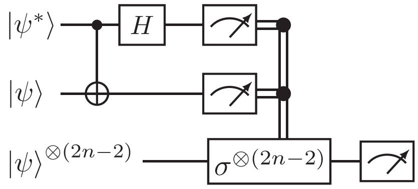

FIG. 1. Measurement protocol for Algorithm 2.

computer and transform into the Bell basis $|\eta\rangle = U_{\mathrm{Bell}}^{\otimes N}|\psi^*\rangle \otimes |\psi\rangle$. Next, we sample from $2N$-qubit state $|\eta\rangle$ in the computational basis, gaining outcome $\boldsymbol{r} \in \{0,1\}^{2N}$. As shown in [50,51], we have $\Xi(\sigma_{\boldsymbol{r}}) = |\langle \boldsymbol{r}|\eta\rangle|^2$, where $|\boldsymbol{r}\rangle$ is the computational basis state corresponding to bitstring $\boldsymbol{r}$. Thus, sampling $\boldsymbol{r}$ from $|\eta\rangle$ corresponds to sampling Pauli strings $\sigma_{\boldsymbol{r}} \sim \Xi(\sigma_{\boldsymbol{r}})$. Then, we perform $2n-2$ measurements on $|\psi\rangle$ in the eigenbasis of the sampled $\sigma_{\boldsymbol{r}}$ and multiply the measured eigenvalues λ_k, gaining an unbiased estimator of $\langle\psi|\sigma_{\boldsymbol{r}}|\psi\rangle^{2n-2}$.

The measured eigenvalues $\prod_{k=1}^{2n-2}\lambda_k \in \{+1,-1\}$ have a range $\Delta\omega_n = 2$, thus according to Hoeffding's inequality we require at most $C = O(n\epsilon^{-2})$ copies of $|\psi\rangle$ and $O(\epsilon^{-2})$ copies of $|\psi^*\rangle$ for any integer $n > 1$. Note that $|\psi^*\rangle$ cannot be efficiently prepared in general with only black-box access to $|\psi\rangle$ [52–54]. However, when we have a circuit description of the unitary preparing the state, $|\psi^*\rangle$ is constructed by an element-wise conjugation of the coefficients of the unitary [55].

Tsallis SE.—We now define a measure of nonstabilizerness that we call the Tsallis-n SE [56],

$$T_n(|\psi\rangle) = -(1-n)^{-1}\left(1 - \sum_{\sigma\in\mathcal{P}} 2^{-N}\langle\psi|\sigma|\psi\rangle^{2n}\right). \quad (6)$$

They are a generalization of the linear SE T_2 [15] and the von Neumann SE $T_1 = M_1 = -2^{-N}\sum_{\sigma\in\mathcal{P}}\langle\psi|\sigma|\psi\rangle^2 \times \ln(\langle\psi|\sigma|\psi\rangle^2)$ [17]. T_n can be efficiently measured for integer $n > 1$ using our protocols. They are faithful measures of nonstabilizerness that are invariant under Clifford unitaries and related to Rényi SEs via $M_n = (1-n)^{-1}\ln[1 + (1-n)T_n]$. Tsallis SEs lack the additive property of the Rényi SE; however our numerics suggest that Tsallis SEs may be a strong monotone that is a not necessary but highly desirable property for resource measures [57]. Within extensive numerical optimization for $N \leq 6$ qubits we were unable to find states that could violate strong monotonicity for the Tsallis-n SE for $n \geq 2$ (see SM D [40]).

Note that measuring the Rényi SE $M_n \sim \ln(A_n)$ with precision ϵ_M requires $O[n\exp(M_n)\epsilon_M^{-2}]$ samples due to the logarithm (see SM D [40]). Thus, M_n is efficiently measurable as long as $M_n = O[\log(N)]$.

ALGORITHM 2. SE with complex conjugate.

Input: Integer $n > 1$; L repetitions
Output: $A_n(|\psi\rangle)$
$A_n = 0$
for $k = 1, \ldots, L$ **do**
  Prepare $|\eta\rangle = U_{\mathrm{Bell}}^{\otimes N}|\psi^*\rangle \otimes |\psi\rangle$
  Sample $\boldsymbol{r} \sim |\langle \boldsymbol{r}|\eta\rangle|^2$
  $b = 1$
  for $\ell = 1, \ldots, 2n-2$ **do**
    Prepare $|\psi\rangle$ and measure in eigenbasis of Paulistring $\sigma_{\boldsymbol{r}}$ for eigenvalue $\lambda \in \{+1,-1\}$
    $b = b \cdot \lambda$
  end
  $A_n = A_n + b/L$
end

Clifford-averaged OTOCs.—We now show how to efficiently measure $4n$-point OTOCs of unitary U averaged over the Clifford group. The $4n$-point OTOC for N-qubit Pauli strings σ and σ' is given by [30]

$$\mathrm{otoc}_{4n}(U,\sigma,\sigma') = \left(2^{-N}\mathrm{tr}(\sigma U \sigma' U^\dagger)\right)^{2n}. \quad (7)$$

We find that otoc_{4n} averaged over the group of Clifford unitaries $\mathcal{C}_N$ can be related to SE of U, which we define via the Choi state $|U\rangle = I_N \otimes U|\Phi\rangle$, where $|\Phi\rangle = 2^{-N/2}\sum_{i=0}^{2^N-1}|i\rangle \otimes |i\rangle$. In particular, we have

$$\mathop{\mathbb{E}}_{U_{\mathrm{C}},U'_{\mathrm{C}}\in\mathcal{C}_N}\left[\mathrm{otoc}_{4n}(U_{\mathrm{C}}UU'_{\mathrm{C}},\sigma,\sigma')\right] = \frac{A_n(|U\rangle)4^N - 1}{(4^N-1)^2}, \quad (8)$$

where the Pauli strings $\sigma, \sigma' \in \mathcal{P}/\{I_N\}$ exclude the identity and the proof is found in SM K [40] using results of Ref. [30]. For odd $n > 1$, we can efficiently measure Eq. (8) via Algorithm 1. For even $n > 1$, we additionally require the complex conjugate $|U^*\rangle$ for Algorithm 2. The complex conjugate of the Choi state can be efficiently prepared with access to U^* or $U^\dagger$ due to the ricochet property $|U^*\rangle = I_N \otimes U^*|\Phi\rangle = U^\dagger \otimes I_N|\Phi\rangle$ [55].

Multifractal flatness.—The participation entropy is given by $\mathcal{I}_q(|\psi\rangle) = \sum_k |\langle k|\psi\rangle|^{2q}$, where $|k\rangle$ are computational basis states, $q > 0$ and $0 \leq \mathcal{I}_q \leq 1$ [32]. The participation entropy quantifies the spread of the wave function over basis states, i.e., $\mathcal{I}_q = 1$ for computational basis states, while it is small when the state is delocalized over many computational basis states. The multifractal flatness $\mathcal{F}(|\psi\rangle) = \mathcal{I}_3(|\psi\rangle) - \mathcal{I}_2^2(|\psi\rangle)$ measures the flatness of the distribution $|\langle k|\psi\rangle|^2$, i.e., we have $\mathcal{F} = 0$ when the distribution $|\langle k|\psi\rangle|^2$ is constant over its support, else we have $\mathcal{F} > 0$. In particular, stabilizer states have $\mathcal{F} = 0$ [58].

Recently, $\mathcal{F}$ averaged over $\mathcal{C}_N$ has been proposed as $\bar{\mathcal{F}}$ [31]. This quantity describes the participation ratio averaged over all possible choices of basis states. $\bar{\mathcal{F}}$ has been connected to SEs as follows [31]:

$$\bar{\mathcal{F}}(|\psi\rangle) = \mathop{\mathbb{E}}_{U_C \in \mathcal{C}_N}[\mathcal{F}(U_C|\psi\rangle)] = \frac{2[1 - A_2(|\psi\rangle)]}{(2^N + 1)(2^N + 2)}. \qquad (9)$$

Thus, Algorithm 2 allows us to efficiently measure $\bar{\mathcal{F}}(|\psi\rangle)$ directly without the need of averaging over $\mathcal{C}_N$.

Bounds on nonstabilizerness.—We now provide efficient bounds on three magic monotones, namely the robustness of magic R [8], stabilizer extent ξ [59], and the stabilizer fidelity $F_{\rm STAB} = \max_{|\phi\rangle \in {\rm STAB}} |\langle\psi|\phi\rangle|^2$ [59] (see SM E [40]). Computing R, ξ, and $F_{\rm STAB}$ requires solving an optimization program over the set of pure N-qubit stabilizer states. As the number of stabilizer states scales as $O(2^{N^2})$, these three measures in general become numerically infeasible beyond five qubits [8,60].

Our algorithms provide efficient bounds for integer $n > 1$ (see [17] or SM E [40]),

$$R \geq \xi \geq F_{\rm STAB}^{-1} \geq A_n^{-\frac{1}{2n}}. \qquad (10)$$

The bound can be tightened for $n \geq \frac{1}{2}$ to $R \geq A_n^{[1/2(1-n)]}$ [8,15]. With methods from Ref. [49,61], we also prove a lower bound on $F_{\rm STAB}$ for $n > 1$ (see SM F [40]),

$$A_n^{\frac{1}{2n}} \geq F_{\rm STAB} \geq \frac{A_n - 2^{1-n}}{1 - 2^{1-n}}. \qquad (11)$$

The min-relative entropy of magic $D_{\min} = -\ln(F_{\rm STAB})$ can be seen as the distance to the nearest stabilizer state. We now argue that $D_{\min}$, Rényi SEs with $n \geq 2$ and the recently introduced additive Bell magic $\mathcal{B}_{\rm a}$ [14] are closely related. In particular, we find evidence for respective upper and lower bounds independent of qubit number N (see SM G [40]). Via numerical optimization we find $1.7M_2 \gtrsim D_{\min} \geq \frac{1}{4}M_2$ as well as $3.5M_2 \gtrsim \mathcal{B}_{\rm a} \gtrsim 2.88M_2$ for at least $N \leq 4$, while similar bounds can also be found for larger n. Thus, $D_{\min}$, $M_{n\geq 2}$, and $\mathcal{B}_{\rm a}$ can be seen as measures of nonstabilizerness that relate to the distance to the nearest stabilizer state. In contrast, the robustness of magic R and stabilizer extent ξ relate to the degree a state can be approximated by a combination of stabilizer states. They belong to a different class of nonstabilizerness measures as they cannot be upper bounded with $D_{\min}$ or M_n for $n > 1/2$ [17].

Demonstration.—We now study SEs with Bell measurements on the IonQ quantum computer [14] using Algorithm 1 in Fig. 2. We investigate random Clifford circuits $U_{\rm C}$ doped with $N_{\rm T}$ non-Clifford gates,

$$|\psi(N_{\rm T})\rangle = U_{\rm C}^{(0)} \prod_{k=1}^{N_{\rm T}} V_{\rm T}^{(k)} U_{\rm C}^{(k)} |0\rangle, \qquad (12)$$

where $V_{\rm T}^{(k)} = \exp[-i(\pi/8)\sigma^z_{g(k)}]$ is the T gate of the kth layer acting on a randomly chosen qubit $g(k)$. With increasing $N_{\rm T}$ these states transition from efficiently simulable stabilizer states to intractable quantum states [34,62]. To reduce noise, we compress the circuits into layered circuits composed of single-qubit operations and CNOT gates arranged in a nearest-neighbor chain configuration [14]. The state prepared by the quantum computer is not pure but degraded by noise. However, SEs are faithful measures of nonstabilizerness only for pure states. Using measurements on the noisy state, we mitigate A_n and T_n from measurements on noisy states by assuming a global depolarization error model (see SM H [40]).

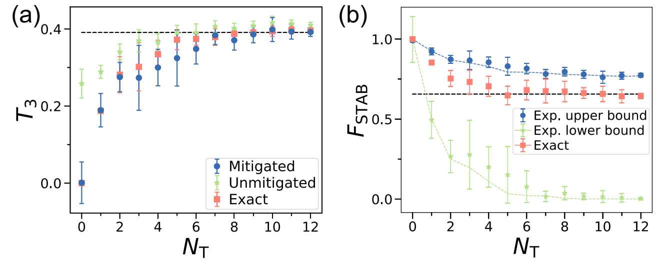

FIG. 2. Measurement of nonstabilizerness for quantum states generated by Eq. (12) with random Clifford circuits doped with $N_{\rm T}$ T gates on the IonQ quantum computer. (a) We show Tsallis SE T_3 with and without error mitigation as well as exact simulation. Dashed line is average value for Haar random states. Dots represent mean value and error bars the standard deviation taken over 6 random instances of the circuit. We have $N = 3$ qubits, 10^3 Bell measurements, and a measured depolarization error of $p \approx 0.1$. (b) We show upper and lower bounds on $F_{\rm STAB}$ via Eq. (11) evaluated using the error mitigated T_3 as blue and green dots as well as simulation of the bound as dashed lines. The orange dots show simulations of $F_{\rm STAB}$.

In Fig. 2(a), we show T_3 with and without error mitigation for different $N_{\rm T}$, where for each value we average over six random instances of Eq. (12). We find that the results on the IonQ quantum computer with error mitigation closely match the simulated values. The Tsallis SE is zero for $N_{\rm T} = 0$, then increases with $N_{\rm T}$ until it converges to the average value of Haar random states indicated as black dashed line. In Fig. 2(b), we use the mitigated results for A_3 to compute upper and lower bounds for the stabilizer fidelity $F_{\rm STAB}$ using Eq. (11). The measured result indeed gives valid bounds of the exactly simulated $F_{\rm STAB}$. We find that the upper bound is relatively tight, while the lower bound is nontrivial only for small $N_{\rm T}$. In SM I [40], we provide additional results for the IonQ quantum computer on measures of nonstabilizerness. While our error mitigation scheme assumes global depolarization noise, it works well on actual quantum computers that have more complicated noise profiles. In SM J [40], we simulate our error mitigation scheme for various unital and nonunital noise models, and find very good performance. As SEs are moments of exponentially many Pauli strings, self-averaging effects may explain the good performance.

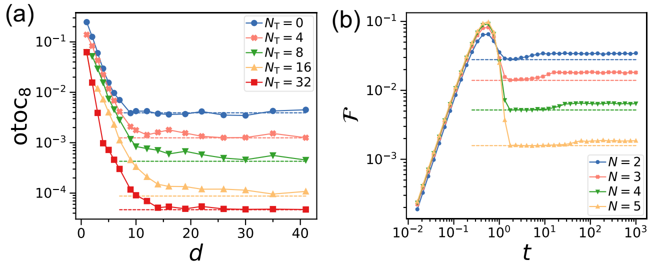

FIG. 3. (a) $\mathrm{otoc}_8(U, \sigma_1^x, \sigma_1^x)$ against d layers of single-qubit Clifford gates and CNOT gates arranged in a nearest-neighbor chain, doped with N_T T gates and $N = 4$ qubits. Dashed line is the Clifford-averaged OTOC Eq. (8). (b) Multifractal flatness $\mathcal{F}$ for evolution in time t with random Hamiltonian $\exp(-iH_\mathrm{GUE}t)|0\rangle$. Dashed line is the Clifford-averaged multifractal flatness Eq. (9).

Scrambling.—We now study scrambling using the multifractal flatness $\mathcal{F}$ and OTOCs. In Fig. 3(a), we show $\mathrm{otoc}_8(U, \sigma_1^x, \sigma_1^x)$ against d layers of Clifford gates doped with N_T T gates. We find that the OTOC decreases with d, converging to a minimum once the unitary is fully scrambled. This minimum is given by the Clifford averaged OTOC Eq. (8) and depends on N_T. The d needed to converge depends on the number of T gates, where for $N_\mathrm{T} = 0$ convergence is achieved for $d \sim 10$, while higher N_T requires larger d to converge. We observe similar convergence for $\mathcal{F}$ and other OTOCs in SM K [40].

In Fig. 3(b), we study $\mathcal{F}$ for the evolution of a state $|\psi(t)\rangle = \exp(-iH_\mathrm{GUE}t)|0\rangle$ in time t using a random Hamiltonian H_GUE drawn from the Gaussian unitary ensemble (GUE). We observe that $\mathcal{F}$ initially increases, reaching a maximum at $t \sim 1$. This is followed by a sudden dip to the Clifford-averaged multifractal flatness Eq. (9). This is hallmark of reaching deep thermalization or unitary design, where the system is indistinguishable from Haar-random dynamics [63]. Counterintuitively, for longer (exponential) times $\mathcal{F}$ ramps up again, converging to a value above the Clifford-average. Here, the system stops being fully random due to dephasing of energy eigenvalues [63]. In SM K [40], we show how to measure $\mathcal{F}$ and approximate GUE Hamiltonians using a Hamiltonian of random Pauli strings that can be implemented in experiment.

Discussions.—We show how to efficiently measure SEs with a cost independent of qubit number N, which is an exponential improvement over previous protocols [14,39]. For integer $n > 1$, our protocol is asymptotically optimal with the number of copies scaling as $O(n\epsilon^{-2})$ and the classical postprocessing time as $O(nN\epsilon^{-2})$ with additive error ϵ. The protocol is easy to implement using Bell measurements that have been demonstrated for quantum computers and simulators [64–66]. We note that our approach is distinct from the previously introduced Bell magic [14] as shown in SM L [40].

Our measurement protocol allows for efficient experimental characterization of different important properties of quantum states. We demonstrate an efficient bound on nonstabilizerness monotones that otherwise are hard to compute beyond a few qubits. These monotones serve as lower bounds on state preparation complexity and characterize the runtime of classical simulation algorithms [8,59]. Further, we show how to efficiently measure Clifford-averaged $4n$-point OTOCs. Our protocol has the advantage that it does not require implementing time reversal for odd $n > 1$, which can be a challenge [36]. Our protocol can measure higher order OTOCs that promise to reveal more features compared to the usually considered four-point OTOCs [67,68]. Our methods allow direct experimental study phase transitions in SE that have been found for purity testing [29] and quantum error correction [25]. Finally, we enable certification of magic gates in fault-tolerant quantum computers, where the SE could be directly evaluated from recent experimental data [69].

We use our methods to study scrambling in Clifford circuits doped with T gates and random Hamiltonian evolution. OTOCs not only measure scrambling, but also depend on nonstabilizerness in a nontrivial way [70]. We can disentangle these two effects by measuring the Clifford-averaged OTOC. We study when Clifford circuits doped with T gates become fully scrambled, revealing that the depth depends on the number of T gates. We also study the scrambling with random Hamiltonians. Notably, random Hamiltonian evolution deep thermalizes at intermediate times, becoming indistinguishable from Haar-random unitaries [63] that we observe via the convergence of the multifractal flatness to its Clifford-averaged value. Counterintuitively, the evolution becomes less random again for long times due to the dephasing of its energy eigenstates [63]. While nonlocal OTOCs and SEs lack clear signatures of this effect (see SE L), we find that multifractal flatness and same-site OTOCs exhibit clear gaps to their Clifford-average.

Future work could find efficient protocols for even n without the need of complex conjugation and tighten the lower bound of SEs for the stabilizer fidelity.

Note added.—Before acceptance of this Letter, the monotonicity of Rényi SE and strong monotonicity of Tsallis SE have been proven for $n \geq 2$ [72].

The code for this work is available on GitHub [71].

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