CHARACTERIZING SLOPES FOR 5_2

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Abstract. We prove that all rational slopes are characterizing for the knot 5_2, except possibly for positive integers. Along the way, we classify the Dehn surgeries on knots in S^3 that produce the Brieskorn sphere $\Sigma(2,3,11)$, and we study knots on which large integral surgeries are almost L-spaces.

1. Introduction

Let $K \subset S^3$ be a knot. Then a rational number r is said to be a *characterizing slope* for K if the result $S^3_r(K)$ of Dehn surgery on K with slope r does not arise as r-surgery on any other knot: in other words, if whenever there is an orientation-preserving homeomorphism

$$S^3_r(K) \cong S^3_r(K'),$$

the knot K' must be isotopic to K.

All rational numbers are characterizing slopes for the unknot, as well as for the trefoils and the figure eight knot. These are theorems of Kronheimer–Mrowka–Ozsváth–Szabó [KMOS07] and of Ozsváth–Szabó [OS19], respectively, each relying on a theorem (due to Ghiggini [Ghi08] in the latter case) asserting that some form of Floer homology detects the knot in question. Ni–Zhang and McCoy [NZ14, McC20, McC21] have proved that many slopes are characterizing for torus knots, especially $T_{2,5}$ [NZ23]. More generally, Lackenby [Lac19] has shown that every knot has infinitely many characterizing slopes, and McCoy [McC19] has strengthened this in the hyperbolic case.

Our main result, Theorem 1.1, says that almost all slopes are characterizing for the knot 5_2, shown in Figure 1. This is strongest result to date for any non-fibered knot and for any hyperbolic knot other than the figure eight:

Theorem 1.1. *Let r be any rational number other than a positive integer. If for some knot $K \subset S^3$ there is an orientation-preserving homeomorphism*

$$S^3_r(K) \cong S^3_r(5_2),$$

then K is isotopic to 5_2. In other words, every $r \in \mathbb{Q} \setminus \mathbb{Z}_{>0}$ is characterizing for 5_2.

It is possible that no positive integer is characterizing for 5_2 (and hence that Theorem 1.1 is optimal). Indeed, Baker–Motegi [BM18] have exhibited hyperbolic knots such as 8_6 with no integral characterizing slopes, and Abe–Tagami [AT21] proved similar results for many other low-crossing knots. At the very least, Proposition 8.3 says that the positive integer 1 is not characterizing for 5_2:

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FIGURE 1. The knot 5_2 (left), and its mirror $\overline{5_2}$ (right).

Proposition 1.2. *There is an orientation-preserving homeomorphism*

$$S^3_1(5_2) \cong S^3_1(P(-3,3,8)),$$

so 1 *is not a characterizing slope for* 5_2.

This fact was originally discovered by Akbulut [Akb91], who also showed that the traces of the corresponding surgeries are homeomorphic but not diffeomorphic.

Remark 1.3. The orientation-preserving condition is a necessary part of Theorem 1.1. For example, there are homeomorphisms

$$S^3_{1/2}(5_2) \cong -S^3_{1/2}(6_1), \qquad\qquad S^3_1(5_2) \cong -S^3_1(\overline{6_1}).$$

This can be deduced from [BS21, Proposition 7.2], in which $5_2 = K(2,4)$ and $6_1 = K(-2,4)$.

As an application, we determine all of the ways in which the Brieskorn sphere $\Sigma(2,3,11)$ can arise from Dehn surgery on a knot in S^3:

Theorem 1.4. *Given a knot $K \subset S^3$ and a rational number r, there exists an orientation-preserving homeomorphism*

$$S^3_r(K) \cong \Sigma(2,3,11)$$

if and only if (K,r) is either $(T_{-2,3}, -\frac{1}{2})$ or $(5_2,-1)$.

Similar results have been achieved for $\Sigma(2,3,5)$ by Ghiggini [Ghi08, Corollary 1.7], and for $\Sigma(2,3,7)$ by Ozsváth–Szabó [OS19, Corollary 1.3].

The proof of Theorem 1.1 relies on our recent classification [BS22] of genus-1 knots which are *nearly fibered* from the point of view of knot Floer homology:

Theorem 1.5 ([BS22, Theorem 1.2])**.** *Let $K \subset S^3$ be a knot of Seifert genus 1 such that*

$$\dim_{\mathbb{Q}} \widehat{HFK}(K,1;\mathbb{Q}) = 2.$$

Then K is one of the knots

$$5_2,\ 15n_{43522},\ \mathrm{Wh}^-(T_{2,3},2),\ \mathrm{Wh}^+(T_{2,3},2),\ P(-3,3,2n+1)\ (n \in \mathbb{Z})$$

or their mirrors; the knot Floer homologies of these knots are given in Table 1.

Theorem 1.1 is then a combination of Theorems 1.6 and 1.7 below. By way of notation, whenever we discuss an isomorphism between Heegaard Floer homologies of the form

$$HF^+(Y;\mathbb{Q}) \cong HF^+(Y';\mathbb{Q})$$

in this paper, we will always mean an isomorphism of $\mathbb{Q}[U]$-modules which respects a decomposition of each side into summands indexed by Spin^c structures on Y and Y', respectively.

K	$\widehat{HFK}(K,1;\mathbb{Q})$	$\widehat{HFK}(K,0;\mathbb{Q})$	$\widehat{HFK}(K,-1;\mathbb{Q})$
5_2	$\mathbb{Q}^2_{(2)}$	$\mathbb{Q}^3_{(1)}$	$\mathbb{Q}^2_{(0)}$
$15n_{43522}$	$\mathbb{Q}^2_{(0)}$	$\mathbb{Q}^4_{(-1)} \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}^2_{(-2)}$
$\mathrm{Wh}^-(T_{2,3},2)$	$\mathbb{Q}^2_{(0)}$	$\mathbb{Q}^4_{(-1)} \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}^2_{(-2)}$
$P(-3,3,2n+1)$	$\mathbb{Q}^2_{(1)}$	$\mathbb{Q}^5_{(0)}$	$\mathbb{Q}^2_{(-1)}$
$\mathrm{Wh}^+(T_{2,3},2)$	$\mathbb{Q}^2_{(-1)}$	$\mathbb{Q}^4_{(-2)} \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}^2_{(-3)}$

Table 1. The knot Floer homologies of the knots in Theorem 1.5, grouped by whether the Alexander polynomial is $2t-3+2t^{-1}$ or $-2t+5-2t^{-1}$. The subscripts denote Maslov gradings.

Theorem 1.6 (Theorem 4.1)**.** *Suppose for some knot $K \subset S^3$ and rational number $r \geq 0$ that there is an isomorphism*

$$HF^+(S^3_r(K);\mathbb{Q}) \cong HF^+(S^3_r(\overline{5_2});\mathbb{Q})$$

of graded $\mathbb{Q}[U]$-modules. Then K is isotopic to $\overline{5_2}$.

Theorem 1.6 immediately implies the case $r \leq 0$ of Theorem 1.1, via the relation

$$S^3_r(K) \cong -S^3_{-r}(\overline{K}),$$

and the relationship between the Heegaard Floer homologies of Y and $-Y$. For the case $r > 0$, we prove the following:

Theorem 1.7. *Suppose for some knot $K \subset S^3$ and rational number $r > 0$ that there is an orientation-preserving homeomorphism*

$$S^3_r(K) \cong S^3_r(5_2),$$

but that K is not isotopic to 5_2. Then r is a positive integer, and $g = g(K)$ is at least 2; if g is even then r divides $g-1$, while if g is odd then r divides $2g-2$. Moreover, K has Alexander polynomial

$$\Delta_K(t) = t^g - 2t^{g-1} + t^{g-2} + 1 + t^{2-g} - 2t^{1-g} + t^{-g},$$

and the knot Floer homology $\widehat{HFK}(K;\mathbb{Q})$ is completely determined as a bigraded $\mathbb{Q}$-vector space by r and g: it is 9-dimensional, and there is a $\mathbb{Q}$ summand in Alexander–Maslov bigrading $(0,0)$ while the rest is supported in bigradings $(a,m) = (a, a+\delta)$, where

$$\delta = 2 - g + \begin{cases} -(g-1)\left(\frac{g-1}{r} - 1\right), & r \mid g-1 \\ -\frac{1}{4r}(2g-2-r)^2, & r \nmid g-1. \end{cases}$$

Most of the content of Theorem 1.7 is in Theorem 6.13, which makes heavy use of the Heegaard Floer mapping cone formula for Dehn surgeries. However, the latter assumes that $g \geq 2$, and it only concludes that r divides $2g-2$. We use an obstruction due to Ito [Ito20] involving finite type invariants to handle the case $g = 1$ in Proposition 7.6, and to improve the condition $r \mid 2g-2$ to $r \mid g-1$ for even g in Proposition 7.7.

Remark 1.8. In fact, the proof of Theorem 1.7 shows that

$$HF^+(S^3_r(K);\mathbb{Q}) \not\cong HF^+(S^3_r(5_2);\mathbb{Q})$$

in nearly all cases where it asserts that $S^3_r(K) \not\cong S^3_r(5_2)$. The exceptions are when $g(K) \geq 2$ is even and r divides $2g(K)-2$ but not $g(K)-1$, and when $g(K)=1$ and K is one of the knots listed in Theorem 1.5 with Alexander polynomial $2t-3+2t^{-1}$. In the latter case, we require the full strength of Theorem 1.5, rather than just the claim that $\widehat{HFK}$ detects 5_2, in order to enumerate the remaining cases and to rule them out one by one in Proposition 7.6.

Remark 1.9. If $g(K)=2$ and $S^3_r(K) \cong S^3_r(5_2)$, then Theorem 1.7 says that $r=1$ and $\delta=0$. This implies that K has the same knot Floer homology as of any of the pretzel knots $P(-3,3,2n)$, where $n \in \mathbb{Z}$. We conjecture that it must then actually be isotopic to $P(-3,3,2n)$ for some n, in which case Remark 7.8 will show that it is $P(-3,3,8)$.

The proofs of Theorems 1.6 and 1.7 rely heavily on formulas which determine the Heegaard Floer homology of Dehn surgeries on a knot K in terms of the canonical $\mathbb{Z}\oplus\mathbb{Z}$ filtration on $CFK^\infty(K)$. This includes both the "large surgeries" formula of [OS04b], which applies to surgeries of integral slope $n \geq 2g(K)-1$, and the "mapping cone" formula of [OS11], which applies to surgeries of any positive rational slope. This should come as no surprise to readers familiar with previous works on characterizing slopes such as [OS19], although the application of these formulae to the problem considered here is substantially more involved. We briefly outline their uses below.

First, for Theorem 1.6, we are able to avoid the heavy machinery of the mapping cone formula by making use of the fact that $\overline{5_2}$ is very nearly an L-space knot.

Definition 1.10. We say that a closed 3-manifold Y is an *almost L-space* if it is a rational homology 3-sphere and satisfies

$$\dim \widehat{HF}(Y;\mathbb{Q}) = |H_1(Y;\mathbb{Z})| + 2.$$

We say that a nontrivial knot $K \subset S^3$ is an *almost L-space knot* if

$$\dim \widehat{HF}(S^3_n(K);\mathbb{Q}) = n+2$$

(that is, if $S^3_n(K)$ is an almost L-space) for some integer $n \geq 2g(K)-1$, in which case one can show that it holds for all $n \geq 2g(K)-1$.

Then $\overline{5_2}$ is an almost L-space knot, and there are very few other examples with genus 1. The following is a combination of Propositions 3.9 and 3.10.

Theorem 1.11. *If $K \subset S^3$ is an almost L-space knot, then one of the following is true:*

(1) K is the left-handed trefoil, figure eight knot, or $\overline{5_2}$.
(2) $g(K) \geq 2$, and K is fibered and strongly quasipositive.

With this theorem at hand, we are able to show quickly that if there is an isomorphism

$$HF^+(S^3_r(K);\mathbb{Q}) \cong HF^+(S^3_r(\overline{5_2});\mathbb{Q})$$

for some rational $r \geq 0$, then K must also be an almost L-space knot of genus 1, and then we only have to rule out the left-handed trefoil and the figure eight. The following is also a straightforward consequence of Theorem 1.11.

Theorem 1.12 (Theorem 3.14)**.** *Let $K \subset S^3$ be a knot. Then $\dim_{\mathbb{Q}} \widehat{HF}(S^3_1(K);\mathbb{Q}) = 3$ if and only if K is either the left-handed trefoil, figure eight, or $\overline{5_2}$.*

Theorem 1.7 requires substantially more effort than Theorem 1.6. The key input is a computation in §6.2 showing that for any $r > 0$ and any Spin^c structure $\mathfrak{s}$ on $S^3_r(5_2)$, the Heegaard Floer homology $HF^+(S^3_r(5_2), \mathfrak{s}; \mathbb{Q})$ is always isomorphic to something of the form

$$\mathcal{T}^+_{(0)} \oplus \mathbb{Q}^{2n}_{(0)}$$

as a relatively graded $\mathbb{Q}[U]$-module. Here,

$$\mathcal{T}^+ \cong \frac{\mathbb{Q}[U, U^{-1}]}{U \cdot \mathbb{Q}[U]},$$

the U-action lowers the grading by 2, and the "(0)" subscripts indicate that the element $1 \in \mathcal{T}$ and the $\mathbb{Q}^{2n}$ summand both lie in grading 0. If $S^3_r(K) \cong S^3_r(5_2)$ for some $r > 0$, then in §5 we find that this imposes strong restrictions on $CFK^\infty(K)$. In the case $g(K) \geq 2$, we see in §6 that these restrictions often imply that for some $\mathfrak{s} \in \mathrm{Spin}^c(S^3_r(K))$, either:

- $\ker(U) \subset HF^+(S^3_r(K), \mathfrak{s}; \mathbb{Q})$ cannot lie in a single grading, or
- $HF^+(S^3_r(K), \mathfrak{s}; \mathbb{Q}) \cong \mathcal{T}^+ \oplus \mathbb{Q}$.

The first of these applies when $0 < r < 1$, or when $r = p/q \geq 1$ is non-integral and $p \mid 2g(K) - 2$, and the second applies when $r = p/q \geq 1$ and $p \nmid 2g(K) - 2$. Both of these contradict the computation of $HF^+(S^3_r(5_2); \mathbb{Q})$, completing the proof in these cases.

1.1. **Notation.** All Floer homologies in this paper will be taken with coefficients in $\mathbb{Q}$. We will therefore omit the coefficients from the notation going forward.

1.2. **Organization.** In §2, we review some facts about knot Floer homology and the large surgery and mapping cone formulas, and then carry out some computations for the knots of Theorem 1.5. In §3, we use this to study the dimension of $\widehat{HF}$ of Dehn surgeries, proving Theorem 1.11 about almost L-space knots. We apply this in §4 to prove Theorem 1.6. In §5, we begin to work toward Theorem 1.7, eliminating all but finitely many K in the case $g(K) = 1$ and then obtaining some restrictions in the case $g(K) \geq 2$, and in §6 we apply the mapping cone formula together with these restrictions to complete the proof of Theorem 1.7 for $g(K) \geq 2$, modulo the modest improvement of Proposition 7.7. In §7, we use finite type invariants to achieve that improvement and to finish off the case $g(K) = 1$, completing the proof of Theorem 1.7 and hence of Theorem 1.1.

In the last few sections we study some specific examples of surgeries. In §8, we prove Proposition 8.3, asserting that 1 is not a characterizing slope for 5_2, and then in §9 we prove Theorem 1.4 on the Dehn surgery characterization of $\Sigma(2, 3, 11)$.

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2. Heegaard Floer homology of surgeries on knots

2.1. **The Heegaard Floer mapping cone formula.** Knot Floer homology [OS04b, Ras03] assigns to any nullhomologous knot $K \subset S^3$ a graded, $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex

$$(CFK^\infty(K), \partial^\infty),$$

whose filtered chain homotopy type completely determines the Heegaard Floer homology of Dehn surgeries on K, where we recall that we are working with coefficients in $\mathbb{Q}$ throughout.

As a matter of convention, we use coordinates (i, j) to refer to the two filtration levels, and notation like

$$C\{i = 0, j \leq 1\} \subset CFK^{\infty}(K)$$

to refer to the subquotient spanned by generators which lie in the indicated subset of the (i, j)-plane. We will also use the shorthand

$$C\{i_0, j_0\} := C\{i = i_0, j = j_0\}.$$

The differential lowers the grading by 1 and does not increase either filtration, meaning that each $C_*\{i \leq i_0,\ j \leq j_0\}$ is a subcomplex: we have

$$\partial^{\infty}\left(C_*\{i \leq i_0,\ j \leq j_0\}\right) \subset C_{*-1}\{i \leq i_0,\ j \leq j_0\}$$

for all (i_0, j_0).

With this in mind, following [Ras03, §4.5 and §5.1], one can take CFK^{∞} to be freely generated over $\mathbb{Q}[U, U^{-1}]$ by $\widehat{HFK}(K; \mathbb{Q})$. We take

$$C_*\{0, a\} \cong \widehat{HFK}_*(K, a; \mathbb{Q}),$$

and the U-action gives isomorphisms

$$U^k : C_*\{0, a\} \xrightarrow{\cong} C_{*-2k}\{-k, a - k\}$$

for all $k \in \mathbb{Z}$. In the form specified here, the restriction of the differential ∂^{∞} to each $C\{i_0, j_0\}$ is zero. See the "reduction lemma" of [HW18, §2.1] for details.

Given this, there are by definition a pair of chain homotopy equivalences

$$C\{i = 0\} \simeq \widehat{CF}(S^3),$$

so the induced complex $(C\{i = 0\}, \partial')$ has homology $\widehat{HF}(S^3) \cong \mathbb{Q}$ supported in grading 0. The Ozsváth–Szabó tau invariant $\tau(K)$ [OS03c] is the minimum j-filtration level at which this generator appears. Similarly, we have a chain homotopy equivalence

$$C\{i \geq 0\} \simeq CF^+(S^3)$$

and then

$$H_*\{i \geq 0\} \cong HF^+(S^3) \cong \mathcal{T}^+ := \frac{\mathbb{Q}[U, U^{-1}]}{U \cdot \mathbb{Q}[U]}.$$

Definition 2.1. Given $CFK^{\infty}(K)$ as above, we define subquotient complexes

$$\begin{aligned} A_s^+ &= C\{\max(i, j - s) \geq 0\}, & B^+ &= C\{i \geq 0\} \\ \hat{A}_s &= C\{\max(i, j - s) = 0\}, & \hat{B} &= C\{i = 0\} \end{aligned}$$

with differentials induced by ∂^{∞}, for all $s \in \mathbb{Z}$. These come with chain maps

$$v_s^+ : A_s^+ \to B^+, \qquad h_s^+ : A_s^+ \to B^+$$

in which v_s^+ is defined by projection onto $C\{i \geq 0\}$, and h_s^+ is a composition

$$A_s^+ \xrightarrow{\text{proj}} C\{j \geq s\} \xrightarrow{U^s} C\{j \geq 0\} \xrightarrow{\simeq} C\{i \geq 0\} = B^+.$$

The last arrow is a homotopy equivalence which exchanges the i and j filtrations; we omit its definition.

Remark 2.2. The projection v_s^+ is an isomorphism at the chain level for all $s \geq g(K)$, since the kernel consists of the direct sum of subspaces

$$C\{i,j\} \cong C\{0, j-i\} \cong \widehat{HFK}(K, j-i)$$

with $i \leq -1$ and $j \geq s \geq g(K)$, and then $\widehat{HFK}(K, j-i) = 0$ because $j - i \geq g(K) + 1$. Similarly, each h_s^+ is an isomorphism for all $s \leq -g(K)$.

These complexes determine the Heegaard Floer homology of "large" surgeries on K, in the following sense.

Theorem 2.3 ([OS04b, Theorem 4.4])**.** *Choose a positive integer $p \geq 2g(K) - 1$. Then there is a canonical affine map $\mathrm{Spin}^c(S^3_p(K)) \cong \mathbb{Z}/p\mathbb{Z}$ (see* [OS08, Lemma 2.2]*) such that we have relatively graded isomorphisms*

$$HF^+(S^3_p(K), s) \cong H_*(A_s^+) \quad \textit{and} \quad \widehat{HF}(S^3_p(K), s) \cong H_*(\hat{A}_s)$$

for any integer s with $|s| \leq \frac{p}{2}$.

Remark 2.4. The definition of the map $\mathrm{Spin}^c(S^3_p(K)) \cong \mathbb{Z}/p\mathbb{Z}$ in [OS08, Lemma 2.2] implies that if $\mathfrak{s} \in \mathrm{Spin}^c(S^3_p(K))$ is identified with $s \in \mathbb{Z}/p\mathbb{Z}$, then the conjugate Spin^c structure $\bar{\mathfrak{s}}$ is identified with $-s$.

They also determine the invariants of arbitrary Dehn surgery, though in a more complicated way. Given relatively prime integers $p, q > 0$ and arbitrary $i \in \mathbb{Z}$, we define

$$\mathbb{A}_i^+ = \bigoplus_{s \in \mathbb{Z}} \left(s, A^+_{\lfloor (i+ps)/q \rfloor}\right), \qquad \mathbb{B}_i^+ = \bigoplus_{s \in \mathbb{Z}} (s, B^+)$$

and a chain map

$$\begin{aligned} D^+_{i,p/q} &: \mathbb{A}_i^+ \to \mathbb{B}_i^+ \\ (s, a_s) &\mapsto \left(s, v^+_{\lfloor (i+ps)/q \rfloor}(a_s)\right) + \left(s+1, h^+_{\lfloor (i+ps)/q \rfloor}(a_s)\right). \end{aligned}$$

The various A^+ and B^+ summands each inherit relative gradings from $CFK^\infty(K)$. We place a relative grading on their direct sums $\mathbb{A}_i^+$ and $\mathbb{B}_i^+$, respecting the relative gradings on each individual summand, so that $D^+_{i,p/q}$ lowers the grading by 1.

Theorem 2.5 ([OS11, Theorem 1.1])**.** *Let $\mathbb{X}^+_{i,p/q}$ denote the mapping cone of the chain map $D^+_{i,p/q} : \mathbb{A}_i^+ \to \mathbb{B}_i^+$. Then there is a natural identification $\mathrm{Spin}^c(S^3_{p/q}(K)) \cong \mathbb{Z}/p\mathbb{Z}$ for which we have a relatively graded isomorphism*

$$H_*(\mathbb{X}^+_{i,p/q}) \cong HF^+(S^3_{p/q}(K), i)$$

for all $i \in \mathbb{Z}/p\mathbb{Z}$.

The A_s^+ complexes have homology of the form

$$H_*(A_s^+) \cong \mathcal{T}^+ \oplus H_{\mathrm{red}}(A_s^+),$$

where $H_{\mathrm{red}}(A_s^+)$ is finitely generated over $\mathbb{Q}$, and the maps v_s^+ and h_s^+ restrict to surjections

$$(v_s^+)_*, (h_s^+)_* : \mathcal{T}^+ \to H_*(B^+) \cong \mathcal{T}^+.$$

Each of these maps is then multiplication by some nonnegative power of U, and we define

$$V_s(K), H_s(K) \in \mathbb{Z}_{\geq 0}$$

to be these exponents.

Proposition 2.6 ([NW15, HLZ15]). *The invariants $V_s = V_s(K)$ and $H_s = H_s(K)$ satisfy the following constraints.*

(1) $V_s \geq V_{s+1}$ *and* $H_s \leq H_{s+1}$ *for all* $s \in \mathbb{Z}$. [NW15, Lemma 2.4]
(2) $V_s = 0$ *for all* $s \geq g(K)$. [NW15, §2.2]
(3) $V_{-s} = V_s + s$ *for all* $s \in \mathbb{Z}$. [HLZ15, Lemma 2.5]
(4) $H_{-s} = V_s$ *for all* $s \in \mathbb{Z}$. [HLZ15, Lemma 2.3]
(5) $V_{s+1} \leq V_s \leq V_{s+1} + 1$ *for all* $s \in \mathbb{Z}$.

Proof. Only the inequality $V_s \leq V_{s+1} + 1$ of item (5) needs to be proved. Combining the other parts of the proposition, we have

$$V_s = V_{-s} - s \leq V_{-s-1} - s = \big(V_{-(s+1)} - (s+1)\big) + 1 = V_{s+1} + 1$$

as desired. □

The following results relate the invariants $V_s(K)$ and $H_s(K)$ to $HF^+(S^3_{p/q}(K))$.

Theorem 2.7 ([NW15, Proposition 1.6]). *Given relatively prime $p, q > 0$ and an integer i with $0 \leq i \leq p-1$, we have*

$$d(S^3_{p/q}(K), i) - d(S^3_{p/q}(U), i) = -2\max\big(V_{\lfloor \frac{i}{q} \rfloor}(K), H_{\lfloor \frac{i-p}{q} \rfloor}(K)\big).$$

Lemma 2.8. *If $K \subset S^3$ has genus $g \geq 1$, then there is a short exact sequence*

$$0 \to \widehat{HFK}_{*+2}(K,g) \to H_*(A^+_{g-1}) \xrightarrow{(v^+_{g-1})_*} H_*(B^+) \to 0$$

of $\mathbb{Q}[U]$-modules, in which $\widehat{HFK}(K,g)$ has trivial U-action and A^+_{g-1} and B^+ are equipped with absolute gradings as quotients of $CFK^\infty(K)$. In particular, for $N \geq 2g-1$ we have $U \cdot HF^+_{\mathrm{red}}(S^3_N(K), g-1) = 0$ and

$$\dim HF^+_{\mathrm{red}}(S^3_N(K), g-1) = \dim \widehat{HFK}(K,g) - V_{g-1}(K).$$

Proof. The short exact sequence is [OS19, Lemma 3.3]. To prove it, we use the short exact sequence of chain complexes

$$0 \to C\{-1, g-1\} \to A^+_{g-1} \xrightarrow{v^+_{g-1}} B^+ \to 0 \tag{2.1}$$

defined by the natural inclusion and projection maps, which induces a long exact sequence

$$\cdots \to H_*(C\{-1, g-1\}) \to H_*(A^+_{g-1}) \xrightarrow{(v^+_{g-1})_*} H_*(B^+) \to \dots$$

on homology. The complex $C\{-1, g-1\}$ has zero differential and trivial U action, and it is the image under U of

$$C\{0, g\} \cong \widehat{HFK}(K,g),$$

hence its homology is just $\widehat{HFK}_{*+2}(K,g)$. Meanwhile we know that $H_*(B^+) \cong \mathcal{T}^+$, and v^+_{g-1} is an isomorphism in all sufficiently large gradings, so it follows that $H_*(A^+_{g-1})$ also contains a tower $\mathcal{T}^+$ which surjects onto $H_*(B^+)$. Thus the long exact sequence splits.

The claim about $\dim HF^+_{\mathrm{red}}(S^3_N(K), g-1)$ now follows quickly from Theorem 2.3, because we can identify $\ker v^+_{g-1}$ with all of $HF^+_{\mathrm{red}}(S^3_N(K), g-1)$ plus whatever portion of $\mathcal{T}^+ \subset H_*(A^+_{g-1})$ is in the kernel, and the latter has dimension V_{g-1} by definition. □

Although Theorem 2.5 as stated only determines the relative grading on $HF^+(S^3_{p/q}(K))$, we can use the integers V_s and H_s to recover the absolute grading by Theorem 2.7.

Proposition 2.9. *Suppose for some knots $K, K' \subset S^3$ and some relatively prime $p, q > 0$ that*
$$HF^+(S^3_{p/q}(K)) \cong HF^+(S^3_{p/q}(K'))$$
as graded $\mathbb{Q}[U]$-modules. Then we have $\Delta''_K(1) = \Delta''_{K'}(1)$. Moreover, if $g(K) = 1$ then $V_0(K) = V_0(K')$, and if in addition $\frac{p}{q} > 1$ then $V_s(K') = 0$ for all $s \geq 1$.

Proof. Let Y be the rational homology 3-sphere $S^3_{p/q}(K)$. Rustamov [Rus04, Theorem 3.3] proved that its Casson–Walker invariant satisfies
$$|H_1(Y;\mathbb{Z})|\lambda(Y) = \sum_{\mathfrak{s}\in\mathrm{Spin}^c(Y)} \left(\chi(HF^+_{\mathrm{red}}(Y,\mathfrak{s})) - \tfrac{1}{2}d(Y,\mathfrak{s})\right),$$
and the right hand side is completely determined by $HF^+(Y)$, hence so is $\lambda(Y)$. The surgery formula for the Casson–Walker invariant [Wal92, Theorem 4.2] then says that
$$\lambda(Y) - \lambda(S^3_{p/q}(U)) = \frac{q}{p}\frac{\Delta''_K(1)}{2},$$
so we conclude that $\Delta''_K(1)$ is determined by $\frac{p}{q}$ and $HF^+(S^3_{p/q}(K))$. By hypothesis the same data determines $\Delta''_{K'}(1)$ in exactly the same way, so these second derivatives are equal.

Now suppose that $g(K) = 1$. Then Proposition 2.6 says that $V_s(K) = H_{-s}(K) = 0$ for all $s \geq 1$, and then that $V_0(K)$ is either 0 or 1 since $V_1(K) = 0$. We therefore have
$$d(S^3_{p/q}(K), i) - d(S^3_{p/q}(U), i) = \begin{cases} -2V_0(K), & 0 \leq i \leq \min(p,q) - 1 \\ 0, & \min(p,q) \leq i \leq p - 1 \end{cases}$$
by Theorem 2.7. It follows that
$$\sum_{i\in\mathbb{Z}/p\mathbb{Z}} \Big(d(S^3_{p/q}(K), i) - d(S^3_{p/q}(U), i)\Big) = -2V_0(K)\cdot\min(p,q). \tag{2.2}$$
By the same argument we have
$$\sum_{i\in\mathbb{Z}/p\mathbb{Z}} \Big(d(S^3_{p/q}(K'), i) - d(S^3_{p/q}(U), i)\Big) \leq -2V_0(K')\cdot\min(p,q), \tag{2.3}$$
and the left sides of (2.2) and (2.3) are equal, so $V_0(K') \leq V_0(K) \leq 1$. If $V_0(K') = 0$ then $V_s(K') = 0$ for all $s \geq 0$, so the left side of (2.3) is equal to 0, hence $V_0(K) = 0$ as well. Otherwise $V_0(K') = 1$ implies that $V_0(K) = 1$, so in any case we have $V_0(K) = V_0(K')$.

Finally, if $V_0(K) = V_0(K') = 1$ and $p > q$ then we have by Theorem 2.7 that
$$\begin{aligned} d(S^3_{p/q}(K'), q) - d(S^3_{p/q}(U), q) &= -2\max(V_1(K'), H_{\lfloor\frac{q-p}{q}\rfloor}(K')) \\ &\leq -2V_1(K'), \end{aligned}$$
which implies that the left side of (2.3) is at most $-2qV_0(K') - 2V_1(K')$. But this is equal to the left side of (2.2), which is equal to
$$-2V_0(K)\cdot q = -2qV_0(K'),$$
so we must have $V_1(K') = 0$. Then $V_s(K') = 0$ for all $s \geq 1$ by Proposition 2.6. $\square$

2.2. **Computations for nearly fibered knots.** In this subsection we work out some examples of the large surgery formula. Let K be a genus-1 knot for which $\widehat{HFK}(K,1)$ is 2-dimensional. Then K is one of the knots listed in Theorem 1.5, with $\widehat{HFK}(K)$ shown in Table 1, and in every case there is some integer $m \in \mathbb{Z}$ such that

$$\widehat{HFK}(K,1) \cong \mathbb{Q}^2_{(m)},$$

where the subscripts denote the Maslov grading. (For the mirrors of the knots in Table 1, this follows from the relation $\widehat{HFK}_m(\overline{K},a) \cong \widehat{HFK}_{-m}(K,-a)$.)

We first determine $HF^+(S^3_1(K))$ in the cases where K is either 5_2 or its mirror.

Proposition 2.10. *We have*

$$HF^+(S^3_1(5_2)) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^2_{(0)} \quad \textit{and} \quad HF^+(S^3_1(\overline{5_2})) \cong \mathcal{T}^+_{(-2)} \oplus \mathbb{Q}_{(-2)}$$

as graded $\mathbb{Q}[U]$*-modules.*

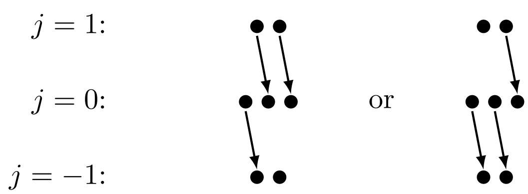
Proof. In these cases K is alternating, so $\widehat{HFK}(K)$ is *thin* — there is some $s \in \mathbb{Z}$ such that each $\widehat{HFK}(K,a)$ is supported in homological grading $a - s$ — and for alternating knots we have $s = -\frac{1}{2}\sigma(K)$ [OS03b, Theorem 1.3], where $\sigma(K)$ is the signature. (This uses the convention that positive knots such as $\overline{5_2}$ have negative signature, so $\sigma(\overline{5_2}) = -2$ and $\sigma(5_2) = 2$.) In this case the differential on $CFK^\infty(K)$ has a fairly simple form, namely

$$\partial^\infty\left(C\{i_0,j_0\}\right) \subset C\{i_0-1,j_0\} \oplus C\{i_0,j_0-1\},$$

by the fact that $\deg(\partial^\infty) = -1$. Since $H_*(C\{i=0\}) \cong \mathbb{Q}$ is supported at Alexander grading $j = \tau(K)$ in homological grading 0, we have $\tau(K) = -\frac{1}{2}\sigma(K)$, so

$$\tau(5_2) = -1 \quad \text{and} \quad \tau(\overline{5_2}) = 1.$$

We can therefore find bases for the complexes $(C\{i=0\}, \partial')$ so that they are represented by the diagrams



for 5_2 and $\overline{5_2}$ respectively. (Here each dot represents a generator of a $\mathbb{Q}$ summand, and an arrow of the form "$\bullet \to \bullet$" means that the corresponding generators x and y satisfy $\partial' x = y$.) In turn, this together with the chain homotopy equivalence $C\{i=0\} \simeq C\{j=0\}$ and the requirement that $(\partial^\infty)^2 = 0$ completely determines $CFK^\infty(K)$ for each of these knots K.

Now by inspecting Figure 2 we see that

$$H_*(A^+_0(5_2)) \cong \frac{\mathbb{Q}[U,U^{-1}]}{U \cdot \mathbb{Q}[U]}\langle d-a\rangle \oplus \mathbb{Q}\langle a,b\rangle \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^2_{(0)}, \tag{2.4}$$

since the indicated elements a,b,d all have homological grading $-1 + \frac{\sigma(5_2)}{2} = 0$. The homology $H_*(B^+(5_2)) \cong \mathcal{T}^+$ has bottom-most element $[d] = (v^+_0)_*([d-a])$, so then $(v^+_0)_*|_{\mathcal{T}^+} : \mathcal{T}^+ \to \mathcal{T}^+$ is an isomorphism and we have $V_0(5_2) = 0$. Now Theorem 2.3 says that $HF^+(S^3_1(5_2)) \cong H_*(A^+_0(5_2))$ as relatively graded groups, while Theorem 2.7 says

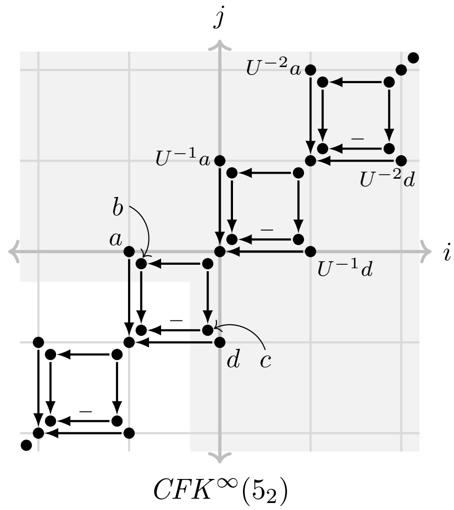


FIGURE 2. The complexes $(CFK^\infty(K), \partial^\infty)$ for $K = 5_2$ and $K = \overline{5_2}$, with A_0^+ shaded. The dots represent generators of $C\{i,j\}$, all of which lie in grading $i + j + \frac{\sigma(K)}{2}$. Minus signs on arrows indicate a coefficient of -1.

that the tower $\mathcal{T}^+$ in $HF^+(S_1^3(5_2))$ has bottom-most grading $d(S_1^3(5_2)) = -2V_0(5_2) = 0$, so we conclude that $HF^+(S_1^3(5_2))$ is exactly as claimed.

Similarly, we see from Figure 2 that

$$H_*(A_0^+(\overline{5_2})) \cong \frac{\mathbb{Q}[U,U^{-1}]}{U \cdot \mathbb{Q}[U]}\langle x\rangle \oplus \mathbb{Q}\langle y\rangle \cong \mathcal{T}^+_{(-2)} \oplus \mathbb{Q}_{(-2)},$$

since the indicated elements $x, y \in C\{-1, 0\}$ have homological grading $-1 + \frac{\sigma(\overline{5_2})}{2} = -2$. The kernel of $(v_0^+)_*$ contains $[x]$ but not $[U^{-1}x]$, so the restriction $(v_0^+)_*|_{\mathcal{T}^+} : \mathcal{T}^+ \to \mathcal{T}^+$ is multiplication by U, hence $V_0(\overline{5_2}) = 1$. Now we conclude exactly as before that $d(S_1^3(\overline{5_2})) = -2V_0(\overline{5_2}) = -2$ and hence that $HF^+(S_1^3(\overline{5_2}))$ is exactly as claimed. $\square$

For the knots of Theorem 1.5 other than 5_2 and $\overline{5_2}$, it is a little bit harder to determine $CFK^\infty(K)$. We will avoid this problem by using the large surgery formula to compute $\widehat{HF}(S_1^3(K))$, and then deducing $HF^+(S_1^3(K))$ from this in Proposition 2.14.

Proposition 2.11. *Let K be a genus-1 knot for which $\widehat{HFK}(K,1) \cong \mathbb{Q}^2_{(m_0+1)}$. If K is neither 5_2 nor its mirror, then $\tau(K) = 0$ and*

$$\widehat{HF}(S_1^3(K)) \cong \mathbb{Q}_{(0)} \oplus \left(\mathbb{Q}_{(m_0)} \oplus \mathbb{Q}_{(m_0-1)}\right)^{\oplus 2}$$

as relatively graded $\mathbb{Q}$-vector spaces.

Proof. We attempt to construct the full knot Floer complex $CFK^\infty(K)$. The relation

$$\widehat{HFK}_m(K,a) \cong \widehat{HFK}_{m-2a}(K,-a)$$

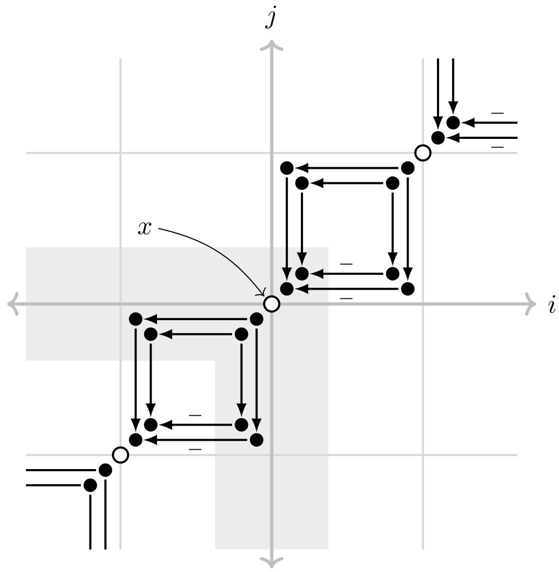


FIGURE 3. The complex $(CFK^\infty(K), \partial^\infty)$, with $\hat{A}_0$ shaded and possible diagonal arrows omitted. The black dots represent generators of $C\{i,j\}$ in grading m_0+i+j, while the white dots represent generators $U^{-i}x$ in grading $i+j$. The minus signs on some arrows indicate a coefficient of -1 in ∂^∞.

tells us that if $\widehat{HFK}(K,1) \cong \mathbb{Q}^2_{(m_0+1)}$ then $\widehat{HFK}(K,-1) \cong \mathbb{Q}^2_{(m_0-1)}$, so the model complex $(C\{i=0\}, \partial')$ for $\widehat{CF}(S^3)$ has the form

$$\begin{tikzcd} \mathbb{Q}^2_{(m_0+1)} \arrow[d, "\partial'_1"] \arrow[dd, bend right=60, "\partial'_2"'] \\ \widehat{HFK}(K,0) \arrow[d, "\partial'_0"] \\ \mathbb{Q}^2_{(m_0-1)}, \end{tikzcd}$$

and in fact the ∂'_2 component of the differential must be zero since it cannot lower the grading by 2.

If K is neither 5_2 nor its mirror, then we can read $\dim \widehat{HFK}(K,0) = 5$ off of Table 1, and so $H_*(C\{i=0\}, \partial') \cong \mathbb{Q}$ is only possible if ∂'_1 is injective and ∂'_2 is surjective. Moreover, the homology is necessarily supported in Alexander grading $j=0$, so $\tau(K)=0$. This completely determines the i-preserving (vertical) component of ∂^∞, as illustrated in Figure 3. The chain homotopy equivalence $C\{i=0\} \simeq C\{j=0\}$ and the relation $(\partial^\infty)^2 = 0$ then nearly suffice to determine ∂^∞; the only ambiguity is whether there are any arrows involving the generators $U^k x \in C\{i=j=-k\}$, and these must be diagonal (meaning neither vertical nor horizontal) if they exist.

This discussion completely determines the subquotient complex $\hat{A}_0$, which is shaded in Figure 3, since it does not see any diagonal arrows that might exist in $CFK^\infty(K)$. The complex has nine generators, only two of which have nonzero differentials, and the hat version of the large surgery formula in Theorem 2.3 tells us that

$$\widehat{HF}(S^3_1(K)) \cong H_*(\hat{A}_0) \cong \mathbb{Q}_0 \oplus \mathbb{Q}^2_{(m_0)} \oplus \mathbb{Q}^2_{(m_0-1)}$$

as relatively graded vector spaces. $\square$

If Y is an arbitrary 3-manifold with torsion Spin^c structure $\mathfrak{s}$, so that its homological grading is $\mathbb{Z}$-valued, then the short exact sequence of complexes

$$0 \to \widehat{CF}_*(Y,\mathfrak{s}) \to CF^+_*(Y,\mathfrak{s}) \xrightarrow{U} CF^+_{*-2}(Y,\mathfrak{s}) \to 0$$

turns into a long exact sequence of $\mathbb{Q}[U]$-modules

$$\cdots \to HF^+_{*+1}(Y,\mathfrak{s}) \xrightarrow{U} HF^+_{*-1}(Y,\mathfrak{s}) \to \widehat{HF}_*(Y,\mathfrak{s}) \to HF^+_*(Y,\mathfrak{s}) \xrightarrow{U} HF^+_{*-2}(Y,\mathfrak{s}) \to \dots,$$

from which we can extract a short exact sequence

$$0 \to \frac{HF^+_{*-1}(Y,\mathfrak{s})}{U \cdot HF^+_{*+1}(Y,\mathfrak{s})} \to \widehat{HF}_*(Y,\mathfrak{s}) \to \ker(U|_{HF^+_*(Y,\mathfrak{s})}) \to 0. \tag{2.5}$$

Equation (2.5) immediately implies the following.

Lemma 2.12. *If $U \cdot HF^+_{\mathrm{red}}(Y,\mathfrak{s}) = 0$ and we have an isomorphism*

$$HF^+(Y) \cong \mathcal{T}^+_{(d)} \oplus \bigoplus_{i=1}^k \mathbb{Q}_{(n_i)}$$

of graded $\mathbb{Q}[U]$-modules, where the (d) subscript denotes the grading of $\ker(U) \subset \mathcal{T}^+$, then

$$\widehat{HF}(Y) \cong \mathbb{Q}_{(d)} \oplus \bigoplus_{i=1}^k \left(\mathbb{Q}_{(n_i+1)} \oplus \mathbb{Q}_{(n_i)}\right).$$

This implies in particular that $\dim \widehat{HF}(Y,\mathfrak{s}) = 1 + 2\dim HF^+_{\mathrm{red}}(Y,\mathfrak{s})$.

Corollary 2.13. *If $K \subset S^3$ has genus $g \geq 1$, then*

$$\frac{\dim \widehat{HF}(S^3_{2g-1}(K), g-1) - 1}{2} = \dim \widehat{HFK}(K,g) - V_{g-1}(K).$$

Proof. Lemma 2.8 says that $U \cdot HF^+_{\mathrm{red}}(S^3_{2g-1}(K), g-1) = 0$ and that

$$\dim HF^+_{\mathrm{red}}(S^3_{2g-1}(K), g-1) = \dim \widehat{HFK}(K,g) - V_{g-1}(K).$$

Now apply Lemma 2.12. $\square$

Proposition 2.14. *Let K be a genus-1 knot for which $\dim_{\mathbb{Q}} \widehat{HFK}(K,1) = 2$. If K is neither 5_2 nor its mirror, then $V_0(K) = 0$ and*

$$HF^+(S^3_1(K)) \cong \mathcal{T}^+_{(0)} \oplus \widehat{HFK}(K,-1)$$

as graded $\mathbb{Q}[U]$-modules.

Proof. We write $\widehat{HFK}(K,1) \cong \mathbb{Q}^2_{(m_0+1)}$ as before, and then the symmetry

$$\widehat{HFK}_m(K,a) \cong \widehat{HFK}_{m-2a}(K,-a)$$

of [OS04b, Equation (2)] implies that $\widehat{HFK}(K,-1) \cong \mathbb{Q}^2_{(m_0-1)}$.

We observe from Lemma 2.8 that $U \cdot HF^+_{\mathrm{red}}(S^3_1(K)) = 0$, since $g(K) = 1$. In Proposition 2.11 we saw that $\dim \widehat{HF}(S^3_1(K)) = 5$, so Lemma 2.12 says that $\dim HF^+_{\mathrm{red}}(S^3_1(K)) = 2$. But then

$$V_0(K) = \dim \widehat{HFK}(K,1) - \dim HF^+_{\mathrm{red}}(S^3_1(K)) = 2 - 2 = 0$$

by another application of Lemma 2.8. With this information at hand, Theorem 2.7 tells us that

$$d(S^3_1(K)) = d(S^3_1(U)) - 2V_0(K) = 0.$$

Now if we write

$$HF^+(S^3_1(K)) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}_{(d)} \oplus \mathbb{Q}_{(e)}$$

for some integers d and e, then Lemma 2.12 says that

$$\widehat{HF}(S^3_1(K)) \cong \mathbb{Q}_{(0)} \oplus \mathbb{Q}_{(d)} \oplus \mathbb{Q}_{(d+1)} \oplus \mathbb{Q}_{(e)} \oplus \mathbb{Q}_{(e+1)}.$$

Up to translation by an overall constant, Proposition 2.11 says that these gradings are $0, m_0, m_0, m_0 - 1, m_0 - 1$ in some order. This is only possible if that constant is zero and $d = e = m_0 - 1$, except possibly if $m_0 = -1$ and $\{d,e\} = \{0,1\}$. But we can rule out this last case because it would imply that $S^3_1(K)$ has Casson invariant

$$\lambda(S^3_1(K)) = \chi(HF^+_{\mathrm{red}}(S^3_1(K))) - \frac{1}{2}d(S^3_1(K)) = 0 - 0 = 0$$

by [OS03a, Theorem 1.3], and yet $\lambda(S^3_1(K)) = \frac{\Delta''_K(1)}{2} = \pm 2$ by the surgery formula for the Casson invariant. This completes the proof. $\square$

3. The dimension of $\widehat{HF}$

3.1. **The invariants $\hat{r}_0$ and $\hat{\nu}$.** For a fixed knot $K \subset S^3$, the dimension of $\widehat{HF}(S^3_{p/q}(K))$ varies in a predictable way with p and q. We will make use of this where possible, since it is easier to apply in practice than the mapping cone formula.

Proposition 3.1. *Let $K \subset S^3$ be a knot. Then there are integers $\hat{r}_0(K)$ and $\hat{\nu}(K)$ such that*

$$\dim_{\mathbb{Q}} \widehat{HF}(S^3_{p/q}(K)) = q \cdot \hat{r}_0(K) + |p - q\hat{\nu}(K)|$$

for all coprime integers $p \neq 0$ and $q > 0$.

Hanselman [Han23, Proposition 15] proved a version of Proposition 3.1 with coefficients in $\mathbb{Z}/2\mathbb{Z}$, though he pointed out that it can be extracted from [OS11, Proposition 9.6], where it is proved with the desired $\mathbb{Q}$ coefficients. (It can also be proved in exactly the same way as its instanton Floer analogue [BS21, Theorem 1.1], using only the surgery exact triangle and an adjunction inequality.) In fact, if the Heegaard Floer ν invariant of K [OS11, Definition 9.1] satisfies $\nu(K) \geq \nu(\overline{K})$, then [OS11, Equation (40)] implies the relation

$$\hat{r}_0(K) - \hat{\nu}(K) = \sum_{s \in \mathbb{Z}} \left(\dim H_*(\hat{A}_s) - 1 \right).$$

Moreover, we know from [BS21, Lemma 10.4] that

$$\hat{\nu}(K) = \begin{cases} \max(2\nu(K)-1,0), & \nu(K) \geq \nu(\overline{K}) \\ -\max(2\nu(\overline{K})-1,0), & \nu(K) \leq \nu(\overline{K}). \end{cases} \tag{3.1}$$

Proposition 3.2. *The invariants $\hat{r}_0(K)$ and $\hat{\nu}(K)$ satisfy the following properties.*

(1) The invariants of K and its mirror are related by $(\hat{r}_0(\overline{K}), \hat{\nu}(\overline{K})) = (\hat{r}_0(K), -\hat{\nu}(K))$.
(2) The difference $\hat{r}_0(K) - |\hat{\nu}(K)|$ is a nonnegative even integer.
(3) $\hat{\nu}$ is a smooth concordance invariant, and

$$|\hat{\nu}(K)| \leq \max(2g_4(K)-1,0)$$

where g_4 denotes the smooth 4-ball genus.
(4) The invariant $\hat{\nu}(K)$ is either odd or zero.
(5) If $V_0(K)=0$ then $\hat{\nu}(K) \leq 0$.
(6) If $\hat{\nu}(K) \leq 0$ then $\tau(K) \leq 0$, and if $\hat{\nu}(K)=0$ then $\tau(K)=0$.

Proof. Claim (1) is immediate from Proposition 3.1 and the relation $S^3_r(\overline{K}) \cong -S^3_{-r}(K)$, together with the fact that $\dim \widehat{HF}(Y) = \dim \widehat{HF}(-Y)$ for all Y. For (2), we choose a positive integer $p > \hat{\nu}(K)$ and apply Proposition 3.1 to get

$$\dim \widehat{HF}(S^3_p(K)) = p + (\hat{r}_0(K) - \hat{\nu}(K)),$$

so by [OS04c, Proposition 5.1] we have

$$\dim \widehat{HF}(S^3_p(K)) - \chi(\widehat{HF}(S^3_p(K))) = \hat{r}_0(K) - \hat{\nu}(K).$$

The left hand side is twice the dimension of the odd-graded part of $\widehat{HF}(S^3_p(K))$, so it is evidently nonnegative and even. The same is true of

$$\hat{r}_0(\overline{K}) - \hat{\nu}(\overline{K}) = \hat{r}_0(K) + \hat{\nu}(K),$$

so in either case $\hat{r}_0(K) - |\hat{\nu}(K)|$ is nonnegative and even as well. Since $\nu(K)$ and $\nu(\overline{K})$ are smooth concordance invariants, claims (3) and (4) follow immediately from (3.1) and the fact that $|\nu(K)| \leq g_4(K)$.

In order to prove (5), we use the invariant $\nu^+(K)$ [HW16], which is by definition the smallest s such that $V_s(K)=0$. If $V_0(K)=0$ then [HW16, Proposition 2.3] tells us that

$$\tau(K) \leq \nu(K) \leq \nu^+(K) = 0,$$

and since $\nu(\overline{K})$ is equal to either $\tau(\overline{K})$ or $\tau(\overline{K})+1$ (see [OS11, Equation (34)]) we have

$$\nu(\overline{K}) \geq \tau(\overline{K}) = -\tau(K) \geq 0 \geq \nu(K).$$

Now (3.1) tells us that $\hat{\nu}(K) = -\max(2\nu(\overline{K})-1,0) \leq 0$. We prove the contrapositive of the first part of (6) similarly: if $\tau(K) \geq 1$ then $\nu(K) \geq \tau(K) \geq 1$ while

$$\nu(\overline{K}) \leq \tau(\overline{K}) + 1 = -\tau(K) + 1 \leq 0,$$

so $\nu(K) > \nu(\overline{K})$, and then (3.1) gives us $\hat{\nu}(K) \geq 2\nu(K) - 1 \geq 1$. Moreover, if $\hat{\nu}(K)=0$ then $\hat{\nu}(\overline{K})=0$ as well, so we have just shown that $\tau(K) \leq 0$ and $-\tau(K) = \tau(\overline{K}) \leq 0$, hence $\tau(K)=0$ as claimed. □

Proposition 3.2 can also be proved by repeating arguments from [BS21] nearly verbatim, but applied to $\widehat{HF}(Y)$ rather than $I^{\#}(Y)$. These arguments rely only on the fact that $\dim \widehat{HF}(S^3) = 1$, together with the surgery exact triangle and adjunction inequality for $\widehat{HF}$.

We note the following examples for later use.

Lemma 3.3. *Suppose that K is one of the genus-1 knots appearing in Theorem 1.5 other than 5_2 and its mirror. Then*

$$(\hat{r}_0(K), \hat{\nu}(K)) = (4, 0).$$

We also have $(\hat{r}_0(\overline{5_2}), \hat{\nu}(\overline{5_2})) = (3, 1)$ and $(\hat{r}_0(5_2), \hat{\nu}(5_2)) = (3, -1)$.

Proof. Proposition 2.11 applies to both K and $\overline{K}$ to tell us that

$$\dim \widehat{HF}(S^3_1(K)) = 5 \quad \text{and} \quad \dim \widehat{HF}(S^3_{-1}(K)) = \dim \widehat{HF}(S^3_1(\overline{K})) = 5.$$

By Proposition 3.1, these can only be equal if $\hat{\nu}(K) = 0$, and then

$$5 = \dim \widehat{HF}(S^3_1(K)) = 1 \cdot \hat{r}_0(K) + |1 - 0 \cdot \hat{\nu}(K)|$$

implies that $\hat{r}_0(K) = 4$.

Similarly, we note from Proposition 2.10 and Lemma 2.12 that

$$\dim \widehat{HF}(S^3_1(\overline{5_2})) = 3 \quad \text{and} \quad \dim \widehat{HF}(S^3_{-1}(\overline{5_2})) = \dim \widehat{HF}(S^3_1(5_2)) = 5.$$

Now Proposition 3.1 only tells us that $\hat{\nu}(\overline{5_2}) \geq 1$, but Proposition 3.2 also bounds it above by 1 and so $\hat{\nu}(\overline{5_2}) = 1$ after all. It now follows immediately that $\hat{r}_0(\overline{5_2}) = 3$, and similarly for 5_2. □

3.2. **Almost L-space knots.** A nontrivial knot $K \subset S^3$ is said to be an *L-space knot* if $S^3_r(K)$ is an L-space for some rational slope $r > 0$, meaning that $\dim \widehat{HF}(S^3_r(K)) = |H_1(S^3_r(K); \mathbb{Z})|$. This places strong restrictions on K.

Theorem 3.4 ([OS05, Ghi08, Ni07, Hed10, OS11])**.** *If K is an L-space knot, then K is fibered and strongly quasipositive, and r-surgery on K is an L-space if and only if $r \geq 2g(K) - 1$.*

Remark 3.5. It follows quickly that a knot K of genus $g \geq 1$ is an L-space knot if and only if $\hat{r}_0(K) = \hat{\nu}(K) = 2g - 1$.

In this section, we develop similar restrictions on knots which fall just short of being L-space knots. We recall the following from Definition 1.10.

Definition 3.6. A knot $K \subset S^3$ is an *almost L-space knot* if

$$\dim_{\mathbb{Q}} \widehat{HF}(S^3_n(K)) = n + 2$$

for some $n \geq 2g(K) - 1$.

Lemma 3.7. *A knot $K \subset S^3$ is an almost L-space knot if and only if $\hat{r}_0(K) - \hat{\nu}(K) = 2$.*

Proof. We note that K must be nontrivial since all surgeries on the unknot are L-spaces. Using the inequality

$$\hat{\nu}(K) \leq \max(2g_4(K) - 1, 0) \leq 2g(K) - 1$$

of Proposition 3.2, it follows that if $n \geq 2g(K)-1$ then

$$\dim \widehat{HF}(S^3_n(K)) = \hat{r}_0(K) + |n - \hat{\nu}(K)| = n + (\hat{r}_0(K) - \hat{\nu}(K)).$$

By assumption the left side is $n+2$ for some such n, which proves the lemma. $\square$

Lemma 3.8. *If $K \subset S^3$ is an almost L-space knot of genus $g \geq 1$, then*

$$\widehat{HF}(S^3_{2g-1}(K), s) \cong \begin{cases} \mathbb{Q}^3 & s = 0 \\ \mathbb{Q} & 1 \leq |s| \leq g-1, \end{cases}$$

and similarly there is some $n \geq 1$ such that

$$HF^+(S^3_{2g-1}(K), s) \cong \begin{cases} \mathcal{T}^+ \oplus \mathbb{Q}[U]/U^n & s = 0 \\ \mathcal{T}^+ & 1 \leq |s| \leq g-1 \end{cases}$$

as $\mathbb{Q}[U]$-modules.

Proof. Let $Y = S^3_{2g-1}(K)$. By Lemma 3.7 and $\hat{\nu}(K) \leq 2g-1$ we have

$$\sum_{s \in \mathbb{Z}/(2g-1)\mathbb{Z}} \dim \widehat{HF}(Y, s) = \dim \widehat{HF}(Y) = 2g+1.$$

Each $\widehat{HF}(Y,s)$ has Euler characteristic 1 [OS04c, Proposition 5.1] and hence odd dimension. Since the total dimension is $2g+1$ there must be a unique s_0 with

$$\dim \widehat{HF}(Y, s_0) = 3$$

and $\dim \widehat{HF}(Y,s) = 1$ for all other $s \neq s_0$. But we have

$$\widehat{HF}(Y, s_0) \cong \widehat{HF}(Y, -s_0)$$

by conjugation symmetry [OS04c, Theorem 2.4], recalling from Remark 2.4 that s and $-s$ determine conjugate Spin^c structures, so $-s_0 \equiv s_0 \pmod{2g-1}$ and therefore $s_0 = 0$.

In order to pass from $\widehat{HF}$ to HF^+, we use the exact triangle (2.5) to see that if

$$HF^+(Y,s) \cong \mathcal{T}^+ \oplus \left(\bigoplus_{i=1}^{k} \mathbb{Q}[U]/U^{n_i} \right)$$

as $\mathbb{Q}[U]$-modules for some $k \geq 0$ and $n_1, \dots, n_k \geq 1$, then

$$\dim \widehat{HF}(Y,s) = \dim \operatorname{coker}(U) + \dim \ker(U) = k + (k+1) = 2k+1.$$

From this we conclude that $k = 1$ if $s \equiv 0 \pmod{2g-1}$ and $k = 0$ otherwise, proving the lemma. $\square$

Proposition 3.9. *Let K be an almost L-space knot of genus $g \geq 1$. Then exactly one of the following must hold.*

- *$g = 1$, and K is the left-handed trefoil, figure eight, or $\overline{5_2}$.*
- *$g \geq 2$, and K is fibered with $V_{g-1}(K) = 1$.*

Proof. According to Lemma 2.8 we have

$$\dim HF^+_{\mathrm{red}}(S^3_{2g-1}(K), g-1) = \dim \widehat{HFK}(K, g) - V_{g-1}(K). \tag{3.2}$$

We also recall from Proposition 2.6 that $V_g(K) = 0$ and $V_g(K) \leq V_{g-1}(K) \leq V_g(K)+1$, so $V_{g-1}(K)$ is either 0 or 1.

Now suppose that $g = 1$. In this case, we know by Lemma 3.8 that

$$HF^+(S^3_1(K)) \cong \mathcal{T}^+ \oplus \mathbb{Q}[U]/U^n$$

for some $n \geq 1$, and Lemma 2.8 says that the U-action on $HF^+_{\mathrm{red}}(S^3_1(K)) \cong \mathbb{Q}[U]/U^n$ is trivial, so $n = 1$. Then $\dim HF^+_{\mathrm{red}}(S^3_1(K), 0) = 1$, and (3.2) becomes

$$\dim \widehat{HFK}(K, 1) = \begin{cases} 1, & V_0(K) = 0 \\ 2, & V_0(K) = 1. \end{cases}$$

Thus if $V_0(K) = 0$ then K is fibered [Ghi08], and the right-handed trefoil is an L-space knot, so K must be the left-handed trefoil or the figure eight instead; and in the remaining cases we have $V_0(K) = 1$ and $\dim \widehat{HFK}(K, 1) = 2$. In these cases, Propositions 2.10 and 2.14 tell us that

$$V_0(K) = -\frac{1}{2}d(S^3_1(K)) = \begin{cases} 1, & K \cong \overline{5_2} \\ 0, & K \not\cong \overline{5_2}, \end{cases} \tag{3.3}$$

so K must be $\overline{5_2}$.

From now on we suppose that $g \geq 2$. Here the Spin^c structures 0 and $g-1$ on $S^3_{2g-1}(K)$ are different, so by Lemma 3.8 we have

$$HF^+_{\mathrm{red}}(S^3_{2g-1}(K), g-1) = 0$$

and so (3.2) becomes $0 = \dim \widehat{HFK}(K, g) - V_{g-1}(K)$. Thus

$$\dim \widehat{HFK}(K, g) = V_{g-1}(K) \leq 1.$$

But this dimension must be positive [OS04a, Theorem 1.2], so it is equal to 1, and then this implies that K is fibered [Ni07]. □

Proposition 3.10. *If K is an almost L-space knot of genus $g \geq 2$, then $\tau(K) = g$ and so K is strongly quasipositive.*

Proof. Proposition 3.9 says that K is fibered, and that $V_{g-1}(K) = 1$. Since K is fibered, it is strongly quasipositive if and only if $\tau(K) = g$ [Hed10, Theorem 1.2]. Thus we will suppose that $\tau(K) \leq g-1$ and show that this leads to a contradiction.

The assumption that $\tau(K) \leq g-1$ is equivalent to the assertion that the map

$$H_*(C\{i = 0, j \leq g-1\}) \to H_*(C\{i = 0\}) \cong \widehat{HF}(S^3) \cong \mathbb{Q}$$

is surjective. In this case the short exact sequence of complexes

$$0 \to C\{i = 0, j \leq g-1\} \to C\{i = 0\} \to C\{0, g\} \to 0$$

gives rise to a long exact sequence in homology which splits as

$$0 \to \underbrace{H_{*+1}(C\{0, g\})}_{\cong \widehat{HFK}(K,g) \cong \mathbb{Q}} \to H_*(C\{i = 0, j \leq g-1\}) \to \underbrace{H_*(C\{i = 0\})}_{\cong \widehat{HF}(S^3) \cong \mathbb{Q}} \to 0,$$

so $H_*(C\{i = 0, j \leq g-1\}) \cong \mathbb{Q}^2$.

We now consider the short exact sequence of complexes

$$0 \to C\{i < 0, j = g-1\} \xrightarrow{\iota} \hat{A}_{g-1} \to C\{i = 0, j \leq g-1\} \to 0,$$

whose first term is equal to

$$C\{-1,g-1\} \cong C\{0,g\} \cong \widehat{HFK}(K,g) \cong \mathbb{Q}.$$

The hat version of the large surgeries formula (Theorem 2.3) tells us that

$$H_*(\hat{A}_{g-1}) \cong \widehat{HF}(S^3_{2g-1}(K), g-1) \cong \mathbb{Q}$$

by Lemma 3.8, so we get a long exact sequence

$$\cdots \to \underbrace{H_*(C\{-1,g-1\})}_{\cong\mathbb{Q}} \xrightarrow{\iota_*} \underbrace{H_*(\hat{A}_{g-1})}_{\cong\mathbb{Q}} \to \underbrace{H_*(C\{i=0, j\le g-1\})}_{\mathbb{Q}^2} \to \dots,$$

from which the map $\iota_* : H_*(C\{-1,g-1\}) \to H_*(\hat{A}_{g-1})$ is zero.

Finally, the inclusion map $C\{-1,g-1\} \hookrightarrow A^+_{g-1}$ factors through ι as

$$C\{-1,g-1\} \xrightarrow{\iota} \hat{A}_{g-1} \hookrightarrow A^+_{g-1},$$

so the induced map

$$H_*(C\{-1,g-1\}) \to H_*(A^+_{g-1})$$

on homology must be zero, since it factors through $\iota_* = 0$. But this map belongs to the short exact sequence

$$0 \to H_*(C\{-1,g-1\}) \to H_*(A^+_{g-1}) \xrightarrow{(v^+_{g-1})_*} H_*(B^+)$$

of Lemma 2.8, so it must also be injective, and since $H_*(C\{-1,g-1\}) \cong \mathbb{Q}$ is nonzero, we have a contradiction. Therefore $\tau(K) = g$, as desired. □

Corollary 3.11. *If K is an almost L-space knot of genus $g \ge 2$, then $(\hat{r}_0(K), \hat{\nu}(K)) = (2g+1, 2g-1)$.*

Proof. Proposition 3.10 says that $\tau(K) = g$. The invariant $\nu(K)$ of [OS11, Definition 9.1] is equal to either $\tau(K)$ or $\tau(K)+1$ by [OS11, Equation (34)], but it is also at most g by definition, so we have

$$\nu(K) = g \quad \text{and} \quad \nu(\overline{K}) \le \tau(\overline{K}) + 1 = -g+1.$$

Since $\nu(K) > \nu(\overline{K})$, we apply (3.1) to get $\hat{\nu}(K) = \max(2\nu(K)-1, 0) = 2g-1$. Then $\hat{r}_0(K) = 2g+1$ as well by Lemma 3.7. □

Remark 3.12. Let K be an almost L-space knot of genus $g \ge 2$. Then Lemma 3.8 and the large surgeries formula imply that $H_*(\hat{A}_s) \cong \mathbb{Q}$ for all $s \ge 1$, and so one can repeat the proof of [OS05, Theorem 1.2] to show, among other things, that

$$\dim \widehat{HFK}(K,a) = 0 \text{ or } 1 \quad \text{for all } a \ge 2,$$

hence by symmetry whenever $|a| \ge 2$; the corresponding t^a-coefficients of $\Delta_K(t)$ must then be either 0 or ± 1. We will not pursue this further here.

We conclude by noting the following consequences, which we will not use in this paper.

Theorem 3.13. *We have $\hat{r}_0(K) \le 3$ if and only if K has crossing number at most 5.*

Proof. We replace K with its mirror as needed to ensure that $\hat{\nu}(K) \geq 0$, since this does not change $\hat{r}_0(K)$. Now by Proposition 3.2 the difference $\hat{r}_0(K) - \hat{\nu}(K)$ is nonnegative and even, and we have

$$0 \leq \hat{r}_0(K) - \hat{\nu}(K) \leq \hat{r}_0(K) \leq 3,$$

so it must be either 0 or 2.

Supposing that the difference is 2, then K is an almost L-space knot by Lemma 3.7. If $g(K) \geq 2$ then Corollary 3.11 says that $\hat{r}_0(K) = 2g(K) + 1 \geq 5$, which cannot happen. So $g(K) = 1$, and then Proposition 3.9 says that K is either $T_{-2,3}$, a figure eight, or $\overline{5_2}$.

Otherwise we have $\hat{r}_0(K) = \hat{\nu}(K)$, so by Remark 3.5, if K is nontrivial then it must be a nontrivial L-space knot satisfying $\hat{r}_0(K) = 2g(K) - 1$. But then $\hat{r}_0(K) \leq 3$ implies that $g(K)$ is either 1 or 2, so K must be a right-handed trefoil [Ghi08] or a $(2, 5)$ torus knot [FRW22]. Up to mirroring we have now accounted for all knots of at most five crossings and ruled out everything else, so this completes the proof. □

Theorem 3.14. *If $\dim_{\mathbb{Q}} \widehat{HF}(S^3_1(K)) = 3$, then K is either the left-handed trefoil, figure eight, or $\overline{5_2}$.*

Proof. Proposition 3.1 says that

$$3 = \dim_{\mathbb{Q}} \widehat{HF}(S^3_1(K)) = \hat{r}_0(K) + |1 - \hat{\nu}(K)|, \tag{3.4}$$

so $\hat{r}_0(K) \leq 3$ with equality if and only if $\hat{\nu}(K) = 1$. If $\hat{\nu}(K) > 1$ then we have $3 > \hat{r}_0(K) \geq \hat{\nu}(K) > 1$, so $\hat{\nu}(K) = 2$ and this contradicts Proposition 3.2. Thus $\hat{\nu}(K) \leq 1$ and now (3.4) becomes $\hat{r}_0(K) - \hat{\nu}(K) = 2$. So K is an almost L-space knot, with genus 1 by Corollary 3.11, and now Proposition 3.9 says that it must be one of the knots claimed above. □

4. The mirror of 5_2

Our goal in this section is to prove that not only are nonnegative slopes characterizing for $\overline{5_2}$, but in fact the Heegaard Floer homology of such surgeries characterizes $\overline{5_2}$.

Theorem 4.1. *Suppose for some rational number $r \geq 0$ and knot $K \subset S^3$ that there is an isomorphism*

$$HF^+(S^3_r(K)) \cong HF^+(S^3_r(\overline{5_2}))$$

of graded $\mathbb{Q}[U]$-modules. Then K is isotopic to $\overline{5_2}$.

We recall from Lemma 3.3 that $\hat{r}_0(\overline{5_2}) = 3$ and $\hat{\nu}(\overline{5_2}) = 1$. Thus if p and q are relatively prime, with $p \neq 0$ and $q > 0$, then

$$\dim \widehat{HF}(S^3_{p/q}(\overline{5_2})) = 3q + |p - q| = \begin{cases} p + 2q, & p \geq q \\ 4q - p, & p \leq q. \end{cases} \tag{4.1}$$

Throughout this section we will make implicit use of the fact that $HF^+(Y)$ completely determines $\widehat{HF}(Y)$.

Lemma 4.2. *Suppose that $0 < \frac{p}{q} \leq 1$ and that there is an isomorphism*

$$HF^+(S^3_{p/q}(K)) \cong HF^+(S^3_{p/q}(\overline{5_2}))$$

of graded $\mathbb{Q}[U]$-modules. Then K is an almost L-space knot of genus 1.

Proof. By equation (4.1) we have

$$\begin{aligned} 4q-p &= q\cdot \hat{r}_0(K) + |p - q\hat{\nu}(K)| \\ &= \begin{cases} p + q(\hat{r}_0(K) - \hat{\nu}(K)), & \frac{p}{q} \geq \hat{\nu}(K) \\ q(\hat{r}_0(K) + \hat{\nu}(K)) - p, & \frac{p}{q} < \hat{\nu}(K). \end{cases} \end{aligned}$$

In the case $\frac{p}{q} \leq \hat{\nu}(K)$ this simplifies to $\hat{r}_0(K) + \hat{\nu}(K) = 4$, and given that

$$\hat{r}_0(K) \geq \hat{\nu}(K) \geq \frac{p}{q} > 0,$$

Proposition 3.2 says that this is only possible if $\hat{r}_0(K) = 3$ and $\hat{\nu}(K) = 1$.

Now we suppose instead that $\frac{p}{q} > \hat{\nu}(K)$, and then we have

$$4q - p = p + q(\hat{r}_0(K) - \hat{\nu}(K)),$$

or

$$\frac{p}{q} = 2 - \frac{\hat{r}_0(K) - \hat{\nu}(K)}{2}.$$

Since $0 < \frac{p}{q} \leq 1$, and $\frac{1}{2}(\hat{r}_0(K) - \hat{\nu}(K))$ is a nonnegative integer, it follows that $\frac{p}{q} = 1$ and that $\hat{r}_0(K) - \hat{\nu}(K) = 2$. But then $\hat{\nu}(K) < \frac{p}{q} = 1$, and $r_0(K) \geq |\hat{\nu}(K)|$ by Proposition 3.2, so $(\hat{r}_0(K), \hat{\nu}(K))$ must be either $(2, 0)$ or $(1, -1)$.

In all cases we have shown that K is an almost L-space knot and $|\hat{\nu}(K)| \leq 1$. According to Corollary 3.11, if $g(K) \geq 2$ then $\hat{\nu}(K) = 2g(K) - 1 \geq 3$, which is impossible, so in fact $g(K) = 1$ and the proof is complete. □

Lemma 4.3. *Suppose that $\frac{p}{q} > 1$ and that there is an isomorphism*

$$HF^+(S^3_{p/q}(K)) \cong HF^+(S^3_{p/q}(\overline{5_2}))$$

of graded $\mathbb{Q}[U]$-modules. Then K is an almost L-space knot of genus 1.

Proof. By equation (4.1) we have

$$\begin{aligned} p + 2q &= q\cdot \hat{r}_0(K) + |p - q\hat{\nu}(K)| \\ &= \begin{cases} p + q(\hat{r}_0(K) - \hat{\nu}(K)), & \frac{p}{q} \geq \hat{\nu}(K) \\ q(\hat{r}_0(K) + \hat{\nu}(K)) - p, & \frac{p}{q} \leq \hat{\nu}(K). \end{cases} \end{aligned}$$

Now if $\frac{p}{q} \geq \hat{\nu}(K)$ then this immediately reduces to

$$\hat{r}_0(K) - \hat{\nu}(K) = 2,$$

so K is an almost L-space knot by Lemma 3.7.

In the remaining case we have $\hat{\nu}(K) > \frac{p}{q} > 1$, and so the above equation becomes

$$p + 2q = q(\hat{r}_0(K) + \hat{\nu}(K)) - p,$$

or equivalently

$$\frac{p}{q} = \frac{\hat{r}_0(K) + \hat{\nu}(K)}{2} - 1. \tag{4.2}$$

Now we combine this with $\frac{p}{q} < \hat{\nu}(K)$ and rearrange to get

$$\hat{r}_0(K) - 2 < \hat{\nu}(K),$$

and then by Proposition 3.2 it follows that $\hat{r}_0(K) = \hat{\nu}(K)$ and so K is an L-space knot. Remark 3.5 says that $\hat{r}_0(K) = \hat{\nu}(K) = 2g(K) - 1$, so in fact (4.2) becomes

$$\frac{p}{q} = 2g(K) - 2.$$

By the assumption $\frac{p}{q} > 1$ it follows that $g(K) \geq 2$.

Now in either case, if we suppose that $g(K) = g \geq 2$, then we have $V_{g-1}(K) = 1$. Indeed, if K is an almost L-space knot then this is part of Proposition 3.9. If instead K is an L-space knot then it is strongly quasipositive by Theorem 3.4, so the invariant $\nu^+(K)$ of [HW16] is equal to $g(K)$ by [HW16, Proposition 3]; this is by definition the least s such that $V_s(K) = 0$, so in particular $V_{g-1}(K) = 1$ as claimed. Either way, we have $V_1(K) \geq 1$ by Proposition 2.6. But then Proposition 2.9 says that if $\frac{p}{q} > 1$ and

$$HF^+(S^3_{p/q}(\overline{5_2})) \cong HF^+(S^3_{p/q}(K))$$

then $V_s(K) = 0$ for all $s \geq 1$, so this is a contradiction. Thus $g = 1$.

We conclude that K cannot be an L-space knot, since that would have implied that $g(K) \geq 2$, and so K must be an almost L-space knot of genus 1 after all. $\square$

Combining the above lemmas yields the following.

Proposition 4.4. *Suppose that $\frac{p}{q} > 0$ and that there is an isomorphism*

$$HF^+(S^3_{p/q}(K)) \cong HF^+(S^3_{p/q}(\overline{5_2}))$$

of graded $\mathbb{Q}[U]$-modules. Then K is isotopic to $\overline{5_2}$.

Proof. We know that K is an almost L-space knot of genus 1, by Lemma 4.2 if $0 < \frac{p}{q} \leq 1$ and by Lemma 4.3 if $\frac{p}{q} > 1$. Then its Alexander polynomial must have the form

$$\Delta_K(t) = at + (1 - 2a) + at^{-1}$$

for some $a \in \mathbb{Z}$. We have $\Delta''_K(1) = 2a$, whereas $\Delta''_{\overline{5_2}}(t) = 4$, so $a = 2$ by Proposition 2.9. This proves that

$$\Delta_K(t) = \Delta_{\overline{5_2}}(t) = 2t - 3 + 2t^{-1}.$$

But none of the genus-1 knots in Proposition 3.9 have this Alexander polynomial except for $\overline{5_2}$ itself, so $K \cong \overline{5_2}$. $\square$

We can also handle zero-surgery by a somewhat different argument.

Proposition 4.5. *Suppose for some knot $K \subset S^3$ that there is an isomorphism*

$$HF^+(S^3_0(K)) \cong HF^+(S^3_0(\overline{5_2}))$$

of graded $\mathbb{Q}[U]$-modules. Then $K \cong \overline{5_2}$. Similarly, if we have an isomorphism

$$HF^+(S^3_0(K)) \cong HF^+(S^3_0(5_2))$$

then $K \cong 5_2$.

Proof. We show first that $g(K) \leq 1$. Supposing instead that K has genus $g \geq 2$, there is a non-torsion Spin^c structure $\mathfrak{s}_{g-1}$ for which $HF^+(S^3_0(K), \mathfrak{s}_{g-1}) \neq 0$, namely the one specified by $\langle c_1(\mathfrak{s}_{g-1}), [\hat{\Sigma}] \rangle = 2g - 2$ for a capped-off Seifert surface $\hat{\Sigma}$, by the isomorphism $HF^+(S^3_0(K), \mathfrak{s}_{g-1}) \cong \widehat{HFK}(K, g)$ of [OS04b, Corollary 4.5] together with the fact that $\widehat{HFK}$

detects genus [OS04a, Theorem 1.2]. On the other hand, since 5_2 and its mirror both have genus 1, we have

$$HF^+(S^3_0(5_2),\mathfrak{s}) \cong HF^+(S^3_0(\overline{5_2}),\mathfrak{s}) \cong 0$$

in all non-torsion Spin^c structures, by the adjunction inequality [OS04c, Theorem 7.1]. Thus $g \leq 1$ as claimed.

Next, we recall from Lemma 3.3 that $(\hat{r}_0(\overline{5_2}), \hat{\nu}(\overline{5_2})) = (3,1)$, so we have

$$\dim \widehat{HF}(S^3_1(\overline{5_2})) = 3, \qquad \dim \widehat{HF}(S^3_{-1}(\overline{5_2})) = 5$$

and so $\dim \widehat{HF}(S^3_0(\overline{5_2})) = 4$ by the surgery exact triangle for $\widehat{HF}$, since it differs by 1 from each of these other dimensions. We also have $\dim \widehat{HF}(S^3_0(5_2)) = 4$ by the same argument, so in either case $\dim \widehat{HF}(S^3_0(K)) = 4$, and then

$$\dim \widehat{HF}(S^3_1(K)) = 3 \text{ or } 5$$

again by the surgery exact triangle. This means that K cannot be unknotted, so $g(K) = 1$. We apply Corollary 2.13 to get

$$\dim \widehat{HFK}(K,1) - V_0(K) = \frac{\dim \widehat{HF}(S^3_1(K)) - 1}{2} = 1 \text{ or } 2,$$

and $g(K) = 1$ implies that $0 \leq V_0(K) \leq 1$ by Proposition 2.6, hence $\dim \widehat{HFK}(K,1) \leq 3$.

Now we use the fact that $HF^+(S^3_0(K))$ determines the Alexander polynomial $\Delta_K(t)$, by [OS04c, Proposition 10.14] and [OS04c, Theorem 10.17], to see that

$$\Delta_K(t) = \Delta_{5_2}(t) = 2t - 3 + 2t^{-1}.$$

But the linear coefficient 2 is equal to the Euler characteristic $\chi(\widehat{HFK}(K,1))$, so in particular $\dim \widehat{HFK}(K,1)$ must be even. It follows from the above bound that

$$\dim \widehat{HFK}(K,1) = 2$$

and so K must be one of the knots listed in Theorem 1.5.

Finally, we can read the correction terms $d_{\pm 1/2}(S^3_0(K))$ off of $HF^+(S^3_0(K))$, since they are defined as the grading of the bottom-most element of a tower $\mathcal{T}^+$ in grading $\pm\frac{1}{2} \pmod 2$. According to [OS03a, Proposition 4.12], these are determined by the formulas

$$\begin{aligned} d_{1/2}(S^3_0(K)) &= d(S^3_1(K)) + \tfrac{1}{2}, \\ d_{-1/2}(S^3_0(K)) &= d(S^3_{-1}(K)) - \tfrac{1}{2} = d(-S^3_1(\overline{K})) - \tfrac{1}{2} \\ &= -d(S^3_1(\overline{K})) - \tfrac{1}{2}. \end{aligned}$$

Now Theorem 2.7 tells us that

$$d_{1/2}(S^3_0(K)) = -2V_0(K) + \tfrac{1}{2}, \qquad d_{-1/2}(S^3_0(K)) = 2V_0(\overline{K}) - \tfrac{1}{2}$$

and so $HF^+(S^3_0(K))$ determines both $V_0(K)$ and $V_0(\overline{K})$. But we saw in (3.3) that if K is one of the knots in Theorem 1.5, then

$$(V_0(K), V_0(\overline{K})) = \begin{cases} (1,0), & K \cong \overline{5_2} \\ (0,1), & K \cong 5_2 \\ (0,0), & \text{otherwise,} \end{cases}$$

so $HF^+(S^3_0(\overline{5_2}))$ and $HF^+(S^3_0(5_2))$ are different from each other and from each of the invariants $HF^+(S^3_0(K))$ where K is another of the knots in Theorem 1.5. This completes the proof. □

Combining Proposition 4.4 in the case $r > 0$ and Proposition 4.5 for $r = 0$, this completes the proof of Theorem 4.1. □

5. The knot 5_2

In this section we start to consider whether positive slopes are characterizing slopes for 5_2. We will achieve partial results in this direction without using the mapping cone formula (Theorem 2.5), which we then apply in Section 6.

Lemma 5.1. *Suppose that there is some knot K and some rational $r > 0$ such that*

$$HF^+(S^3_r(5_2)) \cong HF^+(S^3_r(K))$$

as graded $\mathbb{Q}[U]$-modules. Then $V_s(K) = 0$ for all $s \geq 0$. In addition, if $g(K) = 1$ then $\Delta_K(t) = \Delta_{5_2}(t) = 2t - 3 + 2t^{-1}$.

Proof. We recall from Proposition 2.10 that

$$V_0(5_2) = -\frac{1}{2}d(S^3_1(5_2)) = 0,$$

and then Propositions 2.9 and 2.6 say that $V_0(K) = 0$ and that the sequence of $V_s(K)$ is nonincreasing, proving the first claim. The second claim also follows from Proposition 2.9, once we use $g(K) = 1$ to write $\Delta_K(t) = at + (1 - 2a) + at^{-1}$ for some a and then observe that

$$a = \frac{\Delta''_K(1)}{2} = \frac{\Delta''_{5_2}(1)}{2} = 2.$$ □

Lemma 5.2. *Suppose for some knot $K \not\cong 5_2$ and some rational $r > 0$ that*

$$HF^+(S^3_r(5_2)) \cong HF^+(S^3_r(K))$$

as graded $\mathbb{Q}[U]$-modules. Then $\hat{r}_0(K) = 4$ and $\hat{\nu}(K) = 0$.

Proof. Write $r = \frac{p}{q}$ for some coprime $p, q > 0$. We note that since $\frac{p}{q} > 0 > \hat{\nu}(5_2)$, we have

$$\begin{aligned}\dim \widehat{HF}(S^3_{p/q}(5_2)) &= q \cdot \hat{r}_0(5_2) + |p - q\hat{\nu}(5_2)| \\ &= 3q + |p + q| = p + 4q,\end{aligned}$$

and by hypothesis this is equal to $\dim \widehat{HF}(S^3_{p/q}(K))$.

We next observe that $\hat{\nu}(K) \leq 0$: according to Proposition 3.2, it is enough to show that $V_0(K) = 0$, and this was already proved in Lemma 5.1. Thus $\frac{p}{q} > \hat{\nu}(K)$, and we have

$$\begin{aligned}\dim \widehat{HF}(S^3_{p/q}(K)) &= q \cdot \hat{r}_0(K) + (p - q\hat{\nu}(K)) \\ &= p + q(\hat{r}_0(K) - \hat{\nu}(K)).\end{aligned}$$

This is equal to $\dim \widehat{HF}(S^3_{p/q}(5_2)) = p + 4q$, so we must have $\hat{r}_0(K) - \hat{\nu}(K) = 4$.

Now since $0 \leq \hat{r}_0(K) = \hat{\nu}(K) + 4 \leq 4$ and $\hat{r}_0(K) \geq |\hat{\nu}(K)|$, the only possibilities for these invariants are

$$(\hat{r}_0(K), \hat{\nu}(K)) = (4, 0) \text{ or } (3, -1) \text{ or } (2, -2),$$

and the last is impossible because Proposition 3.2 says that $\hat{\nu}(K)$ must be 0 or odd. If $(\hat{r}_0(K), \hat{\nu}(K)) = (3,-1)$ then $(\hat{r}_0(\overline{K}), \hat{\nu}(\overline{K})) = (3,1)$, so $\overline{K}$ is an almost L-space knot by Lemma 3.7, and then it must have genus 1 by Corollary 3.11. Now Proposition 3.9 says that either $\overline{K} \cong \overline{5_2}$, or $\Delta_K(t) = \Delta_{\overline{K}}(t)$ is different from $\Delta_{5_2}(t)$. But the first option is ruled out by the assumption $K \not\cong 5_2$, and the second by Lemma 5.1. We conclude that $(\hat{r}_0(K), \hat{\nu}(K))$ cannot be $(3,-1)$, and so the only remaining possibility is $(4,0)$. □

Proposition 5.3. *Suppose for some rational $r > 0$ and some knot $K \not\cong 5_2$ that*

$$HF^+(S^3_r(K)) \cong HF^+(S^3_r(5_2))$$

as graded $\mathbb{Q}[U]$-modules. Then $\tau(K) = 0$, and the following must hold.

- *If $g(K) = 1$ then K is either $15n_{43522}$ or $\mathrm{Wh}^-(T_{2,3}, 2)$, up to mirroring.*
- *If $g(K) \geq 2$ then K is fibered, and*

$$H_*(A^+_s(K)) \cong \begin{cases} \mathcal{T}^+ \oplus \mathbb{Q}, & |s| = g(K) - 1 \\ \mathcal{T}^+, & \textit{otherwise} \end{cases}$$

  for all $|s| \leq g(K) - 1$. In this case the maps

$$v^+_s : A^+_s(K) \to B^+(K) \quad \textit{and} \quad h^+_{-s} : A^+_{-s}(K) \to B^+(K)$$

  are quasi-isomorphisms for $0 \leq s \leq g(K) - 2$.

Proof. Let $g = g(K)$. Lemma 5.2 tells us that $\hat{r}_0(K) = 4$ and $\hat{\nu}(K) = 0$, so $\tau(K) = 0$ by Proposition 3.2, and we also have

$$\dim \widehat{HF}(S^3_{2g-1}(K)) = 4 + |(2g-1) - 0| = 2g + 3.$$

Lemma 5.1 says that $V_{g-1}(K) = 0$, so Corollary 2.13 becomes

$$\dim \widehat{HFK}(K, g) = \frac{\dim \widehat{HF}(S^3_{2g-1}(K), g-1) - 1}{2}. \tag{5.1}$$

We will use this to bound $\dim \widehat{HFK}(K, g)$ from above.

We suppose first that $g = 1$. In this case we have

$$\dim \widehat{HF}(S^3_1(K), 0) = \dim \widehat{HF}(S^3_1(K)) = 2g + 3 = 5,$$

so (5.1) becomes $\dim \widehat{HFK}(K, 1) = 2$. From Lemma 5.1 we have $\Delta_K(t) = 2t - 3 + 2t^{-1}$, so Theorem 1.5 now tells us that K must be one of 5_2, $15n_{43522}$, or $\mathrm{Wh}^-(T_{2,3}, 2)$ up to mirroring. But we have assumed that K is not 5_2, and it cannot be $\overline{5_2}$ since $V_0(\overline{5_2}) = 1$, so this leaves only the knots named in the proposition.

Now we suppose instead that $g \geq 2$. In this case, the unique self-conjugate element of

$$\mathrm{Spin}^c(S^3_{2g-1}(K)) \cong \mathbb{Z}/(2g-1)\mathbb{Z}$$

is identified with 0, and in particular it is different from $g - 1$, which is conjugate to $1 - g$. Since $\dim \widehat{HFK}(S^3_{2g-1}(K), s)$ is odd for all s, we use the conjugation symmetry of $\widehat{HF}$ (see

Remark 2.4) to show that

$$\begin{aligned}
2g+3 &= \dim \widehat{HF}(S^3_{2g-1}(K)) \\
&= \sum_{s\in\mathbb{Z}/(2g-1)\mathbb{Z}} \dim \widehat{HF}(S^3_{2g-1}(K),s) \\
&= 2\dim \widehat{HF}(S^3_{2g-1}(K),g-1) + \sum_{|s|\leq g-2} \dim \widehat{HF}(S^3_{2g-1}(K),s) \\
&\geq 2\dim \widehat{HF}(S^3_{2g-1}(K),g-1) + (2g-3),
\end{aligned}$$

since there are $2g-3$ different summands on the right. This shows that

$$\dim \widehat{HF}(S^3_{2g-1}(K),g-1) \leq 3,$$

and then (5.1) becomes $\dim \widehat{HFK}(K,g) \leq 1$. But $\dim \widehat{HFK}(K,g)$ must be positive, so equality holds, which implies that

- $\dim \widehat{HFK}(K,g) = 1$, and then K must be fibered [Ni07]; and
- $\dim \widehat{HF}(S^3_{2g-1}(K),s)$ is 3 if $s \equiv \pm(g-1) \pmod{2g-1}$, and 1 otherwise.

Applying Lemmas 2.8 and 2.12, we conclude that

$$HF^+_{\mathrm{red}}(S^3_{2g-1}(K),s) \cong \begin{cases} \mathbb{Q}, & s=\pm(g-1) \\ 0, & 2-g\leq s\leq g-2. \end{cases}$$

The large surgery formula (Theorem 2.3) says that

$$H_*(A^+_s) \cong HF^+(S^3_{2g-1}(K),s)$$

whenever $|s|\leq g-1$, so this completes the description of $H_*(A^+_s)$.

Now if $0\leq s\leq g-2$ then $v^+_s : A^+_s \to B^+$ induces a map on homology of the form

$$(v^+_s)_* : \mathcal{T}^+ \cong H_*(A^+_s) \to H_*(B^+) \cong \mathcal{T}^+,$$

and this map is multiplication by $U^{V_s(K)}$, but Lemma 5.1 says that $V_s(K)=0$ and so $(v^+_s)_*$ is an isomorphism. The map $(h^+_{-s})_*$ has the same form and is identified with multiplication by $U^{H_{-s}(K)}$, but Proposition 2.6 says that $H_{-s}(K)=V_s(K)=0$, so $(h^+_{-s})_*$ is an isomorphism as well. $\square$

5.1. $\widehat{HFK}$ **in the higher genus case.** Suppose that we have a homeomorphism

$$S^3_r(K) \cong S^3_r(5_2)$$

for some slope $r>0$ and some knot K of genus $g\geq 2$. Then Proposition 5.3 says that K is fibered, that $\tau(K)=0$, and that

$$H_*(A^+_s) \cong \begin{cases} \mathcal{T}^+\oplus\mathbb{Q}, & |s|=g-1 \\ \mathcal{T}^+, & \text{otherwise.} \end{cases}$$

In addition, Lemma 5.1 together with Proposition 2.6 tells us that

$$V_s(K) = \begin{cases} 0, & s\geq 0 \\ |s|, & s<0 \end{cases}$$

for all $s \in \mathbb{Z}$. We will use all of this information to determine $\widehat{HFK}(K)$ as a bigraded vector space.

Lemma 5.4. *There is some integer $d \in \mathbb{Z}$ such that*

$$H_*(A_s^+) \cong \begin{cases} \mathcal{T}^+_{(0)} \oplus \mathbb{Q}_{(d)}, & s = g-1 \\ \mathcal{T}^+_{(2-2g)} \oplus \mathbb{Q}_{(d+2-2g)}, & s = 1-g \\ \mathcal{T}^+_{(\min(0,2s))}, & \textit{otherwise.} \end{cases}$$

Proof. We consider each of the maps

$$(v_s^+)_* : H_*(A_s^+) \to H_*(B^+) \cong \mathcal{T}^+_{(0)},$$

which are induced by projections at the chain level. For $s \geq 0$ we have $V_s(K) = 0$, so these maps restrict to graded isomorphisms on the towers $\mathcal{T}^+ \subset H_*(A_s^+)$; thus these towers have their bottom-most elements in grading 0. By contrast, for $s < 0$ the maps $(v_s^+)_*$ are modeled on multiplication by $U^{V_s(K)} = U^{|s|}$, so the element of $\mathcal{T}^+ \subset H_*(A_s^+)$ in grading 0 is at height $|s|$ in the tower, meaning that the bottom element has grading $-2|s| = 2s$.

Having determined the grading on each tower, we set d equal to the grading of the $\mathbb{Q}$ summand of $H_*(A_{g-1}^+)$. Then it only remains to identify the grading on the $\mathbb{Q}$ summand of $H_*(A_{1-g}^+)$. We apply the large surgery formula, Theorem 2.3, to get relatively graded isomorphisms

$$\begin{aligned} HF^+(S^3_{2g-1}(K), g-1) &\cong H_*(A_{g-1}^+), \\ HF^+(S^3_{2g-1}(K), 1-g) &\cong H_*(A_{1-g}^+). \end{aligned}$$

By conjugation symmetry these HF^+ invariants are isomorphic to each other, so we also have a relatively graded isomorphism

$$H_*(A_{g-1}^+) \cong H_*(A_{1-g}^+).$$

But this means that the grading of the $\mathbb{Q}$ summand of $H_*(A_{1-g}^+)$ must be d greater than that of the bottom of the tower $\mathcal{T}^+_{(2-2g)}$, so its grading is $d + 2 - 2g$ as claimed. $\square$

We now start with the top-most Alexander grading of $\widehat{HFK}(K)$, which we already know to be 1-dimensional because K is fibered.

Lemma 5.5. *We have $\widehat{HFK}(K, g) \cong \mathbb{Q}_{(d+2)}$ and $\widehat{HFK}(K, -g) \cong \mathbb{Q}_{(d+2-2g)}$, where d is the integer from Lemma 5.4.*

Proof. Lemma 2.8 gives us a short exact sequence

$$0 \to \widehat{HFK}_{*+2}(K, g) \to \mathcal{T}^+_{(0)} \oplus \mathbb{Q}_{(d)} \xrightarrow{(v_{g-1}^+)_*} \mathcal{T}^+_{(0)} \to 0,$$

where $(v_{g-1}^+)_*$ has kernel $\mathbb{Q}_{(d)}$. The grading on $\widehat{HFK}(K, -g)$ now comes from the symmetry

$$\widehat{HFK}_m(K, a) \cong \widehat{HFK}_{m-2a}(K, -a)$$

of [OS04b, Equation (3)]. $\square$

Throughout the remainder of this section we write
$$\mathcal{F}_s = C\{i = 0, j \le s\}$$
to denote the filtration
$$0 \subset \mathcal{F}_{-g} \subset \mathcal{F}_{1-g} \subset \cdots \subset \mathcal{F}_g$$
of $\widehat{CF}(S^3)$ whose associated graded groups are the various $\widehat{HFK}(K, a)$. In particular the short exact sequence
$$0 \to \mathcal{F}_{s-1} \hookrightarrow \mathcal{F}_s \to C\{0, s\} \to 0$$
of chain complexes gives rise to a long exact sequence
$$\cdots \to H_*(\mathcal{F}_{s-1}) \to H_*(\mathcal{F}_s) \to \widehat{HFK}_*(K, s) \to H_{*-1}(\mathcal{F}_{s-1}) \to \cdots . \tag{5.2}$$

Lemma 5.6. *For all $s \in \mathbb{Z}$, there is a long exact sequence*
$$\cdots \to H_{*-(2s-2)}(\mathcal{F}_{-s}) \to H_*(A^+_{s-1}) \xrightarrow{(\pi_s)_*} H_*(A^+_s) \to H_{*-(2s-1)}(\mathcal{F}_{-s}) \to \cdots ,$$
and $(v^+_{s-1})_*$ is equal to the composition
$$H_*(A^+_{s-1}) \xrightarrow{(\pi_s)_*} H_*(A^+_s) \xrightarrow{(v^+_s)_*} H_*(B^+).$$

Proof. There is a short exact sequence of chain complexes
$$0 \to C\{i \le -1, j = s-1\} \to A^+_{s-1} \xrightarrow{\pi_s} A^+_s \to 0 \tag{5.3}$$
in which π_s is projection. Then $v^+_{s-1} = v^+_s \circ \pi_s$ at the chain level, hence $(v^+_{s-1})_*$ factors as claimed. We also have a chain homotopy equivalence
$$\begin{aligned} C_*\{i \le -1, j = s-1\} &\xrightarrow{U^{s-1}} C_{*-(2s-2)}\{i \le -s, j = 0\} \\ &\longrightarrow C_{*-(2s-2)}\{i = 0, j \le -s\} \end{aligned}$$
so the long exact sequence of homology groups associated to (5.3) takes the form promised by the lemma. ☐

Lemma 5.7. *We have*
$$H_*(\mathcal{F}_0) \cong \begin{cases} \mathbb{Q}_{(0)}, & g \ge 3 \\ \mathbb{Q}_{(0)} \oplus \mathbb{Q}_{(d)}, & g = 2. \end{cases}$$

Proof. We apply Lemma 5.6 with $s = 0$: supposing for now that $g \ge 3$, the composition
$$\begin{tikzcd} H_*(A^+_{-1}) \arrow[r, "(\pi_0)_*"] \arrow[d, "\cong"] & H_*(A^+_0) \arrow[r, "(v^+_0)_*"] \arrow[d, "\cong"] & H_*(B^+) \arrow[d, "\cong"] \\ \mathcal{T}^+_{(-2)} \arrow[r, "(\pi_0)_*"] & \mathcal{T}^+_{(0)} \arrow[r, "U^{V_0(K)}=1"] & \mathcal{T}^+_{(0)} \end{tikzcd}$$
is equal to $(v^+_{-1})_*$ and hence identified with multiplication by $U^{V_{-1}(K)} = U$. In particular, the map $(\pi_0)_*$ is surjective and also identified with multiplication by U, so the long exact sequence of Lemma 5.6 splits as
$$0 \to H_{i+2}(\mathcal{F}_0) \to H_i(A^+_{-1}) \xrightarrow{(\pi_0)_*} H_i(A^+_0) \to 0$$

for each i, and we have

$$H_{i+2}(\mathcal{F}_0) \cong \ker((\pi_0)_*) \cong \begin{cases} \mathbb{Q}, & i = -2 \\ 0, & \text{otherwise} \end{cases}$$

since -2 is the grading of the bottom-most element of $H_*(A^+_{-1}) \cong \mathcal{T}^+_{(-2)}$.

Now suppose that $g = 2$. Then we factor $(v^+_{-1})_*$ as

$$\begin{array}{ccccc} H_*(A^+_{-1}) & \xrightarrow{(\pi_0)_*} & H_*(A^+_0) & \xrightarrow{(v^+_0)_*} & H_*(B^+) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \mathcal{T}^+_{(-2)} \oplus \mathbb{Q}_{(d-2)} & \xrightarrow{(\pi_0)_*} & \mathcal{T}^+_{(0)} & \xrightarrow{U^{V_0(K)}=1} & \mathcal{T}^+_{(0)} \end{array}$$

where the gradings on $H_*(A^+_{-1})$ come from Lemma 5.4. In this case $(\pi_0)_*$ is still surjective, so once again we identify its kernel $\mathbb{Q}_{(-2)} \oplus \mathbb{Q}_{(d-2)}$ with $H_{*+2}(\mathcal{F}_0)$. ☐

Proposition 5.8. *We have* $\widehat{HFK}(K,-g) \cong \mathbb{Q}_{(d+2-2g)}$, *and* $\widehat{HFK}(K,1-g) \cong \mathbb{Q}^2_{(d+3-2g)}$. *If* $g \geq 3$ *then*

$$\widehat{HFK}(K,s) \cong \begin{cases} \mathbb{Q}_{(d+4-2g)}, & s = 2-g \\ 0, & 3-g \leq s \leq -1 \\ \mathbb{Q}_{(0)}, & s = 0. \end{cases}$$

If $g = 2$ *instead, then* $\widehat{HFK}(K,0) \cong \mathbb{Q}_{(0)} \oplus \mathbb{Q}^2_{(d)}$.

Proof. The computation of $\widehat{HFK}(K,-g)$ is Lemma 5.5. When $s = g-1$, we can factor $(v^+_{g-2})_*$ as

$$\begin{array}{ccccc} H_*(A^+_{g-2}) & \xrightarrow{(\pi_{g-1})_*} & H_*(A^+_{g-1}) & \xrightarrow{(v^+_{g-1})_*} & H_*(B^+) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \mathcal{T}^+_{(0)} & \xrightarrow{(\pi_{g-1})_*} & \mathcal{T}^+_{(0)} \oplus \mathbb{Q}_{(d)} & \xrightarrow{U^{V_s(K)}=1} & \mathcal{T}^+_{(0)}, \end{array}$$

and the composition is an isomorphism $\mathcal{T}^+_{(0)} \to \mathcal{T}^+_{(0)}$ since $V_{g-2}(K) = 0$. Thus $(\pi_{g-1})_*$ is injective, with cokernel $\mathbb{Q}_{(d)}$. Now the sequence of Lemma 5.6 splits as

$$0 \to H_*(A^+_{g-2}) \xrightarrow{(\pi_{g-1})_*} H_*(A^+_{g-1}) \to H_{*-(2g-3)}(\mathcal{F}_{1-g}) \to 0,$$

so we have $H_*(\mathcal{F}_{1-g}) \cong \mathbb{Q}_{(d-(2g-3))}$. But we also know that

$$H_*(\mathcal{F}_{-g}) \cong \widehat{HFK}(K,-g) \cong \mathbb{Q}_{(d+2-2g)}$$

by Lemma 5.5, so the induced map $H_*(\mathcal{F}_{-g}) \to H_*(\mathcal{F}_{1-g})$ must be zero for grading reasons. Thus when $s = 1-g$ the exact sequence (5.2) splits and we have

$$\widehat{HFK}_*(K,1-g) \cong H_*(\mathcal{F}_{1-g}) \oplus H_{*-1}(\mathcal{F}_{-g}) \cong \mathbb{Q}^2_{(d+3-2g)}.$$

Now if $g \geq 3$ then we consider the map $(v^+_{s-1})_*$ for each of $s = 1, 2, \dots, g-2$ in turn. In each case $(v^+_{s-1})_*$ factors as

$$\begin{tikzcd} H_*(A^+_{s-1}) \arrow[r, "(\pi_s)_*"] \arrow[d, "\cong"] & H_*(A^+_s) \arrow[r, "(v^+_s)_*"] \arrow[d, "\cong"] & H_*(B^+) \arrow[d, "\cong"] \\ \mathcal{T}^+_{(0)} \arrow[r, "(\pi_s)_*"] & \mathcal{T}^+_{(0)} \arrow[r, "U^{V_s(K)}=1"] & \mathcal{T}^+_{(0)}, \end{tikzcd}$$

and is an isomorphism, since it is identified with multiplication by $U^{V_{s-1}(K)} = 1$ as a map $\mathcal{T}^+_{(0)} \to \mathcal{T}^+_{(0)}$. It follows that each $(\pi_s)_*$ is an isomorphism, so the exact sequence of Lemma 5.6 tells us that

$$H_*(\mathcal{F}_{-s}) = 0, \quad s = 1, 2, \dots, g-2.$$

Applying the long exact sequence (5.2) for $s = 3-g, 4-g, \dots, 0$, we know that $H_*(\mathcal{F}_{s-1}) = 0$ for each s, and so

$$\widehat{HFK}_*(K, s) \cong H_*(\mathcal{F}_s) \cong \begin{cases} \mathbb{Q}_{(0)}, & s = 0 \\ 0, & 3-g \leq s \leq -1, \end{cases}$$

the case $s = 0$ having been computed in Lemma 5.7.

Similarly, if we take $s = 2-g$ in (5.2) then we get a long exact sequence

$$\cdots \to H_*(\mathcal{F}_{1-g}) \to H_*(\mathcal{F}_{2-g}) \to \widehat{HFK}_*(K, 2-g) \to H_{*-1}(\mathcal{F}_{1-g}) \to \cdots. \tag{5.4}$$

For $g \geq 3$ we have seen that $H_*(\mathcal{F}_{2-g}) = 0$, and so

$$\widehat{HFK}_*(K, 2-g) \cong H_{*-1}(\mathcal{F}_{1-g}) \cong \mathbb{Q}_{(d+4-2g)}.$$

If $g = 2$ instead, then we have computed above that

$$H_*(\mathcal{F}_{-1}) = H_*(\mathcal{F}_{1-g}) \cong \mathbb{Q}_{(d-1)}$$

while $H_*(\mathcal{F}_0) \cong \mathbb{Q}_{(0)} \oplus \mathbb{Q}_{(d)}$ by Lemma 5.7, so it remains to be seen whether the map $\iota : H_*(\mathcal{F}_{-1}) \to H_*(\mathcal{F}_0)$ is zero or not.

Assuming that $g = 2$, we now consider the inclusion-induced maps

$$\begin{tikzcd} H_*(\mathcal{F}_{-1}) \arrow[r, "\iota"] \arrow[d, "\cong"] & H_*(\mathcal{F}_0) \arrow[r] \arrow[d, "\cong"] & \widehat{HF}(S^3) \arrow[d, "\cong"] \\ \mathbb{Q}_{(d-1)} \arrow[r] & \mathbb{Q}_{(0)} \oplus \mathbb{Q}_{(d)} \arrow[r] & \mathbb{Q}_{(0)}, \end{tikzcd}$$

where $H_*(\mathcal{F}_{-1}) = H_*(\mathcal{F}_{1-g})$ was computed above, and we used Lemma 5.7 to identify $H_*(\mathcal{F}_0)$. If ι is nonzero then for degree reasons we must have $d = 1$, and then its image is the $\mathbb{Q}_{(0)}$ summand of $H_*(\mathcal{F}_0)$. But the map $H_*(\mathcal{F}_0) \to \widehat{HF}(S^3)$ is surjective since $\tau(K) \leq 0$, so it must be nonzero on this $\mathbb{Q}_{(0)}$ summand, in which case the composition across the top row is also surjective. This would in turn imply that $\tau(K) \leq -1$, contradicting Proposition 5.3. We conclude that $\iota = 0$, so (5.4) splits as

$$0 \to H_*(\mathcal{F}_0) \to \widehat{HFK}_*(K, 0) \to H_{*-1}(\mathcal{F}_{-1}) \to 0.$$

Thus $\widehat{HFK}_*(K, 0) \cong \mathbb{Q}_{(0)} \oplus \mathbb{Q}^2_{(d)}$, completing the proof. $\square$

6. The mapping cone formula and 5_2

Suppose for some knot $K \not\cong 5_2$ and some rational slope $r > 0$ that $S^3_r(K) \cong S^3_r(5_2)$. In this section we will apply the mapping cone formula, Theorem 2.5, to compare $HF^+(S^3_r(K))$ to $HF^+(S^3_r(5_2))$.

Throughout this section we will assume that K has genus $g \geq 2$. Then Proposition 5.3 says that K is fibered, and that we can write

$$H_*(A^+_{g-1}(K)) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}_{(d)}$$

for some integer $d \in \mathbb{Z}$. Proposition 5.8 then describes $\widehat{HFK}(K)$ completely in terms of g and d.

We also record from Lemma 5.1, together with Proposition 2.6, the values

$$V_s(K) = \begin{cases} 0, & s \geq 0 \\ -s, & s < 0, \end{cases} \qquad\qquad H_s(K) = \begin{cases} s, & s \geq 0 \\ 0, & s < 0. \end{cases}$$

The values of $V_s(5_2)$ and $H_s(5_2)$ are identical, so we will refer to these throughout without reference to the particular knot.

6.1. Preliminaries.

We begin by recording some facts about the mapping cone formula which will simplify our computations.

Proposition 6.1. *Let $K \subset S^3$ be a nontrivial knot of genus $g \geq 1$, and let $p, q > 0$ be relatively prime integers. Fix an integer i, and suppose there are some integers $s \leq s'$ such that*

- $h^+_{\lfloor \frac{i+pt}{q} \rfloor}$ *is a quasi-isomorphism for all $t < s$, and*
- $v^+_{\lfloor \frac{i+pt}{q} \rfloor}$ *is a quasi-isomorphism for all $t > s'$.*

Define truncated complexes

$$\mathbb{A}^{[s,s']}_{i,p/q} = \bigoplus_{s \leq t \leq s'} \left(t, A^+_{\lfloor \frac{i+pt}{q} \rfloor}\right), \qquad\qquad \mathbb{B}^{[s,s']}_{i,p/q} = \bigoplus_{s < t \leq s'} \left(t, B^+\right),$$

and a map

$$\begin{aligned} D^{[s,s']}_{i,p/q} &: \mathbb{A}^{[s,s']}_{i,p/q} \to \mathbb{B}^{[s,s']}_{i,p/q} \\ &(t, a_t) \mapsto (t, v^+_{\lfloor \frac{i+pt}{q} \rfloor}(a_t)) + (t+1, h^+_{\lfloor \frac{i+pt}{q} \rfloor}(a_t)) \end{aligned}$$

where we interpret $(s, v^+_{\lfloor \frac{i+ps}{q} \rfloor}(a_s))$ and $(s'+1, h^+_{\lfloor \frac{i+ps'}{q} \rfloor}(a_{s'}))$ as zero. Then there is an isomorphism

$$HF^+(S^3_{p/q}(K), i) \cong \ker\left((D^{[s,s']}_{i,p/q})_* : H_*(\mathbb{A}^{[s,s']}_{i,p/q}) \to H_*(\mathbb{B}^{[s,s']}_{i,p/q})\right)$$

of relatively graded $\mathbb{Q}[U]$-modules.

Proof. Theorem 2.5 gives a relatively graded isomorphism between $HF^+(S^3_{p/q}(K), i)$ and the homology of the mapping cone $\mathbb{X}^+_{i,p/q}$, which we can write as

$$\begin{tikzcd}
\cdots \arrow[dr, "h^+"] & A^+_{\lfloor \frac{i+(t-1)p}{q} \rfloor} \arrow[d, "v^+"] \arrow[dr, "h^+"] & A^+_{\lfloor \frac{i+tp}{q} \rfloor} \arrow[d, "v^+"] \arrow[dr, "h^+"] & A^+_{\lfloor \frac{i+(t+1)p}{q} \rfloor} \arrow[d, "v^+"] \arrow[dr, "h^+"] & \cdots \\
\cdots & B^+ & B^+ & B^+ & \cdots
\end{tikzcd}$$

where we understand each h^+ or v^+ with domain A^+_j to mean h^+_j or v^+_j respectively. We observe that the subcomplex

$$\begin{tikzcd}
\cdots \arrow[dr, "h^+"] & A^+_{\lfloor \frac{i+(s-1)p}{q} \rfloor} \arrow[d, "v^+"] \arrow[dr, "h^+"] & \\
\cdots & B^+ & B^+,
\end{tikzcd}$$

consisting of all summands $(t, A^+_{\lfloor \frac{i+tp}{q} \rfloor})$ with $t < s$ and all (t, B^+) with $t \leq s$, is acyclic because each of its h^+ maps is a quasi-isomorphism. Similarly, the subcomplex

$$\begin{tikzcd}
A^+_{\lfloor \frac{i+(s'+1)p}{q} \rfloor} \arrow[d, "v^+"] \arrow[dr, "h^+"] & \cdots \\
B^+ & \cdots,
\end{tikzcd}$$

consisting of all summands $(t, A^+_{\lfloor \frac{i+tp}{q} \rfloor})$ and (t, B^+) with $t > s'$, is acyclic because each of its v^+ maps is a quasi-isomorphism. Thus we may take the quotient of $\mathbb{X}^+_{i,p/q}$ by each of these subcomplexes in turn, and the projection maps are both quasi-isomorphisms. But this leaves the truncated complex

$$\begin{tikzcd}
A^+_{\lfloor \frac{i+ps}{q} \rfloor} \arrow[dr, "h^+"] & A^+_{\lfloor \frac{i+p(s+1)}{q} \rfloor} \arrow[d, "v^+"] \arrow[dr, "h^+"] & \cdots \arrow[dr, "h^+"] & A^+_{\lfloor \frac{i+p(s'-1)}{q} \rfloor} \arrow[d, "v^+"] \arrow[dr, "h^+"] & A^+_{\lfloor \frac{i+ps'}{q} \rfloor} \arrow[d, "v^+"] \\
 & B^+ & \cdots & B^+ & B^+,
\end{tikzcd}$$

which is precisely the mapping cone $\mathbb{X}^{[s,s']}_{i,p/q}$ of $D^{[s,s']}_{i,p/q}$, and so we have

$$HF^+(S^3_{p/q}(K), i) \cong H_*(\mathbb{X}^{[s,s']}_{i,p/q})).$$

The truncated mapping cone fits into a long exact sequence

$$\cdots \to H_{*+1}(\mathbb{X}^{[s,s']}_{i,p/q}) \to H_*(\mathbb{A}^{[s,s']}_{i,p/q}) \xrightarrow{(D^{[s,s']}_{i,p/q})_*} H_*((\mathbb{B}^{[s,s']}_{i,p/q}) \to \ldots,$$

and so it now suffices to prove that $(D^{[s,s']}_{i,p/q})_*$ is surjective, cf. [NW15, Lemma 2.8]. But the restriction of $(D^{[s,s']}_{i,p/q})_*$ to all of the tower summands

$$\mathcal{T}^+ \subset H_*(A^+_{\lfloor \frac{i+pt}{q} \rfloor}) \subset \bigoplus_{s\le t\le s'} H_*(A^+_{\lfloor \frac{i+pt}{q} \rfloor}) \cong H_*(\mathbb{A}^{[s,s']}_{i,p/q})$$

has the form

$$\begin{tikzcd}
\mathcal{T}^+ \arrow[dr, "h_*^+"] & \mathcal{T}^+ \arrow[d, "v_*^+"] \arrow[dr, "h_*^+"] & \cdots \arrow[dr, "h_*^+"] & \mathcal{T}^+ \arrow[d, "v_*^+"] \arrow[dr, "h_*^+"] & \mathcal{T}^+ \arrow[d, "v_*^+"] & \subset & H_*(\mathbb{A}^{[s,s']}_{i,p/q}) \arrow[d, "(D^{[s,s']}_{i,p/q})_*"] \\
& \mathcal{T}^+ & \cdots & \mathcal{T}^+ & \mathcal{T}^+ & \cong & H_*(\mathbb{B}^{[s,s']}_{i,p/q}),
\end{tikzcd}$$

and each of the v_*^+ and h_*^+ components are surjective, so it follows that the total map is surjective as well. This identifies $H_*(\mathbb{X}^{[s,s']}_{i,p/q})$, and hence $HF^+(S^3_{p/q}(K), i)$, with the kernel of $(D^{[s,s']}_{i,p/q})_*$ up to an overall grading shift, as promised. □

Corollary 6.2. *Let $K \subset S^3$ be a nontrivial knot of genus $g \ge 1$, and let $p, q > 0$ be relatively prime integers. Fix an integer i, and suppose there is some $s \in \mathbb{Z}$ such that*

- $h^+_{\lfloor \frac{i+pt}{q} \rfloor}$ *is a quasi-isomorphism for all $t < s$, and*
- $v^+_{\lfloor \frac{i+pt}{q} \rfloor}$ *is a quasi-isomorphism for all $t > s$.*

Then $HF^+(S^3_{p/q}(K), i) \cong H_*(A^+_{\lfloor \frac{i+ps}{q} \rfloor})$ as relatively graded $\mathbb{Q}[U]$-modules.

Proof. We apply Proposition 6.1 to identify $HF^+(S^3_{p/q}(K), i)$ with the kernel of the map

$$(D^{[s,s]}_{i,p/q})_* : H_*(A^+_{\lfloor \frac{i+ps}{q} \rfloor}) \to 0.$$ □

Proposition 6.3. *Let $K \subset S^3$ be a knot of genus $g \ge 1$, and fix $i \in \mathbb{Z}$ and $\frac{p}{q} \ge 2g-1$. Then there is at most one $s \in \mathbb{Z}$ such that*

$$1 - g \le \left\lfloor \frac{i+ps}{q} \right\rfloor \le g - 1,$$

and we have

$$HF^+(S^3_{p/q}(K), i) = \begin{cases} H_*(A^+_{\lfloor \frac{i+ps}{q} \rfloor}) & \text{if } s \text{ exists} \\ \mathcal{T}^+ & \text{otherwise} \end{cases}$$

as relatively graded $\mathbb{Q}[U]$-modules.

Proof. Suppose first that s exists. The desired inequality is equivalent to

$$q(1-g) \le i + ps < qg.$$

Thus if there is a solution s, then for all integers $t > s$ we have

$$i + pt \ge i + p(s+1) \ge q(1-g) + p \ge q(1-g) + q(2g-1) = qg,$$

while for all integers $t < s$ we have

$$i + pt \le i + p(s-1) < qg - p \le qg - q(2g-1) = q(1-g).$$

In either case t cannot be a solution, so if s exists then it is unique. But then we know that

- $h^+_{\lfloor\frac{i+pt}{q}\rfloor}$ is a quasi-isomorphism for all $t < s$, since $\lfloor\frac{i+pt}{q}\rfloor \leq -g$; and
- $v^+_{\lfloor\frac{i+pt}{q}\rfloor}$ is a quasi-isomorphism for all $t > s$, since $\lfloor\frac{i+pt}{q}\rfloor \geq g$.

So Corollary 6.2 tells us that $HF^+(S^3_{p/q}(K), i) \cong H_*(A^+_{\lfloor\frac{i+ps}{q}\rfloor})$, as claimed.

Now if no such s exists, then we let σ be the least integer such that $\lfloor\frac{i+p\sigma}{q}\rfloor \geq 0$. It follows that $\lfloor\frac{i+pt}{q}\rfloor \leq -g$ for all $t < \sigma$, and that $\lfloor\frac{i+pt}{q}\rfloor \geq g$ for all $t > \sigma$, so now Corollary 6.2 says that

$$HF^+(S^3_{p/q}(K), i) \cong H_*(A^+_{\lfloor\frac{i+p\sigma}{q}\rfloor}).$$

But in fact $\lfloor\frac{i+p\sigma}{q}\rfloor \geq g$, so $H_*(A^+_{\lfloor\frac{i+p\sigma}{q}\rfloor}) \cong H_*(B^+) \cong \mathcal{T}^+$ and this completes the proof. $\square$

6.2. **Computations for 5_2.** We begin by computing $HF^+(S^3_{p/q}(5_2), i)$ for all slopes $\frac{p}{q} \geq 1$. We recall from (2.4) that

$$H_*(A^+_0(5_2)) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^2_{(0)}.$$

Lemma 6.4. *If $\frac{p}{q} \geq 1$ and $0 \leq i \leq p-1$, then we have*

$$HF^+(S^3_{p/q}(5_2), i) \cong \begin{cases} \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^2_{(0)}, & i = 0, 1, \dots, q-1 \\ \mathcal{T}^+_{(0)}, & \textit{otherwise} \end{cases}$$

as relatively graded $\mathbb{Q}[U]$-modules.

Proof. The condition $\lfloor\frac{i+ps}{q}\rfloor = 0$ is equivalent to

$$0 \leq i + ps < q,$$

so we can find such an s if and only if $i \equiv 0, 1, \dots, q-1 \pmod{p}$, or (since we assumed $0 \leq i \leq p-1$) if and only if $0 \leq i \leq q-1$. If s does not exist then $HF^+(S^3_{p/q}(K), i) \cong \mathcal{T}^+$ by Proposition 6.3 (applied with $g = g(5_2) = 1$). If instead s exists, then we must have $0 \leq i \leq q-1$, and now Proposition 6.3, together with (2.4), says that

$$HF^+(S^3_{p/q}(5_2), i) \equiv H_*(A^+_0(5_2)) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^2_{(0)}$$

as relatively graded $\mathbb{Q}[U]$-modules. $\square$

Proposition 6.5. *Suppose that $0 < \frac{p}{q} < 1$. Then*

$$HF^+(S^3_{p/q}(5_2), i) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^{2n_i}_{(0)}$$

as relatively graded $\mathbb{Q}[U]$-modules, where n_i is the number of $t \in \mathbb{Z}$ such that $0 \leq i + pt < q$.

Proof. We define a pair of integers s, s' by

$$s = \min\{t \in \mathbb{Z} \mid i + pt \geq 0\},$$
$$s' = \max\{t \in \mathbb{Z} \mid i + pt \leq q-1\}.$$

Then $p < q$ implies that $s \leq s'$, and for all $t \in \mathbb{Z}$ we have

$$\left\lfloor \frac{i+pt}{q} \right\rfloor = 0 \quad \Longleftrightarrow \quad s \leq t \leq s',$$

so $n_i = s' - s + 1$.

Now Proposition 6.1 says that $HF^+(S^3_{p/q}(5_2), i)$ is isomorphic to the kernel of $(D^{[s,s']}_{i,p/q})_*$. Recalling again from (2.4) that $H_*(A^+_0) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^2_{(0)}$, this map has the form

$$\begin{tikzcd}
\mathcal{T}^+_{(0)} \oplus \mathbb{Q}^2_{(0)} \arrow[dr, "h^+_*"] & \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^2_{(0)} \arrow[d, "v^+_*"] \arrow[dr, "h^+_*"] & \cdots \arrow[dr, "h^+_*"] & \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^2_{(0)} \arrow[d, "v^+_*"] \arrow[dr, "h^+_*"] & \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^2_{(0)} \arrow[d, "v^+_*"] \\
& \mathcal{T}^+_{(-1)} & \cdots & \mathcal{T}^+_{(-1)} & \mathcal{T}^+_{(-1)}.
\end{tikzcd}$$

Here we are able to assign these gradings to each summand because $V_0(5_2) = H_0(5_2) = 0$, and so each of the maps $v^+_* = (v^+_0)_*$ and $h^+_* = (h^+_0)_*$ gives a degree-(-1) isomorphism between the respective towers.

We see by inspection that $\ker(D^{[s,s']}_{i,p/q})_*$ contains a tower $\mathcal{T}^+$ whose bottom-most element is in grading 0, as an alternating sum of the bottom-most elements of the towers $\mathcal{T}^+_{(0)} \subset H_*(A^+_{\lfloor \frac{i+pt}{q} \rfloor})$, $s \le t \le s'$. The map $(D^{[s,s']}_{i,p/q})_*$ also sends

$$H_0(\mathbb{A}^{[s,s']}_{i,p/q}) \cong \mathbb{Q}^{3(s'-s+1)}$$

onto

$$H_{-1}(\mathbb{B}^{[s,s']}_{i,p/q}) \cong \mathbb{Q}^{s'-s},$$

so its kernel has total dimension $2(s'-s)+3$ in degree zero. We conclude that

$$HF^+(S^3_{p/q}(K), i) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^{2(s'-s+1)}_{(0)}$$

as relatively graded $\mathbb{Q}[U]$-modules. □

6.3. **General facts about the kernel of U.** We will show that under most circumstances, a positive r-surgery on a knot of genus at least 2 cannot have the same Heegaard Floer homology as the corresponding surgery on 5_2. We will handle the cases $r < 1$ and $r \ge 1$ in the next few subsections; before that, we prepare for this work here by proving some general facts about the kernel of the U-action on HF^+ of these surgeries.

Lemma 6.6. *Let K be a knot of genus $g \ge 2$, and suppose for some relatively prime integers $p, q > 0$ that*

$$HF^+(S^3_{p/q}(K)) \cong HF^+(S^3_{p/q}(5_2))$$

as absolutely graded $\mathbb{Q}[U]$-modules. Fix an integer i, and lift the relative gradings on the complexes $\mathbb{A}^+_{i,p/q}$ and $\mathbb{B}^+_{i,p/q}$ to absolute $\mathbb{Z}$-gradings so that $D^+_{i,p/q}$ has degree -1. Let d_s denote the grading of the bottom-most element of the tower

$$\mathcal{T}^+ \subset (s, H_*(A^+_{\lfloor \frac{i+ps}{q} \rfloor})) \subset H_*(\mathbb{A}^+_{i,p/q})$$

for each s.

(1) *If $\lfloor \frac{i+ps}{q} \rfloor \ge 0$, then $d_{s+1} = d_s + 2\lfloor \frac{i+ps}{q} \rfloor$.*
(2) *If $\lfloor \frac{i+ps}{q} \rfloor \le 0$, then $d_s = d_{s-1} + 2\lfloor \frac{i+ps}{q} \rfloor$.*
(3) *If $\lfloor \frac{i+ps}{q} \rfloor \le 0$ and $\lfloor \frac{i+p(s+1)}{q} \rfloor \ge 0$, then $d_s = d_{s+1}$.*

Proof. If $\lfloor \frac{i+ps}{q} \rfloor \geq 0$, then the map $(D^+_{i,p/q})_*$ on homology restricts to the sum of all of the towers $(s, \mathcal{T}^+_{(d_s)}) \subset H_*(\mathbb{A}^+_{i,p/q})$ as

$$\begin{tikzcd}
\cdots & t=s-1 & s & s+1 & \cdots \\
\cdots \arrow[dr, "h^+_*"] & \mathcal{T}^+_{(d_{s-1})} \arrow[d, "v^+_*"] \arrow[dr, "h^+_*"] & \mathcal{T}^+_{(d_s)} \arrow[d, "v^+_*"] \arrow[dr, "h^+_*"] & \mathcal{T}^+_{(d_{s+1})} \arrow[d, "v^+_*"] \arrow[dr, "h^+_*"] & \cdots \\
\cdots & \mathcal{T}^+_{(e_{s-1})} & \mathcal{T}^+_{(e_s)} & \mathcal{T}^+_{(e_{s+1})} & \cdots
\end{tikzcd}$$

for some integers $e_{s-1},\ e_s,\ e_{s+1}$.

Let $n = \lfloor \frac{i+ps}{q} \rfloor$. If $n \geq 0$ then $H_n(K) = n$, so the h^+_* map with domain in column s above has the form

$$(h^+_n)_* : \mathcal{T}^+_{(d_s)} \xrightarrow{U^n} \mathcal{T}^+_{(e_{s+1})},$$

sending a generator in degree $d_s + 2n$ to one in degree e_{s+1}, so we have

$$(d_s + 2n) - 1 = e_{s+1}.$$

But then $\lfloor \frac{i+p(s+1)}{q} \rfloor \geq n \geq 0$, so the v^+_* map in column $s+1$ is identified with the identity map $\mathcal{T}^+_{(d_{s+1})} \to \mathcal{T}^+_{(e_{s+1})}$ and thus

$$d_{s+1} = e_{s+1} + 1 = d_s + 2n.$$

Similarly, if $n \leq 0$ then we have $H_{\lfloor \frac{i+p(s-1)}{q} \rfloor} = 0$ and $V_n = -n$, hence

$$(d_s + 2(-n)) - 1 = e_s = d_{s-1} - 1,$$

or $d_s = d_{s-1} + 2n$.

In the case where $\lfloor \frac{i+ps}{q} \rfloor \leq 0$ and $\lfloor \frac{i+p(s+1)}{q} \rfloor \geq 0$, we note that the h^+_* and v^+_* maps into the $\mathcal{T}^+_{(e_{s+1})}$ tower in column $s+1$ are both modeled on multiplication by 1, since $H_{\lfloor \frac{i+ps}{q} \rfloor}(K) = 0$ and $V_{\lfloor \frac{i+p(s+1)}{q} \rfloor}(K) = 0$. Thus

$$d_s = e_{s+1} + 1 = d_{s+1},$$

completing the proof. $\square$

Lemma 6.7. *Assume the hypotheses and notation of Lemma 6.6, and let*

$$s_0 = \min \left\{ t \in \mathbb{Z} \,\middle|\, \left\lfloor \frac{i+pt}{q} \right\rfloor \geq 0 \right\}.$$

Fix integers s and s' satisfying the hypotheses of Proposition 6.1, and consider the map

$$(D^{[s,s']}_{i,p/q})_* : H_*(\mathbb{A}^{[s,s']}_{i,p/q}) \to H_*(\mathbb{B}^{[s,s']}_{i,p/q})$$

between the homologies of the corresponding truncated complexes. If $s \leq s_0 \leq s'$, then

$$\ker(D^{[s,s']}_{i,p/q})_* \cap \ker(U)$$

contains a $\mathbb{Q}$ submodule in grading d_{s_0}.

Proof. Consider the restriction of $(D^{[s,s']}_{i,p/q})_*$ to the sum of all the towers $(t, \mathcal{T}^+_{(d_t)}) \subset H_*(\mathbb{A}^{[s,s']}_{i,p/q})$. By hypothesis we have

$$\left\lfloor \frac{i+p(s_0-1)}{q} \right\rfloor < 0 \quad \text{and} \quad \left\lfloor \frac{i+ps_0}{q} \right\rfloor \geq 0,$$

so Lemma 6.6 says that the sequence of gradings $\big(d_t\big)$ satisfies

$$\cdots > d_s > d_{s+1} > \cdots > d_{s_0-1} = d_{s_0} \leq d_{s_0+1} \leq \cdots \leq d_{s'} \leq \cdots .$$

Let

$$s_1 = \max\{t \in \mathbb{Z} \mid d_t = d_{s_0}\},$$

so that for all $t \in \{s, \ldots, s'\}$, we have $d_t = d_{s_0}$ if and only if $s_0 - 1 \leq t \leq s_1$. Then near the indices $[s_0-1, s_1+1]$, the restriction of $(D^{[s,s']}_{i,p/q})_*$ has the form

$$\begin{tikzcd}
t = s_0 - 1 & s_0 & \cdots & s_1 & s_1 + 1 \\
\mathcal{T}^+_{(d_{s_0})} \arrow[d, "v^+_*"] \arrow[dr, "1"] & \mathcal{T}^+_{(d_{s_0})} \arrow[d, "1"] \arrow[dr, "1"] & \cdots \arrow[dr, "1"] & \mathcal{T}^+_{(d_{s_0})} \arrow[d, "1"] \arrow[dr, "h^+_*"] & \mathcal{T}^+_{(d_{s_1+1})} \arrow[d, "v^+_*"] \\
\mathcal{T}^+ & \mathcal{T}^+_{(d_{s_0-1})} & \cdots & \mathcal{T}^+_{(d_{s_0-1})} & \mathcal{T}^+
\end{tikzcd}$$

in which we omit any columns at either end whose indices are not in $[s, s']$.

To see that the maps labeled "1" are indeed modeled on multiplication by $U^0 = 1$, we note that they are one of

- an h^+_* map with domain in column $s_0 - 1$, and then since $\lfloor \frac{i+p(s_0-1)}{q} \rfloor < 0$ we have $H_{\lfloor \frac{i+p(s_0-1)}{q} \rfloor}(K) = 0$;
- a v^+_* map with domain in column $t \geq s_0$, and then since $\lfloor \frac{i+pt}{q} \rfloor \geq 0$ we have $V_{\lfloor \frac{i+pt}{q} \rfloor}(K) = 0$; or
- an h^+_* map from column $t \geq s_0$ to column $t+1$ where $d_t = d_{t+1} = d_{s_0}$, and then Lemma 6.6 says that
$$0 = d_{t+1} - d_t = 2 \left\lfloor \frac{i+pt}{q} \right\rfloor,$$
so that $H_{\lfloor \frac{i+pt}{q} \rfloor}(K) = H_0(K) = 0$.

Moreover, the v^+_* map in column $s_0 - 1$ is modeled on multiplication by U^a, where

$$a = V_{\lfloor \frac{i+p(s_0-1)}{q} \rfloor}(K) = -\left\lfloor \frac{i+p(s_0-1)}{q} \right\rfloor \geq 1.$$

Similarly the h^+_* map in column s_1 is modeled on multiplication by U^b, where

$$\begin{aligned} b = H_{\lfloor \frac{i+ps_1}{q} \rfloor}(K) &= \left\lfloor \frac{i+ps_1}{q} \right\rfloor \\ &= \tfrac{1}{2}\left(d_{s_1+1} - d_{s_1}\right) > 0, \end{aligned}$$

by Lemma 6.6 and the definition of s_1.

We now label generators at the bottom of each tower by

$$x_t \in (t, \mathcal{T}^+_{(d_t)}) \subset H_*(\mathbb{A}^{[s,s']}_{i,p/q}), \qquad\qquad y_t \in (t, \mathcal{T}^+) \subset H_*(\mathbb{B}^{[s,s']}_{i,p/q}),$$

so that $Ux_t = 0$ and $Uy_t = 0$ for all t, and the various v^+_* and h^+_* maps carry elements of the form $U^i x_t$ to elements of the form $U^j y_t$ and $U^k y_{t+1}$ respectively. We then define

$$z = \sum_{t=s_0-1}^{s_1} (-1)^t x_t \subset H_{d_{s_0}}(\mathbb{A}^{[s,s']}_{i,p/q}),$$

treating any terms whose indices are not in $[s,s']$ as zero, and it follows from the above discussion that $Uz = 0$ and that

$$(D^{[s,s']}_{i,p/q})_*(z) = (-1)^{s_0-1} y_{s_0} + \left(\sum_{t=s_0}^{s_1-1} (-1)^t (y_t + y_{t+1}) \right) + (-1)^{s_1} y_{s_1} = 0.$$

Thus z generates the desired $\mathbb{Q}$ summand. $\square$

Lemma 6.8. *Assume the hypotheses and notation of Lemma 6.6, and let $s \le s'$ be integers satsifying the hypotheses of Proposition 6.1. Suppose that*

$$\left\lfloor \frac{i+ps'}{q} \right\rfloor = g-1.$$

If $d \in \mathbb{Z}$ denotes the integer such that $H_*(A^+_{g-1}) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}_{(d)}$, as in Lemma 5.4, then we can write

$$(s', H_*(A^+_{g-1})) \cong \mathcal{T}^+_{(d_{s'})} \oplus \mathbb{Q}_{(d_{s'}+d)}$$

as $\mathbb{Q}[U]$-modules such that the $\mathbb{Q}_{(d_{s'}+d)}$ summand lies in

$$\ker(D^{[s,s']}_{i,p/q})_* \cap \ker(U).$$

Proof. The rightmost portion of the truncated mapping cone complex has the form

$$\begin{tikzcd}
\cdots & t = s'-1 & s' \\
\cdots \arrow[dr, "h^+_*"'] & H_*(A^+_{\lfloor \frac{i+p(s'-1)}{q} \rfloor}) \arrow[d, "v^+_*"] \arrow[dr, "h^+_*"] & \mathcal{T}^+_{(d_{s'})} \oplus \mathbb{Q}_{(d_{s'}+d)} \arrow[d, "(v^+_{g-1})_*"] \\
\cdots & \mathcal{T}^+ & \mathcal{T}^+_{(d_{s'}-1)},
\end{tikzcd}$$

where the grading on the bottom $\mathcal{T}^+$ in column s' follows from $V_{g-1}(K) = 0$. Let $x_{s'}$ and $y_{s'}$ be bottom-most elements of the towers at the top and bottom of column s', chosen so that $(v^+_{g-1})_*(x_{s'}) = y_{s'}$.

Let z generate the $\mathbb{Q}_{(d_{s'}+d)}$ summand in column s', so that $Uz = 0$. If $(v^+_{g-1})_*(z) = 0$ then we are done, since z generates the desired submodule. Otherwise, we observe that

$$U \cdot (v^+_{g-1})_*(z) = (v^+_{g-1})_*(Uz) = (v^+_{g-1})_*(0) = 0,$$

so $(v^+_{g-1})_*(z)$ must be a nonzero element at the bottom of the $\mathcal{T}^+_{(d_{s'}-1)}$ tower. In this case, we can write

$$(v^+_{g-1})_*(z) = \lambda y_{s'} = \lambda \cdot (v^+_{g-1})_*(x_{s'})$$

for some nonzero $\lambda \in \mathbb{Q}$. For grading reasons it now follows that $d = 0$, since z must lie in grading $d_{s'}$, and so

$$z - \lambda x_{s'} \in \ker(v^+_{g-1})_*.$$

Now we can write the $H_*(A^+_{g-1})$ in column s' as the $\mathbb{Q}[U]$-module

$$\mathcal{T}^+\langle x_{s'}\rangle \oplus \mathbb{Q}\langle z - \lambda x_{s'}\rangle,$$

and the $\mathbb{Q}$ summand is in $\ker(v^+_{g-1})_* = \ker(D^{[s,s']}_{i,p/q})_*$ as well as $\ker(U)$, as desired. □

6.4. **Small positive surgeries.** In Proposition 6.5 we showed that if $0 < r < 1$, then there is a relatively graded isomorphism of the form

$$HF^+(S^3_r(5_2), i) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}^{2n_i}_{(0)}$$

for all i. We will show that this cannot be the case for $HF^+(S^3_r(K))$ if K is a knot of genus at least 2 that satisfies the hypotheses of Proposition 5.3.

Proposition 6.9. *Let K be a knot of genus $g \geq 2$, and fix relatively prime positive integers $q > p > 0$. Then*

$$HF^+(S^3_{p/q}(K)) \not\cong HF^+(S^3_{p/q}(5_2))$$

as absolutely graded $\mathbb{Q}[U]$-modules.

Proof. If $HF^+(S^3_{p/q}(K)) \cong HF^+(S^3_{p/q}(5_2))$, then K satisfies the conclusions of Proposition 5.3. In this case, Proposition 6.5 says that for all i, the submodule

$$\ker(U) \subset HF^+(S^3_{p/q}(5_2), i)$$

lies in a single homological grading. Thus the same must be true for

$$\ker(U) \subset HF^+(S^3_{p/q}(K), i),$$

so we will find an integer i for which this is not the case, giving a contradiction.

We fix an integer i between 0 and $p-1$, inclusive, such that

$$i \equiv gq - 1 \pmod{p}.$$

We then define

$$s = \min\{t \in \mathbb{Z} \mid i + pt \geq (1-g)q\}, \qquad\qquad s' = \frac{gq-1-i}{p}.$$

By construction we have

$$\left\lfloor \frac{i+ps'}{q} \right\rfloor = g-1 \quad \text{and} \quad \left\lfloor \frac{i+p(s'+1)}{q} \right\rfloor \geq g,$$

and since $1 \leq p+1 \leq q$ we have

$$\left\lfloor \frac{i+p(s'-1)}{q} \right\rfloor = \left\lfloor \frac{gq-(p+1)}{q} \right\rfloor = g-1$$

as well. We also observe that $\lfloor \frac{i+pt}{q} \rfloor \geq 0$ if and only if $t \geq 0$, and so $s \leq t \leq s'$.

According to Proposition 6.1, we can identify $HF^+(S^3_{p/q}(K), i)$ with the kernel of

$$(D^{[s,s']}_{i,p/q})_* : H_*(\mathbb{A}^{[s,s']}_{i,p/q}) \to H_*(\mathbb{B}^{[s,s']}_{i,p/q})$$

up to an overall grading shift, so it will suffice to show that

$$\ker(D^{[s,s']}_{i,p/q})_* \cap \ker(U)$$

does not lie in a single homological grading. Supposing otherwise, we choose an arbitrary lift of the relative gradings on $\mathbb{A}^{[s,s']}_{i,p/q}$ and $\mathbb{B}^{[s,s']}_{i,p/q}$ to an absolute $\mathbb{Z}$-grading, and let $d_t \in \mathbb{Z}$ denote the bottom-most grading in each tower

$$\mathcal{T}^+_{(d_t)} \subset (t, H_*(A^+_{\lfloor \frac{i+pt}{q} \rfloor})) \subset H_*(\mathbb{A}^{[s,s']}_{i,p/q}).$$

Lemma 6.7 now says that there is a $\mathbb{Q}$-submodule of $\ker(D^{[s,s']}_{i,p/q})_*$ in grading d_{s_0}, and Lemma 6.8 says that there is also a $\mathbb{Q}$-submodule in grading $d_{s'} + d$, hence

$$d_{s'} + d = d_{s_0}$$

by hypothesis. But according to Lemma 6.6, we also have

$$\begin{aligned} d_{s'} &= d_{s'-1} + 2\left\lfloor \frac{i+p(s'-1)}{q} \right\rfloor \\ &= d_{s'-1} + 2(g-1) \\ &\geq d_{s_0} + 2(g-1), \end{aligned}$$

so then $d = d_{s_0} - d_{s'} \leq 2 - 2g$.

We now examine the rightmost portion of the truncated complex $\mathbb{X}^{[s,s']}_{i,p/q}$. Since $\lfloor \frac{i+p(s'-1)}{q} \rfloor = g-1$, the last two columns have the form

$$\begin{tikzcd}
\cdots & t = s'-1 & s' \\
\cdots \arrow[dr, "h^+_*"'] & \mathcal{T}^+_{(d_{s'-1})} \oplus \mathbb{Q}_{(d_{s'-1}+d)} \arrow[d, "(v^+_{g-1})_*"] \arrow[dr, "(h^+_{g-1})_*"] & \mathcal{T}^+_{(d_{s'})} \oplus \mathbb{Q}_{(d_{s'}+d)} \arrow[d, "(v^+_{g-1})_*"] \\
\cdots & \mathcal{T}^+_{(d_{s'-1}-1)} & \mathcal{T}^+_{(d_{s'}-1)},
\end{tikzcd}$$

with $d_{s'} = d_{s'-1} + 2(g-1)$ as above. Since $d \leq 2 - 2g \leq -2$, the map $(v^+_{g-1})_*$ in column $s'-1$ must send the $\mathbb{Q}_{(d_{s'-1}+d)}$ submodule to zero for grading reasons. That same submodule must be sent by $(h^+_{g-1})_*$ into column s', in grading

$$\begin{aligned} d_{s'-1} + d - 1 &= (d_{s'} - 2(g-1)) + d - 1 \\ &= (d_{s'} + d) + (1 - 2g) \\ &\leq d_{s'} - 1 - 2g. \end{aligned}$$

This is strictly less than the bottom-most grading of the corresponding tower, so this image also must be zero, and it follows that in column $s'-1$ we have

$$\mathbb{Q}_{(d_{s'-1}+d)} \subset \ker(D^{[s,s']}_{i,p/q})_* \cap \ker(U)$$

as well. Since

$$d_{s'-1} + d = (d_{s'} + d) - (2g-2) < d_{s'} + d,$$

it follows that $\ker(D^{[s,s']}_{i,p/q})_* \cap \ker(U)$ is not supported in a single grading, and so we have a contradiction. $\square$

6.5. **Large positive surgeries.** In this subsection we attempt to understand when there can be a homeomorphism

$$S^3_r(K) \cong S^3_r(5_2)$$

for some slope $r \geq 1$ and some knot K of genus at least 2. We implicitly identify

$$\mathrm{Spin}^c(S^3_{p/q}(K)) \cong \mathbb{Z}/p\mathbb{Z}$$

throughout, as in the statement of Theorem 2.5.

The following lemma will help us find Spin^c structures $\mathfrak{s}$ where $HF^+(S^3_{p/q}(K), \mathfrak{s})$ differs from HF^+ of any Spin^c structure on $S^3_{p/q}(5_2)$.

Lemma 6.10. *Let $g \geq 2$ be an integer, and let $p > q > 0$ be relatively prime positive integers such that p does not divide $2g-2$. Then there exists an integer $i \in \mathbb{Z}$ for which the equation*

$$\left\lfloor \frac{i+ps}{q} \right\rfloor = g-1$$

has an integer solution $s \in \mathbb{Z}$, but the equation

$$\left\lfloor \frac{i+ps}{q} \right\rfloor = 1-g$$

does not.

Proof. We note that $\lfloor \frac{i+ps}{q} \rfloor = g-1$ admits a solution $s \in \mathbb{Z}$ if and only if

$$q(g-1) \leq i + ps \leq qg - 1,$$

or equivalently if and only if

$$i \equiv qg - j \pmod{p} \text{ for some } j \in \{1, 2, \dots, q\}. \tag{6.1}$$

Similarly, the equation $\lfloor \frac{i+ps}{q} \rfloor = 1-g$ has a solution $s \in \mathbb{Z}$ if and only if

$$q(1-g) \leq i + ps \leq q(2-g) - 1,$$

or equivalently if and only if

$$i \equiv q(2-g) - k \pmod{p} \text{ for some } k \in \{1, 2, \dots, q\}. \tag{6.2}$$

Each of (6.1) and (6.2) is solved by exactly q residue classes modulo p, and these solutions coincide if and only if

$$qg \equiv q(2-g) \pmod{p},$$

which is equivalent to $2g-2 \equiv 0 \pmod{p}$ since p and q are coprime. But we have assumed that this is not the case, so the set of i in (6.1) is not a subset of the set in (6.2), and hence there is some i which satisfies (6.1) but not (6.2). This is the desired i. ☐

Proposition 6.11. *Let $K \subset S^3$ be a nontrivial knot of genus $g \geq 2$, and let $p \geq q > 0$ be relatively prime positive integers. If there is an isomorphism*

$$HF^+(S^3_{p/q}(K)) \cong HF^+(S^3_{p/q}(5_2))$$

of graded $\mathbb{Q}[U]$-modules, then p divides $2g-2$.

Proof. We suppose that $p \nmid 2g-2$. Then Lemma 6.4 says that

$$\dim_{\mathbb{Q}} HF^+_{\mathrm{red}}(S^3_{p/q}(5_2), i) \cong 0 \text{ or } 2 \text{ for all } i,$$

so for the sake of a contradiction it will suffice to find i such that $HF^+_{\mathrm{red}}(S^3_{p/q}(K), i)$ is 1-dimensional. We start by applying Lemma 6.10 to find $i \in \mathbb{Z}$ and $s' \in \mathbb{Z}$ such that

$$\left\lfloor \frac{i+ps'}{q} \right\rfloor = g-1$$

and such that $\lfloor \frac{i+pt}{q} \rfloor = 1-g$ has no solutions $t \in \mathbb{Z}$; this will be the desired i.

Let s be the least integer satisfying

$$\left\lfloor \frac{i+ps}{q} \right\rfloor \geq 0.$$

Then $h^+_{\lfloor \frac{i+pt}{q} \rfloor}$ is a quasi-isomorphism for all $t < s$, since then $\lfloor \frac{i+pt}{q} \rfloor$ is negative but not equal to $1-g$; if $1-g < \lfloor \frac{i+pt}{q} \rfloor < 0$ then this is part of Proposition 5.3, and if $\lfloor \frac{i+pt}{q} \rfloor < 1-g$ then this is true for arbitrary genus-g knots. Likewise $v^+_{\lfloor \frac{i+pt}{q} \rfloor}(K)$ is a quasi-isomorphism for all $t > s'$, since then $\lfloor \frac{i+pt}{q} \rfloor \geq g$. Thus Proposition 6.1 says that $HF^+(S^3_{p/q}(K), i)$ is isomorphic to the kernel of the truncated map

$$(D^{[s,s']}_{i,p/q})_* : H_*(\mathbb{A}^{[s,s']}_{i,p/q}) \to H_*(\mathbb{B}^{[s,s']}_{i,p/q}).$$

The domain is a sum of relatively graded $\mathbb{Q}[U]$-modules of the form

$$H_*(A^+_{\lfloor \frac{i+pt}{q} \rfloor}) \cong \begin{cases} \mathcal{T}^+, & s \leq t < s' \\ \mathcal{T}^+ \oplus \mathbb{Q}, & t = s', \end{cases}$$

and we know that $H_*(B^+) \cong \mathcal{T}^+$, so $(D^{[s,s']}_{i,p/q})_*$ has the form

$$\begin{tikzcd}
t=s & s+1 & \dots & s'-1 & s' \\
\mathcal{T}^+ \arrow[dr, "h_*^+"] & \mathcal{T}^+ \arrow[d, "v_*^+"] \arrow[dr, "h_*^+"] & \cdots \arrow[dr, "h_*^+"] & \mathcal{T}^+ \arrow[d, "v_*^+"] \arrow[dr, "h_*^+"] & \mathcal{T}^+ \oplus \mathbb{Q} \arrow[d, "(v_{g-1}^+)_*"] \\
& \mathcal{T}^+ & \cdots & \mathcal{T}^+ & \mathcal{T}^+.
\end{tikzcd}$$

Lemma 6.8 says that we can arrange for the $\mathbb{Q}$ summand in column s' above to belong to $\ker(D^{[s,s']}_{i,p/q})_*$. Having done so, we see that $\ker(D^{[s,s']}_{i,p/q})_*$ is isomorphic as a $\mathbb{Q}[U]$-module to the direct sum of that $\mathbb{Q}$ with the kernel of

$$\begin{tikzcd}
t=s & s+1 & \dots & s'-1 & s' \\
\mathcal{T}^+ \arrow[dr, "h_*^+"] & \mathcal{T}^+ \arrow[d, "1"] \arrow[dr, "h_*^+"] & \cdots \arrow[dr, "h_*^+"] & \mathcal{T}^+ \arrow[d, "1"] \arrow[dr, "h_*^+"] & \mathcal{T}^+ \arrow[d, "1"] \\
& \mathcal{T}^+ & \cdots & \mathcal{T}^+ & \mathcal{T}^+.
\end{tikzcd}$$

(Here each v_*^+ map is identified with multiplication by $U^0 = 1$, since $t \geq s$ implies that $\lfloor \frac{i+pt}{q} \rfloor \geq 0$ and hence $V_{\lfloor \frac{i+pt}{q} \rfloor}(K) = 0$.) But this kernel can be identified with the $\mathcal{T}^+$ in column s, so now we apply Proposition 6.1 to conclude that

$$HF^+(S^3_{p/q}(K), i) \cong \ker(D^{[s,s']}_{i,p/q})_* \cong \mathcal{T}^+ \oplus \mathbb{Q}$$

up to an overall grading shift. This says that $HF^+_{\mathrm{red}}(S^3_{p/q}(K), i) \cong \mathbb{Q}$, which gives the desired contradiction. $\square$

Proposition 6.11 takes care of most slopes $r \geq 1$ (for knots of a fixed genus g) without making use of gradings on the mapping cone complex. By being careful about gradings, we can handle the remaining non-integral cases as well.

Proposition 6.12. *Let K be a nontrivial knot of genus $g \geq 2$, and let $p \geq q > 0$ be relatively prime positive integers. If there is an isomorphism*

$$HF^+(S^3_{p/q}(K)) \cong HF^+(S^3_{p/q}(5_2))$$

of graded $\mathbb{Q}[U]$-modules, then $q = 1$ and p divides $2g - 2$.

Proof. Proposition 6.11 tells us that p divides $2g-2$, so it remains to be seen that $q = 1$. We will assume to the contrary that $q \geq 2$. If we write $e = \frac{2g-2}{p}$ then $\frac{p}{q} = \frac{2g-2}{qe}$, and the assumption $q \geq 2$ means that $\frac{p}{q}$ is neither $2g-2$ nor $g-1$, so it follows that $qe \geq 3$, or $\frac{p}{q} \leq \frac{2g-2}{3}$.

As usual, we will take $d \in \mathbb{Z}$ such that

$$H_*(A^+_{g-1}) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}_{(d)},$$

as guaranteed by Lemma 5.4. This integer d depends only on K, which is the key fact we will use below to rule out any case where $q \geq 2$.

Fixing some choice of

$$i = q(g-1) + j, \quad j = 0, 1, \dots, q-1,$$

we take $s = -qe$ and $s' = 0$, and then we have

$$\left\lfloor \frac{i+ps}{q} \right\rfloor = \left\lfloor \frac{(q(g-1)+j) - pqe}{q} \right\rfloor = \left\lfloor g - 1 - pe + \frac{j}{q} \right\rfloor = 1 - g$$

and $\left\lfloor \frac{i+ps'}{q} \right\rfloor = g - 1$, while (since $\frac{p}{q} \geq 1$) $\lfloor \frac{i+p(s-1)}{q} \rfloor \leq -g$ and $\lfloor \frac{i+p(s'+1)}{q} \rfloor \geq g$. Thus

$$HF^+(S^3_{p/q}(K), i) \cong \ker\left((D^{[s,s']}_{i,p/q})_* : H_*(\mathbb{A}^{[s,s']}_{i,p/q}) \to H_*(\mathbb{B}^{[s,s']}_{i,p/q})\right)$$

by Proposition 6.1. We put an absolute $\mathbb{Z}$-grading on the truncated mapping cone complex $\mathbb{X}^{[s,s']}_{i,p/q}$, with d_t denoting the bottom-most grading for the tower in each summand $(t, H_*(A^+_{\lfloor \frac{i+pt}{q} \rfloor}))$ as usual, and we let

$$s_0 = \min\left\{ t \in \mathbb{Z} \,\middle|\, \left\lfloor \frac{i+pt}{q} \right\rfloor \geq 0 \right\}.$$

Then Lemmas 6.7 and 6.8 tell us that

$$\ker(D^{[s,s']}_{i,p/q})_* \cap \ker(U)$$

contains $\mathbb{Q}$ submodules in gradings d_{s_0} and $d_{s'} + d$ respectively. But by Proposition 6.4 these gradings must be the same, so we have

$$-d = d_{s'} - d_{s_0}.$$

We remark that since $\frac{p}{q} \leq \frac{2g-2}{3}$, it follows that $s_0 \leq s' - 1$.

We now attempt to work out this value in more detail. According to Lemma 6.6, we have

$$-d = d_{s'} - d_{s_0} = \sum_{t=s_0}^{s'-1} (d_{t+1} - d_t) = 2 \sum_{t=s_0}^{s'-1} \left\lfloor \frac{i+pt}{q} \right\rfloor,$$

which, since $i = q(g-1) + j$, can be written as

$$-d = 2 \sum_{t=s_0}^{s'-1} \left((g-1) + \left\lfloor \frac{j+pt}{q} \right\rfloor \right). \tag{6.3}$$

We note that

$$\begin{aligned} \left\lfloor \frac{i+pt}{q} \right\rfloor \geq 0 &\iff (q(g-1)+j) + pt \geq 0 \\ &\iff t \geq -q\left(\frac{g-1}{p}\right) - \frac{j}{p} = -q\left(\frac{e}{2}\right) - \frac{j}{p} \end{aligned}$$

and so we have

$$s_0 = \left\lceil -\frac{qe}{2} - \frac{j}{p} \right\rceil = -\left\lfloor \frac{qe}{2} + \frac{j}{p} \right\rfloor, \tag{6.4}$$

while $s' - 1 = -1$ since $s' = 0$ by definition. This makes it clear that while the various d_t may have depended on our choice of i and on the absolute grading on $\mathbb{X}^{[s,s']}_{i,p/q}$, the expression (6.3) for d depends only on p, q, g, and our choice of $j \in \{0, 1, \ldots, q-1\}$. But we have already remarked that d depends only on K, so we will show that different choices of j lead to different values of d and thus get a contradiction.

Supposing first that $q \cdot e$ is even, we have $\frac{qe}{2} \in \mathbb{Z}$ while $0 \leq \frac{j}{p} \leq \frac{q-1}{p} < 1$, and so

$$s_0 = -q\left(\frac{e}{2}\right) \quad \text{for } j = 0, 1, 2, \ldots, q-1.$$

In particular, the indices in the sum (6.3) are the same for each such choice of j, and the individual summands are monotonically increasing in j. But the value of d must be independent of j, so the sum in (6.3) must be the same term-by-term for $j = 0$ as it is for $j = q - 1$. Thus we have

$$\left\lfloor \frac{0+pt}{q} \right\rfloor = \left\lfloor \frac{(q-1)+pt}{q} \right\rfloor \quad \text{for } s_0 \leq t \leq s' - 1.$$

And this in turn requires that $0 + pt$ be a multiple of q: otherwise, there will be some $j \in \{1, \ldots, q-1\}$ such that $j + pt$ is a multiple of q, and then we have

$$\left\lfloor \frac{0+pt}{q} \right\rfloor \leq \left\lfloor \frac{j-1+pt}{q} \right\rfloor < \left\lfloor \frac{j+pt}{q} \right\rfloor \leq \left\lfloor \frac{(q-1)+pt}{q} \right\rfloor.$$

In the case $t = -1$ it follows that $-p$ is a multiple of q, but since p and q are coprime and $q \geq 2$ this is impossible.

In the remaining case both q and $e = \frac{2g-2}{p}$ are odd, so in particular p must be even. In this case $\frac{qe}{2}$ is a half-integer, with floor $\frac{qe-1}{2} \geq 1$ since $q > 1$, so we compute from (6.4) that

$$s_0 = \begin{cases} -\lfloor \frac{qe}{2} \rfloor, & 0 \leq j \leq \frac{p}{2} - 1 \\ -\lfloor \frac{qe}{2} \rfloor - 1, & \frac{p}{2} \leq j \leq q-1. \end{cases}$$

(We note that $q - 1 < p$, so that $0 \leq \frac{j}{p} < 1$ for all such j.) If the second possibility occurs then the $t = s_0$ term in the sum (6.3) is

$$\begin{aligned} g - 1 + \left\lfloor \frac{j + ps_0}{q} \right\rfloor &= g - 1 + \left\lfloor \frac{j + p(-\frac{qe+1}{2})}{q} \right\rfloor \\ &= g - 1 + \left\lfloor \frac{j - q(g-1) - \frac{p}{2}}{q} \right\rfloor \\ &= \left\lfloor \frac{j - \frac{p}{2}}{q} \right\rfloor = 0, \end{aligned}$$

so we may as well omit it and begin with $t = s_0 + 1 = -\lfloor \frac{qe}{2} \rfloor$. Thus either way (6.3) becomes

$$-d = 2 \sum_{t=-\frac{qe-1}{2}}^{-1} \left((g-1) + \left\lfloor \frac{j + pt}{q} \right\rfloor \right)$$

for any of $j = 0, 1, \dots, q-1$. Now by exactly the same argument as in the case $\frac{qe}{2} \in \mathbb{Z}$, we set $t = -1$ and let j be either of 0 and $q - 1$, and we conclude that

$$\left\lfloor \frac{-p}{q} \right\rfloor = \left\lfloor \frac{(q-1) - p}{q} \right\rfloor$$

and then that $-p$ is a multiple of q, giving a contradiction.

We have now found a contradiction in all cases where $p \mid 2g-2$ and $q \geq 2$, so we conclude that $q = 1$ after all. □

6.6. **Conclusion.** Combining earlier results throughout this section and Section 5, we have nearly proved the following.

Theorem 6.13. *Let $K \not\cong 5_2$ be a knot of genus $g \geq 2$ in S^3, and suppose for some rational $r > 0$ that*

$$S^3_r(K) \cong S^3_r(5_2).$$

Then r is an integer dividing $2g-2$. Moreover, in these cases $\widehat{HFK}(K)$ is completely determined by the integers g and

$$d = \begin{cases} -(g-1)\left(\frac{g-1}{r} - 1\right), & r \mid g-1 \\ -\frac{(2g-2-r)^2}{4r}, & r \nmid g-1 \end{cases}$$

as in Proposition 5.8. In particular, K has Alexander polynomial

$$\Delta_K(t) = t^g - 2t^{g-1} + t^{g-2} + 1 + t^{2-g} - 2t^{1-g} + t^{-g}.$$

Proof. We have shown that

$$HF^+(S^3_r(K)) \not\cong HF^+(S^3_r(5_2))$$

in each of the following cases:

- when $0 < r < 1$, by Proposition 6.9;
- when $r = \frac{p}{q} \geq 1$ with $p \nmid 2g-2$, by Proposition 6.11;
- when $r = \frac{p}{q} \geq 1$ is non-integral and $p \mid 2g-2$, by Proposition 6.12.

This leaves only the cases where r is an integer dividing $2g-2$.

In the remaining cases, we once again write $H_*(A^+_{g-1}) \cong \mathcal{T}^+_{(0)} \oplus \mathbb{Q}_{(d)}$, and then $\widehat{HFK}(K)$ is determined by g and d according to Proposition 5.8. Following the argument and notation from the proof of Proposition 6.12, with $(p,q,i,j) = (r,1,g-1,0)$, we set $s' = 0$ and

$$s_0 = -\left\lfloor \frac{e}{2} \right\rfloor = -\left\lfloor \frac{g-1}{p} \right\rfloor$$

as in (6.4). Then by (6.3) we see that d is even, hence Proposition 5.8 determines the Alexander polynomial of K as promised; and we have

$$\begin{aligned}
-d &= 2\sum_{t=s_0}^{-1} \left((g-1) + \left\lfloor \frac{j+pt}{q} \right\rfloor \right) \\
&= (2g-2)|s_0| + 2\sum_{t'=1}^{|s_0|} r \cdot (-t') \\
&= (2g-2)|s_0| - r|s_0|(|s_0|+1).
\end{aligned}$$

When p divides $g-1$ we have $|s_0| = \frac{g-1}{r}$, and thus

$$\begin{aligned}
-d &= \frac{2(g-1)^2}{r} - (g-1)\left(\frac{g-1}{r} + 1\right) \\
&= (g-1)\left(\frac{g-1}{r} - 1\right).
\end{aligned}$$

Otherwise, since p divides $2g-2$ it follows that $\frac{g-1}{p}$ is a half-integer; then

$$s_0 = -\left(\frac{g-1}{p} - \frac{1}{2}\right) = -\frac{2g-2-p}{2p}$$

and p is an even integer. Since $r = p$ we have

$$\begin{aligned}
-d &= \frac{(2g-2)(2g-2-r)}{2r} - \left(\frac{2g-2-r}{2}\right)\left(\frac{2g-2+r}{2r}\right) \\
&= \tfrac{1}{4r}\left(\left(2(2g-2)^2 - 2r(2g-2)\right) - \left((2g-2)^2 - r^2\right)\right) \\
&= \frac{(2g-2-r)^2}{4r}.
\end{aligned}$$

Thus d is exactly as claimed. □

Remark 6.14. We can collapse the Alexander–Maslov bigrading (a,m) on $\widehat{HFK}(K)$ into a single grading $\delta = m - a$. If $S^3_r(K) \cong S^3_r(5_2)$ for some $r > 0$, then according to Proposition 5.8, all of $\widehat{HFK}(K)$ except for a $\mathbb{Q}_{(0)}$ summand in Alexander grading 0 must be supported in δ-grading $d + 2 - g$. Using Theorem 6.13 (for which we recall the assumption $g \geq 2$), we see that if $r \mid g-1$ then

$$d \leq 2 - 2g < g - 2,$$

whereas if $r \nmid g-1$ then

$$d \le 0 \le g-2$$

with equality on the left and on the right if and only if $r = 2g-2$ and $g = 2$, respectively. In any case $\widehat{HFK}(K)$ is supported in non-positive δ-gradings, and it is thin if and only if $g(K) = r = 2$.

7. Quantum obstructions to surgery

Ito [Ito20] used the LMO invariant of closed 3-manifolds to produce obstructions to cosmetic and other surgeries in terms of finite type invariants. These include the coefficients $a_{2n}(K)$ of the Conway polynomial

$$\nabla_K(z) = a_0(K) + a_2(K)z^2 + a_4(K)z^4 + \dots,$$

as well as an invariant $v_3(K) \in \frac{1}{4}\mathbb{Z}$ which is determined by the Jones polynomial of K. In particular, he proved the following, which we will apply to improve Theorem 6.13.

Theorem 7.1 ([Ito20, Corollary 1.3(iv)])**.** *Suppose for some knots $K, K' \subset S^3$ and rational $r \neq 0$ that $S^3_r(K) \cong S^3_r(K')$. Then either*

(1) $a_4(K) = a_4(K')$ and $v_3(K) = v_3(K')$, or
(2) $a_4(K) \neq a_4(K')$ and $v_3(K) \neq v_3(K')$, in which case

$$r = -\frac{5(a_4(K) - a_4(K'))}{4(v_3(K) - v_3(K'))}. \tag{7.1}$$

Remark 7.2. The sign in front of the right side of (7.1) was omitted in [Ito20]. In fact, [Ito20, Theorem 1.2] gives a surgery formula for the degree-2 part $\lambda_2(S^3_r(K))$ of the LMO invariant, in which one of the terms is $-\frac{5a_4(K)}{4} \cdot \frac{1}{r^2}$. In the proof of [Ito20, Corollary 1.3(iv)] this term appears without the minus sign, which accounts for the discrepancy.

In order to apply Theorem 7.1 to a potential surgery $S^3_r(K) \cong S^3_r(5_2)$, we first recall that the Conway polynomial can be recovered from the Alexander polynomial by the relation

$$\Delta_K(t^2) = \nabla_K(t - t^{-1}).$$

In particular, we have

$$\nabla_{5_2}(t - t^{-1}) = 2t^2 - 3 + 2t^{-2} = 1 + 2(t - t^{-1})^2,$$

so $\nabla_{5_2}(z) = 1 + 2z^2$ and thus $a_4(5_2) = 0$. The computation of $a_4(K)$ is more involved.

Lemma 7.3. *Suppose for some knot $K \ncong 5_2$ and $r \in \mathbb{Q}$ that $S^3_r(K) \cong S^3_r(5_2)$. If $g(K) \ge 2$, then $a_2(K) = 2$ and $a_4(K) = (g(K)-1)^2$.*

Proof. Theorem 6.13 tells us that r is a positive integer dividing $2g(K) - 2$, and that if we write

$$f_g(t) = t^g - 2t^{g-1} + t^{g-2} + 1 + t^{2-g} - 2t^{1-g} + t^{-g}$$

for all integers $g \ge 2$, then $\Delta_K(t) = f_{g(K)}(t)$. These polynomials satisfy the relation

$$(f_g(t) - 1)(t + t^{-1}) = (f_{g+1}(t) - 1) + (f_{g-1}(t) - 1)$$

for all $g \ge 3$, and if we write $t = s^2$ then this becomes

$$(f_g(s^2) - 1)\big((s - s^{-1})^2 + 2\big) = f_{g+1}(s^2) + f_{g-1}(s^2) - 2. \tag{7.2}$$

Define polynomials $p_g(z)$ for all $g \geq 2$ such that

$$p_g(s - s^{-1}) = f_g(s^2).$$

We can check that

$$\begin{aligned} p_2(z) &= 1 + 2z^2 + z^4 \\ p_3(z) &= 1 + 2z^2 + 4z^4 + z^6, \end{aligned}$$

and then (7.2) becomes

$$\big(p_g(s - s^{-1}) - 1\big)\big((s - s^{-1})^2 + 2\big) = p_{g+1}(s + s^{-1}) + p_{g-1}(s + s^{-1}) - 2.$$

Substituting $z = s - s^{-1}$, we have

$$p_{g+1}(z) = (z^2 + 2)(p_g(z) - 1) - p_{g-1}(z) + 2 \tag{7.3}$$

for all $g \geq 3$, and moreover $p_{g(K)}(z)$ is the Conway polynomial $\nabla_K(z)$.

We now claim by induction that

$$p_g(K) = 1 + 2z^2 + (g-1)^2 z^4 + O(z^6)$$

for all $g \geq 2$. It is certainly true for $g = 2$ and $g = 3$, and then for $g \geq 3$ we examine (7.3) modulo z^6 to get

$$\begin{aligned} p_{g+1}(z) &\equiv (z^2 + 2)\left(2z^2 + (g-1)^2 z^4\right) - \left(1 + 2z^2 + (g-2)^2 z^4\right) + 2 \\ &\equiv \left((2g^2 - 4g + 4)z^4 + 4z^2\right) - \left((g^2 - 4g + 4)z^4 + 2z^2\right) + 1 \\ &\equiv g^2 z^4 + 2z^2 + 1 \pmod{z^6} \end{aligned}$$

exactly as claimed. But this means that the coefficients $a_2(K)$ and $a_4(K)$ of z^2 and z^4 in $\nabla_K(z) = p_{g(K)}(z)$ are 2 and $(g(K) - 1)^2$ respectively, proving the lemma. $\square$

We can evaluate $v_3(K)$ in terms of the Jones polynomial $V_K(q)$ as follows.

Lemma 7.4. *We have* $4v_3(K) = -\frac{1}{36}(V_K'''(1) + 3V_K''(1))$.

Proof. We note from [Ito20, Lemma 2.1] that if we evaluate the Jones polynomial

$$V_K(q) = \sum_{i \in \mathbb{Z}} c_i q^i$$

at $q = e^h$ and write the corresponding power series as

$$\sum_{n=0}^{\infty} j_n(K) h^n = V_K(e^h) = \sum_{i \in \mathbb{Z}} c_i \left(\sum_{n=0}^{\infty} \frac{(ih)^n}{n!} \right),$$

then $v_3(K) = -\frac{1}{24} j_3(K)$. Comparing h^3-coefficients gives us

$$4v_3(K) = -\frac{1}{6} j_3(K) = -\frac{1}{36} \sum_{i \in \mathbb{Z}} c_i \cdot i^3.$$

At the same time, we have

$$V_K'''(1) + 3V_K''(1) + V_K'(1) = \sum_{i \in \mathbb{Z}} c_i \cdot \big((i^3 - 3i^2 + 2i) + 3(i^2 - i) + i\big) = \sum_{i \in \mathbb{Z}} c_i \cdot i^3,$$

and we know that $V_K'(1) = 0$ [Jon87, §12], so the lemma follows. $\square$

Example 7.5. We know that

$$V_{5_2}(q) = q^{-1} - q^{-2} + 2q^{-3} - q^{-4} + q^{-5} - q^{-6}$$

and since $V'''_{5_2}(1) = 144$ and $V''_{5_2}(1) = -12$, we get $4v_3(5_2) = -3$.

We can use this obstruction to prove that non-characterizing slopes for 5_2 cannot arise from other knots of genus 1.

Proposition 7.6. *Suppose for some knot K of genus 1 and some $r \in \mathbb{Q}$ that $S^3_r(K) \cong S^3_r(5_2)$. Then K is isotopic to 5_2.*

Proof. If $K \not\cong 5_2$ then Proposition 5.3 says that K is either $15n_{43522}$ or $\mathrm{Wh}^-(T_{2,3}, 2)$, up to mirroring. But in these cases we have

$$a_4(K) = a_4(5_2) = 0,$$

since $\Delta_K(t) = \Delta_{5_2}(t) = 2t - 3 + 2t^{-1}$, and yet we can compute from Lemma 7.4 that

$$4v_3(K) = \pm 7 \text{ or } \pm 1$$

respectively, while $4v_3(5_2) = -3$. Thus Theorem 7.1 says that $S^3_r(K) \not\cong S^3_r(5_2)$ after all. □

We can now use Lemmas 7.3 and 7.4 to identify potentially non-characterizing slopes.

Proposition 7.7. *Suppose that $S^3_r(K) \cong S^3_r(5_2)$ for some integer $r \geq 1$, and that K is not isotopic to 5_2. Then the Jones polynomial $V_K(q)$ satisfies $\frac{1}{36}V'''_K(1) \in \mathbb{Z}$, and we have*

$$r = \frac{5(g(K)-1)^2}{\frac{1}{36}V'''_K(1) - 4}.$$

Moreover, if $g(K)$ is even then r divides $g(K) - 1$.

Proof. Write $g = g(K)$. We know that $g \geq 2$ by Proposition 7.6, hence Lemma 7.3 says that $a_4(K) = (g-1)^2$, which is different from $a_4(5_2) = 0$. We thus apply Theorem 7.1 to see that

$$r = -\frac{5(a_4(K) - a_4(5_2))}{4(v_3(K) - v_3(5_2))} = -\frac{5(g-1)^2}{4v_3(K) + 3}.$$

Proposition 2.9 tells us that $\Delta''_K(1) = \Delta''_{5_2}(1) = 4$, so $V''_K(1) = -3\Delta''_K(1) = -12$, again by [Jon87, §12]. Thus

$$4v_3(K) - 4v_3(5_2) = -\frac{1}{36}(V'''_K(1) - 36) + 3 = 4 - \frac{V'''_K(1)}{36},$$

which must be an integer since $4v_3(K)$ is, and this completes the determination of r.

Now supposing that g is even, we have expressed r as a divisor of the odd integer $5(g-1)^2$. Thus r is odd, and it divides $2g-2$ by Theorem 6.13, so it must in fact divide $g-1$ as claimed. □

This last result allows us to complete the proof of Theorem 1.7.

Proof of Theorem 1.7. If $S^3_r(K) \cong S^3_r(5_2)$ but $K \not\cong 5_2$, then Proposition 7.6 says that K has genus $g \geq 2$. In this case Theorem 6.13 says that r is a positive integer dividing $2g-2$, and that $\widehat{HFK}(K)$ has the claimed form. The only remaining claim is that if g is even then r divides $g-1$, and this is part of Proposition 7.7. □

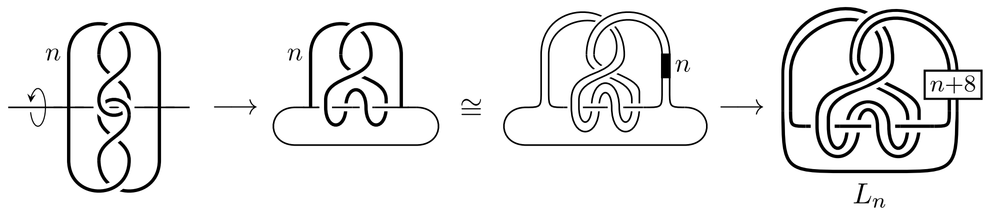


FIGURE 4. A link L_n whose branched double cover is $S^3_n(5_2)$. We quotient 5_2 by a rotation τ around the indicated axis of symmetry, simplify the resulting diagram by an isotopy, and then replace the arc $5_2/\tau$ by a rational tangle. The box labeled "$n+8$" corresponds to $n+8$ signed half-twists.

Remark 7.8. As a final example of the effectiveness of Proposition 7.7, let us suppose that $S^3_r(5_2) \cong S^3_r(P(-3,3,2n))$ for some integers $r \geq 1$ and n. Since $P(-3,3,2n)$ has genus 2, Proposition 7.7 says that $r = 1$. Moreover, an exercise with the skein relation for the Jones polynomial shows that

$$\begin{aligned} V_{P(-3,3,2n)}(q) &= q^{-2n} V_{P(-3,3,0)}(q) + (1 - q^{-2n}) \\ &= -q^{-2n-3} + q^{-2n-2} - q^{-2n-1} + 2q^{-2n} - q^{-2n+1} + q^{-2n+2} - q^{-2n+3} + 1. \end{aligned}$$

(We note that $P(-3,3,0) \cong T_{2,3} \# T_{-2,3}$.) From this one can show that

$$\tfrac{1}{36} V'''_{P(-3,3,2n)}(1) - 4 = 2n - 3,$$

so $r = 1 = \frac{5}{2n-3}$ implies that $2n = 8$.

In Section 8 we will see that $S^3_1(5_2)$ is in fact homeomorphic to $S^3_1(P(-3,3,8))$.

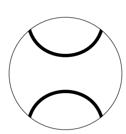
8. Non-characterizing slopes for 5_2

In this section we prove that 1 is not a characterizing slope for 5_2.

Proposition 8.1. *For any integer $n \in \mathbb{Z}$, the 3-manifold $S^3_n(5_2)$ is the branched double cover of the link L_n shown in Figure 4.*

Proof. The knot 5_2 is strongly invertible, meaning that there is an involution $\tau : S^3 \to S^3$ such that $\tau(5_2) = 5_2$, and the fixed set of τ is an unknot U meeting 5_2 in two points. In the quotient $S^3/\tau \cong S^3$, we remove a neighborhood of $5_2/\tau$; this turns U/τ into a tangle with four endpoints, whose branched double cover is $S^3 \setminus N(5_2)$, and we can fill in this tangle by gluing in a rational tangle to get a link L_r whose branched double cover is any Dehn surgery $S^3_r(5_2)$.

This process is illustrated in Figure 4. In order to determine that the box with $n+8$ twists actually corresponds to $S^3_n(5_2)$, we observe that replacing it with the rational tangle



turns L_n into an unknot, whose branched double cover S^3 is the result of $\frac{1}{0}$-surgery on 5_2. Then each possible number of half-twists corresponds to a surgery with slope at distance 1

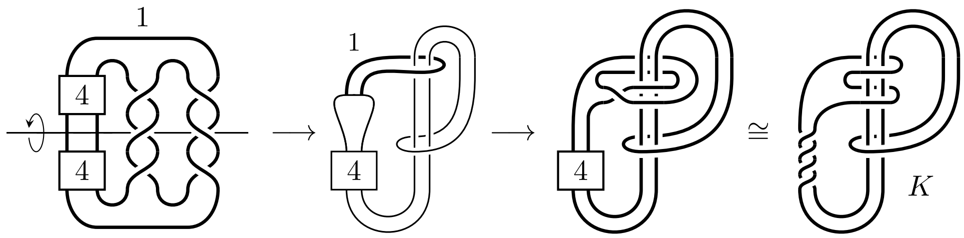


FIGURE 5. Identifying $S^3_1(P(-3,3,8))$ as a branched double cover $\Sigma_2(K)$. We quotient $P(-3,3,8)$ by a rotation τ around an axis of symmetry and simplify by an isotopy, following [BS21, Figure 7]. We then replace a neighborhood of the arc $P(-3,3,8)/\tau$ with a rational tangle, and isotope further to get the desired knot K.

from $\frac{1}{0}$, so these are exactly the integral slopes. We can finally compute that $\det(L_n) = |n|$, so that $\Sigma_2(L_n)$ is identified with $S^3_n(5_2)$ as claimed. □

Remark 8.2. Another construction of links with branched double cover $S^3_n(5_2)$ was given in [BS21, Lemma 8.3], where the argument was specialized to $n = -3$ but works for arbitrary integers. That construction uses a different involution, and hence produces different links (illustrated in [BS21, Figure 12]) in general.

Proposition 8.3. *There is an orientation-preserving homeomorphism*

$$S^3_1(5_2) \cong S^3_1(P(-3,3,8)).$$

Proof. Let $P = P(-3,3,8)$ for convenience. Then P is strongly invertible, and we can adapt the proof of [BS21, Proposition 7.6], which was originally due to Ken Baker, to realize $S^3_1(P)$ as the branched double cover of a knot $K \subset S^3$, as shown in Figure 5.

We now claim that K is isotopic to the knot L_1 from Figure 4, and so

$$S^3_1(P) \cong \Sigma_2(K) \cong \Sigma_2(L_1) \cong S^3_1(5_2)$$

by Proposition 8.1. Rather than find this isotopy explicitly, we observe that SnapPy recognizes each of K and L_1 as either $14n_{14254}$ or its mirror, so that

$$S^3_1(P) \cong \Sigma_2(K) \quad \text{and} \quad S^3_1(5_2) \cong \Sigma_2(L_1)$$

are homeomorphic up to orientation. But we cannot have $S^3_1(5_2) \cong -S^3_1(P)$, since their Casson invariants satisfy

$$\begin{aligned} \lambda(S^3_1(5_2)) &= \tfrac{1}{2}\Delta''_{5_2}(1) = 2, \\ \lambda(-S^3_1(P)) &= \lambda(S^3_{-1}(\overline{P})) = -\tfrac{1}{2}\Delta''_{\overline{P}}(1) = -2. \end{aligned}$$

(This computation follows from $\Delta_{\overline{P}}(t) = t^2 - 2t + 3 - 2t^{-1} + t^{-2}$.) Thus $S^3_1(5_2) \cong S^3_1(P)$ as oriented 3-manifolds. □

9. The $\Sigma(2,3,11)$ realization problem

Let $Y = -\Sigma(2,3,11)$. Then we have orientation-preserving homeomorphisms

$$Y \cong S^3_{1/2}(T_{2,3}) \cong S^3_1(\overline{5_2}).$$

(Up to an overall orientation reversal, the latter identification is the case $S^3_{-1}(K(2,4)) \cong S^3_{-1/2}(K(2,2))$ of [BS21, Proposition 7.2], for example.) Our goal in this section is to prove that these are the only ways to express Y as Dehn surgery on a knot in S^3.

Theorem 9.1. *Suppose for some knot $K \subset S^3$ and some rational $r \in \mathbb{Q}$ that*
$$S^3_r(K) \cong -\Sigma(2,3,11).$$
Then (K,r) is either $(T_{2,3}, \frac{1}{2})$ or $(\overline{5_2}, 1)$.

This is equivalent to Theorem 1.4, as can be seen by the identity $S^3_r(K) \cong -S^3_{-r}(\overline{K})$.

Proof of Theorem 9.1. Since Y is a homology sphere, we can write $r = \frac{1}{n}$ for some nonzero $n \in \mathbb{Z}$. If $n = 1$ and hence $r = 1$, we have
$$HF^+(S^3_1(K)) \cong HF^+(Y) \cong HF^+(S^3_1(\overline{5_2})).$$
We then apply Theorem 1.6 to conclude that $K \cong \overline{5_2}$. Similarly, if $r = \frac{1}{2}$ then we must have $K \cong T_{2,3}$, since all slopes are characterizing slopes for the right-handed trefoil [OS19].

Supposing from now on that n is neither 1 nor 2, we first claim that $n \geq 3$. Indeed, we know that
$$d(S^3_{1/(-n)}(\overline{K})) = d(-S^3_{1/n}(K)) = d(-Y) = 2,$$
where we have read $d(Y) = d(S^3_1(\overline{5_2})) = -2$ off of Proposition 2.10. But if $n < 0$, or equivalently $-n > 0$, then Theorem 2.7 says that
$$d(S^3_{1/(-n)}(\overline{K})) \leq d(S^3_{1/(-n)}(U)) = d(S^3) = 0.$$
This would be a contradiction, so we must have $n > 0$ and hence $n \geq 3$ as claimed.

Now that we have $n \geq 3$, we compute that $\dim \widehat{HF}(Y) = \dim \widehat{HF}(S^3_1(\overline{5_2})) = 3$ from Proposition 2.10 and Lemma 2.12. Thus
$$\begin{aligned} 3 = \dim \widehat{HF}(S^3_{1/n}(K)) &= n \cdot \hat{r}_0(K) + |1 - n\hat{\nu}(K)| \\ &\geq 3 \cdot \hat{r}_0(K) + 1, \end{aligned}$$
since $\hat{r}_0(K) \geq |\hat{\nu}(K)| \geq 0$ and since $1 - n\hat{\nu}(K) \equiv 1 \pmod{n}$ is nonzero. This is only possible if $\hat{r}_0(K) = 0$, in which case $\hat{\nu}(K) = 0$ as well and then $\dim \widehat{HF}(S^3_{1/n}(K))$ must be 1 rather than 3, so we have a contradiction. This completes the proof. ☐

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